ENDO-TRIVIAL MODULES: A REDUCTION TO p'-CENTRAL EXTENSIONS

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Abstract. We examine the behavior of the group of endo-trivial modules under inflation from a quotient modulo a normal subgroup of order prime to p. We prove that everything is controlled by a representation group of the quotient. Examples show that this inflation map is in general not surjective.

1. Introduction

Endo-trivial modules play an important role in the representation theory of finite groups in prime characteristic p. They have been classified in a number of special cases (see the recent papers [CMN15, LM15b] and the references therein). Over an algebraically closed field k of prime characteristic p, endo-trivial modules for a finite group G form an abelian group $T(G)$, which is known to be finitely generated. The main question is to understand the structure of $T(G)$ and of its torsion subgroup $TT(G)$.

We analyze how $T(G)$ and $T(G/O_{p'}(G))$ are related, where $O_{p'}(G)$ denotes as usual the largest normal subgroup of G of order prime to p. Setting $Q := G/O_{p'}(G)$ for simplicity, there is an inflation map

$$\mathrm{Inf}_Q^G : T(Q) \longrightarrow T(G)$$

which is easily seen to be injective. But examples show that it is in general not surjective, so we cannot expect an isomorphism between $T(G)$ and $T(Q)$.

We let $TT(G)$ be the torsion subgroup of $T(G)$ and $X(G)$ be the subgroup of $TT(G)$ consisting of all one-dimensional representations, that is, $X(G) \cong \mathrm{Hom}(G, k^\times)$. We also let $K(G)$ be the kernel of the restriction map $\mathrm{Res}_P^G : T(G) \longrightarrow T(P)$ to a Sylow p-subgroup P of G. It is known that $X(G) \subseteq K(G) \subseteq TT(G)$ and that $K(G) = TT(G)$ in almost all cases (namely if we exclude the cases when a Sylow p-subgroup of G is cyclic, generalized quaternion or semi-dihedral). Moreover, there are numerous cases where $K(G)$ is just equal to $X(G)$, but not always. However, the structure of $K(G)$ is not known in general.

Associated with Q, there is a p'-representation group $\widetilde{Q}$ which is a central extension with kernel of order prime to p. This controls the behavior of projective representations of Q (in the sense of Schur). When Q is a perfect group, then $\widetilde{Q}$ is unique and is also called the universal p'-central extension of Q. When Q is not perfect, then $\widetilde{Q}$ may not be unique.

Our main result is as follows:

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Theorem 1.1. *Let G be a finite group with p-rank at least 2 and no strongly p-embedded subgroups. Let $\widetilde{Q}$ be any p'-representation group of the group $Q := G/O_{p'}(G)$.*

(a) *There exists an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \longrightarrow T(\widetilde{Q})/X(\widetilde{Q}) .$$

In particular, $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$, for any endo-trivial kQ-module W.

(b) *The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \longrightarrow K(\widetilde{Q})/X(\widetilde{Q}) .$$

(c) *In particular, if $K(\widetilde{Q}) \cong X(\widetilde{Q})$, then $K(G) \cong X(G)$.*

Examples show that $\Phi_{G,\widetilde{Q}}$ is in general not surjective (see Section 7), but the theorem provides some information on $K(G)$, for all groups such that $G/O_{p'}(G) = Q$, provided $K(\widetilde{Q})$ is known for the single group $\widetilde{Q}$. We also conjecture that $\Phi_{G,\widetilde{Q}}$ induces an isomorphism on the torsion-free part of $T(G)$ and $T(\widetilde{Q})$ (see Section 5). Moreover, in case Q is perfect, then there is an alternative approach to $\Phi_{G,\widetilde{Q}}$ which we present in Section 6.

The two main assumptions on G in Theorem 1.1 are needed for applying a result of Koshitani and Lassueur [KL15]. However, these assumptions are not really restrictive because endo-trivial modules are completely understood in the two excluded cases: they are classified if the p-rank is 1, by [MT07, CMT13], and $T(G) \cong T(H)$ if G has a strongly p-embedded subgroup H, see [MT07, Lemma 2.7]. The two assumptions also allow us to prove that $T(G) \cong T(G/[G, A])$, where $A = O_{p'}(G)$, or in other words that the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with kernel A of order prime to p can always be replaced by the central extension

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \longrightarrow Q \longrightarrow 1 .$$

This is explained in Section 3.

2. Notation and Preliminaries

Throughout, unless otherwise specified, we use the following notation. We let k denote an algebraically closed field of prime characteristic p. We assume that groups are finite, and that all modules over group algebras are finitely generated. We set $\otimes := \otimes_k$. If G is an arbitrary finite group and V is a kG-module, we denote by $\rho_V : G \longrightarrow \mathrm{GL}(V)$ the corresponding k-representation, and by $\pi_V : \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V)$ the canonical surjection.

Assuming moreover that $p \mid |G|$, we recall that a kG-module V is called *endo-trivial* if there is an isomorphism of kG-modules $\mathrm{End}_k(V) \cong k \oplus (\mathrm{proj})$, where k denotes the trivial kG-module and (proj) some projective kG-module, which might be zero. Any endo-trivial kG-module V splits as a direct sum $V = V_0 \oplus (\mathrm{proj})$ where V_0, the projective-free part of V, is indecomposable and endo-trivial. The relation

$$U \sim V \Longleftrightarrow U_0 \cong V_0$$

is an equivalence relation on the class of endo-trivial kG-modules, and $T(G)$ denotes the resulting set of equivalence classes (which we denote by square brackets). Then $T(G)$, endowed with the law $[U] + [V] := [U \otimes V]$, is an abelian group called the *group of endo-trivial modules of* G. The zero element is the class $[k]$ of the trivial module and $-[V] = [V^*]$, the class of the dual module. By a result of Puig, the group $T(G)$ is known to be a finitely generated abelian group, see e.g. [CMN06, Corollary 2.5].

We let $X(G)$ denote the group of one-dimensional kG-modules endowed with the tensor product $\otimes$, and recall that $X(G) \cong \mathrm{Hom}(G, k^\times) \cong (G/[G,G])_{p'}$. Identifying a one-dimensional module with its class in $T(G)$, we consider $X(G)$ as a subgroup of $T(G)$.

Furthermore, if P is a Sylow p-subgroup of G, we set

$$K(G) = \mathrm{Ker}\left(\mathrm{Res}^G_P : T(G) \longrightarrow T(P)\right) .$$

In other words, the class of an indecomposable endo-trivial kG-module V belongs to $K(G)$ if and only if $V\downarrow^G_P \cong k \oplus (\mathrm{proj})$, that is, in other words, V has trivial source. We have $X(G) \subseteq K(G)$ because any one-dimensional kP-module is trivial. Moreover, $K(G) \subseteq TT(G)$ (see [CMT11a, Lemma 2.3]), and $K(G) = TT(G)$ unless P is cyclic, generalized quaternion, or semi-dihedral, by the main result of [CT05].

By a central extension (E, π) of Q, we mean a group extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \mathrm{Ker}\,\pi$ central in E. Recall that (E, π) is said to have the *projective lifting property (relative to k)* if, for every finite-dimensional k-vector space V, every group homomorphism $\theta : Q \longrightarrow \mathrm{PGL}(V)$ can be completed to a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda|_Z} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \theta} & & \\
1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) & \longrightarrow & 1
\end{array}$$

In general, the homomorphism λ is not uniquely defined. However, by commutativity of the diagram, the following holds:

Lemma 2.1. *In the above situation, if $\lambda, \lambda' : E \longrightarrow \mathrm{GL}(V)$ are two liftings of θ to E, then there exists a degree one representation $\mu : E \longrightarrow \mathrm{GL}(k)$ such that $\lambda' = \lambda \otimes \mu$.*

By results of Schur (slightly generalized for dealing with the case of characteristic p), given a finite group Q, there always exists a central extension (E, π) of Q, with kernel $M_k(Q) := H^2(Q, k^\times)$, which has the projective lifting property. A *p'-representation group of Q* (or a *representation group of Q relative to k*) is a central extension $(\widetilde{Q}, \pi)$ of Q of minimal order with the projective lifting property. In this case $M_k(Q) \cong \mathrm{Ker}\,\pi \leq [\widetilde{Q}, \widetilde{Q}]$. We recall that $M_k(Q) \cong H^2(Q, \mathbb{C}^\times)_{p'}$, the p'-part of the Schur multiplier of Q, and that in general a group Q with $X(Q) \neq 1$ may have several nonisomorphic p'-representation groups. Furthermore, fixing a p'-representation group $(\widetilde{Q}, \pi)$ of Q, the abelian group $M_k(Q)$ becomes isomorphic to its $k^\times$-dual via the transgression homomorphism

$$\mathrm{tr} : \mathrm{Hom}(M_k(Q), k^\times) \longrightarrow H^2(Q, k^\times)$$

defined by $\mathrm{tr}(\varphi) = [\varphi \circ \alpha]$, where the cocycle $\alpha \in Z^2(Q, M_k(Q))$ is in the cohomology class corresponding to the central extension $1 \longrightarrow M_k(Q) \longrightarrow \widetilde{Q} \xrightarrow{\pi} Q \longrightarrow 1$. For further details and proofs we refer the reader to [NT89, Chap. 3, §5] and [CR90, §11E].

If V, W are two finite-dimensional k-vector spaces, then the tensor product of linear maps induces a tensor product $- \otimes - : \mathrm{PGL}(V) \times \mathrm{PGL}(W) \longrightarrow \mathrm{PGL}(V \otimes W)$ via $\pi_V(\alpha) \otimes \pi_W(\beta) := \pi_{V\otimes W}(\alpha \otimes \beta)$ for any $\alpha \in \mathrm{GL}(V)$ and any $\beta \in \mathrm{GL}(W)$. Therefore, if $\mu : Q \longrightarrow \mathrm{PGL}(V)$ and $\nu : Q \longrightarrow \mathrm{PGL}(W)$ are group homomorphisms, we may define a group homomorphism $\mu \otimes \nu : Q \longrightarrow \mathrm{PGL}(V \otimes W)$ via $(\mu \otimes \nu)(q) := \mu(q) \otimes \nu(q)$ for every $q \in Q$. We shall use the following well-known results throughout:

Lemma 2.2. *Let $1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1$ be an arbitrary group extension.*

(a) *Whenever V is a kG-module such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, the group homomorphism $\rho_V : G \to \mathrm{GL}(V)$ induces a uniquely defined group homomorphism $\theta_V : Q \longrightarrow \mathrm{PGL}(V)$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \rho_V|_A} & & \downarrow{\scriptstyle \rho_V} & & \downarrow{\scriptstyle \theta_V} & & \\
1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) & \longrightarrow & 1
\end{array}$$

(b) *If V, W are kG-modules such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$ and $\rho_W(A) \subseteq k^\times \cdot \mathrm{Id}_W$, then $\rho_{V\otimes W}(A) \subseteq k^\times \cdot \mathrm{Id}_{V\otimes W}$ and we have $\theta_{V\otimes W} = \theta_V \otimes \theta_W$.*

Proof. (a) Choose a set-theoretic section $s : Q \longrightarrow G$ for π and define $\theta_V := \pi_V \circ \rho_V \circ s$. Since $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, the map θ_V is a group homomorphism making the diagram commute. Clearly θ_V is uniquely defined since π is an epimorphism.

(b) This is a straightforward computation.

□

3. Endo-trivial modules and central extensions

We now fix G to be a finite group of order divisible by p, we set $A := O_{p'}(G)$ and $Q := G/A$, and we denote by $\pi_G : G \longrightarrow Q$ the quotient map. Moreover, we let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a fixed p'-representation group of Q.

Since A is a p'-subgroup of G, inflation induces an injective group homomorphism

$$\mathrm{Inf}_Q^G : T(Q) \longrightarrow T(G)\,, \qquad [V] \longrightarrow [\mathrm{Inf}_Q^G(V)]\,.$$

This is because the inflation of a projective module remains projective when the kernel A is a p'-group. We emphasize that endo-trivial kG-modules cannot be recovered from endo-trivial kQ-modules, as in general the inflation map Inf_Q^G is not an isomorphism (see Section 7).

Hypothesis 3.1. Assume G is a finite group fulfilling the following two conditions:

(1) the p-rank of G is greater than or equal to 2; and
(2) G has no strongly p-embedded subgroups.

Lemma 3.2. *Suppose that G satisfies Hypothesis 3.1.*

(a) *If V is an indecomposable endo-trivial kG-module, then $V{\downarrow}^G_A \cong Y \oplus \cdots \oplus Y$, where Y is a one-dimensional kA-module.*
(b) *If V is an indecomposable endo-trivial kG-module, then $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$.*
(c) *The inflation map*
$$\mathrm{Inf}^G_{G/[G,A]} : T(G/[G,A]) \longrightarrow T(G)$$
is a group isomorphism.

Proof. (a) Since G satisfies Hypothesis 3.1, any composition factor Y of $V{\downarrow}^G_A$ is G-invariant, by [KL15, Lemma 4.3]. Therefore $V{\downarrow}^G_A \cong Y \oplus \cdots \oplus Y$ and [KL15, Theorem 4.4] proves that $\dim Y = 1$ (using a result of Navarro and Robinson [NR12]).

(b) This is a restatement of (a).

(c) Since $[G,A]$ is a normal p'-subgroup of G, the inflation map $\mathrm{Inf}^G_{G/[G,A]}$ is a well-defined injective group homomorphism. In order to prove that it is surjective, it suffices to prove that $[G,A]$ acts trivially on any indecomposable endo-trivial kG-module V. But by (b) we have
$$\rho_V([G,A]) \subseteq [\rho_V(G), \rho_V(A)] \subseteq [\rho_V(G), k^\times \cdot \mathrm{Id}_V] = \{\mathrm{Id}_V\}\,.$$
Hence $[G,A]$ acts trivially on V. □

Part (c) of Lemma 3.2 is a reduction to the case of central extensions. Explicitly, for the study of endo-trivial modules, we may always replace the given extension
$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$
and consider instead the central extension
$$1 \longrightarrow A/[G,A] \longrightarrow G/[G,A] \longrightarrow Q \longrightarrow 1\,.$$
We shall in fact not use this reduction for the proof of our main result, but rather apply directly Lemma 3.2(b).

Lemma 3.3. *Let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a p'-representation group of Q. Then $X(\widetilde{Q}) = \mathrm{Inf}^{\widetilde{Q}}_Q(X(Q))$, hence $X(\widetilde{Q}) \cong X(Q)$.*

Proof. We apply the fact, mentioned in Section 2, that $\mathrm{Ker}\,\pi_{\widetilde{Q}} \subseteq [\widetilde{Q}, \widetilde{Q}]$. This implies that any one-dimensional representation of $\widetilde{Q}$ has $\mathrm{Ker}\,\pi_{\widetilde{Q}}$ in its kernel, hence is inflated from $\widetilde{Q}/\,\mathrm{Ker}\,\pi_{\widetilde{Q}} \cong Q$.

Another way of seeing the same thing is to associate to the central extension
$$1 \longrightarrow M_k(Q) \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$
the Hochschild-Serre five-term exact sequence
$$1 \longrightarrow \mathrm{Hom}(Q,k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(\widetilde{Q},k^\times) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(M_k(Q),k^\times) \xrightarrow{\mathrm{tr}} H^2(Q,k^\times) \xrightarrow{\mathrm{Inf}} H^2(\widetilde{Q},k^\times)\,.$$
Since the transgression map tr is an isomorphism, the first map Inf must be an isomorphism as well. □

4. Proof of Theorem 1.1

Keep the notation of the previous section. Moreover, given an endo-trivial kG-module V such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, we let

$$\theta_V : Q \longrightarrow \mathrm{PGL}(V)$$

denote the induced homomorphism constructed in Lemma 2.2(a). The projective lifting property for the central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ allows us to fix a representation

$$\rho_{V_{\widetilde{Q}}} : \widetilde{Q} \longrightarrow \mathrm{GL}(V)$$

lifting θ_V to $\widetilde{Q}$. We denote by $V_{\widetilde{Q}}$ the corresponding $k\widetilde{Q}$-module.

Lemma 4.1. *Let V be an endo-trivial kG-module such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. Then $V_{\widetilde{Q}}$ is an endo-trivial $k\widetilde{Q}$-module.*

Proof. We have to work with two group extensions

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi_G} Q \longrightarrow 1 \qquad \text{and} \qquad 1 \longrightarrow M \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1\,,$$

where $M := M_k(Q)$. Both A and M have order prime to p.

Let $P \in \mathrm{Syl}_p(G)$, set $\overline{P} := AP/A \in \mathrm{Syl}_p(Q)$, and let $\imath_P : P \longrightarrow AP$ be the inclusion map, so that $\varphi := \pi_G \circ \imath_P : P \longrightarrow \overline{P}$ is an isomorphism. Next choose $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{Q})$ such that $M\widetilde{P}/M = \overline{P} \in \mathrm{Syl}_p(Q)$. Let $\imath_{\widetilde{P}} : \widetilde{P} \longrightarrow M\widetilde{P}$ be the inclusion map, so that $\psi := \pi_{\widetilde{Q}} \circ \imath_{\widetilde{P}} : \widetilde{P} \longrightarrow \overline{P}$ is an isomorphism. Consider now the two commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & \overline{P} \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \theta_V} \\ \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{P} & \xrightarrow{\psi} & \overline{P} \\ {\scriptstyle \rho_{\widetilde{Q}}}\downarrow & & \downarrow{\scriptstyle \theta_V} \\ \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) \end{array}$$

where we write $\rho_{\widetilde{Q}} := \rho_{V_{\widetilde{Q}}}$ for simplicity. Since ϕ and ψ are isomorphisms, for any $u \in \overline{P}$, we have

$$\pi_V \rho_V \phi^{-1}(u) = \theta_V(u) = \pi_V \rho_{\widetilde{Q}} \psi^{-1}(u)\,.$$

We claim that if two elements $u_1, u_2 \in \mathrm{GL}(V)$ have p-power order and satisfy $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = u_2$. Postponing the proof of the claim, we deduce that

$$\rho_V \phi^{-1}(u) = \rho_{\widetilde{Q}} \psi^{-1}(u)\,,$$

because they have p-power order. This means that the representations $(\rho_V)_{|P}$ and $(\rho_{\widetilde{Q}})_{|\widetilde{P}}$, transported via isomorphisms to representations of $\overline{P}$, are equal. Now, a module is endo-trivial if and only if its restriction to a Sylow p-subgroup is (see [CMN06, Proposition 2.6]). Moreover, this property is preserved when transported via group isomorphisms. Since V is endo-trivial, so is $V{\downarrow_P}$, hence so is $V_{\widetilde{Q}}{\downarrow_{\widetilde{P}}}$, and it follows that $V_{\widetilde{Q}}$ is endo-trivial.

We are left with the proof of the claim. If $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = \alpha u_2$ where $\alpha \in k^\times$. For some large enough power p^n, we have $u_1^{p^n} = u_2^{p^n} = 1$. Therefore we obtain

$$1 = u_1^{p^n} = (\alpha u_2)^{p^n} = \alpha^{p^n} u_2^{p^n} = \alpha^{p^n}\,.$$

But there are no nontrivial p-th roots of unity in $k^\times$, so we get $\alpha = 1$, hence $u_1 = u_2$. $\square$

Proposition 4.2. *Assume G satisfies Hypothesis 3.1. Then there is an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \longrightarrow T(\widetilde{Q})/X(\widetilde{Q})$$

defined by $\Phi_{G,\widetilde{Q}}([V] + X(G)) := [V_{\widetilde{Q}}] + X(\widetilde{Q})$ for any indecomposable endo-trivial kG-module V. Moreover, for any endo-trivial kQ-module W, the homomorphism $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$.

Proof. First, Lemma 4.1 allows us to define a map $\phi : T(G) \longrightarrow T(\widetilde{Q})/X(\widetilde{Q})$ by setting $\phi([V]) := [V_{\widetilde{Q}}] + X(\widetilde{Q})$ for any $[V] \in T(G)$ such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. The definition of $\phi([V])$ does not depend on the choice of $V_{\widetilde{Q}}$, for if $\rho_{V'_{\widetilde{Q}}}$ is a second lifting of θ_V to $\widetilde{Q}$, then by Lemma 2.1 there exists $X \in X(\widetilde{Q})$ such that $V'_{\widetilde{Q}} \cong V_{\widetilde{Q}} \otimes X$, hence $\phi([V_{\widetilde{Q}}]) = \phi([V'_{\widetilde{Q}}])$.

Next, let V, W be two indecomposable endo-trivial kG-modules. Part (b) of Lemma 3.2 implies that $\rho_{V\otimes W}(A) = (\rho_V \otimes \rho_W)(A) \subseteq k^\times \cdot \mathrm{Id}_{V\otimes W}$. Thus, by Lemma 2.2(b), $\theta_{V\otimes W} = \theta_V \otimes \theta_W$, and it is easy to verify that $\rho_{V_{\widetilde{Q}}} \otimes \rho_{W_{\widetilde{Q}}}$ lifts $\theta_V \otimes \theta_W$ to $\widetilde{Q}$. Therefore, by Lemma 2.1, there exists $X \in X(\widetilde{Q})$ such that $(V \otimes W)_{\widetilde{Q}} \cong V_{\widetilde{Q}} \otimes W_{\widetilde{Q}} \otimes X$. This shows that ϕ is a group homomorphism.

It is clear that $\mathrm{Ker}\,\phi = X(G)$, since by construction $\dim V_{\widetilde{Q}} = \dim V$ for any indecomposable endo-trivial kG-module V. As a result, ϕ induces the required homomorphism $\Phi_{G,\widetilde{Q}}$.

Finally, if W is any endo-trivial kQ-module, then the $k\widetilde{Q}$-module constructed from $V = \mathrm{Inf}_Q^G(W)$ is easily seen to be the inflated module $V_{\widetilde{Q}} = \mathrm{Inf}_Q^{\widetilde{Q}}(W)$, because the map $\theta_V : Q \to \mathrm{PGL}(V)$ comes from a group homomorphism $Q \to \mathrm{GL}(V)$. This shows that the class of $\mathrm{Inf}_Q^G(W)$ is mapped to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$ under the map $\Phi_{G,\widetilde{Q}}$, proving the additional statement. $\square$

Corollary 4.3. *Assume G satisfies Hypothesis 3.1. If $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are two non-isomorphic p'-representation groups of Q, then*

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2} : T(\widetilde{Q}_1)/X(\widetilde{Q}_1) \longrightarrow T(\widetilde{Q}_2)/X(\widetilde{Q}_2)$$

is an isomorphism.

Proof. Let V be an indecomposable $k\widetilde{Q}_1$-module. By construction

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2}([V] + X(\widetilde{Q}_1)) = [W] + X(\widetilde{Q}_2)\,,$$

where $W := V_{\widetilde{Q}_2}$ is a $k\widetilde{Q}_2$-module such that ρ_W lifts $\theta_V : Q \longrightarrow \mathrm{PGL}(V)$ to $\widetilde{Q}_2$. But then ρ_V lifts $\theta_W = \theta_V$ to $\widetilde{Q}_1$, so that by construction

$$\Phi_{\widetilde{Q}_2,\widetilde{Q}_1}([W] + X(\widetilde{Q}_2)) = [V] + X(\widetilde{Q}_1)\,.$$

In other words, $\Phi_{\widetilde{Q}_1,\widetilde{Q}_2} \circ \Phi_{\widetilde{Q}_2,\widetilde{Q}_1} = \mathrm{Id}$. Similarly $\Phi_{\widetilde{Q}_2,\widetilde{Q}_1} \circ \Phi_{\widetilde{Q}_1,\widetilde{Q}_2} = \mathrm{Id}$. $\square$

Corollary 4.4. *Assume G satisfies Hypothesis 3.1. The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \longrightarrow K(\widetilde{Q})/X(\widetilde{Q}) \,.$$

In particular, if $K(\widetilde{Q}) \cong X(\widetilde{Q})$, then $K(G) \cong X(G)$.

Proof. Let $P \in \mathrm{Syl}_p(G)$ and let V be an indecomposable endo-trivial kG-module. As in the proof of Lemma 4.1, the two modules $V\!\downarrow^G_P$ and $V_{\widetilde{Q}}\!\downarrow^{\widetilde{Q}}_{\widetilde{P}}$ are isomorphic, provided we view them as modules over the group $\overline{P}$ via the isomorphisms $P \cong \overline{P}$ and $\widetilde{P} \cong \overline{P}$. It follows that V has a trivial source if and only if $V_{\widetilde{Q}}$ has. Therefore $\Phi_{G,\widetilde{Q}}$ restricts to an injective group homomorphism

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \longrightarrow K(\widetilde{Q})/X(\widetilde{Q}) \,.$$

The special case follows. □

Proposition 4.2 together with Corollary 4.4 prove Theorem 1.1.

5. Conjecture on the torsion-free part

We keep the notation of the previous sections. Let $TF(G) = T(G)/TT(G)$, the torsion-free part of the group of endo-trivial modules. Since $X(G) \subseteq TT(G)$, the map

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \longrightarrow T(\widetilde{Q})/X(\widetilde{Q})$$

induces an injective group homomorphism

$$\Psi_{G,\widetilde{Q}} : TF(G) \longrightarrow TF(\widetilde{Q}) \,.$$

We know that $\Phi_{G,\widetilde{Q}}$ is in general not surjective, but we conjecture that $\Psi_{G,\widetilde{Q}}$ is surjective.

Conjecture 5.1. (a) *The map $\mathrm{Inf}^G_Q : TF(Q) \longrightarrow TF(G)$ is an isomorphism.*
(b) *The map $\Psi_{G,\widetilde{Q}} : TF(G) \longrightarrow TF(\widetilde{Q})$ is an isomorphism.*

Note that (b) follows from (a), by applying (a) to both $\mathrm{Inf}^G_Q : TF(Q) \longrightarrow TF(G)$ and $\mathrm{Inf}^{\widetilde{Q}}_Q : TF(Q) \longrightarrow TF(\widetilde{Q})$ and composing, because the map $\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q})$ is the identity on modules inflated from Q.

Part (a) of Conjecture 5.1 is in fact a consequence of any of the two conjectures made in [CMT11b]. First, Conjecture 10.1 in [CMT11b] asserts that, if a group homomorphism $\phi : G \to G'$ induces an isomorphism between the corresponding p-fusion systems, then ϕ should induce an isomorphism $TF(G') \xrightarrow{\sim} TF(G)$. In the special case where ϕ is the quotient map $\phi : G \to Q = G/O_{p'}(G)$, it is well-known that the fusion systems are isomorphic, so we would obtain the isomorphism $TF(Q) \xrightarrow{\sim} TF(G)$ of Conjecture 5.1 above. This special case is explicitly mentioned at the end of Section 10 in [CMT11b].

Conjecture 9.2 in [CMT11b] asserts that the group $TF(G)$ should be generated by endo-trivial modules lying in the principal block. Since $O_{p'}(G)$ acts trivially on any module lying in the principal block of G, such a module is inflated from Q, so the inflation map $\mathrm{Inf}^G_Q : TF(Q) \longrightarrow TF(G)$ in Conjecture 5.1 above should be an isomorphism.

Example 7.3 below illustrates a method allowing one to prove that the maps in Conjecture 5.1 are isomorphisms in specific cases.

6. The perfect case

When the group $Q = G/O_{p'}(G)$ is perfect, there is an alternative approach to the construction of the injective group homomorphism of Theorem 1.1(a) using universal central extensions.

Recall that a *universal* p'*-central extension* of an arbitrary finite group Q is by definition a central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

with $M_{p'} = \operatorname{Ker} \pi_{\widetilde{Q}}$ of order prime to p and satisfying the following universal property: For any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \operatorname{Ker} \pi$ of order prime to p, there exists a unique group homomorphism $\phi : \widetilde{Q} \to E$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{p'} & \longrightarrow & \widetilde{Q} & \xrightarrow{\pi_{\widetilde{Q}}} & Q & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi|_{M_{p'}}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1
\end{array}$$

A standard argument shows that if a universal p'-central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ exists, then it is unique up to isomorphism.

Lemma 6.1. *If $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a universal p'-central extension of a finite group R, then $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is p'-representation group of Q.*

Proof. Let $(\check{Q}, \pi_{\check{Q}})$ be an arbitrary p'-representation group of Q. Let V be a finite-dimensional k-vector space and $\theta : Q \longrightarrow \mathrm{PGL}(V)$ a group homomorphism. Because $(\check{Q}, \pi_{\check{Q}})$ has the projective lifting property and $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, there exist group homomorphisms $\widetilde{\theta} : \check{Q} \longrightarrow \mathrm{GL}(V)$ and $\phi : \widetilde{Q} \longrightarrow \check{Q}$ such that $\widetilde{\theta} \circ \phi$ lifts θ. Therefore $(\widetilde{Q}, \pi_{\widetilde{Q}})$ has the projective lifting property as well.

Now, because $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, it is easy to see that $X(\widetilde{Q}) = X(Q) = 1$. Therefore the Hochschild-Serre 5-term exact sequence associated to $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is:

$$1 \longrightarrow 1 \longrightarrow 1 \longrightarrow \mathrm{Hom}(M_{p'}, k^\times) \xrightarrow{\mathrm{tr}} H^2(Q, k^\times) \xrightarrow{\mathrm{Inf}} H^2(\widetilde{Q}, k^\times)$$

Thus the transgression map $\mathrm{tr} : \mathrm{Hom}(M_{p'}, k^\times) \longrightarrow H^2(Q, k^\times) = M_k(Q)$ is injective. But $M_{p'} \cong \mathrm{Hom}(M_{p'}, k^\times)$, therefore by minimality of $(\check{Q}, \pi_{\check{Q}})$, we have $|M_{p'}| = |M_k(Q)|$ and $|\widetilde{Q}| = |\check{Q}|$, proving that $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a p'-representation group of Q. □

Lemma 6.2. *Any finite perfect group Q admits a universal p'-central extension.*

Proof. Since Q is a perfect group, it is well-known that Q has a representation group relative to $\mathbb{C}$, say $(\widehat{Q}, \pi_{\widehat{Q}})$, which is unique up to isomorphism and that

$$\mathrm{Ker}(\pi_{\widehat{Q}}) =: M \cong M_{\mathbb{C}}(Q) = H^2(Q, \mathbb{C}^\times)\,,$$

the Schur multiplier of Q. Moreover, $(\widehat{Q}, \pi_{\widehat{Q}})$ is a universal central extension of Q, in particular perfect, see [Rot95, Theorem 11.11]. Thus, for any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

where $Z = \operatorname{Ker} \pi$, there exists a unique group homomorphism $\psi : \widehat{Q} \to E$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & \widehat{Q} & \xrightarrow{\pi_{\widehat{Q}}} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi|_M} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array}$$

If Z has order prime to p, then the p-part M_p of M lies in the kernel of $\psi|_M$. Passing to the quotient by M_p, we define $\widetilde{Q} := \widehat{Q}/M_p$ and denote by $\phi : \widetilde{Q} \to E$ the map induced by ψ. Thus we obtain an induced central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

where $M_{p'} := M/M_p$. This is a universal p'-central extension of Q by construction. □

Given an arbitrary group extension $1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$ with perfect quotient Q and kernel A of order prime to p, there is an induced p'-central extension:

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \xrightarrow{\pi_G} Q \longrightarrow 1$$

Moreover, by the above, Q admits a universal p'-central extension, which is in fact a p'-representation group $(\widetilde{Q}, \pi_{\widetilde{Q}})$ of Q. Therefore, by the universal property, there exists a unique group homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ lifting the identity on Q.

Lemma 6.3. *The homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ induces a group homomorphism*

$$\phi_G^* : T(G/[G, A]) \longrightarrow T(\widetilde{Q})$$

such that $\phi_G^* = \operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} \circ \operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. Moreover, both $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$ preserve indecomposability of endo-trivial modules.

Proof. The kernel of ϕ_G is contained in $\operatorname{Ker} \pi_{\widetilde{Q}} = M_{p'}$, which is a p'-group. Therefore, there is an induced inflation map $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} : T(\operatorname{Im}(\phi_G)) \to T(\widetilde{Q})$, preserving indecomposability of endo-trivial modules.

Since $\operatorname{Im}(\phi_G)$ maps onto Q via π_G, the group $G/[G, A]$ is the product of $\operatorname{Im}(\phi_G)$ and the central p'-subgroup $A/[G, A]$. It follows that $\operatorname{Im}(\phi_G)$ is a normal subgroup of $G/[G, A]$ of index prime to p. Therefore, by [CMN09, Prop. 3.1], the restriction to $\operatorname{Im}(\phi_G)$ of any indecomposable endo-trivial $k(G/[G, A])$-module remains indecomposable and is endo-trivial.

We define ϕ_G^* to be the composite of $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. □

Composing the group homomorphism

$$\phi_G^* : T(G/[G, A]) \longrightarrow T(\widetilde{Q})$$

with the inverse of the isomorphism

$$\mathrm{Inf}_{G/[G,A]}^{G} : T(G/[G,A]) \longrightarrow T(G)$$

of Lemma 3.2, we obtain a group homomorphism

$$\Phi : T(G) \longrightarrow T(\widetilde{Q})\,.$$

We now show that this provides the alternative approach to the map of Theorem 1.1.

Proposition 6.4. *Suppose that G satisfies Hypothesis 3.1 and that Q is perfect.*
(a) $\operatorname{Ker}\Phi = X(G)$.
(b) *The induced injective group homomorphism*

$$\overline{\Phi} : T(G)/X(G) \longrightarrow T(\widetilde{Q}) = T(\widetilde{Q})/X(\widetilde{Q})$$

coincides with the map $\Phi_{G,\widetilde{Q}}$ of Theorem 1.1.

Proof. Consider the map $\phi_G^* : T(G/[G,A]) \to T(\widetilde{Q})$ of Lemma 6.3. It is clear that the image of a one-dimensional module is one-dimensional, hence trivial since $X(\widetilde{Q}) = 1$ by Lemma 3.3. Therefore $X(G) \subseteq \operatorname{Ker}\Phi$. It follows that Φ induces a group homomorphism $\overline{\Phi}$ as in the statement.

Our assumption on G implies that, if V is an endo-trivial kG-module, then $[G,A]$ acts trivially on V (Lemma 3.2). Moreover, $\rho_V : G/[G,A] \to \mathrm{GL}(V)$ lifts $\theta_V : Q \to \mathrm{PGL}(V)$, as in Section 4. It is then clear that $\rho_V\phi_G : \widetilde{Q} \to \mathrm{GL}(V)$ also lifts $\theta_V : Q \to \mathrm{PGL}(V)$. Therefore, the definition of $\Phi_{G,\widetilde{Q}}$ (see Proposition 4.2) shows that the class of V is mapped by $\Phi_{G,\widetilde{Q}}$ to the class of the module $V_{\widetilde{Q}}$ corresponding to the representation $\rho_V\phi_G$. In other words, $[V_{\widetilde{Q}}] = \Phi([V])$ and this shows that $\Phi_{G,\widetilde{Q}}$ coincides with $\overline{\Phi}$.

Finally, since $\Phi_{G,\widetilde{Q}}$ is injective and is equal to $\overline{\Phi}$, we have $\operatorname{Ker}\overline{\Phi} = \{0\}$. Therefore we obtain $\operatorname{Ker}\Phi = X(G)$. □

Remark 6.5. The proof we give above shows that Proposition 6.4 remains valid if the assumption that Q is perfect is replaced with the assumption that Q admits a universal p'-central extension. It is possible to prove that this happens if and only if $X(Q) = 1$, although we could not find a reference to such a statement in the literature. Hence, for simplicity, we restrict ourselves to the perfect case in this article.

7. Examples

In this final section, we provide various examples, in particular illustrating cases where the morphism $\Phi_{G,\widetilde{Q}}$ is not surjective.

Example 7.1. Suppose that Q is simple and take $G = Q$, hence $A = O_{p'}(G) = \{1\}$. Then $\Phi_{Q,\widetilde{Q}}$ is just the inflation map $T(Q) \to T(\widetilde{Q})$. If Q is a finite simple group listed in the table below, then it is known that its unique p'-representation group $\widetilde{Q}$ has indecomposable endo-trivial modules lying in faithful p-blocks, namely not inflated from Q.

The results concerning the sporadic groups can be found in [LM15b, Table 3], and those about the alternating group $\mathfrak{A}_6$ in [LM15c, Theorem A & Theorem B] together with [CMN09, Theorem A & Theorem B].

Q	p	$\widetilde{Q}$	$T(Q)$	$T(\widetilde{Q})$
$\mathfrak{A}_6$	3	$2.\mathfrak{A}_6$	$\mathbb{Z}\oplus\mathbb{Z}/4$	$\mathbb{Z}\oplus\mathbb{Z}/8$
$\mathfrak{A}_6$	2	$3.\mathfrak{A}_6$	$\mathbb{Z}^2$	$\mathbb{Z}^2\oplus\mathbb{Z}/3$
M_{22}	3	$4.M_{22}$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4$
J_3	2	$3.J_3$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/3$
Ru	3	$2.Ru$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/4$
Fi_{22}	5	$6.Fi_{22}$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/6\oplus\mathbb{Z}/2$

Further examples are given by the exceptional covering group $2.F_4(2)$ of the exceptional group of Lie type $F_4(2)$, which possesses simple torsion endo-trivial modules lying in faithful blocks in characteristics 5 and 7 (see [LM15a, Proposition 5.5]), although the full structure of the group of endo-trivial modules has not been determined in these cases.

Example 7.2. Assume $p > 2$, let $n \geq max\{2p, p+4\}$ be an integer and denote by $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ the two isoclinic p'-representation groups of the symmetric group $\mathfrak{S}_n$. Corollary 4.3 yields

$$T(\widetilde{\mathfrak{S}}_n)/X(\widetilde{\mathfrak{S}}_n) \cong T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n)\,.$$

However, [LM15c, Thm. B(1),(2)] proves a stronger result, namely $T(\widetilde{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widetilde{\mathfrak{S}}_n}(T(\mathfrak{S}_n))$ and $T(\widehat{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widehat{\mathfrak{S}}_n}(T(\mathfrak{S}_n))$.

Consequently, given any finite group G such that $G/O_{p'}(G)$ is isomorphic to one of $\mathfrak{S}_n$, $\widetilde{\mathfrak{S}}_n$ or $\widehat{\mathfrak{S}}_n$ (with $n \geq max\{2p, p+4\}$), by Theorem 1.1 there exist injective group homomorphisms

$$T(\mathfrak{S}_n)/X(\mathfrak{S}_n) \longrightarrow T(G)/X(G) \overset{\Phi_{G,\widehat{\mathfrak{S}}_n}}{\longrightarrow} T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n) \overset{\sim}{\longrightarrow} T(\mathfrak{S}_n)/X(\mathfrak{S}_n)\,,$$

where the first map is induced by inflation. Hence we have $T(G)/X(G) \cong T(\mathfrak{S}_n)/X(\mathfrak{S}_n)$. We recall that the structure of $T(\mathfrak{S}_n)$ is known by [CMN09].

Example 7.3. In this final example, we outline a method which allows us to show that the maps $\mathrm{Inf}_Q^{\widetilde{Q}}$ is an isomorphism on the torsion-free part of the groups of endo-trivial modules of Q and $\widetilde{Q}$ in some concrete cases.

Specifically, we may use the fact that endo-trivial modules are liftable to characteristic zero, and afford characters taking root-of-unity values at p-singular conjugacy classes, see [LMS13, Theorem 1.3 and Corollary 2.3]. Therefore, if for every faithful p-block B of $k\widetilde{Q}$ (of full defect) no elements of $\mathbb{Z}\,\mathrm{Irr}_{\mathbb{C}}(B)$ take root-of-unity values at p-singular conjugacy classes of $\widetilde{Q}$, then any endo-trivial $k\widetilde{Q}$-module is inflated from Q, hence

$$\mathrm{Inf}_Q^{\widetilde{Q}} : TF(Q) \longrightarrow TF(\widetilde{Q})$$

is an isomorphism.

This was used in [LM15c, Theorem B] in the case that $Q = \mathfrak{S}_n$, $n \geq max\{2p, p+4\}$ (as mentioned in Example 7.2 above), as well as in [LM15b, Lemma 4.3 and Lemma 6.2] for a large number of sporadic simple groups Q. More precisely, in characteristic $p = 2$

for $Q = M_{12}$, M_{22}, J_2, HS, McL, Ru, Suz, ON, Fi_{22}, Co_1, Fi'_{24}, or B, in characteristic $p = 3$ for $Q = M_{12}$, J_2, HS, Suz, Fi_{22}, Co_1, or B, in characteristic $p = 5$ for $Q = J_2$, HS, Ru, Suz, Co_1, Fi'_{24}, or B, and in characteristic $p = 7$ for $Q = Co_1$, Fi'_{24}, or B.

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