Partial Differential Equations

On a new class of functions related to VMO

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1. Main results

The principal motivation of this note comes from the study of the topological degree of maps from the sphere $S^d$ into itself. It was proved in [2] that the degree is well-defined for maps $u \in VMO(S^d, S^d)$. In fact it suffices to assume that

$$
\limsup_{|Q| \to 0} \int_Q \left| u(x) - \int_Q u(y) \, dy \right| \, dx < 1;
$$

(1.1)
and the constant 1 is optimal. Note that (1.1) is satisfied in particular if
\[
\limsup_{|Q| \to 0} \iint_{Q \times Q} |u(x) - u(y)| \, dx \, dy < 1/2.
\]

On the other hand, it was proved in [6] that the degree of \( u \) is well-defined when
\[
\iint_{|u(x) - u(y)| > \delta} \frac{1}{|x - y|^{2d}} \, dx \, dy < +\infty \quad \text{for some} \quad \delta \in (0, \ell_d),
\]
(1.2)
where \( \ell_d = \sqrt{2 + 2/(d + 1)} \); and moreover
\[
|\text{deg } u| \leq C_d \iint_{|u(x) - u(y)| \geq \delta} \frac{1}{|x - y|^{2d}} \, dx \, dy.
\]
(1.3)
Therefore it is natural to investigate the possible connection between the spaces \( VMO, BMO \), and the class of functions satisfying conditions of the type (1.2). We introduce the following definitions. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^d \), and \( 0 < p < +\infty \). Set
\[
I_p = \left\{ g \in L^1(\Omega; \mathbb{R}) ; \iint_{\Omega \times \Omega} \frac{1}{|x - y|^{d + p}} \, dx \, dy < +\infty \quad \forall \delta > 0 \right\}
\]
and
\[
J_p = \left\{ g \in L^1(\Omega; \mathbb{R}) ; \iint_{\Omega \times \Omega} \frac{1}{|x - y|^{d + p}} \, dx \, dy < +\infty \quad \text{for some} \quad \delta > 0 \right\}.
\]
The case \( p < 0 \) is not interesting because \( I_p = J_p \) coincides with \( L^1(\Omega) \).

Here is a brief list of properties:

A) \( I_p \) and \( J_p \) are vector spaces.
B) \( I_p \subset I_q \) and \( J_p \subset J_q \) if \( p \geq q \).
C) \( C(\Omega) \subset I_p \subset J_p \) for all \( p > 0 \).
D) \( W^{s,p} \subset I_p \) for all \( s > 0 \) and \( p > 1 \).

We recall here that, for \( 0 < s < 1 \) and \( p > 1 \),
\[
W^{s,p}(\Omega) := \left\{ g \in L^p(\Omega) ; \|g\|_{W^{s,p}} < +\infty \right\},
\]
where
\[
\|g\|^p_{W^{s,p}} := \iint_{\Omega \times \Omega} \frac{|g(x) - g(y)|^p}{|x - y|^{d + sp}} \, dx \, dy.
\]
E) \( W^{1,p} \subset I_p \) for all \( p > 1 \). More precisely (see [5]), for \( p > 1 \) and \( g \in W^{1,p}(\Omega) \), we have
\[
\delta^p \iint_{\Omega \times \Omega} \frac{1}{|x - y|^{d + p}} \, dx \, dy \leq C_{d,p,\Omega} \int_{\Omega} |\nabla g|^p \, dx.
\]
The constant \( C_{d,p,\Omega} \) blows up as \( p \to 1 \) and in fact \( W^{1,1} \not\subset I_1 \) (an example due to A. Ponce is presented in [5]).
F) \( J_p \subset L^p \) with \( 1 < p < d \) and \( \frac{d}{p} = \frac{d}{p} - \frac{1}{2} \) (see [7]). This is an extension of the classical Sobolev embedding \( W^{1,p} \subset L^p \).

It is not true that \( I_d \subset L^\infty \) (clearly \( W^{s,p} \subset I_d \) and it is known that \( W^{s,p} \not\subset L^\infty \) for \( sp = d \)). Even when \( p > d \), it is not true that \( I_p \subset L^\infty \) (see [7]); this is in contrast with the Morrey-Sobolev embedding.

It is known that \( W^{s,p} \subset VMO \) for all \( s > 0 \), \( 0 < s \leq 1 \), and \( sp = d \); see e.g. [2]. In view of D), one may wonder whether the larger space \( I_d \) is also contained in \( VMO \). The answer is positive:
Theorem 1. Let $d \geq 1$. Then

a) $I_d \subset BMO$.
b) $I_d \subset VMO$.

Remark 1. The exponent $d$ in Theorem 1 is optimal in the following sense: if $d \geq 1$ and $0 \leq p < d$ then $I_p \not\subset BMO$. Indeed, let $q > 1$ and $0 < s < 1$ be such that $p < sq < d$. Then

\[ W^{s,q} \subset I_{sq} \subset I_p \text{ and } W^{s,q} \not\subset BMO. \]

This implies $I_p \not\subset BMO$.

The proof of Theorem 1 is essentially based on the following proposition which is proved in [7]. In what follows, we denote by $Q$ the unit cube in $\mathbb{R}^d$.

Proposition 1. Let $d \geq 1$, $p \geq 1$, $\delta > 0$, and $g \in L^1(Q)$. Then

\[
\int_Q \int_Q \frac{|g(x) - g(y)|^p}{|x - y|^{d+p}} \, dx \, dy \leq C_{d,p} \int_Q \int_Q \frac{\delta^p}{|x - y|^{d+p}} \, dx \, dy + \delta^p,
\]

for some positive constant $C_{d,p}$ depending only on $d$ and $p$.

Remark 2. The proof of Proposition 1 is quite delicate and it would be desirable to find a more elementary argument, even for $d = 1$. It makes use of ideas introduced in Bourgain–Nguyen [1]. It also relies on the John–Nirenberg inequality [4]. Some inequalities related to (1.4) and their applications in the theory of Sobolev spaces are presented in [7].

One may ask whether the inclusions in Theorem 1 are strict. It turns out that $VMO$ is “much bigger” than $I_d$. In fact, we have a stronger assertion:

Theorem 2. Let $d \geq 1$. Then there exists $g \in VMO$ such that $g \in W^{s,p}$ for all $s \in (0, 1)$, $p > 1$ with $sp < 1$, and $g \notin I_s$, i.e.,

\[
\int_Q \int_Q \frac{1}{|x - y|^{d+1}} \, dx \, dy = +\infty, \quad \forall \delta > 0.
\]

Remark 3. Let $0 \leq t < 1$ and $d \geq 1$. We have not been able to construct a function $g \in VMO$ such that $g \notin J_t$. It might be true, for example, that $VMO \subset J_0$; this is an open problem.

We next present a variant of Proposition 1.

Theorem 3. Let $1 \leq p < +\infty$ and $0 < \delta < \sqrt{3}$. We have, for all $g \in C(Q, \mathbb{R})$,

\[
\int_Q \int_Q \frac{|g(x) - g(y)|^p}{|x - y|^{d+p}} \, dx \, dy \leq C_{d,p,\delta} \left( \int_Q \int_Q \frac{1}{|x - y|^{d+p}} \, dx \, dy + 1 \right).
\]

Moreover, the restriction that $\delta < \sqrt{3}$ is optimal.

Theorem 3 has been proved in [3] when $p = 1$ and $d = 1$. Already in this case the proof is quite elaborate. The case $d = 1$ and $p > 1$ can be proved using exactly the same argument as in the case $d = 1$ and $p = 1$. The proof in the case $d > 1$ is a consequence of the 1-d case using the argument in Step 2 of the proof of [7, Theorem 1].

Theorem 3 fails if we delete the assumption that $g \in C(Q)$. In fact, for each $n \in \mathbb{N}_+$, take $g_n(x) = 0$ on $(0, 1/2) \times (0, 1)^{N-1}$ and $g_n(x) = 2\pi n$ for $x \in (1/2, 1) \times (0, 1)^{N-1}$. Then

\[
\int_Q \int_Q |g_n(x) - g_n(y)|^p \, dx \, dy \to \infty \text{ as } n \to \infty.
\]
\[
\int_{Q} \int_{Q} \frac{1}{|x-y|^{d+p}} \, dx \, dy = 0, \quad \forall \delta > 0.
\]

Theorem 3 implies Proposition 1 when \( g \in C(\overline{Q}) \). However we do not know how to deduce Proposition 1 from Theorem 3 for a general function \( g \in L^1(Q) \) because we are not able to pass to the limit in the RHS of (1.4) when \( g \) is regularized.

Another natural question is whether (1.5) holds for \( g \in VMO(Q) \). We know that the answer is positive if \( d = 1 \) and \( p = 1 \) (see [3]). By the same method as in [3], one can prove that the answer holds for \( d = 1 \) and \( p > 1 \).

We also have

**Theorem 4.** Let \( d \geq 1 \) and \( k \in \mathbb{N}_+ \) be such that \( 1 \leq k \leq d \). Then there exists \( g \in VMO(Q) \) such that \( g \in W^{s,p}(Q) \) for all \( s \in (0, 1) \), \( p > 1 \), and \( sp < k \), and

\[
\int_{Q} \int_{Q} \frac{dx \, dy}{|x-y|^{d+k}} = +\infty, \quad \forall 0 < \delta < 2.
\]

Detailed proofs of these results will be presented elsewhere.

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**References**