Further characterizations of Sobolev spaces

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Abstract. Let \((F_n)_{n \in \mathbb{N}}\) be a sequence of non-decreasing functions from \([0, +\infty)\) into \([0, +\infty)\). Under some suitable hypotheses on \((F_n)_{n \in \mathbb{N}}\), we prove that if \(g \in L^p(\mathbb{R}^N), 1 < p < +\infty\), satisfies
\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{N+p}} \, dx \, dy < +\infty,
\]
then \(g \in W^{1,p}(\mathbb{R}^N)\) and moreover
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{N+p}} \, dx \, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx,
\]
where \(K_{N,p}\) is a positive constant depending only on \(N\) and \(p\). This extends some results in J. Bourgain and H.-M. Nguyen [A new characterization of Sobolev spaces, C. R. Math. Acad. Sci. Paris 343, 75–80 (2006)] and H.-M. Nguyen [Some new characterizations of Sobolev spaces, J. Funct. Anal. 237, 689–720 (2006)]. We also present some partial results concerning the case \(p = 1\) and various open problems.

Keywords. Sobolev spaces

1. Introduction

In [7], we established the following characterizations of Sobolev spaces:

**Proposition 1** ([7 Theorem 2]). Let \(1 < p < +\infty\). Then

(a) There exists a constant \(C_{N,p}\), depending only on \(N\) and \(p\), such that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dy \, dx \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx, \quad \forall \delta > 0, \forall g \in W^{1,p}(\mathbb{R}^N).
\]
If \( g \in L^p(\mathbb{R}^N) \) satisfies
\[
\sup_{0<\delta<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy < +\infty,
\]
then \( g \in W^{1,p}(\mathbb{R}^N) \).

Moreover, for any \( g \in W^{1,p}(\mathbb{R}^N) \),
\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx,
\]
where \( K_{N,p} \) is defined by
\[
K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p \, d\sigma, \tag{1.1}
\]
for any \( e \in \mathbb{S}^{N-1} \).

**Proposition 2** ([7, Theorem 3]). Let \( 1 < p < +\infty \). Then

(a) For every \( g \in W^{1,p}(\mathbb{R}^N) \),
\[
\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon |g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy + \int_{\mathbb{R}^N} \int_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}} \, dx \, dy 
\leq C_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx,
\]
where \( C_{N,p} \) is a positive constant depending only on \( N \) and \( p \).

(b) If \( g \in L^p(\mathbb{R}^N) \) satisfies
\[
\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N} \int_{|g(x)-g(y)|\leq 1} \frac{\varepsilon |g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy + \int_{\mathbb{R}^N} \int_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}} \, dx \, dy < +\infty,
\]
then \( g \in W^{1,p}(\mathbb{R}^N) \).

(c) Moreover, for any \( g \in W^{1,p}(\mathbb{R}^N) \),
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|g(x)-g(y)|\leq 1} \frac{\varepsilon |g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx,
\]
where \( K_{N,p} \) is defined by \((1.1)\).

A sharper version of assertion (b) of Proposition 1 was established by J. Bourgain and H.-M. Nguyen in [2].
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Proposition 3 ([2, Theorem 1]). Let \( g \in L^p(\mathbb{R}^N) \), \( 1 < p < +\infty \), be such that

\[
\begin{align*}
&\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta_n^p}{|x-y|^{N+p}} \, dx \, dy < +\infty \\
&\text{for some sequence } (\delta_n)_{n \in \mathbb{N}} \text{ of positive numbers with } \lim_{n \to \infty} \delta_n = 0. \text{ Then } g \in W^{1,p}(\mathbb{R}^N).
\end{align*}
\]

When \( p = 1 \), we have

Proposition 4. Let \( g \in L^1(\mathbb{R}^N) \) be such that

\[
\begin{align*}
&\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta_n}{|x-y|^{N+1}} \, dx \, dy < +\infty \\
&\text{for some sequence } (\delta_n)_{n \in \mathbb{N}} \text{ of positive numbers with } \lim_{n \to \infty} \delta_n = 0. \text{ Then } g \in BV(\mathbb{R}^N); \text{ moreover, there exists a constant } c_N, \text{ depending only on } N, \text{ such that}
\end{align*}
\]

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta_n}{|x-y|^{N+1}} \, dx \, dy \geq c_N \int_{\mathbb{R}^N} |\nabla g| \, dx.
\]

Remark 1. Proposition 4 is not stated explicitly in [2], but its proof is implicit there (see the proof of [2, Theorem 1]).

The proof of Proposition 3 is much more involved than the one of Propositions 1 and 2 (see [2]).

In this paper, we generalize Propositions 1–3 as follows:

Theorem 1. Let \( 1 < p < +\infty \) and \((F_n)_{n \in \mathbb{N}}\) be a sequence of functions from \([0, +\infty)\) into \([0, +\infty)\) such that

(i) \( F_n(t) \) is a non-decreasing function with respect to \( t \) on \([0, +\infty)\), for all \( n \in \mathbb{N} \).

(ii) \( \int_0^1 F_n(t)t^{-(p+1)} \, dt = 1 \) for all \( n \in \mathbb{N} \).

(iii) \( F_n(t) \) converges uniformly to 0 on every compact subset of \((0, +\infty)\) as \( n \) goes to infinity.

Then

(a) If \( g \in W^{1,p}(\mathbb{R}^N) \), then for every \( n \in \mathbb{N} \),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(g(x)-g(y))}{|x-y|^{N+p}} \, dx \, dy \leq C_{N,p} \int_0^\infty F_n(t)t^{-(p+1)} \, dt \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx,
\]

where \( C_{N,p} \) is a positive constant depending only on \( N \) and \( p \).
(b) If \( g \in L^p(\mathbb{R}^N) \) satisfies
\[
\lim\inf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \frac{dx}{|x-y|^{N+p}} dy < +\infty, \tag{1.6}
\]
then \( g \in W^{1,p}(\mathbb{R}^N) \) and
\[
\lim\inf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \frac{dx}{|x-y|^{N+p}} dy \geq K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx. \tag{1.7}
\]

(c) Moreover, if
\[
\limsup_{n \to \infty} \int_0^\infty F_n(t)t^{-(p+1)} dt < +\infty, \tag{1.8}
\]
then
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \frac{dx}{|x-y|^{N+p}} dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N). \tag{1.9}
\]

Here \( K_{N,p} \) is defined by (1.1).

**Remark 2.** Many ideas used in the proof of Theorems 1 and 2 are borrowed from the method of J. Bourgain and H.-M. Nguyen in [2].

**Remark 3.** Propositions 1 and 3 follow from Theorem 1 by choosing
\[
F_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \delta_n, \\ \frac{p\delta_n^p}{1 - \delta_n^p} & \text{otherwise.} \end{cases} \tag{1.10}
\]

To deduce Proposition 2 we choose
\[
F_n(t) = \begin{cases} \epsilon_n t^{p+\epsilon_n} & \text{if } 0 \leq t \leq 1, \\ \epsilon_n & \text{otherwise.} \end{cases}
\]

**Remark 4.** We now make some comments about hypotheses (i)–(iii) on the sequence \((F_n)\). The conclusion of Theorem 1 may fail if we do not assume (i). For example, let
\[
F_n(t) = \begin{cases} nt^{p+1} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{otherwise,} \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Clearly,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \frac{dx}{|x-y|^{N+p}} dy = 0, \quad \forall n \geq 1, \forall p > 1,
\]
but \( g \not\in W^{1,p}(\mathbb{R}^N) \) for all \( p > 1 \).
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Condition (ii) is a normalization condition. Indeed, if we assume
\[ \lim_{n \to \infty} \int_0^1 F_n(t) t^{-(p+1)} \, dt = +\infty, \]
then \( g \) is a constant function (see Corollary \[1\]).

Condition (iii) is also important. Indeed, the sequence \( F_n(t) = t^{p+1}, \) for all \( n \geq 1, \) satisfies conditions (i) and (ii). However, condition (1.6) is equivalent to \( g \in W^{1/(p+1), p+1}(\mathbb{R}^N). \)

The analogue of assertion (b) in Theorem \[1\] for \( p = 1 \) is the following

**Theorem 2.** Let \( (F_n)_{n \in \mathbb{N}} \) be a sequence of functions from \( [0, +\infty) \) into \( [0, +\infty) \) satisfying (i), (ii) with \( p = 1 \) and (iii). Assume that \( g \in L^1(\mathbb{R}^N) \) and \( g \) satisfies (1.6) with \( p = 1. \) Then \( g \in BV(\mathbb{R}^N). \) Moreover, there exists a constant \( c_N, \) depending only on \( N, \) such that
\[ \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{N+1}} \, dx \, dy \geq c_N \int_{\mathbb{R}^N} |\nabla g| \, dx. \] (1.11)

Comparing with (1.7), we have

**Question 1.** Can one replace \( c_N \) by \( K_{N,1} \) in (1.11)?

The reader can find further questions in Section 4.

**Remark 5.** Proposition \[4\] follows from Theorem \[2\] by choosing \( F_n \) as in (1.10) with \( p = 1. \)

For what concerns the analogues of assertions (a) and (c), A. Ponce has constructed a function \( g \in W^{1,1}(\mathbb{R}) \) such that
\[ \lim_{\delta \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta}{|g(x) - g(y)|} \, dx \, dy = +\infty. \]

Hence the analogues of these assertions for \( p = 1 \) do not hold (see \[7\]).

The proof of assertion (a) in Theorem \[1\] is similar to one in \[7\]; it is based on maximal functions. We present two methods of proof of assertion (b) in Theorem \[1\]. The first one, is based on Proposition \[3\] The second one which relies heavily on Lemma \[2\] below, is more complicated but is interesting in its own right. For what concerns Theorem \[2\] we are able to apply the first method, but not the second due to lack of an analogue of Lemma \[2\] for \( p = 1. \) Lemma \[2\] is closely related to Proposition \[3\]; its proof uses many ideas of J. Bourgain and H.-M. Nguyen from \[3\]. The proof of assertion (c) in Theorem \[1\] is also much more delicate than the one of assertion (c) in Propositions \[1\] and \[2\].

The paper is organized as follows. In Section 2 we will prove Theorems \[1\] and \[2\]. In Section 3 we present another proof of assertion (b) in Theorem \[1\]. Finally, in Section 4 we will discuss problems related to \( \Gamma \)-convergence.
2. Proofs of Theorems 1 and 2

2.1. Proof of assertion (a) in Theorem 1

Using the change of variables formula and Fubini’s theorem, one gets

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(\frac{|g(x) - g(y)|}{|x - y|^{N+p}}) \, dx \, dy = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty F_n(\frac{|g(x + h\sigma) - g(x)|}{h^{p+1}}) \, dh \, dx \, d\sigma.
$$

Consequently, to prove (1.5), it suffices to show that

$$
\int_{\mathbb{R}^N} \int_0^\infty F_n(\frac{|g(x + h\sigma) - g(x)|}{h^{p+1}}) \, dh \, dx \leq C_p \int_0^\infty F_n(t)t^{-(p+1)} \, dt \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx
$$

for all \( n \in \mathbb{N} \), where \( C_p \) is a positive constant depending only on \( p \).

Without loss of generality we may assume that \( \sigma = e_N \). Since \( g \in W^{1,p}(\mathbb{R}^N) \), \( g(x', \cdot) \in W^{1,p}(\mathbb{R}) \) for almost every \( x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \).

Fix \( x' \in \mathbb{R}^{N-1} \) such that \( g(x', \cdot) \in W^{1,p}(\mathbb{R}) \). Then

$$
|g(x + he_N) - g(x)| \leq h \int_{x_N}^{x_N+h} \left| \frac{\partial g}{\partial x_N}(x', s) \right| \, ds \leq h M_N\left( \frac{\partial g}{\partial x_N} \right)(x)
$$

for almost every \( (x_N, h) \in \mathbb{R} \times (0, +\infty) \), where \( M_N(f) \) denotes the maximal function of \( f \) with respect to the variable \( x_N \) in the positive direction, i.e.,

$$
M_N(f)(x', x_N) = \sup_{h>0} \int_{x_N}^{x_N+h} |f(x', s)| \, ds.
$$

Hence, since \( F_n(t) \) is a non-decreasing function with respect to \( t \),

$$
F_n(|g(x + he_N) - g(x)|) \leq F_n\left( h M_N\left( \frac{\partial g}{\partial x_N} \right)(x) \right) \quad \text{for a.e. } (x_N, h) \in \mathbb{R} \times (0, +\infty),
$$

which shows that

$$
\int_{\mathbb{R}^N} \int_0^\infty F_n(\frac{|g(x + h\sigma) - g(x)|}{h^{p+1}}) \, dh \, dx \leq \int_{\mathbb{R}^N} \int_0^\infty F_n(h M_N(\frac{\partial g}{\partial x_N})(x)) \frac{h}{h^{p+1}} \, dh \, dx.
$$

A direct computation yields

$$
\int_{\mathbb{R}^N} \int_0^\infty F_n(\frac{|g(x + he_N) - g(x)|}{h^{p+1}}) \, dh \, dx \leq \int_{\mathbb{R}^N} \left| M_N\left( \frac{\partial g}{\partial x_N} \right)(x) \right|^p \, dx \int_0^\infty F_n(h)h^{-(p+1)} \, dh.
$$
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On the other hand, using the theory of maximal functions (see, e.g., [9, Chapter 1]), one finds

\[ \left| M_N^N \left( \frac{\partial g}{\partial x_N} \right)(x) \right|^p \, dx_N \, dx' \leq C_p \int_{\mathbb{R}^{N-1}} \left| \frac{\partial g}{\partial x_N} \right|^p \, dx_N \, dx'. \]

Consequently,

\[ \int_{\mathbb{R}^N} \left| M_N^N \left( \frac{\partial g}{\partial x_N} \right)(x) \right|^p \, dx \leq C_p \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx. \]

Therefore (2.1) is proved and (1.5) follows. \(\square\)

2.2. Proof of assertion (c) in Theorem 1

The following lemma is useful in the proof of assertion (c) in Theorem 1.

**Lemma 1.** Assume that \( g \in W^{1,p}(\mathbb{R}) \) and \((F_n)_{n \in \mathbb{N}}\) satisfies hypotheses (i)–(iii) of Theorem 1 and (1.8). Then

\[ \lim_{n \to \infty} \int_{\mathbb{R}} \int_{0}^{\infty} F_n(|g(t+h)-g(t)|) \frac{1}{h^{p+1}} \, dh \, dt = \int_{\mathbb{R}} |g'(t)|^p \, dt. \tag{2.3} \]

**Proof.** Since \( g \in W^{1,p}(\mathbb{R}) \),

\[ |g(t+h) - g(t)| \leq h \int_{t}^{t+h} |g'(s)| \, ds \leq h|M_+(g')(t)| \quad \text{for a.e. } (t, h) \in \mathbb{R} \times (0, +\infty), \]

where \( M_+(g') \) denotes the maximal function of \( g' \) in the positive direction, i.e.,

\[ M_+(g')(t) = \sup_{h>0} \int_{t}^{t+h} |g'(s)| \, ds. \]

Hence, since \( F_n \) is a non-decreasing function, it follows that, for all measurable sets \( A \subset \mathbb{R} \),

\[ \int_{A} \int_{0}^{\infty} F_n(|g(t+h)-g(t)|) \frac{1}{h^{p+1}} \, dh \, dt \leq \sup_{n \in \mathbb{N}} \int_{0}^{\infty} F_n(h)h^{-(p+1)} \, dh \int_{A} |M_+(g')(t)|^p \, dt. \tag{2.4} \]

On the other hand, since \( g \in W^{1,p}(\mathbb{R}) \) and \( 1 < p < +\infty \), applying the theory of maximal functions (see [9, Chapter 1]), one gets \( M_+(g') \in L^p(\mathbb{R}) \) and

\[ \int_{\mathbb{R}} |M_+(g')(x)|^p \, dx \leq C \int_{\mathbb{R}} |g'(x)|^p \, dx. \tag{2.5} \]
Hereafter in this proof \( C \) will denote a positive constant depending only on \( p \). Thus it follows from (2.4) that for each \( \varepsilon \in (0, 1) \), there exists a positive constant \( k = k(\varepsilon) \geq 1 \) such that

\[
\int_{\mathbb{R} \setminus B} \int_0^\infty F_n(\frac{|g(t + h) - g(t)|}{h}) \frac{dh}{h^{p+1}} \, dt + \int_{\mathbb{R} \setminus B} |g'(t)|^p \, dt \leq \frac{\varepsilon}{2},
\]

(2.6)

where

\[
B := \{ t \in [-k, k]; |g'(t)| \leq k \}.
\]

Set, for each \( \tau > 0 \),

\[
A_\tau = \left\{ t \in B; q(t) \leq \frac{|g(t + h) - g(t)|}{h} \leq |g'(t)| + \gamma \text{ for a.e. } h \in [0, \tau] \right\},
\]

where \( \gamma = \varepsilon/k^p \) and \( q(t) \) is defined as follows:

\[
q(t) = \begin{cases} |g'(t)| - \gamma & \text{if } |g'(t)| \geq \gamma, \\ 0 & \text{otherwise}. \end{cases}
\]

(2.7)

Since \( g \in W^{1,p}(\mathbb{R}) \), it follows from (2.4) and (2.5) that one can choose \( \tau \) sufficiently small such that

\[
\int_{B \setminus A_\tau} \int_0^\infty F_n(\frac{|g(t + h) - g(t)|}{h}) \frac{dh}{h^{p+1}} \, dt + \int_{B \setminus A_\tau} |g'(t)|^p \, dt \leq \frac{\varepsilon}{2}.
\]

(2.8)

On the other hand, since \( F_n \) is a non-decreasing function,

\[
\int_{A_\tau} \int_0^\tau F_n(\frac{|g(t + h) - g(t)|}{h}) \frac{dh}{h^{p+1}} \, dt \leq \int_{A_\tau} \int_0^\tau \frac{F_n((|g'(t)| + \gamma)h)}{h^{p+1}} \, dh \, dt.
\]

A direct computation yields

\[
\int_{A_\tau} \int_0^\tau \frac{F_n((|g'(t)| + \gamma)h)}{h^{p+1}} \, dh \, dt = \int_{A_\tau} (|g'(t)| + \gamma)^p \int_0^{(|g'(t)| + \gamma)\tau} F_n(s)s^{-(p+1)} \, ds \, dt.
\]

Moreover, since \( \int_0^1 F_n(t)^{-(p+1)} \, dt = 1 \) and \( F_n(t) \) converges uniformly to 0 on every compact subset of \((0, +\infty)\) as \( n \) goes to infinity,

\[
\lim_{n \to \infty} \int_{A_\tau} (|g'(t)| + \gamma)^p \int_0^{(|g'(t)| + \gamma)\tau} F_n(s)s^{-(p+1)} \, ds \, dt = \int_{A_\tau} (|g'(t)| + \gamma)^p \, dt.
\]

Therefore,

\[
\limsup_{n \to \infty} \int_{A_\tau} \int_0^\tau \frac{F_n(|g(t + h) - g(t)|)}{h^{p+1}} \, dh \, dt \leq \int_{A_\tau} (|g'(t)| + \gamma)^p \, dt.
\]

(2.9)
Furthermore,

\[
\int_{\mathbb{R}} \left( \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh - |g'(t)|^p \right) \, dt \\
= \int_{\mathbb{R} \setminus B} \left( \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh - |g'(t)|^p \right) \, dt \\
+ \int_{B \setminus A_t} \left( \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh - |g'(t)|^p \right) \, dt \\
+ \int_{A_t} \left( \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh - |g'(t)|^p \right) \, dt.
\]

Thus combining (2.6), (2.8), and (2.9) yields

\[
\int_{\mathbb{R}} \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \\
\leq 3\varepsilon + \int_{A_t} ((|g'(t)+\gamma|)^p - |g'(t)|^p) \, dt, \tag{2.10}
\]

when \( n \geq n_\varepsilon \).

Since \((a+\gamma)^p \leq a^p(1+C\gamma/a)\) if \(\gamma \leq a\) and \((a+\gamma)^p \leq C\gamma\) if \(a \leq \gamma \leq 1\), one has

\[
\int_{A_t} ((|g'(t)+\gamma|)^p - |g'(t)|^p) \, dt \leq C\gamma k^p = C\varepsilon.
\]

Here we use the fact that \(A_t \subset B\) and the choice of \(\gamma = \varepsilon/k^p\). Hence it follows from (2.10) that

\[
\int_{\mathbb{R}} \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \leq C\varepsilon, \quad \forall n \geq n_\varepsilon. \tag{2.11}
\]

Similarly,

\[
\int_{\mathbb{R}} \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \\
\geq -3\varepsilon + \int_{A_t} (q^p(t) - |g'(t)|^p) \, dt \tag{2.12}
\]

for all \( n \geq n_\varepsilon \). Recall here that the function \(q\) is defined by (2.7). Since \((a-\gamma)^p + C\gamma a^{-1} \geq a^p\) for all \(a \geq \gamma > 0\) and \(A_t \subset B\), one deduces that

\[
\int_{A_t} (q^p(t) - |g'(t)|^p) \, dt \geq -C\gamma k^p = -C\varepsilon.
\]

Thus, from (2.12),

\[
\int_{\mathbb{R}} \int_{0}^{\tau} \frac{F_n(||g(t+h)-g(t)||)}{h^{p+1}} \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \geq -C\varepsilon, \quad \forall n \geq n_\varepsilon. \tag{2.13}
\]
Combining (2.11) and (2.13) yields

\[ \left| \int_{\mathbb{R}} \int_{\tau}^{\tau} F_n\left( \frac{|g(t + h) - g(t)|}{h^{p+1}} \right) \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \right| \leq C\varepsilon, \quad \forall n \geq n_\varepsilon. \]  

(2.14)

On the other hand, since \( g \in W^{1,p}(\mathbb{R}) \), it follows that \( g \in L^\infty(\mathbb{R}) \). Thus since \( F_n(t) \) converges uniformly to 0 on every compact subset of \((0, +\infty)\), applying Lebesgue’s dominated convergence theorem, one obtains

\[
\lim_{n \to \infty} \int_{|t| \leq k} \int_{\tau}^{m} \frac{F_n(|g(t + h) - g(t)|)}{h^{p+1}} \, dh \, dt = 0, \quad \forall m > 0.
\]

Moreover, since \( F_n \) is a non-decreasing function, it follows that

\[
\int_{|t| \leq k} \int_{m}^{\infty} \frac{F_n(|g(t + h) - g(t)|)}{h^{p+1}} \, dh \, dt \leq 2k \int_{m}^{\infty} \frac{\|g'\|_{L^p} h^{(p-1)/p}}{h^{p+1}} \, dh \leq 2k m \|g'\|_{L^p} \int_{0}^{\infty} F_n(t) r^{-(p+1)} \, dt.
\]

Hence using (1.8), one gets

\[
\lim_{n \to \infty} \int_{|t| \leq k} \int_{\tau}^{m} \frac{F_n(|g(t + h) - g(t)|)}{h^{p+1}} \, dh \, dt = 0.
\]

(2.15)

Therefore the conclusion of Lemma 1 follows from (2.6), (2.14), and (2.15).

Proof of assertion (c). We claim that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{0}^{\infty} F_n(|g(x + h\sigma) - g(x)|) \, dh \, dx = \int_{\mathbb{R}^N} |\nabla g(x) \cdot \sigma|^p \, dx.
\]

(2.16)

Without loss of generality, one may assume that \( \sigma = e_N \). Take \( x' \in \mathbb{R}^{N-1} \) such that \( g(x', \cdot) \in W^{1,p}(\mathbb{R}) \). As in (2.4), one gets

\[
\int_{\mathbb{R}} \int_{0}^{\infty} F_n(|g(x', x_N + h) - g(x', x_N)|) \, dh \, dx_N \leq \int_{0}^{\infty} F_n(t) r^{-(p+1)} \, dt \int_{\mathbb{R}} |M_N(\partial g/\partial x_N)(x', x_N)|^p \, dx_N.
\]

Here \( M_N \) is defined by (2.2). On the other hand, by Lemma 1

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{0}^{\infty} F_n(|g(x', x_N + h) - g(x', x_N)|) \, dh \, dx_N = \int_{\mathbb{R}^N} |\partial g / \partial x_N(x', x_N)|^p \, dx_N.
\]

(2.17)

Thus, applying Lebesgue’s dominated convergence theorem, one obtains

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{0}^{\infty} F_n(|g(x + h e_N) - g(x)|) \, dh \, dx = \int_{\mathbb{R}^N} |\nabla g(x) \cdot e_N|^p \, dx.
\]

Therefore the conclusion of assertion (c) in Theorem 1 follows from (2.1), (2.16) and Lebesgue’s dominated convergence theorem.
2.3. Proof of assertion (b) in Theorem

Without loss of generality, we may assume that
\[ M := 1 + \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{p+N}} \, dx \, dy < +\infty. \]

Thus since \( F_n(t) \) is a non-decreasing function with respect to \( t \),
\[
\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} F_n(2^{-(k+1)}) \int_{2^{-(k+1)} < |g(x)-g(y)| \leq 2^{-k}} \frac{1}{|x - y|^{p+N}} \, dx \, dy < M. \tag{2.18}
\]

On the other hand, by (i)–(iii) it follows that for each \( s > 0 \) there exists \( n \) such that \( F_n(s) > 0 \). Thus since \( F_n \) is a non-decreasing function,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{p+N}} \, dx \, dy < +\infty, \quad \forall s > 0.
\]

Hence
\[
\sum_{k=1}^{\infty} F_n(2^{-(k+1)}) \int_{|g(x)-g(y)| > 2^{-(k+1)}} \frac{1}{|x - y|^{p+N}} \, dx \, dy
\]
\[
= \sum_{k=1}^{\infty} F_n(2^{-(k+1)})
\]
\[
\times \left( \int_{|g(x)-g(y)| > 2^{-(k+1)}} \frac{1}{|x - y|^{p+N}} \, dx \, dy - \int_{|g(x)-g(y)| > 2^{-k}} \frac{1}{|x - y|^{p+N}} \, dx \, dy \right).
\]

However, the right hand side above equals
\[
\sum_{k=1}^{\infty} 2^{p(k+1)} F_n(2^{-(k+1)}) \int_{|g(x)-g(y)| > 2^{-(k+1)}} \frac{2^{-(k+1)p}}{|x - y|^{p+N}} \, dx \, dy
\]
\[
- \sum_{k=1}^{\infty} 2^{pk} F_n(2^{-(k+1)}) \int_{|g(x)-g(y)| > 2^{-(k+1)}} \frac{2^{-kp}}{|x - y|^{p+N}} \, dx \, dy
\]
\[
= \sum_{k=2}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) \int_{|g(x)-g(y)| > 2^{-k}} \frac{2^{-kp}}{|x - y|^{p+N}} \, dx \, dy
\]
\[
- F_n(1/4) \int_{|g(x)-g(y)| > 1/2} \frac{1}{|x - y|^{p+N}} \, dx \, dy,
\]
and, from hypothesis (iii) on \( F_n \),

\[
\lim_{n \to \infty} F_n(1/4) = 0.
\]

Thus it follows from (2.18) that

\[
\limsup_{n \to \infty} \sum_{k=2}^{\infty} 2^{pk} (F_n(2^{-k}) - F_n(2^{-(k+1)})) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-kp}}{|x-y|^{p+N}} \,dx \,dy \leq M. \tag{2.19}
\]

We claim that

\[
\liminf_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-kp}}{|x-y|^{p+N}} \,dx \,dy < +\infty. \tag{2.20}
\]

We prove (2.20) by contradiction. Suppose it does not hold. Then there exists \( k_M \in \mathbb{N} \) such that for all \( k \geq k_M \),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-kp}}{|x-y|^{p+N}} \,dx \,dy > 2^{p+2}M.
\]

Hence it follows from (2.19) that

\[
\limsup_{n \to \infty} 2^{p+2}M \sum_{k=k_M}^{\infty} 2^{pk} (F_n(2^{-k}) - F_n(2^{-(k+1)})) \leq M,
\]

which shows that

\[
\limsup_{n \to \infty} \sum_{k=k_M}^{\infty} 2^{pk} (F_n(2^{-k}) - F_n(2^{-(k+1)})) \leq 2^{-(p+2)}. \tag{2.21}
\]

On the other hand, since \( p \geq 1 \),

\[
\sum_{k=k_M}^{\infty} 2^{pk} (F_n(2^{-k}) - F_n(2^{-(k+1)})) = \sum_{k=k_M}^{\infty} 2^{pk} F_n(2^{-k}) - \sum_{k=1+k_M}^{\infty} 2^{pk} F_n(2^{-k}) \\
\geq \frac{1}{2} \sum_{k=1+k_M}^{\infty} 2^{pk} F_n(2^{-k}),
\]

and, since \( F_n \) is a non-decreasing function,

\[
\int_{0}^{2^{-(1+k_M)}} F_n(t) t^{-(p+1)} \,dt = \sum_{k=1+k_M}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} F_n(t) t^{-(p+1)} \,dt \\
\leq \sum_{k=1+k_M}^{\infty} F_n(2^{-k}) \int_{2^{-(k+1)}}^{2^{-k}} t^{-(p+1)} \,dt \\
\leq \frac{2^p}{p} \sum_{k=1+k_M}^{\infty} 2^{pk} F_n(2^{-k}).
\]
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Thus it follows from (2.21) that
\[
\limsup_{n \to \infty} 2^{-(p+1)} \int_0^{2^{-(1+4k)}} F_n(t) t^{-(p+1)} dt \leq 2^{-(p+2)}.
\]
This implies
\[
\limsup_{n \to \infty} \int_0^{2^{-(1+4k)}} F_n(t) t^{-(p+1)} dt \leq \frac{1}{2}.
\] (2.22)
However, by (ii) and (iii), one gets
\[
\lim_{n \to \infty} \int_0^{2^{-(1+4k)}} F_n(t) t^{-(p+1)} dt = 1.
\]
This contradicts (2.22), and proves (2.10). Thus by Proposition 3, it follows that \(g \in W^{1,p}(\mathbb{R}^N)\).

In order to prove (1.7), we consider the sequence of functions \(G_n\) defined by
\[
G_n(t) = \begin{cases} F_n(t) & \text{if } 0 \leq t \leq 1, \\ F_n(1) & \text{otherwise}. \end{cases}
\]
This sequence satisfies hypotheses (i)–(iii) of Theorem 1 and (1.8). By assertion (c) of Theorem 1 (1.7) follows. \(\square\)

2.4. Proof of Theorem 2

Applying the same method as in Section 2.3, one can prove that
\[
\liminf_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-k}}{|x - y|^{N+1}} \, dx \, dy < +\infty.
\]
Thus by Proposition 4 one has \(g \in BV(\mathbb{R}^N)\) and
\[
\liminf_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-k}}{|x - y|^{N+1}} \, dx \, dy \geq c \int_{\mathbb{R}^N} |\nabla g| \, dx.
\]
Hereafter in this proof \(c\) denotes a constant depending only on \(N\). Thus there exists a constant \(k_0 \in \mathbb{N}\) such that for all \(k \geq k_0\),
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2^{-k}}{|x - y|^{N+1}} \, dx \, dy \geq c \int_{\mathbb{R}^N} |\nabla g| \, dx.
\]
Applying the method of Section 2.3, one gets

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n \left( \frac{|g(x) - g(y)|}{|x - y|^{N+1}} \right) dx \, dy \geq c \int_0^{2-(1+k_0)} F_n(t) t^{-2} \, dt \int_{\mathbb{R}^N} |\nabla g| \, dx \\
+ F_n(1/4) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N+1}} \, dx \, dy.
\]

On the other hand, since \( F_n \) converges uniformly to 0 on every compact subset of \((0, +\infty)\) and

\[
\int_0^1 F_n(t) t^{-2} \, dt = 1,
\]

it follows that

\[
\lim_{n \to \infty} \int_0^{2-(1+k_0)} F_n(t) t^{-2} \, dt = 1.
\]

Therefore

\[
\lim \inf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n \left( \frac{|g(x) - g(y)|}{|x - y|^{N+1}} \right) dx \, dy \geq c \int_{\mathbb{R}^N} |\nabla g| \, dx. \quad \square
\]

Theorems 1 and 2 have the following interesting consequence. It is motivated by the work of J. Bourgain, H. Brezis and P. Mironescu in [1] and [5].

**Corollary 1.** Let \( p \geq 1 \) and \((F_n)_{n \in \mathbb{N}}\) be a sequence of non-decreasing functions from \([0, +\infty)\) into \([0, +\infty)\) such that \( F_n(1) \) is bounded and

\[
\lim \sup_{n \to \infty} \int_0^1 F_n(t) t^{-(p+1)} \, dt = +\infty.
\]

Assume that \( g \in L^p(\mathbb{R}^N) \) and \( g \) satisfies \((1.6)\). Then \( g \) is a constant function.

**Proof.** Without loss of generality, one may assume that

\[
\lim_{n \to \infty} \int_0^1 F_n(t) t^{-(p+1)} \, dt = +\infty.
\]

For each \( n \in \mathbb{N} \), set

\[
G_n(t) = \begin{cases} 
  F_n(t) & \text{if } 0 \leq t \leq 1, \\
  \frac{\int_0^1 F_n(t) t^{-(p+1)} \, dt}{\int_0^1 F_n(t) t^{-(p+1)} \, dt} F_n(1) & \text{otherwise}.
\end{cases}
\]

Then \((G_n)_{n \in \mathbb{N}}\) satisfies hypotheses (i)–(iii).
By Theorems 1 and 2 there exists a constant $c_{N,p} > 0$ such that
\[ c_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_n(|g(x) - g(y)|) \, dx \, dy. \]

However,
\[ \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_n(|g(x) - g(y)|) \, dx \, dy \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \, dx \, dy = 0. \]

Thus it follows that $\int_{\mathbb{R}^N} |\nabla g|^p \, dx = 0$. Therefore $g$ is a constant function.

**Remark 6.** The conditions of Corollary 1 are satisfied by $F_n(t) = t^p$ for all $n \in \mathbb{N}$ with $p \geq 1$. Hence any function $g \in L^p(\mathbb{R}^N)$ satisfying
\[ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |g(x) - g(y)|^p \, dx \, dy < +\infty \]

must be a constant. This was already observed in [5].

3. Another proof of assertion (b) in Theorem 1

First in Section 3.1 we present a fundamental lemma. Then in Section 3.2 we discuss a new proof of assertion (b) in Theorem 1.

3.1. A fundamental lemma

The following lemma will play an important role in this section.

**Lemma 2** (Fundamental lemma). Let $g \in L^p(\mathbb{R}^N), 1 < p < +\infty$. Assume that
\[ \iint_{K \times K} \frac{1}{|x - y|^{N+1}} \, dx \, dy < +\infty, \quad \forall K \subset \subset \mathbb{R}^N, \forall \varepsilon > 0 \quad (3.1) \]

and
\[ \liminf_{\varepsilon \to 0^+} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varepsilon^p}{|x - y|^{N+p}} \, dx \, dy < +\infty. \quad (3.2) \]

Then $g \in W^{1,p}(\mathbb{R}^N)$. 

Remark 7. Condition (3.2) alone is not sufficient to show that $g \in W^{1,p}(\mathbb{R}^N)$ (in contrast with condition (1.2)). For example

$$g(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Surprisingly, the mild additional assumption (3.1) together with (3.2) implies that $g \in W^{1,p}(\mathbb{R}^N)$.

In order to prove Lemma 2, we need some useful lemmas. The first lemma, which was used in [2], is a direct consequence of a result due to J. Bourgain, H. Brezis, and P. Mironescu (see [1]).

Lemma 3. Let $g$ be a measurable function on the interval $[a, b]$ ($-\infty < a < b < +\infty$), $y \in \mathbb{R}$, and $\delta > 0$. Set

$$B = \{x \in [a, b]; g(x) < y\}.$$

Assume that

$$0 < \frac{|[a, b] \cap B|}{b - a} < 1 \quad (3.3)$$

and

$$\int_a^b \int_a^b \frac{1}{|x - y|^2} \, dx \, dy < +\infty. \quad (3.4)$$

Then

$$|[a, b] \cap A_\tau| > 0, \quad \forall \tau > \delta,$$

where $A_\tau := \{x \in [a, b]; y \leq g(x) < y + \tau\}$.

Hereafter $|A|$ denotes the Lebesgue measure of $A$ for any measurable set $A \subset \mathbb{R}$.

Proof. We prove Lemma 3 by contradiction. Suppose that $|[a, b] \cap A_\tau| = 0$ for some $\tau > \delta$. Then from (3.4),

$$\int_B \int_{[a, b] \setminus B} \frac{1}{|x - y|^2} \, dx \, dy < +\infty.$$

This implies (see [1])

$$|B| = 0 \quad \text{or} \quad |[a, b] \setminus B| = 0,$$

which contradicts (3.3). □

The following lemma will be useful to prove Lemma 5. Estimate (3.6) was mentioned and used in [2]. Estimate (3.7) was also hidden there. It will play a role in the proof of Lemma 5. For the convenience of the reader, we will reproduce the proof.
Lemma 4. Let \( g \) be a measurable function on the interval \([a, b]\) \((\infty < a < b < \infty)\), \( y \in \mathbb{R} \), \( r > 0 \), \( s > 0 \), and \( \tau > \delta > 0 \). Set

\[
B = \{ x \in \mathbb{R}; g(x) < y \}, \quad A = \{ x \in \mathbb{R}; y \leq g(x) < y + \tau \}.
\]

Assume that

\[
\frac{|[a, b] \cap B|}{b - a} = r, \quad \frac{|[a, b] \cap A|}{b - a} \leq s, \quad r + s < 1,
\]

and

\[
\int_a^b \int_a^b \frac{1}{|x - y|^2} \, dx \, dy < +\infty. \tag{3.5}
\]

Then there exists a subinterval \([c, d]\) \(\subset [a, b]\) \((a \leq c < d \leq b)\) such that

\[
\frac{|[c, d] \cap B|}{d - c} = r, \quad s/4 \leq \frac{|[c, d] \cap A|}{d - c} \leq s, \tag{3.6}
\]

and

\[
\frac{d - c}{b - a} \leq 4 \frac{|[a, b] \cap A|}{s(b - a)}. \tag{3.7}
\]

Proof. Set \([a_1, b_1] = [a, b]\). Suppose that there exists \([a_k, b_k] \subset [a, b], k \geq 1\), such that

\[
\frac{|[a_k, b_k] \cap B|}{b_k - a_k} = r \quad \text{and} \quad \frac{|[a_k, b_k] \cap A|}{b_k - a_k} \leq s.
\]

If

\[
\frac{|[a_k, b_k] \cap A|}{b_k - a_k} \geq s/4,
\]

then take \([c, d] = [a_k, b_k]\). Otherwise, by Lemma 3, one has

\[
0 < \frac{|[a_k, b_k] \cap A|}{b_k - a_k} < s/4.
\]

Take \(s_k > 0\) such that \(s/s_k \in \mathbb{Z^+}\) and

\[
s_k/2 < \frac{|[a_k, b_k] \cap A|}{b_k - a_k} \leq s_k. \tag{3.8}
\]

Then

\[
s_k < s/2. \tag{3.9}
\]

Set

\[
\lambda_k = \frac{(b_k - a_k)s_k}{2s}. \tag{3.10}
\]

Consider the function \(\psi_k(t)\) defined as follows:

\[
\psi_k(t) = |[t - \lambda_k, t + \lambda_k] \cap B|, \quad \forall t \in [a_k + \lambda_k, b_k - \lambda_k].
\]

We claim that there exists \(t_k \in [a_k + \lambda_k, b_k - \lambda_k]\) such that \(\psi_k(t_k)/(2\lambda_k) = r\).
To see this, we argue by contradiction. Suppose that \( \psi_k(t)/(2\lambda_k) \neq r \) for all \( t \in [a_k + \lambda_k, b_k - \lambda_k] \). Since \( \psi_k \) is a continuous function on \([a_k + \lambda_k, b_k - \lambda_k]\), we assume as well that \( \psi_k(t)/(2\lambda_k) < r \) for all \( t \in [a_k + \lambda_k, b_k - \lambda_k] \). Since \( (b_k - a_k)/(2\lambda_k) = s/s_k \in \mathbb{Z}_+ \), it follows that

\[
[a_k, b_k] \cap B < 2r\lambda_k \frac{b_k - a_k}{2\lambda_k} = r(b_k - a_k).
\]

This contradicts the fact that \( |a_k, b_k| \cap B|/(b_k - a_k) = r \).

Set \( [a_{k+1}, b_{k+1}] = [t_k - \lambda_k, t_k + \lambda_k] \subseteq [a_k, b_k] \). Then

\[
\frac{|[a_{k+1}, b_{k+1}] \cap B|}{b_{k+1} - a_{k+1}} = r \quad \text{and} \quad \frac{|[a_{k+1}, b_{k+1}] \cap A|}{b_{k+1} - a_{k+1}} \leq \frac{|[a_k, b_k] \cap A|}{2\lambda_k}.
\]

Thus it follows from (3.8) and (3.10) that

\[
\frac{|[a_{k+1}, b_{k+1}] \cap A|}{b_{k+1} - a_{k+1}} \leq \frac{|[a_k, b_k] \cap A|}{2\lambda_k} = \frac{s_k b_k - a_k}{2\lambda_k} = s.
\]

Moreover,

\[
\frac{b_{k+1} - a_{k+1}}{b_k - a_k} = \frac{2\lambda_k}{s_k} = \frac{s_k}{s}.
\]

Thus from (3.9), this implies

\[
\frac{b_{k+1} - a_{k+1}}{b_k - a_k} \leq \frac{1}{2}. \tag{3.11}
\]

On the other hand,

\[
\int_{a_k}^{b_k} \int_{a_k}^{b_k} \frac{1}{|x - y|} \, dx \, dy \geq \int_{x \in [a_k, b_k] \cap B} \int_{y \in [a_k, b_k] \cap (A \cup B)} \frac{1}{|x - y|} \, dx \, dy,
\]

which shows

\[
\int_{a_k}^{b_k} \int_{a_k}^{b_k} \frac{1}{|x - y|} \, dx \, dy \geq r(1 - r - s). \tag{3.12}
\]

Combining (3.5), (3.11), and (3.12) shows that the above process will stop at some \( k \in \mathbb{Z}_+ \). Then \([c, d] \subseteq [a, b] \setminus \{c, d\} = [a_k, b_k]\), and

\[
\frac{|[c, d] \cap B|}{d - c} = r, \quad \frac{s}{4} \leq \frac{|[c, d] \cap A|}{d - c} \leq s.
\]

If \( k \geq 2 \), then it follows from (3.8) and (3.10) that

\[
\frac{d - c}{b - a} \leq \frac{b_2 - a_2}{b_1 - a_1} = \frac{2\lambda_1}{s} = \frac{s_1}{s} \leq 2 \frac{|[a, b] \cap A|}{s(b - a)}.
\]

Otherwise \((k = 1)\), the estimate (3.7) holds clearly.

The proof is complete. \( \square \)

The following lemma plays an important role in the proof of Lemma [9]
Lemma 5. Let $g$ be a measurable function on the interval $[a, b)$ $(\infty < a < b < +\infty)$, \( y \in \mathbb{R} \), and $\tau > \delta > 0$. Set

\[
\begin{align*}
B_j &= \{ x \in \mathbb{R}; \ g(x) < y + j \tau \}, \\
A_j &= \{ x \in \mathbb{R}; \ y + j \tau \leq g(x) < y + (j + 1) \tau \}, \quad \forall j \in \mathbb{Z}.
\end{align*}
\]

Assume that

\[
\frac{|[a, b] \cap B_0|}{b - a} = \frac{1}{2}, \quad \frac{|[a, b] \cap A_0|}{b - a} \leq \frac{1}{8},
\]

and

\[
\int_a^b \int_{|g(x) - g(y)| > \delta} \frac{1}{|x - y|^2} \, dx \, dy < +\infty. \tag{3.13}
\]

Then for each $r > 8$, there exist $m \in \mathbb{Z}_+$, $l_m \in \mathbb{Z}$, and $[c, d] \subset [a, b)$ ($c < d$) such that

\[
\begin{align*}
&\left\{ \begin{array}{l}
|l_m| \leq 2m, \\
|[c, d] \cap A_m| \geq |[c, d] \cap A_{m+2}| \geq \frac{1}{4 \cdot 8m+1} \cdot r^{m+1}, \\
d - c \leq 4^m (8/r)^{m(m-1)/2} (b - a),
\end{array} \right.
\end{align*}
\]

Proof. Set $j_1 = 0$. By Lemma 4, there exists $[a_1, b_1] \subset [a, b]$ such that

\[
\frac{|[a_1, b_1] \cap B_{j_1}|}{b_1 - a_1} = \frac{1}{2} \quad \text{and} \quad \frac{1}{4 \cdot 8} \leq \frac{|[a_1, b_1] \cap A_{j_1}|}{b_1 - a_1} \leq \frac{1}{8}.
\]

Suppose that there exist $[a_k, b_k]$ and $j_k$ ($k \geq 1$) such that

\[
\begin{align*}
&\left\{ \begin{array}{l}
|j_k| \leq 2(k - 1), \\
\frac{1}{2} - 2 \sum_{i=1}^{k-1} \frac{1}{8^i} \leq \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} \leq \frac{1}{2} + 2 \sum_{i=1}^{k-1} \frac{1}{8^i}, \\
\frac{1}{4 \cdot 8^k} \leq \frac{|[a_k, b_k] \cap A_{j_k}|}{b_k - a_k} \leq \frac{1}{8^k}.
\end{array} \right.
\end{align*}
\]

Then we have the following cases:

Case 1:

\[
\frac{|[a_k, b_k] \cap A_{j_k+1}|}{b_k - a_k} \geq \frac{1}{8^{k+1}} \quad \text{and} \quad \frac{|[a_k, b_k] \cap A_{j_k-1}|}{b_k - a_k} \geq \frac{1}{8^{k+1}}.
\]

Set

\[
m = k, \quad l_m = j_k - 1, \quad [c, d] = [a_k, b_k].
\]

Then

\[
\frac{|[c, d] \cap A_{l_m}|}{d - c} \geq \frac{1}{8^{m+1}} \quad \text{and} \quad \frac{|[c, d] \cap A_{l_m+2}|}{d - c} \geq \frac{1}{8^{m+1}}. \tag{3.15}
\]

Case 2:

\[
\frac{|[a_k, b_k] \cap A_{j_k+1}|}{b_k - a_k} \leq \frac{1}{8^{k+1}} \quad \text{or} \quad \frac{|[a_k, b_k] \cap A_{j_k-1}|}{b_k - a_k} \leq \frac{1}{8^{k+1}}.
\]
Case 2.1:
\[
\frac{|[a_k, b_k] \cap A_{j_k+1}|}{b_k - a_k} \leq \frac{1}{8^k + 1}.
\]

Case 2.1.1:
\[
\frac{|[a_k, b_k] \cap A_{j_k+2}|}{b_k - a_k} \geq \frac{1}{r^k + 1}.
\]

Set
\[
m = k, \quad l_m = j_k, \quad [c, d] = [a_k, b_k].
\]

Then
\[
\frac{|[c, d] \cap A_{j_k}|}{d - c} \geq \frac{1}{4 \cdot 8^m + 8^{m+1}}.
\]

Case 2.1.2:
\[
\frac{|[a_k, b_k] \cap A_{j_k+2}|}{b_k - a_k} < \frac{1}{r^k + 1}.
\]

Set
\[
j_{k+1} = j_k + 2.
\]

Then, from the first inequality of (3.14),
\[
|j_{k+1}| \leq 2k.
\]

Applying Lemma 4 with \( s = 1/8^k + 1 \) and \( B = B_{j_k+1} \), one gets \([a_{k+1}, b_{k+1}] \subset [a_k, b_k]\) such that
\[
\begin{align*}
\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_k+1}|}{b_{k+1} - a_{k+1}} &= \frac{|[a_k, b_k] \cap B_{j_k+1}|}{b_k - a_k}, \\
\frac{1}{4 \cdot 8^k + 1} \leq \frac{|[a_{k+1}, b_{k+1}] \cap A_{j_k+1}|}{b_{k+1} - a_{k+1}} &\leq \frac{1}{8^k + 1}, \\
\frac{|[a_{k+1}, b_{k+1}] \cap A_{j_k+2}|}{b_{k+1} - a_{k+1}} &\leq \frac{4 \cdot 8^k + 1}{r^k + 1}.
\end{align*}
\]

Thus from (3.14) and (3.19),
\[
\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_k+1}|}{b_{k+1} - a_{k+1}} \geq \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} \geq \frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{8^i}
\]
and, since \( j_{k+1} = j_k + 2 \) (see (3.17)),
\[
\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_k+1}|}{b_{k+1} - a_{k+1}} = \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} = \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} + \frac{|[a_k, b_k] \cap (A_{j_k} \cup A_{j_k+1})|}{b_k - a_k}
\leq \frac{1}{2} + 2 \sum_{i=1}^{k-1} \frac{1}{8^i} + \frac{1}{8^k} + \frac{1}{8^k + 1}.
\]
Hence
\[
\frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{8^i} \leq \frac{|a_{k+1}, b_{k+1} \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \leq \frac{1}{2} + 2 \sum_{i=1}^{k} \frac{1}{8^i}.
\] (3.20)

Therefore from (3.18), (3.20) and the last two estimates of (3.19), one gets
\[
\begin{cases}
|j_{k+1}| \leq 2k, \\
\frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{8^i} \leq \frac{|a_{k+1}, b_{k+1} \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \leq \frac{1}{2} + 2 \sum_{i=1}^{k} \frac{1}{8^i}, \\
\frac{1}{4 \cdot 8^{k+1}} \leq \frac{|a_{k+1}, b_{k+1} \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \leq \frac{1}{8^{k+1}}, \\
\frac{b_{k+1} - a_{k+1}}{b_{k} - a_{k}} \leq \frac{4 \cdot 8^{k+1}}{r^{k+1}}.
\end{cases}
\] (3.21)

**Case 2.2:**
\[
\frac{|a_{k}, b_{k}| \cap A_{j_{k+1}}}{b_{k} - a_{k}} \geq \frac{1}{8^{k+1}} \quad \text{and} \quad \frac{|a_{k}, b_{k}| \cap A_{j_{k+1}}}{b_{k} - a_{k}} \leq \frac{1}{8^{k+1}}.
\]

**Case 2.2.1:**
\[
\frac{|a_{k}, b_{k}| \cap A_{j_{k+2}}}{b_{k} - a_{k}} \geq \frac{1}{r^{k+1}}.
\]

Set
\[
m = k, \quad l_m = j_k - 2, \quad [c, d] = [a_k, b_k].
\]

Then
\[
\frac{|c, d| \cap A_{l_m}}{d - c} \geq \frac{1}{4 \cdot 8^{m+l} r^{m+1}}.
\] (3.22)

**Case 2.2.2:**
\[
\frac{|a_{k}, b_{k}| \cap A_{j_{k-2}}}{b_{k} - a_{k}} \leq \frac{1}{r^{k+1}}.
\]

Set
\[
j_{k+1} = j_k - 2.
\] (3.23)

Then from the first inequality of (3.14),
\[
|j_{k+1}| \leq 2k.
\] (3.24)

Applying Lemma 4 with \( s = 1/8^{k+1} \) and \( B = B_{j_{k+1}} \), one gets \([a_{k+1}, b_{k+1}] \subset [a_k, b_k]\) such that
\[
\begin{cases}
\frac{|a_{k+1}, b_{k+1} \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} = \frac{|a_{k}, b_{k} \cap B_{j_{k+1}}|}{b_{k} - a_{k}}, \\
\frac{1}{4 \cdot 8^{k+1}} \leq \frac{|a_{k+1}, b_{k+1} \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \leq \frac{1}{8^{k+1}}, \\
\frac{b_{k+1} - a_{k+1}}{b_{k} - a_{k}} \leq \frac{4 \cdot 8^{k+1}}{r^{k+1}}.
\end{cases}
\] (3.25)
Thus from the second estimate of (3.14),
\[
\frac{\|a_{k+1}, b_{k+1}\| \cap B_{b_{k+1}}}{b_{k+1} - a_{k+1}} \leq \frac{\|a_k, b_k\| \cap B_{b_k}}{b_k - a_k} \leq \frac{1}{2} + 2 \sum_{i=1}^{k} \frac{1}{g^i},
\]
and, since \(j_{k+1} = j_k - 2\) (see (3.24)),
\[
\frac{\|a_{k+1}, b_{k+1}\| \cap B_{b_{k+1}}}{b_{k+1} - a_{k+1}} = \frac{\|a_k, b_k\| \cap B_{b_k}}{b_k - a_k} = \frac{\|a_k, b_k\| \cap (A_{j_k-1} \cup A_{j_k-2})}{b_k - a_k} \geq \frac{1}{2} - 2 \sum_{i=1}^{k-1} \frac{1}{g^i} - \frac{1}{8^{k+1}} = \frac{1}{r^{k+1}}.
\]
Hence
\[
\frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{g^i} \leq \frac{\|a_{k+1}, b_{k+1}\| \cap B_{b_{k+1}}}{b_{k+1} - a_{k+1}} \leq \frac{1}{2} + 2 \sum_{i=1}^{k} \frac{1}{g^i}.
\] (3.26)

Therefore from (3.24), (3.26) and the last two estimates of (3.25), one has
\[
\begin{align*}
\left\{ \begin{array}{l}
j_{k+1} \leq 2k, \\
\frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{g^i} \leq \frac{\|a_{k+1}, b_{k+1}\| \cap B_{b_{k+1}}}{b_{k+1} - a_{k+1}} \leq \frac{1}{2} + 2 \sum_{i=1}^{k} \frac{1}{g^i}, \\
\frac{1}{4 \cdot 8^{k+1}} \leq \frac{\|a_{k+1}, b_{k+1}\| \cap A_{j_k+1}}{b_{k+1} - a_{k+1}} \leq \frac{1}{8^{k+1}}, \\
\frac{b_{k+1} - a_{k+1}}{b_k - a_k} \leq \frac{4 \cdot 8^{k+1}}{r^{k+1}}.
\end{array} \right.
\] (3.27)

On the other hand, from (3.14),
\[
\int_{a_k}^{b_k} \int_{a_k}^{b_k} \frac{1}{|x - y|^d} dx dy \geq \iiint_{x \in [a_k, b_k] \cap B_{b_k}, \ y \in [a_k, b_k] \cap (B_{b_k} \cup A_{j_k})} \frac{1}{|x - y|^d} dx dy \geq 1.
\] (3.28)

Thus it follows from inequality (3.28), the last inequalities of (3.21) and (3.27), and (3.13) that this process will stop at some \(k \in \mathbb{Z}_+\). Thus from (3.15), (3.16) and (3.22), it suffices to show that \((|c, d| = [a_m, b_m])
\[
d - c \leq 4^m (8/r)^{m(m-1)/2} (b - a).
\] (3.29)

In fact, from the last inequality of (3.21) and (3.27),
\[
b_i - a_i \leq \frac{4 \cdot 8^{i}}{r^i} (b_i - a_i - 1), \quad \forall 2 \leq i \leq m.
\]
which shows, since \( d - c = b_m - a_m \),

\[
d - c \leq 4^m (8/r)^{m(m+1)/2-1} (b - a) \leq 4^m (8/r)^{m(m-1)/2} (b - a)
\]

(the above inequality is evident when \( m = 1 \)).

The proof is complete. \(\square\)

Lemma 5 has the following consequence which is one of the main ingredients to establish Lemma 6.

**Corollary 2.** Let \( 1 < p < +\infty \). Under the hypotheses of Lemma 5 there exist \( m \in \mathbb{Z}_+ \) and \( l_m \in \mathbb{Z} \) such that

\[
|l_m| \leq 2m
\]

and

\[
\iint_{x \in [a, b] \cap A_{l_m}, y \in [a, b] \cap A_{l_m+2}} \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq c_p m (b - a)^{1-p}
\]

for some positive constant \( c_p \) depending only on \( p \).

**Proof.** Take \( r = 16 \). By Lemma 5 there exist \( m \in \mathbb{Z}_+, l_m \in \mathbb{Z}, \) and \([c, d] \subset [a, b]\) such that

\[
\begin{aligned}
|l_m| &\leq 2m, \\
&

\frac{||[c, d] \cap A_{l_m}| - ||[c, d] \cap A_{l_m+2}|}{d - c} \geq \frac{1}{4 \cdot 8^{m+1} p m + 1}, \\
\end{aligned}
\]

\[
d - c \leq 4^m (8/r)^{m(m-1)/2} (b - a).
\]

Hence

\[
\frac{||[c, d] \cap A_{l_m}| - ||[c, d] \cap A_{l_m+2}|}{(d - c)^{p+1}} \geq \frac{1}{4 \cdot 8^{m+1} p m + 1} (d - c)^{1-p}
\]

\[
\geq \frac{1}{4 \cdot 8^{m+1} p m + 1} 4^m (1-p) (r/8)^{m(m-1)(p-1)/2} (b - a)^{1-p},
\]

which shows that (since \( p > 1 \))

\[
\frac{||[c, d] \cap A_{l_m}| - ||[c, d] \cap A_{l_m+2}|}{(d - c)^{p+1}} \geq c_p m (b - a)^{1-p}
\]

for some positive constant \( c_p \) depending only on \( p \).

On the other hand,

\[
\iint_{x \in [c, d] \cap A_{l_m}, y \in [c, d] \cap A_{l_m+2}} \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq \frac{||[c, d] \cap A_{l_m}| - ||[c, d] \cap A_{l_m+2}|}{(d - c)^{p+1}}.
\]
Therefore,
\[
\int_{x \in [a,b] \cap A_{lm}, \ y \in [a,b] \cap A_{lm+2}} \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq c_p m(b-a)^{1-p}. \quad \square
\]

The following lemma plays a crucial role in this section.

**Lemma 6.** Let \( g \) be a measurable function on a bounded interval \( I \) and \( 1 < p < +\infty \). Assume that
\[
\int_{I \times I} \frac{1}{|x - y|^p} \, dx \, dy < +\infty, \quad \forall \varepsilon > 0. \tag{3.30}
\]

Then
\[
\lim_{\varepsilon \to 0_+} \int_{I \times I} \frac{\varepsilon^p}{|x - y|^{p+1}} \, dx \, dy \geq c_p \frac{1}{|I|^{p-1}} (\text{ess sup}_I g - \text{ess inf}_I g)^p \tag{3.31}
\]
for some positive constant \( c_p \) depending only on \( p \).

**Proof.** *Case 1: \( f \) is bounded.* We first follow the idea and the notations used in the proof of [2, Lemma 1].

By rescaling, we may assume \( I = [0, 1] \). Set \( s_+ = \text{ess sup}_I g, \ s_- = \text{ess inf}_I g \).

Rescaling and translating \( g \), one may also assume
\[
s_+ = 1 \quad \text{and} \quad s_- = 0 \tag{3.32}
\]
(unless \( g \) is constant on \( I \) in which case there is nothing to prove).

Take \( 0 < \delta \ll 1 \) small enough to ensure that there are (density) points \( t_+, t_- \in [40\delta, 1 - 40\delta] \subset [0, 1] \) with
\[
\begin{align*}
\left[ t_+ - \tau, t_+ + \tau \right] \cap \left[ g > \frac{3}{4} \right] &> \frac{9}{5} \tau, \\
\left[ t_- - \tau, t_- + \tau \right] \cap \left[ g < \frac{1}{4} \right] &> \frac{9}{5} \tau,
\end{align*} \tag{3.33}
\]

Take \( K \in \mathbb{Z}_+ \) such that
\[
\delta < 2^{-K} \leq 2\delta \tag{3.34}
\]
and define
\[
J = \{ j \in \mathbb{Z}_+: 1/4 < j 2^{-K} < 3/4 \}.
\]

Then
\[
|J| \geq 2^{K-1} - 2 \approx 1/\delta. \tag{3.35}
\]
For each \( j \), set
\[
A_j = \{ x \in [0, 1]; (j - 1)2^{-K} \leq g(x) < j2^{-K} \} \quad \text{and} \quad B_j = \bigcup_{j' < j} A_{j'}.
\]
Since \( A_j \cap A_k = \emptyset \) for \( j \neq k \), it follows from (3.35) that
\[
\text{card}(G) \geq 2^{K-2} - 3 \approx 1/\delta, \quad \text{where} \quad G = \{ j \in J; |A_j| < 2^{-K+2} \}. \quad (3.36)
\]
For each \( j \in G \), set \( \lambda_j = |A_j| \) and consider the function \( \psi_j(t) \) defined as follows:
\[
\psi_j(t) = |[t - 4\lambda_j, t + 4\lambda_j] \cap B_j|, \quad \forall t \in [40\delta, 1 - 40\delta].
\]
Then, from (3.33),
\[
\psi_j(t_+ < 4\lambda_j \quad \text{and} \quad \psi_j(t_-) > 4\lambda_j.
\]
Thus, since \( \psi_j \) is a continuous function on the interval \([40\delta, 1 - 40\delta]\) containing two points \( t_+ \) and \( t_- \), there exists \( t_j \in [40\delta, 1 - 40\delta] \) such that
\[
\psi_j(t_j) = 4\lambda_j.
\]
In the rest of the proof we introduce a new way to estimate the left side of (3.31). Since \( \lambda_j \lesssim \delta \), it follows from Corollary 2 that there exist \( m_j \in \mathbb{Z}_+ \) and \( l_j \in \mathbb{Z} \) such that
\[
|l_j - j| \leq 2m_j
\]
and
\[
\iint_{x \in I \cap A_{l_j} \atop y \in I \cap A_{l_j+2}} \frac{1}{|x - y|^{p+1}} \, dx \, dy \geq c_p m_j \delta^{1-p}, \quad (3.37)
\]
for some positive constant \( c_p \) depending only on \( p \).

Set \( i_0 = -1 \) and
\[
C_i = \{ j \in G; l_j = i \}, \quad \forall i \in \mathbb{Z}.
\]
For each \( n \geq 1 \), if
\[
\{ i \in \mathbb{Z}; i \geq i_{n-1} + 1 \text{ and } C_i \neq \emptyset \} \neq \emptyset,
\]
then set
\[
i_n = \inf\{ i \in \mathbb{Z}; i \geq i_{n-1} + 1 \text{ and } C_i \neq \emptyset \},
\quad k_n = \max\{ m_j; j \in G \text{ and } l_j = i_n \}.
\]
Then
\[
k_n \gtrsim \text{card}\{ j \in G; l_j = i_n \}.
\]
Hence it follows from (3.36) that
\[
\sum_{n \geq 1, \text{ exists}} k_n \gtrsim \text{card}(G) \approx \frac{1}{\delta}. \quad (3.38)
\]
On the other hand, from (3.37),
\[ \int_{-K}^{K} \int_{-K}^{K} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \geq \sum_{n \geq 1, k_n \text{ exists}} \sum_{y \in I \cap A_{n+2}} \int_{I \cap A_n} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \]
\[ \geq c_p \sum_{n \geq 1, k_n \text{ exists}} k_n \delta. \]  
(3.39)
Therefore the conclusion of Lemma 6 follows from (3.38), (3.39), and (3.34).

**Case 2:** \( f \) is unbounded. By the method used in Case 1, one has
\[ \liminf_{\varepsilon \to 0_+} \int_{I \times I} \frac{\varepsilon^p}{|x-y|^{p+1}} \, dx \, dy = +\infty. \]
\[ \varepsilon < |g(x) - g(y)| < 10^{-10} \varepsilon. \]

**Remark 8.** It is interesting to compare Lemma 6 with Lemma 2 in [2] which asserts that, for each \( p \geq 1 \), there exists a positive constant \( c_p \) depending only on \( p \) such that, for any bounded interval \( I \) and for any measurable function \( g \) defined on \( I \),
\[ \liminf_{\varepsilon \to 0_+} \int_{I \times I} \frac{\varepsilon^p}{|x-y|^{p+1}} \, dx \, dy \geq c_p \frac{1}{|I|^{p-1}} (\text{ess sup}_{I} g - \text{ess inf}_{I} g)^p. \]
Obviously, Lemma 6 implies this assertion for the case \( p > 1 \).

We are now ready to prove Lemma 2.

**Proof of Lemma 2** Without loss of generality, we assume that
\[ \sup_{n \in \mathbb{N}} \int_{\mathbb{R} \times \mathbb{R}} \frac{\varepsilon_n^p}{|x-y|^{p+n}} \, dx \, dy < +\infty \]  
(3.40)
for some sequence of positive numbers \( \varepsilon_n \) such that \( \lim_{n \to \infty} \varepsilon_n = 0 \).

**Step 1:** Proof of Lemma 2 when \( N = 1 \). This proof is similar to the one of [2] Theorem 1 for the case \( N = 1 \). We reproduce it here for the convenience of the reader.
Set \( r_h(g)(x) = \frac{g(x+h) - g(x)}{h} \), \( \forall x \in \mathbb{R}, \forall 0 < h < 1 \).

For each \( m \geq 2 \), take \( K \in \mathbb{R}_+ \) such that \( Kh > m \); then
\[ \int_{-m}^{m} |r_h(g)(x)|^p \, dx \leq \sum_{k=-K}^{(k+1)h} \int_{kh}^{(k+1)h} |r_h(g)(x)|^p \, dx. \]
Thus, since
\[ \int_a^{a+h} |u(x)|^p \, dx \leq \int_a^{a+h} \frac{1}{h^p} \, \text{ess sup}_{x \in (a,a+2h)} g - \text{ess inf}_{x \in (a,a+2h)} g \, dx, \]
it follows from Lemma 6 that, for some constant \( c_p > 0 \),
\[ \int_m^{-m} |u(x)|^p \, dx \leq c_p \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \frac{\varepsilon_n^p}{|x-y|^{p+1}} \, dx \, dy. \tag{3.41} \]

Since \( m \geq 2 \) is arbitrary, we deduce from (3.41) that
\[ \int_{\mathbb{R}^N} |u(x)|^p \, dx \leq c_p \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \frac{\varepsilon_n^p}{|x-y|^{p+1}} \, dx \, dy. \tag{3.42} \]
Therefore since (3.42) holds for all \( 0 < h < 1 \), it follows that \( g \in W^{1,p}(\mathbb{R}) \) (see [4, Chapter 8]).

**Step 2:** Proof of Theorem 1 for \( N \geq 2 \). Using the change of variables formula and Fubini’s theorem, one gets
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{p+N}} \, dx \, dy = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{1}{h^{p+1}} \, dh \, dx \, d\sigma. \]

Hence, it follows from (3.2) that
\[ \lim_{\varepsilon \to 0^+} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{\varepsilon_n^p}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty. \]
Applying Fatou’s lemma, one has
\[ \int_{\mathbb{S}^{N-1}} \liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{\varepsilon_n^p}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty. \]
Thus for almost every \( \sigma \in \mathbb{S}^{N-1} \),
\[ \liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{\varepsilon_n^p}{h^{p+1}} \, dh \, dx < +\infty. \tag{3.43} \]
On the other hand, from (3.1),
\[ \int_{\mathbb{S}^{N-1}} \int_{|x|<r} \int_0^{r} \frac{1}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty, \quad \forall r > 0, \forall \varepsilon > 0. \]
Hence

\[
\int_{|x|<r} \int_{0}^{r} \frac{1}{h^{p+1}} \, dh \, dx < +\infty, \quad \forall r > 0, \quad \forall \varepsilon > 0,
\]

(3.44)
for almost every \( \sigma \in \mathbb{S}^{N-1} \).

Fix \( \sigma \in \mathbb{S}^{N-1} \) such that conditions (3.43) and (3.44) are satisfied. We claim that

\[
\frac{\partial g}{\partial \sigma} \in L^p(\mathbb{R}^N).
\]

In fact, without loss of generality, suppose that \( \sigma = e_N := (0, \ldots, 0, 1) \). Then from (3.43), we have

\[
\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\varepsilon}^{\infty} \frac{\varepsilon^p}{h^{p+1}} \, dh \, dx' < +\infty.
\]

Hence applying Fatou’s lemma, one gets

\[
\int_{\mathbb{R}^{N-1}} \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\varepsilon}^{\infty} \frac{\varepsilon^p}{|x_N - y_N|^{p+1}} \, dx' \, dy_N < +\infty.
\]

On the other hand, (3.44) gives

\[
\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\varepsilon}^{\infty} \frac{\varepsilon^p}{|x_N - y_N|^{p+1}} \, dx' \, dy_N < +\infty, \quad \forall K \subseteq \mathbb{R}, \forall \varepsilon > 0,
\]

for almost every \( x' \in \mathbb{R}^{N-1} \). Therefore applying Lemma 2 for the case \( N = 1 \), one has \( g(x', \cdot) \in W^{1,p}(\mathbb{R}) \) for almost every \( x' \in \mathbb{R}^{N-1} \) and moreover (see [7]),

\[
\int_{\mathbb{R}^{N-1}} \frac{\left| \frac{\partial g}{\partial x_N} (x') \right|^p}{|x_N - x'|^{p+1}} \, dx < +\infty.
\]

Since \( \frac{\partial g}{\partial \sigma} \in L^p(\mathbb{R}^N) \) for almost every \( \sigma \in \mathbb{S}^{N-1} \), we conclude that \( g \in W^{1,p}(\mathbb{R}^N) \).

This completes the proof of the fundamental lemma 2. \( \square \)

**Remark 9.** The constant 10 which appears in the condition “\( \varepsilon < |g(x) - g(y)| < 10\varepsilon \)” is a technical constant. We believe that 10 can be replaced by any positive constant strictly greater than 1, but we have not been able to prove this.

**Remark 10.** Lemma 2 is only proved in the case \( 1 < p < +\infty \). Lemma 2 clearly implies Proposition 3.

When \( p = 1 \) we have the following
Question 2. Assume that \( g \in L^1(\mathbb{R}^N) \) satisfies
\[
\iint_{K \times K} \frac{1}{|x - y|^{N+1}} \, dx \, dy < +\infty, \quad \forall K \subset \subset \mathbb{R}^N, \forall \varepsilon > 0
\]
and
\[
\liminf_{\varepsilon \to 0^+} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varepsilon}{|x - y|^{N+1}} \, dx \, dy < +\infty.
\]
Does \( g \) belong to \( BV(\mathbb{R}^N) \)?

3.2. Proof of assertion (b) in Theorem 1

Without loss of generality, we may assume that
\[
M := 1 + \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_n(|g(x) - g(y)|) \frac{10^{-p(k+1)}}{|x - y|^{p+N}} \, dx \, dy < +\infty.
\]
Thus since \( F_n(t) \) is a non-decreasing function with respect to \( t \),
\[
\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} 10^{p(k+1)} F_n(10^{-(k+1)}) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{10^{-p(k+1)}}{|x - y|^{p+N}} \, dx \, dy \leq M. \quad (3.45)
\]
We claim that
\[
\liminf_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{10^{-p(k+1)}}{|x - y|^{p+N}} \, dx \, dy < +\infty. \quad (3.46)
\]
We prove (3.46) by contradiction. Suppose it does not hold. Then there exists \( k_M \in \mathbb{N} \) such that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{10^{-p(k+1)}}{|x - y|^{p+N}} \, dx \, dy > 2 \cdot 10^{p+1} M, \quad \forall k \geq k_M.
\]
Hence (3.45) implies
\[
\sup_{n \in \mathbb{N}} 2 \cdot 10^{p+1} M \sum_{k=k_M}^{\infty} 10^{p(k+1)} F_n(10^{-(k+1)}) \leq M,
\]
which shows that
\[
\sup_{n \in \mathbb{N}} \sum_{k=k_M}^{\infty} 10^{-(k+1)} 10^{(k+2)(p+1)} F_n(10^{-(k+1)}) \leq \frac{1}{2}. \quad (3.47)
\]
On the other hand, since $F_n$ converges uniformly to 0 on any compact subset of $(0, +\infty)$ and

$$
\int_0^1 F_n(t)t^{-(p+1)} \, dt = 1
$$

(see hypotheses (ii) and (iii) on $F_n$), one gets

$$
1 = \lim_{n \to \infty} \int_0^{10^{-kM+1}} F_n(t)t^{-(p+1)} \, dt = \lim_{n \to \infty} \sum_{k=kM}^{\infty} \int_{10^{-(k+1)}}^{10^{-(k+2)}} F_n(t)t^{-(p+1)} \, dt
$$

$$
\leq \limsup_{n \to \infty} \sum_{k=kM}^{\infty} 10^{-(k+1)} 10^{(k+2)(p+1)} F_n(10^{-(k+1)})
$$

(since $F_n(t)$ is a non-decreasing function with respect to $t$). This contradicts (3.47), and proves (3.46).

By (i)–(iii) it follows that for each $s > 0$ there exists $n$ such that $F_n(s) > 0$. Hence since $F_n$ is a non-decreasing function,

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{p+N}} \, dxdy < +\infty, \quad \forall s > 0.
$$

(3.48)

Therefore by Lemma 2, it follows from (3.46) and (3.48) that $g \in W^{1,p}(\mathbb{R}^N)$.

4. $\Gamma$-convergence

In this section we investigate some questions relating to $\Gamma$-convergence. In [8], A. C. Ponce studied similar questions in the context of [1].

We first recall the concept of $\Gamma$-convergence (see [3, 6]). One says that a sequence of functionals $(I_n)$, with values in $[0, +\infty]$, $\Gamma$-converges to a functional $I$ on $L^p(\mathbb{R}^N)$ when the following two conditions are satisfied:

(A) For every $g \in L^p(\mathbb{R}^N)$, there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}^N)$ converging to $g$ in $L^p(\mathbb{R}^N)$ and

$$
\limsup_{n \to \infty} I_n(g_n) \leq I(g).
$$

(B) For every $g \in L^p(\mathbb{R}^N)$ and for every sequence $(g_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}^N)$ converging to $g$ in $L^p(\mathbb{R}^N)$, we have

$$
I(g) \leq \liminf_{n \to \infty} I_n(g_n).
$$

We now take

$$
I_{p,n}(g) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta_n^p}{|x-y|^{N+p}} \, dxdy
$$

where

$$
|g(x) - g(y)| > \delta_n
$$
Further characterizations of Sobolev spaces

for some sequence \((\delta_n)\) converging to 0, and

\[
I_p(g) = \begin{cases} 
\frac{K_{N,p}}{p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx 
& \text{if } g \in W^{1,p}(\mathbb{R}^N) \text{ for } p > 1, \text{ resp. } g \in BV(\mathbb{R}^N) \text{ for } p = 1, \\
+\infty & \text{otherwise.}
\end{cases}
\]

When \(1 < p < \infty\), property (A) is satisfied with \(g_n = g\) by Proposition 1. A very interesting open question is

**Question 3.** When \(1 < p < \infty\), does \((I_{p,n})_{\Gamma_1}\) -converge to \(I_p\) on \(L^p(\mathbb{R}^N)\)?

In order to give a positive answer to Question 3, it would suffice to prove property (B), i.e., that one can take \(c_{N,p} = (1/p)K_{N,p}\) in (4.3) below. This problem is open even for \(N = 1\). A partial answer is given in Theorem 3 below.

The same question can be asked for \(p = 1\): **Question 4.** Does \((I_{1,n})_{\Gamma_1}\) -converge to \(I_1\) on \(L^1(\mathbb{R}^N)\)?

Here the situation is more delicate. As pointed out in Remark 5, there exists a function \(g \in W^{1,1}(\mathbb{R})\) such that \(\lim_{n \to \infty} I_{1,n}(g) = +\infty\) (while \(I_1(g) < +\infty\)). Hence we cannot argue as above by taking \(g_n = g\) to prove property (A). However, (A) is still true:

**Proposition 5.** Given any \(g \in BV(\mathbb{R}^N)\) there exists a sequence \(g_n \in C^\infty_c(\mathbb{R}^N)\) converging to \(g\) in \(L^1(\mathbb{R}^N)\) and such that

\[
\lim_{n \to \infty} I_{1,n}(g_n) \leq K_{N,1} \int_{\mathbb{R}^N} |
abla g| \, dx.
\]

**Proof.** Let \((h_k)_{k \in \mathbb{N}}\) be a sequence in \(C^\infty_c(\mathbb{R}^N)\) converging to \(g\) in \(L^1(\mathbb{R}^N)\) and such that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla h_k| \, dx = \int_{\mathbb{R}^N} |\nabla g| \, dx.
\] (4.1)

Using the same method as in the proof of [7, Lemma 3], one gets

\[
\lim_{\delta \to 0} I_1(h_k, \delta) = K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx.
\] (4.2)

Thus there exists an increasing sequence \(n_k\) such that for all \(n \geq n_k\),

\[
I_{1,n}(h_k) \leq K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx + \frac{1}{k}.
\]

Define the sequence \(g_n\) by \(g_n = h_k\) if \(n_k < n \leq n_{k+1}\), where \(n_0 = 0\). Then for all \(n_k < n \leq n_{k+1}\),

\[
I_{1,n}(g_n) \leq K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx + \frac{1}{k}.
\]
which shows that
\[
\limsup_{n \to \infty} I_{1,n}(g_n) \leq K_N \int_{\mathbb{R}^N} |\nabla g| \, dx.
\]
\qed

We now prove the following result which is a partial answer to Questions 3 and 4. It was announced in [7] for the one-dimensional case.

**Theorem 3.** Let \((g_n)_{n \in \mathbb{N}}\) be a sequence in \(L^p(\mathbb{R}^N)\), \(1 \leq p < +\infty\), converging in \(L^p(\mathbb{R}^N)\) to some \(g \in L^p(\mathbb{R}^N)\), and \((\delta_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers with \(\lim_{n \to \infty} \delta_n = 0\). Suppose that
\[
\sup_{n \in \mathbb{N}} \int_{|g_n(x) - g_n(y)| > \delta_n} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy < +\infty.
\]
Then \(g \in W^{1,p}(\mathbb{R}^N)\) if \(p > 1\) and \(g \in BV(\mathbb{R}^N)\) if \(p = 1\). Moreover,
\[
\liminf_{n \to \infty} \int_{|g_n(x) - g_n(y)| > \delta_n} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy \geq c_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx \tag{4.3}
\]
for some positive constant \(c_{N,p}\).

We first prove a technical lemma which plays the same role as [2, Lemma 2] in the proof of [2, Theorem 1]. Its proof is based on that of [2, Lemma 2] and Egorov’s theorem (see e.g. [10]).

**Lemma 7.** Let \(I\) be a bounded interval, \((g_n)_{n \in \mathbb{N}}\) be a sequence in \(L^p(I)\), \(1 \leq p < +\infty\), converging in \(L^p(I)\) to some \(g \in L^p(I)\), and \((\delta_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers with \(\lim_{n \to \infty} \delta_n = 0\). Then
\[
\liminf_{n \to \infty} \int_{|g_n(x) - g_n(y)| > \delta_n} \frac{\delta_n^p}{|x - y|^{p+1}} \, dx \, dy \geq c_p \frac{1}{|I|^{p-1}} |\text{ess sup}_I g - \text{ess inf}_I g|^p,
\]
for some positive constant \(c_p\) depending only on \(p\).

**Proof.** Without loss of generality suppose that \(I = [0, 1]\), \(\text{ess sup}_I g = s_+\), \(\text{ess inf}_I g = s_-\) (\(-\infty < s_- < s_+ < +\infty\)) and \(s_+ - s_- = 1\).

Take \(0 < \delta \ll 1\) small enough to ensure that there are (density) points \(t_+, t_- \in [40\delta, 1 - 40\delta] \subset [0, 1]\) with
\[
\begin{align*}
&\left[ [t_+ - 40\delta, t_+ + 40\delta] \cap \left[ g > \frac{4}{5}s_+ + \frac{4}{5}s_- \right] \right] > 70\delta, \\
&\left[ [t_- - 40\delta, t_- + 40\delta] \cap \left[ g < \frac{4}{5}s_- + \frac{1}{5}s_+ \right] \right] > 70\delta.
\end{align*}
\]
We will assume as well that $g_n$ converges to $g$ for almost every $x \in I$. Thus, by Egorov’s theorem (see [10]), there exists a constant $n_0$ such that for all $n \geq n_0$, we have $\delta_n \leq \delta$ and

\[
\begin{cases}
[t_+ - 40\delta, t_+ + 40\delta] \cap \left\{g_n > \frac{3}{4}s_+ + \frac{1}{4}s_-\right\} > 60\delta, \\
[t_- - 40\delta, t_- + 40\delta] \cap \left\{g_n < \frac{3}{4}s_- + \frac{1}{4}s_+\right\} > 60\delta.
\end{cases}
\]

Fix $n \geq n_0$, take $K \in \mathbb{Z}_+$ such that $\delta_n < 2^{-K} \leq 2\delta_n$ and set

\[
J = \left\{ j \in \mathbb{Z}_+; \ \frac{3}{4}s_- + \frac{1}{4}s_+ < j2^{-K} < \frac{3}{4}s_+ + \frac{1}{4}s_- \right\}.
\] (4.4)

Then $|J| \geq 2^{K-1} - 2 \approx 1/\delta_n$.

Define (for notational ease)

\[
A_j = \{x \in [0, 1]; \ (j - 1)2^{-K} \leq g_n(x) < j2^{-K}\}, \quad B_j = \bigcup_{j' < j} A_{j'}, \quad \forall j \in \mathbb{Z}.
\]

Since the sets $A_j$ are disjoint, it follows from (4.4) that

\[
\text{card}(G) \geq 2^{K-2} - 3 \approx 1/\delta_n,
\]

where (for notational ease)

\[
G = \{ j \in J; \ |A_j| < 2^{-K+2} \}.
\]

For each $j \in G$, set $\lambda_{1,j} = |A_j|$ and consider the function $\psi_j(t)$ defined as follows:

\[
\psi_j(t) = \|[t - 40\delta, t + 40\delta] \cap B_j|, \quad \forall t \in [40\delta, 1 - 40\delta].
\]

Applying the same method as in [2, Lemma 2], we deduce that there exist $\lambda_j > 0$ and $t_j \in [t - 40\delta, t + 40\delta]$ such that

\[
\begin{cases}
||t_j - 40\lambda_j, t_j + 40\lambda_j| \cap B_j| = 40\lambda_j, \\
\lambda_j/4 \leq ||t_j - 40\lambda_j, t_j + 40\lambda_j| \cap A_j| \leq \lambda_j.
\end{cases}
\]

It follows that $\lambda_j \lesssim \delta_n$. The rest of the proof is similar to the one of [2, Lemma 2]. \[\square\]

**Proof of Theorem 3.** The proof is the same as that of [2, Theorem 1], with the use of Lemma 7 in place of [2, Lemma 1]. The details are left to the reader. \[\square\]

Finally, we present a special case where the $\Gamma$-convergence of a sequence of functionals can be established.

Consider the functionals $J_n$ and $J$ on $L^p(\mathbb{R}^N)$ defined as follows:

\[
J_n(g) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{N+p}} \, dx \, dy, \quad \forall n \in \mathbb{N},
\] (4.5)
and

\[
J(g) = \begin{cases} 
K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx & \text{if } g \in W^{1,p}(\mathbb{R}^N) \text{ for } p > 1, \text{ resp. } g \in BV(\mathbb{R}^N) \text{ for } p = 1, \\
+\infty & \text{otherwise},
\end{cases}
\]

(4.6)

for all \( g \in L^p(\mathbb{R}^N) \).

We have the following

**Theorem 4.** Let \( 1 \leq p < +\infty \) and \( F_n : [0, +\infty) \to [0, +\infty) \) is convex on the interval \([0, 1]\) for all \( n \in \mathbb{N} \), and satisfies hypotheses (i)-(iii) of Theorem 1. Then \( (J_n) \) \( \Gamma \)-converges to \( J \) on \( L^p(\mathbb{R}^N) \).

As a consequence of Theorem 4, we have

**Corollary 3.** Let \( F_n \) be defined as follows:

\[
F_n(t) = \begin{cases} 
\varepsilon_n t^{p+\varepsilon_n} & \text{if } 0 \leq t \leq 1, \\
\varepsilon_n & \text{otherwise},
\end{cases}
\]

for some sequence \((\varepsilon_n)\) converging to 0. Then \( (J_n) \) \( \Gamma \)-converges to \( J \).

In order to prove Theorem 4 one needs the following two lemmas.

**Lemma 8.** Let \( 1 \leq p < +\infty \) and \( F_n : [0, +\infty) \to [0, +\infty) \) satisfy hypotheses (i)-(iii) of Theorem 1 and \((g_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}) \cap C^2_b(\mathbb{R}) \). Assume that \( g_n \) converges to \( g \) in \( L^p(\mathbb{R}) \) and \( C^2_b(\mathbb{R}) \). Then

\[
\liminf_{n \to \infty} \int_\mathbb{R} \int_0^\infty F_n(|g_n(x+h) - g_n(x)|) \frac{dh}{h^{p+1}} \, dx \geq \int_\mathbb{R} |g'|^p \, dx.
\]

We recall that

\[
C^2_b(\mathbb{R}^N) = \{ g \in C^2(\mathbb{R}^N); \|g\|_{L^\infty} + \|Dg\|_{L^\infty} + \|D^2g\|_{L^\infty} < +\infty \},
\]

with the norm

\[
\|g\|_{C^2_b} = \|g\|_{L^\infty} + \|Dg\|_{L^\infty} + \|D^2g\|_{L^\infty}, \quad \forall g \in C^2_b(\mathbb{R}^N).
\]

**Proof.** Fix \( \varepsilon > 0 \) arbitrary. Take \( \delta > 0 \) such that

\[
\int_{A_\delta} |g'|^p \, dx > (1 - \varepsilon) \int_\mathbb{R} |g'|^p \, dx,
\]

(4.7)

where

\[
A_\delta := \{ x \in \mathbb{R}; \ |g'(x)| > 2\delta \}.
\]

Since \( g_n \) converges to \( g \) in \( C^1(\mathbb{R}) \), there exists some \( n_\varepsilon \in \mathbb{N} \) such that

\[
|g_n'(x)| > \delta, \quad \forall x \in A_\delta.
\]
Hence since \((g_n)\) is bounded in \(C^2_0(\mathbb{R})\), it follows that
\[
|g_n(x + h) - g_n(x)| \geq (1 - \varepsilon)h|g'(x)|, \quad \forall x \in A_\delta, \forall 0 < h < \tau,
\]
for some \(\tau > 0\) and for all \(n \geq n_\varepsilon\). Thus since \(F_n\) is non-decreasing, one gets
\[
\liminf_{n \to \infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x + h) - g_n(x)|)}{h^{p+1}} \, dh \, dx \\
\geq \liminf_{n \to \infty} \int_{A_\delta} \int_0^\tau \frac{F_n((1 - \varepsilon)h|g'_n(x)|)}{h^{p+1}} \, dh \, dx. \tag{4.8}
\]

On the other hand, from (i),
\[
\liminf_{n \to \infty} \int_{A_\delta} \int_0^\tau \frac{F_n((1 - \varepsilon)h|g'_n(x)|)}{h^{p+1}} \, dh \, dx \geq \liminf_{n \to \infty} (1 - \varepsilon)^p \int_{A_\delta} |g'_n|^p \, dx \int_0^{\tau/2} F_n(t)t^{-(p+1)} \, dt.
\]

Thus it follows from (ii) and (iii) that
\[
\liminf_{n \to \infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x + h) - g_n(x)|)}{h^{p+1}} \, dh \, dx \geq (1 - \varepsilon)^p \liminf_{n \to \infty} \int_{A_\delta} |g'_n|^p \, dx,
\]
which implies, from (4.7),
\[
\liminf_{n \to \infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x + h) - g_n(x)|)}{h^{p+1}} \, dh \, dx \geq (1 - \varepsilon)^{p+1} \int_{\mathbb{R}} |g'|^p \, dx.
\]
Therefore, since \(\varepsilon > 0\) is arbitrary,
\[
\liminf_{n \to \infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x + h) - g_n(x)|)}{h^{p+1}} \, dh \, dx \geq \int_{\mathbb{R}} |g'|^p \, dx.
\]

**Lemma 9.** Let \(1 \leq p < +\infty\) and \(F_n : [0, +\infty) \to [0, +\infty)\) satisfy hypotheses (i)-(iii) of Theorem 1 and \(g \in C^2_0(\mathbb{R}^N)\). Then
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{p+1}} \, dx \, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.
\]

**Proof.** Without loss of generality, one may assume that
\[
\text{supp } g \subset \{x \in \mathbb{R}^N; \ |x| \leq 1\}. \tag{4.9}
\]

Using the change of variables formula one has
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{p+1}} \, dx \, dy = \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma.
\]
Thus by Lemma 8 and Fatou’s lemma, one gets

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{p+1}} \, dx \, dy \geq K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.
\]

Hence it suffices to prove that

\[
\limsup_{n \to \infty} \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.
\]

Fix \( \varepsilon > 0 \) (arbitrary). Define

\[A_\varepsilon := \{(\sigma, x) \in S^{N-1} \times \mathbb{R}^N; |\nabla g(x) \cdot \sigma| > 2\varepsilon\}.
\]

Choose \( 0 < \tau < 1 \) such that \( \tau \|g\|_{C^2}^\varepsilon < \varepsilon^2 \). Then

\[|g(x + h\sigma) - g(x)| < (1 + \varepsilon)h|\nabla g(x) \cdot \sigma|, \quad \forall 0 < h < \tau, (\sigma, x) \in A_\varepsilon.
\]

Thus it follows from (i) that

\[
\limsup_{n \to \infty} \int_{A_\varepsilon} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq \limsup_{n \to \infty} \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_0^{\tau} \frac{F_n(|(1 + \varepsilon)|\nabla g(x) \cdot \sigma|)}{h^{p+1}} \, dh \, dx \, d\sigma,
\]

which implies

\[
\limsup_{n \to \infty} \int_{A_\varepsilon} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq (1 + \varepsilon)^p K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.
\]

On the other hand, from the choice of \( \tau \),

\[
\int_{A_\varepsilon^c} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \leq \int_{S^{N-1}} \int_{|x| < 2} \int_0^{\tau} \frac{F_n(3\varepsilon h)}{h^{p+1}} \, dh \, dx \leq C\varepsilon^p,
\]

where \( A_\varepsilon^c \) denotes the complement of \( A_\varepsilon \) in \( S^{N-1} \times \mathbb{R}^N \). Hereafter in this proof, \( C \) denotes a constant independent of \( n \). Thus

\[
\limsup_{n \to \infty} \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq (1 + \varepsilon)^p K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx + C\varepsilon^p. \tag{4.10}
\]

On the other hand, it follows from (iii) that

\[
\lim_{n \to \infty} \int_{|x| < 2} \int_0^{\tau} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma = 0. \tag{4.11}
\]
From (4.9), one gets
\[ \int_{S^{N-1}} \int_{|x| \geq 2} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} dh \, dx \, d\sigma = \int_{S^{N-1}} \int_{|x| \geq 2} \frac{F_n(|g(x + h\sigma)|)}{h^{p+1}} dh \, dx \, d\sigma. \]

Also, from (4.9),
\[ \int_{S^{N-1}} \int_{|x| \geq 2} \frac{F_n(|g(x + h\sigma)|)}{h^{p+1}} dh \, dx \, d\sigma \leq \int_{S^{N-1}} \int_{R^N} \frac{F_n(|g(x)|)}{h^{p+1}} dh \, dx \, d\sigma. \]

Thus
\[ \lim_{n \to \infty} \int_{S^{N-1}} \int_{|x| \geq 2} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} dh \, dx \, d\sigma = 0. \quad (4.12) \]

Combining (4.11) and (4.12) yields
\[ \lim_{n \to \infty} \int_{S^{N-1}} \int_{R^N} \int_{|x| \geq 2} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} dh \, dx \, d\sigma = 0. \quad (4.13) \]

Hence it follows from (4.10) and (4.13) that
\[ \limsup_{n \to \infty} \int_{S^{N-1}} \int_{R^N} \int_{0}^{\infty} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} dh \, dx \, d\sigma \leq (1 + \varepsilon)^p K_{N,p} \int_{R^N} |\nabla g|^p \, dx + C\varepsilon^p. \]

Therefore, since \( \varepsilon > 0 \) is arbitrary, one has
\[ \limsup_{n \to \infty} \int_{S^{N-1}} \int_{R^N} \int_{0}^{\infty} \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} dh \, dx \, d\sigma \leq K_{N,p} \int_{R^N} |\nabla g|^p \, dx. \quad \square \]

Proof of Theorem 4

Step 1: Proof of property (A). By the same method as in the proof of Proposition 5, property (A) follows from Lemma 3.

Step 2: Proof of property (B). We use some ideas of A. Ponce in the proof of [8, Lemma 12.2]. Let \((g_n)\) be a sequence converging to \(g\) in \(L^p(R^N)\). Let \((\rho_n)\) be a sequence of smooth mollifiers. Set
\[ g_{n,k}(x) = \begin{cases} g_n(x) & \text{if } |g_n(x)| \leq k, \\ kg_n(x) & \text{otherwise}, \end{cases} \]
for all \(k \in \mathbb{N}\), and
\[ g_{n,k,\delta} = g_{n,k} * \rho_\delta, \quad \forall n \in \mathbb{N}, \forall k \in \mathbb{N}. \]
Then, since $F_n$ is convex on $[0, 1]$, it follows from (i) and (iii) that

$$
\liminf_{n \to \infty} J_n(g_{n,k}, \delta) \leq \liminf_{n \to \infty} J_n(g_{n,k}).
$$

(4.14)

On the other hand, from Fatou’s lemma and Lemma 8, one gets

$$
\liminf_{n \to \infty} J_n(g_{n,k}, \delta) \geq J(h_k, \delta),
$$

(4.15)

where

$$
h_k(x) = \begin{cases} 
g(x) & \text{if } |g(x)| \leq k, \\
k|g(x)| & \text{otherwise,}
\end{cases}
$$

and

$$
h_{k,\delta} = h_k \ast \rho_{\delta}.
$$

Thus, since

$$
\liminf_{\delta \to 0} J(h_{k,\delta}) \geq J(h_k),
$$

it follows from (4.14) and (4.15) that

$$
\liminf_{n \to \infty} J_n(g_{n,k}) \geq J(h_k).
$$

On the other hand, from (i),

$$
J_n(g_{n,k}) \leq J(g_n).
$$

Therefore,

$$
\liminf_{n \to \infty} J_n(g_n) \geq J(h_k), \quad \forall k \in \mathbb{N}.
$$

This implies

$$
\liminf_{n \to \infty} J_n(g_n) \geq J(g).
$$

However, without the assumption on the convexity of $(F_n)$ on $[0, 1]$ in Theorem 4, one has

**Question 5.** Assume that $F_n : [0, +\infty) \to [0, +\infty)$ satisfies hypotheses (i)–(iii) of Theorem 1 and $J_n$ and $J$ are defined by (4.5) and (4.6). Does $(J_n)$ $\Gamma$-converge to $J$ on $L^p(\mathbb{R}^N)$ for all $p \geq 1$?

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References


