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Research Article

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Superlensing using complementary media and reflecting complementary media for electromagnetic waves

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Abstract: In this paper, we present the proof of superlensing an arbitrary object using complementary media and we study reflecting complementary media for electromagnetic waves. The analysis is based on the reflecting technique and new results on the compactness, existence, and stability for the Maxwell equations with low regularity data.

Keywords: Negative index materials, superlensing, complementary media, cloaking, localized resonance, electromagnetic waves

MSC 2010: 35B34, 35B35, 35B40, 35J05, 78A25

1 Introduction

Negative index materials (NIMs) were first investigated theoretically by Veselago in [36]. The existence of NIMs was confirmed experimentally by Shelby, Smith and Schultz in [35]. The study of NIMs has attracted a lot of attention in the scientific community thanks to their interesting properties and many possible applications, such as superlensing using complementary media, see [13, 19, 28, 30, 31, 33, 34], cloaking using complementary media, see [10, 21, 23, 25], cloaking a source via anomalous localized resonance, see [2, 4, 9, 12, 16–18, 26] and references therein, and cloaking an object via anomalous localized resonance, see [20]. A survey for recent mathematics progress on these applications can be found in [24]. In this paper, we present the proof of superlensing using complementary media for electromagnetic waves.

Superlensing using complementary media was suggested by Veselago in [36] for a slab lens (a slab of index −1) using the ray theory. Later, cylindrical lenses in the two-dimensional quasistatic regime, the Veselago slab and cylindrical lenses in the finite frequency regime, and spherical lenses in the finite frequency regime were studied by Nicorovici, McPhedran and Milton in [28], Pendry in [30, 31], and Pendry and Ramakrishna in [34] respectively for constant isotropic objects. Superlensing arbitrary inhomogeneous objects using complementary media in the acoustic setting was established in [19] for schemes inspired from [28, 31, 34] and guided by the concept of reflecting complementary media in [15]. The proof of superlensing arbitrary inhomogeneous objects using complementary media for electromagnetic waves presented in this paper therefore represents the natural completion of this line of work.

Let us describe how to magnify m times (m is a given real number greater than 1) the region B_{r_0} for some $r_0 > 0$ in which the medium is characterized by a pair of two uniformly elliptic matrix-valued functions (ε_O, μ_O) using complementary media. The idea suggested by Pendry and Ramakrishna in [34] is to put a lens in $B_{r_2} \setminus B_{r_0}$ whose medium is characterized by $(-(r_2^2/|x|^2)I, -(r_2^2/|x|^2)I)$; the loss is ignored. Our lens

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construction is as follows. Let $\alpha, \beta > 1$ be such that

$$\alpha\beta - \alpha - \beta = 0. \tag{1.1}$$

Set

$$r_1 = m^{1-\frac{1}{\alpha}} r_0, \quad r_2 = m r_0, \quad \text{and} \quad r_3 = m^{2-\frac{1}{\alpha}} r_0, \tag{1.2}$$

and define $F : B_{r_2} \setminus \{0\} \to \mathbb{R}^3 \setminus \bar{B}_{r_2}$ and $G : \mathbb{R}^3 \setminus \bar{B}_{r_3} \to B_{r_3} \setminus \{0\}$ by

$$F(x) = \frac{r_2^\alpha x}{|x|^\alpha} \quad \text{and} \quad G(x) = \frac{r_3^\beta x}{|x|^\beta}.$$

Our lens contains two parts (see Figure 1). The first one of NIMs is given by

$$(F_*^{-1} I, F_*^{-1} I) \quad \text{in } B_{r_2} \setminus B_{r_1} \tag{1.3}$$

(see (1.6) below for the explicit formula) and the second one is

$$(mI, mI) \quad \text{in } B_{r_1} \setminus B_{r_0}. \tag{1.4}$$

Given a diffeomorphism $\mathcal{T}$ from D onto D', the following standard notations are used:

$$\mathcal{T}_* a(x') = \frac{\nabla \mathcal{T}(x) a(x) \nabla \mathcal{T}^T(x)}{\det \nabla \mathcal{T}(x)} \quad \text{and} \quad \mathcal{T}_* j(x') = \frac{\nabla \mathcal{T}(x) j(x)}{\det \nabla \mathcal{T}(x)}, \tag{1.5}$$

with $x' = \mathcal{T}(x)$, for a matrix-valued function a and a vector-valued function j defined in D.

As showed later in Section 3, we have

$$F_*^{-1} I = -\frac{r_2^\alpha}{r^\alpha} \left[\frac{1}{\alpha - 1} e_r \otimes e_r + (\alpha - 1)(e_\theta \otimes e_\theta + e_\theta \otimes e_\varphi) \right] \quad \text{in } B_{r_2} \setminus B_{r_1}. \tag{1.6}$$

Letting $\alpha = \beta = 2$, one rediscovers the construction suggested by Pendry and Ramakrishna in $B_{r_2} \setminus B_{r_1}$. Note that even in this case, our lens construction contains two layers and is different from theirs where one layer is used. We emphasize here that the lens-construction is independent of the object. Taking into account the loss, the medium is characterized by $(\varepsilon_\delta, \mu_\delta)$, where

$$(\varepsilon_\delta, \mu_\delta) = \begin{cases} (F_*^{-1} I + i\delta I, F_*^{-1} I + i\delta I) & \text{in } B_{r_2} \setminus B_{r_1}, \\ (mI, mI) & \text{in } B_{r_1} \setminus B_{r_0}, \\ (\varepsilon_O, \mu_O) & \text{in } B_{r_0}, \\ (I, I) & \text{otherwise.} \end{cases}$$

Some comments on the construction are necessary. The media (ε_O, μ_O) in $B_{r_2} \setminus B_{r_1}$ and (I, I) in $B_{r_3} \setminus B_{r_2}$ are complementary or more precisely reflecting complementary (see Section 2). For a given r_2, we choose r_1 and r_3 such that $\frac{r_3}{r_1} = m$ and $F(\partial B_{r_1}) = \partial B_{r_3}$, since a superlens of m times magnification is considered. The choice of $(\varepsilon_\delta, \mu_\delta) = (\varepsilon_O, \mu_O) = (mI, mI)$ in $B_{r_1} \setminus B_{r_0}$ and $r_2 = m r_0$ is to ensure, by (1.1), that

$$(G_* F_* \varepsilon_O, G_* F_* \mu_O) = (I, I) \quad \text{in } B_{r_3} \setminus B_{r_2}. \tag{1.7}$$

Fix $k > 0$. Given $j \in L^2_c(\mathbb{R}^3)$ with $\operatorname{supp} j \subset\subset \mathbb{R}^3 \setminus B_{r_3}$, let $(E_\delta, H_\delta), (\hat{E}, \hat{H}) \in [H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)]^2$ $(\delta > 0)$ be respectively the unique outgoing solution to

$$\begin{cases} \nabla \times E_\delta = ik\mu_\delta H_\delta & \text{in } \mathbb{R}^3, \\ \nabla \times H_\delta = -ik\varepsilon_\delta E_\delta + j & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \nabla \times \hat{E} = ik\hat{\mu}\hat{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{H} = -ik\hat{\varepsilon}\hat{E} + j & \text{in } \mathbb{R}^3, \end{cases} \tag{1.9}$$

where

$$(\hat{\varepsilon}, \hat{\mu}) = \begin{cases} (I, I) & \text{in } \mathbb{R}^3 \setminus B_{mr_0}, \\ (m^{-1}\varepsilon_O(\cdot/m), m^{-1}\mu_O(\cdot/m)) & \text{otherwise.} \end{cases}$$

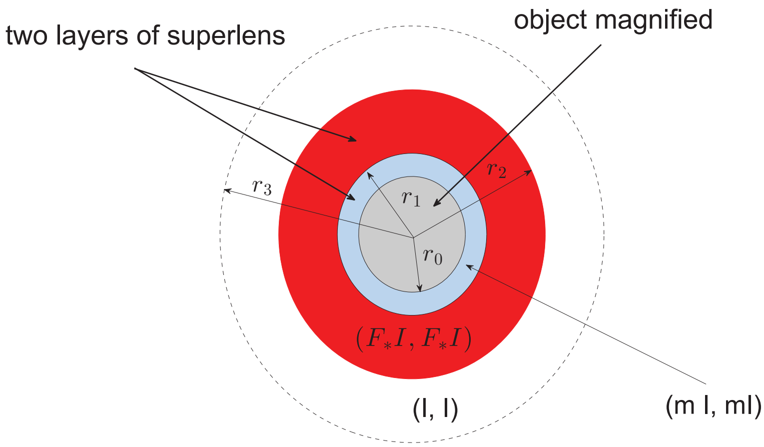

Figure 1: Lens contains two layers: the outer layer using NIMs is given by (1.3), the inner layer is given by (1.4).

Recall that a solution $(E, H) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus B_R)]^2$ (for some $R > 0$) to the system

$$\begin{cases} \nabla \times E = ikH & \text{in } \mathbb{R}^3 \setminus B_R, \\ \nabla \times H = -ikE & \text{in } \mathbb{R}^3 \setminus B_R, \end{cases}$$

is said to satisfy the outgoing condition (or the Silver-Müller radiation condition) if

$$E \times x + rH = O\Big(\frac{1}{r}\Big) \quad \text{as } r = |x| \to +\infty.$$

Our result on superlensing is the following theorem.

Theorem 1. *Let $j \in [L^2(\mathbb{R}^3)]^3$ with $\operatorname{supp} j \subset B_{R_0} \setminus B_{r_3}$ for some $R_0 > 0$. Let $(E_\delta, H_\delta), (\hat{E}, \hat{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ be the unique outgoing solutions to* (1.8) *and* (1.9)*, respectively. We have, for $R > 0$,*

$$\|(E_\delta, H_\delta) - (\hat{E}, \hat{H})\|_{H(\mathrm{curl}, B_R \setminus B_{r_3})} \le C_R \delta^{\frac{1}{2}} \|j\|_{L^2}$$

for some positive constant C_R independent of δ and j. In particular,

$$(E_\delta, H_\delta) \to (\hat{E}, \hat{H}) \quad \text{in } [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus B_{r_3})]^2 \quad \text{as } \delta \to 0. \tag{1.10}$$

For an observer outside B_{r_3}, the object (ε_O, μ_O) in B_{r_0} would act like

$$(m^{-1}\varepsilon_O(\cdot/m), m^{-1}\mu_O(\cdot/m)) \quad \text{in } B_{mr_0}$$

by (1.10): one has a superlens whose magnification is m.

The proof of Theorem 1 given in Section 3 is derived from Theorem 2 in Section 2. Section 2 is devoted to the concept of reflecting complementary media (Definition 1) and their properties (Theorem 2). This concept appears naturally in the study of superlensing mentioned above and is inspired from [15]. The analysis of Theorem 2 is based on the reflecting technique which has root from [15] and a number of new results on the compactness, existence, and stability for the Maxwell equations with low regularity data.

The paper is organized as follows. In Section 2, we discuss reflecting complementary media. Proof of Theorem 1 is given in Section 3.

2 Reflecting complementary media

Let $\Omega_1 \subset\subset \Omega_2$ be smooth simply connected bounded open subsets of $\mathbb{R}^3$. Let ε and μ be two *real* measurable matrix-valued functions defined in $\mathbb{R}^3$. We assume that ε, μ are *bounded* in $\mathbb{R}^3$ and uniformly elliptic in $\mathbb{R}^3 \setminus (\Omega_2 \setminus \Omega_1)$, i.e., for some $1 \le \Lambda < +\infty$,

$$\frac{1}{\Lambda}|\xi|^2 \le \langle \varepsilon(x)\xi, \xi\rangle \le \Lambda|\xi|^2, \quad \frac{1}{\Lambda}|\xi|^2 \le \langle \mu(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3, \text{ a.e. } x \in \mathbb{R}^3 \setminus (\Omega_2 \setminus \Omega_1), \tag{2.1}$$

and

$$\varepsilon = \mu = I \quad \text{in } \mathbb{R}^3 \setminus B_{R_0}, \tag{2.2}$$

for some $R_0 > 0$ with $\Omega_2 \subset\subset B_{R_0}$. Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. We also assume that [1]

$$(\varepsilon, \mu) \text{ is piecewise } C^1. \tag{2.3}$$

Set, for $\delta \ge 0$,

$$(\varepsilon_\delta, \mu_\delta) = \begin{cases} (\varepsilon + i\delta I, \mu + i\delta I) & \text{if } x \in \Omega_2 \setminus \Omega_1, \\ (\varepsilon, \mu) & \text{otherwise.} \end{cases} \tag{2.4}$$

It is clear that $(\varepsilon_0, \mu_0) = (\varepsilon, \mu)$ in $\mathbb{R}^3$. Note that we do not impose the ellipticity of ε and μ in $\mathbb{R}^3$. In fact, as seen later, in the setting of reflecting complementary media, they are negative in $\Omega_2 \setminus \Omega_1$ (see Remark 2). Fix $k > 0$. Given $j \in L^2_c(\mathbb{R}^3)$, we are interested in the behavior of the unique outgoing solution $(E_\delta, H_\delta) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ ($\delta > 0$) to the Maxwell system

$$\begin{cases} \nabla \times E_\delta = ik\mu_\delta H_\delta & \text{in } \mathbb{R}^3, \\ \nabla \times H_\delta = -ik\varepsilon_\delta E_\delta + j & \text{in } \mathbb{R}^3, \end{cases} \tag{2.5}$$

as $\delta \to 0$ in the case (ε, μ) satisfies the reflecting complementary property, a concept introduced in Definition 1 below.

For an open subset Ω of $\mathbb{R}^3$, the following standard notations are used:

$$H(\mathrm{curl}, \Omega) := \{u \in [L^2(\Omega)]^3 : \nabla \times u \in [L^2(\Omega)]^3\},$$
$$\|u\|_{H(\mathrm{curl},\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla \times u\|_{L^2(\Omega)},$$
$$H_{\mathrm{loc}}(\mathrm{curl}, \Omega) := \{u \in [L^2_{\mathrm{loc}}(\Omega)]^3 : \nabla \times u \in [L^2_{\mathrm{loc}}(\Omega)]^3\}.$$

We are ready to introduce:

Definition 1 (Reflecting complementary media). Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ be smooth simply connected bounded open subsets of $\mathbb{R}^3$. The media (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary if there exists a diffeomorphism $F : \Omega_2 \setminus \bar{\Omega}_1 \to \Omega_3 \setminus \bar{\Omega}_2$ such that

$$(\varepsilon, \mu) = (F_*\varepsilon, F_*\mu) \quad \text{in } \Omega_3 \setminus \Omega_2, \tag{2.6}$$
$$F(x) = x \quad \text{on } \partial\Omega_2, \tag{2.7}$$

and the following two conditions hold:
(1) There exists a diffeomorphism extension of F, which is still denoted by F, from $\Omega_2 \setminus \{x_1\}$ onto $\mathbb{R}^3 \setminus \bar{\Omega}_2$ for some $x_1 \in \Omega_1$.
(2) There exists a diffeomorphism $G : \mathbb{R}^3 \setminus \bar{\Omega}_3 \to \Omega_3 \setminus \{x_1\}$ such that $G \in C^1(\mathbb{R}^3 \setminus \Omega_3)$, $G(x) = x$ on $\partial\Omega_3$, and $G \circ F : \Omega_1 \to \Omega_3$ is a diffeomorphism if one sets $G \circ F(x_1) = x_1$.

Here and in what follows, when we mention a diffeomorphism $F : \Omega \to \Omega'$ for two open *smooth* subsets Ω, Ω' of $\mathbb{R}^d$, we mean that F is a diffeomorphism, $F \in C^1(\bar{\Omega})$, and $F^{-1} \in C^1(\bar{\Omega}')$.

The illustration of reflecting complementary media is given in Figure 2. Note that the superlensing setting in Theorem 1 has this property. Theorem 1 will be derived from Theorem 2-below, on properties of the reflecting complementary media.

Remark 1. We emphasize here that in (1.5), $\det DT(x)$ is used and not $|\det DT(x)|$, and in (2.6), one requires that $(\varepsilon, \mu) = (F_*\varepsilon, F_*\mu)$ not $(\varepsilon, \mu) = (-F_*\varepsilon, -F_*\mu)$. These conventions are different from the ones in the acoustic setting, see, e.g., [15], and are more convenient in the study of Maxwell equations.

1 This condition is used for various uniqueness statements obtained by the unique continuation principle.

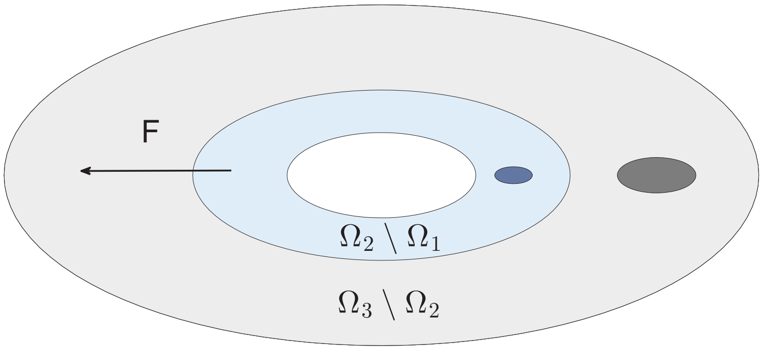

Figure 2: The media (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary media if roughly speaking $(F_*\varepsilon, F_*\mu) = (\varepsilon, \mu)$ in $\Omega_3 \setminus \Omega_2$ for some diffeomorphism F from $\Omega_2 \setminus \bar{\Omega}_1$ to $\Omega_3 \setminus \bar{\Omega}_2$ such that $F(x) = x$ on $\partial\Omega_2$

Remark 2. Assume that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary and (ε, μ) is positive in $\Omega_3 \setminus \Omega_2$. Since $F(x) = x$ on $\partial\Omega_2$ and $F : \Omega_2 \setminus \Omega_1 \to \Omega_3 \setminus \Omega_2$ is a diffeomorphism, it follows that $\det \nabla F(x) < 0$ in $\Omega_2 \setminus \Omega_1$. Therefore, (ε, μ) is negative in $\Omega_2 \setminus \Omega_1$.

We next make some comments on the definition. Condition (2.6) implies that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are complementary in the "usual" sense.[2] The term "reflecting" in the definition comes from (2.7) and the assumption $\Omega_1 \subset \Omega_2 \subset \Omega_3$. Conditions (2.6) and (2.7) are the main assumptions in the definition. They are motivated by the following observation. Assume that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary and suppose that there exists a solution $(E_0, H_0) \in [H(\operatorname{curl}, \Omega_3 \setminus \Omega_1)]^2$ of

$$\begin{cases} \nabla \times E_0 = ik\mu H_0 & \text{in } \Omega_3 \setminus \Omega_1, \\ \nabla \times H_0 = -ik\varepsilon E_0 & \text{in } \Omega_3 \setminus \Omega_1. \end{cases}$$

For $x' \in \Omega_3 \setminus \Omega_2$, define $(E_0^{(1)}(x'), H_0^{(1)}(x')) = (\nabla F^{-T}(x) E_0(x), \nabla F^{-T}(x) H_0(x))$, where $x = F^{-1}(x')$. Conditions (2.6) and (2.7) imply that, by the rule of change of variables (see, e.g., Lemma 7 in Section 2.1),

$$\begin{cases} \nabla \times (E_0 - E_0^{(1)}) = ik\mu(H_0 - H_0^{(1)}) & \text{in } \Omega_3 \setminus \Omega_2, \\ \nabla \times (H_0 - H_0^{(1)}) = -ik\varepsilon(E_0 - E_0^{(1)}) & \text{in } \Omega_3 \setminus \Omega_2, \\ (E_0 - E_0^{(1)}) \times \nu = (E_0 - E_0^{(1)}) \times \nu = 0 & \text{on } \partial\Omega_2. \end{cases}$$

Hence, $(E_0, H_0) = (E_0^{(1)}, H_0^{(1)})$ in $\Omega_3 \setminus \Omega_2$ if (ε, μ) is uniformly elliptic in $\Omega_3 \setminus \Omega_2$ by the unique continuation principle; this is the main motivation for conditions (2.6) and (2.7). Conditions (1) and (2) are mild assumptions. Introducing G in the definition makes the analysis more accessible; see Sections 2.2.

Here and in what follows, we denote

$$(\hat{\varepsilon}, \hat{\mu}) := \begin{cases} (\varepsilon, \mu) & \text{if } x \in \mathbb{R}^3 \setminus \Omega_3, \\ (G_* F_* \varepsilon, G_* F_* \mu) & \text{if } x \in \Omega_3. \end{cases} \tag{2.8}$$

The following definition is used in the statement of Theorem 2 below.

Definition 2 (Compatibility condition). Assume that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^3$. Then $j \in [L^2_c(\mathbb{R}^3)]^3$ *with* $\operatorname{supp} j \cap \Omega_3 = \emptyset$ is said to be compatible if and only if there exists $(\mathbf{E}, \mathbf{H}) \in [H(\operatorname{curl}, \Omega_3 \setminus \Omega_2)]^2$ such that

$$\begin{cases} \nabla \times \mathbf{E} = ik\mu \mathbf{H} & \text{in } \Omega_3 \setminus \Omega_2, \\ \nabla \times \mathbf{H} = -ik\varepsilon \mathbf{E} & \text{in } \Omega_3 \setminus \Omega_2, \\ \mathbf{E} \times \nu = \hat{E} \times \nu, \quad \mathbf{H} \times \nu = \hat{H} \times \nu & \text{on } \partial\Omega_3, \end{cases} \tag{2.9}$$

2 In fact, the concept of complementary media have not be defined in a precise manner. Property (2.6) mentioned here is the common point in some examples discussed in the literature.

where $(\hat{E}, \hat{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique outgoing solution to the system

$$\begin{cases} \nabla \times \hat{E} = ik\hat{\mu}\hat{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{H} = -ik\hat{\varepsilon}\hat{E} + j & \text{in } \mathbb{R}^3. \end{cases}$$

Remark 3. It is important to note that $\hat{\varepsilon}$ and $\hat{\mu}$ are uniformly elliptic in $\mathbb{R}^3$ by (2.8) since $\det \nabla F$ and $\det \nabla G$ are both negative. The existence and uniqueness of $(\hat{E}, \hat{H})$ then follow from Lemma 4 in Section 2.1. The uniqueness of $(\mathbf{E}, \mathbf{H})$ is a consequence of the unique continuation principle (see [3, 27]).

Remark 4. Note that (2.9) is a Cauchy problem: the uniqueness is ensured by the unique continuation principle but the existence is not; hence the resonance might appear.

Our main result on the reflecting complementary media for electromagnetic waves is:

Theorem 2. *Let $k > 0$, $0 < \delta < 1$, $j \in [L^2(\mathbb{R}^3)]^3$ with compact support, and let $(E_\delta, H_\delta) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ be the unique outgoing solution of (2.5). Assume that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^3$ and $\operatorname{supp} j \cap \Omega_3 = \emptyset$. We have:*

(a) *Case 1: j is compatible. There exists a unique outgoing solution $(E_0, H_0) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ to*

$$\begin{cases} \nabla \times E_0 = ik\mu H_0 & \text{in } \mathbb{R}^3, \\ \nabla \times H_0 = -ik\varepsilon E_0 + j & \text{in } \mathbb{R}^3. \end{cases} \tag{2.10}$$

Moreover,

$$(E_0, H_0) = (\hat{E}, \hat{H}) \quad \text{in } \mathbb{R}^3 \setminus \Omega_3,$$

and, for all $R > 0$,

$$\|(E_\delta, H_\delta) - (E_0, H_0)\|_{H(\mathrm{curl}, B_R)} \le C_R \delta^{\frac{1}{2}} \|(E_0, H_0)\|_{L^2((\Omega_2 \setminus \Omega_1) \cup B_R)}$$

for some positive constant C_R independent of j and δ.

(b) *Case 2: j is not compatible. We have*

$$\lim_{\delta \to 0} \|(E_\delta, H_\delta)\|_{H(\mathrm{curl}, B_R)} = +\infty$$

for $R > 0$ such that $\bar{\Omega}_2 \subset B_R$.

The implication of Theorem 1 from Theorem 2 is given in Section 3.

The rest of this section containing two subsections is devoted to the proof of Theorem 2. In the first one, we presents some lemmas used in the proof of Theorem 2. The proof of Theorem 2 is given in the second subsection.

2.1 Some useful lemmas

In this subsection, we present some technical lemmas which are used in the proof of Theorem 2.

The following compactness result plays an important role in our analysis.

Lemma 1. *Let D be a bounded smooth open subset of $\mathbb{R}^3$, $(u^{(n)}) \subset H(\mathrm{curl}, D)$, and let ε be a symmetric uniformly elliptic matrix-valued function defined in D. Assume that*

$$\sup_{n \in \mathbb{N}} \|u^{(n)}\|_{H(\mathrm{curl}, D)} < +\infty,$$

and[3]

$$(\nabla \cdot (\varepsilon u^{(n)})) \text{ converges in } H^{-1}(D) \text{ and } (u^{(n)} \times \nu) \text{ converges in } [H^{-\frac{1}{2}}(\partial D)]^3. \tag{2.11}$$

There exists a subsequence of $(u^{(n)})$ which converges in $[L^2(D)]^3$.

3 $H^{-1}(D)$ denotes the duality of $H^1_0(D)$.

Proof. We first assume that D is simply connected. Let B be an open ball such that $\bar{D} \subset B$. Let $\varphi^{(n)} \in H^1(B \setminus D)$ be the unique solution with zero mean, i.e., $\int_{B\setminus D} \varphi_n = 0$, to

$$\begin{cases} -\Delta\varphi^{(n)} = 0 & \text{in } B \setminus D, \\ \partial_\nu \varphi^{(n)} = (\nabla \times u^{(n)}) \cdot \nu & \text{on } \partial D, \\ \partial_\nu \varphi^{(n)} = 0 & \text{on } \partial B. \end{cases}$$

The existence of $\varphi^{(n)}$ is a consequence of the fact

$$\int_{\partial D} (\nabla \times u^{(n)}) \cdot \nu = 0,$$

since $\nabla \cdot (\nabla \times u^{(n)}) = 0$ in D. Set

$$\chi^{(n)} = \begin{cases} \nabla \times u^{(n)} & \text{in } D, \\ \nabla \varphi^{(n)} & \text{in } B \setminus D, \\ 0 & \text{in } \mathbb{R}^3 \setminus B. \end{cases}$$

It is clear that $\nabla \cdot \chi^{(n)} = 0$ in $\mathbb{R}^3$. Set[4]

$$\Psi^{(n)} = G * \chi^{(n)} \quad \text{in } \mathbb{R}^3,$$

where G is the fundamental solution to the Laplace equation in $\mathbb{R}^3$; this implies $-\Delta\Psi^{(n)} = \chi^{(n)}$ in $\mathbb{R}^3$ and

$$\|\Psi_n\|_{H^2(B)} \le C\|\chi_n\|_{L^2}. \tag{2.12}$$

Here and in what follows, C denotes a positive constant depending only on B and D. Since $\nabla \cdot \chi^{(n)} = 0$ in $\mathbb{R}^3$, it follows that

$$\nabla \cdot \Psi^{(n)} = 0 \quad \text{in } \mathbb{R}^3. \tag{2.13}$$

Set

$$w^{(n)} = \nabla \times \Psi^{(n)} \quad \text{in } D.$$

We derive from (2.12) that $(w^{(n)})$ is bounded in $[H^1(D)]^3$. Without loss of generality, one may assume that

$$(w^{(n)}) \text{ converges in } [L^2(D)]^3. \tag{2.14}$$

Using the fact that

$$\nabla \times (\nabla \times \Psi^{(n)}) = \nabla(\nabla \cdot \Psi^{(n)}) - \Delta\Psi^{(n)} \quad \text{in } D,$$

we derive from (2.13) that

$$\nabla \times w^{(n)} = \nabla \times u^{(n)} \quad \text{in } D.$$

Since D is simply connected, one has

$$u^{(n)} = w^{(n)} + \nabla p^{(n)} \quad \text{in } D$$

for some $p^{(n)} \in H^1(D)$ such that $\int_{\partial D} p^{(n)} = 0$ (see, e.g., [14, Theorem 3.37]); hence

$$\nabla \cdot (\varepsilon \nabla p^{(n)}) = \nabla \cdot (\varepsilon u^{(n)}) - \nabla \cdot (\varepsilon w^{(n)}) \quad \text{in } D.$$

A combination of (2.11) and (2.14) yields

$$(\nabla \cdot (\varepsilon \nabla p^{(n)})) \text{ converges in } H^{-1}(D). \tag{2.15}$$

On the other hand,

$$\|p^{(n)} - p^{(m)}\|_{H^{\frac{1}{2}}(\partial D)} \le C\|\nabla p^{(n)} \times \nu - \nabla p^{(m)} \times \nu\|_{H^{-\frac{1}{2}}(\partial D)}$$

4 The notation $*$ here means the convolution.

since $\int_{\partial D} p^{(n)} = 0$; which implies

$$\|p^{(n)} - p^{(m)}\|_{H^{\frac{1}{2}}(\partial D)} \leq C\|u^{(n)} \times \nu - u^{(m)} \times \nu\|_{H^{-\frac{1}{2}}(\partial D)} + C\|w^{(n)} \times \nu - w^{(m)} \times \nu\|_{H^{-\frac{1}{2}}(\partial D)}.$$

Since $(w^{(n)})$ is bounded in $[H^1(D)]^3$ and converges in $[L^2(D)]^3$, it follows from (2.11) that $(p^{(n)})$ converges in $H^{\frac{1}{2}}(\partial D)$. Combining this, (2.11), and (2.15), we derive that $(p^{(n)})$ converges in $H^1(D)$. Since $u^{(n)} = w^{(n)} + \nabla p^{(n)}$ and (w^n) converges in $[L^2(D)]^3$, we derive that $(u^{(n)})$ converges in $[L^2(D)]^3$. The proof is complete in the case D is simply connected. The proof in the general case follows by using local charts. □

Remark 5. Lemma 1 is known if instead of (2.11) one assumes that

$$(\nabla \cdot (\varepsilon u^{(n)})) \text{ is bounded in } L^2(D) \text{ and } (u^{(n)} \times \nu) \text{ is bounded in } [L^2(\partial D)]^3$$

(see [37]). It is clear that Lemma 1 implies the known compactness result. The case $\varepsilon = I$ was established in [8, Lemma A5] under the additional assumption $(\nabla \cdot (\varepsilon u^{(n)}))$ is bounded in L^2. The proof presented here is in the same spirit of the one given in [8], which has roots from [7]. Condition (2.11) appears naturally when one studies the existence and the stability for Maxwell equations (see Lemmas 3, 4, and 5).

The second lemma is a known result on the trace of $H(\mathrm{curl}, D)$ (see [1, 5, 29]).

Lemma 2. *Let D be a smooth open bounded subset of $\mathbb{R}^3$ and set $\Gamma = \partial D$. The tangential trace operator*

$$\gamma_0 : H(\mathrm{curl}, D) \to H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \quad u \mapsto u \times \nu$$

is continuous. Moreover, for all $\phi \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, there exists $u \in H(\mathrm{curl}, D)$ such that $\gamma_0(u) = \phi$ and

$$\|u\|_{H(\mathrm{curl},D)} \leq C\|\phi\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}$$

for some positive constant C independent of ϕ.

Here

$$H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) := \{\phi \in [H^{-\frac{1}{2}}(\Gamma)]^3 : \phi \cdot \nu = 0 \text{ and } \mathrm{div}_\Gamma\, \phi \in H^{-\frac{1}{2}}(\Gamma)\}$$

and

$$\|\phi\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} := \|\phi\|_{H^{-\frac{1}{2}}(\Gamma)} + \|\, \mathrm{div}_\Gamma\, \phi\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Using Lemmas 1 and 2, we can easily reach the following result, which is used in the proof of Lemma 6 to establish the stability of (2.5).

Lemma 3. *Let $k > 0$, D a smooth open bounded subset of $\mathbb{R}^3$, $f, g \in [L^2(D)]^3$, and $h_1, h_2 \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \partial D)$, and let ε and μ be two symmetric uniformly elliptic matrix-valued functions defined in D such that* (2.3) *holds. Assume that $(\mathcal{E}, \mathcal{H}) \in [H(\mathrm{curl}, D)]^2$ is a solution to*

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} + f & \text{in } D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} + g & \text{in } D, \\ \mathcal{H} \times \nu = h_1, \quad \mathcal{E} \times \nu = h_2 & \text{on } \partial D. \end{cases}$$

Then

$$\|(\mathcal{E}, \mathcal{H})\|_{H(\mathrm{curl},D)} \leq C\big(\|(f, g)\|_{L^2(D)} + \|(h_1, h_2)\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial D)}\big) \tag{2.16}$$

for some positive constant C depending on D, ε, μ, and k but independent of f, g, h_1, and h_2.

Proof. Using Lemma 2, without loss of generality, one may assume that $h_1 = h_2 = 0$. We prove (2.16) by contradiction. Assume that there exist $f_n, g_n \in L^2(D)$ such that

$$\|(\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\|_{H(\mathrm{curl},D)} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \|(f_n, g_n)\|_{L^2(D)} = 0. \tag{2.17}$$

Here $(\mathcal{E}^{(n)}, \mathcal{H}^{(n)})$ is the unique solution to

$$\begin{cases} \nabla \times \mathcal{E}^{(n)} = ik\mu\mathcal{H}^{(n)} + f_n & \text{in } D, \\ \nabla \times \mathcal{H}^{(n)} = -ik\varepsilon\mathcal{E}^{(n)} + g_n & \text{in } D, \\ \mathcal{H}^{(n)} \times \nu = \mathcal{E}^{(n)} \times \nu = 0 & \text{on } \partial D. \end{cases} \tag{2.18}$$

Applying Lemma 1, one may assume that $(\mathcal{E}^{(n)}, \mathcal{H}^{(n)}) \to (\mathcal{E}, \mathcal{H})$ in $[L^2(D)]^6$ and hence in $[H(\operatorname{curl}, D)]^2$ by (2.18). Moreover,

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} & \text{in } D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} & \text{in } D, \\ \mathcal{H} \times \nu = \mathcal{E} \times \nu = 0 & \text{on } \partial D. \end{cases}$$

This implies $\mathcal{E} = \mathcal{H} = 0$ by the unique continuation principle [27, Theorem 1]. This contradicts the fact

$$\|(\mathcal{E}, \mathcal{H})\|_{H(\operatorname{curl}, D)} = 1,$$

by (2.17). The conclusion follows. □

We next deal with the existence, uniqueness, and stability of outgoing solutions defined in the whole space.

Lemma 4. *Let $k > 0$, let D be a smooth open bounded subset of $\mathbb{R}^3$, $f, g \in [L^2(\mathbb{R}^3)]^3$, $h_1, h_2 \in H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial D)$. Assume that $\bar{D}$, $\operatorname{supp} f$, $\operatorname{supp} g \subset B_{R_0}$ for some $R_0 > 0$. Let ε, μ be two symmetric uniformly elliptic matrix-valued functions defined in $\mathbb{R}^3$ such that (2.2) and (2.3) hold. There exists $(\mathcal{E}, \mathcal{H}) \in [\bigcap_{R>0} H(\operatorname{curl}, B_R \setminus \partial D)]^2$ the unique outgoing solution to* [5]

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} + f & \text{in } \mathbb{R}^3 \setminus \partial D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} + g & \text{in } \mathbb{R}^3 \setminus \partial D, \\ [\mathcal{H} \times \nu] = h_1, \quad [\mathcal{E} \times \nu] = h_2 & \text{on } \partial D. \end{cases} \tag{2.19}$$

Moreover,

$$\|(\mathcal{E}, \mathcal{H})\|_{H(\operatorname{curl}, B_R \setminus \partial D)} \le C_R\big(\|(f, g)\|_{L^2} + \|(h_1, h_2)\|_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial D)}\big) \tag{2.20}$$

for some positive constant C_R depending on R, R_0, D, ε, μ, and k, but independent of f, g, h_1, and h_2.

The well-posedness of (2.19) is known for $h_1 = h_2 = 0$ and $f, g \in H(\operatorname{div}, \mathbb{R}^3)$ (in this case, $\|(f, g)\|_{L^2}$ is replaced by $\|(f, g)\|_{H(\operatorname{div})}$ in (2.20) since the standard compactness criterion was used).[6] To our knowledge, Lemma 4 is new and the proof requires the new compactness criterion in Lemma 1.

Proof. Using Lemma 2, without loss of generality, one may assume that $h_1 = h_2 = 0$. The uniqueness is a consequence of Rellich's lemma (see, e.g., [6, Theoren 6.1]) and the unique continuation principle [27, Theorem 1.1]. The details are left to the reader. The existence and the stability can be derived from the uniqueness using the limiting absorption principle in the spirit of [11] and the compactness result in Lemma 1 as follows. For $0 < \tau < 1$, let $(\mathcal{E}^\tau, \mathcal{H}^\tau) \in [H(\operatorname{curl}, \mathbb{R}^3)]^2$ be the unique solution to

$$\begin{cases} \nabla \times \mathcal{E}^\tau = ik(1 + i\tau)\mu\mathcal{H}^\tau + f & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H}^\tau = -ik(1 + i\tau)\varepsilon\mathcal{E}^\tau + g & \text{in } \mathbb{R}^3. \end{cases} \tag{2.21}$$

This implies

$$\nabla \times \Big(\frac{1}{1 + i\tau}\mu^{-1}\nabla \times \mathcal{E}^\tau\Big) - k^2(1 + i\tau)\mathcal{E}^\tau = ikg + \nabla \times \Big(\frac{1}{1 + i\tau}\mu^{-1} f\Big) \quad \text{in } \mathbb{R}^3.$$

Multiplying the equation by $\bar{\mathcal{E}}^\tau$ (the conjugate of $\mathcal{E}^\tau$), integrating on $\mathbb{R}^3$, and considering the imaginary part, we have

$$\|(\mathcal{E}^\tau, \mathcal{H}^\tau)\|_{H(\operatorname{curl}, \mathbb{R}^3)} \le \frac{C}{\tau}\|(f, g)\|_{L^2}.$$

Here and in what follows in this proof, C denotes a positive constant independent of f, g, and τ. We claim that

$$\|(\mathcal{E}^\tau, \mathcal{H}^\tau)\|_{H(\operatorname{curl}, B_{R_0+2})} \le C\|(f, g)\|_{L^2}. \tag{2.22}$$

5 Here $[\,\cdot\,]$ denotes the jump across the boundary.
6 Note that $H(\operatorname{div}, \mathbb{R}^3) := \{u \in [L^2(\mathbb{R}^3)]^3 : \operatorname{div} u \in L^2(\mathbb{R}^3)\}$ and $\|u\|_{H(\operatorname{div})} := \|u\|_{L^2} + \|\operatorname{div} u\|_{L^2}$.

We prove this by contradiction. To this end, assume that there exist $\tau_n \to 0_+$ and $f_n, g_n \in L^2(\mathbb{R}^3)$ with $\operatorname{supp} f_n, \operatorname{supp} g_n \subset B_{R_0}$ such that

$$\|(\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\|_{H(\mathrm{curl}, B_{R_0+2})} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \|(f_n, g_n)\|_{L^2} = 0.$$

Here $(\mathcal{E}^{(n)}, \mathcal{H}^{(n)}) \in [H(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique outgoing solution to (2.21) with $f = f_n$, $g = g_n$, and $\tau = \tau_n$. The Stratton–Chu formula (see, e.g., [6, Theorem 6.6]), gives, for $|x| > R_0 + 1$,

$$\mathcal{E}^{(n)}(x) = -\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{E}^{(n)} \times \nu) G_n(x, y)\, dy + \frac{1}{ik_n} \operatorname{curl}\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{H}^{(n)} \times \nu) G_n(x, y)\, dy \tag{2.23}$$

and

$$\mathcal{H}^{(n)}(x) = -\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{H}^{(n)} \times \nu) G_n(x, y)\, dy - \frac{1}{ik_n} \operatorname{curl}\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{E}^{(n)} \times \nu) G_n(x, y)\, dy. \tag{2.24}$$

Here $k_n = k(1 + i\tau_n)$ and $G_n(x, y) = \frac{e^{ik_n|x-y|}}{4\pi|x-y|}$. Since $\|(\mathcal{E}_n, \mathcal{H}_n)\|_{L^2(B_{R_0+2})} = 1$, it follows from (2.23) and (2.24) that

$$\|(\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\|_{H(\mathrm{curl}, B_R)} \le C_R \quad \text{for all } R > 0.$$

Applying Lemma 1, without loss of generality, one may assume that $(\mathcal{E}^{(n)}, \mathcal{H}^{(n)}) \to (\mathcal{E}, \mathcal{H})$ in $[L^2_{\mathrm{loc}}(\mathbb{R}^3)]^6$, and hence in $[H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ by (2.21). Moreover, $(\mathcal{E}, \mathcal{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ satisfies

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} & \text{in } \mathbb{R}^3. \end{cases}$$

Letting $n \to \infty$ in (2.23) and (2.24), we derive that $(\mathcal{E}, \mathcal{H})$ satisfies the Stratton–Chu formula

$$\mathcal{E}(x) = -\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{E} \times \nu) G(x, y)\, dy + \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{H} \times \nu) G(x, y)\, dy \tag{2.25}$$

and

$$\mathcal{H}(x) = -\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{H} \times \nu) G(x, y)\, dy - \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial B_{R_0+1}} (\mathcal{E} \times \nu) G(x, y)\, dy, \tag{2.26}$$

where $G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$. Hence $(\mathcal{E}, \mathcal{H})$ satisfies the outgoing condition. The uniqueness of the outgoing solutions yields

$$\mathcal{E} = \mathcal{H} = 0 \quad \text{in } \mathbb{R}^3.$$

This contradicts the fact $\|(\mathcal{E}, \mathcal{H})\|_{H(\mathrm{curl}, B_{R_0+2})} = \lim_{n\to\infty} \|(\mathcal{E}^{(n)}, \mathcal{H}^{(n)})\|_{H(\mathrm{curl}, B_{R_0+2})} = 1$. Hence (2.22) is proved. From (2.22), (2.23) and (2.24), we obtain

$$\|(\mathcal{E}^{\tau}, \mathcal{H}^{\tau})\|_{H(\mathrm{curl}, B_R)} \le C_R \|(f, g)\|_{L^2}.$$

Applying Lemma 1, without loss of generality, one may assume that $(\mathcal{E}^{\tau}, \mathcal{H}^{\tau}) \to (\mathcal{E}, \mathcal{H})$ in $[H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ as $\tau \to 0$; moreover, $(\mathcal{E}, \mathcal{H})$ is a solution to

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} + f & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} + g & \text{in } \mathbb{R}^3. \end{cases}$$

We also have (2.25) and (2.26) for $(\mathcal{E}, \mathcal{H})$. Therefore, $(\mathcal{E}, \mathcal{H})$ satisfies the outgoing condition. The estimate of $(\mathcal{E}, \mathcal{H})$ follows from the estimate of $(\mathcal{E}^{\tau}, \mathcal{H}^{\tau})$. The proof is complete. □

Remark 6. The unique continuation of the Maxwell equations has a long story, see, e.g., [3, 11, 27, 32] and the references therein. It has been known from [11] that the principle holds for ε, μ in C^2. However, under the assumption ε, μ in C^1, it was proved recently in [27] (see also [3] for a more general setting) using the fact the Maxwell equations can be reduced to a weakly coupled second order elliptic equations see, e.g., [11, p. 168].

Similarly, we obtain the following result on the exterior Dirichlet boundary problem.

Lemma 5. *Let $k > 0$, let D be a smooth open bounded subset of $\mathbb{R}^3$, $f, g \in [L^2(\mathbb{R}^3 \setminus D)]^3$, and $h \in H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial D)$. Let ε, μ be two symmetric uniformly elliptic matrix-valued functions defined in $\mathbb{R}^3 \setminus D$ such that conditions (2.2) and (2.3) hold. Assume that $\mathbb{R}^3 \setminus D$ is connected and $\operatorname{supp} f, \operatorname{supp} g \subset B_{R_0} \setminus D$ for some $R_0 > 0$. Let $(\mathcal{E}, \mathcal{H}) \in [H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus D)]^2$ be the unique outgoing solution to*

$$\begin{cases} \nabla \times \mathcal{E} = ik\mu\mathcal{H} + f & \text{in } \mathbb{R}^3 \setminus D, \\ \nabla \times \mathcal{H} = -ik\varepsilon\mathcal{E} + g & \text{in } \mathbb{R}^3 \setminus D, \\ \mathcal{E} \times \nu = h & \text{on } \partial D. \end{cases}$$

Then

$$\|(\mathcal{E}, \mathcal{H})\|_{H(\operatorname{curl}, B_R \setminus D)} \le C_R\big(\|(f, g)\|_{L^2} + \|h\|_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial D)}\big)$$

for some positive constant C_R depends on R, R_0, D, ε, μ, and k, but independent of f, g, and h.

Remark 7. The same result holds if the condition $\mathcal{E} \times \nu = h$ on ∂D is replaced by the condition $\mathcal{H} \times \nu = h$ on ∂D.

Proof. The proof of Lemma 5 is similar to the one of Lemma 4. The details are left to the reader. □

We are ready to state and prove the stability result for (2.5).

Lemma 6. *Let $0 < \delta < 1$, $f, g \in [L^2(\mathbb{R}^3)]^3$, and let $(\varepsilon_\delta, \mu_\delta)$ be defined in (2.4). Assume that ε and μ are bounded in $\mathbb{R}^d$ and satisfy (2.1), (2.2) and (2.3), and $\operatorname{supp} f$, $\operatorname{supp} g$, $\bar{\Omega}_2 \subset B_{R_0}$. There exists a unique outgoing solution $(\mathcal{E}_\delta, \mathcal{H}_\delta) \in [H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)]^2$ to*

$$\begin{cases} \nabla \times \mathcal{E}_\delta = ik\mu_\delta\mathcal{H}_\delta + f & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H}_\delta = -ik\varepsilon_\delta\mathcal{E}_\delta + g & \text{in } \mathbb{R}^3. \end{cases}$$

Moreover,

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\operatorname{curl}, B_R)} \le \frac{C_R}{\delta}\|(f, g)\|_{L^2}. \tag{2.27}$$

Assume in addition that $\operatorname{supp} f \subset \bar{D}$, $\operatorname{supp} g \subset \bar{D}$, and $\bar{D} \cap \Omega_2 = \emptyset$ for some smooth open subset D of $\mathbb{R}^3$. Then

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\operatorname{curl}, B_R)} \le \frac{C_R}{\delta}\|(f, g)\|_{L^2}\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\operatorname{curl}, D)} + C_R\|(f, g)\|^2_{L^2}, \tag{2.28}$$

Here C_R denotes a positive constant depending on R, R_0, ε, μ, and D but independent of f, g, and δ.

Remark 8. Lemma 6 does not require any assumptions on the reflecting complementary property. In the proof of Theorem 2, we apply Lemma 6 with $D = B_R \setminus \Omega_2$ for some $R > 0$.

Proof. For $\delta > 0$ fixed, the existence and uniqueness of $(\mathcal{E}_\delta, \mathcal{H}_\delta)$ can be obtained as in the proof of Lemma 4. The details are omitted. We only give the proof of (2.27) and (2.28). We have, in $\mathbb{R}^3$,

$$\nabla \times (\mu_\delta^{-1}\nabla \times \mathcal{E}_\delta) - k^2\varepsilon_\delta\mathcal{E}_\delta = \nabla \times (\mu_\delta^{-1} f) + ikg.$$

Set

$$M_\delta = \frac{1}{\delta}\|(f, g)\|_{L^2}\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{L^2(B_{R_0})} + \|(f, g)\|^2_{L^2}.$$

Multiplying the equation by $\bar{\mathcal{E}}_\delta$, integrating in B_R, and using the fact $\operatorname{supp} f \subset B_{R_0}$, we have, for $R > R_0$,

$$\int_{B_R} \langle \mu_\delta^{-1}\nabla \times \mathcal{E}_\delta, \nabla \times \mathcal{E}_\delta\rangle - \int_{\partial B_R} \langle (\mu_\delta^{-1}\nabla \times \mathcal{E}_\delta) \times \nu, \mathcal{E}_\delta\rangle - k^2 \int_{B_R} \langle \varepsilon_\delta\mathcal{E}_\delta, \mathcal{E}_\delta\rangle = \int_{B_R} \langle \mu_\delta^{-1} f, \nabla \times \mathcal{E}_\delta\rangle + \int_{B_R} \langle ikg, \mathcal{E}_\delta\rangle.$$

Since $\mu_\delta = I$, $f = 0$, and $\nabla \times \mathcal{E}_\delta = ik\mathcal{H}_\delta$ in $\mathbb{R}^3 \setminus B_{R_0}$, we derive that, for $R > R_0$,

$$\int_{B_R} \langle \mu_\delta^{-1}\nabla \times \mathcal{E}_\delta, \nabla \times \mathcal{E}_\delta\rangle + \int_{\partial B_R} \langle ikH_\delta, \mathcal{E}_\delta \times \nu\rangle - k^2 \int_{B_R} \langle \varepsilon_\delta\mathcal{E}_\delta, \mathcal{E}_\delta\rangle = \int_{B_R} \langle \mu_\delta^{-1} f, \nabla \times \mathcal{E}_\delta\rangle + \int_{B_R} \langle ikg, \mathcal{E}_\delta\rangle.$$

Letting $R \to +\infty$, using the outgoing condition, and considering the imaginary part, we obtain

$$\|\mathcal{E}_\delta\|^2_{H(\operatorname{curl}, \Omega_2 \setminus \Omega_1)} \le CM_\delta. \tag{2.29}$$

This implies, by Lemma 2,

$$\|\mathcal{E}_\delta \times \nu\|^2_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial\Omega_2)} + \|\mathcal{E}_\delta \times \nu\|^2_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial\Omega_1)} \le CM_\delta.$$

Similarly, we have

$$\|\mathcal{H}_\delta\|^2_{H(\operatorname{curl}, \Omega_2\setminus\Omega_1)} \le CM_\delta, \tag{2.30}$$

which yields, by Lemma 2 again,

$$\|\mathcal{H}_\delta \times \nu\|^2_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial\Omega_2)} + \|\mathcal{H}_\delta \times \nu\|^2_{H^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \partial\Omega_1)} \le CM_\delta.$$

Applying Lemma 5, we have

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\operatorname{curl}, B_R\setminus\Omega_2)} \le C_R M_\delta, \tag{2.31}$$

and applying Lemma 3, we obtain

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|^2_{H(\operatorname{curl}, \Omega_1)} \le CM_\delta. \tag{2.32}$$

A combination of (2.29), (2.30), (2.31), and (2.32) yields

$$\|(\mathcal{E}_\delta, \mathcal{H}_\delta)\|_{H(\operatorname{curl}, B_R)} \le C_R M_\delta. \tag{2.33}$$

This implies (2.27). Inequality (2.28) follows from (2.33) by noting that in the definition of M_δ, one can replace $\|(E_\delta, H_\delta)\|_{L^2}$ by $\|(E_\delta, H_\delta)\|_{L^2(D)}$ if $\operatorname{supp} f \subset \bar{D}$, $\operatorname{supp} g \subset \bar{D}$, and $\bar{D} \cap \Omega_2 = \emptyset$. □

The following change of variables for the Maxwell equations motivates the definition of reflecting complementary media.

Lemma 7. *Let D, D' be two open bounded connected subsets of $\mathbb{R}^3$ and let $\mathcal{T}: D \to D'$ be bijective such that $\mathcal{T} \in C^1(\bar{D})$ and $\mathcal{T}^{-1} \in C^1(\bar{D}')$. Assume that $\varepsilon, \mu \in [L^\infty(D)]^{3\times 3}$, $j \in [L^2(D)]^3$ and $(E, H) \in [H(\operatorname{curl}, D)]^2$ is a solution to*

$$\begin{cases} \nabla \times E = ik\mu H & \text{in } D, \\ \nabla \times H = -ik\varepsilon E + j & \text{in } D. \end{cases}$$

Define (E', H') in D' as follows:

$$E'(x') = \mathcal{T} * E(x') := \nabla\mathcal{T}^{-T}(x)E(x) \quad \text{and} \quad H'(x') = \mathcal{T} * H(x') := \nabla\mathcal{T}^{-T}(x)H(x), \tag{2.34}$$

with $x' = \mathcal{T}(x)$ and set $\varepsilon' = \mathcal{T}_*\varepsilon$, $\mu' = \mathcal{T}_*\mu$, and $j' = T_* j$ by (1.5). Then (E', H') is a solution to

$$\begin{cases} \nabla' \times E' = ik\mu' H' & \text{in } D', \\ \nabla' \times H' = -ik\varepsilon' E' + j' & \text{in } D'. \end{cases} \tag{2.35}$$

Assume in addition that D is of class C^1 and $\mathbf{T} = \mathcal{T}|_{\partial D} : \partial D \to \partial D'$ is a diffeomorphism. Let ν and ν' denote the outward unit normal vector on ∂D and $\partial D'$. We have

$$\text{if } E \times \nu = g \text{ and } H \times \nu = h \text{ on } \partial D, \text{ then } E' \times \nu' = \mathbf{T}_* g \text{ and } H' \times \nu' = \mathbf{T}_* h \text{ on } \partial D', \tag{2.36}$$

where $\mathbf{T}_*$ is defined by, for a tangential vector field φ defined in ∂D,

$$\mathbf{T}_*\varphi(x') = \operatorname{sign} \cdot \frac{\nabla_{\partial D}\mathbf{T}(x)\varphi(x)}{|\det \nabla_{\partial D}\mathbf{T}(x)|} \quad \text{with } x' = \mathbf{T}(x),$$

where sign $:= \det \nabla\mathcal{T}(x)/|\det \nabla\mathcal{T}(x)|$ *for some $x \in D$. In particular, if $D \cap D' = \emptyset$ and $\mathcal{T}(x) = x$ on $\partial D \cap \partial D'$, then*

$$H' \times \nu = H \times \nu \quad \text{and} \quad E' \times \nu = E \times \nu \quad \text{on } \partial D \cap \partial D'. \tag{2.37}$$

Assertion (2.37) immediately follows from (2.36) by noting that sign $= -1$, $\nu' = -\nu$, and $\mathbf{T} = I$ on $\partial D \cap \partial D'$ in this case. This assertion is used several times in the proof of Theorem 2.

Remark 9. Note that the definition of $\mathcal{T}*$ is different from $\mathcal{T}_*$ for a field in $\mathbb{R}^3$. It is helpful to remember that for electromagnetic fields (2.34) is used whereas for sources (1.5) is involved.

Remark 10. System (2.35) is known for a smooth pair (E, H). Statement (2.36) might be known; however, we cannot find a reference for it. For the convenience of the reader, we give the details of the proof in Appendix A for the form stated here.

2.2 Proof of Theorem 2

The proof is divided into three steps.

- Step 1: Assume that there exists an outgoing solution $(E_0, H_0) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ of (2.10). Then j is compatible and $(E_0, H_0) = (\mathscr{E}_0, \mathscr{H}_0)$, where

$$(\mathscr{E}_0, \mathscr{H}_0) := \begin{cases} (\hat{E}, \hat{H}) & \text{in } \mathbb{R}^3 \setminus \Omega_3, \\ (\mathbf{E}, \mathbf{H}) & \text{in } \Omega_3 \setminus \Omega_2, \\ (F^{-1} * \mathbf{E}, F^{-1} * \mathbf{H}) & \text{in } \Omega_2 \setminus \Omega_1, \\ (F^{-1} * G^{-1} * \hat{E}, F^{-1} * G^{-1} * \hat{H}) & \text{in } \Omega_1. \end{cases} \tag{2.38}$$

- Step 2: Assume that j is compatible. Then $(\mathscr{E}_0, \mathscr{H}_0)$ given in (2.38) is the unique outgoing solution of (2.10). Moreover,

$$\|(E_\delta, H_\delta) - (\mathscr{E}_0, \mathscr{H}_0)\|_{H(\text{curl}, B_R)} \le C_R \delta^{\frac{1}{2}} \|(\mathscr{E}_0, \mathscr{H}_0)\|_{L^2((\Omega_2 \setminus \Omega_1) \cup B_R)}.$$

- Step 3: Assume that j is not compatible. Then

$$\lim_{\delta \to 0_+} \|(E_\delta, H_\delta)\|_{H(\text{curl}, B_R)} = +\infty$$

for $R > 0$ such that $\bar{\Omega}_2 \subset B_R$.

It is clear that the conclusion follows after Step 3. We now proceed these steps.

Step 1. Let $(E_0^{(1)}, H_0^{(1)})$ be the reflection of (E_0, H_0) in Ω_2 through $\partial\Omega_2$ by F, i.e.,[7]

$$(E_0^{(1)}, H_0^{(1)}) = (F * E_0, F * H_0) \quad \text{in } \mathbb{R}^3 \setminus \Omega_2,$$

which implies

$$(E_0, H_0) = (F^{-1} * E_0^{(1)}, F^{-1} * H_0^{(1)}) \quad \text{in } \Omega_2 \setminus \{x_1\}. \tag{2.39}$$

Recall that $(\varepsilon_0, \mu_0) = (\varepsilon, \mu)$ in $\mathbb{R}^3$. It follows from Lemma 7 that

$$\begin{cases} \nabla \times E_0^{(1)} = ikF_*\mu H_0^{(1)} & \text{in } \mathbb{R}^3 \setminus \Omega_2, \\ \nabla \times H_0^{(1)} = -ikF_*\varepsilon E_0^{(1)} & \text{in } \mathbb{R}^3 \setminus \Omega_2, \end{cases} \tag{2.40}$$

and

$$E_0^{(1)} \times \nu = E_0 \times \nu \quad \text{and} \quad H_0^{(1)} \times \nu = H_0 \times \nu \quad \text{on } \partial\Omega_2.$$

Since $(F_*\varepsilon, F_*\mu) = (\varepsilon, \mu)$ in $\Omega_3 \setminus \Omega_2$, it follows from the unique continuation principle that

$$(E_0^{(1)}, H_0^{(1)}) = (E_0, H_0) \quad \text{in } \Omega_3 \setminus \Omega_2. \tag{2.41}$$

Let $(E_0^{(2)}, H_0^{(2)})$ be the reflection of $(E_0^{(1)}, H_0^{(1)})$ in $\mathbb{R}^3 \setminus \Omega_3$ through $\partial\Omega_3$ by G, i.e.,

$$(E_0^{(2)}, H_0^{(2)}) = (G * E_0^{(1)}, G * H_0^{(1)}) \quad \text{in } \Omega_3 \setminus \{x_1\},$$

which implies

$$(E_0^{(1)}, H_0^{(1)}) = (G^{-1} * E_0^{(2)}, G^{-1} * H_0^{(2)}) \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}_3. \tag{2.42}$$

Set

$$(\mathcal{E}, \mathcal{H}) = \begin{cases} (E_0, H_0) & \text{in } \mathbb{R}^3 \setminus \Omega_3, \\ (E_0^{(2)}, H_0^{(2)}) & \text{in } \Omega_3. \end{cases}$$

We have, by applying Lemma 7 and using (2.40),

$$\begin{cases} \nabla \times E_0^{(2)} = ik\hat{\mu} H_0^{(2)} & \text{in } \Omega_3, \\ \nabla \times H_0^{(2)} = -ik\hat{\varepsilon} E_0^{(2)} & \text{in } \Omega_3, \end{cases}$$

7 The definition of $F * E$ and $F * H$ are given in (2.34).

and, by applying Lemma 7 and using (2.41),

$$E_0^{(2)} \times \nu = E_0^{(1)} \times \nu = E_0 \times \nu \quad \text{and} \quad H_0^{(2)} \times \nu = H_0^{(1)} \times \nu = H_0 \times \nu \quad \text{on } \partial\Omega_3.$$

It follows that $(\mathcal{E}, \mathcal{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$, $(\mathcal{E}, \mathcal{H})$ satisfies the outgoing condition, and

$$\begin{cases} \nabla \times \mathcal{E} = ik\hat{\mu}\mathcal{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H} = -ik\hat{\varepsilon}\mathcal{E} + j & \text{in } \mathbb{R}^3. \end{cases}$$

We derive that

$$(\mathcal{E}, \mathcal{H}) = (\hat{E}, \hat{H}) \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad (\mathbf{E}, \mathbf{H}) = (E_0, H_0) \quad \text{in } \Omega_3 \setminus \Omega_2.$$

From the definitions of $(\mathscr{E}_0, \mathscr{H}_0)$ and $(\mathcal{E}, \mathcal{H})$, (2.39), and (2.42), we obtain

$$(E_0, H_0) = (\mathscr{E}_0, \mathscr{H}_0) \quad \text{in } \mathbb{R}^3.$$

Step 2. It is clear that

$$(\mathscr{E}_0, \mathscr{H}_0) \in [H(\mathrm{curl}, B_R \setminus (\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3)) \quad \text{for all } R > 0. \tag{2.43}$$

Using the fact $F(x) = x$ on $\partial\Omega_2$ and $G(x) = x$ on $\partial\Omega_3$ and applying Lemma 7, we have

$$[\mathscr{E}_0 \times \nu] = [\mathscr{H}_0 \times \nu] = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \tag{2.44}$$

From the definition of $(\mathbf{E}, \mathbf{H})$ and $(\mathscr{E}, \mathscr{H})$, we obtain

$$[\mathscr{E}_0 \times \nu] = [\mathscr{H}_0 \times \nu] = 0 \quad \text{on } \partial\Omega_3. \tag{2.45}$$

Applying Lemma 7 again, we get

$$\begin{cases} \nabla \times \mathscr{E}_0 = ik\mu_0\mathscr{H}_0 & \text{in } \mathbb{R}^3 \setminus (\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3), \\ \nabla \times \mathscr{H}_0 = -ik\varepsilon_0\mathscr{E}_0 + j & \text{in } \mathbb{R}^3 \setminus (\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3). \end{cases} \tag{2.46}$$

We derive from (2.43), (2.44), (2.45), and (2.46) that $(\mathscr{E}_0, \mathscr{H}_0) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is an outgoing solution of (2.10) and hence the unique outgoing solution by Step 1. We have

$$\begin{cases} \nabla \times (E_\delta - \mathscr{E}_0) = ik\mu_\delta(H_\delta - \mathscr{H}_0) + ik(\mu_0 - \mu_\delta)\mathscr{H}_0 & \text{in } \mathbb{R}^3, \\ \nabla \times (H_\delta - \mathscr{H}_0) = -ik\varepsilon_\delta(E_\delta - \mathscr{E}_0) + ik(\varepsilon_\delta - \varepsilon_0)\mathscr{E}_0 & \text{in } \mathbb{R}^3. \end{cases}$$

Moreover, $(E_\delta - \mathscr{E}_0, H_\delta - \mathscr{H}_0)$ satisfies the outgoing condition. Applying (2.27) in Lemma 6, we have, for $R > 0$,

$$\|(E_\delta - \mathscr{E}_0, H_\delta - \mathscr{H}_0)\|_{H(\mathrm{curl}, B_R)} \le C_R \|(\mathscr{E}_0, \mathscr{H}_0)\|_{L^2(\Omega_2 \setminus \Omega_1)},$$

which implies

$$\|(E_\delta, H_\delta)\|_{H(\mathrm{curl}, B_R)} \le C_R \|(\mathscr{E}_0, \mathscr{H}_0)\|_{L^2((\Omega_2 \setminus \Omega_1) \cup B_R)}. \tag{2.47}$$

Applying (2.28) in Lemma 6 for $(E_\delta - \mathscr{E}_0, H_\delta - \mathscr{H}_0)$ and using (2.47), we obtain this time, for $R > 0$,

$$\|(E_\delta - \mathscr{E}_0, H_\delta - \mathscr{H}_0)\|_{H(\mathrm{curl}, B_R)} \le C_R \delta^{\frac{1}{2}} \|(E_0, H_0)\|_{L^2((\Omega_2 \setminus \Omega_1) \cup B_R)}.$$

Step 3. We prove Step 3 by contradiction. Assume that there exists $R > 0$ such that $\bar{\Omega}_2 \subset B_R$, and, for some $(\delta_n) \to 0$,

$$\sup_n \|(E_{\delta_n}, H_{\delta_n})\|_{H(\mathrm{curl}, B_R)} < +\infty.$$

Applying Lemma 5, we have

$$\sup_n \|(E_{\delta_n}, H_{\delta_n})\|_{H(\mathrm{curl}, B_r)} < +\infty \text{ for all } r > 0.$$

Without loss of generality, one may assume that $(E_{\delta_n}, H_{\delta_n}) \rightharpoonup (E_0, H_0)$ weakly in $[H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$. Then (E_0, H_0) is an outgoing solution to (2.10). Therefore, j is compatible by Step 1. We have a contradiction. The proof of Step 3 is complete. □

Remark 11. It is clear from the proof that in the case j is compatible, (E_0, H_0) can be determined by

$$(E_0, H_0) = \begin{cases} (\hat{E}, \hat{H}) & \text{in } \mathbb{R}^3 \setminus \Omega_3, \\ (\mathbf{E}, \mathbf{H}) & \text{in } \Omega_3 \setminus \Omega_2, \\ (F^{-1} * \mathbf{E}, F^{-1} * \mathbf{H}) & \text{in } \Omega_2 \setminus \Omega_1, \\ (F^{-1} * G^{-1} * \hat{E}, F^{-1} * G^{-1} * \hat{H}) & \text{in } \Omega_1. \end{cases} \tag{2.48}$$

This is the key observation for the setting of Theorem 2.

Remark 12. In this paper, we confine ourself to the case $\operatorname{supp} j \cap \Omega_3 = \emptyset$ for simple presentation. In fact, the proof of Theorem 2 can be extended to cover the case where no condition on $\operatorname{supp} j$ is required. The details are left to the reader (see [15] for a complete account in the acoustic setting).

Remark 13. In [22, Theorems 2 and 3 and Proposition 2] we showed that in the acoustic setting the complementary property is "necessary" to the appearance of resonance in the sense that the field can blow up in L^2-norm even in the region away from the interface of sign changing coefficients. This property would hold in the electromagnetic setting and will be considered elsewhere.

3 Proof of Theorem 1

Theorem 1 is a consequence of Theorem 2. In fact, we can derive from Theorem 2 the following more general result:

Proposition 1. *Let $0 < \delta < 1$, $j \in L^2_c(\mathbb{R}^3)$ and let $(E_\delta, H_\delta) \in [H_{\rm loc}(\operatorname{curl}, \mathbb{R}^3)]^2$ be the unique outgoing solution of (2.5). Assume that (ε, μ) in $\Omega_2 \setminus \Omega_1$ and (ε, μ) in $\Omega_3 \setminus \Omega_2$ are reflecting complementary for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \mathbb{R}^3$,* and *$(\hat{\varepsilon}, \hat{\mu}) = (\varepsilon, \mu)$ in $\Omega_3 \setminus \Omega_2$. Then j with $\operatorname{supp} j \cap \Omega_3 = \emptyset$ is compatible and there exists a unique outgoing solution $(E_0, H_0) \in [H_{\rm loc}(\operatorname{curl}, \mathbb{R}^3)]^2$ to*

$$\begin{cases} \nabla \times E_0 = ik\mu H_0 & \text{in } \mathbb{R}^3, \\ \nabla \times H_0 = -ik\varepsilon E_0 + j & \text{in } \mathbb{R}^3. \end{cases}$$

Moreover,

$$(E_0, H_0) = (\hat{E}, \hat{H}) \quad \text{in } \mathbb{R}^3 \setminus \Omega_3,$$

and, for all $R > 0$,

$$\|(E_\delta, H_\delta) - (E_0, H_0)\|_{H(\operatorname{curl}, B_R)} \le C_R \delta^{\frac{1}{2}} \|j\|_{L^2}$$

for some positive constant C_R independent of j and δ.

Proof. Since $(\hat{\varepsilon}, \hat{\mu}) = (\varepsilon, \mu)$ in $\Omega_3 \setminus \Omega_2$, we have

$$\begin{cases} \nabla \times \hat{E} = ik\mu \hat{H} & \text{in } \Omega_3 \setminus \Omega_2, \\ \nabla \times \hat{H} = -ik\varepsilon \hat{E} & \text{in } \Omega_3 \setminus \Omega_2. \end{cases}$$

Hence $(\mathbf{E}, \mathbf{H})$ exists in $\Omega_3 \setminus \Omega_2$ and $(\mathbf{E}, \mathbf{H}) = (\hat{E}, \hat{H})$ in $\Omega_3 \setminus \Omega_2$. We derive from (2.48) that

$$(E_0, H_0) = \begin{cases} (\hat{E}, \hat{H}) & \text{in } \mathbb{R}^3 \setminus \Omega_2, \\ (F^{-1} * \hat{E}, F^{-1} * \hat{H}) & \text{in } \Omega_2 \setminus \Omega_1, \\ (F^{-1} * G^{-1} * \hat{E}, F^{-1} * G^{-1} * \hat{H}) & \text{in } \Omega_1. \end{cases} \tag{3.1}$$

On the other hand, from the definition of $(\hat{E}, \hat{H})$ and Lemma 4, we have

$$\|(\hat{E}, \hat{H})\|_{H(\operatorname{curl}, B_R)} \le C_R \|j\|_{L^2}.$$

It follows from (3.1) that

$$\|(E_0, H_0)\|_{H(\operatorname{curl}, B_R)} \le C_R \|j\|_{L^2}.$$

The conclusion now follows from Theorem 2. □

We are now ready to give the

Proof of Theorem 1. Applying Proposition 1 with $\Omega_j = B_{r_j}$ and noting that $(\hat{\varepsilon}, \hat{\mu}) = (\varepsilon, \mu)$ in $B_{r_3} \setminus B_{r_2}$ by (1.7), one obtains the conclusion of Theorem 1. □

Remark 14. It follows from (1.2) that $r_3 \to m r_0$ as $\alpha \to 1_+$. Therefore, for any $\varepsilon > 0$, there exists a lens-construction such that m times magnification for an object in B_{r_0} takes place for any f with $\operatorname{supp} f \cap B_{mr_0+\varepsilon} = \emptyset$.

Remark 15. The construction of lenses is not restricted to the symmetric geometry considered here: the geometry of lenses can be quite arbitrary (see Proposition 1). This is one of the motivations of the study reflecting complementary media in a general form given in Section 2.

Remark 16. In comparison with the lens construction in [19] for the acoustic setting, we assume here that $mr_0 = r_2$ instead of the condition $mr_0 < \sqrt{r_2 r_3}$. A rate of the convergence which is $\delta^{\frac{1}{2}}$ is obtained in this case and the proof does not involve the removing localized singularity technique.

In the rest of this section, we give the

Proof of (1.6). We have

$$F^{-1}(x) = \frac{r_3^\beta}{m} K(x), \quad \text{where } K(x) = \frac{x}{|x|^\beta}.$$

We claim that

$$\frac{\nabla K(x) \nabla K(x)^T}{\det(\nabla K(x))} = -|x|^\beta \Big[\frac{1}{\alpha - 1} e_r \otimes e_r + (\alpha - 1)(e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) \Big]. \tag{3.2}$$

By using rotations, it suffices to prove (3.2) for $x = (x_1, 0, 0)$. We have

$$\frac{\partial}{\partial x_j} \Big(\frac{x_i}{|x|^\beta} \Big) = \frac{\delta_{ij}}{|x|^\beta} - \beta \frac{x_i x_j}{|x|^{\beta+2}}.$$

It follows that, for $x = (x_1, 0, 0)$,

$$\frac{\nabla K(x) \nabla K(x)^T}{\det(\nabla K(x))} = \frac{|x_1|^\beta}{(1-\beta)} \begin{pmatrix} (1-\beta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

hence (3.2) is proved since $(\alpha - 1)(\beta - 1) = 1$. Thus, for $x' = F^{-1}(x)$, we have

$$F_*^{-1} I(x') = -\frac{m}{r_3^\beta} \frac{r_2^{\alpha\beta}}{|x'|^{\beta(\alpha-1)}} \Big[\frac{1}{\alpha - 1} e_{r'} \otimes e_{r'} + (\alpha - 1)(e_{\theta'} \otimes e_{\theta'} + e_{\varphi'} \otimes e_{\varphi'}) \Big],$$

since $(e_{r'}, e_{\theta'}, e_{\varphi'}) = (e_r, e_\theta, e_\varphi)$. Using the fact that $\beta(\alpha - 1) = \alpha$ and

$$\frac{m r_2^{\alpha\beta}}{r_3^\beta} = r_2^\alpha \frac{m r_2^{\alpha(\beta-1)}}{r_3^\beta} = r_2^\alpha \frac{m r_2^\beta}{r_3^\beta} = r_2^\alpha m \Big(\frac{1}{m^{\frac{\alpha-1}{\alpha}}} \Big)^\beta = r_2^\alpha,$$

we obtain

$$F_*^{-1} I(x') = -\frac{r_2^\alpha}{|x'|^\alpha} \Big[\frac{1}{\alpha - 1} e_{r'} \otimes e_{r'} + (\alpha - 1)(e_{\theta'} \otimes e_{\theta'} + e_{\varphi'} \otimes e_{\varphi'}) \Big],$$

which is (1.6). □

A Appendix: Proof of Lemma 7

Set

$$J(x) = \nabla T(x) \quad \text{for } x \in D.$$

We first prove $\nabla' \times H' = -ik\varepsilon' E' + j'$ for smooth (E, H). We have, for indices in $\{1, 2, 3\}$,

$$(\nabla \times H)_c = \epsilon_{abc} \partial_a H_b, \quad \partial_a = J_{da} \partial'_d,$$

and

$$H_b = (J^T H')_b = J_{db} H'_d, \quad E_c = (J^T E')_c = J_{dc} E'_d.$$

Here ϵ_{abc} denotes the usual Levi-Civita permutation, i.e.,

$$\epsilon_{abc} = \begin{cases} \operatorname{sign}(abc) & \text{if } abc \text{ is a permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(\nabla \times H)_c = \epsilon_{abc} \partial_a H_b = \epsilon_{abc} J_{da} \partial'_d (J_{eb} H'_e) = \epsilon_{abc} J_{da} J_{eb} \partial'_d H'_e \tag{A.1}$$

and

$$-ik(\varepsilon E)_c = -ik\varepsilon_{cd} E_d = -ik\varepsilon_{cd} J_{ed} E'_e. \tag{A.2}$$

Here in the last identity of (A.1), we used the fact that

$$\epsilon_{abc} J_{da} \partial'_d (J_{eb}) = 0.$$

From (A.1), we derive that

$$J_{fc} (\nabla \times H)_c = \epsilon_{abc} J_{fc} J_{da} J_{eb} \partial'_d H'_e = \det J \, \epsilon_{def} \partial'_d H'_e = \det J (\nabla' \times H')_f,$$

which yields

$$J(\nabla \times H) = \det J (\nabla' \times H'). \tag{A.3}$$

From (A.2), we obtain

$$-ikJ_{fc}(\varepsilon E)_c = -ikJ_{fc}\varepsilon_{cd} J_{ed} E'_e = -ik \det J (\varepsilon' E')_f,$$

which implies

$$-ikJ\varepsilon E = -ik \det J \varepsilon' E'. \tag{A.4}$$

It is clear that

$$J_{fc} j_c = \det J j'_f. \tag{A.5}$$

Identity $\nabla' \times H' = -ik\varepsilon' E' + j'$ for smooth (E, H, j) now follows from (A.3), (A.4), and (A.5). The proof of $\nabla' \times H' = -ik\varepsilon' E' + j'$ in the general case $(E, H) \in [H(\operatorname{curl}, D)]^2$ can be proceeded as follows. We have, for $\varphi \in [C^1_c(D)]^3$,

$$\int_D (\nabla \times H) J^T \varphi = \int_D (-ik\varepsilon E + j) J^T \varphi. \tag{A.6}$$

It is clear that

$$\int_D (-ik\varepsilon E + j) J^T \varphi = \int_D (-ikJ\varepsilon J^T J^{-T} E + Jj)\varphi = \operatorname{sign} \cdot \int_{D'} (-ik\varepsilon' E' + j')\varphi_1, \tag{A.7}$$

where $\varphi_1(x') = \varphi(x)$. In the last inequality, we made a change of variable $x' = T(x)$. On the other hand,

$$\int_D (\nabla \times H) J^T \varphi = -\int_D H \nabla \times (J^T \varphi).$$

We have, as in (A.3),

$$\nabla \times (J^T \varphi) = \det J J^{-1} (\nabla' \times \varphi_1).$$

It follows that, after a change of variables,

$$\int_D (\nabla \times H) J^T \varphi = -\int_D \det J J^{-T} H (\nabla' \times \varphi_1) = -\operatorname{sign} \cdot \int_{D'} H' (\nabla' \times \varphi_1),$$

which implies

$$\int_D (\nabla \times H) J^T \varphi = \operatorname{sign} \cdot \int_{D'} (\nabla' \times H') \varphi_1. \tag{A.8}$$

A combination of (A.6), (A.7), and (A.8) yields

$$\int_{D'} (\nabla' \times H')\varphi_1 = \int_{D'} (-ik\varepsilon' E' + j')\varphi_1.$$

Since φ is arbitrary, so is φ_1, we derive that $\nabla' \times H' = -ik\varepsilon' E' + j'$. Identity $\nabla' \times E' = ik\mu' H'$ can be obtained form $\nabla' \times H' = -ik\varepsilon' E' + j'$ by interchanging the role of E and H, ε and μ, replacing k by $-k$, and taking $j = 0$.

We next prove (2.36). Fix $\varphi \in [C^1(\bar{D})]^3$. We have

$$\int_D (\nabla \times H) J^T \varphi = \int_D (-ik\varepsilon E + j) J^T \varphi.$$

This implies, by an integration by parts,

$$\int_D H \nabla \times (J^T \varphi) - \int_{\partial D} h(J^T \varphi) = \int_D (-ik\varepsilon E + j) J^T \varphi.$$

Since (A.7) also holds for $\varphi \in C^1(\bar{D})$, we derive that

$$\operatorname{sign} \cdot \int_{D'} H'(\nabla' \times \varphi_1) - \int_{\partial D} h(J^T \varphi) = \operatorname{sign} \cdot \int_{D'} (-ik\varepsilon' E' + j')\varphi_1,$$

where $\varphi_1(x') = \varphi(x)$. Integration by parts gives

$$\operatorname{sign} \cdot \int_{D'} (\nabla' \times H')\varphi_1 + \operatorname{sign} \cdot \int_{\partial D'} \varphi_1 (H' \times \nu') - \int_{\partial D} h(J^T \varphi) = \operatorname{sign} \cdot \int_{D'} (-ik\varepsilon' E' + j')\varphi_1.$$

We obtain

$$\int_{\partial D'} (H' \times \nu')\varphi_1 = \operatorname{sign} \cdot \int_{\partial D} \mathbf{J} h \varphi,$$

where $\mathbf{J} = \nabla_{\partial D} \mathbf{T}$ since $h \cdot \nu = 0$ on ∂D. A change of variable yields

$$H' \times \nu' = \mathbf{T}_* h \quad \text{on } \partial D'.$$

Similarly, we obtain $E' \times \nu' = \mathbf{T}_* g$ on $\partial D'$. The proof is complete.

References

[1] A. Alonso and A. Valli, Some remarks on the characterization of the space of tangential traces of $H(\mathrm{rot}; \Omega)$ and the construction of an extension operator, *Manuscripta Math.* **89** (1996), 159–178.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G. W. Milton, Spectral theory of a Neumann–Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, *Arch. Ration. Mech. Anal.* **208** (2013), no. 2, 667–692.

[3] J. M. Ball, Y. Capdeboscq and B. Tsering-Xiao, On uniqueness for time harmonic anisotropic Maxwell's equations with piecewise regular coefficients, *Math. Models Methods Appl. Sci.* **22** (2012), no. 11, Article ID 1250036.

[4] G. Bouchitté and B. Schweizer, Cloaking of small objects by anomalous localized resonance, *Quart. J. Mech. Appl. Math.* **63** (2010), no. 4, 437–463.

[5] A. Buffa, M. Costabel and D. Sheen, On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.* **276** (2002), no. 2, 845–867.

[6] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Appl. Math. Sci. 93, Springer., Berlin, 1998.

[7] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Methods Appl. Sci.* **12** (1990), no. 4, 365–368.

[8] H. Haddar, P. Joly and H.-M. Nguyen, Generalized impedance boundary conditions for scattering problems from strongly absorbing obstacles: The case of Maxwell's equations, *Math. Models Methods Appl. Sci.* **18** (2008), no. 10, 1787–1827.

[9] R. V. Kohn, J. Lu, B. Schweizer and M. I. Weinstein, A variational perspective on cloaking by anomalous localized resonance, *Comm. Math. Phys.* **328** (2014), no. 1, 1–27.

[10] Y. Lai, H. Chen, Z. Zhang and C. T. Chan, Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell, *Phys. Rev. Lett.* **102** (2009), Article ID 093901.
[11] R. Leis, *Initial-Boundary Value Problems in Mathematical Physics*, B. G. Teubner, Stuttgart, 1986.
[12] G. W. Milton and N.-A. P. Nicorovici, On the cloaking effects associated with anomalous localized resonance, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** (2006), no. 2074, 3027–3059.
[13] G. W. Milton, N.-A. P. Nicorovici, R. C. McPhedran and V. A. Podolskiy, A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461** (2005), no. 2064, 3999–4034.
[14] P. Monk, *Finite Element Methods for Maxwell's Equations*, Numer. Math. Sci. Comput., Oxford University Press, New York, 2003.
[15] H.-M. Nguyen, Asymptotic behavior of solutions to the Helmholtz equations with sign changing coefficients, *Trans. Amer. Math. Soc.* **367** (2015), no. 9, 6581–6595.
[16] H.-M. Nguyen, Cloaking via anomalous localized resonance. A connection between the localized resonance and the blow up of the power for doubly complementary media, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 1, 41–46.
[17] H.-M. Nguyen, Cloaking via anomalous localized resonance for doubly complementary media in the finite frequency regime, preprint (2015), http://arxiv.org/abs/1511.08053; to appear in *J. Anal. Math.*
[18] H.-M. Nguyen, Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime, *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 6, 1327–1365.
[19] H.-M. Nguyen, Superlensing using complementary media, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32** (2015), no. 2, 471–484.
[20] H.-M. Nguyen, Cloaking an arbitrary object via anomalous localized resonance: The cloak is independent of the object: The acoustic case, preprint (2016), https://arxiv.org/abs/1607.06492; to appear in *SIAM J. Math. Anal.*
[21] H.-M. Nguyen, Cloaking using complementary media in the quasistatic regime, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), no. 6, 1509–1518.
[22] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients, *J. Math. Pures Appl. (9)* **106** (2016), no. 2, 342–374.
[23] H.-M. Nguyen, Cloaking using complementary media for electromagnetic waves, preprint (2017), https://arxiv.org/abs/1701.02339.
[24] H.-M. Nguyen, Negative index materials and their applications: Recent mathematics progress, *Chin. Ann. Math. Ser. B* **38** (2017), no. 2, 601–628.
[25] H.-M. Nguyen and L. H. Nguyen, Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations, *Trans. Amer. Math. Soc. Ser. B* **2** (2015), 93–112.
[26] H.-M. Nguyen and L. H. Nguyen, Localized and complete resonance in plasmonic structures, *ESAIM Math. Model. Numer. Anal.* **49** (2015), no. 3, 741–754.
[27] T. Nguyen and J.-N. Wang, Quantitative uniqueness estimate for the Maxwell system with Lipschitz anisotropic media, *Proc. Amer. Math. Soc.* **140** (2012), no. 2, 595–605.
[28] N. A. Nicorovici, R. C. McPhedran and G. W. Milton, Optical and dielectric properties of partially resonant composites, *Phys. Rev. B* **49** (1994), 8479–8482.
[29] L. Paquet, Problèmes mixtes pour le système de Maxwell, *Ann. Fac. Sci. Toulouse Math. (5)* **4** (1982), no. 2, 103–141.
[30] J. B. Pendry, Negative refraction makes a perfect lens, *Phys. Rev. Lett.* **85** (2000), 3966–3969.
[31] J. B. Pendry, Perfect cylindrical lenses, *Opt. Express* **1** (2003), 755–760.
[32] M. H. Protter, Unique continuation for elliptic equations, *Trans. Amer. Math. Soc.* **95** (1960), 81–91.
[33] S. A. Ramakrishna and J. B. Pendry, Focusing light using negative refraction, *J. Phys.* **14** (2002), 8463–8479.
[34] S. A. Ramakrishna and J. B. Pendry, Spherical perfect lens: Solutions of Maxwell's equations for spherical geometry, *Phys. Rev. B* **69** (2004), Article ID 115115.
[35] R. A. Shelby, D. R. Smith and S. Schultz, Experimental verification of a negative index of refraction, *Sci.* **292** (2001), 77–79.
[36] V. G. Veselago, The electrodynamics of substances with simultaneously negative values of ϵ and μ, *Usp. Fiz. Nauk* **92** (1964), 517–526.
[37] C. Weber, A local compactness theorem for Maxwell's equations, *Math. Methods Appl. Sci.* **2** (1980), no. 1, 12–25.