Γ-convergence, Sobolev norms, and BV functions

Hoai-Minh Nguyen *

June 22, 2010

Abstract

We prove that the family of functionals \((I_\delta)\) defined by

\[
I_\delta(g) = \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \quad \forall \, g \in L^p(\mathbb{R}^N),
\]

for \(p \geq 1\) and \(\delta > 0\), Γ-converges in \(L^p(\mathbb{R}^N)\), as \(\delta\) goes to 0, when \(p \geq 1\). Hereafter \(|\cdot|\) denotes the Euclidean norm of \(\mathbb{R}^N\). We also introduce a characterization for BV functions which has some advantages in comparison with the classic one based on the notion of essential variation on a.e. line.

Contents

1 Introduction 2

2 Preliminaries 5

3 Proof of Property (G2) 11

4 Proof of Proposition 2 in the case \(p > 1\) 17

5 A characterization of BV functions 20

6 Proof of Proposition 2 in the case \(p = 1\) 24

6.1 Another definition of \(C_{N,1}^1\) 24

6.2 Some useful lemmas 27

6.3 Proof of Proposition 2 in the case \(p = 1\) 31

*Courant Institute, New York University, 251 Mercer St., New York, NY, 10012, hoaiminh@cims.nyu.edu
1 Introduction

Recently, the following new characterization of Sobolev spaces was established in [12, Theorem 2] and [4, Theorem 1].

**Theorem 1** Let $N \geq 1$, $1 < p < + \infty$, and $g \in L^p(\mathbb{R}^N)$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if

$$\lim_{\delta \to 0^+} I_\delta(g) < +\infty.$$

Moreover,

$$\lim_{\delta \to 0^+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

where $K_{N,p}$ is given by

$$K_{N,p} = \int_{S^{N-1}} |e \cdot \sigma|^p \, d\sigma,$$

for any $e \in S^{N-1}$.

We recall that when $p = 1$:

a) If $g \in L^1(\mathbb{R}^N)$ and $\lim_{\delta \to 0^+} I_\delta(g) < +\infty$, then $g \in BV(\mathbb{R}^N)$ (see [4, 15]).

b) There exists $g \in W^{1,1}(\mathbb{R})$ such that $\lim_{\delta \to 0^+} I_\delta(g) = +\infty$ (example communicated to us by A. Ponce; see [12]).

This characterization is distinct from the one of J. Bourgain, H. Brezis, and P. Mironescu [2] (see also [3]) but it is inspired by the results of [2]. Quantities similar to $I_\delta$ appear in new estimates for the degree (see [3, 14, 7, 8]). Further results related to Theorem 1 are presented in [15] and in recent work of D. Chiron [9]. In [16], some results in the spirit of the Poincaré inequality and the Sobolev inequality, where $\int_{\mathbb{R}^N} |\nabla g|^p$ is replaced by $I_\delta(g)$, are established.

Let $p \geq 1$ and $N \geq 1$. Define, for $g \in L^p(\mathbb{R}^N)$,

$$J(g) = \begin{cases} \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } p = 1 \text{ and } g \in BV(\mathbb{R}^N), \end{cases}$$

A natural question raised by H. Brezis (personal communication) is whether $(I_\delta)$ $\Gamma$-converges in $L^p(\mathbb{R}^N)$ to $J$ in the sense of E. De Giorgi for $p > 1$ (see e.g. [5, 11] for an introduction of $\Gamma$-convergence). We recall that a family $(I_\delta)_{\delta \in (0,1)}$ of functionals defined on $L^p(\mathbb{R}^N)$ $\Gamma$-converges in $L^p(\mathbb{R}^N)$ ($p \geq 1$), as $\delta$ goes to 0, to a functional $I$ defined on $L^p(\mathbb{R}^N)$ if and only if the following two conditions are satisfied:

(G1) For each $g \in L^p(\mathbb{R}^N)$ and for every family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, one has

$$\lim_{\delta \to 0} I_\delta(g_\delta) \geq I(g).$$
(G2) For each \( g \in L^p(\mathbb{R}^N) \), there exists a family \((g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)\) such that \( g_\delta \) converges to \( g \) in \( L^p(\mathbb{R}^N) \) as \( \delta \) goes to 0, and
\[
\lim_{\delta \to 0} I_\delta(g_\delta) \leq I(g).
\]

Surprisingly, \((I_\delta)\) does not \( \Gamma \)-converge to \( J \) in \( L^p(\mathbb{R}^N) \) for \( p > 1 \) but it \( \Gamma \)-converges to \( \lambda J \) for some \( 0 < \lambda < 1 \); the same fact holds for the case \( p = 1 \). More precisely, we have

**Theorem 2** Let \( p \geq 1 \) and \( N \geq 1 \). Then \((I_\delta) \Gamma \)-converges in \( L^p(\mathbb{R}^N) \) to the functional \( I \) defined by, for any \( g \in L^p(\mathbb{R}^N) \),
\[
I(g) = \begin{cases} 
C_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \\
+p & \text{otherwise.}
\end{cases}
\]

Here the constant \( C_{N,p} \) is defined by (1.3) below and satisfies
\[
0 < C_{N,p} < \frac{1}{p} K_{N,p}.
\]

For \( p \geq 1 \) and \( N \geq 1 \), the definition of the constant \( C_{N,p} \) is the following
\[
C_{N,p} := \inf \lim_{\delta \to 0} \iint_{Q^2} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy,
\]

where the infimum is taken over all families of measurable functions \((h_\delta)_{\delta\in(0,1)}\) defined on the unit open cube \( Q \) of \( \mathbb{R}^N \) such that \( h_\delta \) converges to \( h(x) = \frac{x_1 + \cdots + x_N}{\sqrt{N}} \) in (Lebesgue) measure on \( Q \) as \( \delta \) goes to 0. We recall here that a family of measurable functions \((h_\delta)_{\delta\in(0,1)}\) defined on a measurable subset \( B \) of \( \mathbb{R}^N \) is said to converge in measure on \( B \) to a measurable function \( h \) defined on \( B \) if and only if for any \( \varepsilon > 0 \),
\[
\lim_{\delta \to 0} |\{x \in B; |h_\delta(x) - h(x)| > \varepsilon\}| = 0.
\]

Henceforth, for a measurable subset \( A \) of \( \mathbb{R}^N \), \(|A|\) denotes the Lebesgue measure of \( A \).

The proof of Theorem 2 is divided into three steps:

**Step 1:** Proof of Property (G2).

The main steps of the proof of Property (G2) are:

(a) We show that (Lemma 2 in Section 2) there exists a family \((h_\delta)_{\delta\in(0,1)} \subset L^p(Q)\) such that \( h_\delta \) converges to \( h(x) = \frac{x_1 + \cdots + x_N}{\sqrt{N}} \) in \( L^p(Q) \) and (compare with (1.3))
\[
\lim_{\delta \to 0} \iint_{|h_\delta(x)-h_\delta(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq C_{N,p}.
\]
We prove Property (G2) in the case $g$ is “continuous and piecewise linear” with compact support (Lemma 7) and then obtain Property (G2) in the general case by a density argument. The proof of (b) is presented in Section 3.

**Step 2**: Proof of Property (G1).

Property (G1) is a consequence of the following two propositions:

**Proposition 1** Let $p \geq 1$, $N \geq 1$, and $g \in L^p(\mathbb{R}^N)$. Assume that there exists a family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ and

$$\lim_{\delta \to 0} I_\delta(g_\delta) < +\infty.$$ 

Then $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ (resp. $g \in BV(\mathbb{R}^N)$ if $p = 1$); moreover

$$J(g) \leq C \lim_{\delta \to 0} I_\delta(g_\delta),$$

for some $C > 0$ depending only on $N$ and $p$.

Proposition 1 has been proved in [15] (see [15, Theorem 3]); the proof in [15] relies heavily on the ideas of [4].

**Proposition 2** Let $p \geq 1$ and $N \geq 1$. Then for any $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ or $g \in BV(\mathbb{R}^N)$ if $p = 1$, and for any family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, we have

$$\lim_{\delta \to 0} I_\delta(g_\delta) \geq I(g).$$

The proof of Proposition 2 for the case $p > 1$, which is presented in Section 4, follows from the definition of $C_{N,p}$ and the fact that any function in $W^{1,p}(\mathbb{R})$ is locally approximately linear in the sense of measure (see e.g. [10, Theorem 4 on page 223] and the remark below it). When $p = 1$, we can not directly apply the method used in the case $p > 1$. In this case, the proof, which is presented in Section 6, relies on Proposition 4, Lemma 14 and a new characterization of BV functions which we introduce in Section 5 (see Proposition 5). This characterization is based on and generalizes the notion of essential variation in the one dimensional case. It has some advantages in comparison with the classic one (see e.g. [10, Theorem 2 on page 220], [11, Section 3.11]), which is based on the notion of essential variation on a.e. line.

**Step 3**: Proof of (1.2). Inequality (1.2) is proved in [13].

Concerning the constant $C_{N,p}$, we have the following questions

**Open question 1** What is the explicit value of $C_{N,p}$?

When $N = 1$, we made a guess in [13]:

**Open question 2** Does

$$C_{1,p} = \begin{cases} 
\frac{2}{p(p-1)} \left(1 - \frac{1}{2^{p-1}}\right) & \text{if } p > 1, \\
2 \ln 2 & \text{if } p = 1.
\end{cases}$$

The results of this paper were announced in [13].
2 Preliminaries

In this section, we prove some useful lemmas which will be used later.

Lemma 1 Let \( N \geq 1, \ p \geq 1, \ A \) be a measurable subset of \( \mathbb{R}^N \), and \( f \) and \( g \) be two measurable functions defined on \( A \). Define \( h_1 = \min(f, g) \) and \( h_2 = \max(f, g) \). Then

\[
\int_A \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq \int_A \frac{\delta^p}{|f(x)-f(y)|^{N+p}} \, dx \, dy
\]

and

\[
\int_A \frac{\delta^p}{|y-x|^{N+p}} \, dx \, dy \leq \int_A \frac{\delta^p}{|f(x)-f(y)|^{N+p}} \, dx \, dy
\]

where

\[
B_1 = \{ x \in A; f(x) \leq g(x) \} \quad \text{and} \quad B_2 = \{ x \in A; f(x) \geq g(x) \}.
\]

Moreover, if \( g \) is Lipschitz on \( A \) with a Lipschitz constant \( L \), then

\[
\int_A \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq \int_A \frac{\delta^p}{|f(x)-f(y)|^{N+p}} \, dx \, dy + CL|A \setminus B_1| \tag{2.3}
\]

and

\[
\int_A \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq \int_A \frac{\delta^p}{|f(x)-f(y)|^{N+p}} \, dx \, dy + CL|A \setminus B_2| \tag{2.4}
\]

Hereafter in this paper, \( C \) denotes a positive constant depending only on \( N \) and \( p \).

Proof: It suffices to prove (2.1) and (2.3) since (2.2) and (2.4) follow from (2.1) and (2.3) easily.

We first prove (2.1). If \( x, y \in B_1 \) then \( |h_1(x) - h_1(y)| = |f(x) - f(y)| \). Otherwise \( x \notin B_1 \) or \( y \notin B_1 \). Then

\[
|h_1(x) - h_1(y)| \leq \max(|f(x) - f(y)|, |g(x) - g(y)|). \tag{2.1}
\]

Hence (2.1) follows.

To obtain (2.3) from (2.1), we remark that

\[
\int_{A \setminus B_1} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq 2 \int_{A \setminus \{B_1 \}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy,
\]
and
\[ \int\int_{A \times (A \setminus B_1)} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \int\int_{(A \setminus B_1) \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq CL^p|A \setminus B_1|, \]
for any Lipschitz function \( g \) on \( A \) with a Lipschitz constant \( L \).

Here is an obvious consequence of Lemma 1 which will be used several times later.

**Corollary 1** Let \( N \geq 1, p \geq 1, -\infty \leq m_1 < m_2 \leq +\infty \), \( A \) be a measurable subset of \( \mathbb{R}^N \), and \( f \) be a measurable function defined on \( A \). Define \( h = \min(\max(f, m_1), m_2) \). Then
\[ \int\int_{A^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \int\int_{A^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy. \] (2.5)

**Remark 1** Estimate (2.5) was observed and used in [12] and [4].

Another useful and obvious consequence of Lemma 1 is the following.

**Corollary 2** Let \( N \geq 1, p \geq 1 \), \( A \) be a measurable subset of \( \mathbb{R}^N \), \( f \) and \( g \) be two measurable functions defined on \( A \), and \( c \) be a positive number. Define \( h = \min(\max(f, g - c), g + c) \). Suppose that \( g \) is Lipschitz on \( A \) with a Lipschitz constant \( L \). Then
\[ \int\int_{A^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \int\int_{A^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy + CL^p|B|, \]
where
\[ B := \{ x \in A; |f(x) - g(x)| > c \}. \]

An important application of Corollary 2 is the following.

**Corollary 3** Let \( N \geq 1, p \geq 1 \), \( A \) be a measurable subset of \( \mathbb{R}^N \), \( g \) be a Lipschitz function defined on \( A \), \( (\delta_n)_{n \in \mathbb{N}} \) be a sequence of positive numbers converging to 0, and \( (g_n)_{n \in \mathbb{N}} \) be a sequence of measurable functions defined on \( A \) such that \( g_n \) converges to \( g \) in measure on \( A \). Then there exists a sequence of measurable functions \( h_n \) defined on \( A \) such that \( h_n \) converges to \( g \) uniformly on \( A \) and
\[ \lim_{n \to \infty} \int\int_{|h_n(x) - h_n(y)| > \delta_n} \frac{\delta^p_n}{|x - y|^{N+p}} \, dx \, dy \leq \lim_{n \to \infty} \int\int_{|g_n(x) - g_n(y)| > \delta_n} \frac{\delta^p_n}{|x - y|^{N+p}} \, dx \, dy. \]

**Proof:** Since \( g_n \) converges to \( g \) in measure on \( A \), there exists a sequence of positive numbers \( (c_n)_{n \in \mathbb{N}} \) converging to 0 such that
\[ \lim_{n \to \infty} |A_n| = 0, \]
where
\[ A_n := \{x \in A; |g_n(x) - g(x)| > c_n\}. \]
Define \( h_n = \min(\max(g_n, g - c_n), g + c_n) \). Applying Corollary 2, we have
\[ \int\int_{A_n^2} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy \leq \int\int_{A_n^2} \frac{\delta_n^p}{|g_n(x) - g_n(y)|^{N+p}} \, dx \, dy + CL^p|A_n|, \]
where \( L \) is a Lipschitz constant of \( g \). Therefore,
\[ \lim_{n \to \infty} \int\int_{Q} \frac{\delta_n^p}{|x - y|^{N+p}} \, dx \, dy \leq \lim_{n \to \infty} \int\int_{Q_n^2} \frac{\delta_k^p}{|x - y|^{N+p}} \, dx \, dy. \]

Using Corollary 3, we can prove the following lemma which plays an important role in the proof of Property (G2).

**Lemma 2** Let \( N \geq 1, p \geq 1, Q \) be the unit cube of \( \mathbb{R}^N \), and \( g \equiv \sum_{i=1}^{N} x_i \). Then there exist a family of measurable functions \( (g_\delta)_{\delta \in (0,1)} \) defined on \( Q \) and a family of positive numbers \( (c_\delta)_{\delta \in (0,1)} \) converging to 0 such that \( c_\delta \geq \sqrt{\delta}, |g_\delta(x) - g(x)| \leq 2Nc_\delta, g_\delta \) is Lipschitz on \( Q_{c_\delta} \) with a Lipschitz constant 1, and
\[ \lim_{\delta \to 0} \int\int_{Q} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p}. \]

Hereafter, for \( c > 0, Q_c \) is defined by
\[ Q_c := \{x \in Q; \text{dist}_{\infty}(x, \partial Q) \leq c\}, \]
with
\[ \text{dist}_{\infty}(x, A) := \inf_{y \in A} \sup_{i=1, \ldots, N} |x_i - y_i|. \]
for any set \( A \subset \mathbb{R}^N \).

**Proof:** It is standard to see that there exist a sequence of positive numbers \( (\delta_k)_{k \in \mathbb{N}} \) converging to 0, and a sequence of measurable functions \( (g_k)_{k \in \mathbb{N}} \) converging to \( g \) in measure on \( Q \) such that
\[ \lim_{k \to \infty} \int\int_{Q} \frac{\delta_k^p}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p}. \quad (2.6) \]
Using Corollary 3, one may assume that \( g_k \) converges to \( g \) uniformly on \( Q \) as \( k \) goes to infinity. Set
\[ c_k = \max(\sup_{x \in Q} |g_k(x) - g(x)|, \sqrt{\delta_k}). \quad (2.7) \]
Define
\[
\begin{align*}
g_{1,k} &= \min \left( \max \left( g_{0,k}(x), g(0, x_2, \ldots, x_N) + 2c_k \right), g(1, x_2, \ldots, x_N) - 2c_k \right), \\
g_{2,k} &= \min \left( \max \left( g_{1,k}(x), g(x_1, 0, \ldots, x_N) + 4c_k \right), g(x_1, 1, \ldots, x_N) - 4c_k \right), \\
&\quad \vdots \\
g_{N,k} &= \min \left( \max \left( g_{N-1,k}(x), g(x_1, \ldots, x_{N-1}, 0) + 2Nc_k \right), g(x_1, \ldots, x_{N-1}, 1) - 2Nc_k \right),
\end{align*}
\]
with the notation \( g_{0,k} = g_k \). Then, since
\[
\begin{align*}
g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) + 2ic_k &\leq g(x) + 2ic_k \\
g(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_N) - 2ic_k &\geq g(x) - 2ic_k,
\end{align*}
\]
it follows that
\[
g_{i,k}(x) \geq \min \left( g_{i-1,k}(x), g(x) - 2ic_k \right)
\]
and
\[
g_{i,k}(x) \leq \max \left( g_{i-1,k}(x), g(x) + 2ic_k \right).
\]
Then, since \( g(x) - c_k \leq g_{0,k}(x) \leq g(x) + c_k \), we have
\[
g(x) - 2ic_k \leq g_{i,k}(x) \leq g(x) + 2ic_k,
\]
for \( 1 \leq i \leq N \). Since \( c_k \) is small if \( k \) is large, it follows from (2.8) and (2.9) that
\[
g_{i,k}(x) = \begin{cases} 
  g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) + 2ic_k & \text{if } 0 \leq x_i \leq c_k, \\
  g(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_N) - 2ic_k, & \text{if } 1 - c_k \leq x_i \leq 1,
\end{cases}
\]
for large \( k \). Thus from (2.8), \( g_{N,k} \) is Lipschitz on \( Q_{c_k} \) with a Lipschitz constant 1 (= \(|Dg|\)).

On the other hand, applying Corollary 2 we have
\[
\lim_{k \to \infty} \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} \; dx \; dy \leq \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} \; dx \; dy,
\]
for all \( 1 \leq i \leq N \), which implies
\[
\lim_{k \to \infty} \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} \; dx \; dy \leq \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} \; dx \; dy. \quad (2.11)
\]

Hence using the arguments above, it suffices to construct a family \((h_\delta)\) such that \((h_\delta)\) converging to \( g \) uniformly on \( Q \) and
\[
\lim_{\delta \to 0} \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} \; dx \; dy \leq C_{N,p}.
\]
For this end, let $(\tau_k)_{k\in\mathbb{N}}$ be a strictly decreasing positive sequence such that $\tau_k \leq c_k \delta_k$. For each $\delta$ small, let $k$ be such that $\tau_{k+1} < \delta \leq \tau_k$ ($k$ is large), define $m_1 = \delta_k / \delta$ and $m = [m_1]$. Hereafter $[a]$ denotes the largest integer less than $a$.

Define $h_{\delta}^{(1)} : [0, m]^N \to \mathbb{R}$ as follows:

$$h_{\delta}^{(1)}(y) = \sum_{i=1}^{N} \frac{[y_i]}{\sqrt{N}} + g_{N,k}(x),$$

where $y = (y_1, \cdots, y_N)$ and $x = (x_1, \ldots, x_N)$ with $x_i = y_i - [y_i]$.

For $\alpha \in \mathbb{N}^N$ and $c > 0$, set

$$Q_{[\alpha]} := Q + (\alpha_1, \cdots, \alpha_N), \quad Q_{[\alpha],c} := Q_c + (\alpha_1, \cdots, \alpha_N),$$

and

$$D_{[\alpha],c} := Q_{[\alpha]} \setminus Q_{[\alpha],c}.$$

Here $A + \alpha := \{x + \alpha; x \in A\}$.

We claim that

$$\text{Lip}(h_{\delta}^{(1)}, B) \leq C,$$

where $B = \bigcup_{\alpha \in \{0, \ldots, m-1\}^N} Q_{[\alpha],c} \setminus Q_{[\alpha],c_k/2}$.

Hereafter Lip($f, A$) denotes the Lipschitz constant of $f$ on $A$ for a function $f$ defined on a subset $A$ of $\mathbb{R}^N$. We recall that $C$ denotes a positive constant depending only on $N$ and $p$.

Indeed, since Lip($g_{N,k}, Q_{c_k} \setminus Q_{c_k/2}$) $\leq 1$, it is clear that

$$\text{Lip}(h_{\delta}^{(1)}, Q_{[\alpha],c} \setminus Q_{[\alpha],c_k/2}) \leq 1, \quad \forall \alpha \in \{0, \cdots, m-1\}^N.$$  \hfill (2.14)

On the other hand,

$$\left| h_{\delta}^{(1)}(x) - \sum_{i=1}^{N} \frac{x_i}{\sqrt{N}} \right| \leq Cc_k \quad \forall x \in [0, m]^N$$

and if $\alpha \neq \alpha'$,

$$|x - y| \geq Cc_k \quad \forall x \in Q_{[\alpha],c_k} \setminus Q_{[\alpha],c_k/2}, \quad y \in Q_{[\alpha'],c_k} \setminus Q_{[\alpha'],c_k/2}. \hfill (2.16)$$

Combining (2.14), (2.15), and (2.16) yields (2.13).

From (2.13), there exists $h_{\delta}^{(2)} : \mathbb{R}^N \to \mathbb{R}^N$ such that $h_{\delta}^{(2)} = h_{\delta}^{(1)}$ on $B$ and

$$\text{Lip}(h_{\delta}^{(2)}, \mathbb{R}^N) \leq C.$$  \hfill (2.17)

Define

$$h_{\delta}^{(3)} = \begin{cases} 
  h_{\delta}^{(1)}(x) & \text{if } x \in D_{[\alpha],c_k/2} \text{ for some } \alpha \in \{0, \cdots, m-1\}^N, \\
  h_{\delta}^{(2)}(x) & \text{otherwise.} 
\end{cases} \hfill (2.18)$$
and

\[ h_\delta(x) = \frac{1}{m_1} h_\delta^{(3)}(mx), \]

We have

\[ \int \int_{Q^2} \frac{\delta^p}{|x - y|^{N+p}} dx \, dy = \frac{m^{p-N}}{m_1^p} \int \int_{[0,m]^N \times [0,m]^N} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \quad (2.19) \]

and

\[ \int \int_{[0,m]^N \times [0,m]^N} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \leq \sum_{\alpha \in \mathbb{N}^N} \int \int_{D_\alpha_1, c_{\alpha_1}/2} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \]

\[ + \sum_{1 \leq \alpha_i \leq m-1} \int \int_{D_{[\alpha], c_{\alpha}/2} \times [0,m]^N \setminus D_{[\alpha], c_{\alpha}/2}} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy. \quad (2.20) \]

It is clear from (2.12) and (2.18) that

\[ \int \int_{D_{[\alpha], c_{\alpha}/2} \times [0,m]^N \setminus D_{[\alpha], c_{\alpha}/2}} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \leq \int \int_{Q^2} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy. \quad (2.21) \]

From (2.13), (2.17), and (2.18), we have \( \text{Lip}(h_\delta^{(3)}, [0, m]^N \setminus \bigcup_{\alpha \in \{0, \ldots, m-1\}^N} Q_{[\alpha], c_{\alpha}}) \leq C. \) This implies, after using (2.7),

\[ \int \int_{D_{[\alpha], c_{\alpha}/2} \times [0,m]^N \setminus D_{[\alpha], c_{\alpha}/2}} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \leq C(c_k + \delta_k^p / \delta_k^p) \leq C(c_k + \delta_k^{\frac{q}{p}}). \quad (2.22) \]

Combining (2.20), (2.21), and (2.22) yields

\[ \int \int_{[0,m]^N} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \leq m^N \int \int_{[0,1]^N} \frac{\delta_k^p}{|x - y|^{N+p}} dx \, dy \\
+ C m^N (c_k + \delta_k^{\frac{q}{p}}). \quad (2.23) \]
Since \( m \leq m_1 \), we deduce from (2.19) and (2.23) that
\[
\int_{Q^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \int_{Q^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy + C(c_k + \delta_k^p). \quad (2.24)
\]
Therefore, it follows from (2.11), (2.6), and (2.24) that
\[
\lim_{\delta \to 0} \int_{Q^2} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p},
\]
since \( \lim_{k \to \infty} \delta_k = \lim_{k \to \infty} c_k = 0 \).

From (2.18), it is clear that \( h_\delta \) converges to \( g \) uniformly on \( Q \). \( \square \)

3 Proof of Property (G2)

The proof of Property (G2) will be derived after establishing several lemmas. The first one, which will be used in the proof of Lemma 4, deals with a covering result which is quite classic for experts; however we cannot find any reference for it. For the convenience of the reader, the proof is presented.

**Lemma 3** Let \( \Omega \) be a nonempty open subset of \( \mathbb{R}^N \) and \( B \) be a nonempty bounded open subset of \( \mathbb{R}^N \) with \( |\partial B| = 0 \). Then there exists a collection of open subsets \( (B_i)_{i \in \mathbb{N}} \) such that \( B_i \subset \Omega \), \( B_i \) is an image of \( B \) by a dilatation and a translation, \( B_i \cap B_j = \emptyset \) for \( i \neq j \), and \( \sum_{i \in \mathbb{N}} |B_i| = |\Omega| \).

**Proof:** Let \( \bar{B} \) be an image of \( B \) by a dilatation and a translation such that the closure of \( \bar{B} \) is included in \( Q \). Since \( \bar{B} \) is open, \( |\bar{B}| = c > 0 \). Set \( \Omega_0 = \Omega \). Consider a collection \( (Q_{1,i})_{i \in \mathbb{N}} \) such that \( Q_{1,i} \subset \Omega_0 \), \( Q_{1,i} \) is an image of \( Q \) by a dilatation and a translation, \( Q_{1,i} \cap Q_{1,j} = \emptyset \) for \( i \neq j \), and \( |\Omega| = \sum_{i \in \mathbb{N}} |Q_{1,i}| \). The existence of this collection follows from [17] the assertion d in page 50. Then there exists a collection of disjoint sets \( (B_{1,i})_{i \in \mathbb{N}} \) such that \( B_{1,i} \) is an image of \( B \) by a translation and a dilatation, \( B_{1,i} \subset Q_{1,i} \) and \( |B_{1,i}| = c|Q_{1,i}| \). This implies that \( \sum_{i \in \mathbb{N}} |B_{1,i}| = c|\Omega_0| \). Set \( \Omega_1 = \bigcup_{i \in \mathbb{N}} (Q_{1,i} \setminus B_{1,i}) \). Then \( \Omega_1 \) is open and \( |\Omega_1| = (1-c)|\Omega_0| \) (since \( |\partial B| = 0 \)). Continuing this process, we find collections of sets \( (Q_{k,i})_{(k,i) \in \mathbb{N}^2} \) and \( (B_{k,i})_{(k,i) \in \mathbb{N}^2} \), and open subsets \( (\Omega_k)_{k \in \mathbb{N}} \) of \( \mathbb{R}^N \) such that \( Q_{k,i} \) and \( B_{k,i} \) are images of \( Q \) and \( B \) respectively by a dilatation and a translation, \( Q_{k,i} \subset \Omega_{k-1} \), \( Q_{k,i} \cap Q_{k,j} = \emptyset \) for \( i \neq j \), \( \sum_{i \in \mathbb{N}} |Q_{k,i}| = |\Omega_{k-1}| \), \( B_{k,i} \subset Q_{k,i} \), \( |B_{k,i}| = c|Q_{k,i}| \), and \( \Omega_k = \bigcup_{i \in \mathbb{N}} (Q_{k,i} \setminus B_{k,i}) \). Set
\[
a_m = \sum_{k=1}^{m} \sum_{i \in \mathbb{N}} |B_{k,i}|.
\]
Then since
\[
\sum_{k=1}^{m} \sum_{i \in \mathbb{N}} |B_{k,i}| = c(|\Omega| - \sum_{k=1}^{m-1} \sum_{i \in \mathbb{N}} |B_{k,i}|) + \sum_{k=1}^{m-1} \sum_{i \in \mathbb{N}} |B_{k,i}|,
\]
we have
\[ a_m = c(\Omega - a_{m-1}) + a_{m-1}. \]  
(3.1)

It is easy to see that \(a_m\) is increasing and bounded from above. Hence \(a_m\) converges to \(a\). Thus from (3.1), \(a = |\Omega|\). The conclusion follows by taking the collection \((B_{i,k})_{(i,k)\in \mathbb{N}^2}\).

**Lemma 4** Let \(S\) be an open subset of \(\mathbb{R}^N\) with \(|\partial S| = 0\) and \(g\) be an affine function defined on \(S\). Then
\[
\inf_{\delta \to 0} \lim_{\delta \to 0} \int_{S \times S} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = C_{N,p}|Dg|^p|S|,
\]
where the infimum is taken over all families of measurable functions \((g_\delta)_{\delta \in (0,1)}\) defined on \(S\) such that \(g_\delta\) converges to \(g\) in measure on \(S\) as \(\delta\) goes to \(0\). Moreover, there exists a family of measurable functions \((h_\delta)_{\delta \in (0,1)}\) such that \(h_\delta\) converges to \(g\) uniformly on \(S\) and
\[
\lim_{\delta \to 0} \int_{S \times S} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = C_{N,p}|Dg|^p|S|.
\]

**Proof:** After using a rotation, a dilatation, and a translation, one may assume that
\[ g = \sum_{i=1}^N x_i. \sqrt{N}. \]

Set
\[
\tilde{C}_{N,p} = \inf_{\delta \to 0} \lim_{\delta \to 0} \int_{S \times S} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy,
\]
(3.2)
where the infimum is taken over all families of measurable functions \((g_\delta)_{\delta \in (0,1)}\) defined on \(S\) such that \(g_\delta\) converges to \(g\) in measure as \(\delta\) goes to \(0\).

**Claim 1:** \(\tilde{C}_{N,p} \geq C_{N,p}|S|\).

**Claim 2:** \(\tilde{C}_{N,p} \leq C_{N,p}|S|\) and there exists a family of measurable functions \((h_\delta)_{\delta \in (0,1)}\) defined on \(S\) such that \(h_\delta\) converges to \(g\) uniformly on \(S\), and
\[
\lim_{\delta \to 0} \int_{S \times S} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = C_{N,p}|S|.
\]

It is clear that the conclusion follows from Claims 1 and 2.

**Proof of Claim 1:** Let \((g_\delta)_{\delta \in (0,1)}\) be a family of measurable functions which converges to \(g\) in measure on \(S\). By Lemma 3, there exists a sequence of sets \((Q_i)_{i \in \mathbb{N}}\) such that \(Q_i\) is an image of \(Q\) by a dilation and a translation, \(Q_i \cap Q_j = \emptyset\) for \(i \neq j\), \(Q_i \subset S\), and
\[ |S| = \sum_{i \in \mathbb{N}} |Q_i|. \]
From the definition of \( C_{N,p} \), by a change of variables, we have
\[
\lim_{\delta \to 0} \int_{|g_\delta(x) - g_\delta(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq C_{N,p} |Q_i|,
\]
which implies
\[
\lim_{\delta \to 0} \int_{|g_\delta(x) - g_\delta(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq C_{N,p} \sum_{i \in \mathbb{N}} |Q_i| = C_{N,p} |S|. \tag{3.3}
\]
Claim 1 now follows from (3.2) and (3.3).

**Proof of Claim 2:** We prove Claim 2 by contradiction. Suppose that this is not true. Then there exists \( \varepsilon_0 > 0 \) such that
\[
\lim_{\delta \to 0} \int_{|h_\delta(x) - h_\delta(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq (C_{N,p} + \varepsilon_0)|S|, \tag{3.4}
\]
for any family of measurable functions \((h_\delta)_{\delta \in (0,1)}\) such that \(h_\delta\) uniformly converges to \(g\) on \(S\). Let \((g_\delta)_{\delta \in (0,1)}\) be a family of measurable functions defined on \(Q\) such that \(g_\delta\) converges to \(g\) uniformly on \(Q\) and
\[
\lim_{\delta \to 0} \int_{Q} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = C_{N,p}.
\]
The existence of \((g_\delta)_{\delta \in (0,1)}\) is affirmed by Lemma 2. From Lemma 3, there exists a collection of sets \((S_i)_{i \in \mathbb{N}}\) such that \(S_i\) is an image of \(S\) by a dilatation and a translation for \(i \in \mathbb{N}\), \(S_i \cap S_j = \emptyset\) for \(i \neq j\), \(S_i \subset Q\), and
\[
|Q| = \sum_{i \in \mathbb{N}} |S_i|.
\]
Then, by a change of variables, it follows from (3.4) that
\[
\lim_{\delta \to 0} \int_{|g_\delta(x) - g_\delta(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq (C_{N,p} + \varepsilon_0)|S_i|.
\]
This implies
\[
\lim_{\delta \to 0} \int_{Q} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq (C_{N,p} + \varepsilon_0) \sum_{i \in \mathbb{N}} |S_i| = (C_{N,p} + \varepsilon_0)|Q|.
\]
This contradicts the choice of \((g_\delta)\). \( \square \)

We next introduce the following notation:
**Definition 1** Let \( A_1, A_2, \ldots, A_m \) be disjoint open \((N + 1)\)-simplices in \( \mathbb{R}^N \) such that every coordinate component of any vertex of \( A_i \) is equal to 0 or 1, \( A_i \cap A_j = \emptyset \) for \( i \neq j \),

\[
Q = \bigcup_{i=1}^{m} A_i,
\]

and

\[
A_1 = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N; x_i > 0 \text{ for all } 1 \leq i \leq N, \text{ and } \sum_{i=1}^{N} x_i < 1 \}.
\]

The following lemma is a variant of Lemma 2 for \( \{ A_\ell \}_{\ell=1}^{m} \).

**Lemma 5** Let \( \ell \in \{1, \ldots, m\} \) and \( g \) be an affine function defined on \( A_\ell \) such that \( \frac{\partial g}{\partial n} \neq 0 \) along the boundary of \( A_\ell \). Then there exists a family of measurable functions \((g_\delta)_{\delta \in (0,1)}\) defined on \( A_\ell \) and a family of positive numbers \((c_\delta)_{\delta \in (0,1)}\) converging to 0 such that

\[
|g_\delta(x) - g(x)| \leq 8N(|Dg| + 1)c_\delta \text{ for all } x \in A_\ell,
\]

\( g_\delta \) is Lipschitz on \( A_\ell \), \( c_\delta \), where

\[
A_{\ell,c_\delta} := \{ x \in A_\ell; \text{dist}_\infty(x, A_{\ell,c}) \leq c_\delta \},
\]

with a Lipschitz constant \(|Dg|\), and

\[
\lim_{\delta \to 0} \iint_{A_\ell \times A_\ell} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p}|Dg|^p|A_\ell|.
\]

**Proof:** It suffices to prove the case \( \ell = 1 \). We adapt here the idea used in the proof of Lemma 2. By Lemma 4 there exists a family of measurable functions \((g_\delta)_{\delta \in (0,1)}\) such that \( g_\delta \) converges to \( g \) uniformly on \( A_1 \) as \( \delta \) goes to 0, and

\[
\lim_{\delta \to 0} \iint_{A_1 \times A_1} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = C_{N,p}|Dg|^p|A_1|.
\]

Set

\[
c_\delta = \max(\|g_\delta - g\|_{L^\infty(A_1)}, \sqrt{\delta}), \quad l_\delta = 2(|Dg|c_\delta + c_\delta),
\]

and

\[
g_{0,\delta} = g_\delta,
\]

for \( \delta \in (0,1) \). For \( i = 1, 2, \ldots, N \), define

\[
g_{i,\delta}(x) = \begin{cases} 
\max\left(g_{i-1,\delta}(x), g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) + il_\delta \right) & \text{if } \frac{\partial g}{\partial x_i} > 0, \\
\min\left(g_{i-1,\delta}(x), g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) - il_\delta \right) & \text{if } \frac{\partial g}{\partial x_i} < 0.
\end{cases}
\]
Set $e = (\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}})$ and define

$$g_{N+1,\delta}(x) = \begin{cases} \max (g_{N,\delta}(x), g(z(x)) + (N + 1)l_\delta) & \text{if } \frac{\partial g}{\partial e} < 0 \\ \min (g_{N,\delta}(x), g(z(x)) - (N + 1)l_\delta) & \text{if } \frac{\partial g}{\partial e} > 0. \end{cases}$$

(3.6)

Here for each $x \in \mathbb{R}^N$, $z(x) = x - \langle x, e \rangle e + e$, i.e. $z(x)$ denotes the projection of $x$ on the hyperplane $P$ which is orthogonal to $e$ and contains $e$. By the same way to obtain (2.10) in the proof of Lemma 1, we have, for $\delta$ small (this will be assumed in what follows),

$$|g_{i,\delta}(x) - g(x)| \leq il_\delta \quad \forall x \in A_1.$$

This implies, by (3.5) and (3.6),

$$g_{i,\delta}(x) = \begin{cases} g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) + il_\delta & \text{if } \frac{\partial g}{\partial x_i} > 0 \\ g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) - il_\delta & \text{if } \frac{\partial g}{\partial x_i} < 0, \end{cases}$$

for $1 \leq i \leq N$, for any $x \in A_1$ such that $0 \leq x_i \leq c_\delta$, and

$$g_{N+1,\delta}(x) = \begin{cases} g(z(x)) + (N + 1)l_\delta & \text{if } \frac{\partial g}{\partial e} < 0 \\ g(z(x)) - (N + 1)l_\delta & \text{if } \frac{\partial g}{\partial e} > 0, \end{cases}$$

for any $x \in A_1$ such that $0 \leq |x - z(x)| \leq c_\delta$, where $z(x) = x - \langle x, e \rangle e + e$. Then $g_{N+1,\delta}$ is Lipschitz on $A_{1,c_\delta}$ with a Lipschitz constant $|Dg|.$

It remains to prove

$$\lim_{\delta \to 0} \int_{A_1 \times A_1} \mathbb{1}_{|g_{N+1,\delta}(x) - g_{N+1,\delta}(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \lim_{\delta \to 0} \int_{A_1 \times A_1} \mathbb{1}_{|g_i(x) - g_i(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy. \quad (3.7)$$

Since $\frac{\partial g}{\partial n} \neq 0$, it follows from (3.5) and (3.6) that $g_{i,\delta}$ converges to $g$ in measure in $A_1$. Applying Lemma 2 we have, for $1 \leq i \leq N + 1,$

$$\lim_{\delta \to 0} \int_{A_1 \times A_1} \mathbb{1}_{|g_{i,\delta}(x) - g_{i,\delta}(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq \lim_{\delta \to 0} \int_{A_1 \times A_1} \mathbb{1}_{|g_{i-1,\delta}(x) - g_{i-1,\delta}(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy,$$

which implies (3.7).

To approximate a smooth function by a family of continuous piecewise linear functions and to be able to apply Lemma 3 we introduce the following

**Definition 2** For each $k \in \mathbb{N}$, $K$ is called a $k$-net of $\mathbb{R}^N$ if and only if there exist $z \in \mathbb{Z}^N$ and $\ell \in \{1, 2, \ldots, m\}$ such that $K = \frac{1}{2^\ell} A_\ell + \frac{z}{2^\ell}.$
Hereafter, for any two subsets $A$ and $B$ of $\mathbb{R}^N$ and a real number $c$, we define
\[ cA = \{ca \in \mathbb{R}^N; a \in A\} \]
and
\[ A + B := \{a + b \in \mathbb{R}^N; a \in A \text{ and } b \in B\}. \]
When $B$ is a set containing only a vector $v$, we write $A + v$ instead of $A + \{v\}$.

**Definition 3** A continuous function $g$ on $\mathbb{R}^N$ is said to be a continuous piecewise linear function defined on $k$-nets if and only if $g$ is affine on each $k$-net of $\mathbb{R}^N$.

The following result follows immediately from Lemma 5 after a change of variables.

**Lemma 6** Let $K$ be a $k$-net of $\mathbb{R}^N$ and $g$ be an affine function defined on $K$ such that $\frac{\partial g}{\partial n} \neq 0$ along the boundary of $K$. Then there exists a family of measurable functions $(g_\delta)_{\delta \in (0,1)}$ defined on $K$ and a family of positive numbers $(c_\delta)_{\delta \in (0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|g_\delta(x) - g(x)| \leq 2^{-k+3}N(|Dg| + 1)c_\delta$ for all $x \in K$, $g_\delta$ is Lipschitz function on $K_{2^{-k}c_\delta}$ with a Lipschitz constant $|Dg|$, and
\[
\lim_{\delta \to 0} \int\int_{K \times K} \frac{\delta^p}{|x - y|^{N+p}} dx \, dy \leq C_{N,p} |Dg|^p |K|.
\]

Hereafter
\[ K_\tau := \{x \in K; \text{dist}_\infty(x, K^c) \leq \tau\}, \quad \forall \tau > 0. \]

We are now ready to prove Property (G2) for a “continuous piecewise linear function” with compact support.

**Lemma 7** Let $g$ be a continuous piecewise linear function on $k$-nets with compact support such that on each $k$-net $\frac{\partial g}{\partial n} \neq 0$ along the boundary of that $k$-net unless $g$ is constant on this one. Then there exists a family of measurable functions $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and
\[
\lim_{\delta \to 0} \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} dx \, dy \leq C_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx.
\]

**Proof:** For each $k$-net $K$, if $g$ is not constant on $K$, by Lemma 6 there exist a family of measurable functions $(h_{i,\delta})_{\delta \in (0,1)}$ defined on $K$ and a family of positive numbers $(c_{K,\delta})_{\delta \in (0,1)}$ converging to 0 such that $c_{K,\delta} \geq \sqrt{\delta}$, $|h_{K,\delta}(x) - g(x)| \leq 2^{-k+3}N(|Dg|_\infty + 1)c_{K,\delta}$ for $x \in K$, $h_{K,\delta}$ is Lipschitz on $K_{2^{-k}c_{K,\delta}}$ with a Lipschitz constant $|Dg|_{L^\infty(\mathbb{R}^N)}$, and
\[
\lim_{\delta \to 0} \int\int_{K^2} \frac{\delta^p}{|x - y|^{N+p}} dx \, dy \leq C_{N,p} \int_K |Dg|^p dx.
\]
If \( g \) is constant on \( K \), define \( h_{K,\delta} = g \) on \( K \), \( c_{K,\delta} = \sqrt{\delta} \).

We now follow the ideas used in the proof of Lemma 2. Define \( g^{(1)}_\delta : \mathbb{R}^N \to \mathbb{R} \) by \( g_\delta(x) = h_{K,\delta}(x) \) if \( x \in K \). By the same way to construct \( h^{(3)}_\delta \) from \( h^{(1)}_\delta \) in the proof of Lemma 2, from \( g^{(1)}_\delta \) we can construct \( g_\delta \) such that \( g_\delta(x) = g^{(1)}_\delta(x) \) if \( x \in K_{2^{-k-1}c_{K,\delta}} \), \( g_\delta \) is Lipschitz on \( \mathbb{R}^N \setminus \bigcup K_{2^{-k-1}c_{K,\delta}} \) with a Lipschitz constant \( C(\|Dg\|_\infty + 1) \). Moreover, one can assume that \( \text{supp} \ g_\delta \subset \text{supp} \ g + B_1 \) (the unit ball of \( \mathbb{R}^N \)) as \( \delta \) is small. Hence similar to (2.23), we have

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq \sum_{K} \int_{K^2} \frac{\delta^p}{|h_{K,\delta}(x) - h_{K,\delta}(y)|^p} \, dx \, dy
\]

\[
+ C(k, g)(\max_K c_{K,\delta} + \delta^\frac{p}{2}).
\]

This implies

\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \leq C_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx.
\]

\( \square \)

**Proof of Property (G2):** Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of smooth functions with compact support in \( \mathbb{R}^N \) such that \( g_n \) converges to \( g \) in \( L^p(\mathbb{R}^N) \) and \( \|Dg_n\|_{L^p(\mathbb{R}^N)} \) converges to \( \|Dg\|_{L^p(\mathbb{R}^N)} \) (when \( p = 1 \), the \( L^1 \)-norm is replaced by the total mass). For each \( n \in \mathbb{N} \), let \( (g_{k,n})_{k \in \mathbb{N}} \) be a sequence of functions defined on \( \mathbb{R}^N \) such that \( g_{k,n} \) is a continuous piecewise linear function with compact support defined on \( k \)-nets, \( g_{k,n} \) converges to \( g_n \) in \( W^{1,p}(\mathbb{R}^N) \). Without loss of generality, one may assume that \( \frac{\partial g_{k,n}}{\partial n} \neq 0 \) along the boundary of each \( k \)-net unless \( g_{k,n} \) is constant on this one. Applying Lemma 7, we find a family \( (g_{\delta,k,n})_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N) \) such that \( g_{\delta,k,n} \) converges to \( g_{k,n} \) in \( L^p(\mathbb{R}^N) \), as \( \delta \) goes to 0, and

\[
\lim_{\delta \to 0} I_\delta(g_{\delta,k,n}) \leq C_{N,p} \int_{\mathbb{R}^N} |Dg_{k,n}|^p \, dx.
\]

The rest of the proof, which is quite standard, is left to the reader. \( \square \)

### 4 Proof of Proposition 2 in the case \( p > 1 \)

We begin this section with the following result which is a consequence of Lemma 4

**Lemma 8** Let \( p \geq 1, \varepsilon > 0 \) and \( \tilde{Q} \) be an image of \( Q \), the unit cube of \( \mathbb{R}^N \), by a translation and a dilatation. Let \( l \) be the edge length of \( Q \). Then there exist three positive numbers \( \delta_1, \delta_2, \) and \( \delta_3 \) depending only on \( \varepsilon \) such that if \( g \) is a measurable function defined on \( \tilde{Q} \),

\[
\left| \{x \in \tilde{Q} : |g(x) - ((a, x) + b)| > l|a|\delta_1 \} \right| < 2|\tilde{Q}|,
\]

17
and $\delta < l|a|\delta_3$, for some $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$, then

$$\int_{|g(x)−g(y)|>\delta} \frac{\delta^p}{|x−y|^{N+p}} \, dx \, dy \geq (C_{N,p}−\varepsilon)|a|^p|Q|.$$ 

Hereafter $\langle.,.\rangle$ denotes the usual scalar product in $\mathbb{R}^N$.

**Proof:** By a change of variables, without loss of generality, it suffices to prove Lemma 8 in the case $Q = \bar{Q}$ and $|a| = 1$. We prove this by contradiction. Suppose that this is not true. Then there exist $\varepsilon_0 > 0$, a sequence of measurable functions $(g_n)_{n \in \mathbb{N}}$, a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$, a sequence $(b_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, and a sequence $(\delta_n)_{n \in \mathbb{N}}$ converging to 0 such that $|a_n| = 1$,

$$\frac{\left\{ x \in Q; |g_n(x)−(⟨a_n, x⟩+b_n)| > \frac{1}{n}\right\}}{n} < \frac{1}{n},$$

and

$$\int_{|g_n(x)−g_n(y)|>\delta_n} \frac{\delta_n^p}{|x−y|^{N+p}} \, dx \, dy < C_{N,p}−\varepsilon_0.$$ 

Without loss of generality, one may assume that $b_n = 0$ for $n \in \mathbb{N}$. Since $|a_n| = 1$, there exist $a \in \mathbb{R}^N$ and a subsequence $(a_{n_k})$ of $(a_n)$ such that $a_{n_k}$ converges to $a$ and $|a| = 1$. Then $g_{n_k}$ converges to $⟨a, .⟩$ in measure on $Q$ and

$$\int_{|g_{n_k}(x)−g_{n_k}(y)|>\delta_{n_k}} \frac{\delta_{n_k}^p}{|x−y|^{N+p}} \, dx \, dy < C_{N,p}−\varepsilon_0.$$ 

This contradicts Lemma 8.

We are ready to give the

**Proof of Proposition 2 in the case $p > 1$:** In the proof we essentially use the following result (see e.g., [10, Theorem 1 page 228]): Let $f \in W^{1,p}_{loc}(\mathbb{R}^N)$. Then for a.e. $x \in \mathbb{R}^N$,

$$\lim_{r \to 0} \frac{1}{r} \int_{B(x,r)} |f(y)−f(x)−⟨Df(x), y−x⟩| \, dy = 0. \quad (4.1)$$ 

Here $B(x,r)$ denotes the ball of $\mathbb{R}^N$ centered at $x$ with radius $r$.

For $x \in \mathbb{R}^N$ and $r > 0$, let $Q(x,r)$ denote the open cube centered at $x$ with edge length $2r$, i.e.,

$$Q(x,r) := \{ y = (y_1, \ldots, y_N) \in \mathbb{R}^N; |y_i−x_i| < r \text{ for all } 1 \leq i \leq N \}.$$ 

For $n = 1, 2, 3, \ldots$, set $P_n = 2^{−n} \mathbb{Z}^N$ and let $\Omega_n$ be the collection of all open cubes with edge length $2^{−n}$ whose corners belong to $P_n$. For $x \in \mathbb{R}^N$, define

$$\rho_n(x) := \sup_{0<r<2^{−n+1}} \frac{1}{r} \int_{Q(x,r)} |g(y)−g(x)−⟨Dg(x), y−x⟩| \, dy, \quad (4.2)$$

18
and
\[ \tau_n(x) := \sup_{Q' \in \Omega_k; k \geq n; Q' \text{ contains } x} \int_{Q'} |Dg(y) - Dg(x)|^p \, dy, \quad (4.3) \]
and, for \( m \in \mathbb{N} \), set
\[ A_m = \left\{ x \in [-m, m]^N; 1/m \leq |Dg(x)| \leq m, \lim_{n \to \infty} \rho_n(x) = 0, \text{ and } \lim_{n \to \infty} \tau_n(x) = 0 \right\}. \quad (4.4) \]

Fix \( \varepsilon > 0 \) (arbitrary) and let \( \delta_1, \delta_2, \) and \( \delta_3 \) be three positive constants corresponding to \( \varepsilon \) in Lemma 8. By (4.1), the theory of maximal functions, and the Egorov theorem, it follows from (4.4) that there exist \( m \in \mathbb{N}_+ \) and a compact set \( B_m \subset A_m \) such that \( \rho_n \) and \( \tau_n \) converge to 0 uniformly on \( B_m \), and
\[ \int_{\mathbb{R}^N \setminus B_m} |Dg|^p \, dx \leq \varepsilon \int_{\mathbb{R}^N} |Dg|^p \, dx. \quad (4.5) \]

For \( k \in \mathbb{N} \), define
\[ J_k = \{ Q' \in \Omega_k; Q' \cap B_m \neq \emptyset \}. \]
Take \( Q' \in J_k \) and \( x \in Q' \cap B_m \). From (4.2),
\[ \frac{1}{|Q'|^{1/p}} \int_{Q'} |g(y) - g(x) - \langle Dg(x), y - x \rangle| \, dy \leq \rho_k(x) \quad (4.6) \]
and from (4.3),
\[ \int_{Q'} |Dg(y) - Dg(x)|^p \, dy \leq \tau_k(x) \quad (4.7) \]

Since \( \rho_n \) and \( \tau_n \) go to 0 uniformly on \( B_m \), it follows from (4.6) and (4.7) that there exists \( k \) such that if \( Q' \in J_k \) and \( x \in Q' \cap B_m \), then
\[ \left| \left\{ y \in Q'; |g(y) - g(x) - Dg(x)(y - x)| > \frac{\delta_1}{2m} |Q'|^{1/p} \right\} \right| \leq \frac{\delta_2}{2} |Q'|, \quad (4.8) \]
and
\[ |Dg(x)|^p |Q'| \geq (1 - \varepsilon) \int_{Q'} |Dg|^p \, dy. \quad (4.9) \]

Since \( g_\delta \) converges to \( g \) in measure, we have
\[ \left| \left\{ y \in Q'; |g_\delta(y) - g(y)| > \frac{\delta_1}{2m} |Q'|^{1/p} \right\} \right| \leq \frac{\delta_2}{2} |Q'|, \quad (4.10) \]
as \( \delta \) is small. We deduce from (4.8) and (4.10) that
\[ \left| \left\{ y \in Q'; |g_\delta(y) - g(x) - Dg(x)(y - x)| > \frac{\delta_1}{m} |Q'|^{1/p} \right\} \right| \leq \delta_2 |Q'|, \]
as \( \delta \) is small. Applying Lemma 8, we obtain
\[ \lim_{\delta \to 0} \int_{|g_\delta(x) - g_\delta(y)| > \delta} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \geq (C_{N,p} - \varepsilon) |Dg(x)|^p |Q'|, \]

Since \( g_\delta \) converges to \( g \) in measure, we have
which implies, by (4.9),

\[
\lim_{\delta \to 0} \int_Q \delta P \frac{1}{|x-y|^{N+p}} \, dx \, dy \geq (C_{N,p} - \varepsilon)(1 - \varepsilon) \int_{Q'} |Dg|^p \, dy. \tag{4.11}
\]

Since

\[
\lim_{\delta \to 0} \int_{R^N \times \R^N} \delta P \frac{1}{|x-y|^{N+p}} \, dx \, dy = \sum_{Q' \in \mathcal{J}_k} \lim_{\delta \to 0} \int_{Q'} \delta P \frac{1}{|x-y|^{N+p}} \, dx \, dy,
\]

it follows from (4.5) and (4.11) that

\[
\lim_{\delta \to 0} \int_{R^N \times \R^N} \frac{1}{|x-y|^{N+p}} \, dx \, dy \geq (C_{N,p} - \varepsilon)(1 - \varepsilon)^2 \int_{R^N} |Dg|^p \, dx.
\]

Since \(\varepsilon > 0\) is arbitrary,

\[
\lim_{\delta \to 0} \int_{R^N \times \R^N} \frac{1}{|x-y|^{N+p}} \, dx \, dy \geq C_{N,p} \int_{R^N} |Dg|^p \, dx.
\]

\[\square\]

5 A characterization of BV functions

In this section, we introduce a characterization of BV functions which will be useful in the proof of Proposition 2 in the case \(p = 1\). As mentioned in the introduction, this characterization is motivated from the one based on the notion of essential variation on a.e. line. We first present the following notion, which is motivated by the concept of Lebesgue points.

**Definition 4** Let \(g \in L^1(\prod_{i=1}^N (a_i, b_i)) (a_i < b_i)\) and \(t \in (a_i, b_i)\). Then the surface \(x_1 = t\) is said to be a Lebesgue surface of \(g\) if and only if for almost every \(z' \in \prod_{i=2}^N (a_i, b_i), (t, z')\) is a Lebesgue point of \(g\), the restriction of \(g\) on the surface \(x_1 = t\) is integrable with respect to \((N-1)\)-Hausdorff measure, and

\[
\lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \left| g(s, z') - g(t, z') \right| dz' \, ds = 0. \tag{5.1}
\]

For \(i = 2, \ldots, N\), we also define the notion of the Lesbegue surface for surfaces \(x_i = t\) with \(t \in (a_i, b_i)\) by the similar manner.
The following lemma is a consequence of the theory of maximal functions (see e.g. [18]) and Fubini’s theorem. The details of the proof are left to the reader.

**Lemma 9** Let \( g \in L^1(\prod_{i=1}^{N}(a_i, b_i)) \) and \( j \in \{1, \ldots, N\} \). Then for almost every \( t \in (a_j, b_j) \), the surface \( x_j = t \) is a Lebesgue surface of \( g \).

The following definition will be used in Proposition 3 which deals with a characterization for \( BV \) functions.

**Definition 5** Let \( g \in L^1(\prod_{i=1}^{N}(a_i, b_i)) \). The essential variation of \( g \) in the first direction is defined as follows

\[
\text{ess } V(g, 1) = \sup \left\{ \sum_{i=1}^{m} \int_{\prod_{i=2}^{N}(a_i, b_i)} |g(t_{i+1}, x') - g(t_i, x')| \, dx' \right\},
\]

where the supremum is taken over all finite partitions \( \{a_1 < t_1 < \cdots < t_{m+1} < b_1\} \) such that the surface \( x_1 = t_k \) is a Lebesgue surface of \( g \) for \( 1 \leq k \leq m + 1 \). For \( 2 \leq j \leq N \), we also define \( \text{ess } V(g, j) \) the essential variation of \( g \) in the \( j \)th direction by the similar manner.

The following proposition, which gives a characterization for \( BV \) functions, is the main result of this section.

**Proposition 3** Let \( g \in L^1(\prod_{i=1}^{N}(a_i, b_i)) \). Then \( g \in BV(\prod_{i=1}^{N}(a_i, b_i)) \) if and only if

\[
\text{ess } V(g, j) < +\infty, \quad \forall 1 \leq j \leq N.
\]

Moreover, for \( g \in BV(\prod_{i=1}^{N}(a_i, b_i)) \),

\[
\text{ess } V(g, j) = \|Dg \cdot e_j\|((\prod_{i=1}^{N}(a_i, b_i)), \quad \forall 1 \leq j \leq N.
\]

Here \( Dg \cdot e_j \) denotes the derivative of \( g \) with respect to the variable \( x_j \) and \( \|Dg \cdot e_j\|((\prod_{i=1}^{N}(a_i, b_i)) \) denotes its total mass.

**Remark 2** This characterization is well-known in the case \( N = 1, 2 \).

**Proof:** The proof we will present below is quite standard. Suppose that \( \text{ess } V(g, j) < +\infty \) for all \( 1 \leq j \leq N \). We claim that \( g \in BV(\prod_{i=1}^{N}(a_i, b_i)) \) and

\[
\|Dg \cdot e_j\|((\prod_{i=1}^{N}(a_i, b_i)) \leq \text{ess } V(g, j), \quad \forall 1 \leq j \leq N. \tag{5.2}
\]
Let \((\rho_\varepsilon)\) be a standard sequence of smooth mollifiers on \(\mathbb{R}\) such that \(\text{supp} \rho_\varepsilon \subset (-\varepsilon, \varepsilon)\). Fix \(\varepsilon > 0\) and set \(g_\varepsilon(x) = \int_{\mathbb{R}} g(x_1 - s, x') \rho_\varepsilon(s) \, ds\). Choose arbitrarily \(a_1 + \varepsilon < t_1 < \cdots < t_{m+1} < b_1 - \varepsilon\). We have

\[
\sum_{k=1}^m \int_{\prod_{i=2}^N [a_i, b_i]} |g_\varepsilon(t_{k+1}, x') - g_\varepsilon(t_k, x')| \, dx' = \sum_{k=1}^m \int_{\prod_{i=2}^N [a_i, b_i]} \left| \int_{-\varepsilon}^\varepsilon \rho_\varepsilon(s) [g(t_{k+1} - s, x') - g(t_k - s, x')] \, ds \right| \, dx' \\
\leq \sum_{k=1}^m \int_{\prod_{i=2}^N [a_i, b_i]} \int_{-\varepsilon}^\varepsilon \rho_\varepsilon(s) |g(t_{k+1} - s, x') - g(t_k - s, x')| \, ds \, dx' \\
\leq \int_{-\varepsilon}^\varepsilon \rho_\varepsilon(s) \sum_{k=1}^m \int_{\prod_{i=2}^N [a_i, b_i]} |g(t_{k+1} - s, x') - g(t_k - s, x')| \, dx' \, ds. \tag{5.3}
\]

On the other hand, for almost every \(t \in (a_1, b_1)\), the surface \(x_1 = t\) is a Lebesgue surface of \(g\) (see Lemma 9). Hence

\[
\int_{-\varepsilon}^\varepsilon \rho_\varepsilon(s) \sum_{k=1}^m \int_{\prod_{i=2}^N [a_i, b_i]} |g(t_{k+1} - s, x') - g(t_k - s, x')| \, dx' \, ds \\
\leq \int_{-\varepsilon}^\varepsilon \rho_\varepsilon(s) \text{ess} V(g, 1) \, ds \leq \text{ess} V(g, 1). \tag{5.4}
\]

Combining (5.3) and (5.4) yields

\[
\int_{a_1 + \varepsilon}^{b_1 - \varepsilon} \int_{\prod_{i=2}^N [a_i, b_i]} |Dg_\varepsilon \cdot e_1| \, dx' \, ds \leq \text{ess} V(g, 1).
\]

This implies

\[
\|Dg \cdot e_1\| \left( \prod_{i=1}^N (a_i, b_i) \right) \leq \text{ess} V(g, 1).
\]

Similarly,

\[
\|Dg \cdot e_j\| \left( \prod_{i=1}^N (a_i, b_i) \right) \leq \text{ess} V(g, j), \quad \forall 2 \leq j \leq N.
\]

Therefore, \(g \in BV \left( \prod_{i=1}^N (a_i, b_i) \right)\) and (5.2)
We now suppose that $g \in BV \left( \prod_{i=1}^{N} (a_i, b_i) \right)$. We claim that

$$\text{ess } V(g, j) \leq \|Dg \cdot e_j\| \left( \prod_{i=1}^{N} (a_i, b_i) \right), \quad \forall 1 \leq j \leq N. \quad (5.5)$$

In fact, consider $\{a_1 < t_1 < \cdots < t_{m+1} < b_1\}$ such that the surface $x_1 = t_k$ is a Lebesgue surface of $g$ for $k = 1, \cdots, m + 1$. Then, by (5.1),

$$\sum_{k=1}^{m} \int_{\prod_{i=2}^{N} (a_i, b_i)} |g(t_{k+1}, x') - g(t_k, x')| \, dx' \geq \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \int_{\prod_{i=2}^{N} (a_i, b_i)} |g_{\varepsilon}(t_{k+1}, x') - g_{\varepsilon}(t_k, x')| \, dx'. $$

However,

$$\sum_{k=1}^{m} \int_{\prod_{i=2}^{N} (a_i, b_i)} |g_{\varepsilon}(t_{k+1}, x') - g_{\varepsilon}(t_k, x')| \, dx' \leq \int_{t_1}^{t_{m+1}} \int_{\prod_{i=2}^{N} (a_i, b_i)} |Dg_{\varepsilon} \cdot e_1| \, dx$$

and

$$\int_{t_1}^{t_{m+1}} \int_{\prod_{i=2}^{N} (a_i, b_i)} |Dg_{\varepsilon} \cdot e_1| \, dx \leq \|Dg \cdot e_1\| \left( \prod_{i=1}^{N} (a_i, b_i) \right),$$

when $\varepsilon$ is small. It follows that

$$\sum_{k=1}^{m} \int_{\prod_{i=2}^{N} (a_i, b_i)} |g(t_{k+1}, x') - g(t_k, x')| \, dx' \leq \|Dg \cdot e_1\| \left( \prod_{i=1}^{N} (a_i, b_i) \right),$$

which implies

$$\text{ess } V(g, 1) \leq \|Dg \cdot e_1\| \left( \prod_{i=1}^{N} (a_i, b_i) \right).$$

Similarly,

$$\text{ess } V(g, j) \leq \|Dg \cdot e_j\| \left( \prod_{i=1}^{N} (a_i, b_i) \right), \quad \forall 2 \leq j \leq N.$$

Thus (5.5) is proved.

The conclusion of Proposition 3 now follows from (5.2) and (5.5). □

**Remark 3** We do not use any property of Lebesgue points in the definition of Lebesgue surfaces in the proof, but it will be useful in the proof of Proposition 2 in the case $p = 1$ (see the proof of Lemma 14).
6 Proof of Proposition 2 in the case $p = 1$

6.1 Another definition of $C_{N, 1}$

Define

$$b_{N, 1} := \inf_{\delta \to 0} \lim_{\delta \to 0} \int_{Q^2} \delta \frac{\delta}{|x - y|^{N+1}} \, dx \, dy.$$  \hspace{1cm} (6.1)

where the infimum is taken over all family of measurable functions $(g_{\delta})_{\delta \in (0, 1)}$ such that $g_{\delta}$ converges to $H_{\frac{1}{2}}$ in measure as $\delta$ goes to 0. Here and afterwards $H_c(x) := H(x_1 - c, x')$ for any $c \in \mathbb{R}$, where $H$ is the function defined on $\mathbb{R}^N$ by

$$H(x) = \begin{cases} 0 & \text{if } x_1 < 0, \\ 1 & \text{otherwise.} \end{cases}$$

This section is devoted to prove

Proposition 4

$$b_{N, 1} = C_{N, 1}.$$  

The proof of Proposition 4 is based on two lemmas. The first one will be used to prove that $b_{N, 1} \geq C_{N, 1}$.

Lemma 10 There exist a sequence of measurable functions $(\psi_k)$ and a sequence of positive numbers $(\tau_k)$ converging to 0 such that $\psi_k$ converges to $g \equiv x_1$ in measure on $Q$, and

$$\lim_{k \to \infty} \int_{\{\psi_k(x) - \psi_k(y) > \tau_k\}} \frac{\tau_k}{|x - y|^{N+1}} \, dx \, dy = b_{N, 1}.$$  \hspace{1cm} (6.2)

Proof: From the definition of $b_{N, 1}$, there exist a sequence $(\delta_k)$ converging to 0 and a sequence of measurable functions $(g_k)$ converging in measure to $H_{\frac{1}{2}}$ as $k$ goes to infinity such that

$$\lim_{k \to \infty} \int_{\{g_k(x) - g_k(y) > \delta_k\}} \frac{\delta_k}{|x - y|^{N+1}} \, dx \, dy = b_{N, 1}.$$  \hspace{1cm} (6.2)

Since $g_k$ converges to $H_{\frac{1}{2}}$ in measure on $Q$, there exists a sequence of positive numbers $(c_k)_{k \in \mathbb{N}}$ converging to 0 such that

$$\lim_{k \to \infty} \frac{|\{x \in Q; |g_k(x) - H_{\frac{1}{2}}(x)| \geq c_k\}|}{c_k} = 0.$$  \hspace{1cm} (6.3)

Define $h_{1,k}, h_{2,k} : Q \to \mathbb{R}$ as follows

$$h_{1,k}(x) = \begin{cases} c_k & \text{if } x_1 < \frac{1}{2} - c_k, \\ 1 + c_k & \text{if } x_1 > \frac{1}{2}, \\ \frac{1}{c_k}(x_1 - \frac{1}{2} + c_k) + c_k & \text{otherwise}, \end{cases}$$
and

\[ h_{2,k}(x) = \begin{cases} 
- c_k & \text{if } x_1 < 1/2, \\
1 - c_k & \text{if } x_1 > 1/2 + c_k, \\
\frac{1}{c_k}(x_1 - 1/2) - c_k & \text{otherwise.} 
\end{cases} \]

Set

\[ g_{1,k} = \min \left( \max(g_k, h_{2,k}), h_{1,k} \right), \]

and

\[ g_{2,k} = \min \left( \max(g_{1,k}, c_k), 1 - c_k \right). \]

We claim that

\[ g_{2,k}(x) = c_k \quad \forall \ x \in Q; x_1 < 1/2 - c_k, \quad g_{2,k}(x) = 1 - c_k \quad \forall \ x \in Q; x_1 > 1/2 + c_k, \] (6.4)

and

\[ \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy = b_{N,1}. \] (6.5)

In fact, it suffices to prove (6.5). By Corollary 1, we have

\[ \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy \leq \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy. \] (6.6)

Since \( |Dh_{1,k}(x)| \leq \frac{1}{c_k} \) and \( |Dh_{2,k}(x)| \leq \frac{1}{c_k} \) for all \( x \in Q \), it follows from (6.3), and Lemma 1 that

\[ \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy \leq \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy. \] (6.7)

Combining (6.2), (6.6), and (6.7) yields

\[ \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy \leq b_{N,1}. \] (6.8)

From (6.8) and (6.1) we obtain (6.5).

Let \( h_k : Q \to \mathbb{R} \) be defined by

\[ h_k(x) = \frac{g_{2,k}(x) - c_k}{1 - 2c_k}, \] (6.9)

and set \( \varepsilon_k = \delta_k/(1 - 2c_k) \). It is clear that \( \varepsilon_k \) converges to 0 as \( k \) goes to infinity, and

\[ \lim_{k \to \infty} \int_{Q^2} \frac{\varepsilon_k}{|x - y|^{N+1}} \ dx \ dy = \lim_{k \to \infty} \int_{Q^2} \frac{\delta_k}{|x - y|^{N+1}} \ dx \ dy. \] (6.10)
We deduce from (6.5) and (6.10) that

\[
\lim_{k \to \infty} \int_{Q^2} \frac{\varepsilon_k}{|x - y|^{N+1}} \, dx \, dy = b_{N,1}. \tag{6.11}
\]

For each \( n \in \mathbb{N} \) (arbitrary), consider the sequence \((f_k) : Q \mapsto \mathbb{R}\) which is defined as follows

\[ f_k(x) = \frac{1}{n} h_k \left( x_1 - \frac{i}{n} + \frac{1}{2n}, x_2, \ldots, x_N + \frac{1}{n} \right) \quad \text{if} \quad x_1 \in \left[ \frac{i}{n}, \frac{i+1}{n} \right], \quad 0 \leq i \leq n - 1. \]

We claim that

\[
\lim_{k \to \infty} \int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy = b_{N,1}, \tag{6.12}
\]

and

\[
\int_{Q} |f_k(x) - x_1| \, dx \leq \frac{1}{n}. \tag{6.13}
\]

Indeed, (6.13) is clear from the definition of \( f_k \) and the fact that \( 0 \leq h_k(x) \leq 1 \) for all \( x \in Q \). It suffices to prove (6.12). We have

\[
\int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy = \sum_{i=0}^{n-1} \int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy
\]

\[
= \sum_{i=0}^{n-1} \int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy + \sum_{i=0}^{n-1} \int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy. \tag{6.14}
\]

On the other hand, since \( h_k(x) = 0 \) if \( x_1 < \frac{1}{2} - c_k \), \( h_k(x) = 1 \) if \( x_1 > \frac{1}{2} + c_k \), and \( c_k \) converges to 0 as \( k \) goes to infinity (by (6.4) and (6.9)), it follows from (6.11) that

\[
\int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy = \frac{1}{n} \int_{Q^2} \frac{\varepsilon_k}{|x - y|^{N+1}} \, dx \, dy, \tag{6.15}
\]

\[
\lim_{k \to \infty} \int_{Q^2} \frac{\varepsilon_k}{|x - y|^{N+1}} \, dx \, dy = b_{N,1}, \tag{6.16}
\]
and
\[
\lim_{k \to \infty} \int_{Q^2} \frac{\varepsilon_k/n}{|x - y|^{N+1}} \, dx \, dy = 0. \tag{6.17}
\]
Combining (6.14), (6.15), (6.16), and (6.17) yields (6.12).

The conclusion now follows from (6.12) and (6.13). \[\square\]

We also have the following result which implies \( C_{N,1} \geq b_{N,1} \).

**Lemma 11** There exist a sequence of measurable functions \((\psi_k)\) and a sequence of positive numbers \((\tau_k)\) converging to 0 such that \(\psi_k\) converges to \(H_{1/2}\) in measure on \(Q\) and

\[
\lim_{k \to \infty} \int_{Q^2} \frac{\tau_k}{|x - y|^{N+1}} \, dx \, dy = C_{N,1}.
\]

**Proof:** Let \((g_n)\) be a sequence of functions defined on \(Q\) as follows

\[
g_n(x) = \begin{cases} 
0 & \text{if } x^1 \leq \frac{1}{2} - \frac{1}{n}, \\
n[x^1 - \frac{1}{2} + \frac{1}{n}] & \text{if } \frac{1}{2} - \frac{1}{n} < x^1 \leq \frac{1}{2}, \quad \forall \, n \in \mathbb{N}. \\
1 & \text{otherwise},
\end{cases}
\]

For each \(n\), by Lemma 4 there exists a family of measurable functions \((g_n,\delta)_{\delta \in (0,1)}\) defined on \(Q\) such that \(g_n,\delta\) converges to \(g_n\) in measure and

\[
\lim_{\delta \to 0} \int_{Q^2} \frac{\delta}{|x - y|^{N+1}} \, dx \, dy = C_{N,1}
\]

Therefore, the conclusion follows. \[\square\]

We are ready to give

**Proof of Proposition 4** We have \(C_{N,1} \leq b_{N,1}\) by Lemma 10 and the definition of \(C_{N,1}\); and \(C_{N,1} \geq b_{N,1}\) by Lemma 11 and the definition of \(b_{N,1}\). This implies \(b_{N,1} = C_{N,1}\). \[\square\]

### 6.2 Some useful lemmas

In this section, we prove some useful lemmas which will be used in the proof of Proposition 2 in the case \(p = 1\). Our main goal is to prove Lemma 14. From the definition of \(b_{N,1}\), we have
Lemma 12 For any $\varepsilon > 0$, there exist three positive numbers $\delta_1$, $\delta_2$, and $\delta_3$ such that if $g \in L^1\left(\prod_{i=1}^{N}(a_i, b_i)\right)$ $(a_i < b_i)$,
\[
\left\{ x \in \prod_{i=1}^{N}(a_i, b_i) \mid |g(x) - \frac{cH_{a_1+b_1-a_1}(x) + d}{2}| > |c|\delta_1 \right\} < \delta_2 \prod_{i=1}^{N}(b_i - a_i),
\]
and $\delta < |c|\delta_3$, for some $c$ and $d$ in $\mathbb{R}$, then
\[
\int_{\prod_{i=1}^{N}(a_i, b_i) \times \prod_{i=1}^{N}(a_i, b_i)} \frac{\delta}{|x - y|^{N+1}} dx dy \geq |c|(b_{N,1} - \varepsilon).
\]
Proof: The proof is similar to the one of Lemma 8. The details are left to the reader. □

The following lemma, in which we do not require any condition on the convergence of $(g_\delta)$, will play an important role in the proof of Lemma 14.

Lemma 13 Let $(g_\delta)_{\delta \in (0,1)} \subset L^1\left(\prod_{i=1}^{N}(a_i, b_i)\right)$. Assume that $g_\delta(x) \leq 0$ for $x$ with $x_1 \leq a_1 + \delta$ and $g_\delta(x) \geq c$ $(c > 0)$ for $x$ such that $x_1 \geq b_1 - \delta$. Then
\[
\lim_{\delta \to 0} \int_{\prod_{i=1}^{N}(a_i, b_i) \times \prod_{i=1}^{N}(a_i, b_i) \mid |g(x) - g(y)| > \delta} \frac{\delta}{|x - y|^{N+1}} dx dy \geq b_{N,1}.
\]
Proof: Without loss of generality, by Corollary 1 one may assume that $g_\delta(x) = 0$ for all $x$ such that $x_1 < a_1 + \delta$ and $g(x) = c$ for all $x$ such that $x_1 \geq b_1 - \delta$. For $\varepsilon > 0$, let $\delta_2$ be a positive constant corresponding to $\varepsilon$ in Lemma 12. Set $Q_\delta := [a_1 - \frac{b_1 - a_1}{2\delta_2}, b_1 + \frac{b_1 - a_1}{2\delta_2}] \times \prod_{i=2}^{N}(a_i, b_i)$. Define $h_\delta : Q_\delta \mapsto \mathbb{R}$ by
\[
h_\delta(x) = \begin{cases} 
0 & \text{if } x_1 \in (a_1 - \frac{b_1 - a_1}{2\delta_2}, a_1), \\
g_\delta(x) & \text{if } x_1 \in (a_1, b_1), \\
c & \text{if } x_1 \in (b_1, b_1 + \frac{b_1 - a_1}{2\delta_2}).
\end{cases}
\]
Applying Lemma 12 for the function $h_\delta$, we have
\[
\lim_{\delta \to 0} \int_{Q_\delta} \frac{\delta}{|x - y|^{N+1}} dx dy \geq c(b_{N,1} - \varepsilon).
\]
Since $g_\delta(x) = 0$ if $x_1 < a_1 + \delta$ and $g_\delta(x) = c$ if $x_1 > b_1 - \delta$, it follows that
\[
\lim_{\delta \to 0} \int_{\prod_{i=1}^{N}(a_i, b_i) \times \prod_{i=1}^{N}(a_i, b_i) \mid |g_\delta(x) - g_\delta(y)| > \delta} \frac{\delta}{|x - y|^{N+1}} dx dy \geq c(b_{N,1} - \varepsilon).
\]
Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

The following lemma plays a crucial role in the proof of Proposition 2 in the case $p = 1$. 28
Lemma 14 Let \( g \in L^1\left( \prod_{i=1}^N (a_i, b_i) \right) \) and \( (g_\delta)_{\delta \in (0, 1)} \subset L^1\left( \prod_{i=1}^N [a_i, b_i] \right) \) \( (a_i < b_i) \) such that \( g_\delta \) converges to \( g \) in measure on \( \prod_{i=1}^N [a_i, b_i] \). Then for any \( t_1 \) and \( t_2 \) in \( (a_1, b_1) \) \( (t_1 < t_2) \) such that the surface \( x_1 = t_j \) \( (j = 1, 2) \) is a Lebesgue surface of \( g \), we have

\[
\lim_{\delta \to 0} \iint_{\int (t_1, t_2) \times \prod_{i=2}^N (a_i, b_i) \bigg| \int_{g_\delta(x) - g_\delta(y)} > \delta \bigg|} \frac{\delta}{|x - y|^{N+1}} \, dx \, dy \geq b_{N,1} \int_{\prod_{i=2}^N (a_i, b_i)} |g(t_2, x') - g(t_1, x')| \, dx'.
\]

Proof: Fix \( \tau > 0 \) (arbitrary). Let \( A \) be the set of all elements \( z' \in \prod_{i=2}^N [a_i, b_i] \) such that \( (t_1, z') \) is a Lebesgue point of \( g|_{x_1=t_1} \), \( (t_2, z') \) is a Lebesgue point of \( g|_{x_1=t_2} \); \( (t_1, z') \) and \( (t_2, z') \) are Lebesgue points of \( g \). For each \( z' \in A \), let \( Q'(z') \subset \mathbb{R}^{N-1} \) be a closed cube center at \( z' \) such that

\[
\begin{align*}
\{ & \left. (x_1, y') \in (t_1, t_1 + 2l) \times Q'(z') : |g(x_1, y') - g(t_1, z')| \geq \tau/2 \right| \leq \tau |Q'(z')|, \\
\{ & \left. (x_1, y') \in (t_2 - 2l, t_2) \times Q'(z') : |g(x_1, y') - g(t_2, z')| \geq \tau/2 \right| \leq \tau |Q'(z')|,
\end{align*}
\]

(6.19)

where \( l = \frac{\tau}{2} |Q'(z')|^{\frac{1}{N-1}} \). Hereafter in the proof, \( |Q'(z')| \) denotes the \( (N - 1) \)-dimensional Hausdorff measure of \( Q'(z') \).

Since \( g_\delta \) converges to \( g \) in measure, it follows from (6.19) that, when \( \delta \) is small,

\[
\begin{align*}
\{ & \left. (x_1, y') \in (t_1, t_1 + 2l) \times Q'(z') : |g_\delta(x_1, y') - g(t_1, z')| \geq \tau \right| \leq 2\tau |Q'(z')|, \\
\{ & \left. (x_1, y') \in (t_2 - 2l, t_2) \times Q'(z') : |g_\delta(x_1, y') - g(t_2, z')| \geq \tau \right| \leq 2\tau |Q'(z')|.
\end{align*}
\]

(6.20)

We claim that

\[
\lim_{\delta \to 0} \iint_{\int (t_1, t_2) \times Q'(z') \bigg| \int_{g_\delta(x) - g_\delta(y)} > \delta \bigg|} \frac{\delta}{|x - y|^{N+1}} \, dx \, dy \geq b_{N,1} |g(t_2, z') - g(t_1, z')| - C\tau |Q'(z')|,
\]

(6.21)

for some constant \( C \) depending only on \( N \).

Without loss of generality, one may assume that \( g(t_1, z') < g(t_2, z') \). Define \( f_1 : (t_1, t_2) \times Q'(z') \to \mathbb{R} \) by

\[
f_1(y) = \begin{cases} 
  g(t_1, z') & \text{if } y_1 \leq t_1 + l, \\
  g(t_2, z') & \text{if } y_1 \geq t_1 + 2l, \\
  \frac{1}{l} \left[ g(t_2, z') - g(t_1, z') \right] y_1 + 2g(t_1, z') - g(t_2, z') & \text{otherwise.}
\end{cases}
\]

29
and set
\[ h_{1,\delta} = \min \left\{ \max \left( \min \left( g_{\delta}, g(t_2, z') \right), g(t_1, z) \right), f_1 \right\}. \]

Since \( f_1 \) is a Lipschitz function with a Lipschitz constant \( \frac{1}{2}[g(t_2, z') - g(t_1, z')] \), it follows from (6.20) and Lemma 1 that
\[
\int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy \leq \int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy
\]
\[
+ C\tau [g(t_2, z') - g(t_1, z')] |Q'(z')|. \tag{6.22}
\]

Similarly, define \( f_2 : (t_1, t_2) \times Q'(z') \rightarrow \mathbb{R} \) by
\[
f_2(y) = \begin{cases} 
  g(t_1, z') & \text{if } y_1 \leq t_2 - 2l, \\
  g(t_2, z') & \text{if } y_1 \geq t_2 - l,
  \\
  \frac{1}{l} [g(t_2, z') - g(t_1, z')] (y_1 - t_2 + 2l) + g(t_1, z') & \text{otherwise},
\end{cases}
\]
and set
\[ h_{2,\delta} = \max(g_{1,\delta}, f_2). \]

We have
\[
\int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy \leq \int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy
\]
\[
+ C\tau [g(t_2, z') - g(t_1, z')] |Q'(z')|. \tag{6.23}
\]

Combining (6.22) and (6.23) yields
\[
\int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy \leq \int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy
\]
\[
+ C\tau [g(t_2, z') - g(t_1, z')] |Q'(z')|. \tag{6.24}
\]

On the other hand, by Lemma 13,
\[
\lim_{\delta \to 0} \int \int \frac{\delta}{|x-y|^{N+1}} \, dx \, dy \geq b_{N,1} [g(t_2, z') - g(t_1, z')] |Q'(z')|. \tag{6.25}
\]

From (6.24) and (6.25), (6.21) holds.
From (6.18-ii) and (6.21), we deduce that
\[
\lim_{\delta \to 0} \iint_{[(t_1,t_2) \times Q(x')]^2 \mid g_\delta(x) - g_\delta(y) \geq \delta} \frac{\delta}{|x - y|^{N+1}} \, dx \, dy \geq b_{N,1}[1 - 2\tau] \int_{Q(x')} \mid g(t_2, x') - g(t_1, x') \mid \, dx' - C\tau |Q(x')|.
\] (6.26)

Applying Besicovitch’s covering theorem, it follows from (6.26) that
\[
\lim_{\delta \to 0} \iint_{[(t_1,t_2) \times \Pi_{i=2}^{N} (a_i, b_i)]^2 \mid g_\delta(x) - g_\delta(y) \geq \delta} \frac{\delta}{|x - y|^{N+1}} \, dx \, dy \geq b_{N,1}[1 - 2\tau] \int_{\Pi_{i=2}^{N} (a_i, b_i)} \mid g(t_2, x') - g(t_1, x') \mid \, dx' - C\tau \Pi_{i=2}^{N} (b_i - a_i).
\]
Since \(\tau > 0\) is arbitrary, we obtain the conclusion. \(\square\)

**Remark 4** It is surprising that the inequality in Lemma 14 involves the constant \(C_{N,1}\) (since \(b_{N,1} = C_{N,1}\)), although \(C_{N,1}\) is defined by a process depending on a smooth function.

### 6.3 Proof of Proposition 2 in the case \(p = 1\)

We recall that for each \(g \in BV(\mathbb{R}^N)\), \(\|Dg\|\) is a Radon measure on \(\mathbb{R}^N\) and there exist a \(\|Dg\|-\)measurable function \(\sigma : \mathbb{R}^N \mapsto \mathbb{R}^N\) such that
\[
\begin{cases}
|\sigma(x)| = 1 \|Dg\| \text{ a.e.}, \\
\int_{\mathbb{R}^N} f \text{div} \psi = -\int_{\mathbb{R}^N} \psi \cdot \sigma d\|Dg\|, \quad \forall \psi \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)
\end{cases}
\]
(see e.g. [10 Theorem 1 on page 167]). Then for \(\|Dg\|\) a.e. \(x \in \mathbb{R}^N\), we have (see e.g. [10 Theorem 1 on page 38])
\[
\lim_{r \to 0} \frac{\|Dg \cdot \sigma(x)|(Q(x, \sigma(x), r))}{\|Dg\|(Q(x, \sigma(x), r))} = 1.
\] (6.27)

Hereafter for any \((x, \sigma, r) \in \mathbb{R}^N \times S^{N-1} \times (0, +\infty)\), \(Q(x, \sigma, r)\) denotes the closed cube centered at \(x\) with edge length \(2r\) such that one of its faces is orthogonal to \(\sigma\).

Fix \(\varepsilon > 0\) (arbitrary). By Besicovitch’s covering theorem, there exists a family of cubes \(\left(Q(x_i, \sigma(x_i), r_i)\right)_{i \in \mathbb{N}}\) such that
\[
Q(x_i, \sigma(x_i), r_i) \cap Q(x_j, \sigma(x_j), r_j) = \emptyset, \quad \text{for } i \neq j,
\] (6.28)
\[
\frac{\|Dg \cdot \sigma(x)|(Q(x_i, \sigma(x_i), r_i))}{\|Dg\|(Q(x_i, \sigma(x_i), r_i))} \geq 1 - \varepsilon,
\] (6.29)
\[ \| Dg \cdot \sigma(x_i) \| (\partial Q(x_i, \sigma(x_i), r_i)) = 0, \]  
and
\[ \| Dg \| (\mathbb{R}^N) = \| Dg \| \left( \bigcup_{i \in \mathbb{N}} Q(x_i, \sigma(x_i), r_i) \right). \]

Hence it follows (6.29) and (6.31) that
\[ \| Dg \| (\mathbb{R}^N) \leq 1 - \varepsilon \sum_{i \in \mathbb{N}} \| Dg \cdot \sigma(x_i) \| (Q(x_i, \sigma(x_i), r_i)). \] (6.32)

Applying Lemma 14 and Proposition 3, we deduce from (6.30) that
\[ b_{N,1} \| Dg \cdot \sigma(x_i) \| (Q(x_i, \sigma(x_i), r_i)) \leq \lim_{\delta \to 0} \int \int_{[Q(x_i, \sigma(x_i), r_i)]} \frac{\delta}{|x - y|^{N+1}} dx dy. \] (6.33)

Combining (6.32) and (6.33) yields
\[ b_{N,1} \| Dg \| (\mathbb{R}^N) \leq 1 - \varepsilon \lim_{\delta \to 0} \int \int_{[\bigcup_{i=1}^k Q(x_i, \sigma(x_i), r_i)]} \frac{\delta}{|g_\delta(x) - g_\delta(y)| + \delta} dx dy. \] (6.34)

Since \( \varepsilon > 0 \) is arbitrary and \( b_{N,1} = C_{N,1} \) (see Proposition 4), it follows from (6.34) that
\[ \lim_{\delta \to 0} I_\delta(g_\delta) \geq I(g). \]

\[ \square \]

**Acknowledgments.** The author thanks warmly J. Bourgain for the discussion during the preparation of the paper [4]. The main ideas in the proof of Proposition 1 originate in [4]. He is deeply grateful to H. Brezis for conjecturing Theorem 2 and for his encouragement. He also thanks A. Ponce for communicating the example mentioned after Theorem 1 and V. Millot for interesting discussions. Part of this work was done when the author visited the Institute for Advanced Study and Rutgers University; he thanks these Mathematics Departments for the hospitality. The author also thanks the referees who read carefully the paper and made useful comments and suggestions which help him improve the presentation of the paper and the proof of Proposition 2 in the case \( p = 1 \).

**References**


