Partial Differential Equations

Inequalities related to liftings and applications

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Abstract


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J. Bourgain et H. Brezis [1] (voir aussi [2]) ont prouvé

Théorème 1. Soient $d \geq 1$ un entier, $T^d$ le tore de dimension $d$, $\psi \in C^\infty(T^d, \mathbb{R})$, et $g = e^{i\psi}$. Alors il existe $\psi_1, \psi_2 \in C^\infty(T^d, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,

$$|\psi_1|_{W^{1,1}} \leq C|g|_{H^{1/2}}^2 \quad \text{et} \quad |\psi_2|_{H^{1/2}} \leq C|g|_{H^{1/2}},$$

pour une certaine constante $C > 0$ qui ne dépend que de $d$.

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Dans cette Note, on considère la semi-norme suivante de l’espace $W^{s,p}(\mathbb{T}^d)$ $(0 < s < 1$ et $p > 1)$ défini par (1) et on note $H^{\frac{1}{2}} = W^{\frac{1}{2},2}$.

**Question.** Soient $d \geq 1$ un entier, $\mathbb{T}^d$ le tore de dimension $d$, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, et $g = e^{i\psi}$. Est-ce qu’il existe $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,

$$
|\psi_1|_{W^{1,1}} \leq C|g|_{W^{\frac{1}{p},p}}^p \quad \text{et} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leq C|g|_{W^{\frac{1}{p},p}}^p,
$$

pour une certaine constante $C > 0$ qui ne dépend que de $d$ et de $p$ ?

Le but principal de la Note est de donner une réponse positive à cette question au cas $d = 1$. En fait, on prouve :

**Théorème 2.** Soient $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^1, \mathbb{R})$, et $g = e^{i\psi}$. Alors il existe $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^1, \mathbb{R})$ telles que $\psi = \psi_1 + \psi_2$,

$$
|\psi_1|_{W^{1,1}} \leq CT\sqrt{\gamma}(g) \quad \text{et} \quad |\psi_2|_{W^{\frac{1}{p},p}} \leq C|g|_{W^{\frac{1}{p},p}}^p,
$$

pour une certaine constante $C > 0$ qui ne dépend que de $d$ et de $p$. Ici $Tg$ est défini par (2).

Un autre résultat de la Note est

**Théorème 3.** Soient $d \geq 1$, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^d, \mathbb{R})$, et $g = e^{i\psi}$. Supposons que $|g|_{BMO} = \beta < 1$. Alors il existe une constante $C > 0$ qui ne dépend que de $d$ et de $p$ telle que

$$
|\psi|_{W^{\frac{1}{p},p}} \leq \frac{C}{1 - \beta}|g|_{W^{\frac{1}{p},p}}.
$$

Ici la BMO-semi-norme est définie par (4).

1. Introduction and main results

J. Bourgain and H. Brezis [1] (see also [2]) proved

**Theorem 1.** Let $d \geq 1$, $\mathbb{T}^d$ be the $d$-dimensional torus, $\psi \in C^\infty(\mathbb{T}^d, \mathbb{R})$, and $g = e^{i\psi}$. Then there exist $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R})$ such that $\psi = \psi_1 + \psi_2$,

$$
|\psi_1|_{W^{1,1}} \leq C|g|^2_{H^{\frac{1}{2}}} \quad \text{and} \quad |\psi_2|_{H^{\frac{1}{2}}} \leq C|g|^2_{H^{\frac{1}{2}}},
$$

for some constant $C > 0$ depending only on $d$.

In this Note, one considers the following semi-norm of $W^{s,p}(\mathbb{T}^d)$, for $0 < s < 1$ and $p > 1$,

$$
|g|_{W^{s,p}} = \left(\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}} \, dx \, dy\right)^{\frac{1}{p}},
$$

(1)

and one notes $H^{\frac{1}{2}} = W^{\frac{1}{2},2}$.

Their proof is delicate. Using the inequality (see e.g. [7])

$$
\left| \int_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \zeta \, dx \right| \leq C\left(\|\zeta\|_{L^\infty} + |\zeta|_{H^{\frac{1}{2}}}ight)(|g|^2_{H^{\frac{1}{2}}} + |g|_{H^{\frac{1}{2}}})
$$

where $g = e^{i\psi}$ and $\psi \in C^1(\mathbb{T}^d, \mathbb{R})$ and a dual argument one can prove that there exist $\psi_1$ and $\psi_2$ such that $\psi = \psi_1 + \psi_2$, $|\psi_1|_{BV} + |\psi_2|_{H^{\frac{1}{2}}} \leq C(\|g|^2_{H^{\frac{1}{2}}} + |g|_{H^{\frac{1}{2}}})$ in the case $d = 1$ (argument communicated to us by H. Brezis
and P. Mironescu). From these facts, J. Bourgain, H. Brezis, and P. Mironescu suggested the following (see [3, Open Problem 1])

**Question.** Let \( d \geq 1 \) be an integer, \( \mathbb{T}^d \) be the \( d \)-dimensional torus, \( p > 1 \), \( q = \frac{p}{p-1} \), \( \psi \in C^\infty(\mathbb{T}^d, \mathbb{R}) \), and \( g = e^{i\psi} \). Does there exist \( \psi_1, \psi_2 \in C^\infty(\mathbb{T}^d, \mathbb{R}) \) such that \( \psi = \psi_1 + \psi_2 \),

\[
|\psi_1|_{W^{1,1}} \leq C |g|_{W^{1,p}}^{\frac{p}{p-1}} \quad \text{and} \quad |\psi_2|_{W^{1,p}} \leq C |g|_{W^{1,q}},
\]

for some constant \( C > 0 \) depending only on \( d \) and \( p \).

The main goal of this Note is to give an affirmative answer to this question in the case \( d = 1 \). We first introduce

**Definition 1.** Let \( d \geq 1 \) be an integer, \( \mathbb{T}^d \) be the \( d \)-dimensional torus, \( g \in L^\infty(\mathbb{T}^d, S^1) \) and \( \delta > 0 \). Define

\[
T_\delta(g) = \iint_{\mathbb{T}^d \times \mathbb{T}^d} \frac{1}{|x-y|^{d+1}} \, dx \, dy.
\]

We next prove two inequalities related to liftings.

**Lemma 1.** Let \( d \geq 1 \), \( p > 1 \), \( q = \frac{p}{p-1} \), \( \zeta \in C^1(\mathbb{T}^d; \mathbb{R}) \), \( \psi \in C^1(\mathbb{T}^d; \mathbb{R}) \), and \( g = e^{i\psi} \). Then

\[
\left| \int_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \, \zeta \, dx \right| \leq C \left( \|\zeta\|_{L^\infty} T_{\sqrt{\tau}}(g) + |\zeta|_{W^{1,q}} |g|_{W^{1,p}} \right),
\]

for some constant \( C > 0 \) depending only on \( d \) and \( p \).

**Lemma 2.** Let \( d \geq 1 \), \( p > 1 \), \( q = \frac{p}{p-1} \), \( \zeta \in C^1(\mathbb{T}^d; \mathbb{R}) \), \( \psi \in C^1(\mathbb{T}^d; \mathbb{R}) \), and \( g = e^{i\psi} \). Assume that \( |g|_{BMO} = \beta < 1 \). Then there exists a constant \( C > 0 \) depending only on \( d \) and \( p \) such that

\[
\left| \int_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \, \zeta \, dx \right| \leq \frac{C}{1 - \beta} |\zeta|_{W^{1,q}} |g|_{W^{1,p}}.
\]

Hereafter, we use the following BMO-semi-norm:

\[
|f|_{BMO(\Omega)} := \sup_{B(x, r) \subseteq \Omega} \left| \int_{B(x, r)} f(\xi) - \iint_{B(x, r)} f(\eta) \, d\eta \right| \, d\xi, \quad \forall f \in BMO(\Omega).
\]

Here \( B(x, r) \) denotes the ball in \( \Omega \) of radius \( r \) centered at \( x \).

Using Lemma 1 and the dual argument, we can prove

**Theorem 2.** Let \( p > 1 \), \( q = \frac{p}{p-1} \), \( \psi \in C^1(\mathbb{T}^1, \mathbb{R}) \), and \( g = e^{i\psi} \). Then there exist \( \psi_1, \psi_2 \in C^\infty(\mathbb{T}^1, \mathbb{R}) \) such that \( \psi = \psi_1 + \psi_2 \),

\[
|\psi_1|_{W^{1,1}} \leq CT_{\sqrt{\tau}}(g) \quad \text{and} \quad |\psi_2|_{W^{1,p}} \leq C |g|_{W^{1,q}},
\]

for some constant \( C > 0 \) depending only on \( p \).

This theorem answers positively [3, Open Problem 1], which corresponds to the question in the case \( d = 1 \) and \( p > 2 \). Our method does not seem to be generalized to the case \( d \geq 2 \). After our work was finished, P. Mironescu [8] informs us that the answer to the question is also true in higher dimensional cases. The reader can find many interesting questions related to liftings of \( S^1 \)-valued maps in [7].

We next give a useful consequence of Lemma 2.
Theorem 3. Let $d \geq 1$, $p > 1$, $q = \frac{p}{p-1}$, $\psi \in C^1(\mathbb{T}^d; \mathbb{R})$, and $g = e^{i\psi}$. Assume that $|g|_{\text{BMO}} = \beta < 1$. Then there exists a constant $C > 0$ depending only on $d$ and $p$ such that

$$|\psi|_{W^{\frac{1}{p}, \frac{1}{p}}(\mathbb{T}^d)} \leq \frac{C}{1-\beta} |g|_{W^{\frac{1}{p}, \frac{1}{p}}}.$$ 

This result is inspired by the one of R.R. Coifman and Y. Meyer [6] (see also [5]) who proved that there exists a constant $c > 0$ such that if $|g|_{\text{BMO}} < c$ then $|\psi|_{\text{BMO}} \leq 4|g|_{\text{BMO}} (g = e^{i\psi})$.

2. Proofs

2.1. Proof of Lemma 1

Step 1: Proof of Lemma 1 in the one-dimensional case.

Since $\int_{\mathbb{T}^1} \psi' \, ds = 0$, without loss of generality, one may assume that $\int_{\mathbb{T}^1} \zeta \, ds = 0$. Set $\tilde{g}(s) = g(e^{is})$ and $\tilde{\zeta}(s) = \zeta(e^{is})$ for $s \in \mathbb{R}$. Let $B_1$ be the unit ball of $\mathbb{R}^2$ and $\chi \in C^1(B_1)$ be such that $\chi(x) = 1$ for $|x| \geq \frac{3}{4}$, $\chi(x) = 0$ for $|x| \leq \frac{1}{2}$, $0 \leq \chi(x) \leq 1$ and $|D\chi(x)| \leq 1$, for $x \in B_1$. Define $u : B_1 \mapsto \mathbb{R}^2$ and $\eta : B_1 \mapsto \mathbb{R}$ as follows

$$u[(1-h)e^{i\theta}] = \int_0^{\theta+2\pi h} \tilde{g}(s) \, ds \quad \text{and} \quad \eta[(1-h)e^{i\theta}] = \int_0^{\theta+2\pi h} \tilde{\zeta}(s) \, ds,$$

for $h \in [0, 1]$ and $\theta \in [0, 2\pi]$. Set $\tilde{\eta} = \chi \eta$ and define

$$\tilde{u}(X) = \begin{cases} u(X)/|u(X)| & \text{if } |u(X)| \geq \alpha, \\ u(X)/\alpha & \text{otherwise}, \end{cases} \quad \forall X \in B_1,$$

for some small positive regular value $\alpha$ of $|u|$. Then $\tilde{u} \in W^{1, \infty}(B_1)$ and $\tilde{\eta} \in W^{1, \infty}(B_1)$. Since $|\tilde{u}| \leq 1$ and $|D\tilde{u}| \lesssim |Du|$ on $B_1$, integrating by part, one has

$$\left| \int_{\mathbb{T}^1} \psi' \zeta \, ds \right| \lesssim \|\eta\|_{L^\infty} \int_{B_1 \setminus B_{\frac{1}{2}}} |\det D\tilde{u}| \, dX + \int_{B_1} |D\tilde{\eta}||Du| \, dX. \quad (5)$$

Hereafter $B_{\frac{1}{2}} := B(0, \frac{1}{2})$. Applying the method used in [9] (see also [4] in the case $\sqrt{3}$ is replaced by $\delta$ for $\delta < \sqrt{2}$), one obtains

$$\int_{B_1 \setminus B_{\frac{1}{2}}} |\det D\tilde{u}| \, dX \lesssim T_{\sqrt{3}}(g). \quad (6)$$

On the other hand, since $\frac{1}{p} + \frac{1}{q} = 1$, by Holder’s inequality, since $\tilde{\eta} = 0$ in $B_{\frac{1}{2}}$, one has

$$\int_{B_1} |D\tilde{\eta}||Du| \, dX \leq \left( \int_{B_1 \setminus B_{\frac{1}{2}}} h^{p-2}|Du|^p \, dX \right)^{\frac{1}{p}} \left( \int_{B_1 \setminus B_{\frac{1}{2}}} h^{q-2}|Du|^q \, dX \right)^{\frac{1}{q}}. \quad (7)$$

However, for $h < \frac{1}{2}$,

$$|Du|[(1-h)e^{i\theta}] \lesssim \frac{1}{h} |\tilde{g}(\theta + 2\pi h) - \tilde{g}(\theta)| + \frac{1}{h} \int_0^{\theta+2\pi h} |\tilde{g}(s) - \tilde{g}(\theta + 2\pi h)| \, ds \quad (8)$$

and
2.2. Proof of Lemma 2

Similarly,\[\int_{0}^{2\pi} \int_{0}^{\frac{1-h}{h^2}} |\tilde{g}(\theta + 2\pi h) - \tilde{g}(\theta)|^p \, dh \, d\theta + \int_{0}^{2\pi} \int_{0}^{\frac{1-h}{h^2}} |\tilde{g}(s) - \tilde{g}(\theta + 2\pi h)|^p \, dx \, d\theta \lesssim |g|^p_{W^\frac{1}{2},p}.\]

Hence, it follows that
\[\int_{B \setminus B_{\frac{1}{2}}} h^{p-2} |Du|^p \, dX \lesssim |g|^p_{W^\frac{1}{2},p}. \tag{9}\]

Similarly,
\[\int_{B \setminus B_{\frac{1}{2}}} h^{q-2} |D\eta|^q \, dX \lesssim |\xi|^q_{W^\frac{1}{2},q}. \tag{10}\]

The conclusion in the case \(N = 1\) follows from (5), (6), (9), and (10), since \(\int_{T^1} \xi \, ds = 0.\)

Step 2: Proof of Lemma 1 in the general case.

Without loss of generality, one may assume that \(i = 1.\) The proof in this case follows as in Step 1 by using the extension \(u : B_1 \times \mathbb{T}^{d-1} \mapsto \mathbb{R}\) defined as follows
\[u[(1-h)e^{i\theta}, x_2, \ldots, x_d] = \int_{B(x, h)} \tilde{g}(s) \, ds,\]

where \(x = (e^{i\theta}, x_2, \ldots, x_d) \in \mathbb{T}^d.\) The details are left to the reader. \(\square\)

2.2. Proof of Lemma 2

Without loss of generality, one may assume that \(i = 1.\) We use the same notation as in the proof of Lemma 1. The essential point in this proof is to choose \(1 - \beta < \alpha < 1 - \beta / 2\) such that \(\alpha\) is a regular value of \(|u|\). Then since \(|\tilde{g}|_{\text{BMO}} = \beta\), it follows that \(|u| > \alpha\) in \(B_1.\) Hence, from the definition of \(\tilde{u}, |\tilde{u}| = 1.\) Thus (see (5) in the case \(d = 1))\n
\[\left|\int_{\mathbb{T}^d} \frac{\partial u}{\partial x_1} \xi \, dx\right| \lesssim \frac{1}{1 - \beta} \int_{B_1 \times \mathbb{T}^{d-1}} |D\tilde{u}| |D\tilde{u}| \, dx.\]

Therefore, as in the proof of Lemma 1, one obtains
\[\left|\int_{\mathbb{T}^d} \frac{\partial u}{\partial x_1} \xi \, dx\right| \lesssim \frac{1}{1 - \beta} |\xi|_{W^\frac{1}{2},q} |g|_{W^\frac{1}{2},p}. \square\]

2.3. Proof of Theorems 2 and 3

Proof of Theorem 2. Set \(D(A) = \{(\xi, \zeta) ; \xi \in C^1(\mathbb{T}^1; \mathbb{R})\}\) and define \(A : D(A) \mapsto \mathbb{R}\) by \(A(\xi, \zeta) = \int_{\mathbb{T}^1} \psi \cdot \zeta.\) By Hahn–Banach's theorem and Theorem 1, there exists \(A : L^\infty(\mathbb{T}^1) \times W^{\frac{1}{2},q}(\mathbb{T}^1; \mathbb{R}) \mapsto \mathbb{R}\) such that \(A(\xi_1, \zeta_2) = A(\xi_1, \zeta_2)\) for \((\xi_1, \zeta_2) \in D(A)\) and \(|A(\xi_1, \zeta_2)| \lesssim T_{\sqrt{\gamma}} |\xi_1|_{L^\infty} + |g|_{W^\frac{1}{2},p} |\zeta_2|_{W^{\frac{1}{2},q}}.\) Define \(\psi_1(\xi) = A(\xi, 0)\) and \(\psi_2(\zeta) = A(0, \zeta)\) for \(\xi \in C^1(\mathbb{T}^1)\). Then \(\psi_1 \in \mathcal{M}(\mathbb{T}^1),\) the space of all Radon measures on \(\mathbb{T}^1,\) and \(\psi_2 \in [W^{\frac{1}{2},q}]^*,\) the duality of \(W^{\frac{1}{2},q},\)
\(|\psi_1|_{\mathcal{M}} \lesssim T_{\sqrt{\gamma}}(g)\) and \(|\psi_2|_{[W^{\frac{1}{2},q}]^*} \lesssim |g|_{W^{\frac{1}{2},q}}.\) Without loss of generality, one may assume that \(\langle \psi, 1 \rangle = \langle \psi_1, 1 \rangle = 0.\)

Then there exist \(\psi_1\) and \(\psi_2\) such that \(\psi_1 = \psi_1, \psi_2 = \psi_2\) and \(\psi = \psi_1 + \psi_2.\) It is clear to see that \(|\psi_1|_{BV} \lesssim T_{\sqrt{\gamma}}(g),\)
\(|\psi_2|_{W^{\frac{1}{2},p}} \lesssim |g|_{W^{\frac{1}{2},p}}.\) Define \(w_{1n} = \psi_1 * \rho_n\) and \(w_{2n} = \psi_1 - w_{1n} + w_{2n}.\) Since \(\psi_1 = \psi_2 = \psi \in W^{\frac{1}{2},p}\), when \(n\) is sufficiently big, one has \(|w_{2n}|_{W^{\frac{1}{2},p}} \lesssim |\psi_1 - \psi_{1,n}|_{W^{\frac{1}{2},p}} + |\psi_2|_{W^{\frac{1}{2},p}} \lesssim |g|_{W^{\frac{1}{2},p}}.\) On the other hand, it is clear to see that \(|\psi_{1,n}|_{BV} \lesssim |\psi_1| \lesssim T_{\sqrt{\gamma}}(g). \square\)
Proof of Theorem 3. For \( \xi \in C^\infty(\mathbb{T}^d) \) with \( \int_{\mathbb{T}^d} \xi \, dx = 0 \), define \( \zeta \) by \( \Delta_{\mathbb{T}^d} \zeta = \xi \) and \( \int_{\mathbb{T}^d} \zeta \, dx = 0 \). Here \( \Delta_{\mathbb{T}^d} \) denotes the Laplace–Beltrami operator on \( \mathbb{T}^d \). Applying Lemma 2, one has

\[
\left| \int_{\mathbb{T}^d} \psi \xi \, dx \right| \leq \sum_{i=1}^d \int_{\mathbb{T}^d} \frac{\partial \psi}{\partial x_i} \frac{\partial \xi}{\partial x_i} \, dx \leq \sum_{i=1}^d \frac{1}{1-\beta} \left| \frac{\partial \zeta}{\partial x_i} \right|_{W^{1,q}_q} |g|_{W^{\frac{1}{q},p}}.
\]

This implies

\[
\left| \int_{\mathbb{T}^d} \psi \xi \, dx \right| \leq \frac{1}{1-\beta} \| \xi \|_{W^{1,q}_q} |g|_{W^{\frac{1}{q},p}}.
\]

Hence,

\[
\left| \int_{\mathbb{T}^d} \left( \psi - \int_{\mathbb{T}^d} \psi \right) \xi \, dx \right| = \left| \int_{\mathbb{T}^d} \left( \xi - \int_{\mathbb{T}^d} \xi \right) \psi \, dx \right| \leq \frac{1}{1-\beta} \| \xi \|_{W^{1,q}_q} |g|_{W^{\frac{1}{q},p}}, \quad \forall \xi \in C^\infty(\mathbb{T}^d).
\]

It follows that

\[
|\psi|_{W^{\frac{1}{p},q}} = \left| \psi - \int_{\mathbb{T}^d} \psi \right|_{W^{\frac{1}{p},q}} \leq \frac{1}{1-\beta} |g|_{W^{\frac{1}{q},p}}.
\]

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References


