Some inequalities related to Sobolev norms

Hoai-Minh Nguyen *

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Abstract

In this paper, we study some properties related to the new characterizations of Sobolev spaces introduced in [16, 4, 18]. More precisely, we establish variants of the Poincaré inequality, the Sobolev inequality, and the Rellich-Kondrachov compactness theorem, where $\int_{\mathbb{R}^N} |\nabla g|^p \, dx$ is replaced by some quantity of the type

$$I_\delta(g) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy.$$

1 Introduction

We first introduce the quantity $I_\delta(g)$, which plays an important role in this paper,

$$I_\delta(g) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \quad \forall \, g \in L^1_{loc}(\mathbb{R}^N).$$

We next recall some new characterizations of Sobolev spaces in [16, 4, 18]. The first one is as follows

Proposition 1 *Let $1 < p < +\infty$. Then*

a) There exists a constant $C_{N,p}$ depending only on N and p such that

$$I_\delta(g) \le C_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx, \quad \forall \, \delta > 0, \forall \, g \in W^{1,p}(\mathbb{R}^N).$$

b) If $g \in L^p(\mathbb{R}^N)$ satisfies

$$\liminf_{\delta \to 0_+} I_\delta(g) < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

c) Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta \to 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx,$$

where $K_{N,p}$ is defined by

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p \, d\sigma, \tag{1.1}$$

for any $e \in \mathbb{S}^{N-1}$.

*Courant Institute, NYU, 251 Mercer Street, New York, NY 10012, USA, hoaiminh@cims.nyu.edu

Remark 1 *Assertions a) and c) are proved in [16] (Theorem 2). Assertion b) is proved by Bourgain-Nguyen in [4]. The proof of Assertion b) is delicate. Under the following stronger assumption*

$$\limsup_{\delta \to 0_+} I_\delta(g) < +\infty,$$

a simple proof is given in [16] (see the proof of Theorem 2).

In [18], we improve statement *b*) in Proposition 1 by proving

Proposition 2 *Let $N \geq 1$, $p > 1$, and $g \in L^p(\mathbb{R}^N)$. Assume that $I_\delta(g) < +\infty$ for all $\delta > 0$, and*

$$\liminf_{\delta \to 0_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \atop \delta < |g(x) - g(y)| < 10\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy < +\infty.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$.

Remark 2 *To prove Proposition 2, we developed the method introduced in [4]. The observation used in the proof Proposition 2 will play an important role in the proof of statement b) in Theorem 1 which is crucial to establish Theorem 3 and Proposition 6.*

The second characterization is a generalization of Proposition 1.

Proposition 3 *[18, Theorem 1] Let $1 < p < +\infty$ and $(F_n)_{n \in \mathbb{N}}$ be a sequence of functions from $[0, +\infty)$ into $[0, +\infty)$ such that*

i) $F_n(t)$ is a non-decreasing function with respect to t on $[0, +\infty)$, for all $n \in \mathbb{N}$.

ii) $\displaystyle\int_0^1 F_n(t) t^{-(p+1)} \, dt = 1$, for all $n \in \mathbb{N}$.

iii) $F_n(t)$ converges uniformly to 0 on every compact subset of $(0, +\infty)$ as n goes to infinity.

Then

a) If $g \in W^{1,p}(\mathbb{R}^N)$, then for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy \leq C_{N,p} \int_0^\infty F_n(t) t^{-(p+1)} \, dt \int_{\mathbb{R}^N} |\nabla g|^p \, dx,$$

where $C_{N,p}$ is a positive constant depending only on N and p.

b) If $g \in L^p(\mathbb{R}^N)$ and g satisfies

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$ and

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy \geq K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

c) Moreover, if

$$\limsup_{n \to \infty} \int_0^\infty F_n(t) t^{-(p+1)} \, dt < +\infty,$$

then, for any $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

Here $K_{N,p}$ *is defined by* (1.1).

Remark 3 *Proposition 1 follows from Proposition 3 by choosing*

$$F_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \delta_n, \\ \dfrac{p\delta_n^p}{1-\delta_n^p} & \text{otherwise.} \end{cases} \tag{1.2}$$

Remark 4 *Assumption i)-iii) of the sequence* (F_n) *are necessary to obtain a), b), and c) (see [18, Remark 4] for detailed discussion).*

In this paper, we establish variants of the Poincaré inequality, the Sobolev inequality, and the Rellich-Kondrachov theorem which are inspired by these characterizations. Our first result motivated by Proposition 1 and the Poincaré inequality is the following theorem, which is proved in Section 2.

Theorem 1 *Let* $N \ge 1$, $p \ge 1$, *and* g *be a real measurable function defined on a ball* $B \subset \mathbb{R}^N$. *Assume that*

$$\mathop{\int_B \int_B}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy < +\infty.$$

Then

a) *If* $p \ge 1$, *we have*

$$\int_B \int_B |g(x)-g(y)|^p \, dx\, dy \le C_{N,p}\Big(|B|^{\frac{N+p}{N}} \mathop{\int_B \int_B}_{|g(x)-g(y)|>\delta} \frac{\delta}{|x-y|^{N+p}}\, dx\, dy + \delta^p |B|^2 \Big), \tag{1.3}$$

for all $\delta > 0$ *and for some positive constant* $C_{N,p}$ *depending only on* N *and* p.

b) *If* $p > 1$, *we have*

$$\int_B \int_B |g(x)-g(y)|^p \, dx\, dy \le C_{N,p}\Big(|B|^{\frac{N+p}{N}} \mathop{\int_B \int_B}_{\delta<|g(x)-g(y)|<10\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + \delta^p |B|^2 \Big), \tag{1.4}$$

for all $\delta > 0$. *Here* $C_{N,p}$ *is a positive constant depending only on* N *and* p.

Remark 5 *We do not know whether* (1.4) *is valid with* $p = 1$.

Remark 6 *Inequality* (1.4) *plays an important role in the proof of Theorem 3 below.*

Remark 7 *A variant of estimate* (1.3) *was established by Bourgain-Brezis-Mironescu [3] as follows. Let* $g \in C(I = [0,1], \mathbb{R})$. *Then*

$$\int_I \int_I |g(x)-g(y)| \, dx\, dy \le C\Big(|I|^2 \mathop{\int_I \int_I}_{|e^{ig(x)}-e^{ig(y)}|>\delta} \frac{1}{|x-y|^2}\, dx\, dy + |I|^2 \Big),$$

for some universal positive constant C, *when* δ *is small. The continuity of* g *is necessary for such a result. Recently in a joint work with Brezis [6], using a completely different argument, we establish the above inequality for any* $\delta < \sqrt{3}$ *and for any* $g \in VMO(I, \mathbb{R})$ ($\sqrt{3}$ *is optimal). The proofs in [3] and [6] are involved. The approach in [6] can also be used to obtain a similar inequality for*

$p > 1$. *These results can be extended to higher dimensions for a smooth function g using the idea in Step 2 of the proof of Theorem 1 in this paper (see [7]). Nevertheless we do not know how to obtain* (1.3) *under the general condition as in statement a) using this approach since the standard density arguments do not work in this context. This is due to the fact that quantity in the RHS of* (1.3) *is "unstable" under the convolution.*

Our next result is a variant of Rellich-Kondrachov theorem, whose proof is presented in Section 3.

Theorem 2 *Let $N \geq 1$, $p \geq 1$, $(g_n) : \mathbb{R}^N \to \mathbb{R}$ be a bounded sequence of functions in $L^p(\mathbb{R}^N)$ and (δ_n) be a sequence of positive numbers converging to 0 such that*

$$\liminf_{n\to\infty} I_{\delta_n}(g_n) = \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \limits_{|g_n(x)-g_n(y)|>\delta_n} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy < +\infty. \tag{1.5}$$

Then there exist a subsequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that $\big(g_{n_k}\big)$ converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$. Moreover, $g \in W^{1,p}(\mathbb{R}^N)$ for $p > 1$ resp. $g \in BV(\mathbb{R}^N)$ for $p = 1$ and there exists a positive constant C, depending only on N and p, such that

$$\int_{\mathbb{R}^N} |\nabla g|^p\,dx \leq C \liminf_{n\to\infty} I_{\delta_n}(g_n). \tag{1.6}$$

Remark 8 *The optimal constant in* (1.6)*, which was discussed in the context of Gamma-convergence in [17], [19], is strictly less than $K_{N,p}/p$.*

Remark 9 *The conclusion of Theorem 2 still holds in the case $p > 1$ if* (1.5) *is replaced by the conditions that $I_{\delta_n}(g_n) < +\infty$ for all $n \in \mathbb{N}$, and*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \limits_{\delta_n<|g_n(x)-g_n(y)|<10\delta_n} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Remark 10 *When $p > 1$, Theorem 2 implies the well-known Rellich-Kondrachov theorem, since $I_\delta(g) \leq C_{N,p}\int_{\mathbb{R}^N} |\nabla g|^p\,dx$.*

A variant of the Sobolev inequality, which is proved in Section 4, is as follows

Theorem 3 *Let $1 < p < N$, $\delta > 0$, and g be a real measurable function defined on $\mathbb{R}^N$ such that*

$$I_\delta(g) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Then there exist two positive constants C and λ, depending only on N and p, such that

$$\Big(\int\limits_{|g|>\lambda\delta} |g|^q\,dx\Big)^{\frac{1}{q}} \leq C\,[I_\delta(g)]^{\frac{1}{p}}, \tag{1.7}$$

with $q = \frac{Np}{N-p}$.

Remark 11 *Letting δ go to 0 in* (1.7)*, we rediscover and extend the Sobolev inequality since $\lim_{\delta\to 0} I_\delta(g) = \frac{1}{p}K_{N,p}\int_{\mathbb{R}^N} |\nabla g|^p\,dx$, $I_\delta(g) \leq C_{N,p}\int_{\mathbb{R}^N} |\nabla g|^p\,dx$ (see Proposition 1), and $\lim_{\delta\to 0}\int_{|g|>\lambda\delta} |g|^q\,dx = \int_{\mathbb{R}^N} |g|^q\,dx$ for $g \in W^{1,p}(\mathbb{R}^N)$. Since $I_\delta(g) \leq \frac{\delta^p}{\delta'^p} I_{\delta'}(g)$ for $\delta \geq \delta'$, Theorem 3 is more interesting when it is applied for large δ.*

When $N \geq 1$ and $p = N$, estimates (1.3) and (1.4) clearly imply that $g \in BMO(\mathbb{R}^N)$, the space of all functions of bounded mean oscillation defined on $\mathbb{R}^N$ if $g \in L^1(\mathbb{R}^N)$ and $I_\delta(g) < +\infty$ for some $\delta > 0$. Moreover, there exists a positive constant C, depending only on N, such that

$$|g|_{BMO} := \sup_Q \fint_Q \fint_Q |g(x) - g(y)| \, dx \, dy \leq C \left(I_\delta^{\frac{1}{N}}(g) + \delta \right),$$

where the supremum is taken over all cubes of $\mathbb{R}^N$. In a joint work with Brezis [7] we also show that if $g \in L^1(\mathbb{R}^N)$ and $I_\delta(g) < +\infty$ ($p = N$) for all $\delta > 0$, then $g \in VMO(\mathbb{R}^N)$, the spaces of all functions of vanishing mean oscillation. More properties in the case $p = N$ can be found in [7]. When $p > N$ and $I_\delta(g) < +\infty$ for some δ, one cannot hope that $g \in L^\infty_{loc}(\mathbb{R}^N)$. This follows from the fact that the function $g(x) := \ln \ln |\ln |x||$ in B_λ (λ is small), the ball centered at the origin with radius λ, does not belong to $L^\infty(B_\lambda)$ and

$$\int_{B_\lambda} \int_{\substack{B_\lambda \\ |g(x)-g(y)|>\delta}} \frac{\delta^r}{|x-y|^{N+r}} \, dx \, dy < +\infty \quad \forall \, r > 1.$$

Applying Theorem 1, we can prove that the sharp function of g belongs to $L^q_w(\mathbb{R}^N)$ with $q = Np/(N-p)$ if $g \in L^p(\mathbb{R}^N)$ ($p \geq 1$) and $I_\delta(g) < +\infty$ for some $\delta > 0$ (see Section 4). In fact we can prove that $g \in L^q(\mathbb{R}^N)$ if $p > 1$ and $I_\delta(g) < +\infty$ for some $\delta > 0$ (see Theorem 3). However, we have the following

Open question 1 *Let $p = 1$ and $N \geq 2$. Is it true that $g \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ if $g \in L^1(\mathbb{R}^N)$ and $I_\delta(g) < +\infty$ for some $\delta > 0$?*

Motivated by Proposition 3, we establish the following results, whose proofs are presented in Section 5.

Proposition 4 *Let g be a real measurable function defined on a ball $B \subset \mathbb{R}^N$ and $F : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function. Then there exists a constant $C > 0$, depending only on N and p, such that*

$$\left(F(1) + \int_0^1 F(t) t^{-(p+1)} \, dt \right) \int_B \int_B |g(x) - g(y)|^p \, dx \, dy$$
$$\leq C \left(|B|^{\frac{N+p}{N}} \int_B \int_B \frac{F(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy + F(1)|B|^2 \right).$$

Proposition 5 *Let $1 \leq p < N$, $(F_n) : [0, +\infty) \to [0, +\infty)$ be a sequence of non-decreasing functions such that $\lim_{n\to\infty} F_n(1) = 0$,*

$$F_n(1) + \int_0^1 F_n(t) t^{-(p+1)} \, dt = 1,$$

and $(g_n) : \mathbb{R}^N \to \mathbb{R}$ be a bounded sequence of real functions in $L^p(\mathbb{R}^N)$. Assume that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g_n(x) - g_n(y)|)}{|x-y|^{N+p}} \, dx \, dy < +\infty.$$

Then there exist a subsequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that (g_{n_k}) converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$.

Proposition 6 *Let $1 < p < N$, $F : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function and g be a real measurable function defined on $\mathbb{R}^N$. Assume that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Then there exist two positive constants C and λ, depending only on N and p, such that

$$\Big(\int\limits_{|g|>\lambda F(2^{-n})} |g|^q\,dx\Big)^{\frac{1}{q}} \le C\Big(\frac{1}{2^{np}F(2^{-n})}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy\Big)^{\frac{1}{p}} \quad \forall\, n \in \mathbb{Z},$$

with $q = \frac{Np}{N-p}$.

Applications of Propositions 4, 5, and 6 will be given in Section 5.3. It would be nice to obtain similar results to Theorems 1, 2, and 3, and Propositions 4, 5, and 6 in a more general setting e.g. in Carnot-Carathéodory spaces or in metric spaces with appropriate properties.

Recently many authors have suggested various definitions of Sobolev spaces and studied the well-known properties of Sobolev spaces in their contexts e.g. Ambrosio [1], Korevaar-Schoen [14], Reshetnyak [21], Hajlaz-Koskela [12], Bourgain-Brezis-Mironescu [2] and references therein. The characterizations mentioned in this paper are quite close to the work of Bourgain-Brezis-Mironescu [2]. However the connection is not transparent.

Theorem 1, whose proof is presented in Section 2, is the starting point of this paper. In the proof of Theorem 1, we use of ideas in [4] and [18], and the John-Nirenberg inequality [13]. Theorem 2 is derived from Theorem 1 by the standard technique used in Bourgain-Brezis-Mironescu [2] (see also [20]). The main ingredient of the proof of Theorem 3 is part *b*) of Theorem 1. The proof also makes use of the theory of sharp functions due to Fefferman and Stein [9] and the method of truncation due to Mazya [15]. Obtaining Sobolev's inequality from Poincaré's inequality previously appeared in the literature see e.g. [22], [10], [11], [12]. However, our approach is different from the works mentioned here, which were inspired by the Riesz potential theory. Moreover, we could not apply their methods in our setting because of the presence of the two terms in the RHS of (1.3) and (1.4). Proposition 4 is derived from Theorem 1 using ideas in [18]. The proofs of Propositions 5 and 6 follow from Proposition 4 by applying the same methods used in the proofs of Theorems 2 and 3.

The paper is organized as follows. In Section 2, we prove Theorem 1. Section 3 is devoted to the proof of Theorem 2. Theorem 3 is proved in Section 4. Section 5 is devoted to the proofs of Propositions 4, 5, and (6), and their applications.

2 A variant of Poincaré's inequality. Proof of Theorem 1

2.1 Preliminaries

In this section, we present some technical lemmas which will be used in the proof of Theorem 1. We first recall some useful results in [18].

Lemma 1 *[18, Lemma 3] Let g be a real measurable function defined on the interval $[a, b]$ $(-\infty < a < b < +\infty)$, $z \in \mathbb{R}$, and $\delta > 0$. Set*

$$B = \{x \in [a, b];\ g(x) < z\}.$$

Assume that

$$0 < \frac{|[a,b] \cap B|}{b-a} < 1,$$

and

$$\int_a^b \int_a^b_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\, dx\, dy < +\infty.$$

Then

$$|[a,b] \cap A_\tau| > 0, \quad \forall\, \tau > \delta,$$

where $A_\tau := \{x \in [a,b];\ z \le g(x) < z+\tau\}$.

Hereafter $|A|$ denotes the Lebesgue measure of A for any measurable set $A \subset \mathbb{R}^N$.

Lemma 2 *[18, Lemma 4] Let g be a real measurable function defined on the interval $[a,b]$ $(-\infty < a < b < +\infty)$, $z \in \mathbb{R}$, $r > 0$, $s > 0$, and $\tau > \delta > 0$. Set*

$$B = \{x \in \mathbb{R};\ g(x) < z\}, \quad A = \{x \in \mathbb{R};\ z \le g(x) < z+\tau\}.$$

Assume that

$$\frac{|[a,b] \cap B|}{b-a} = r, \quad \frac{|[a,b] \cap A|}{b-a} \le s, \quad r+s < 1,$$

and

$$\int_a^b \int_a^b_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\, dx\, dy < +\infty.$$

Then there exists a subinterval $[c,d] \subset [a,b]$ $(a \le c < d \le b)$, such that

$$\frac{|[c,d] \cap B|}{d-c} = r \quad \text{and} \quad s/4 \le \frac{|[c,d] \cap A|}{d-c} \le s.$$

Lemma 3 *[18, Lemma 5] Let g be a real measurable function defined on the interval $[a,b]$ $(-\infty < a < b < +\infty)$, $z \in \mathbb{R}$, $\tau > \delta > 0$, and $0 < \lambda \le 1/2$. Set*

$$\begin{cases} B_j & = \{x \in \mathbb{R};\ g(x) < z + j\tau\}, \\ A_j & = \{x \in \mathbb{R};\ z + j\tau \le g(x) < z + (j+1)\tau\}, \end{cases} \qquad \forall\, j \in \mathbb{Z}.$$

Assume that

$$\frac{|[a,b] \cap B_0|}{b-a} = \lambda, \quad \frac{|[a,b] \cap A_0|}{b-a} \le \lambda/4,$$

and

$$\int_a^b \int_a^b_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\, dx\, dy < +\infty.$$

Then for each $r > 4/\lambda$, there exist $m \in \mathbb{Z}_+$, $l_m \in \mathbb{Z}$, and $[c,d] \subset [a,b]$ $(c < d)$such that

$$\begin{cases} |l_m| \le 2m, \\ \dfrac{|[c,d] \cap A_{l_m}|}{d-c} \dfrac{|[c,d] \cap A_{l_m+2}|}{d-c} \ge \dfrac{1}{4}[\lambda/(4r)]^{m+1}, \\ (d-c) \le 4^m [4/(\lambda r)]^{\frac{m(m-1)}{2}} (b-a). \end{cases}$$

Lemma 4 *[18, Corollary 6] Let $1 < p < +\infty$ and $0 < \lambda_0 \le \lambda \le 1/2$. Under the assumptions of Lemma 3, there exist $m \in \mathbb{Z}_+$ and $l_m \in \mathbb{Z}$ such that*

$$|l_m| \le 2m$$

and

$$\iint\limits_{\substack{x\in[a,b]\cap A_{l_m}\\ y\in[a,b]\cap A_{l_m+2}}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \ge C_{p,\lambda_0} m(b-a)^{1-p},$$

for some positive constant C_{p,λ_0} depending only on p and λ_0.

Remark 12 *Lemmas 3 and 4, which will be used in the proof of part b) of Lemma 5, are presented in [18] (see [18, Lemma 5] and [18, Corollary 6]) only for the case $\lambda = 1/2$. However their proofs are almost the same as the ones of [18, Lemma 5] and [18, Corollary 6]. The details are left to the reader.*

The following lemma is one of the main ingredients in the proof of Theorem 1.

Lemma 5 *Let $p \ge 1$, $0 < \tau_0 < \frac{1}{2}$, and g be a real measurable function defined on a bounded interval I. Suppose that there exist $0 < \tau_0 < \tau < \frac{1}{2}$, $c_1 < c_2$, and two non-empty sub-intervals I_1 and I_2 of I such that*

$$\big|\{x \in I_1;\, g(x) < c_1\}\big| \ge \tau|I_1| \quad \textit{and} \quad \big|\{x \in I_2;\, g(x) > c_2\}\big| \ge \tau|I_2|. \tag{2.1}$$

Then there exists some positive constant C depending only on p and τ_0 such that:

a) If $p \ge 1$, we have

$$\int_I\int_I\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \ge C_{p,\tau_0}(c_2-c_1)^p|I|^{1-p}, \quad \forall\,\delta \in (0,\delta_0). \tag{2.2}$$

b) If $p > 1$, $\delta \in (0,\delta_0)$, and

$$\int_I\int_I\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy < +\infty,$$

we have

$$\int_I\int_I\limits_{\delta<|g(x)-g(y)|<10\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \ge C_{p,\tau_0}(c_2-c_1)^p|I|^{1-p}. \tag{2.3}$$

Here $\delta_0 = \frac{\tau(c_2-c_1)}{200}\min\left\{\frac{|I_1|}{|I|}, \frac{|I_2|}{|I|}\right\}$.

Remark 13 *Lemma 5 is a variant of [4, Lemma 2] and [18, Lemma 6] stating that the limit of the LHS of (2.2) and (2.3) as δ goes to 0 gives upper bounds of $|I|^{1-p}(\operatorname{ess\,sup}_I g - \operatorname{ess\,inf}_I g)$ up to a constant. Lemma 5 gives the range of δ (independent of g) for which (2.2) and (2.3) hold if (2.1) is satisfied. The proof of Lemma 5 completely borrows arguments used in the ones of [4, Lemma 2] and [18, Lemma 6].*

In what follows, the notation $a \lesssim b$ means that there exists a positive constant c depending only on N and p, such that $a \leq cb$. The notation $a \gtrsim b$ means that $b \lesssim a$ and the notation $a \approx b$ means that $a \lesssim b$ and $b \lesssim a$.

Proof. By scaling and translating, one can assume as well that $I = [0, 1]$, $c_1 = 0$, and $c_2 = 1$. Take $\delta \in (0, \frac{\tau}{200}) \min\{|I_1|, |I_2|\}$ and $K \in \mathbb{Z}_+$ such that

$$\delta < 2^{-K} \leq 2\delta. \tag{2.4}$$

Denote

$$J = \Big\{ j \in \mathbb{Z}_+; \frac{1}{4} < j2^{-K} < \frac{3}{4} \Big\}.$$

Then

$$\mathrm{card}(J) \geq 2^{K-1} - 2 \approx \frac{1}{\delta}. \tag{2.5}$$

For each j, define the following sets

$$A_j = \left\{ x \in [0, 1];\ (j-1)2^{-K} \leq g(x) < j2^{-K} \right\},$$

$$B_j = \bigcup_{j' < j} A_{j'}, \text{ and } C_j = \bigcup_{j' > j} A_{j'},$$

so that $B_j \times C_j \subset \left[|g(x) - g(y)| \geq 2^{-K} \right] \subset [|g(x) - g(y)| > \delta]$.
Set

$$G = \{ j \in J;\ |A_j| < 2^{-K+2} \}. \tag{2.6}$$

Since the collection (A_j) is disjoint, it follows from (2.5) that

$$\mathrm{card}(G) \geq 2^{K-2} - 3 \approx \frac{1}{\delta}, \tag{2.7}$$

For each $j \in G$, set $\lambda_{1,j} = |A_j| > 0$ by Lemma 1. We claim that there exist $s_{1,j}$ and $s_{2,j}$ in $[4\lambda_{1,j}, 1 - 4\lambda_{1,j}]$ such that

$$|[s_{1,j} - 4\lambda_{1,j}, s_{1,j} + 4\lambda_{1,j}] \cap B_j| > \tau/2 \quad \text{and} \quad |[s_{2,j} - 4\lambda_{1,j}, s_{2,j} + 4\lambda_{1,j}] \cap (A_j \cup C_j)| > \tau/2. \tag{2.8}$$

We first prove that there exists $s_{1,j} \in [4\lambda_{1,j}, 1 - 4\lambda_{1,j}]$ such that

$$|[s_{1,j} - 4\lambda_{1,j}, s_{1,j} + 4\lambda_{1,j}] \cap B_j| > \tau/2 \tag{2.9}$$

by contradiction. Suppose that

$$|[t - 4\lambda_{1,j}, t + 4\lambda_{1,j}] \cap B_j| < \tau/2 \quad \forall\, t \in [4\lambda_{1,j}, 1 - 4\lambda_{1,j}]. \tag{2.10}$$

Set $t_0 = 4\lambda_{1,j} + \inf_{x \in I_1} x$ and $t_{i+1} = t_i + 8\lambda_{1,j}$ for $i \geq 0$. Let n be such that $t_n + 4\lambda_{1,j} \in I_1$ and $t_{n+1} + 4\lambda_{1,j} \notin I_1$. We have

$$|I_1 \cap B_j| \leq \sum_{i=0}^{n} \left| \left[t_i - 4\lambda_{1,j}, t_i + 4\lambda_{1,j} \right] \cap B_j \right| + 8\lambda_{1,j}.$$

We deduce from (2.10) that

$$|I_1 \cap B_j| \leq \tau |I_1|/2 + 8\lambda_{1,j}. \tag{2.11}$$

However since $j \in G$, $2^{-K} \leq \delta \leq \tau |I_1|/200$, it follows from (2.6) that

$$8\lambda_{1,j} \leq 8.2^{-K+2} = 32.2^{-K} \leq 64\delta < \tau |I_1|/2. \tag{2.12}$$

Combining (2.11) and (2.12) yields that

$$|I_1 \cap B_j| < \tau |I_1|.$$

This contradicts the fact that $|I_1 \cap B_j| \geq |I_1 \cap B_1| > \tau |I_1|$.

Thus there exists $s_{1,j} \in [4\lambda_{1,j}, 1-4\lambda_{1,j}]$ such that $|[s_{1,j} - 4\lambda_{1,j}, s_{1,j} + 4\lambda_{1,j}] \cap B_j| > \tau/2$.

Similarly, since $|I_2 \cap (A_j \cup C_j)| > \tau |I_2|$, there exists $s_{2,j} \in [4\lambda_{1,j}, 1-4\lambda_{1,j}]$ such that $|[s_{2,j} - 4\lambda_{1,j}, s_{2,j} + 4\lambda_{1,j}] \cap (A_j \cup C_j)| > \tau/2$. Therefore (2.8) is proved.

From (2.8) it is clear that there exists $t_{1,j} \in [4\lambda_{1,j}, 1-4\lambda_{1,j}]$ such that

$$\tau/2 \leq |[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap B_j| \leq 1 - \tau/2. \tag{2.13}$$

On the other hand, since $|A_j| \leq 2^{-K+2} \leq 8\delta \leq \tau/8$, it follows that

$$|[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}] \cap A_j| \leq |A_j| \leq \tau/8. \tag{2.14}$$

Combining (2.13) and (2.14) and using Lemma 2, we have, for some $t_j, \lambda_j > 0$,

$$\tau/2 \leq \frac{|[t_j - 4\lambda_j, t_j + 4\lambda_j] \cap B_j|}{8\lambda_j} \leq 1 - \tau/2, \tag{2.15}$$

and

$$\tau/32 \leq \frac{|[t_j - 4\lambda_j, t_j + 4\lambda_j] \cap A_j|}{8\lambda_j} \leq \tau/8. \tag{2.16}$$

Proof of a): $p \geq 1$. Set $\lambda = \inf_{j \in G} \lambda_j$ ($\lambda > 0$ since G is finite). Define P_m as follows

$$P_m = \{j \in G;\, 2^{m-1}\lambda \leq \lambda_j < 2^m \lambda\}, \quad \forall\, m \geq 1. \tag{2.17}$$

Then $G = \bigcup_{i=1}^n P_m$ for some n. From (2.7), we have

$$\sum_{m=1}^n \mathrm{card}(P_m) \gtrsim \frac{1}{\delta}. \tag{2.18}$$

For each m ($1 \leq m \leq n$), since $A_j \cap A_k = \emptyset$ for $j \neq k$, it follows from (2.16) that there exists $J_m \subset P_m$ such that

$$a)\, \mathrm{card}(J_m) \gtrsim \mathrm{card}(P_m) \quad \text{and} \quad b)\, |t_i - t_j| > 2^{m+3}\lambda, \quad \forall\, i, j \in J_m, i \neq j. \tag{2.19}$$

Combining (2.17), (2.18), and (2.19) yields

$$[t_i - 4\lambda_i, t_i + 4\lambda_i] \cap [t_j - 4\lambda_j, t_j + 4\lambda_j] = \emptyset, \quad \forall\, i, j \in J_m, i \neq j, \tag{2.20}$$

and

$$\sum_{m=1}^n \mathrm{card}(J_m) \gtrsim \frac{1}{\delta}. \tag{2.21}$$

Set $U_0 := \emptyset$ and, for $m = 1, 2, \ldots, n$,

$$\begin{cases} L_m = \big\{j \in J_m;\, \big|[t_j - 4\lambda_j, t_j + 4\lambda_j] \setminus U_{m-1}\big| \geq (8 - \tau/16)\lambda_j\big\}, \\ U_m = \big(\bigcup_{j \in L_m} [t_j - 4\lambda_j, t_j + 4\lambda_j]\big) \cup U_{m-1}, \\ a_m = \mathrm{card}(J_m) \quad \text{and} \quad b_m = \mathrm{card}(L_m). \end{cases} \tag{2.22}$$

From (2.20), we have

$$\sum_{j\in J_m\setminus L_m} |[t_j-4\lambda_j, t_j+4\lambda_j]| \lesssim \frac{1}{\tau}|U_{m-1}|.$$

Hence since $L_m \subset J_m \subset P_m$, it follows from (2.17) that

$$2^{m-1}(a_m-b_m) \lesssim \frac{1}{\tau}\sum_{i=1}^{m-1} 2^i b_i,$$

which shows that

$$a_m \lesssim b_m + \frac{8}{\tau}\sum_{i=1}^{m-1} 2^{(i-m)} b_i.$$

Consequently,

$$\sum_{m=1}^n a_m \lesssim \sum_{m=1}^n b_m + \frac{8}{\tau}\sum_{m=1}^n\sum_{i=1}^{m-1} 2^{(i-m)} b_i = \sum_{m=1}^n b_m + \frac{8}{\tau}\sum_{i=1}^n b_i \sum_{m=i+1}^n 2^{(i-m)}.$$

Since $\sum_{i=1}^\infty 2^{-i}=1$, it follows from from (2.18) and (2.21) that

$$\sum_{m=1}^n b_m \gtrsim \tau \sum_{m=1}^n a_m \gtrsim \frac{\tau}{\delta}. \tag{2.23}$$

Combining (2.15), (2.16), (2.22), and (2.23) yields

$$\underset{|g(x)-g(y)|>\delta}{\int_I\int_I} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \ge \sum_{m=1}^n \sum_{j\in L_m} \underset{\substack{([t_j-4\lambda_j,t_j+4\lambda_j]\setminus U_{m-1})^2\\ x\in B_j,\, y\in C_j}}{\iint} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy$$

$$\gtrsim \delta^p \sum_{m=1}^n b_m\tau/\delta^{p-1} \gtrsim \tau^2.$$

This implies the conclusion of Lemma 5 in this case.

Proof of $b)$: $p>1$. Take $j\in G$. By Lemma 4, we deduce from (2.15) and (2.16) that there exist $m_j\in\mathbb{Z}_+$ and $l_j\in\mathbb{Z}$ such that

$$|l_j - j| \le 2m_j \tag{2.24}$$

and

$$\underset{\substack{x\in\mathcal{I}\cap A_{l_j}\\ y\in\mathcal{I}\cap A_{l_j+2}}}{\iint} \frac{1}{|x-y|^{p+1}}\,dx\,dy \gtrsim c_{\tau_0} m_j \lambda_j^{1-p} \gtrsim c_{\tau_0} m_j \delta^{1-p}. \tag{2.25}$$

Hereafter c_{τ_0} denotes a positive constant depending only on τ_0. The last inequality follows from the fact that $\lambda_j \lesssim \delta$.

Set $i_0=-1$ and

$$C_i = \{j\in G;\, l_j = i\}, \quad \forall\, i\in\mathbb{Z}.$$

For each $n\ge 1$, if

$$\Big\{i\in\mathbb{Z};\, i\ge i_{n-1}+1 \text{ and } C_i\ne\emptyset\Big\}\ne\emptyset,$$

then set

$$\begin{cases} i_n &= \inf\Big\{i \in \mathbb{Z};\ i \ge i_{n-1}+1 \text{ and } C_i \ne \emptyset\Big\},\\ k_n &= \max\Big\{m_j;\ j \in G \text{ and } l_j = i_n\Big\}.\end{cases}$$

From (2.24), we have

$$k_n \gtrsim \operatorname{card}\{j \in G;\ l_j = i_n\}.$$

Hence we deduce from (2.7) that

$$\sum_{n\ge 1, k_n \text{ exists}} k_n \gtrsim \operatorname{card}(G) \approx \frac{1}{\delta}. \tag{2.26}$$

On the other hand, from (2.25),

$$\int_{\mathcal{I}}\int_{\mathcal{I}}\limits_{2^{-K}\le|g(x)-g(y)|\le 3.2^{-K}} \frac{\delta^p}{|x-y|^{p-1}}\,dx\,dy \ge \sum_{n\ge1,k_n \text{ exists}} \iint\limits_{\substack{x\in\mathcal{I}\cap A_{i_n}\\ y\in\mathcal{I}\cap A_{i_n+2}}} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy$$

$$\gtrsim c_{\tau_0} \sum_{n\ge1,k_n\text{ exists}} k_n\delta. \tag{2.27}$$

Therefore the conclusion of Lemma 4 in the case $p > 1$ follows from (2.26), (2.27), and (2.4). □

2.2 Proof of Theorem 1

Step 1: $N = 1$. Let I be a bounded interval of $\mathbb{R}$. We first assume that $g \in L^\infty(I)$ and

$$\int_I\int_I\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy < +\infty;$$

and prove, if $p \ge 1$,

$$\int_I\int_I |g(x)-g(y)|^p\,dx\,dy \le C_p\Big(|I|^{p+1}\int_I\int_I\limits_{|g(x)-g(y)|>\delta}\frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p|I|^2\Big), \tag{2.28}$$

and, if $p > 1$,

$$\int_I\int_I |g(x)-g(y)|^p\,dx\,dy \le C_p\Big(|I|^{p+1}\int_I\int_I\limits_{\delta<|g(x)-g(y)|<10\delta}\frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p|I|^2\Big), \tag{2.29}$$

where C_p is a positive constant depending only on p.

By scaling, one may assume that $I = [0,1]$ and

$$|g|_{BMO(I)} = 2. \tag{2.30}$$

We recall the following fact due to John and Nirenberg [13]: There exist two universal constants c_1 and c_2 such that if $-\infty < a < b < +\infty$ and $u \in BMO([a,b])$ then

$$\Big|\{x \in (a,b);\, |u - \fint_a^b u(s)\,ds| > t\}\Big| \le c_1(b-a)\exp\Big(-\frac{c_2 t}{|u|_{BMO([a,b])}}\Big) \quad \forall\, t>0. \tag{2.31}$$

Let $0 < a < b < 1$ be such that

$$\fint_a^b \Big| g(x) - \fint_a^b g(s)\,ds \Big|\,dx \ge 1. \tag{2.32}$$

The existence of a and b follows from (2.30). Without loss of generality, one may assume that

$$\fint_a^b g\,dx = 0. \tag{2.33}$$

By (2.31), it follows from (2.30), (2.32) and (2.33) that there exist two universal constants $\tau_1 < 0$ and $\tau_2 > 0$ such that

$$\frac{1}{b-a}\big|\{x \in (a,b); g(x) < \tau_1\}\big| \gtrsim 1 \quad \text{and} \quad \frac{1}{b-a}\big|\{x \in (a,b); g(x) > \tau_2\}\big| \gtrsim 1.$$

Applying Lemma 5, we have, if $p \ge 1$,

$$\mathop{\int_a^b\int_a^b}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p \gtrsim 1 \quad \forall\,\delta > 0.$$

and, if $p > 1$,

$$\mathop{\int_a^b\int_a^b}_{\delta<|g(x)-g(y)|<10\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p \gtrsim 1 \quad \forall\,\delta > 0.$$

This completes the proof in the case $g \in L^\infty(I)$.

The proof in the general case (without assuming that $g \in L^\infty(I)$) goes as follows. Let $A > 0$ and define

$$g_A = \min\{\max\{g, -A\}, A\}. \tag{2.34}$$

Then $g_A \in L^\infty(I)$. Hence it follows from (2.28) and (2.29) that if $p \ge 1$

$$\int_I\int_I |g_A(x) - g_A(y)|^p\,dx\,dy \le C_p\Big(|I|^{p+1} \mathop{\int_I\int_I}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p|I|^2\Big) \quad \forall\,p \ge 1,$$

and if $p > 1$

$$\int_I\int_I |g_A(x) - g_A(y)|^p\,dx\,dy \le C_p\Big(|I|^{p+1} \mathop{\int_I\int_I}_{\delta<|g_A(x)-g_A(y)|<10\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy + \delta^p|I|^2\Big) \quad \forall\,p > 1.$$

By letting A go to infinity and using Fatou's lemma, the conclusion follows. $\square$

Step 2: $N \ge 2$. Let us sketch the proof in the case $N = 2$. The proof in the general case follows similarly. We present here only the proof of (1.4). The proof of (1.3) is almost the same as the one of (1.4). Without loss of generality, one may assume that $B = B_1$ the unit ball centered at the origin. Let f be an extension of g on B_8, the ball centered at 0 with radius 8, such that

$$\mathop{\int_{B_1}\int_{B_1}}_{\delta<|g(x)-g(y)|<10\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy \approx \mathop{\int_{B_8}\int_{B_8}}_{\delta<|f(x)-f(y)|<10\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy \tag{2.35}$$

and

$$\int_{B_1}\int_{B_1}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy \approx \int_{B_8}\int_{B_8}_{|f(x)-f(y)|>\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy. \tag{2.36}$$

We first note that, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$$\begin{aligned}
&|f\big(sR(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + \hat{t}R(e_2)\big)|^p \\
&\lesssim |f\big(sR(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + tR(e_2)\big)|^p + |f\big(\hat{s}R(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + \hat{t}R(e_2)\big)|^p,
\end{aligned} \tag{2.37}$$

for all $s, t \in \mathbb{R}$ and $R \in SO(2)$ i.e. R is a rotation.

On the other hand, applying Theorem 1 in the case $N = 1$, we have

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |f\big(sR(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + tR(e_2)\big)|^p\,ds\,d\hat{s}\,dt \\
&\qquad \lesssim \int_{-1}^{1} \int_{-1}^{1}\int_{-1}^{1}_{\delta<|f\left(sR(e_1)+tR(e_2)\right)-f\left(\hat{s}R(e_1)+tR(e_2)\right)|<10\delta} \frac{\delta^p}{|s-\hat{s}|^{p+1}}\,ds\,d\hat{s}\,dt + \delta^p.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |f\big(sR(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + tR(e_2)\big)|^p\,ds\,d\hat{s}\,dt \\
&\qquad \lesssim \int_{-2}^{2} \int_{-2}^{2}\int_{-2}^{2}_{\delta<|f\left(hR(e_1)+x\right)-f\left(x\right)|<10\delta} \frac{\delta^p}{|h|^{p+1}}\,dx\,d\,h + \delta^p.
\end{aligned} \tag{2.38}$$

Hereafter $x = (x_1, x_2) = x_1e_1 + x_2e_2$. Similarly, we have

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |f\big(\hat{s}R(e_1) + tR(e_2)\big) - f\big(\hat{s}R(e_1) + tR(e_2)\big)|^p\,dt\,d\hat{t}\,d\hat{s} \\
&\qquad \lesssim \int_{-2}^{2} \int_{-2}^{2}\int_{-2}^{2}_{\delta<|f\left(hR(e_2)+x\right)-f\left(x\right)|<10\delta} \frac{\delta^p}{|h|^{p+1}}\,dx\,d\,h + \delta^p.
\end{aligned} \tag{2.39}$$

Combining (2.35), (2.37), (2.38), and (2.39) yields

$$\begin{aligned}
\int_{B_1}\int_{B_1} |g(x) - g(y)|^p\,dx\,dy &\lesssim \int_{-2}^{2} \int_{-2}^{2}\int_{-2}^{2}_{\delta<|f\left(hR(e_1)+x\right)-f\left(x\right)|<10\delta} \frac{\delta^p}{|h|^{p+1}}\,dx\,d\,h \\
&\quad + \int_{-2}^{2} \int_{-2}^{2}\int_{-2}^{2}_{\delta<|f\left(hR(e_2)+x\right)-f\left(x\right)|<10\delta} \frac{\delta^p}{|h|^{p+1}}\,dx\,d\,h + \delta^p.
\end{aligned}$$

Therefore the conclusion follows after integrating two sides of the above inequality with respect to R. □

3 A variant of Rellich-Kondrachov's theorem. Proof of Theorem 2

In this section, we prove Theorem 2. The following lemma is the key of this section.

Lemma 6 *Let $g : \mathbb{R}^N \to \mathbb{R}$ be a real measurable function and Q be a cube of $\mathbb{R}^N$. Then*

$$\int_Q |g(x) - g_\varepsilon(x)|^p \, dx \lesssim \varepsilon^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q|,$$

where

$$g_\varepsilon = \frac{1}{|\varepsilon Q_1|} g * \chi_\varepsilon.$$

Here Q_1 is the unit cube centered at the origin and χ_ε is the characteristic function of εQ_1.

Henceforth aQ denotes the cube with the same center as Q and a times its length for any cube Q of $\mathbb{R}^N$.

Proof. Let $(Q_i)_{i \in I}$ be a collection of open cubes such that

$$|Q_i| = \varepsilon^N, \quad Q_i \cap Q_j = \emptyset, \quad \forall\, i \neq j, \quad \text{and} \quad Q \subset \cup_{i \in I} \overline{Q_i}. \tag{3.1}$$

Then

$$\int_Q |g(x) - g_\varepsilon(x)|^p \, dx \le \sum_{i \in I} \int_{Q_i} |g(x) - g_\varepsilon(x)|^p \, dx.$$

Hence since

$$\int_{Q_i} |g(x) - g_\varepsilon(x)|^p \, dx \le \frac{1}{\varepsilon^N} \int_{3Q_i} \int_{3Q_i} |g(x) - g(y)|^p \, dx \, dy,$$

it follows that

$$\int_Q |g(x) - g_\varepsilon(x)|^p \, dx \le \frac{1}{\varepsilon^N} \sum_i \int_{3Q_i} \int_{3Q_i} |g(x) - g(y)|^p \, dx \, dy. \tag{3.2}$$

Applying Theorem 1, we deduce from (3.1) that

$$\frac{1}{\varepsilon^N} \sum_i \int_{3Q_i} \int_{3Q_i} |g(x) - g(y)|^p \, dx \, dy \le C_{N,p} \Big(\varepsilon^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q| \Big). \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\int_Q |g(x) - g_\varepsilon(x)|^p \, dx \le C_{N,p} \Big(\varepsilon^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q| \Big).$$

□

We are ready to prove Theorem 2. We follow the standard approach used in [2] (see also [20]).

Proof of Theorem 2. Applying Lemma 6, we have, for each cube Q of $\mathbb{R}^N$,

$$\int_Q |g_n(x) - g_{n,\varepsilon}(x)|^p \, dx \le C_{N,p} \Big(\varepsilon^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g_n(x)-g_n(y)|>\delta_n} \frac{\delta_n^p}{|x-y|^{N+p}} \, dx \, dy + \delta_n^p |Q| \Big),$$

where

$$g_{n,\varepsilon} = \frac{1}{|\varepsilon Q|} g_n * \chi_\varepsilon.$$

Here Q_1 is the unit cube centered at the origin and χ_ε is the characteristic function of εQ_1. Hence

$$\lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \int_Q |g_n(x) - g_{n,\varepsilon}(x)|^p \, dx \right) = 0.$$

Thus, since (g_n) is bounded in $L^p(\mathbb{R}^N)$, by the theorem of Riesz-Frechet-Kolmogorov (see e.g. [5, Theorem IV.25]) and [5, Corollary IV.27], there exists a sub-sequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that g_{n_k} converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$. The second assertion of Theorem 2 follows from [18, Theorem 3]. □

4 A variant of Sobolev's inequality. Proof of Theorem 3

This section will be devoted to the proof of Theorem 3. One of the main ingredients of the proof is the estimate in part *b*) of Theorem 1. The proof also makes use of the theory of sharp functions and the truncation method.

We first recall the definition of the dyadic maximal function $M^\Delta g$ and the dyadic sharp function $g^{\sharp,\Delta}$ associated with g (see e.g. [24]).

Definition 1 *Let $g \in L^1_{loc}(\mathbb{R}^N)$. Then $M^\Delta g$ and $g^{\sharp,\Delta}$ are defined as follows*

$$(M^\Delta g)(x) = \sup_Q \fint_Q |g| \, dy,$$

and

$$g^{\sharp,\Delta}(x) = \sup_Q \fint_Q |g - g_Q| \, dy, \tag{4.1}$$

where the supremum is taken over all dyadic cubes Q containing x and $g_Q := \fint_Q g \, dy$.

The following result which is a consequence of Vitali's covering theorem will be used in the proof of Theorem 3.

Lemma 7 *Let $\delta > 0$, $0 < \theta_1 < \theta_2$, $h \in L^1(\mathbb{R}^N)$, and g be a real measurable function such that*

$$g(x) \le \sup_B |B|^{\theta_1} \left(\fint_B |h| \, dx \right)^{\theta_2} + \delta, \quad \forall \, x \in \mathbb{R}^N, \tag{4.2}$$

where the supremum is taken over all balls B containing x. Then

$$|\{g > t\}| \le C \|h\|_{L^1}^{\frac{\theta_2}{\theta_2 - \theta_1}} / t^{\frac{1}{\theta_2 - \theta_1}}, \quad \forall \, t > 2\delta,$$

for some positive constant C, depending only on θ_1 and θ_2.

Proof. Let $t > 2\delta$. For each $y \in \{g > t\}$, from (4.2), there exists a ball B_y containing y such that

$$t \le 2|B_y|^{\theta_1} \left(\fint_{B_y} |h| \, dx \right)^{\theta_2}.$$

It follows that

$$|B_y|^{\theta_2-\theta_1} \le \frac{2}{t}\Big(\int_{B_y} |h|\,dy\Big)^{\theta_2},$$

which implies

$$|B_y| \le \frac{C}{t^{\frac{1}{\theta_2-\theta_1}}}\Big(\int_{B_y} |h|\,dy\Big)^{\frac{\theta_2}{\theta_2-\theta_1}} < +\infty,$$

since $h \in L^1(\mathbb{R}^N)$. Hereafter in the proof, C denotes a positive constant depending only on θ_1 and θ_2. Applying Vitali's covering theorem (see e.g. [8]), there exists a denumerable collection of balls (B_i) such that $B_i \cap B_j = \emptyset$ for $i \neq j$, $\{g > t\} \subset \bigcup_i 5B_i$ and

$$|B_i| \le \frac{C}{t^{\frac{1}{\theta_2-\theta_1}}}\Big(\int_{B_i} |h|\,dy\Big)^{\frac{\theta_2}{\theta_2-\theta_1}}.$$

Here cB denotes the ball with the same center as B but c times its radius. Thus

$$|\{g > t\}| \le \sum_i \frac{C}{t^{\frac{1}{\theta_2-\theta_1}}}\Big(\int_{B_i} |h|\,dy\Big)^{\frac{\theta_2}{\theta_2-\theta_1}}.$$

Since $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\frac{\theta_2}{\theta_2-\theta_1} \ge 1$, it follows that

$$|\{g > t\}| \le \frac{C}{t^{\frac{1}{\theta_2-\theta_1}}}\Big(\int_{\mathbb{R}^N} |h|\,dy\Big)^{\frac{\theta_2}{\theta_2-\theta_1}}.$$

□

To introduce the truncation method, we need the following definition.

Definition 2 *Let $l < m$, and g be a real measurable function defined on $\mathbb{R}^N$. Consider $h_{l,m} : \mathbb{R} \to \mathbb{R}$ given by*

$$h_{l,m}(t) = \begin{cases} m - l & \text{if } m < t, \\ t - l & \text{if } l < t \le m, \\ 0 & \text{otherwise.} \end{cases}$$

and define the operator $T(l, m, \cdot)$ as follows

$$T(l, m, g)(x) = h_{l,m}\big(g(x)\big).$$

The following lemma plays an important role in the proof of Theorem 3.

Lemma 8 *Let p, $r \ge 1$, $\delta > 0$, and $g \in L^1_{loc}(\mathbb{R}^N)$. Define $g_k = T(10^k, 10^{k+2}, |g|)$ for $k \in \mathbb{Z}$. Assume that there exist a sequence of functions $(v_k) \subset L^1(\mathbb{R}^N)$ and a function $v \in L^1(\mathbb{R}^N)$ such that*

$$\Big|\Big\{g_k^{\sharp,\Delta} > t\Big\}\Big| \le \frac{\|v_k\|^r_{L^1}}{t^p}, \quad \forall\, t > \delta,\, k \in \mathbb{Z} \tag{4.3}$$

and

$$\sum_{k\in\mathbb{Z}} |v_k| \le v. \tag{4.4}$$

Then

$$\int\limits_{|g|>\lambda\delta} |g|^p\,dx \le C\|v\|^r_{L^1},$$

for positive constants C and λ, depending only on N and p.

Proof. We first recall the following estimate [24, Estimate (22), page 153]): Let $0 < b < 1$, $c > 0$, and $f \in L^1_{loc}(\mathbb{R}^N)$. Then

$$|\{M^\Delta f > \alpha,\, f^{\sharp,\Delta} \le c\alpha\}| \le \frac{2^N c}{1-b}|\{M^\Delta f > b\alpha\}|, \quad \forall\, \alpha > 0. \tag{4.5}$$

Applying (4.5) with $f = g_k$, $b = \frac{1}{10}$, $\alpha = 10^k$, and $0 < c < \frac{1}{2}$, which depends only on N and p, and is defined later, we have

$$|\{M^\Delta g_k > 10^k\}| \le c2^{N+1}|\{M^\Delta g_k > 10^{k-1}\}| + |\{g_k^{\sharp,\Delta} > c10^k\}|,$$

which implies

$$10^{kp}|\{M^\Delta g_k > 10^k\}| \le c2^{N+1}10^{kp}|\{M^\Delta g_k > 10^{k-1}\}| + 10^{kp}|\{g_k^{\sharp,\Delta} > c10^k\}|. \tag{4.6}$$

Take $k_0 \in \mathbb{Z}$ such that $c10^{k_0-2} \le \delta < c10^{k_0-1}$. Then (4.6) implies, for $m \ge k_0 + 1$,

$$\sum_{k_0}^m 10^{kp}|\{M^\Delta g_k > 10^k\}| \le c2^{N+1}\sum_{k_0}^m 10^{kp}|\{M^\Delta g_k > 10^{k-1}\}| + \sum_{k_0}^m 10^{kp}|\{g_k^{\sharp,\Delta} > c10^k\}|. \tag{4.7}$$

We first establish a lower bound for the left hand side of (4.7). Since

$$|\{M^\Delta g_k > 10^k\}| \ge |\{g_k > 10^k\}|,$$

it follows from the definition of $M^\Delta g_k$ and g_k that

$$\sum_{k_0}^m 10^{kp}|\{M^\Delta g_k > 10^k\}| \ge C_{N,p}\int_{10^{k_0+1}}^{10^{m+2}} t^{p-1}|\{|g| > t\}|\,dt \quad \forall\, m \ge k_0 + 1. \tag{4.8}$$

We next show an upper bound for the right hand side of (4.7). Using the theory of maximal functions (see e.g. [23, Theorem 1 in page 5]), we have

$$\sum_{k_0}^m 10^{kp}|\{M^\Delta g_k > 10^{k-1}\}| \le C_{N,p}\sum_{k_0}^m \int_{\mathbb{R}^N} |g_k|^p\,dx \le C_{N,p}\int_{10^{k_0}}^{10^{m+2}} t^{p-1}|\{|g| > t\}|\,dt, \tag{4.9}$$

for all $m \ge k_0 + 1$. The last inequality in (4.9) follows from the definition of g_k. On the other hand, since $r \ge 1$, it follows from (4.3) and the definition of v_k that

$$\sum_{k_0}^m 10^{kp}\big|\{g_k^{\sharp,\Delta} > c10^k\}\big| \le \frac{1}{c^p}\sum_{k_0}^m \|v_k\|_{L^1}^r \le \frac{1}{c^p}\|v\|_{L^1}^r. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$c2^{N+1}\sum_{k_0}^m 10^{kp}|\{M^\Delta g_k > 10^{k-1}\}| + \sum_{k_0}^m 10^{kp}|\{g_k^{\sharp,\Delta} > c10^k\}|$$
$$\le C_{N,p}\left(c2^{N+1}\int_{10^{k_0}}^{10^{m+2}} t^{p-1}|\{|g| > t\}|\,dt + \frac{1}{c^p}\|v\|_{L^1}^r\right), \tag{4.11}$$

which is an upper bound for the right hand side of (4.7).

From (4.7), (4.8), and (4.11), we have

$$\int_{10^{k_0+1}}^{10^{m+2}} t^{p-1}|\{|g|>t\}|\,dt \le C_{N,p}\left(c2^{N+1}\int_{10^{k_0}}^{10^{m+2}} t^{p-1}|\{|g|>t\}|\,dt + \frac{1}{c^p}\|v\|_{L^1}^r\right). \tag{4.12}$$

Take c such that $C_{N,p}c2^{N+1} = 1/2$. It follows from (4.12) that

$$\int_{10^{k_0+1}}^{10^{m+2}} t^{p-1}|\{|g|>t\}|\,dt \le C_{N,p}\Big(\int_{10^{k_0}}^{10^{k_0+1}} t^{p-1}|\{|g|>t\}|\,dt + \frac{1}{c^p}\|v\|_{L^1}^r\Big).$$

By (4.3) and (4.4), this implies

$$\int_{10^{k_0+1}}^{10^{m+2}} t^{p-1}|\{|g|>t\}|\,dt \le C_{N,p}\|v\|_{L^1}^r.$$

Letting m go to infinity, the conclusion follows. □

We are ready to give

Proof of Theorem 3. Let $k \in \mathbb{Z}$ be such that $10^k \ge \delta$. Define

$$g_k = T(10^k, 10^{k+2}, |g|)$$

(see Definition 2). From (1.4), we have

$$C_{N,p}\int_B\int_B |g_k(x)-g_k(y)|^p\,dx\,dy \le |B|^{\frac{N+p}{N}} \underset{\delta<|g_k(x)-g_k(y)|<10\delta}{\int_B\int_B} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \delta^p|B|^2, \tag{4.13}$$

for any ball B of $\mathbb{R}^N$. Define

$$h(x) = \underset{|g(y)-g(x)|>\delta}{\int_{\mathbb{R}^N}} \frac{\delta^p}{|y-x|^{N+p}}\,dy,$$

and

$$h_k(x) = h(x)\chi_{10^k<|g|\le 10^{k+2}}(x).$$

Here χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^N$. Since

$$\underset{\delta<|g_k(x)-g_k(y)|<10\delta}{\int_B\int_B} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le 2\int_B \chi_{10^k<|g|\le 10^{k+2}} \underset{|g(x)-g(y)|>\delta}{\int_B} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

it follows from (4.13) that

$$C_{N,p}g_k^{\sharp,\Delta}(x) \le \sup_Q |Q|^{1/N}\left(\fint_Q |h_k|\,dx\right)^{1/p} + \delta,$$

where the supremum is taken over all cubes Q containing x. Applying Lemma 7, we obtain

$$\big|\{g_k^{\sharp,\Delta} > t\}\big| \le C_{N,p}\|h_k\|_{L^1}^{\frac{N}{N-p}}\Big/ t^{\frac{Np}{N-p}}, \quad \forall\, t > C_{N,p}\delta,$$

According to Lemma 8, the conclusion follows. □

5 A general setting

In this section we prove Propositions 4, 5, and 6 and present their applications.

5.1 Proof of Proposition 4

We follows the approach used in the proof of Assertion (b) of [18, Theorem 1]. Since F is a non-decreasing function, we have

$$\int_B \int_B \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy \ge \sum_{n\ge 0} \int_B \int_{B \atop 2^{-n}<|g(x)-g(y)|\le 2^{-n+1}} \frac{F(2^{-n})}{|x-y|^{N+p}}\,dx\,dy. \tag{5.1}$$

On the other hand,

$$\sum_{n\ge 0} \int_B \int_{B \atop 2^{-n}<|g(x)-g(y)|\le 2^{-n+1}} \frac{F(2^{-n})}{|x-y|^{N+p}}\,dx\,dy = \sum_{n\ge 0} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{F(2^{-n})}{|x-y|^{N+p}}\,dx\,dy$$
$$- \sum_{n\ge 0} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{F(2^{-n-1})}{|x-y|^{N+p}}\,dx\,dy - \int_B \int_{B \atop |g(x)-g(y)|>2} \frac{F(1)}{|x-y|^{N+p}}\,dx\,dy.$$

This implies

$$\sum_{n\ge 0} \int_B \int_{B \atop 2^{-n}<|g(x)-g(y)|\le 2^{-n+1}} \frac{F(2^{-n})}{|x-y|^{N+p}}\,dx\,dy$$
$$= \sum_{n\ge 0} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{[F(2^{-n})-F(2^{-n-1})]}{|x-y|^{N+p}}\,dx\,dy - \int_B \int_{B \atop |g(x)-g(y)|>2} \frac{F(1)}{|x-y|^{N+p}}\,dx\,dy. \tag{5.2}$$

Combining (5.1) and (5.2) yields

$$\int_B \int_B \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy + \int_B \int_{B \atop |g(x)-g(y)|>2} \frac{F(1)}{|x-y|^{N+p}}\,dx\,dy$$
$$\ge \sum_{n\ge 0} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{[F(2^{-n})-F(2^{-n-1})]}{|x-y|^{N+p}}\,dx\,dy. \tag{5.3}$$

Applying Theorem 1, we have

$$|B|^{\frac{N+p}{N}} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy + |B|^2 \gtrsim 2^{np} \int_B \int_B |g(x)-g(y)|^p\,dx\,dy.$$

It follows that

$$|B|^{\frac{N+p}{N}} \sum_{n\ge 0} \int_B \int_{B \atop |g(x)-g(y)|>2^{-n}} \frac{[F(2^{-n})-F(2^{-n-1})]}{|x-y|^{N+p}}\,dx\,dy + \sum_{n\ge 0} [F(2^{-n})-F(2^{-n-1})]|B|^2$$
$$\gtrsim \sum_{n\ge 0} 2^{np}[F(2^{-n})-F(2^{-n-1})] \int_B \int_B |g(x)-g(y)|^p\,dx\,dy. \tag{5.4}$$

On the other hand, we have

$$\sum_{n\geq 0} 2^{np}[F(2^{-n})-F(2^{-n-1})] = F(1)+\sum_{n\geq 1}(2^{np}-2^{np-p})F(2^{-n}) \gtrsim F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt, \quad (5.5)$$

since F is non-decreasing, and

$$\sum_{n\geq 0}[F(2^{-n})-F(2^{-n-1})] = F(1). \quad (5.6)$$

Combining (5.4), (5.5), and (5.6) yields

$$|B|^{\frac{N+p}{p}} \sum_{n\geq 0} \underset{|g(x)-g(y)|>2^{-n}}{\int_B\int_B} \frac{[F(2^{-n})-F(2^{-n-1})]}{|x-y|^{N+p}}\,dx\,dy + F(1)|B|^2$$

$$\gtrsim \left(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\right)\int_B\int_B |g(x)-g(y)|^p\,dx\,dy. \quad (5.7)$$

We deduce from (5.3) and (5.7) that

$$\left(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\right)\int_B\int_B |g(x)-g(y)|^p\,dx\,dy$$

$$\lesssim |B|^{\frac{N+p}{N}}\int_B\int_B \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy + |B|^{\frac{N+p}{N}} \underset{|g(x)-g(y)|>2}{\int_B\int_B} \frac{F(1)}{|x-y|^{N+p}}\,dx\,dy + F(1)|B|^2.$$

Since F is non-decreasing, this implies

$$\left(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\right)\int_B\int_B |g(x)-g(y)|^p\,dx\,dy$$

$$\lesssim |B|^{\frac{N+p}{N}}\int_B\int_B \frac{F(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy + F(1)|B|^2.$$

□

5.2 Proofs of Propositions 5 and 6

Proof of Proposition 5. Applying the same method as in Lemma 6, but, using Proposition 4 instead of Theorem 1, we have

$$\int_Q |g_n - g_{n,\varepsilon}|^p\,dx \lesssim \varepsilon^p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g_n(x)-g_n(y)|)}{|x-y|^{N+p}}\,dx\,dy + F_n(1)|Q|.$$

for any cube Q of $\mathbb{R}^N$. Here $g_{n,\varepsilon}$ is defined as in the proof of Lemma 6. It follows that

$$\lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty}\int_Q |g_n(x)-g_{n,\varepsilon}(x)|^p\,dx\right) = 0.$$

Therefore, there exist a sub-sequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that g_{n_k} converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$ (see the proof of Theorem 2). □

Proof of Proposition 6. The conclusion of Proposition 6 follows from Proposition 4 by applying the same method used in the proof of Theorem 3 (see (5.3)). The details of the proof are left for the reader. □

5.3 Applications

In this section, we give some applications of Propositions 4, 5, and 6.

Set

$$F_\varepsilon(t) = \begin{cases} \varepsilon t^{p+\varepsilon} & \text{if } 0 \le t \le 1, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then F_ε is non-decreasing, $\int_0^1 F_\varepsilon(t) t^{-(p+1)}\, dt + F_\varepsilon(1) = 1$, and $\lim_{\varepsilon \to 0} F_\varepsilon(t) = 0$ for all $t > 0$. As a consequences of Propositions 4, 5, and 6, we have

Corollary 1 *Let $p \ge 1$, B be a ball of $\mathbb{R}^N$, and $g \in L^p(B)$. Then there exists a constant $C > 0$, depending only on N and p, such that*

$$\begin{aligned} & C \int_B \int_B |g(x) - g(y)|^p \, dx \, dy \\ & \le |B|^{\frac{N+p}{N}} \underset{|g(x)-g(y)| \le 1}{\int_B \int_B} \frac{\varepsilon |g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy + |B|^{\frac{N+p}{N}} \underset{|g(x)-g(y)| > 1}{\int_B \int_B} \frac{\varepsilon}{|x-y|^{N+p}} \, dx \, dy + \varepsilon |B|^2. \end{aligned}$$

Corollary 2 *Let (g_n) be a sequence of functions in $L^p(\mathbb{R}^N)$ $(1 \le p < N)$ and (ε_n) be a positive sequence converging to 0. Assume that*

$$\liminf_{n \to \infty} \left(\underset{|g(x)-g(y)| \le 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\varepsilon_n |g(x) - g(y)|^{p+\varepsilon_n}}{|x-y|^{N+p}} \, dx \, dy + \underset{|g(x)-g(y)| > 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\varepsilon_n}{|x-y|^{N+p}} \, dx \, dy \right) < +\infty.$$

Then there exist a subsequence (g_{n_k}) of (g_n) and $g \in L^p(\mathbb{R}^N)$ such that (g_{n_k}) converges to g in $L^p_{\text{loc}}(\mathbb{R}^N)$.

Corollary 3 *Let $0 < \varepsilon < 1$, $1 < p < N$ and $g \in L^p(\mathbb{R}^N)$. Assume that*

$$\underset{|g(x)-g(y)| \le 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\varepsilon |g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy + \underset{|g(x)-g(y)| > 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\varepsilon}{|x-y|^{N+p}} \, dx \, dy < +\infty.$$

Then $g \in L^q(\mathbb{R}^N)$ with $q = \frac{Np}{N-p}$ and there exist two positive constants C and λ depending only on N and p, such that

$$\Big(\int_{|g| > \lambda \varepsilon} |g|^q \, dx \Big)^{\frac{1}{q}} \le C \Big(\underset{|g(x)-g(y)| \le 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{|g(x) - g(y)|^{p+\varepsilon}}{|x-y|^{N+p}} \, dx \, dy + \underset{|g(x)-g(y)| > 1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+p}} \, dx \, dy \Big)^{\frac{1}{p}}.$$

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