Mathematical Analysis

A new characterization of Sobolev spaces

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Abstract

Our main result is the following: Let \( g \in L^p(\mathbb{R}^N) \), \( 1 < p < +\infty \), be such that

\[
\sup_{n \in \mathbb{N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy < +\infty,
\]

for some arbitrary sequence of positive numbers \((\delta_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} \delta_n = 0 \). Then \( g \in W^{1,p}(\mathbb{R}^N) \).


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Résumé

Une nouvelle caractérisation des espaces de Sobolev. Notre résultat principal est le suivant : Soit une fonction \( g \in L^p(\mathbb{R}^N) \), \( 1 < p < +\infty \), telle que

\[
\sup_{n \in \mathbb{N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy < +\infty,
\]

où \((\delta_n)_{n \in \mathbb{N}}\) est une suite arbitraire positive telle que \( \lim_{n \to \infty} \delta_n = 0 \). Alors \( g \in W^{1,p}(\mathbb{R}^N) \).


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Our main result is the following:

**Theorem 1.** Let \( g \in L^p(\mathbb{R}^N) \), \( 1 < p < +\infty \), be such that

\[
\sup_{n \in \mathbb{N}} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy < +\infty,
\]

for some sequence of positive numbers \((\delta_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} \delta_n = 0 \). Then \( g \in W^{1,p}(\mathbb{R}^N) \).

This extends a result from [5]. In [5] the second author obtained the same conclusion under the stronger assumption that

\[
\sup_{0 < \delta < 1} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy < +\infty.
\]

(2)

The argument in [5] used an integration of (2) with respect to \( \delta \) and could not be adapted to the assumption (1). The method we present here is totally different and much more delicate.

**2. Proof of Theorem 1**

The following lemma is the main ingredient in the proof of Theorem 1.

**Lemma 2.** Let \( f \) be a measurable function on a bounded nonempty interval \( I \) and \( 1 \leq p < +\infty \). Then

\[
\liminf_{\varepsilon \to 0_+} \int \int_{I \times I} \frac{\varepsilon^p}{|x - y|^{p+1}} \, dx \, dy \geq c \frac{1}{|I|^{p-1}} \left( \text{ess sup}_I f - \text{ess inf}_I f \right)^p,
\]

where \( c = c_p \) is a positive constant depending only on \( p \).

**Proof of Lemma 2.** Step 1. (3) holds if \( f \in L^\infty(I) \).

By rescaling, we may assume \( I = [0, 1] \).

Denote \( s_+ = \text{ess sup}_I f \), \( s_- = \text{ess inf}_I f \). Rescaling \( f \), one may also assume

\[
s_+ - s_- = 1
\]

(4)

(unless \( f \) is constant on \( I \) in which case there is nothing to prove).
Take $0 < \delta \ll 1$ small enough to ensure that there are (density) points $t_+, t_- \in [20\delta, 1 - 20\delta] \subset [0, 1]$ with

$$
\begin{align*}
&\left\{ [t_+ - \tau, t_+ + \tau] \cap \left[ f > \frac{3}{4}s_+ + \frac{1}{4}s_- \right] \right\} > \frac{9}{5}\tau, \\
&\left\{ [t_- - \tau, t_- + \tau] \cap \left[ f < \frac{3}{4}s_+ + \frac{1}{4}s_- \right] \right\} > \frac{9}{5}\tau.
\end{align*}
$$

(5)

Take $K \in \mathbb{Z}_+$ such that $\delta < 2^{-K} \leq 5\delta/4$ and denote

$$
J = \left\{ j \in \mathbb{Z}_+; \frac{3}{4}s_- + \frac{1}{4}s_+ < j2^{-K} < \frac{3}{4}s_+ + \frac{1}{4}s_- \right\}.
$$

Then

$$
|J| > 2^{K-1} - 2 \approx \frac{1}{\delta}.
$$

(6)

For each $j$, define the following sets:

$$
A_j = \left\{ x \in [0, 1]; (j - 1)2^{-K} \leq f(x) < j2^{-K} \right\}, \quad B_j = \bigcup_{j' < j} A_{j'} \quad \text{and} \quad C_j = \bigcup_{j' > j} A_{j'},
$$

so that $B_j \times C_j \subset [\|f(x) - f(y)\| > 2^{-K}] \subset [\|f(x) - f(y)\| > \delta]$.

Since the sets $A_j$ are disjoint, it follows from (6) that

$$
\text{card}(G) \geq 2^{K-2} - 3 \approx \frac{1}{\delta},
$$

(7)

where $G$ is defined by

$$
G = \left\{ j \in J; |A_j| < 2^{-K+2} \right\}.
$$

For each $j \in J$, set $\lambda_{1,j} = |A_j|$ and consider the function $\psi_1(t)$ defined as follows:

$$
\psi_1(t) = \left[ |t - 4\lambda_{1,j}, t + 4\lambda_{1,j}| \cap B_j \right], \quad \forall t \in [20\delta, 1 - 20\delta].
$$

Then, from (5), $\psi_1(t_+) < 4\lambda_{1,j}$ and $\psi_1(t_-) > 4\lambda_{1,j}$. Hence, since $\psi_1$ is a continuous function on the interval $[20\delta, 1 - 20\delta]$ containing the two points $t_+$ and $t_-$, there exists $t_{1,j} \in [20\delta, 1 - 20\delta]$ such that

$$
\psi_1(t_{1,j}) = 4\lambda_{1,j}.
$$

(8)

Since

$$
\iint_{|f(x) - f(y)| > \delta} \frac{1}{|x-y|^2} \, dx \, dy = +\infty,
$$

it follows that $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap A_j| > 0$.

In fact, suppose $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap A_j| = 0$. Then

$$
\iint_{x \in \{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap A_j} \frac{1}{|x-y|^2} \, dx \, dy \leq \iint_{x \in \{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap B_j} \frac{1}{|x-y|^2} \, dx \, dy = +\infty.
$$

Hence $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap B_j| = 0$ or $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \setminus B_j| = 0$ (see [3]). This is a contradiction since $\psi_1(t_{1,j}) = (|t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}| \cap B_j) = 4\lambda_{1,j}$ (see (8)).

If $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap A_j| < \lambda_{1,j}/4$, then take $\lambda_{2,j} > 0$ such that $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$ and

$$
\frac{\lambda_{2,j}}{2} < \left| t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j} \right| \cap A_j \leq \lambda_{2,j}.
$$

Since $|\{t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\} \cap A_j| < \lambda_{1,j}/4$, we infer that $\lambda_{2,j} \leq \lambda_{1,j}/2$.

Set $E_{2,j} = [t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{2,j}] \cap B_j$ and consider the function $\psi_2(t)$ defined as follows

$$
\psi_2(t) = \left[ |t - 4\lambda_{2,j}, t + 4\lambda_{2,j}| \cap B_j \right], \quad \forall t \in E_{2,j}.
$$

We claim that there exists $t_{2,j} \in E_{2,j}$ such that $\psi_2(t_{2,j}) = 4\lambda_{2,j}$. 


To see this, we argue by contradiction. Suppose that \( \psi_2(t) \neq 4\xi, j \), for all \( t \in E_{2 \cdot j} \). Since \( \psi_2 \) is a continuous function on \( E_{2 \cdot j} \), we assume as well that \( \psi_2(t) < 4\xi, j \), for all \( t \in E_{2 \cdot j} \). Since \( \xi, j / \xi, j \in \mathbb{Z}^n \), it follows that \( \psi_1(t, j) < 4\xi_1, j \), hence we have a contradiction to (8).

It is clear that
\[
\begin{align*}
\iint_{[t_{2 \cdot j} - 4\xi, j, t_{2 \cdot j} + 4\xi, j]^2 \setminus [f(x) - f(y)]^2} \frac{1}{|x - y|^{p+1}} \, dx \, dy & \geq \ \int \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq 1.
\end{align*}
\]

If \( |[t_{2 \cdot j} - 4\xi, j, t_{2 \cdot j} + 4\xi, j] \cap A_j| < \lambda, j / 4 \), then take \( \lambda, j (\lambda, j \leq \lambda, j / 2) \) and \( t_{3 \cdot j} \), etc.

On the other hand, since \( \int_{[f(x) - f(y)]^2} \frac{1}{|x - y|^{p+1}} \, dx \, dy < +\infty \), we have
\[
\limsup_{\text{diam}(Q) \to 0} \iint_{Q \times Q} \frac{1}{|x - y|^{p+1}} \, dx \, dy = 0.
\]

Thus, from (9) and the construction of \( t_{k \cdot j} \) and \( \lambda, k \cdot j \), there exist \( t_j \in [2\delta, 1 - 2\delta] \) and \( \lambda_j > 0 \) \( t_j = t_{k \cdot j} \) and \( \lambda_j = \lambda, k \cdot j \) for some \( k \) such that
\[
(a) \ |[t_j - 4\xi, j, t_j + 4\xi, j] \cap B_j| = 4\xi, j \quad \text{and} \quad (b) \ \frac{\lambda_j}{4} \leq |[t_j - 4\xi, j, t_j + 4\xi, j] \cap A_j| \leq \lambda_j.
\]

Set \( \lambda = \inf_{g \in G} \lambda, j \) \( \lambda > 0 \text{ since } G \) is finite. Suppose \( G = \bigcup_{i=1}^{n} I_m \), where \( I_m \) is defined as follows
\[
I_m = \{ j \in G; 2^{m-1} \lambda \leq \lambda_j < 2^m \lambda \}, \quad \forall m \geq 1.
\]

Then it follows from (7) that
\[
\sum_{m=1}^{n} \text{card}(I_m) \geq \frac{1}{\delta}.
\]

For each \( m \) \( 1 \leq m \leq n \), since \( A_j \cap A_k = \emptyset \) for \( j \neq k \), it follows from (10-b) that there exists \( J_m \subset I_m \) such that
\[
(a) \ \text{card}(J_m) \geq \text{card}(I_m) \quad \text{and} \quad (b) \ |t_i - t_j| > 2^{m+2} \lambda, \quad \forall i, j \in J_m.
\]

Then, from (12-b) and the definition of \( I_m \),
\[
[t_i - 4\xi, i, t_i + 4\xi, i] \cap [t_j - 4\xi, j, t_j + 4\xi, j] = \emptyset, \quad \forall i, j \in J_m.
\]

Set \( U_0 := \emptyset \) and
\[
\begin{align*}
L_m &= \{ j \in J_m; |[t_j - 4\xi, j, t_j + 4\xi, j] \setminus U_{m-1}| \geq 6\xi, j \}, \\
U_m &= \left( \bigcup_{j \in L_m} [t_j - 4\xi, j, t_j + 4\xi, j] \right) \cup U_{m-1}, \quad \text{for } m = 1, 2, \ldots, n.
\end{align*}
\]

Set \( a_m = \text{card}(J_m) \) \( b_m = \text{card}(L_m) \).

From (13) and the definitions of \( J_m \) and \( L_m \),
\[
\frac{1}{4} 2^{m-1}(a_m - b_m) \leq \sum_{i=1}^{m-1} 2^i b_i
\]
which shows that
\[
a_m \leq b_m + 8 \sum_{i=1}^{m-1} 2^{(i-m)} b_i.
\]

Consequently,
\[
\sum_{m=1}^{n} a_m \leq \sum_{m=1}^{n} b_m + 8 \sum_{m=1}^{n} \sum_{i=1}^{m-1} 2^{(i-m)} b_i = \sum_{m=1}^{n} b_m + 8 \sum_{i=1}^{n} b_i \sum_{m=i+1}^{n} 2^{(i-m)}.
\]
Since $\sum_{i=1}^{\infty} 2^{-i} = 1$, it follows from (11) and (12-a) that
\[
\sum_{m=1}^{n} b_m \geq \frac{1}{9} \sum_{m=1}^{n} d_m \geq \frac{1}{\delta}.
\]
Therefore, it is easy to see that
\[
\int I \times I \left| f(x) - f(y) \right| > \delta
\]
\[
\geq \frac{1}{|x - y|^{p+1}} \int \int_{|g(x) - g(y)| > \epsilon_n} \epsilon_n^p \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq \frac{n}{b_0^p} \geq \frac{1}{\delta^p},
\]
which yields the conclusion of Lemma 2.

Step 2. Proof of Lemma 2 completed.

Observe that if we define the function
\[
f_A = \left( f \vee (-A) \right) \wedge A,
\]
then
\[
\left| f_A(x) - f_A(y) \right| \leq \left| f(x) - f(y) \right|.
\]
Applying (3) to the sequence $f_A$ and letting $A$ goes to infinity, we deduce that (3) holds for any measurable function $f$ on $I$ (allowing the right-hand side to be $+\infty$).

**Proof of Theorem 1 when $N = 1$.** Set $\tau_h(g)(x) = \frac{g(x+h) - g(x)}{h}$, $\forall x \in \mathbb{R}$, $0 < h < 1$.

For each $m \geq 2$, take $K \in \mathbb{R}^+$ such that $m \leq Kh$, then
\[
\int_{-m}^{m} |\tau_h(g)(x)|^p \, dx \leq \sum_{k=-K}^{K} (k+1)h \int |\tau_h(g)(x)|^p \, dx.
\]
Thus, since
\[
\int_{a}^{a+h} |\tau_h(g)(x)|^p \, dx \leq \int_{a}^{a+h} \left( \frac{1}{h^p} \left[ \sup_{x \in (a, a+2h)} g - \inf_{x \in (a, a+2h)} g \right] \right)^p \, dx,
\]
it follows from Lemma 2 that, for some constant $c = c_p > 0$,
\[
\int_{-m}^{m} |\tau_h(g)(x)|^p \, dx \leq c \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \frac{\epsilon_n^p}{|x - y|^{p+1}} \, dx \, dy.
\]
(14)
Since $m \geq 2$ is arbitrary, (14) shows that
\[
\int |\tau_h(g)(x)|^p \, dx \leq c \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \frac{\epsilon_n^p}{|x - y|^{p+1}} \, dx \, dy.
\]
(15)
Since (15) holds for all $0 < h < 1$, it follows that $g \in W^{1,p}(\mathbb{R})$ (see e.g. [2, Chapter 8]).

In order to establish Theorem 1 in dimension $N \geq 2$, we need the following

**Lemma 3.** Let $g$ be a measurable function on $\mathbb{R}^N$ and $1 \leq p < +\infty$. Then
\[
\int_{\mathbb{R}^N} \int_{|g(x) - g(y)| > \delta} \frac{1}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p} \int_{\mathbb{R}^N} \int_{|g(x) - g(y)| > \delta} \frac{1}{|x - y|^{N+p}} \, dx \, dy, \quad \forall \delta > 0,
\]
where $C_{N,p} > 0$ is a constant depending only on $N$ and $p$. 

Proof of Lemma 3. The method used to prove Lemma 3 is standard (see e.g. [1, Chapter 7]). □

Proof of Theorem 1 completed. Set $\varepsilon_n = 2\delta_n$, for all $n \in \mathbb{N}$. Then it follows from Lemma 3 that

$$
\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \frac{\varepsilon_n^p}{|x_N - y_N|^{p+1}} \, dx \, dy < +\infty.
$$

Using Fatou’s lemma and Theorem 1 in the case $N = 1$, it is not difficult to prove that $g(\cdot, \cdot) \in W^{1,p}(\mathbb{R})$ and moreover

$$
\lim_{n \to \infty} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \frac{\varepsilon_n^p}{|x_N - y_N|^{p+1}} \, dx \, dy = C_p \int_{\mathbb{R}} \left| \frac{\partial g}{\partial x_N}(x', x_N) \right|^p \, dx_N,
$$

for almost everywhere $x' \in \mathbb{R}^{N-1}$ (see [5]).

Thus

$$
\int_{\mathbb{R}^N} \left| \frac{\partial g}{\partial x_N}(x) \right|^p \, dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \left| \frac{\partial g}{\partial x_N}(x', x_N) \right|^p \, dx \, dx' < +\infty.
$$

Similarly,

$$
\int_{\mathbb{R}^N} \left| \frac{\partial g}{\partial x_i}(x) \right|^p \, dx < +\infty, \quad \forall 1 \leq i \leq N - 1.
$$

Therefore, $g \in W^{1,p}(\mathbb{R}^N)$ (see e.g. [4, Chapter 4]). □

References