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H. Haddar — P. Joly — H.M. Nguyen

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Asymptotic Models for Scattering from Strongly Absorbing Obstacles: the Scalar Case

H. Haddar* , P. Joly* , H.M. Nguyen*

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Abstract: We derive different classes of generalized impedance boundary conditions for the scattering problem from highly absorbing obstacles. Compared to existing works, our construction is based on an asymptotic development of the solution with respect to the medium absorption. Error estimates are obtained to mathematically validate the accuracy order of each condition.

Key-words: asymptotic models, general impedance boundary conditions, strongly absorbing mediums

* INRIA-Rocquencourt

Modèles asymptotiques pour la diffraction par des obstacles fortement absorbants: cas scalaire

Résumé : Nous dérivons diverses familles de conditions d'impédances généralisées pour la diffraction d'ondes acoustiques par des obstacles fortement absorbants. Par opposition aux travaux déjà existants, la construction s'appuie sur un développement asymptotique de l'onde diffractée par rapport à l'absorption du milieu. Des estimations d'erreurs sont démontrées pour justifier mathématiquement l'utilisation de ces conditions

Mots-clés : modèles asymptotiques, conditions d'impédances généralisées, obstacles fortement absorbants

1 Introduction

The concept of Generalized Impedance Boundary condition ($GIBC$) is now a rather classical notion in the mathematical modeling of wave propagation phenomena (see for instance [12] and [15]). Such tool is particularly used in electromagnetism for diffraction problems, in the time harmonic regime, by obstacles that are partially or totally penetrable. The general idea is, as soon as it is possible and desirable, to replace the use of an "exact model" inside (the penetrable part of) the obstacle by approximate boundary conditions (also called equivalent or effective conditions) on the boundary of the scatterer. This idea is pertinent in practice when the boundary condition appears to be easy to handle from the numerical point of view, which would be the case if it can be expressed with the help of differential operators. The same type of idea, even though the purpose was different, led to the construction of local absorbing boundary conditions for the wave equation ([10, 8]) or more recently to the construction of On Surface Radiation Conditions ([5, 4]) for pure exterior problems.

The diffraction problem of electromagnetic waves by perfectly conducting obstacles coated with a thin layer of dielectric material is a prototype problem for the use of impedance conditions. Indeed, due to the small (typically with respect to the wavelength) thickness of the coating, the effect of the layer on the exterior medium is, as a first approximation, local (see for instance [15, 12, 9, 7, 2])

Another application, the one we have in mind here, is the diffraction of waves by strongly absorbing obstacles, typically highly conducting bodies in electromagnetism. This time, it is a well-known physical phenomenon, the so-called skin effect, that creates a "thin layer" effect . The conductivity limits the penetrable region to a boundary layer whose depth is inversely proportional to the conductivity of the medium. Then, here again, the effect of the obstacle is, as a first approximation, local. The numerical results presented in Figures 2-5 of section 2.3 illustrate this skin effect phenomena.

As a matter of fact, the research on effective boundary conditions for highly absorbing obstacles began with Leontovich before the apparition of computers and the development of numerical methods. He proposed an impedance boundary condition, that is nowadays known as the Leontovitch boundary condition (and that correspond with the condition of order 1 in this paper). This condition only "sees" locally the tangent plane to the frontier. Later, Rytov [14, 15] proposed an extension of the Leontovitch condition which was already based on the principle of an asymptotic expansion. More recently, Antoine-Barucq-Vernhet [6] proposed a new derivation of impedance boundary conditions based on the technique of expansion of pseudo-differential operators (following there the original ideas of Engquist-Majda [10] for absorbing boundary conditions).

Our purpose in this paper is to revisit the question of $GIBC$'s for the scattering of waves by highly absorbing obstacles with a double objective:

- Propose a new construction of $GIBC$'s which is based, as Rytov's contruction, on an ansatz for the asymptotic expansion of the exact solution but which is technically different: we use a scaling technique and a boundary layer expansion in the neighborhood of the boundary (which is, to our opinion, more adapted to a mathematical analysis) while Rytov uses an ansatz similar to the ansatz for high frequency asymptotics.
- Develop a complete mathematical analysis (existence and uniqueness of the solution, stability and error estimates) for the approximate problems with respect to the medium's absorption.

The second point is probably the main contribution of the present work. It permits in particular to give a sense to the *order* of a given $GIBC$, a notion whose meaning is not always clear (at least not always the same) in the literature (it is sometimes related to the order of the differential operators involved in the condition, sometimes linked to the truncation order of some Taylor expansion,...): a GIBC will be of order k if it provides an error in $O(\varepsilon^{k+1})$. A point deserves to be emphasized in this introduction: for a given order k there is not uniqueness of the $GIBC$. We shall illustrate this fact here by presenting several $GIBC$'s of order 2 and 3 ; for the same order, different $GIBC$s only differ by (maybe important) other features such as their adaptation to a given numerical methods.

It is not surprising to see that, in the mathematical literature, much work is devoted to mathematical analysis or the study of numerical methods for wave propagation models with $GIBC$s (see for instance [1, 16]). Curiously, concerning a rigorous asymptotic analysis of $GIBC$'s for highly absorbing media, it seems that, although some of the works by Artola-Cessenat [3] go in this direction (for different problems than ours, however) there are very few works in the mathematical litterature devoted to such a rigorous asymptotic analysis of $GIBC$'s for highly absorbing media, contrary to the case of thin coatings for which there is an abundant literature.

In this first paper on the subject, we investigate in detail the question of $GIBC$'s for strongly absorbing media in the context of time harmonic acoustic wave in 3 dimensions. The case of Maxwell's equations will be the object of a second paper (note however that, in the degenerate 2D case, we get with this work $GIBC$'s for 2D electromagnetic waves, at least in the case of the TE polarization). The outline of the article is as follows. In section 2, we present the model problem we shall work with and give the main basic mathematical results related to this problem (Theorems 2.1 and 2.2 and corollary 2.1)). We state the main results of our paper in section 3: the presentation of so-called NtD (section 3.1), DtN (section 3.2) and robust (in a sense defined in section 3.2) $GIBC$'s and the approximation theorems 3.1. Section 4 is devoted to the construction of $GIBC$'s (see section 4.4) through the use of a standard scaling technique (cf. section 4.2) in the neighborhood of the boundary that permits an analytic description of the boundary layer (section 4.3) using a system of local coordinates (section 4.1). The central section of the paper is section 5 where we prove error estimates for NtD $GIBC$'s. The analysis is split into two steps: a justification (section

5.1) of the asymptotic expansion of section 4.2 (lemma 5.1 and corollary 5.1) and a second part (section 5.2) linked to the *GIBC* itself (lemmas 5.4 and 5.6). Finally we explain in section 6 how to modify the analysis for DtN and robust *GIBC*'s.

2 Model settings

Let Ω, Ω_i and Ω_e be open domains of $\mathbb{R}^3$ such that $\overline{\Omega} = \overline{\Omega_e} \cup \overline{\Omega_i}$ and $\Omega_i \cap \Omega_e = \emptyset$. We also assume that Ω_i is a simply connected and $\partial\Omega \cup \partial\Omega_i = \emptyset$. In the sequel, we set $\Gamma = \partial\Omega_i$ and, for the simplicity of the exposition, we shall assume that Γ is a C^∞ manifold. (see Fig. 1). We are interested in the acoustic wave propagation inside the domain Ω. We assume

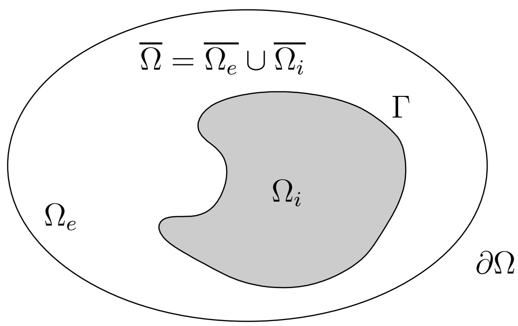

Figure 1: Geometry of the medium

that the time and space scales are chosen in such a way that the speed of waves is 1 and we assume that the medium inside Ω_i is an absorbing medium. In other words, the wave propagation is governed inside Ω, by:

$$\frac{\partial^2 U^\varepsilon}{\partial t^2} + \sigma^\varepsilon(x)\, \frac{\partial U^\varepsilon}{\partial t} - \Delta U^\varepsilon = F, \tag{1}$$

where $\sigma^\varepsilon(x)$ is the function that characterize the absorption of the medium and ε a small parameter defined later:

$$\sigma^\varepsilon(x) = \begin{cases} 0, & \text{in } \Omega_e\ , \\ \sigma^\varepsilon > 0, & \text{in } \Omega_i\ . \end{cases} \tag{2}$$

Considering a time harmonic source $F(x,t) = f(x)\sin\omega t$, where $\omega > 0$ denotes a given frequency, one looks for time harmonic solutions:

$$U^\varepsilon(x,t) = \mathcal{R}e\left(u^\varepsilon(x)\exp i\omega t\right).$$

Then, the function $u^\varepsilon(x)$ is governed by the Helmholtz equation:

$$-\Delta u^\varepsilon - \omega^2 u^\varepsilon + i\omega\, \sigma^\varepsilon(x) u^\varepsilon = f, \quad \text{in } \Omega, \tag{3}$$

where we assume that the support of the function f is confined into Ω_e. Equation (3) has to be complemented with a boundary condition on the exterior boundary $\partial\Omega$. We consider for instance the following absorbing boundary condition (see Remarks 2.1 and 2.2)

$$\partial_n u^\varepsilon + i\omega u^\varepsilon = 0, \qquad \text{on } \partial\Omega. \tag{4}$$

Remark 2.1 *According to (4), the boundary $\partial\Omega$ can be seen as a physical absorbing boundary where a standard impedance condition is applied. The problem (3, 4) can also be seen as an approximation in a bounded domain (namely Ω) of the scattering problem in $\mathbb{R}^3 \setminus \Omega_i$. In such a case, the boundary condition on $\partial\Omega$ has to be understood as an (low order) approximation of the outgoing radiation condition at infinity.*

Remark 2.2 *In this paper, we could have treated as well the scattering problem in $\mathbb{R}^3 \setminus \Omega_i$. The reader will easily be convinced that the obtained results can be extended to this case without any major difficulty. The only difference would lie in the reduction to a bounded domain. This additional difficulty is purely technical and not essential in the context of this paper whose main purpose is the treatment of the "interior boundary" Γ.*

We are interesting in describing the solution behaviour for large σ^ε. For this, it is useful to introduce as a small parameter the quantity:

$$\varepsilon = \sqrt{\omega/\sigma^\varepsilon} \Longleftrightarrow \sigma^\varepsilon = 1/(\omega\,\varepsilon^2)\ . \tag{5}$$

It is easy to see that ε has the same dimension as a length. It represents in fact the width of the penetrable boundary layer inside Ω_i (also called the skin depth).

Our goal in this paper is to characterize, in an approximate way, the restriction u_e^ε of u^ε to the exterior domain Ω_e. In order to do so, it is useful to rewrite problem ((3),(4)) as a transmission problem between $u_i^\varepsilon = u^\varepsilon_{|\Omega_i}$ and $u_e^\varepsilon = u^\varepsilon_{|\Omega_e}$:

$$\left\{\begin{array}{lll}
(i) & -\Delta u_e^\varepsilon - \omega^2 u_e^\varepsilon = f, & \text{in } \Omega_e, \\[2ex]
(ii) & -\Delta u_i^\varepsilon - \omega^2 u_i^\varepsilon + \dfrac{i}{\varepsilon^2} u_i^\varepsilon = 0, & \text{in } \Omega_i, \\[2ex]
(iii) & \partial_n u_e^\varepsilon + i\omega u_e^\varepsilon = 0, & \text{on } \partial\Omega, \\[2ex]
(vi) & u_i^\varepsilon = u_e^\varepsilon, & \text{on } \Gamma, \\[2ex]
(v) & \partial_n u_i^\varepsilon = \partial_n u_e^\varepsilon, & \text{on } \Gamma.
\end{array}\right. \tag{6}$$

2.1 Existence-Uniqueness-Stability

We present here the basic theoretical results relative to problem (3). These results constitute a necessary preliminary step towards the forthcoming asymptotic analysis.

Theorem 2.1 *There exists a unique solution $u^\varepsilon \in H^1(\Omega)$ to problem ((3),(4)). Moreover, there exits a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}\,. \tag{7}$$

Proof. The existence and uniqueness proof is a classical exercise on the use of Fredholm's alternative. Let us simply recall that the uniqueness result resorts to the following identity:

$$\int_\Omega |\nabla u^\varepsilon|^2 - \omega^2 |u^\varepsilon|^2\,dx + i\left(\int_{\partial\Omega} \omega |u^\varepsilon|^2\,ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |u^\varepsilon|^2\,dx\right) = 0.$$

that is valid for any solution u^ε of the homogeneous boundary value problem associated with ((3),(4)) (simply multiply equation (3) by $\bar{u^\varepsilon}$ and integrate by parts over Ω). In particular, $u^\varepsilon = 0$ in Ω_i and by unique continuation $u^\varepsilon = 0$.

Stability estimate (7) is proven by contradiction. Assume the existence of a sequence f^ε with $\|f^\varepsilon\|_{L^2(\Omega)} = 1$ such that the corresponding solution of ((3),(4)), denoted u^ε, is such that $\|u^\varepsilon\|_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$. We set

$$v^\varepsilon = u^\varepsilon / \|u^\varepsilon\|_{L^2(\Omega)} \text{ and } g^\varepsilon = f^\varepsilon / \|u^\varepsilon\|_{L^2(\Omega)}\,.$$

Then $\|v^\varepsilon\|_{L^2(\Omega)} = 1$ and $\|g^\varepsilon\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$. One gets from (3)

$$\left\{\begin{array}{ll} -\Delta v^\varepsilon - \omega^2 v^\varepsilon + i\omega\sigma^\varepsilon v^\varepsilon = g^\varepsilon, & \text{in } \Omega, \\ \partial_n v^\varepsilon + i\omega v^\varepsilon = 0, & \text{on } \partial\Omega. \end{array}\right. \tag{8}$$

Consequently (once again, we multiply the previous equation by $\bar{v}^\varepsilon$ and integrate over Ω)

$$\int_\Omega \left(|\nabla v^\varepsilon|^2 - \omega^2 |v^\varepsilon|^2\right)\,dx + i\left(\omega \int_{\partial\Omega} |v^\varepsilon|^2\,ds + \frac{1}{\varepsilon^2}\int_{\Omega_i} |v^\varepsilon|^2\,dx\right) = \int_{\Omega_e} g^\varepsilon \bar{v}^\varepsilon\,dx. \tag{9}$$

Taking the real part of (9) yields

$$\int_\Omega |\nabla v^\varepsilon|^2\,dx = -\omega^2 \int_\Omega |v^\varepsilon|^2\,dx + \mathcal{R}e \int_{\Omega_e} g^\varepsilon \bar{v}^\varepsilon\,dx.$$

Therefore one deduces that v^ε is bounded in $H^1(\Omega)$. Hence one can assume that, up to the extraction of a subsequence, $v^\varepsilon \to v$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. First we have $\|v\|_{L^2(\Omega)} = 1$. Taking the limit in (8), restricted to Ω_e, yields

$$\left\{\begin{array}{ll} -\Delta v - \omega^2 v = 0, & \text{in } \Omega_e, \\ \partial_n v + i\omega v = 0, & \text{on } \partial\Omega. \end{array}\right. \tag{10}$$

On the other hand, taking the imaginary part in (9) shows in particular that

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} \leq \varepsilon^2 \, \|g^\varepsilon\|_{L^2(\Omega)} \, \|v^\varepsilon\|_{L^2(\Omega)} \, .$$

Thus $v^\varepsilon \to 0$ in $L^2(\Omega_i)$, hence $v = 0$ in Ω_i. In particular

$$v = 0 \quad \text{on} \;\; \partial\Omega_i .$$

Combined with (10), this condition shows that $v = 0$ in Ω_e. We get then $v = 0$ in Ω which is in contradiction with $\|v\|_{L^2(\Omega)} = 1$. □

Corollary 2.1 *There exists a constant $C > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq C \;\, \|f\|_{L^2(\Omega)} \quad \textit{and} \quad \|u^\varepsilon\|_{L^2(\Omega_i)} \leq C\,\varepsilon \;\|f\|_{L^2(\Omega)} \, . \tag{11}$$

Proof. This corollary is a direct consequence of energy identity

$$\int_\Omega \left(\, |\nabla u^\varepsilon|^2 - \omega^2 |u^\varepsilon|^2 \, \right) \, dx + i \left(\int_{\partial\Omega} \omega |u^\varepsilon|^2 \, ds + \frac{1}{\varepsilon^2} \int_{\Omega_i} |u^\varepsilon|^2 \, dx \right) = \int_{\Omega_e} f \, \bar{u^\varepsilon} \, dx$$

and the stability result of Theorem 2.1. □

Corollary 2.1 shows in particular that the solution converges to 0 like $O(\varepsilon)$ inside Ω_i, at least in the L^2 sense. This $O(\varepsilon)$ L^2-interior estimate is in fact not optimal. A sharper result will be given in lemma 5.1, where we show that $\|u^\varepsilon\|_{L^2(\Omega_i)}$ is $O(\varepsilon^{3/2})$ (see Remark 5.2).

2.2 Exponential interior decay of the solution

In this section, we first give an estimate which makes more precise the description of the interior decay results. We show that if we look at the solution in a domain which strictly interior to Ω_i the decay of the solution is more rapid than any power of ε. This is a first way to express that the main part of the interior solution will concentrate near the boundary Γ. The precise result is the following:

Theorem 2.2 *Let $\delta > 0$ and define $\Omega_i^\delta = \{x \in \Omega_i; \mathrm{dist}(x, \partial\Omega_i) > \delta\}$, then there exist two positive constants C^δ and γ^δ independent of ε such that*

$$\|u_i^\varepsilon\|_{H^1(\Omega_i^\delta)} \leq C^\delta \exp\left(-\gamma^\delta/\varepsilon\right) \|f\|_{L^2} \, .$$

Proof. One possible proof of this estimate can be obtained by using the integral representation of the solution inside Ω_i and the result of Corollary 2.1 . We shall present here an alternative variational approach that is also valid in the case of variable coefficients.

Let us introduce a cut-off function $\phi_\delta \in C^\infty(\Omega)$ such that

$$\phi_\delta(x) = 0 \;\text{ in }\; \Omega_e \, , \qquad \phi_\delta(x) = \beta^\delta \;\text{ in }\; \Omega_i^\delta,$$

where the constant $\beta^\delta > 0$ is chosen such that

$$\|\nabla\phi_\delta\|_\infty^2 < \frac{1}{4}\,. \tag{12}$$

We set $v^\varepsilon = \exp(\phi_\delta(x)/\varepsilon)u^\varepsilon$. Straightforward calculations show that

$$\Delta u^\varepsilon = \exp(-\phi_\delta(x)/\varepsilon)\left(\Delta v^\varepsilon - \frac{1}{\varepsilon}(2\nabla\phi_\delta\cdot\nabla v^\varepsilon + \Delta\phi_\delta\, v^\varepsilon) + \frac{|\nabla\phi_\delta|^2}{\varepsilon^2}v^\varepsilon\right).$$

Hence v^ε satisfies

$$\begin{cases} -\Delta v^\varepsilon + \dfrac{2}{\varepsilon}\nabla\phi_\delta\cdot\nabla v^\varepsilon + (-\omega^2 + \dfrac{\Delta\phi_\delta}{\varepsilon} - \dfrac{|\nabla\phi_\delta|^2}{\varepsilon^2} + i\omega\sigma^\varepsilon)\, v^\varepsilon = f & \text{in } \Omega \\ \partial_n v^\varepsilon + i\omega v^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Multiplying the first equation in (13) by $\bar{v}^\varepsilon$ integrating by parts in Ω yields, using the fact that $v^\varepsilon = u_e^\varepsilon$ in Ω_e,

$$\begin{aligned} \int_{\Omega_i}|\nabla v^\varepsilon|^2\,dx + \frac{2}{\varepsilon}\int_{\Omega_i}\nabla\phi_\delta\cdot\nabla v^\varepsilon\,\bar{v}^\varepsilon\,dx + \int_{\Omega_i}\left(-\omega^2 + \frac{\Delta\phi_\delta}{\varepsilon} - \frac{|\nabla\phi_\delta|^2 - i}{\varepsilon^2}\right)|v^\varepsilon|^2\,dx \\ = \int_\Omega f\,\bar{u_e^\varepsilon}\,dx + \int_{\Omega_e}\left(\omega^2|u_e^\varepsilon|^2 - |\nabla u_e^\varepsilon|^2\right)\,dx - i\omega\int_{\partial\Omega}|u_e^\varepsilon|^2\,ds \end{aligned} \tag{14}$$

Let us denote by L_ε the right hand side of the previous equality. According to Corollary 2.1, there exists a constant C independent of ε such that

$$|L_\varepsilon| \le C\;\|f\|^2_{L^2(\Omega)}\,.$$

On the other hand, thanks to inequality (12), using $|b-a| \ge |b| - |a|$, we have the lower bound:

$$\left|\;\frac{|\nabla\phi_\delta|^2 - i}{\varepsilon^2} - \left(\frac{\Delta\phi_\delta}{\varepsilon} - \omega^2\right)\;\right| \;\ge\; \frac{1}{\varepsilon^2} - \frac{|\Delta\phi_\delta|}{\varepsilon} - \omega^2.$$

Therefore, taking the modulus of (14) yields

$$\left|\begin{aligned} \|\nabla v^\varepsilon\|^2_{L^2(\Omega_i)} \;&-\; \frac{2}{\varepsilon}\,\|\nabla\phi_\delta\|_{L^\infty}\,\|\nabla v^\varepsilon\|_{L^2(\Omega_i)}\,\|v^\varepsilon\|_{L^2(\Omega_i)} \\ &+\; \left(\frac{1}{\varepsilon^2} - \frac{\|\Delta\phi_\delta\|_\infty}{\varepsilon} - \omega^2\right)\|v^\varepsilon\|^2_{L^2(\Omega_i)} \le C\;\|f\|^2_{L^2(\Omega)}\,. \end{aligned}\right.$$

Thanks to the inequality,

$$\frac{2}{\varepsilon}\,\|\nabla\phi_\delta\|_{L^\infty}\,\|\nabla v^\varepsilon\|_{L^2(\Omega_i)}\,\|v^\varepsilon\|_{L^2(\Omega_i)} \le \frac{1}{2}\,\|\nabla v^\varepsilon\|^2_{L^2(\Omega_i)} + \frac{2}{\varepsilon^2}\,\|\nabla\phi_\delta\|^2_{L^\infty}\,\|v^\varepsilon\|^2_{L^2(\Omega_i)}\,,$$

and inequality (12),

$$\frac{1}{2}\,\|\nabla v^\varepsilon\|^2_{L^2(\Omega_i)} + \left(\frac{1}{2\varepsilon^2} - \frac{\|\Delta\phi_\delta\|_\infty}{\varepsilon} - \omega^2\right)\|v^\varepsilon\|^2_{L^2(\Omega_i)} \le C\,\|f\|^2_{L^2(\Omega)}\,. \tag{15}$$

Finally, for ε small enough so that $\dfrac{1}{2\varepsilon^2} - \dfrac{\|\Delta\phi_\delta\|_\infty}{\varepsilon} - \omega^2 \ge \dfrac{1}{4\varepsilon^2}$, inequality (15) yields

$$\frac{1}{2}\,\|\nabla v^\varepsilon\|^2_{L^2(\Omega_i)} + \frac{1}{4\varepsilon^2}\,\|v^\varepsilon\|^2_{L^2(\Omega_i)} \le C\,\|f\|^2_{L^2(\Omega)}\,,$$

and the theorem is proven with $\gamma^\delta = \beta^\delta$ since

$$\begin{cases} \|u_i^\varepsilon\|_{L^2(\Omega_i^\delta)} \le \exp\left(-\beta^\delta/\varepsilon\right)\|v^\varepsilon\|_{L^2(\Omega_i)}\;, \\[2ex] \|\nabla u_i^\varepsilon\|_{L^2(\Omega_i^\delta)} \le \exp\left(-\beta^\delta/\varepsilon\right)\,\|\nabla v^\varepsilon\|_{L^2(\Omega_i)}\,. \end{cases}$$

□

2.3 A numerical illustration of the behaviour of the solution

As an illustration of the phenomena we wish to analyse, we present here some numerical results in 2D. More precisely, we compute the diffraction of a incident plane wave propagation along the x_1 axis in the direction $x_1 > 0$ by an absorbing disk Ω_i of center 0 and radius 1, that is we look for a solution of the form:

$$u = \exp i\omega x_1 + u^d, \tag{16}$$

where the total field satisfies the interior equation with $f = 0$ while the diffracted field u^d satisfies the outgoing radiation condition. The pulsation ω is taken equal to 4π which corresponds to a wavelength $\lambda = 0.5$. For the computation, a higher order finite element method with curved elements is used. The effective computations are reduced to the disk of radius 2 thanks to the help of an integral tranparent boundary condition (see [13]).

In figures 2.2 to 2.6, we represent the real part of the solution u. Clearly, the solution penetrates less and less the interior disk when the absorption coefficient σ incerases and the skin effect is clearly visible. To illustrate the boundary layer, we also represent the variations of the modulus of the total field along the line $x_2 = 0$: the exponential decay of the solution inside Ω_i appears clearly.

3 Statement of the main results

In this section, we present various approximate exterior boundary value problem that will characterize various approximations of the "exact" solution u_e^ε in the exterior domain. Each

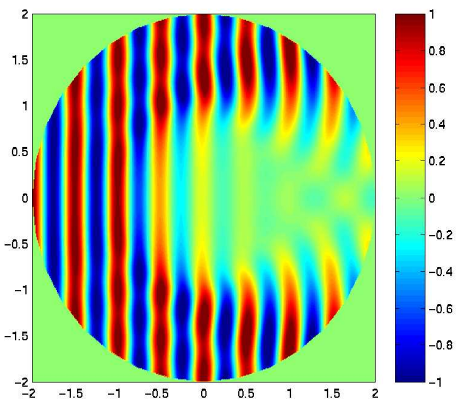

Figure 2: Total field for an absorbing disc: $\sigma = 45$.

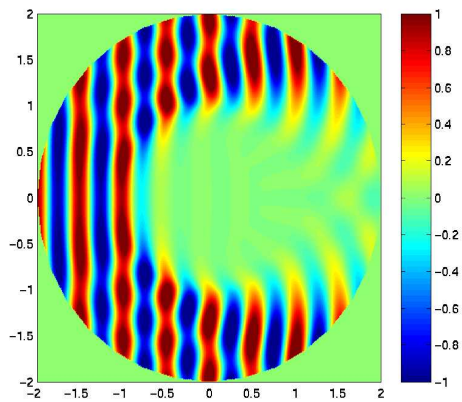

Figure 3: Total field for an absorbing disc: $\sigma = 156$.

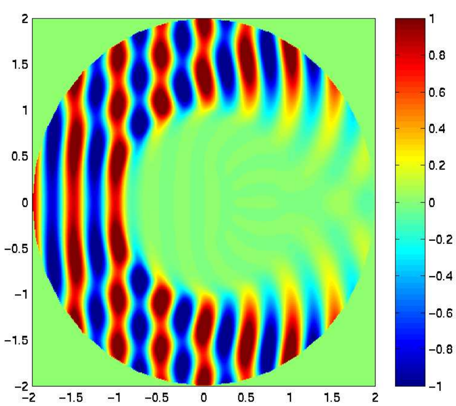

Figure 4: Total field for an absorbing disc: $\sigma = 400$.

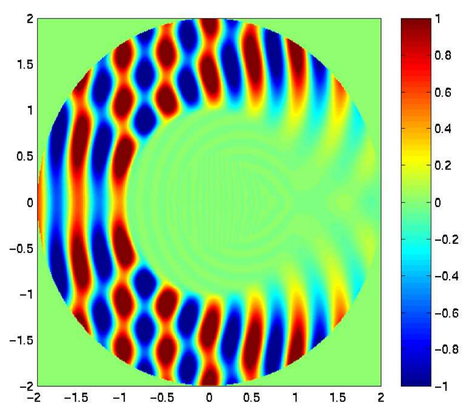

Figure 5: Total field for an absorbing disc: $\sigma = 2500$.

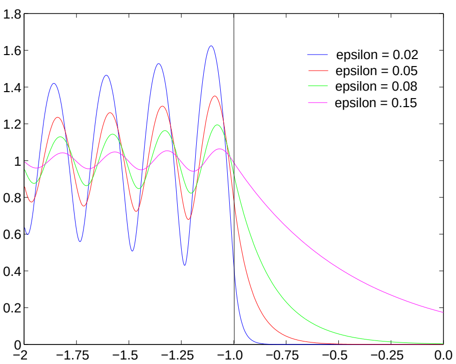

Figure 6: Variation of $x_1 \mapsto |u(x_1,0)|$

of these approximate problem is made of the standard Helmholtz equation in the exterior domain Ω_e, the outgoing impedance condition on $\partial\Omega$,

$$\begin{cases} -\Delta u^{\varepsilon,k} - \omega^2 u^{\varepsilon,k} = f & \text{in } \Omega_e, \\ \partial_n u^{\varepsilon,k} + i\omega u^{\varepsilon,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

and an appropriate $GIBC$ on the interior boundary Γ. We shall denote by $u^{\varepsilon,k}$ the approximate solution, where the integer index k refers to the order of the $GIBC$. The precise mathematical meaning of this order will be clarified with some error estimates (see Theorem 3.1) that completely justify the $GIBC's$. Let us say here that a $GIBC$ of order k is a boundary condition that will provide a (sharp) $O(\varepsilon^{k+1})$ error (in a sense to be given).

Let us mention here that, for a given integer k, there is not a unique way to write a $GIBC$ of order k. The $GIBC$s we will be speaking about in this paper will be of the form of a linear relationship between the Dirichlet and Neumann boundary values, $u^{\varepsilon,k}$ and $\partial_n u^{\varepsilon,k}$, involving local (differential) operators along the boundary Γ. The method that we shall use for deriving these $GIBC$s will naturally lead us to Neumann-to-Dirichlet (NtD) $GIBC$'s. These are the ones that we choose to present first in section 3.1. It will be clear in section 4 that we can derive, at least formally, a $GIBC$ of any order. However, the algebra becomes more and more involved as k increases, and it is perhaps impossible to write a general theory (existence, stability and error analysis) for any k. That is why we shall restrict ourselves, in this paper, to $GIBC$'s of order $k = 0, 1, 2$ and 3.

In section 3.2, we shall show how to easily derive, from (NtD) $GIBC$'s some modified $GIBC$'s that can be of Dirichlet-to-Neumann (DtN) nature (as more commonly presented in the litterature) or of mixed type.

We would not discuss in this paper which GIBC is better for a given order. Several criteria can guide such a choice: the adequation to a particular numerical method, the robustness of the $GIBC$ (this question will be slightly discussed later) or more importantly, its actual accuracy. It appears that a valuable comparison between the accuracy of $GIBC$'s of the same order will rely on numerical computations. That is why this part of the study is delayed to a forthcoming work of more numerical nature.

3.1 Neumann-to-Dirichlet GIBCs

Neumann-to-Dirichlet $GIBC$ can be seen as a (local) approximation of the exact Neumann-to-Dirichlet condition that would characterize u_e^ε, namely:

$$u_e^\varepsilon + \mathcal{D}^\varepsilon \partial_n u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{18}$$

where $\mathcal{D}^\varepsilon \in \mathcal{L}(H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))$ is the boundary operator defined by:

$$\mathcal{D}^\varepsilon \varphi = u_i^\varepsilon(\varphi),$$

where u_i^ε is the unique solution of the interior boundary value problem:

$$\begin{cases} -\Delta u_i^\varepsilon(\varphi) - \omega^2 u_i^\varepsilon(\varphi) + \dfrac{i}{\varepsilon^2} u_i^\varepsilon(\varphi) = 0 & \text{in } \Omega_i, \\ -\partial_n u_i^\varepsilon(\varphi) = \varphi & \text{on } \Gamma. \end{cases} \tag{19}$$

The absorbing nature of the interior medium is equivalent to the following absorbtion property of the operator $\mathcal{D}^\varepsilon$ (this follows from Green's formula):

$$\forall\, \varphi \in H^{-\frac{1}{2}}(\Gamma), \quad \mathcal{I}m \,\langle\, \mathcal{D}^\varepsilon \varphi, \varphi \rangle_\Gamma \;=\; -\frac{1}{\varepsilon^2} \int_{\Omega_i} |u_i^\varepsilon(\varphi)|^2 \, dx \leq 0. \tag{20}$$

It is well known that the operator $\mathcal{D}^\varepsilon$ is a non-local pseudo-differential operator whose explicit expression is not known (see however Remark 3.2 below). Nevertheless as $\varepsilon \to 0$, this operator becomes "almost local" (even differential), which is more or less intuitive according to the exponential interior decay of the solution with respect to ε^{-1}.

We claim that a Neumann-to-Dirichlet $GIBC$ of order k is given by:

$$u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} \partial_n u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{21}$$

where, for $k = 0, 1, 2, 3$, the operator $\mathcal{D}^{\varepsilon,k}$ is given by:

$$\textbf{For k = 0}, \qquad \mathcal{D}^{\varepsilon,0} = 0, \tag{22}$$

$$\textbf{For k = 1}, \qquad \mathcal{D}^{\varepsilon,1} = \frac{\varepsilon}{\alpha}, \tag{23}$$

$$\textbf{For k = 2}, \qquad \mathcal{D}^{\varepsilon,2} = \frac{\varepsilon}{\alpha} + i\mathcal{H}\varepsilon^2, \tag{24}$$

$$\textbf{For k = 3}, \qquad \mathcal{D}^{\varepsilon,3} = \frac{\varepsilon}{\alpha} + i\mathcal{H}\varepsilon^2 - \frac{\alpha\,\varepsilon^3}{2}\left(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma\right), \tag{25}$$

where, in expressions (22) to (25),

- $\alpha := \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ denotes the complex square root of i with positive real part,
- $\mathcal{H}$ and G respectively denotes the mean and Gaussian curvatures of Γ (see section 4.1),
- Δ_Γ denotes the Laplace-Beltrami operator along Γ.

Remark 3.1 *It is worthwhile to remark that:*

- *One recovers of course the Dirichlet condition in the case $k = 0$ (the limit of $\mathcal{D}^{\varepsilon,k}$ when $\varepsilon \to 0$ is 0 for any k): the Dirichlet condition appears as the GIBC of order 0.*
- *The first three conditions are exactly of the same nature and correspond to a purely local impedance condition. Notice that the geometry of Γ only appears in the third condition through the mean curvature $\mathcal{H}$. The numerical approximation of the three conditions has therefore the same cost (provided that $\mathcal{H}$ is easily computable). Thus (24) has to be preferred to (23).*
- *Condition (25) is more complicated. It involves a tangential differential operator along the boundary. This additional complexity will have of course consequences on the numerical approximation but also, as later shown, on the mathematical analysis.*

Remark 3.2 *The operators $\mathcal{D}^{\varepsilon,k}$ are of the form $\sum_{j\leq k} \varepsilon^j\ \mathcal{D}^j$ and thus appear as some truncated Taylor expansions of $\mathcal{D}^\varepsilon$. This is particularly clear in the (very special) case where Ω_i is the half-space $x_3 < 0$. In that case, the symbol of $\mathcal{D}^\varepsilon$ can be computed explicitly. More precisely, if one uses Fourier transform in the variables (x_2, x_2), one gets:*

$$\widehat{\mathcal{D}^\varepsilon\varphi}(\xi) = D^\varepsilon(\xi)\hat{\varphi}(\xi), \quad D^\varepsilon(\xi) = (|\xi|^2 - \omega^2 + i/\varepsilon^2)^{-\frac{1}{2}}, \quad \mathcal{I}m D^\varepsilon(\xi) > 0.$$

It is then straightforward to recover conditions (22) to (25) from successive Taylor expansions (for small ε) of $D^\varepsilon(\xi)$.

The main results of our paper are summarized in the following theorem:

Theorem 3.1 *Let $k = 0, 1, 2$ or 3, then, for sufficiently small ε, the boundary value problem ((17), (21)) has a unique solution $u^{\varepsilon,k} \in H^1(\Omega_e)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(\Omega_e)} \leq C_k\, \varepsilon^{k+1}. \tag{26}$$

Remark 3.3 *Let us mention that:*

- *For $k \leq 2$, the existence and uniqueness proof via Fredholm's alternative is trivial. The uniqueness proof relies on the following inequality (analogous to (20)):*

  $$\forall\, \varphi \in L^2(\Gamma), \quad \mathcal{I}m \int_\Gamma \mathcal{D}^{\varepsilon,k}\varphi \cdot \bar{\varphi}\, ds \leq -\nu \int_\Gamma |\varphi|^2\, ds, \quad (\textit{for some } \nu \leq 0), \tag{27}$$

  that expresses in particular the absorbing nature of the boundary condition and provides a sufficient (but not necessary) condition for the uniqueness of the solution. With this argument, it is easy to see that, for $k = 0, 1$, the existence and uniqueness result holds in fact for any $\varepsilon \geq 0$. For $k = 2$ one easily checks that (27) is true as soon as:

  $$\varepsilon \mathcal{H} \leq \frac{\sqrt{2}}{2}, \quad \textit{a. e. on } \Gamma. \tag{28}$$

  Note that this inequality is algebraic. When Ω_i is convex, $\mathcal{H} \leq 0$ along Γ so that (28) induces no constraints on ε.

- *In the case $k = 3$, the proof is more complicated. In particular, there is no clear equivalent to inequality (27) and the uniqueness proof requires some more sophisticated argument (see lemma 5.4). This explains why in this case, one has no explicit upper bound for ε below which uniqueness is guaranteed.*

3.2 Modified GIBCs

Dirichlet to Neumann $GIBC$'s. If we introduce $\mathcal{N}^\varepsilon := (\mathcal{D}^\varepsilon)^{-1}$, then the exact boundary condition for u_e^ε can be rewritten as :

$$\partial_n u_e^\varepsilon + \mathcal{N}^\varepsilon u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{29}$$

In our terminology a DtN $GIBC$ will be of the form:

$$\partial_n u^{\varepsilon,k} + \mathcal{N}^{\varepsilon,k} u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{30}$$

where $\mathcal{N}^{\varepsilon,k}$ denotes some local approximation of $\mathcal{N}^\varepsilon$. They can be directly obtained from $\mathcal{D}^{\varepsilon,k}$ by seeking local operators $\mathcal{N}^{\varepsilon,k}$ that formally satisfy:

$$\mathcal{D}^{\varepsilon,k} = \left(\mathcal{N}^{\varepsilon,k}\right)^{-1} + O(\varepsilon^{k+1}). \tag{31}$$

The expression of $\mathcal{N}^{\varepsilon,k}$ is derived from formal Taylor expansions of $(\mathcal{D}^{\varepsilon,k})^{-1}$. One gets,

$$\textbf{For k = 2}, \qquad \mathcal{N}^{\varepsilon,2} = \frac{\alpha}{\varepsilon} + \mathcal{H}, \tag{32}$$

$$\textbf{For k = 3}, \qquad \mathcal{N}^{\varepsilon,3} = \frac{\alpha}{\varepsilon} + \mathcal{H} - \frac{\varepsilon}{2\alpha}(\Delta_\Gamma + \mathcal{H}^2 - G + \omega^2). \tag{33}$$

The important point here is that the results (existence, uniqueness and error estimates) stated in theorem 3.1 for problem ((17),(21)) still hold for problem ((17), (30)). We refer to section 6.

Remark 3.4 *Let us notice that:*

- *The difference between conditions (24) and (32) is quite small. In fact (32) can also be seen as a NtD GIBC with* $\mathcal{D}^{\varepsilon,2} = (\frac{\alpha}{\varepsilon} + \mathcal{H})^{-1}$ *! Note however that it is clear that for* $k = 2$, *the problem ((17), (30)) is well posed for any value of* ε. *This is a consequence of the following (uniform) absorption property*

$$\forall\, \varphi \in L^2(\Gamma), \quad \mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,2}\varphi}\, ds \le -\frac{\sqrt{2}}{2\varepsilon} \int_\Gamma |\varphi|^2\, ds. \tag{34}$$

- *The difference between conditions (25) and (33) is much more important. This has consequences on both mathematical and numerical analyses.*

Robust ***GIBC*'s.** As mentioned earlier, an important property of the "exact" impedance condition is what we shall refer to as *absorption property.* It can be formally formulated for $\mathcal{D}^\varepsilon$ (resp. $\mathcal{N}^\varepsilon$) by:

$$\begin{aligned} &\forall\, \varphi, \quad \mathcal{I}m \int_\Gamma \mathcal{D}^\varepsilon \varphi \cdot \overline{\varphi}\, ds \le 0, \\ &\left(\text{resp.} \quad \forall\, \varphi, \quad \mathcal{I}m \int_\Gamma \varphi \cdot \overline{\mathcal{N}^\varepsilon \varphi}\, ds \le 0\right). \end{aligned} \tag{35}$$

It is therefore desirable that NtD *GIBC*'s (resp. DtN *GIBC*'s) preserve (35), i. e. operators $\mathcal{D}^{\varepsilon,k}$ (resp. $\mathcal{N}^{\varepsilon,k}$) appearing in (21) (resp. (30)) possesses an analogous absorption property. In particular, this would automatically imply the well-posedness of the approximate problem for any ε. This is why we shall say that such a boundary condition is *robust.*

Remark 3.5 *As later shown, the robustness, in the sense meant here, is not a necessary condition for a GIBC to work. However, if one thinks of extending these conditions to time dependent problems, then it is more likely that robustness will be required to ensure time stability of the GIBC.*

In this sense, the second order NtD *GIBC* (24) is not robust, while the second order DtN *GIBC* (32) is. In other words (32) is a robust version of (24). Concerning the third order conditions, neither the NtD *GIBC* (25) nor the DtN *GIBC* (33) is robust. Indeed, one has the identities:

$$\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi\cdot\overline{\varphi}\,ds = \frac{\alpha\,\varepsilon^3}{2}\int_\Gamma |\nabla_\Gamma\,\varphi|^2\,ds + \varepsilon\bar{\alpha}\int_\Gamma [\,1+\frac{\varepsilon\mathcal{H}}{\alpha} - i\,\frac{\varepsilon^2}{2}\,(3\mathcal{H}^2 - G + \omega^2)\,]\,|\varphi|^2\,ds,$$
$$\int_\Gamma \varphi\cdot\overline{\mathcal{N}^{\varepsilon,3}\varphi}\,ds = \frac{\alpha\,\varepsilon}{2}\int_\Gamma |\nabla_\Gamma\,\varphi|^2\,ds + \frac{\bar{\alpha}}{\varepsilon}\int_\Gamma [\,1+\frac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\,\frac{\varepsilon^2}{2}\,(\mathcal{H}^2 - G + \omega^2)\,]\,|\varphi|^2\,ds.$$

from which one easily computes that (remember $\alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$):

$$\left\{\begin{aligned} &\mathcal{I}m\int_\Gamma \mathcal{D}^{\varepsilon,3}\varphi\cdot\overline{\varphi}\,ds = \frac{\sqrt{2}\,\varepsilon^3}{4}\int_\Gamma |\nabla_\Gamma\,\varphi|^2\,ds - \frac{\varepsilon\sqrt{2}}{2}\int_\Gamma \rho_1^\varepsilon\,|\varphi|^2\,ds,\\ &\mathcal{I}m\int_\Gamma \varphi\cdot\overline{\mathcal{N}^{\varepsilon,3}\varphi}\,ds = \frac{\sqrt{2}\,\varepsilon}{4}\int_\Gamma |\nabla_\Gamma\,\varphi|^2\,ds - \frac{\sqrt{2}}{2\varepsilon}\int_\Gamma \rho_2^\varepsilon\,|\varphi|^2\,ds.\end{aligned}\right.$$

where the functions ρ_j^ε converge (uniformly on Γ) to 1 when ε tend to 0 (and are thus positive for ε small enough). The problem then is that the integrals in $|\nabla_\Gamma\,\varphi|^2$ come with the wrong sign.

As we shall now explain, it is possible to construct robust *GIBC*s of order 3. The idea is to use some appropriate Padé approximation of the imaginary part of the boundary operators that formally gives the same order of approximation but restore absorption property. Consider for instance the NtD *GIBC* of order 3. Indeed

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3} = -\varepsilon\frac{\sqrt{2}}{2}\left(1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right).$$

One can therefore formally write

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3} = -\varepsilon\,\frac{\sqrt{2}}{2}\,\left(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)\right)\left(1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma\right) + O(\varepsilon^4).$$

Note that, as $\mathcal{H}^2 - G = \frac{1}{4}(c_1 - c_2)^2$, where c_1 and c_2 are the two principal curvatures along Γ (see section 4.1), we have

$$(1 + \frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2)) > 0.$$

It is then sufficient to seek a positive approximation of $(1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma)$ which can be obtained by considering the formal inverse, namely

$$1 - \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}\Delta_\Gamma = \left\{1 + \varepsilon\sqrt{2}\mathcal{H} + \frac{\varepsilon^2}{2}(4\mathcal{H}^2 - \Delta_\Gamma)\right\}^{-1} + O(\varepsilon^3).$$

Therefore,

$$\mathcal{I}m\mathcal{D}^{\varepsilon,3} = -\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}\,(3\mathcal{H}^2 - G + \omega^2))\,(1+\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}(4\mathcal{H}^2-\Delta_\Gamma))^{-1} + O(\varepsilon^4).$$

A robust NtD-like $GIBC$ of order 3 is obtained by replacing $\mathcal{D}^{\varepsilon,3}$ by

$$\begin{aligned}\mathcal{D}_r^{\varepsilon,3} \quad := \quad & \varepsilon\frac{\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right) \\ & -i\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}\,(3\mathcal{H}^2 - G + \omega^2))\,(1+\varepsilon\sqrt{2}\mathcal{H}+\frac{\varepsilon^2}{2}(4\mathcal{H}^2-\Delta_\Gamma))^{-1}.\end{aligned} \tag{36}$$

This expression will de used in practice in the following sense:

$$\mathcal{D}_r^{\varepsilon,3}\varphi := \varepsilon\frac{\sqrt{2}}{2}\left(1-\frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right)\varphi - i\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G + \omega^2))\psi \tag{37}$$

where ψ is solution to:

$$-\frac{\varepsilon^2}{2}\Delta_\Gamma\psi + (1+\varepsilon\sqrt{2}\mathcal{H}+2\varepsilon^2\mathcal{H}^2)\psi = \varphi. \tag{38}$$

One can easily verify that

$$\int_\Gamma \mathcal{D}_r^{\varepsilon,3}\varphi\cdot\overline{\varphi}\,ds = -\varepsilon\frac{\sqrt{2}}{2}(1+\frac{\varepsilon^2}{2}(3\mathcal{H}^2 - G+\omega^2))\int_\Gamma(1+\varepsilon\sqrt{2}\mathcal{H}+\varepsilon^2 2\mathcal{H}^2)|\psi|^2 + \frac{\varepsilon^2}{2}|\nabla_\Gamma\,\psi|^2\,ds. \tag{39}$$

The right hand side is non positive for all ε whence the absorption property for $\mathcal{D}_r^{\varepsilon,3}$.

Of course one can follow a similar procedure to derive robust DtN third order $GIBC$. The expression of this condition is based on the approximation:

$$\mathcal{I}m\mathcal{N}^{\varepsilon,3} = \frac{\sqrt{2}}{2\varepsilon}(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G+\omega^2))\,\{1-\frac{\varepsilon^2}{2}\Delta_\Gamma\}^{-1} + O(\varepsilon^2). \tag{40}$$

Hence, replacing $\mathcal{N}^{\varepsilon,3}$ by

$$\begin{aligned}\mathcal{N}_r^{\varepsilon,3} \quad := \quad & \frac{\sqrt{2}}{2\varepsilon}\left(1+\varepsilon\sqrt{2}\mathcal{H}-\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma)\right) \\ & +i\frac{\sqrt{2}}{2\varepsilon}(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G + \omega^2))\,\{1-\frac{\varepsilon^2}{2}\Delta_\Gamma\}^{-1},\end{aligned} \tag{41}$$

in (33) gives another third order DtN $GIBC$. This condition is robust in view of the following identity, where the right hand side is non positive for all ε,

$$\mathcal{I}m\int_\Gamma \varphi\cdot\overline{\mathcal{N}^{\varepsilon,3}\varphi}\,ds = -\frac{\sqrt{2}}{2\varepsilon}\int_\Gamma(1+\frac{\varepsilon^2}{2}(\mathcal{H}^2 - G+\omega^2))\left(|\psi|^2+\frac{\varepsilon^2}{2}|\nabla_\Gamma\,\psi|^2\right)\,ds \tag{42}$$

where ψ is solution to

$$-\frac{\varepsilon^2}{2}\Delta_\Gamma\psi + \psi = \varphi.$$

Remark 3.6 *As one can notice, there is no unique manner to derive GIBC, and even robust NtD (or DtN) GIBC. Note also that the proposed third order ones involve a fourth order surface differential operator. It does not seem easy to derive a robust third order GIBC with only second order differential operator.*

4 Formal derivation of the GIBC

4.1 Preliminary material

Geometrical tools. Let n be the inward normal field defined on $\partial\Omega_i$ and let δ be a given positive constant chosen to be sufficiently small so that

$$\Omega_i^\delta = \{x \in \Omega_i \ ; \ \mathrm{dist}(x, \partial\Omega_i) < \delta\}$$

can be uniquely parameterized by the tangential coordinate x_Γ on Γ and the normal coordinate $\nu \in (0,\delta)$ through

$$x = x_\Gamma + \nu n, \quad x \in \Omega_i^\delta. \tag{43}$$

Let us now recall some concepts and identities from differential geometry (the notion of surface differential operator is supposed to be known - see [11]). Let $\mathcal{C} := \nabla_\Gamma n$ denote the curvature tensor on Γ. We recall that $\mathcal{C}$ is symmetric and $\mathcal{C}\,n = 0$. We denote c_1, c_2 the eigenvalues of $\mathcal{C}$ (namely the *principal curvatures* associated with tangential eigenvectors τ_1, τ_2. $G := c_1c_2$ and $\mathcal{H} := \frac{1}{2}(c_1 + c_2)$ are respectively the *Gaussian* and *mean curvatures* of Γ. Let us define the tangential operator $\mathcal{R}_\nu$ on Γ by

$$(I + \nu\,\mathcal{C}(x_\Gamma))\,\mathcal{R}_\nu(x_\Gamma) = I_\Gamma(x_\Gamma)$$

where $I_\Gamma(x_\Gamma)$ denotes the projection operator on the tangent plane to Γ at x_Γ. Then one has (see [11])

$$\nabla = \mathcal{R}_\nu\nabla_\Gamma + \partial_\nu\, n, \tag{44}$$

where ∇_Γ is the surface gradient on Γ. If one sets

$$J_\nu := \det(I + \nu\,\mathcal{C}) = 1 + 2\nu\mathcal{H} + \nu^2 G,$$

then, from integration by part formulas and (44), one gets

$$\Delta = \frac{1}{J_\nu}\mathrm{div}_\Gamma\,(\mathcal{R}_\nu J_\nu \mathcal{R}_\nu)\nabla_\Gamma + \frac{1}{J_\nu}\partial_\nu J_\nu \partial_\nu, \tag{45}$$

where div_Γ denotes the surface divergence on Γ. Define the tangential operator $\mathcal{M}$ on Γ by

$$I + \nu M = J_\nu (I + \nu C)^{-1}.$$

then M is independant ν and one has

$$\mathcal{C}\,\mathcal{M} = G\,I$$

Therefore, identity (45) can be transformed to

$$\Delta = \frac{1}{J_\nu}\,\mathrm{div}_\Gamma\,\left(\frac{1}{J_\nu}(I_\Gamma + \nu\mathcal{M})^2\right)\nabla_\Gamma\, + \frac{1}{J_\nu}\partial_\nu J_\nu \partial_\nu,$$

or, in an equivalent form

$$J_\nu^3\,\Delta = J_\nu\,\mathrm{div}_\Gamma\,(I_\Gamma + \nu\mathcal{M})^2\nabla_\Gamma\, - \nabla_\Gamma\, J_\nu\cdot(I_\Gamma + \nu\mathcal{M})^2\nabla_\Gamma\, + J_\nu^3\,\partial^2_{\nu\nu} + 2J_\nu^2\,(\mathcal{H} + \nu G)\,\partial_\nu \quad (46)$$

This latter expression is more convenient for the asymptotic matching procedure, that we shall describe later, because we made the dependence of the operators coefficients polynomial with respect to ν.

The asymptotic ansatz. As it is quite usual, the derivation of the approximate boundary conditions will be based on an ansatz about the solution, that is to say an a priori particular form in which the solution is looked for. To formulate this ansatz, it is useful to introduce a cut off function $\chi \in C^\infty(\Omega_i)$ such that $\chi = 1$ in $\Omega_i^{\delta/2}$ and $\chi = 0$ in $\Omega_i \setminus \Omega_i^\delta$. In our ansatz we are not interested by $(1-\chi)u_i^\varepsilon$ that we know to decrease exponentially to 0 with ε (this is theorem 2.2) . For the remaining part of the solution, we postulate the following expansions:

$$u_e^\varepsilon(x) = u_e^0(x) + \varepsilon u_e^1(x) + \varepsilon^2 u_e^2(x) + \cdots \quad \text{for } x \in \Omega_e \qquad (47)$$

where u_e^ℓ, $\ell = 0,\, 1,\, \cdots$ are functions defined on Ω_e and

$$\chi(x)u_i^\varepsilon(x) = u_i^0(x_\Gamma, \nu/\varepsilon) + \varepsilon u_i^1(x_\Gamma, \nu/\varepsilon) + \varepsilon^2 u_i^2(x_\Gamma, \nu/\varepsilon) + \cdots \quad \text{for } x \in \Omega_i^\delta \qquad (48)$$

where x, x_Γ and ν are as in (43) and where $u_i^\ell(x_\Gamma, \eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ are functions such that

$$\lim_{\eta\to\infty} u_i^\ell(x_\Gamma, \eta) = 0 \quad \text{for a.e. } x_\Gamma \in \Gamma. \qquad (49)$$

The latter condition will ensure that the u_i^ℓ's are exponentially decreasing with respect to η.

Remark 4.1 *Note that the expansion (48) makes sense since the local coordinates* (x_Γ, ν) *can be used inside the support of* χ*.*

In the next section, we shall identify the set of equations satisfied by (u_e^ℓ) and (u_i^ℓ) and the formal expansions (47) and (48) will be justified in section 5.

It will be useful to introduce the notation

$$\widetilde{u}_i^\varepsilon(x_\Gamma,\eta) := u_i^0(x_\Gamma,\eta) + \varepsilon u_i^1(x_\Gamma,\eta) + \varepsilon^2 u_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma \times \mathbb{R}^+, \tag{50}$$

so that ansatz (48) has to be understood as

$$\chi(x) u_i^\varepsilon(x) = \widetilde{u}_i^\varepsilon(x_\Gamma, \nu/\varepsilon) + O(\varepsilon^\infty) \quad \text{for } x \in \Omega_i^\eta. \tag{51}$$

4.2 Asymptotic formal matching

Let us first consider the exterior field u_e^ε, it is clear that each of the terms u_e^k in the expansion satisfies the (outgoing) Helmholtz equation in Ω_e (simply substitute (47) into (6)(i)):

$$\begin{cases} -\Delta u_e^k - \omega^2 u_e^k = 0 & \text{in } \Omega_e, \\ \partial_n u_e^k + i\omega u_e^k = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

Concerning the interior field, from (6-ii), (51) and the substitution $\nu = \varepsilon\eta$ in (46), we obtain the following equation:

$$\left| \begin{aligned} & -\frac{1}{\varepsilon^2} J_{\varepsilon\eta}^3 \, \partial_{\eta\eta}^2 \widetilde{u}_i^\varepsilon - \frac{2}{\varepsilon} J_{\varepsilon\eta}^2 \, (\mathcal{H} + \varepsilon\eta G) \, \partial_\eta \widetilde{u}_i^\varepsilon \\ & \quad - J_{\varepsilon\eta} \operatorname{div}_\Gamma (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \, \nabla_\Gamma \widetilde{u}_i^\varepsilon + \nabla_\Gamma J_{\varepsilon\eta} \cdot (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \, \nabla_\Gamma \widetilde{u}_i^\varepsilon \\ & \qquad\qquad + J_{\varepsilon\eta}^3 (-\omega^2 + \frac{i}{\varepsilon^2}) \, \widetilde{u}_i^\varepsilon = 0 \end{aligned} \right. \tag{53}$$

that can be rearranged in the following form after multiplying by ε^2:

$$\left| \begin{aligned} (-\partial_{\eta\eta}^2 + i) \, \tilde{u}_i^\varepsilon \; &= \; (1 - J_{\varepsilon\eta}^3)(-\partial_{\eta\eta}^2 + i) \, \tilde{u}_i^\varepsilon + 2\varepsilon J_{\varepsilon\eta}^2 (\mathcal{H} + \varepsilon\eta G) \, \partial_\eta \tilde{u}_i^\varepsilon + \varepsilon^2 \omega^2 J_{\varepsilon\eta}^3 \tilde{u}_i^\varepsilon \\ &+ \; \varepsilon^2 J_{\varepsilon\eta} \operatorname{div}_\Gamma (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma \, - \varepsilon^2 \, \nabla_\Gamma J_{\varepsilon\eta} \cdot (I_\Gamma + \varepsilon\eta\mathcal{M})^2 \nabla_\Gamma \, \tilde{u}_i^\varepsilon. \end{aligned} \right. \tag{54}$$

Considering that J_ν is a polynomial of degree 2 in ν, (54) can be rewritten as:

$$(-\partial_{\eta\eta}^2 + i) \, \tilde{u}_i^\varepsilon = \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell \, \tilde{u}_i^\varepsilon, \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{55}$$

where $\mathcal{A}_\ell$ are some partial differential operators in (x_Γ, η) that are independent of ε. Formal identification gives, after rather lengthy than complicated calculations,

$$\mathcal{A}_1 = 2\mathcal{H}\,\partial_\eta - 6\eta\,\mathcal{H}\,(-\partial^2_{\eta\eta} + i) \tag{56}$$

$$\mathcal{A}_2 = \Delta_\Gamma + \omega^2 + 2\eta\,(G + 4\mathcal{H}^2)\partial_\eta - 3\eta^2\,(G + 4\mathcal{H}^2)(-\partial^2_{\eta\eta} + i) \tag{57}$$

$$\begin{aligned}\mathcal{A}_3 &= 2\eta\,\left[\,\mathcal{H}\Delta_\Gamma + \mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) - \nabla_\Gamma\,\mathcal{H}\cdot\nabla_\Gamma\, + 3\omega^2\mathcal{H}\,\right]\\ &+\ 4\eta^2\,\mathcal{H}\,\left[\,(3G + 2\mathcal{H}^2)\,\partial_\eta\,\right] - 4\eta^3\,\mathcal{H}\,(3G + 2\mathcal{H}^2)\,(-\partial^2_{\eta\eta} + i)\end{aligned} \tag{58}$$

$$\begin{aligned}\mathcal{A}_4 &= \eta^2\,\left[\,G\,\Delta_\Gamma + 4\mathcal{H}\,\mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) + \mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,)\,\right]\\ &-\ \eta^2\,\left[\,\nabla_\Gamma\,G\cdot\nabla_\Gamma\, + 4\nabla_\Gamma\,\mathcal{H}\cdot(\mathcal{M}\nabla_\Gamma\,) - 3\omega^2(G + 4\mathcal{H}^2)\,\right]\\ &+\ 4\eta^3\,G\,(G + 4\mathcal{H}^2)\,\partial_\eta - 3\eta^4\,G\,(G + 4\mathcal{H}^2)\,(-\partial^2_{\eta\eta} + i)\end{aligned} \tag{59}$$

$$\begin{aligned}\mathcal{A}_5 &= 2\eta^3\,\left[\,G\,\mathrm{div}_\Gamma\,(\mathcal{M}\nabla_\Gamma\,) + \mathcal{H}\,\mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,)\,\right]\\ &-\ 2\eta^3\,\left[\nabla_\Gamma\,G\cdot(\mathcal{M}\nabla_\Gamma\,) + \nabla_\Gamma\,\mathcal{H}\cdot(\mathcal{M}^2\nabla_\Gamma\,) - 2\omega^2\,\mathcal{H}\,(3G + 2\mathcal{H}^2)\,\right]\\ &+\ 10\eta^4\,G^2\mathcal{H}\,\partial_\eta - 6\eta^5\,G^2\mathcal{H}\,(-\partial^2_{\eta\eta} + i)\end{aligned} \tag{60}$$

$$\begin{aligned}\mathcal{A}_6 &= \eta^4\,\left[\,G\,\mathrm{div}_\Gamma\,(\mathcal{M}^2\nabla_\Gamma\,) - \nabla_\Gamma\,G\cdot(\mathcal{M}^2\nabla_\Gamma\,) + 3\omega^2 G(G + 4\mathcal{H}^2)\,\right]\\ &+\ 2\eta^5\,G^3\partial_\eta - \eta^6\,G^3(-\partial^2_{\eta\eta} + i)\end{aligned} \tag{61}$$

$$\mathcal{A}_7 = 6\eta^5\,\omega^2 G^2\mathcal{H} \tag{62}$$

$$\mathcal{A}_8 = \eta^6\,\omega^2 G^3 \tag{63}$$

Therefore, making the substitution (50) in equation (55) and equating the terms of same order in ε, we obtain an induction on k that allows us to recursively determine the u_i^k's as functions of η. With the convention $u_i^k \equiv 0$ for $k < 0$, one can write it in the form

$$(-\partial^2_{\eta\eta} + i)\,u_i^k = \sum_{\ell=1}^{8} \mathcal{A}_\ell\,u_i^{k-\ell}, \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{64}$$

for all $k \geq 0$. For any $k \geq 0$, one assume that the fields u_i^l and u_e^l are known for $l < k$, (64) can be seen as an ordinary differential equation in η for $\eta \in [0, +\infty[$ whose unknown $\eta \mapsto u_i^k(x_\Gamma, \eta)$ (The variable x_Γ has purely the role of a parameter). As this equation is of order 2, in addition to the condition at infinity (49), the solution of (64) with respect to η requires one initial condition at $\eta = 0$. This condition will be provided by one of the two interface conditions (6-vi) and (6-v). We choose here to use the condition (6-v) which provides us a non homogeneous Neumann condition at $\eta = 0$ whose right hand side will be given by the exterior field u_e^{k-1}, namely (substitute (47)-(48) into (6-v) and identify the series after the change of variable $\nu = \varepsilon\eta$)

$$\partial_\eta u_i^k(x_\Gamma, 0) = \partial_n u_e^{k-1}|_\Gamma(x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{65}$$

With such a choice, the other condition (6-vi) will serve as a non homogeneous Dirichlet boundary condition for the exterior field u_e^k, to complete (52):

$$u_e^k|_\Gamma(x_\Gamma) = u_i^k(x_\Gamma, 0), \quad x_\Gamma \in \Gamma. \tag{66}$$

Remark 4.2 *Choosing (65) as the boundary condition for (64) will naturally lead to NtD GIBC's. The alternative choice (66) would naturally lead to DtN GIBC's. Our choice seems to be more natural because, thanks to the shift of index in (65), the right hand side really appears as something known from previous steps. Condition (66) appears more as a coupling condition!*

4.3 Description of the interior field inside the boundary layer

We are interested in getting analytic expression for the "interior fields" u_i^k by solving the boundary problem (in the variable η) made of (64), (65) and (49). To simplify the notation, we shall set:

$$du_i^k(x_\Gamma) := \partial_\eta u_i^k(x_\Gamma, 0), \quad x_\Gamma \in \Gamma. \tag{67}$$

Using standard techniques for linear differential equations [], it is easy to prove that the solution u_i^k is of the form:

$$u_i^k(x_\Gamma, \eta) = P_{x_\Gamma}^k(\eta)\; e^{-\alpha\,\eta} \tag{68}$$

for all $k \geq 0$, where $P_{x_\Gamma}^k$ is a polynomial with respect to η of degree k whose coefficients are proportional to $du_i^0, \cdots, du_i^{k-1}$ (remember $\alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$). More precisely, these polynomials satisfy a (affine) induction of order 8, of the form:

$$P_{x_\Gamma}^k(\eta) = -\frac{1}{\alpha}\; du_i^{k-1}(x_\Gamma) + \mathcal{L}_k\left(P_{x_\Gamma}^{k-1}(\eta), ..., P_{x_\Gamma}^{k-7}(\eta)\right)$$

where $\mathcal{L}_k$ is a linear form on $\mathbb{C}^7$ whose coefficients are linear in the $du_i^l(x_\Gamma)$'s. We shall not give here the expression of $\mathcal{L}_k$ for any k but restrict ourselves to the first four functions u_i^k

(this is sufficient for $GIBC$'s up to order 3)

$$u_i^0(x_\Gamma,\eta) = 0 \tag{69}$$

$$u_i^1(x_\Gamma,\eta) = -\frac{1}{\alpha}\, du_i^0(x_\Gamma)\, e^{-\alpha\eta} \tag{70}$$

$$u_i^2(x_\Gamma,\eta) = \left\{\left(-\frac{1}{\alpha}\, du_i^1(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2}\, du_i^0(x_\Gamma)\right) + \eta\, \frac{\mathcal{H}}{\alpha}\, du_i^0(x_\Gamma)\right\} e^{-\alpha\eta} \tag{71}$$

$$\begin{aligned} u_i^3(x_\Gamma,\eta) = & \Big\{ -\frac{1}{\alpha}\, du_i^2(x_\Gamma) + \frac{\mathcal{H}}{\alpha^2}\, du_i^1(x_\Gamma) \\ & - \frac{1}{2\alpha^3}\,(3\mathcal{H}^2 - G + \omega^2)\, du_i^0(x_\Gamma) - \frac{1}{2\alpha^3}\,\Delta_\Gamma[du_i^0](x_\Gamma) \\ & + \eta\, [\, \frac{\mathcal{H}}{\alpha}\, du_i^1(x_\Gamma) - \frac{1}{2\alpha^2}\,(\Delta_\Gamma - G + 3\mathcal{H}^2 + \omega^2)\, du_i^0(x_\Gamma)\,] \\ & + \eta^2\, \frac{1}{2\alpha}\,(G - 3\mathcal{H}^2)\, du_i^0(x_\Gamma) \Big\}\; e^{-\alpha\eta} \end{aligned} \tag{72}$$

4.4 Construction of the GIBCs

Let us first check inductively that, starting from $u_i^0 = 0$ and u_e^0 solution of the exterior Dirichlet problem the fields u_e^k and u_i^k are well defined. Assume that u_e^ℓ and u_i^ℓ are known for $\ell \leq k-1$. The the du_i^ℓ's are known by (67) ,u_i^k is determined by the explicit expression (68) (and more precisely (69) to (72) for $k = 0,1,2,3$). Then, u_e^k is determined as the unique solution of the boundary value problem (with $f^0 = f$ and $f^k = 0$ for $k \geq 1$):

$$\begin{cases} -\Delta u_e^k - \omega^2 u_e^k = f^k, & \text{in } \Omega_e, \\ \partial_n u_e^k + i\omega u_e^k = 0, & \text{on } \partial\Omega, \\ u_e^k = u^k_{i\,|\eta=0}\,, & \text{on } \Gamma. \end{cases} \tag{73}$$

Remark 4.3 *Since f is compactly supported in Ω_e, we deduce inductively from standard elliptic regularity that u_e^k is a smooth function in a neighborhood of Γ and $u_e^k(x_\Gamma, 0)$ is also a smooth function.*

The $GIBC$ of order k is obtained by considering the truncated expansion:

$$\widetilde{u}^{\varepsilon,k} := \sum_{\ell=0}^{k} \varepsilon^\ell u_e^\ell \tag{74}$$

as an approximation of order k of u_e^ε.

For example, for $k = 0$, we have $\widetilde{u}^{\varepsilon,k} = u_e^0$ and from equations (66) and (69), we deduce that $\widetilde{u}^{\varepsilon,k} = 0$ on Γ. In this case we set $u^{\varepsilon,k} = \widetilde{u}^{\varepsilon,k}$ and, as emphasized in remark 3.1, the Dirichlet condition:

$$u^{\varepsilon,k} = 0, \quad \text{on } \Gamma, \tag{75}$$

is a the *GIBC* of order 0.

For larger k, another approximation is needed. The principle of the calculation is the following. Using the second interface condition, namely (6-iv), one has

$$\widetilde{u}^{\varepsilon,k}|_\Gamma(x_\Gamma) = \sum_{\ell=0}^{k} \varepsilon^\ell u_i^\ell(x_\Gamma, 0) \quad \text{for} \quad x_\Gamma \in \Gamma. \tag{76}$$

Substituting expressions (69)-(72) into (76) leads to a boundary condition of the form

$$\widetilde{u}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\, \partial_n \widetilde{u}^{\varepsilon,k} = \varepsilon^{k+1}\, g_k^\varepsilon \quad \text{on } \Gamma, \quad \text{with } g_k^\varepsilon = O(1) \tag{77}$$

where $\mathcal{D}^{\varepsilon,k}$ is some boundary operator. The GIBC of order k that defines $u^{\varepsilon,k}$ (not $\widetilde{u}^{\varepsilon,k}$) is then obtained by neglecting the right hand side of (77).

Obtaining (77) is the pure algebraic part of the work and we shall not give the details of the computations which are straightforward and could be automatized. Note however that their complexity increases rapidly with k. For $k \leq 3$, the reader will check easily that the operators $\mathcal{D}^{\varepsilon,k}$'s are the ones announced in section 3.1 and that the g_k^ε's are given by:

$$\left\{ \begin{array}{rcl} g_1^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^1, \\[2ex] g_2^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^2 + i\, \mathcal{H}\, \partial_n(u_e^1 + \varepsilon u_e^2), \\[2ex] g_3^\varepsilon &=& \dfrac{1}{\alpha}\, \partial_n u_e^3 + i\, \mathcal{H} \partial_n(u_e^2 + \varepsilon u_e^3) \\[2ex] && -\; \dfrac{1}{2}\left[\, \Delta_\Gamma \partial_n + (3\mathcal{H}^2 - G + \omega^2)\partial_n \,\right] \left(u_e^1 + \varepsilon u_e^2 + \varepsilon^2 u_e^3\right). \end{array} \right. \tag{78}$$

5 Error analysis of NtD GIBCs

Our goal in this section is to estimate the difference

$$u_e^\varepsilon - u^{\varepsilon,k} \tag{79}$$

where $u^{\varepsilon,k}$ is the solution of the approximate problem ((17), (21)), whose well-posedness will be shown in section 5.2 (lemma 5.4). It appears non trivial to work directly with the difference $u_e^\varepsilon - u^{\varepsilon,k}$, we shall use the truncated series $\widetilde{u}^{\varepsilon,k}$ introduced in section 4.4 as an intermediate quantity. Therefore, the error analysis is split into two steps:

1. Estimate the difference $u_e^\varepsilon - \widetilde{u}^{\varepsilon,k}$; this is the object of section 5.1, and more precisely of lemma 5.1 and corollary 5.1.
2. Estimate the difference $\widetilde{u}^{\varepsilon,k} - u^{\varepsilon,k}$; this is the object of section 5.2 and more precisely of lemma 5.6.

Estimates of theorem 3.1 are then a direct consequence of corollary 5.1 and lemma 5.6.

Remark 5.1 *Note that step 1 of the proof is completely independent on the GIBC and will be valid for any integer k. Also, for $k = 0$, the second step is useless since $\widetilde{u}^{\varepsilon,k} = u^{\varepsilon,k}$.*

5.1 Error analysis of the truncated expansions

Let us introduce the function $\widetilde{u}_\chi^{\varepsilon,k}(x) : \Omega \mapsto \mathbb{C}$ such that

$$\widetilde{u}_\chi^{\varepsilon,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \varepsilon^\ell u_e^\ell(x), & \text{for } x \in \Omega_e, \\ \chi(x) \displaystyle\sum_{\ell=0}^{k} \varepsilon^\ell u_i^\ell(x_\Gamma, \nu/\varepsilon) & \text{for } x \in \Omega_i, \end{cases} \tag{80}$$

where χ, x_Γ and ν are as in section 4.1. The main result of this section is:

Lemma 5.1 *For any integer k, there exists a constant C_k independent of ε such that*

$$\begin{aligned} \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{H^1(\Omega)} &\le C_k\, \varepsilon^{k+\frac{1}{2}}, \\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Omega_i)} &\le C_k\, \varepsilon^{k+\frac{3}{2}}, \\ \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k}\|_{L^2(\Gamma)} &\le C_k\, \varepsilon^{k+1}. \end{aligned} \tag{81}$$

Note that this immediately gives an $O(h^{k+1})$ $H^1(\Omega_e)$-error estimate for the "exterior field":

Corollary 5.1 *For any integer k, there exists a constant $\widetilde{C}_k$ independent of ε such that:*

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le \widetilde{C}_k\, \varepsilon^{k+1}. \tag{82}$$

Proof. Simply write

$$u^\varepsilon - \widetilde{u}^{\varepsilon,k} = u^\varepsilon - \widetilde{u}^{\varepsilon,k+1} + \varepsilon^{k+1} u_e^{k+1}$$

which yields, since $u^{\varepsilon,k+1} = u_\chi^{\varepsilon,k+1}$ in Ω_e,

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \le \|u^\varepsilon - \widetilde{u}_\chi^{\varepsilon,k+1}\|_{H^1(\Omega_e)} + \varepsilon^{k+1}\, \|u_e^{k+1}\|_{H^1(\Omega_e)},$$

that is to say, thanks to the first estimate of lemma 5.1:

$$\|u^\varepsilon - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \leq C_k\ \varepsilon^{k+\frac{3}{2}} +\ \varepsilon^{k+1}\ \|u_e^{k+1}\|_{H^1(\Omega_e)} \leq \widetilde{C}_k\ \varepsilon^{k+1}.$$

□

Remark 5.2 *For $k = 0$, since $\widetilde{u}_\chi^{\varepsilon,0} = 0$ inside Ω_i (cf (70)), one deduces from the second estimate of (81) that: $\|u^\varepsilon\|_{L^2(\Omega_e)} \leq C\ \varepsilon^{\frac{3}{2}}$.*

We shall prove first a technical trace lemma (lemma 5.2) and a fundamental stability estimate (lemma 5.3) that constitutes the basic ingredient to the proof of lemma 5.1.

Lemma 5.2 *Let O be a bounded open set of $\mathbb{R}^n$ with C^1 boundary, then there exist a constant C depending on O only such that*

$$\|u\|^2_{L^2(\partial O)} \leq C\left(\|\nabla u\|_{L^2(O)}\ \|u\|_{L^2(O)} + \|u\|^2_{L^2(O)}\right), \quad \textit{for all } u \in H^1(O). \tag{83}$$

Proof. Assume first that $O = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n\ ;\ x_n > 0\}$ and let u in $C^\infty(\mathbb{R}^n_+)$ with compact support. Obviously

$$|u(x',0)|^2 = -2\int_0^\infty u\ \frac{\partial u}{\partial x_n}\ dx_n.$$

Therefore, using Schwarz inequality,

$$\|u(\cdot,0)\|^2_{L^2(\mathbb{R}^{n-1})} \leq \|\nabla u\|_{L^2(\mathbb{R}^n_+)}\ \|u\|_{L^2(\mathbb{R}^n_+)}. \tag{84}$$

Using the denseness of $C^\infty(\mathbb{R}^n_+)$ functions with compact support we deduce that the previous inequality holds for all $u \in H^1(\mathbb{R}^n_+)$. Now let O be bounded open set of $\mathbb{R}^n$ and let χ a $C^\infty(\Omega)$ cut off function such that

$$\chi(x) = 1 \quad \text{if } \operatorname{dist}(x,\partial O) < \eta/2 \text{ and } \chi(x) = 0 \quad \text{if } \operatorname{dist}(x,\partial O) > \eta$$

for a sufficiently small $\eta > 0$. Using local parametric representations of suppχ, we deduce from (84) the existence of a constant C depending on ∂O and η such that

$$\|u\|^2_{L^2(\partial O)} \leq C\ \|\nabla(\chi u)\|_{L^2(O)}\|\chi u\|_{L^2(O)},$$

whence the result of the lemma with a different constant C. □

Lemma 5.3 *Let $v^\varepsilon \in H^1(\Omega)$ satisfying*

$$\left\{\begin{array}{ll} -\Delta v^\varepsilon - \omega^2 v^\varepsilon = 0, & \textit{in}\ \ \Omega_e, \\ \partial_n v^\varepsilon + i\omega v^\varepsilon = 0, & \textit{on}\ \ \partial\Omega, \end{array}\right. \tag{85}$$

and the a priori estimate

$$\left\{ \begin{array}{rcl} \left| \displaystyle\int_\Omega \left(|\nabla v^\varepsilon|^2 - \omega^2 |v^\varepsilon|^2 \right) dx \right. & + & \left. i \left(\displaystyle\int_{\partial\Omega} \omega |v^\varepsilon|^2 \, ds + \frac{1}{\varepsilon^2} \int_{\Omega_i} |v^\varepsilon|^2 \, dx \right) \right| \\ & \leq & A \left(\varepsilon^{s+\frac{1}{2}} \|v^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^s \, \|v^\varepsilon\|_{L^2(\Omega_i)} \right), \end{array} \right. \tag{86}$$

for some non negative constants A and s independent of ε. Then there exists a constant C independent of ε such that

$$\|v^\varepsilon\|_{H^1(\Omega)} \leq C\,\varepsilon^{s+1}, \quad \|v^\varepsilon\|_{L^2(\Omega_i)} \leq C\,\varepsilon^{s+2}, \quad \|v^\varepsilon\|_{L^2(\Gamma)} \leq C\,\varepsilon^{s+\frac{3}{2}}, \tag{87}$$

for sufficiently small ε.

Proof. We first prove by contradiction that $\|v^\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon^{s+1}$. This is the main step of the proof. Let $w^\varepsilon = v^\varepsilon / \|v^\varepsilon\|_{L^2(\Omega)}$ and assume that $\lambda^\varepsilon := \varepsilon^{-s-1} \|v^\varepsilon\|_{L^2(\Omega)}$ is unbounded as $\varepsilon \to 0$. Estimate (86) (notice it is not homogeneous in v^ε) yields

$$\left\{ \begin{array}{rcl} \left| \displaystyle\int_\Omega \left(|\nabla w^\varepsilon|^2 - \omega^2 |w^\varepsilon|^2 \right) dx \right. & + & \left. i \left(\displaystyle\int_{\partial\Omega} \omega |w^\varepsilon|^2 \, ds + \frac{1}{\varepsilon^2} \int_{\Omega_i} |w^\varepsilon|^2 \, dx \right) \right| \\ & \leq & \dfrac{A}{\lambda^\varepsilon} \left(\varepsilon^{-\frac{1}{2}} \|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1} \, \|w^\varepsilon\|_{L^2(\Omega_i)} \right). \end{array} \right. \tag{88}$$

For sake of conciseness, we will denote by C a positive constant whose value may change from one line to another but remains independent of ε. For instance, (88) yields in particular, since $1/\lambda^\varepsilon$ is bounded,

$$\|w^\varepsilon\|^2_{L^2(\Omega_i)} \leq C\,\varepsilon^{\frac{3}{2}} \|w^\varepsilon\|_{L^2(\Gamma)} + C\,\varepsilon \|w^\varepsilon\|_{L^2(\Omega_i)}.$$

Next, we use Lemma 5.2 with $\mathcal{O} = \Omega_i$ to get

$$\|w^\varepsilon\|^2_{L^2(\Omega_i)} \leq C\,\varepsilon^{\frac{3}{2}} \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} \left(\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} \right) + C\,\varepsilon \|w^\varepsilon\|_{L^2(\Omega_i)},$$

which yields, after division by $\|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}$,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)} \leq C_1\,\varepsilon \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + C_2\,\varepsilon^{\frac{3}{2}} \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}. \tag{89}$$

Using Young's inequality $ab \leq 2/3\; a^{3/2} + 1/3\; b^3$ with $a = K^{-1}\varepsilon$ and $b = K\; \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}$ (where K is a positive constant to be fixed later) we can write

$$\varepsilon \|w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} \leq \frac{2}{3}\, K^{-\frac{3}{2}}\, \varepsilon^{\frac{3}{2}} + \frac{K^3}{3}\, \|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)}. \tag{90}$$

Choosing $C_1 K^3 = 3/2$ and substituting (89) into (90), we deduce a first main inequality,

$$\|w^\varepsilon\|^{\frac{3}{2}}_{L^2(\Omega_i)} \leq C\,\varepsilon^{\frac{3}{2}} \left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} \right). \tag{91}$$

Now, observe that another consequence of (88) is, since $\|w^\varepsilon\|_{L^2(\Omega)} = 1$,

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C\left(1 + \varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} + \varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right). \tag{92}$$

On the other hand, using lemma 5.2 once again, we have

$$\varepsilon^{-\frac{1}{2}}\|w^\varepsilon\|_{L^2(\Gamma)} \leq C\,\varepsilon^{\frac{1}{2}}\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\} + C\,\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\}^{\frac{1}{2}}\,\|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)},$$

which, for ε bounded, implies, using (92),

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C + C\,\left\{\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)}\right\}\left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right). \tag{93}$$

Coming back to (91), we deduce that

$$\varepsilon^{-1}\|w^\varepsilon\|_{L^2(\Omega_i)} \leq C\,\left(1 + \|\nabla w^\varepsilon\|^{\frac{1}{3}}_{L^2(\Omega_i)}\right), \tag{94}$$

that we use in (93) to obtain

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} \leq C\left(1 + \|\nabla w^\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega_i)}\right).$$

This implies in particular that $\|\nabla w^\varepsilon\|_{L^2(\Omega)}$ is uniformly bounded with respect to ε and therefore w^ε is a bounded sequence of $H^1(\Omega)$. Up to an extracted subsequence, one can therefore assume that w^ε converges weakly in $H^1(\Omega)$ and strongly $L^2(\Omega)$ to some w with $\|w\|_{L^2(\Omega)} = 1$.

From (91), we deduce that $w = 0$ in Ω_i. On the other hand, taking the weak limit in the equations satisfied by w^ε in Ω_e and on $\partial\Omega$, then using that $w \in H^1(\Omega)$ one gets

$$\left\{\begin{array}{l} -\Delta w - \omega^2 w = 0, \quad \text{in } \ \Omega_e \\[2mm] \partial_n w + i\omega w = 0 \quad \text{on } \ \partial\Omega, \\[2mm] w = 0 \quad \text{on } \ \Gamma. \end{array}\right. \tag{95}$$

Therefore $w = 0$ in Ω_e, hence $w = 0$ in Ω which contradicts $\|w\|_{L^2(\Omega)} = 1$. Consequently

$$\|v^\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon^{s+1}. \tag{96}$$

Estimate (86) and Lemma 5.2 yields

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} \leq C\left(\varepsilon^{s+\frac{5}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \varepsilon^{s+2}\|v^\varepsilon\|_{L^2(\Omega_i)}\right), \tag{97}$$

and, using (96)

$$\|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \leq C\left(\varepsilon^{2s+2} + \varepsilon^{s+\frac{1}{2}}\|\nabla v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)}\|v^\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \varepsilon^{s}\|v^\varepsilon\|_{L^2(\Omega_i)}\right). \tag{98}$$

Therefore, combining these two estimates, it is not difficult to obtain

$$\|v^\varepsilon\|^2_{L^2(\Omega_i)} + \varepsilon^2\|\nabla v^\varepsilon\|^2_{L^2(\Omega)} \le C\left(\varepsilon^{2s+4} + \varepsilon^{s+2}\left(\|v^\varepsilon\|_{L^2(\Omega_i)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(\Omega)}\right)\right),$$

which yields

$$\|v^\varepsilon\|_{L^2(\Omega_i)} + \varepsilon\|\nabla v^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{s+2}.$$

This corresponds to the first two estimates of (87). The third one is a direct consequence of these two estimates by the application of Lemma 5.2 to Ω_i. □

Remark 5.3 *Notice that since we simply used in the first step of the proof the fact that $1/\lambda^\varepsilon$ is bounded, we have proved in fact that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-(s+1)}\ \|v^\varepsilon\|_{L^2(\Omega)} = 0.$$

Proof of Lemma 5.1. Let us set $e^\varepsilon_k = u^\varepsilon - \widetilde{u}^{\varepsilon,k}_\chi$. The idea of the proof is to show that e^ε_k satisfies an a priori estimate of the type (86) and then to use the stability lemma 5.3. To prove such an estimate, we shall use the equations satisfied by e^ε_k, respectively in Ω_i and Ω_e as well as transmission conditions across Γ.

The exterior equation. By construction, $\widetilde{u}^{\varepsilon,k}_\chi$ satisfies in Ω_e the non homogeneous Helmholtz equation with the radiation boundary condition on $\partial\Omega$ and right hand side f (this is a direct consequence of (52) for each k). Hence, $e^\varepsilon_{e,k} = e^\varepsilon_k|_{\Omega_e}$ satisfies the homogeneous equation:

$$\begin{cases} -\Delta e^\varepsilon_{e,k} - \omega^2 e^\varepsilon_{e,k} = 0, & \text{in } \Omega_e, \\ \partial_n e^\varepsilon_{e,k} + i\omega e^\varepsilon_{e,k} = 0, & \text{on } \partial\Omega. \end{cases} \tag{99}$$

The interior equation. The truncated series $\widetilde{u}^{\varepsilon,k}_\chi$ does not exactly satisfies the damped Helmholtz equation inside Ω_i. They verify this equation with a small right hand side. To see that, let us set:

$$\widetilde{u}^{\varepsilon,k}_i = \sum_{\ell=0}^{k} \varepsilon^\ell\, u^\ell_i, \quad \text{so that } \widetilde{u}^{\varepsilon,k}_\chi = \chi\, \widetilde{u}^{\varepsilon,k}_i \quad \text{in } \Omega_i. \tag{100}$$

Indeed

$$\Delta\widetilde{u}^{\varepsilon,k}_\chi + \omega^2\widetilde{u}^{\varepsilon,k}_\chi - \frac{i}{\varepsilon^2}\,\widetilde{u}^{\varepsilon,k}_\chi = \chi\{\Delta\widetilde{u}^{\varepsilon,k}_i + \omega^2\widetilde{u}^{\varepsilon,k}_i - \frac{i}{\varepsilon^2}\widetilde{u}^{\varepsilon,k}_i\} + 2\nabla\chi\cdot\nabla\widetilde{u}^{\varepsilon,k}_i + \Delta\chi\,\widetilde{u}^{\varepsilon,k}_i.$$

Inside the support of χ the local coordinates $(x_\Gamma, \nu = \varepsilon\eta)$ can be used to make the identification (cf. 55)

$$\Delta + \omega^2 - \frac{i}{\varepsilon^2} \equiv \frac{1}{J^3_\nu\varepsilon^2}\left(-\partial^2_{\eta\eta} + i - \sum_{\ell=1}^{8}\varepsilon^\ell\mathcal{A}_\ell\right). \tag{101}$$

From equation (64), after multiplication by the correct power of ε and summation, it is not difficult to see (the calculations are a little bit long but not difficult) that

$$\left(-\partial^2_{\eta\eta} + i - \sum_{\ell=1}^{8} \varepsilon^\ell \mathcal{A}_\ell\right) \widetilde{u}_i^{\varepsilon,k} = -\varepsilon^{k+1} \sum_{\ell=1}^{8}\sum_{p=0}^{\ell-1} \varepsilon^p \mathcal{A}_{\ell-p-1} u_i^{k+p+1-\ell}. \tag{102}$$

Therefore, thanks to (101) and (102),

$$\Delta \widetilde{u}_\chi^{\varepsilon,k} + \omega^2 \widetilde{u}_\chi^{\varepsilon,k} - \frac{i}{\varepsilon^2} \widetilde{\widetilde{u}_\chi^{\varepsilon,k}} = g^\varepsilon_{k,i}, \quad \text{in } \Omega_i, \tag{103}$$

where the function g_i^ε is given, with obvious notation, by

$$g^\varepsilon_{k,i} = -\varepsilon^{k-1} \chi \sum_{\ell=1}^{8}\sum_{p=0}^{\ell-1} \varepsilon^p \mathcal{A}_{\ell-p-1} u_i^{k+p+1-\ell}(., \nu/\varepsilon) + 2\nabla\chi \cdot \nabla \widetilde{u}_i^{\varepsilon,k} + \Delta\chi\, \widetilde{u}_i^{\varepsilon,k}. \tag{104}$$

From expression (68) and the identity

$$\int_0^{+\infty} \left(\frac{\nu}{\varepsilon}\right)^n e^{-\frac{\nu}{\sqrt{2}\varepsilon}}\, d\nu = C_n\, \varepsilon, \quad \forall\, n \in \mathbb{N}\,,$$

it is not difficult to deduce the following estimate for each u_i^q:

$$\left(\int_{\Omega_i^\delta} |u_i^q(., \nu/\varepsilon)|^2 \, dx \right)^{\frac{1}{2}} \leq C_q(\delta)\, \varepsilon^{\frac{1}{2}}\,. \tag{105}$$

In the same way, one easily shows that:

$$\left(\int_{\Omega_i^\delta \setminus \Omega_i^{\frac{\delta}{2}}} \{\, |\widetilde{u}_i^{\varepsilon,k}|^2 + |\nabla \widetilde{u}_i^{\varepsilon,k}|^2 \,\}\, dx \right)^{\frac{1}{2}} \leq C(\delta)\, \exp(-\delta/\varepsilon)\,. \tag{106}$$

Regrouping estimates (105) and (106) into (104), yields

$$||g^\varepsilon_{k,i}||_{L^2(\Omega_i)} \leq C\, \varepsilon^{k-\frac{1}{2}}\,. \tag{107}$$

Note of course that, by taking the difference between (103) and (6)(ii), $e^\varepsilon_{i,k} := e^\varepsilon_{k|\Omega_i}$ satisfies

$$-\Delta e^\varepsilon_{i,k} + (-\omega^2 + \frac{i}{\varepsilon^2}) e^\varepsilon_{i,k} = g^\varepsilon_{k,i}, \quad \text{in } \Omega_i. \tag{108}$$

The transmission equations. From interface condition (66) it is clear that $\widetilde{u}_\chi^{\varepsilon,k}$, and thus e^ε_k, is continuous across Γ. However, from (65), due to the shift of index between left and right hand sides, the normal derivative of $\widetilde{u}_{\chi}^{\varepsilon,k}$, and thus of e_k^ε, is discontinuous across Γ. More precisely, straightforward calculations lead to the following transmission conditions

$$\left\{\begin{array}{ll} e_{e,k}^\varepsilon - e_{i,k}^\varepsilon = 0, & \text{on } \Gamma, \\ \partial_n\, e_{e,k}^\varepsilon - \partial_n\, e_{i,k}^\varepsilon = \varepsilon^k\, \partial_n\, u_e^k, & \text{on } \Gamma. \end{array}\right. \tag{109}$$

Error estimates. We can now proceed to the final step of the proof. Multiplying equation (99) by $\overline{e_{e,k}^\varepsilon}$ and integrating over Ω_e, we obtain by using Green's formula,

$$\int_{\Omega_e} |\nabla e_{e,k}^\varepsilon|^2\, dx - \omega^2 \int_{\Omega_e} |e_{e,k}^\varepsilon|^2\, dx + i\omega \int_{\partial\Omega} |e_{e,k}^\varepsilon|^2\, d\sigma = \int_\Gamma \partial_n\, e_{e,k}^\varepsilon\, \overline{e_{e,k}^\varepsilon}\, d\sigma\ . \tag{110}$$

In the same way, multiplying equation (108) by $\overline{e_{i,k}^\varepsilon}$ and integrating over Ω_i, one gets

$$\left|\begin{array}{rcl} \displaystyle\int_{\Omega_i} |\nabla e_{i,k}^\varepsilon|^2\, dx - \omega^2 \int_{\Omega_e} |e_{i,k}^\varepsilon|^2\, dx + \frac{i}{\varepsilon^2}\int_{\Omega_i} |e_{i,k}^\varepsilon|^2\, dx & = & \displaystyle -\int_\Gamma \partial_n\, e_{i,k}^\varepsilon\, \overline{e_{i,k}^\varepsilon}\, d\sigma \\ & + & \displaystyle\int_{\Omega_i} g_{k,i}^\varepsilon\, \overline{e_{i,k}^\varepsilon}\, dx\ . \end{array}\right. \tag{111}$$

Adding together (111) and (110) and using (109) and (107), gives

$$\left|\begin{array}{l} \displaystyle\left|\int_\Omega |\nabla e_k^\varepsilon|^2 - \omega^2 \int_\Omega |e_k^\varepsilon|^2 + i\omega \int_{\partial\Omega} |e_k^\varepsilon| + \frac{i}{\varepsilon^2}\int_{\Omega_i} |e_k^\varepsilon|^2\right| \\ \displaystyle\qquad\qquad \leq C_k\, \left(\varepsilon^k\, ||e_k^\varepsilon||_{L^2(\Gamma)} + \varepsilon^{k-\frac{1}{2}}\, ||e_k^\varepsilon||_{L^2(\Omega_i)}\right), \end{array}\right. \tag{112}$$

where C_k is a constant independent of ε. Ones deduces the desired estimates by applying Lemma 5.3. □

5.2 Error estimates for the GIBCs

Existence and uniqueness results for the approximate problems. We shall check here that the $u^{\varepsilon,k}$'s are well defined. This is our next result.

Lemma 5.4 *For $k = 0, 1, 2, 3$, the boundary value problem:*

$$\left\{\begin{array}{l} -\Delta u^{\varepsilon,k} - \omega^2 u^{\varepsilon,k} = f, \quad \textit{in } \Omega_e, \\ \partial_n u^{\varepsilon,k} + i\omega u^{\varepsilon,k} = 0, \quad \textit{on } \partial\Omega, \\ u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k} u^{\varepsilon,k} = 0 \quad \textit{on } \Gamma, \end{array}\right. \tag{113}$$

admits a unique solution in $H^1(\Omega_e)$ provided that $\varepsilon\, \mathcal{H} \leq \sqrt{2}/2$ if $k = 2$ or ε is small enough if $k = 3$.

Proof. Since the proof for $k = 0, 1, 2$ is quite classical, we shall concentrate here on the case $k = 3$. We start by reformulation problem (113) as a system.

New formulation of the problem. Introducing $\varphi^\varepsilon = \partial_n u^{\varepsilon,3}|_\Gamma$ as a new unknown, problem (113) is equivalent, for $k = 3$, to find $(u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(\Omega_e) \times H^1(\Gamma)$ such that

$$\left| \begin{array}{ll} -\Delta u^{\varepsilon,3} - \omega^2 u^{\varepsilon,3} = f, & \text{in } \Omega_e, \\ \partial_n u^{\varepsilon,3} + i\omega u^{\varepsilon,3} = 0, & \text{on } \partial\Omega, \\ \partial_n u^{\varepsilon,3} = \varphi^\varepsilon, & \text{on } \Gamma, \\ -\Delta_\Gamma \, \varphi^\varepsilon - \dfrac{2i}{\varepsilon^2} \, \theta_3(\varepsilon) \, \varphi^\varepsilon = \dfrac{2i\alpha}{\varepsilon^3} \, u^{\varepsilon,3} & \text{on } \Gamma, \end{array} \right. \tag{114}$$

where we have set $\theta_3(\varepsilon) = 1 - \dfrac{\varepsilon \mathcal{H}}{\alpha} - i \, \dfrac{\varepsilon^2 A(\omega)}{2}$ with $A(\omega) = 3\mathcal{H}^2 - G + \omega^2$.

Next we show that problem (114) is of Fredholm type. For this, we first notice that (114) is equivalent to the variational problem:

$$\left\{ \begin{array}{l} \text{Find } (u^{\varepsilon,3}, \varphi^\varepsilon) \in H^1(\Omega_e) \times H^1(\Gamma) \text{ such that } \forall \, (v, \psi) \in H^1(\Omega_e) \times H^1(\Gamma), \\ a_1 \left((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi) \right) + a_2^\varepsilon \left((u^{\varepsilon,3}, \varphi^\varepsilon), (v, \psi) \right) = \displaystyle\int_{\Omega_e} f \overline{v} \, dx. \end{array} \right. \tag{115}$$

where we have set:

$$\left\{ \begin{array}{lcl} a_1 \left((u, \varphi), (v, \psi) \right) & = & \displaystyle\int_{\Omega_e} \nabla u \cdot \nabla \overline{v} \; dx + i\omega \int_{\partial\Omega} u \, \overline{v} \; dx + \int_\Gamma (\nabla_\Gamma \, \varphi \cdot \nabla_\Gamma \, \overline{\phi} + \varphi \, \overline{\psi}) \, ds \\ a_2^\varepsilon \left((u, \varphi), (v, \psi) \right) & = & - \, \omega^2 \displaystyle\int_{\Omega_e} u \, \overline{v} \, dx - \int_\Gamma [\, 1 + \frac{2i}{\varepsilon^2} \, \theta_3(\varepsilon) \,] \varphi \, \overline{\psi} \, ds \\ & - & \dfrac{2i\alpha}{\varepsilon^3} \displaystyle\int_\Gamma u \, \overline{\psi} \; ds \; - \int_\Gamma \varphi \, \overline{v} \; ds. \end{array} \right.$$

One next remarks that $a_1(\cdot, \cdot)$ is coercive in $H^1(\Omega_e) \times H^1(\Gamma)$ while $a_2^\varepsilon(\cdot, \cdot)$ is weakly compact in $H^1(\Omega_e) \times H^1(\Gamma)$:

$$(u^n, \varphi^n) \rightharpoonup (u, \varphi) \text{ in } H^1(\Omega_e) \times H^1(\Gamma) \Longrightarrow a_2^\varepsilon \left((u^n, \varphi^n)(u^n, \varphi^n) \right) \to a_2^\varepsilon \left((u, \varphi)(u, \varphi) \right).$$

Therefore, to prove the existence of the solution of (114) (or (115)), it is sufficient to prove uniqueness.

Uniqueness proof. We prove the uniqueness result for ε small enough by contradiction. If uniqueness fails then, up to the extraction of a sequence of values of ε tending to 0, one can assume that here exists a non trivial solution $(u^{\varepsilon,3}, \varphi^\varepsilon)$ of the homogeneous problem associated with (114), that we can normalize in such a way that:

$$\|u^{\varepsilon,3}\|_{L^2(\Omega_e)} = 1. \tag{116}$$

We multiply the Helmholtz equation by the complex conjugate of $u^{\varepsilon,3}$ and after integration by parts, we replace, in the boundary term on Γ, the trace of $u^{\varepsilon,3}$ by its expression as a function of φ^ε from the last equation of (114). This leads to

$$\int_{\Omega_e} \left(|\nabla u^{\varepsilon,3}|^2 - \omega^2 |u^{\varepsilon,3}|^2\right) dx + \frac{\bar{\alpha}\,\varepsilon^3}{2} \int_\Gamma |\nabla_\Gamma \varphi^\varepsilon|^2 \, ds$$
$$+\varepsilon\alpha \int_\Gamma \theta_3(\varepsilon)\, |\varphi^\varepsilon|^2 \, ds + i\omega \int_{\partial\Omega} |u^{\varepsilon,3}|^2 \, ds = 0.$$

We now take the real part of the last equality (contrary to what is more usual, taking the imaginary part does not provide the desired estimate, the term in $|\nabla_\Gamma \varphi^\varepsilon|^2$ comes with the wrong sign) and use (116) to get

$$\int_{\Omega_e} |\nabla u^{\varepsilon,3}|^2 \, dx + \frac{\varepsilon^3\sqrt{2}}{4} \int_\Gamma |\nabla_\Gamma \varphi^\varepsilon|^2 \, ds + \varepsilon \int_\Gamma \mathcal{R}e(\alpha\theta_3(\varepsilon))|\varphi^\varepsilon|^2 \, ds \leq \omega^2. \tag{117}$$

Since $\mathcal{R}e(\alpha\theta_3(\varepsilon))$ tends to $\sqrt{2}/2$ as ε goes to 0, we deduce that $u^{\varepsilon,3}$ is bounded in $H^1(\Omega_e)$. Therefore, up to the extraction of a subsequence, we can assume that:

$$\begin{cases} u^{\varepsilon,3} \to u, & \text{weakly in } H^1(\Omega_e), \\ u^{\varepsilon,3} \to u, & \text{strongly in } L^2(\Omega_e), \\ \Delta u^{\varepsilon,3} \to \Delta u, & \text{weakly in } L^2(\Omega_e), \end{cases}$$

the latter property being deduced from the Helmholtz equation. By trace theorem, $\partial_n u^{\varepsilon,3}|_\Gamma$ (resp. $\partial_n u^{\varepsilon,3}|_{\partial\Omega}$) converges to $\partial_n u|_\Gamma$ (resp. $\partial_n u|_{\partial\Omega}$) in $H^{-\frac{1}{2}}(\Gamma)$ (resp. $H^{-\frac{1}{2}}(\partial\Omega)$). Of course, at the limit, we have:

$$\begin{cases} -\Delta u - \omega^2 u = 0, & \text{in } \Omega_e, \\ \partial_n u + i\omega u = 0, & \text{in } \partial\Omega. \end{cases} \tag{118}$$

while, passing to the (weak) limit in the last boundary equation of (114) after multiplication by ε^3, we obtain

$$u = 0, \quad \text{on } \Gamma. \tag{119}$$

From (118) and (119), we get $u = 0$ which is in contradiction with $\|u\|_{L^2(\Omega_e)} = 1$. □

Analysis of the difference $u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}$. We now proceed to step 2 of the sketch announced in section 3. From now on, we shall set for $k = 0, 1, 2, 3$,

$$\mathbf{e}^{\varepsilon,k} = u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}. \tag{120}$$

The starting point of the error analysis is to remark that $\mathbf{e}^{\varepsilon,k}$ is a solution of a homogeneous Helmholtz equation with outgoing absorbing condition on $\partial\Omega$,

$$\left\{ \begin{array}{ll} -\Delta \mathbf{e}^{\varepsilon,k} - \omega^2 \mathbf{e}^{\varepsilon,k} = 0 & \text{in } \ \Omega_e, \\ \\ \partial_n \mathbf{e}^{\varepsilon,k} + i\omega \mathbf{e}^{\varepsilon,k} = 0 & \text{on } \ \partial\Omega, \end{array} \right. \tag{121}$$

and satisfies a non homogeneous $GIBC$ boundary condition on Γ with small right hand side. This comes directly from the construction of the $GIBC$ itself and is obtained by making the difference between (77) and (21). Let us formulate this as a lemma:

Lemma 5.5 *For $k = 1, 2, 3$, there exists a smooth function g_k^ε such that*

$$\mathbf{e}^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\, \partial_n \mathbf{e}^{\varepsilon,k} = \varepsilon^{k+1}\, g_k^\varepsilon, \tag{122}$$

with the estimates

$$||g_k^\varepsilon||_{H^{\frac{1}{2}}(\Gamma)} \leq C_k, \qquad \textit{for } k = 1, 2, 3, \tag{123}$$

where C_k is a positive constant independent of ε.

This result can be seen as a *consistency* result for the boundary condition. Combined with a *stability* argument, it is then possible to obtain the following estimates.

Lemma 5.6 *For $k = 1, 2, 3$, there exists a positive constant C_k independent of ε such that*

$$\|u^{\varepsilon,k} - \widetilde{u}^{\varepsilon,k}\|_{H^1(\Omega_e)} \leq \ C_k\, \varepsilon^{k+1}. \tag{124}$$

Proof. From (121) and (122) and Green's formula,

$$\left| \begin{array}{l} \displaystyle\int_{\Omega_e} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2\ )\, dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,k}|^2\, ds \\ \qquad \displaystyle + \int_\Gamma \mathcal{D}^{\varepsilon,k}\, \partial_n \mathbf{e}^{\varepsilon,k} \cdot \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon\, \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds. \end{array} \right. \tag{125}$$

Setting $\varphi_k^\varepsilon = \partial_n \mathbf{e}^{\varepsilon,k}|_\Gamma$, and introducing the functions $\theta_1(\varepsilon) = 1$, $\theta_2(\varepsilon) = 1 - \frac{\varepsilon\mathcal{H}}{\alpha}$, ($\theta_3(\varepsilon)$ has been defined in the proof of Lemma 5.4), one can derive the following general identity by using the explicit expressions of the $\mathcal{D}^{\varepsilon,k}$'s,

$$\left| \begin{array}{l} \displaystyle\int_{\Omega_e} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2\ )\, dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,k}|^2\, ds \\ \displaystyle + \varepsilon\alpha \int_\Gamma \theta_k(\varepsilon)\, |\varphi_k^\varepsilon|^2\, ds + \nu_k\, \frac{\bar{\alpha}\, \varepsilon^3}{2} \int_\Gamma |\nabla_\Gamma\, \varphi_k^\varepsilon|^2\, ds = \varepsilon^{k+1} \int_\Gamma g_k^\varepsilon\, \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\, ds\ , \end{array} \right. \tag{126}$$

where $\nu_k = 0$ for $k = 0, 1, 2$ and $\nu_3 = 1$. Taking the real part,

$$\left| \begin{array}{l} \displaystyle\int_{\Omega_e} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2\ )\ dx +\ \varepsilon \int_\Gamma \mathcal{R}e(\alpha\theta_k(\varepsilon))\ |\varphi_k^\varepsilon|^2\ ds \\ \displaystyle\quad +\nu_k\ \frac{\sqrt{2}\ \varepsilon^3}{4} \int_\Gamma |\nabla_\Gamma\ \varphi_k^\varepsilon|^2\ ds = \varepsilon^{k+1}\ \mathcal{R}e \int_\Gamma g_k^\varepsilon \overline{\partial_n \mathbf{e}^{\varepsilon,k}}\ ds\ . \end{array} \right. \tag{127}$$

In particular, since $\nu_k \geq 0$ and $\mathcal{R}e(\alpha\theta_k(\varepsilon))$ tends to $\sqrt{2}$ as ε tends to 0, we obtain the following estimate, for ε small enough,

$$\left| \begin{array}{lll} \displaystyle\int_{\Omega_e} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,k}|^2\ )\ dx & \leq & \varepsilon^{k+1}\ \|g_k^\varepsilon\|_{H^{\frac{1}{2}}(\Gamma)} \|\partial_n \mathbf{e}^{\varepsilon,k}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ & \leq & C_k\ \varepsilon^{k+1}\ \|\mathbf{e}^{\varepsilon,k}\|_{H^1(\Omega)}, \end{array} \right. \tag{128}$$

where the latter inequality comes from (123) and the fact that $\mathbf{e}^{\varepsilon,k}$ is solution the Helmholtz equation inside Ω_e. The remaining part of the proof in then rather straightforward. We first prove by contradiction that

$$\|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)} \leq C_k\ \varepsilon^{k+1}\ . \tag{129}$$

If (129) is not true, then $\mu_k^\varepsilon = \varepsilon^{-(k+1)} \|\mathbf{e}^{\varepsilon,k}\|$ would blows up (for a subsequence) as ε goes to 0. Then, introducing

$$w^{\varepsilon,k} = \mathbf{e}^{\varepsilon,k}\ /\ \|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)},$$

ones derives from (128)

$$\int_{\Omega_e} |\nabla w^{\varepsilon,k}|^2\ dx \leq \omega^2 + C_k\ (\mu_k^\varepsilon)^{-1}\ \|w^{\varepsilon,k}\|_{H^1(\Omega)} \leq C_k\ (1 + \|w^{\varepsilon,k}\|_{H^1(\Omega)}). \tag{130}$$

Therefore, $w^{\varepsilon,k}$ is bounded in $H^1(\Omega)$ and thus, up to the extraction of a subsequence, converges weakly in $H^1(\Omega_e)$ but strongly in $L^2(\Omega_e)$ to some $w^k \in H^1(\Omega_e)$ that satisfies $\|w^k\|_{L^2(\Omega_e)} = 1$ as well as

$$\left\{ \begin{array}{ll} -\Delta w^k - \omega^2 w^k = 0, & \text{in}\ \ \Omega_e, \\ \partial_n w^k + i\omega w^k = 0, & \text{on}\ \ \partial\Omega. \end{array} \right. \tag{131}$$

Finally, passing to the limit (in the weak sense) in the boundary condition

$$w^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\ \partial_n w^{\varepsilon,k} = g_k^\varepsilon /\ \|\mathbf{e}^{\varepsilon,k}\|_{L^2(\Omega_e)} =\ (\mu_k^\varepsilon)^{-1}\ (g_k^\varepsilon /\ \varepsilon^{k+1})\ , \tag{132}$$

we see ($g_k^\varepsilon /\ \varepsilon^{k+1}$ is bounded and $(\mu_k^\varepsilon)^{-1}$ tends to 0) that w^k also satisfies

$$w^k = 0, \quad \text{on}\ \Gamma. \tag{133}$$

System ((131),(133)) implies that $w^k = 0$, which contradicts $\|w^k\|_{L^2(\Omega)} = 1$. Therefore, (129) holds. The Lemma estimate is now a direct consequence of (129) and (128). □

6 About the analysis of modified GIBC's

The error analysis of modified *GIBC*'s can be done in a similar way as for the NtD *GIBC*s. We shall restrict ourselves to stating the results and indicating the needed modifications in previous section proofs.

6.1 Analysis of DtN *GIBC*s

Theorem 6.1 *Let $k = 1, 2$ or 3, then, assuming ε beeing sufficiently small when $k = 3$, the boundary value problem ((17), (30)) has a unique solution $u^{\varepsilon,k} \in H^1(\Omega_e)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(\Omega_e)} \leq C_k \, \varepsilon^{k+1}. \tag{134}$$

Proof. We shall only treat here the case $k = 3$ (the others are easy) and directly go to the proof of estimate (134) assuming the existence and uniqueness of the solution. Of course, we only need to look at the difference $u^{\varepsilon,3} - \tilde{u}^{\varepsilon,3}$, namely to prove the equivalent to lemma 5.6.

Rather curiously, it appears that treating the boundary condition directly in its DtN form (30) does not lead immediately to the optimal error estimate. This is why we shall rewrite it as an NtD condition by introducing the inverse of the operator $\mathcal{N}^{\varepsilon,3}$ (note that, by Lax-Milgram's lemma, $\mathcal{N}^{\varepsilon,3}$ is an isomorphism from $H^{s+2}(\Gamma)$ onto $H^s(\Gamma)$).

We repeat here the approach of Lemma 5.6. One first checks that the error $\mathbf{e}^{\varepsilon,3}$ satisfies the homogenenous Helmholtz equation in Ω_e together with the non homogeneous boundary condition:

$$\mathbf{e}^{\varepsilon,3} + \left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3} = \varepsilon^4 \, g_3^\varepsilon, \tag{135}$$

where g_3^ε is a smooth function satisfying:

$$||g_3^\varepsilon||_{H^{\frac{1}{2}}(\Gamma)} \leq C_k, \quad \text{for } k = 1, 2, 3, \tag{136}$$

Proceeding as in the proof of Lemma 5.6, we obviously get

$$\left| \begin{aligned} &\int_{\Omega_e} (\ |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,3}|^2\ )\ dx + i\omega \int_{\partial\omega} |\mathbf{e}^{\varepsilon,3}|^2 \ ds \\ &\quad + \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \partial_n \mathbf{e}^{\varepsilon,3}} \cdot \partial_n \mathbf{e}^{\varepsilon,3} \ ds = \varepsilon^{k+1} \int_\Gamma g_3^\varepsilon \ \overline{\partial_n \mathbf{e}^{\varepsilon,3}} \ ds. \end{aligned} \right. \tag{137}$$

The key point is that, at least for ε small enough, for any ψ smooth enough,

$$\mathcal{R}e \int_\Gamma \overline{\left(\mathcal{N}^{\varepsilon,3}\right)^{-1} \psi} \cdot \psi \ dx \leq 0. \tag{138}$$

This is a consequence of

$$\mathcal{R}e \int_\Gamma \mathcal{N}^{\varepsilon,3}\varphi \cdot \overline{\varphi}\, dx \le 0, \quad \text{for any } \varphi \text{ smooth enough,}$$

that follows from the identity (proven in section 3.2)

$$\int_\Gamma \varphi \cdot \overline{\mathcal{N}^{\varepsilon,3}\varphi}\, ds = \frac{\alpha\, \varepsilon}{2}\int_\Gamma |\nabla_\Gamma \varphi|^2\, ds + \frac{\bar{\alpha}}{\varepsilon}\int_\Gamma [\, 1 + \frac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\,\frac{\varepsilon^2}{2}\,(\mathcal{H}^2 - G + \omega^2)\,]\, |\varphi|^2\, ds,$$

and the observation that

$$\begin{cases} \mathcal{R}e\ \alpha = \mathcal{R}e\ \bar{\alpha} = \sqrt{2}/2, \\ \lim_{\varepsilon\to 0}\, [\, 1 + \dfrac{\varepsilon\mathcal{H}}{\bar{\alpha}} + i\,\dfrac{\varepsilon^2}{2}\,(\mathcal{H}^2 - G + \omega^2)\,] = 1. \end{cases}$$

Therefore we have shown that, as soon as ε is small enough,

$$\int_{\Omega_e} (\, |\nabla \mathbf{e}^{\varepsilon,k}|^2 - \omega^2 |\mathbf{e}^{\varepsilon,3}|^2\,)\, dx \le 0, \tag{139}$$

and the conclusion of the proof is identical to the one of Lemma 5.6. □ .

Remark 6.1 *Proceeding as in Section 4.4 (for deriving formulas (78)), one first gets:*

$$\partial_n \mathbf{e}^{\varepsilon,3} + \mathcal{N}^{\varepsilon,3}\mathbf{e}^{\varepsilon,3} = \varepsilon^3\, h_3^\varepsilon,$$

where h_3^ε (as g_3^ε in formula (78)) depends polynomially with respect to ε. It is a polynomial of degree 3 whose coefficients are smooth functions of x_Γ, that can be explicitly expressed in terms of u_e^1, u_e^2 and u_e^3. In particular:

$$h_3^\varepsilon = O(1), \quad \textit{in any Sobolev norm.}$$

One then deduces (135) with:

$$g_3^\varepsilon = (\varepsilon\mathcal{N}^{\varepsilon,3})^{-1}\ h_3^\varepsilon\ .$$

One finally obtains (136) after having noticed that (cf (33):

$$(\varepsilon\mathcal{N}^{\varepsilon,3})^{-1} = \frac{1}{\alpha}\left\{1 + \frac{\varepsilon}{\alpha}\mathcal{H} + i\frac{\varepsilon^2}{2}(\Delta_\Gamma + \mathcal{H}^2 - G + \omega^2)\right\}^{-1} = O(\varepsilon).$$

6.2 Analysis of robust $GIBC$'s

Theorem 6.2 *For any $\varepsilon > 0$, the boundary value problem associated with (17) and the boundary condition:*

$$u^{\varepsilon,3} + \mathcal{D}_r^{\varepsilon,3}\partial_n u^{\varepsilon,3} = 0, \quad \textit{on } \Gamma, \tag{140}$$

where $\mathcal{D}_r^{\varepsilon,3}$ is given by (36), has a unique solution $u^{\varepsilon,3} \in H^1(\Omega_e)$. Moreover, there exists a constant C_3, independent of ε, such that

$$\|u_e^\varepsilon - u^{\varepsilon,3}\|_{H^1(\Omega_e)} \leq C_3\, \varepsilon^4. \tag{141}$$

The same result holds if one replaces (140) by:

$$\partial_n u^{\varepsilon,3} + \mathcal{N}_r^{\varepsilon,3} u^{\varepsilon,3} = 0, \qquad \textit{on } \Gamma, \tag{142}$$

where $\mathcal{N}_r^{\varepsilon,3}$ is given by (41).

We shall note detail the proof of this theorem which is almost identical to the one of theorem 6.1 or lemma 5.6 . Let us simply recall that the existence and uniqueness result is valid for any positive ε is due to properties (39) and (42). The main difference lies in the fact that the algebra to obtain the equivalent to identities (5.5) and (135) is slightly more complicated and the calculations for obtaining the equivalent to property (138) are longer.

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