On the Soundness of Behavioural Abstraction in Hybrid Systems

SIM@SYST.Level, 19th of October, 2014, Cargèse, France

Simon Bliudze and Sébastien Furic

Towards

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S. Bliudze and S. Furic. *An Operational Semantics for Hybrid Systems Involving Behavioral Abstraction*. Proc. of the 10th International Modelica Conference, Lund, Sweden, pp. 693–706. 2014.

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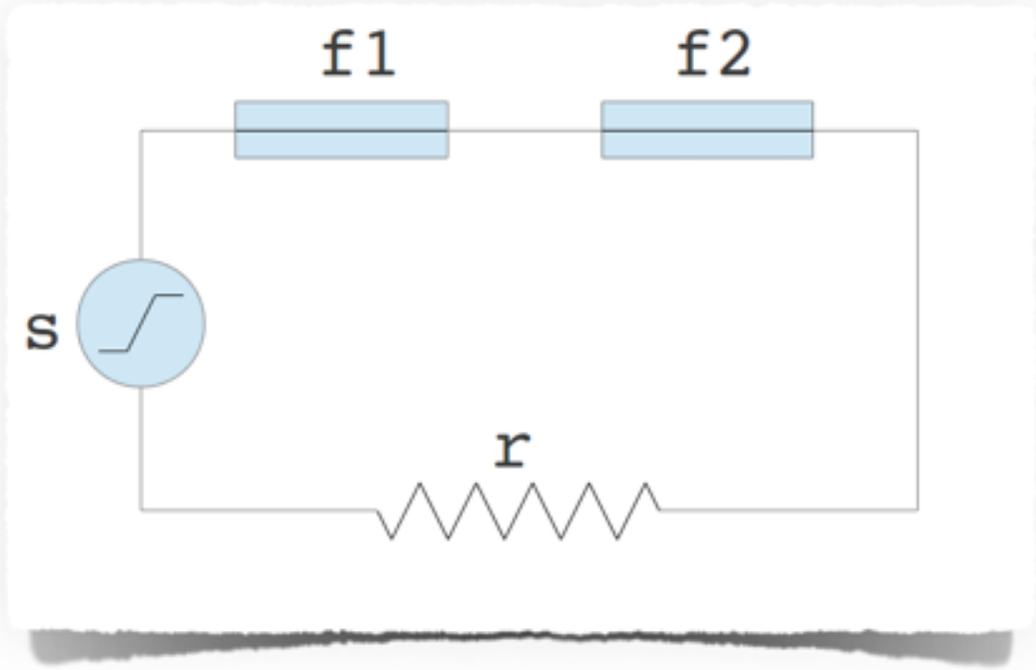
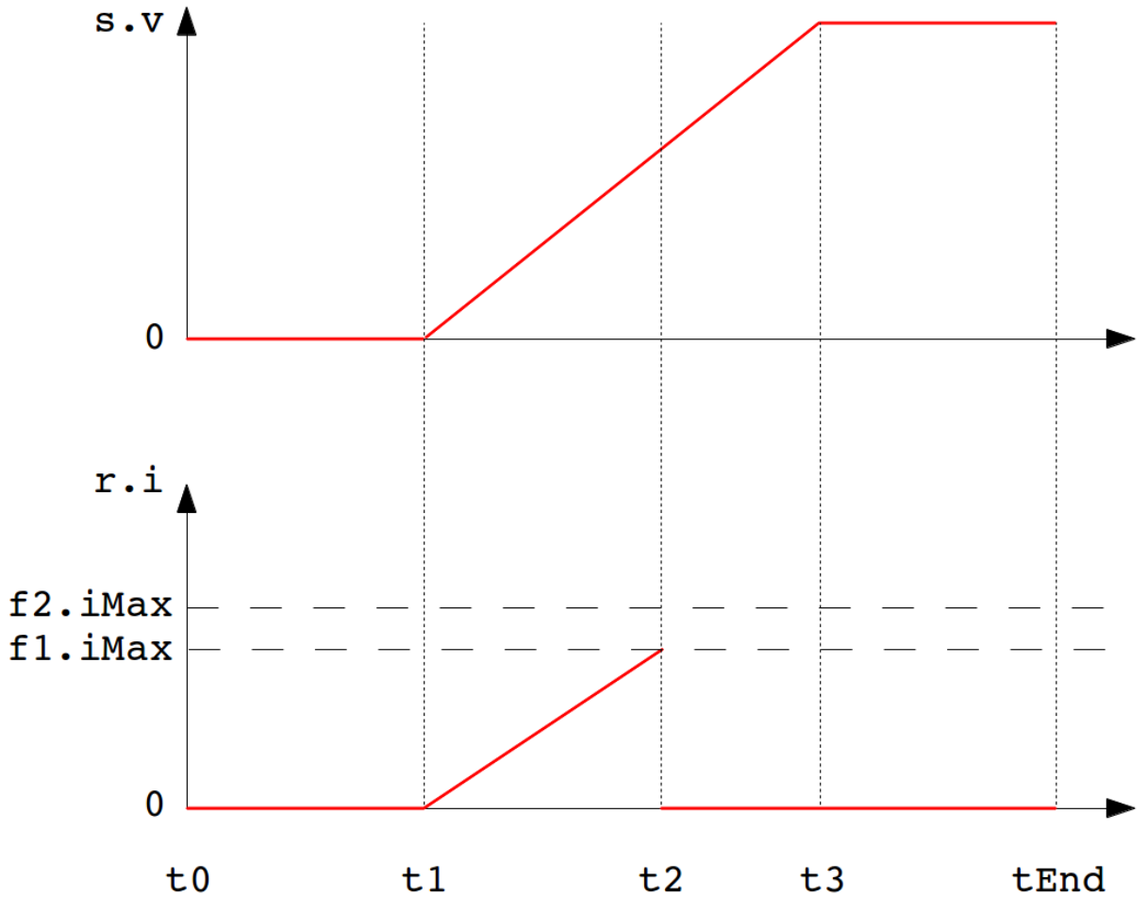
Abstraction

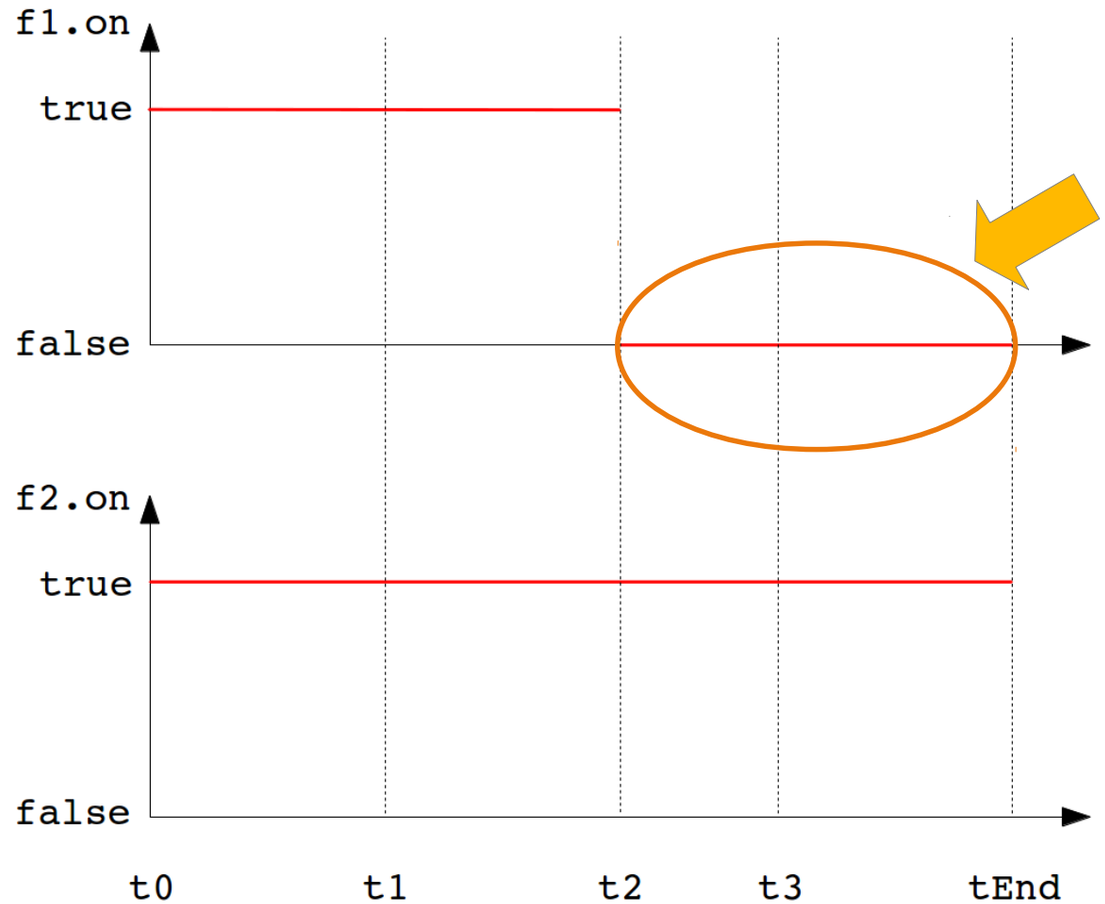
```
model Fuse
    extends Interfaces.OnePort;
    parameter Real iMax;
    parameter Real Ron, Roff;
    Boolean on;
protected Real R;
initial equation
    on = true;
equation
    when i > iMax then
        on = false;
    end when;
    R = if on then Ron else Roff;
    v = R * i;
end Fuse;
```



- The fuse model assumes negligible melting duration
  - In particular w.r.t. the raise duration of the voltage source

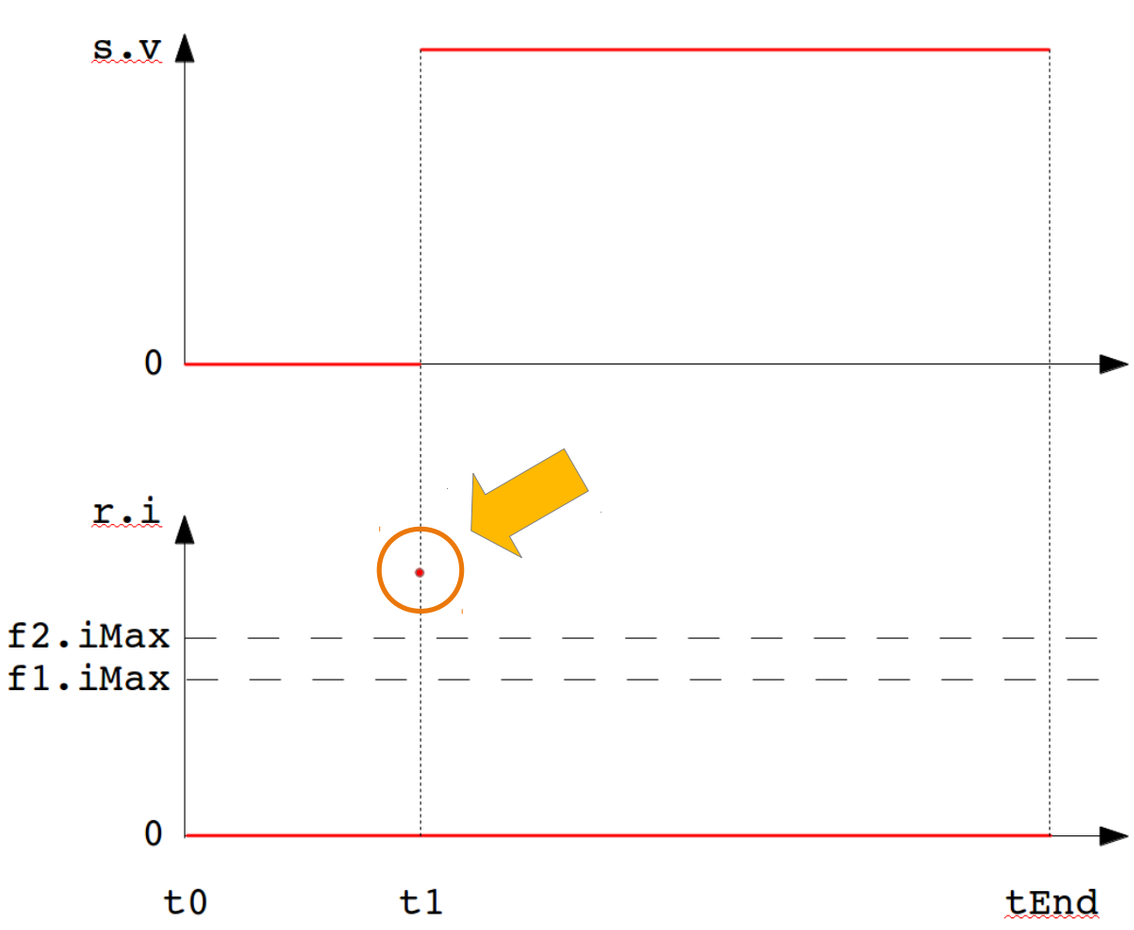
Expected behaviour

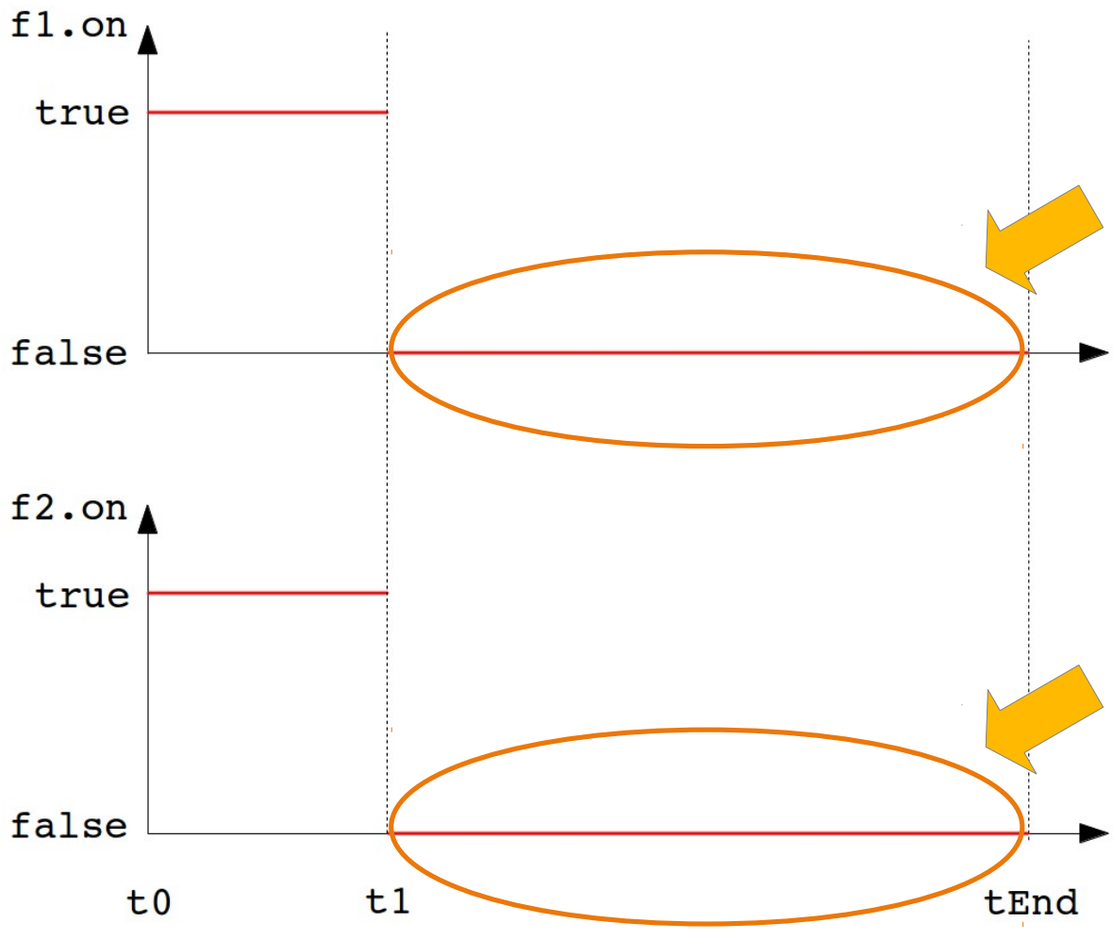




- Only the first fuse melts
  - Independently of the voltage slope

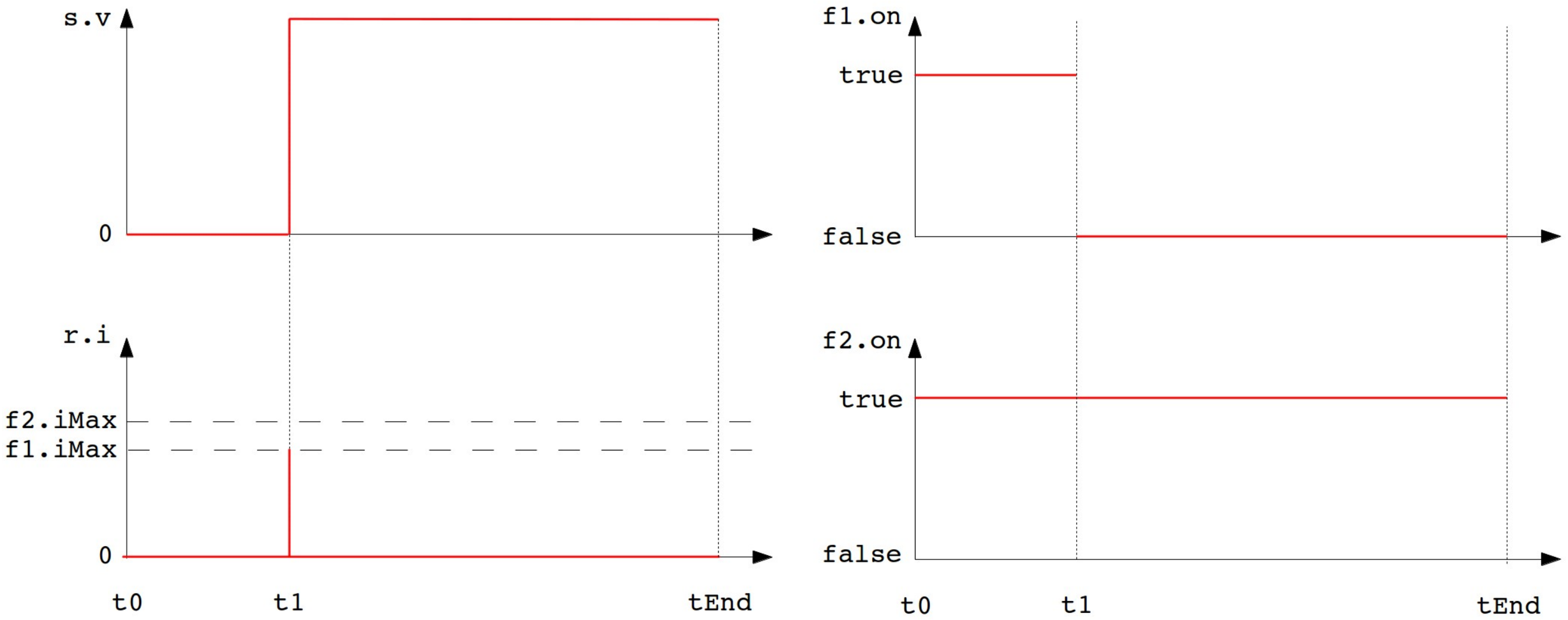
Nested abstraction





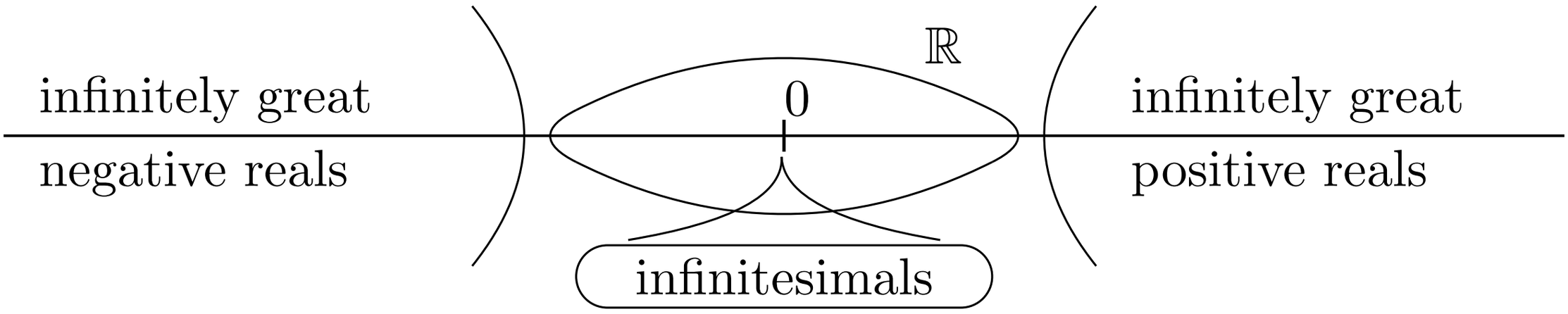
- Suppose we **also** abstract the behaviour of the voltage source
- Both fuses melt due to the loss of signal continuity

Desired behaviour



- Signals are no longer maps from time to values
- We need **infinitesimal** time steps to enable this behaviour

Non-standard analysis



- Used **intuitively** by Leibniz and Newton
- Formalised by Abraham Robinson in the 60s

$$N, N+1, N^2, N/2, e^N, \ldots \qquad \varepsilon = 1/N, \ldots \qquad \varepsilon \approx 0$$

Standardisation

- Every finite non-standard real has a unique standard part

$$x = \mathrm{std}(x) + \varepsilon \qquad \mathrm{std}(x) \in \mathbb{R} \qquad \varepsilon \approx 0$$

- Functions can be standardised

$$\forall x \in \mathbb{R},\ \mathrm{std}\big(f\big)(x) \stackrel{def}{=} \mathrm{std}\big(f(x)\big)$$

- Standardisation of a function is **not** defined on all non-standard reals, but only on the standard ones

$$f : {}^*\mathbb{R} \to {}^*\mathbb{R} \qquad \mathrm{std}\big(f\big) : \mathbb{R} \to \mathbb{R}$$

Examples

- Differentiation

$$\frac{d\left(x^2\right)}{dx} = \frac{(x+dx)^2 - x^2}{dx} = \frac{2x\,dx + dx^2}{dx} = 2x + dx \approx 2x$$

- Integration

$$\int_0^1 f(x)dx \approx \sum_{i=0}^{N-1} f(i\,dx)dx\,,\ \text{where}\ N = 1/dx$$

- Continuity

$$\forall x \in {}^*\mathbb{R}, \quad x \approx a \implies {}^*f(x) \approx {}^*f(a)$$

Everything is a sequence

$$1 = [1, 1, 1, \dots] \qquad {}^*f = [f, f, f, \dots]$$

$$N = [1, 2, 3, \dots] \qquad \varepsilon = 1/N = \left[1, \frac{1}{2}, \frac{1}{3}, \dots\right]$$

$$N + 1 = [2, 3, 4, \dots] \qquad \varepsilon^2 = 1/N^2 = \left[1, \frac{1}{4}, \frac{1}{9}, \dots\right]$$

Quite similar in spirit to the definition of reals using Cauchy sequences

$$x = [x_1, x_2, x_3, \dots] \qquad y = [y_1, y_2, y_3, \dots]$$

$$x < y \overset{def}{\Longleftrightarrow} x_i < y_i \text{ for almost all } i$$

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Transfer principle

- Non-standard reals are a **first-order equivalent** model of the real field
  - Any first-order formula true in $\mathbb{R}$ is true in ${}^*\mathbb{R}$ and vice-versa.

- Example (continuity):

$$\forall \varepsilon \in \mathbb{R}(\varepsilon > 0), \exists \delta \in \mathbb{R}(\delta > 0) :$$
$$\forall x \in \mathbb{R}, \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\right)$$

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$$\forall \varepsilon \in {}^*\mathbb{R}(\varepsilon > 0), \exists \delta \in {}^*\mathbb{R}(\delta > 0):$$
$$\forall x \in {}^*\mathbb{R}, \left(|x-{}^*a| < \delta \Rightarrow |{}^*f(x)-{}^*f({}^*a)| < \varepsilon\right)$$

Łoś' theorem

- Generalisation of the transfer principle
  - Any first-order formula is true in ${}^*\mathbb{R}$ if and only if it is true in $\mathbb{R}$ for almost all indices.

- Example (Archimedean property):

$$\varepsilon = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots], \forall i \in \mathbb{N}, \varepsilon_i \in \mathbb{R}(\varepsilon_i > 0)$$

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} : \; n\varepsilon_i < x \leq (n+1)\varepsilon_i$$

$$\forall x \in {}^*\mathbb{R}, \exists n \in {}^*\mathbb{Z} : \; n\varepsilon < x \leq (n+1)\varepsilon$$

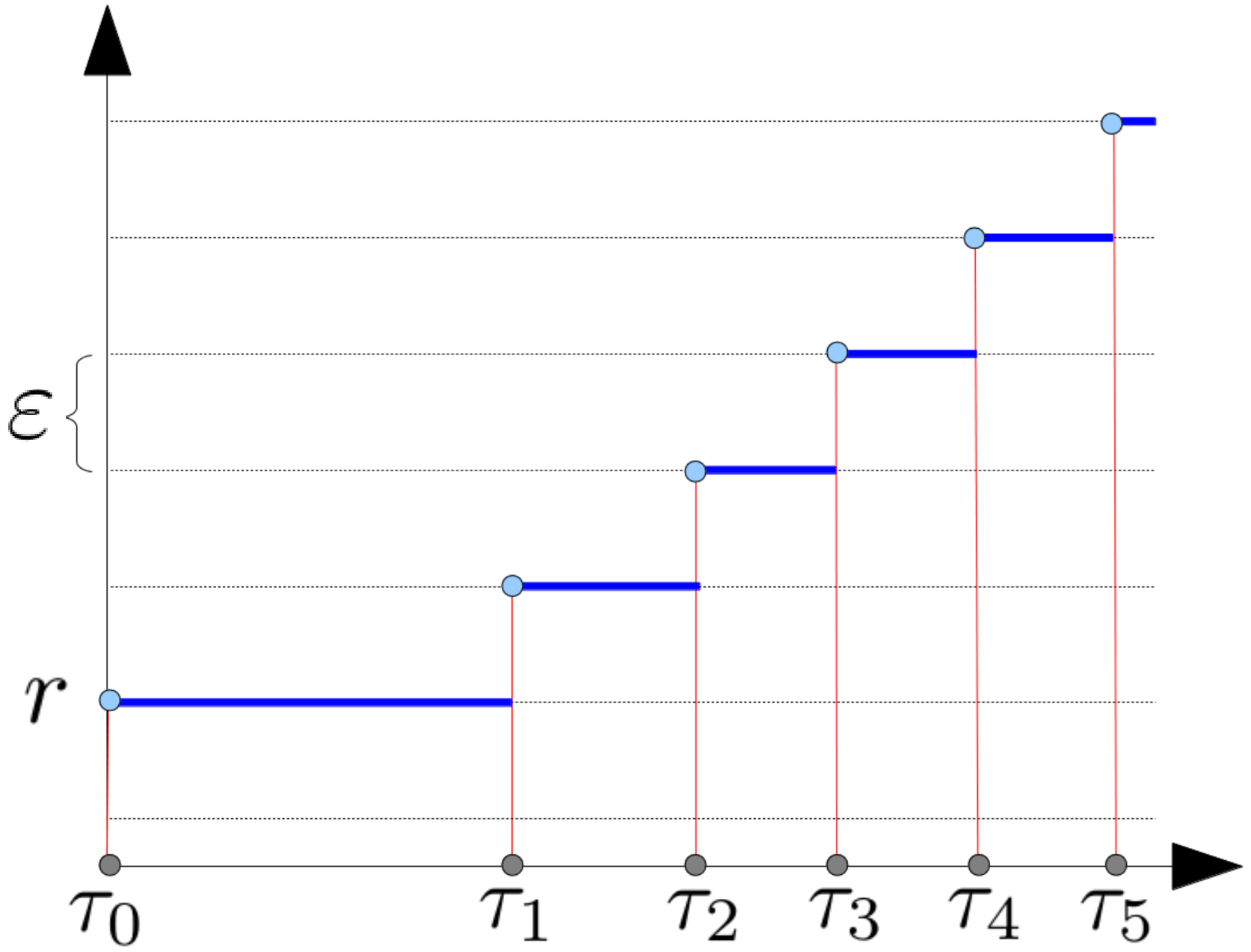

se•man•tic (si man
to meaning or arising from
symbols; semantic change; semanti
semantics. [1655–65; < Gk sēman
mant(ós) marked (sēman-, base o
verbal adj. suffix; akin to sêma si
se•man•tics (si man′tiks), n. (u
linguistics dealing with the stu
meaning is structured in langu
over time. 2. the branch of se
tionship between signs or sy
meaning, or an interpretation
Let's not argue
—se•man′ti•c

QSS approach

$$^*\mathbb{T} \stackrel{def}{=} {}^*\mathbb{R}_0^+$$

$$^*\mathbb{T} \to r + \varepsilon \cdot {}^*\mathbb{Z}$$

$$\varepsilon \approx 0$$

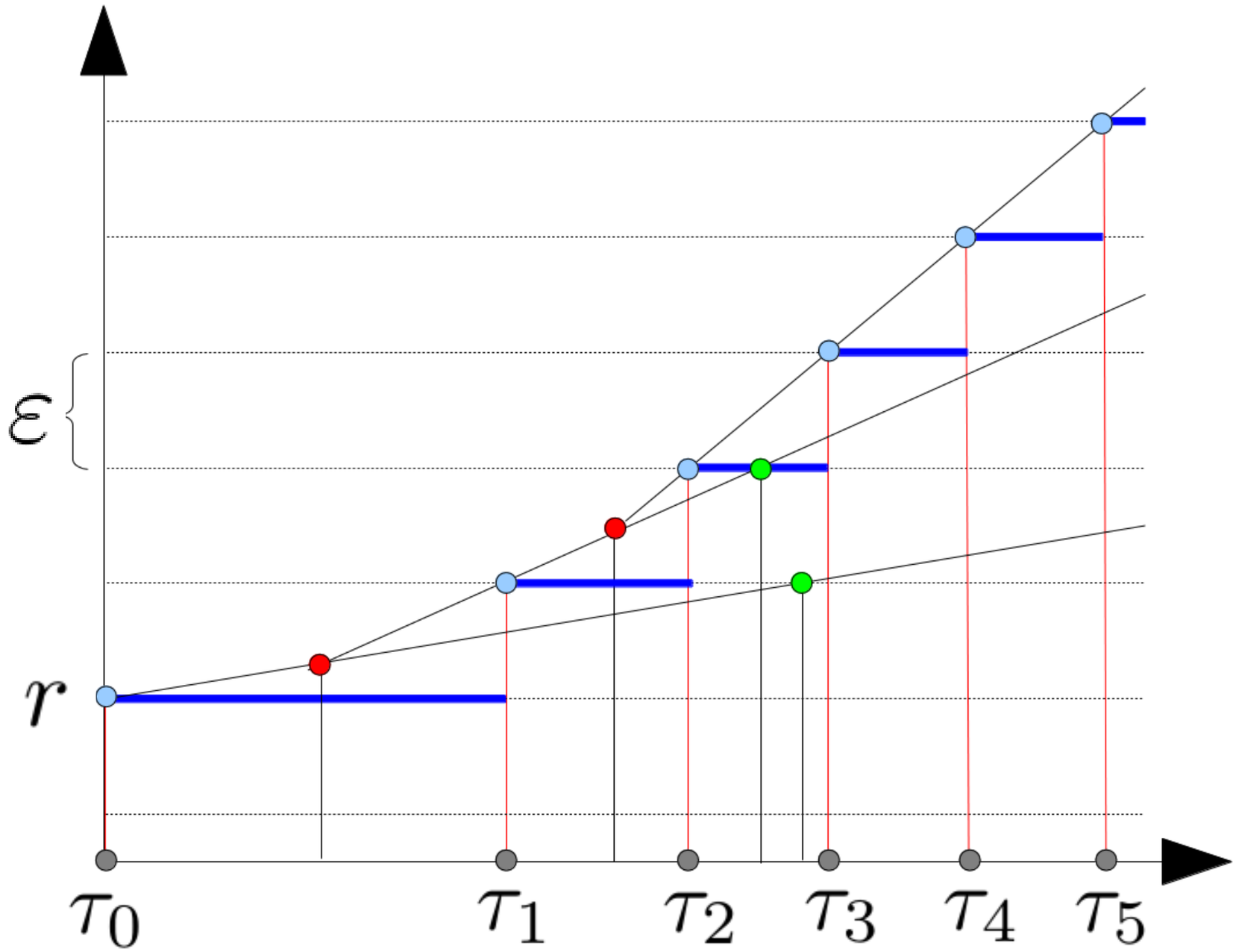


- Force all dense-time signals to have discrete codomains

The meaning of ODE

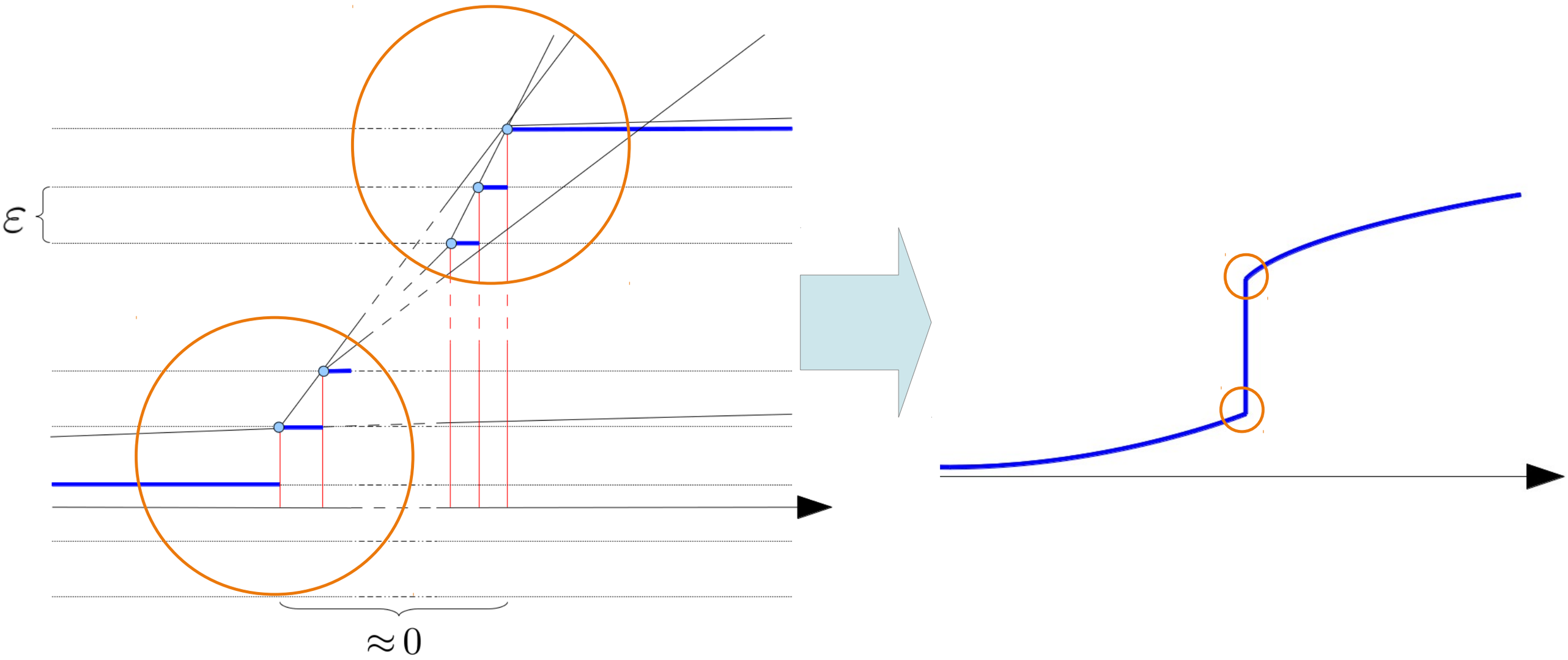
$$\dot{x} = f(x, y)$$

$$x(0) = r$$



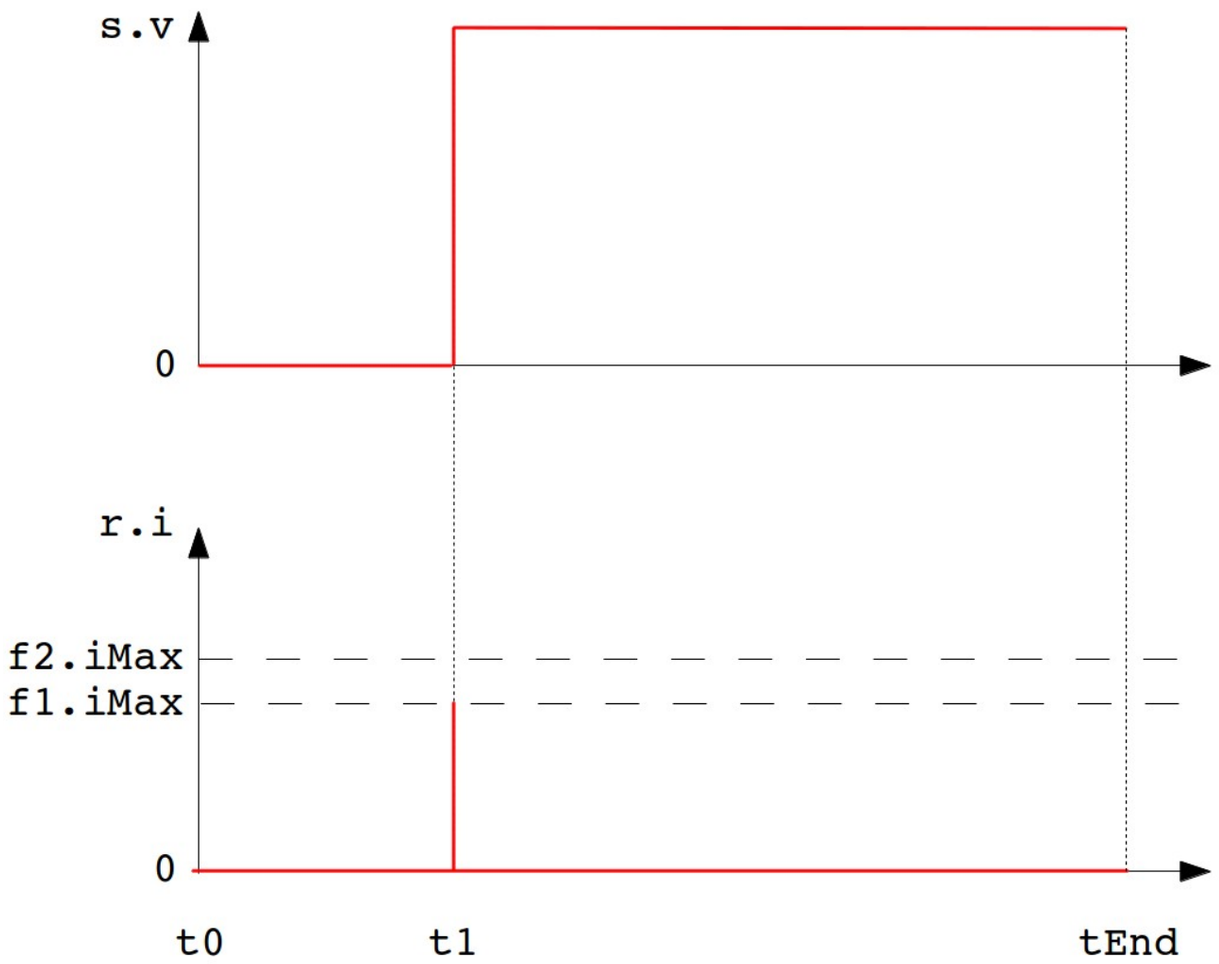
- Red dots indicate events on the input signal

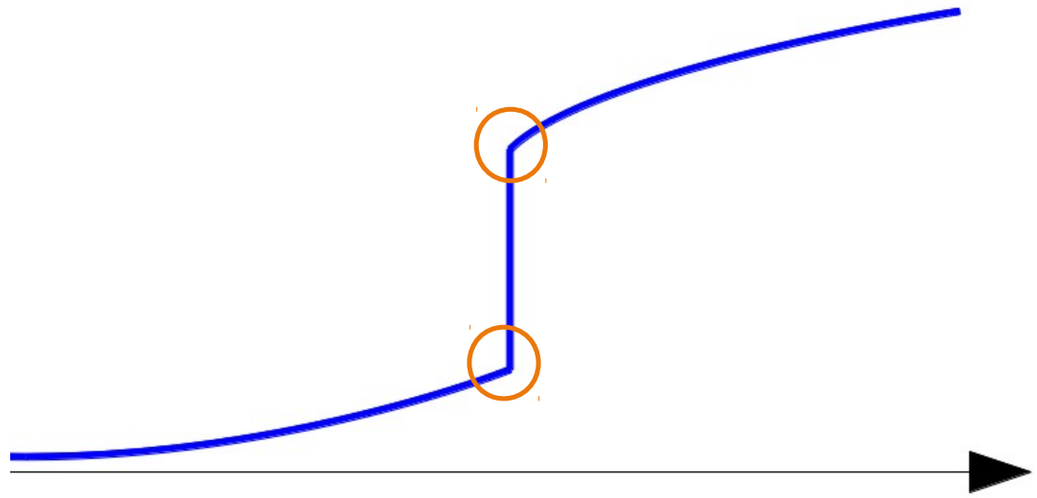
Inifinite slope signals



- After “standardisation” they have vertical slopes

Back to the circuit



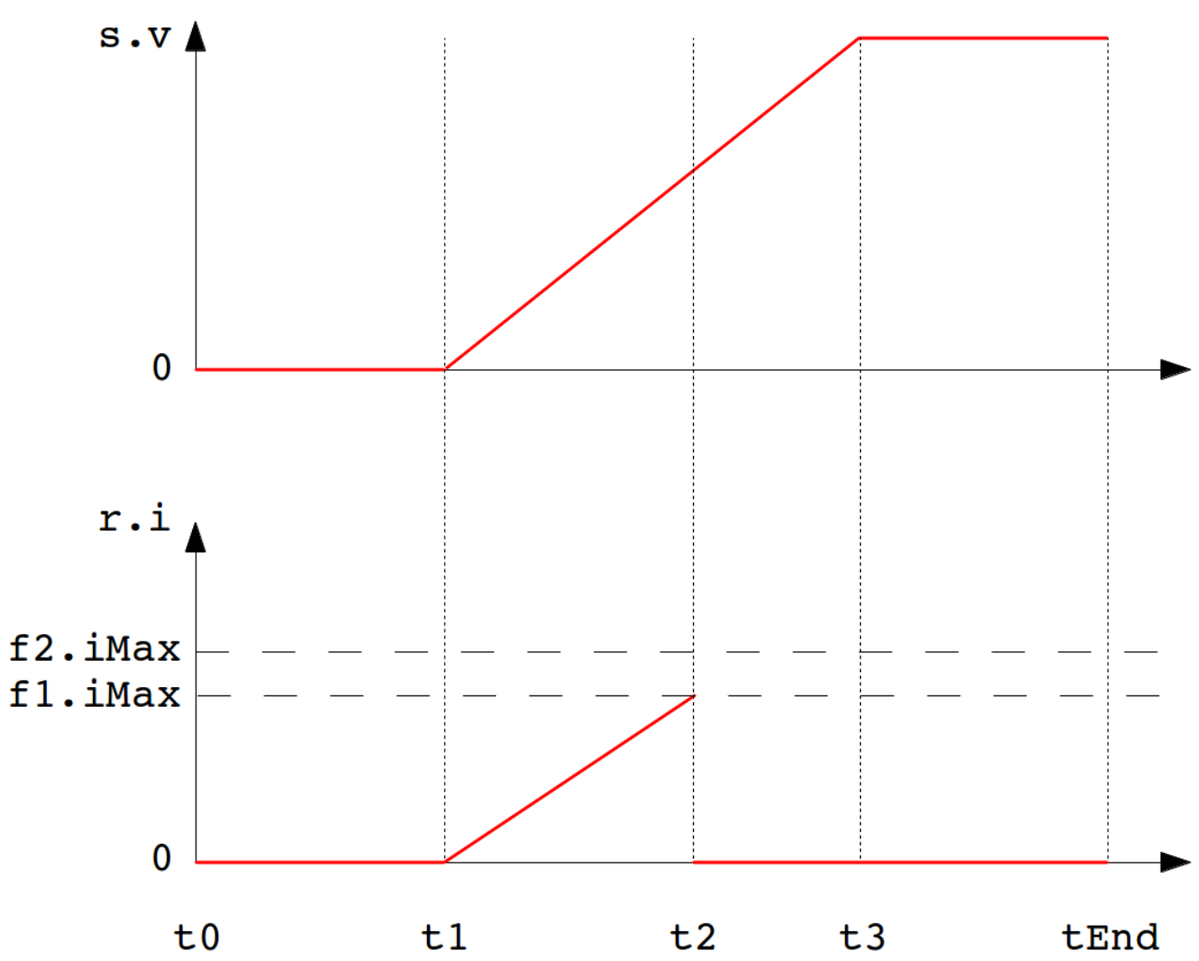


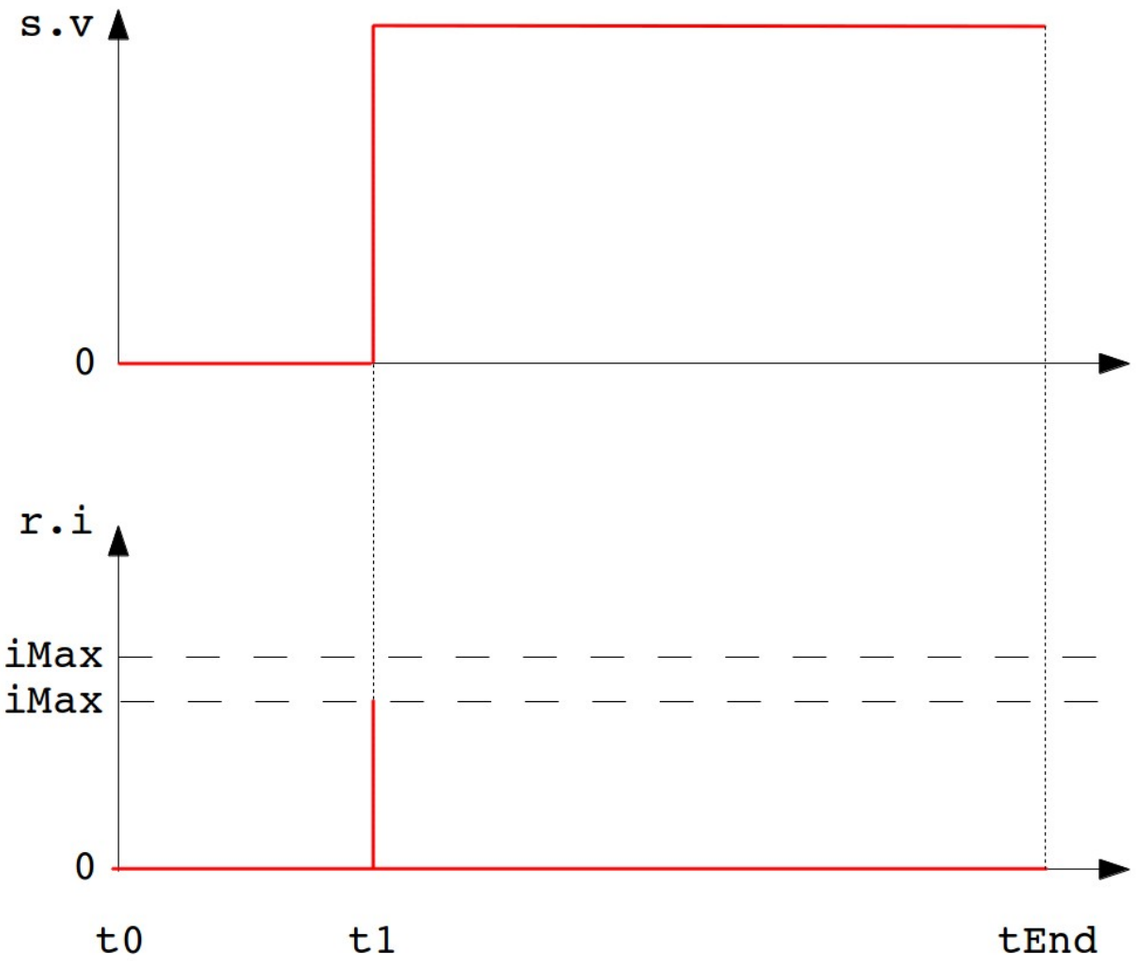
- When the current reaches the rated value of the first fuse, this produces an input event, inverting the slope

ARE WE THERE YET !?!

MATT GROENING

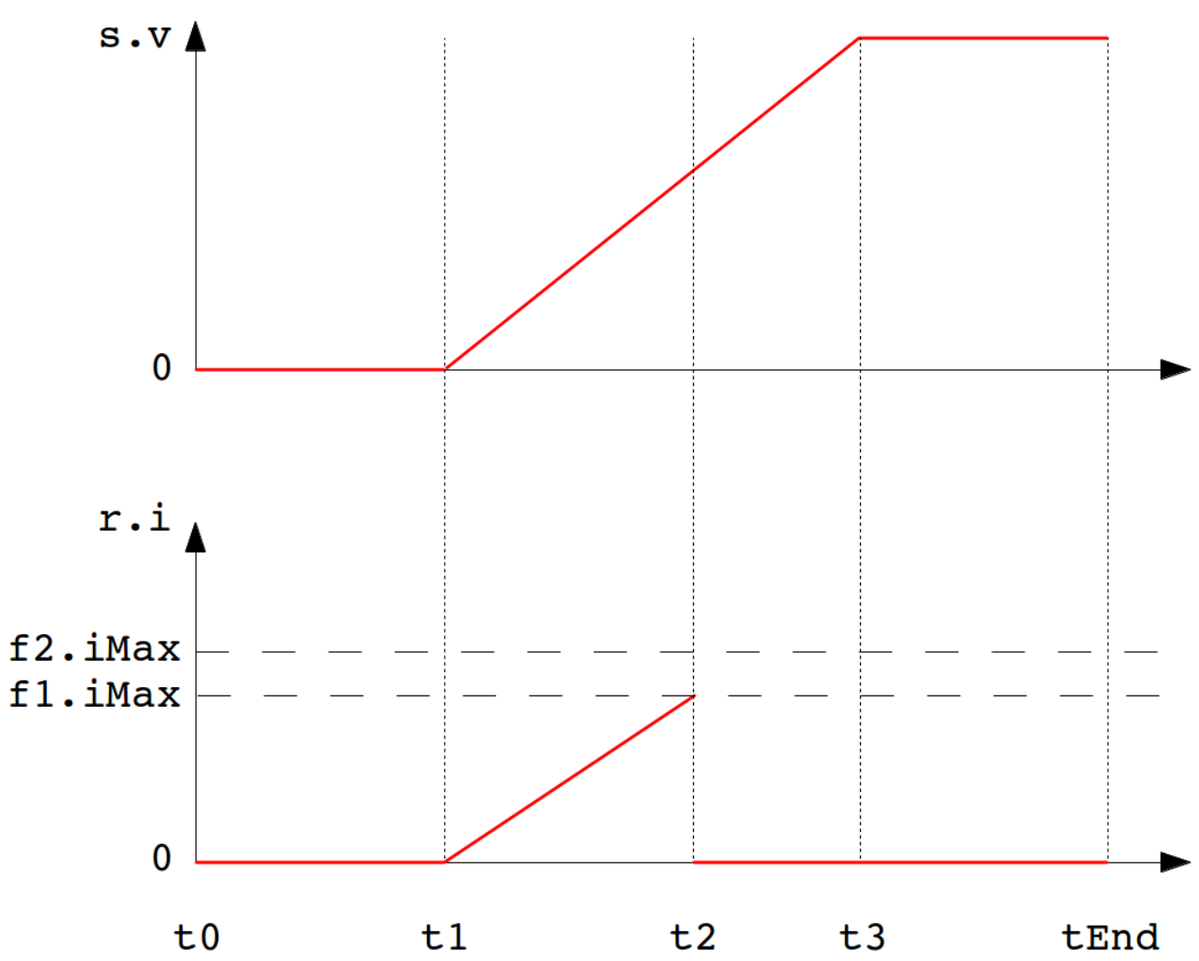
Key assumptions

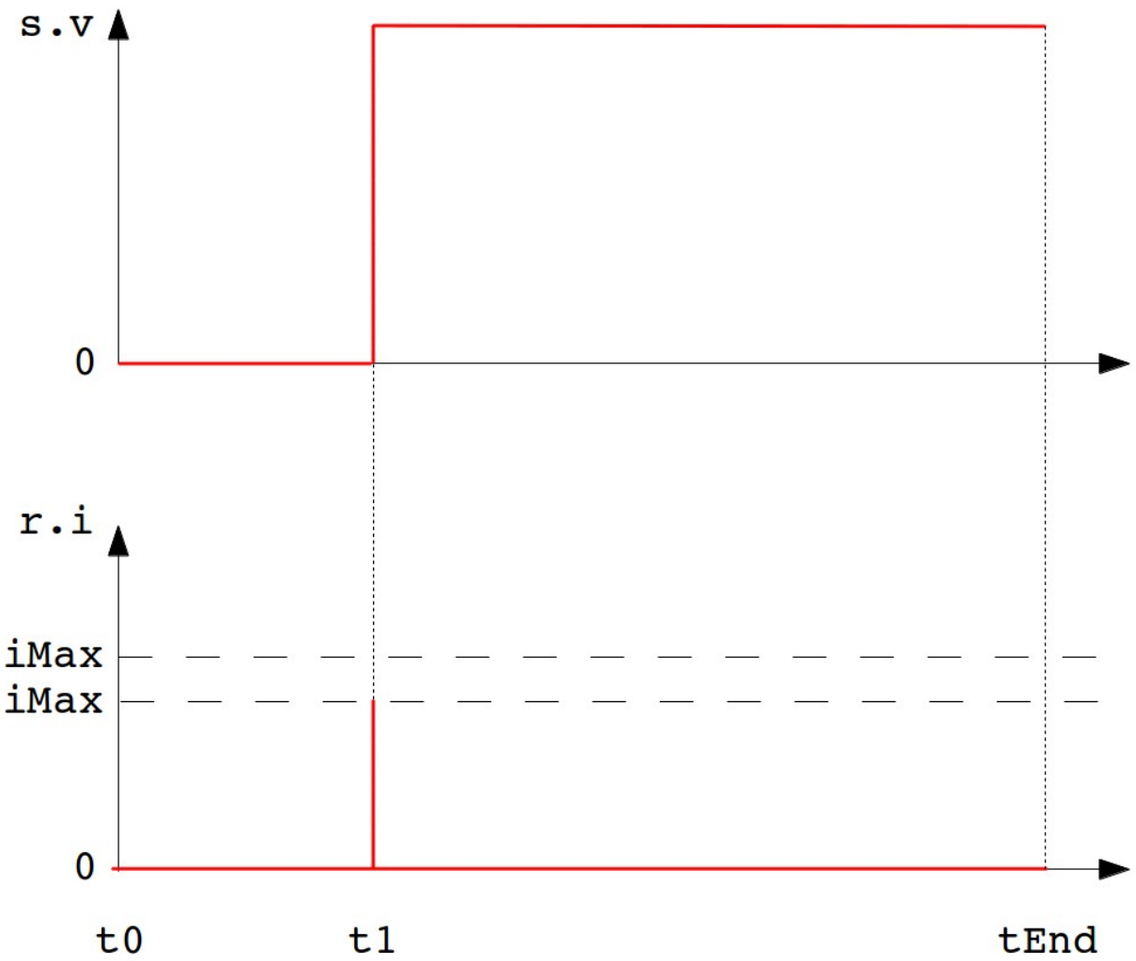




- We rely on two assumptions
  - The signal passes by all intermediate values in the “right order” (continuity)
  - The fuse melts infinitely faster than the voltage increases (model assumption)

Key assumptions





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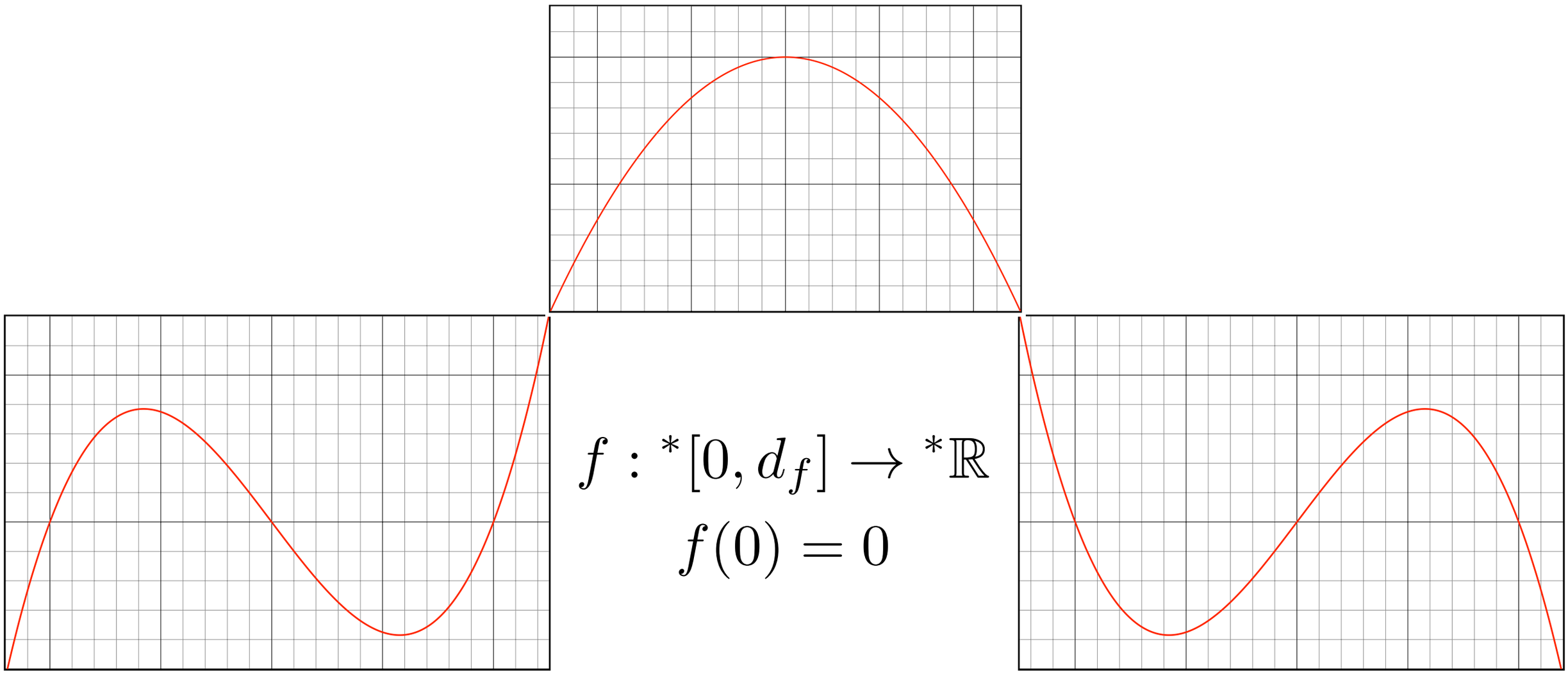
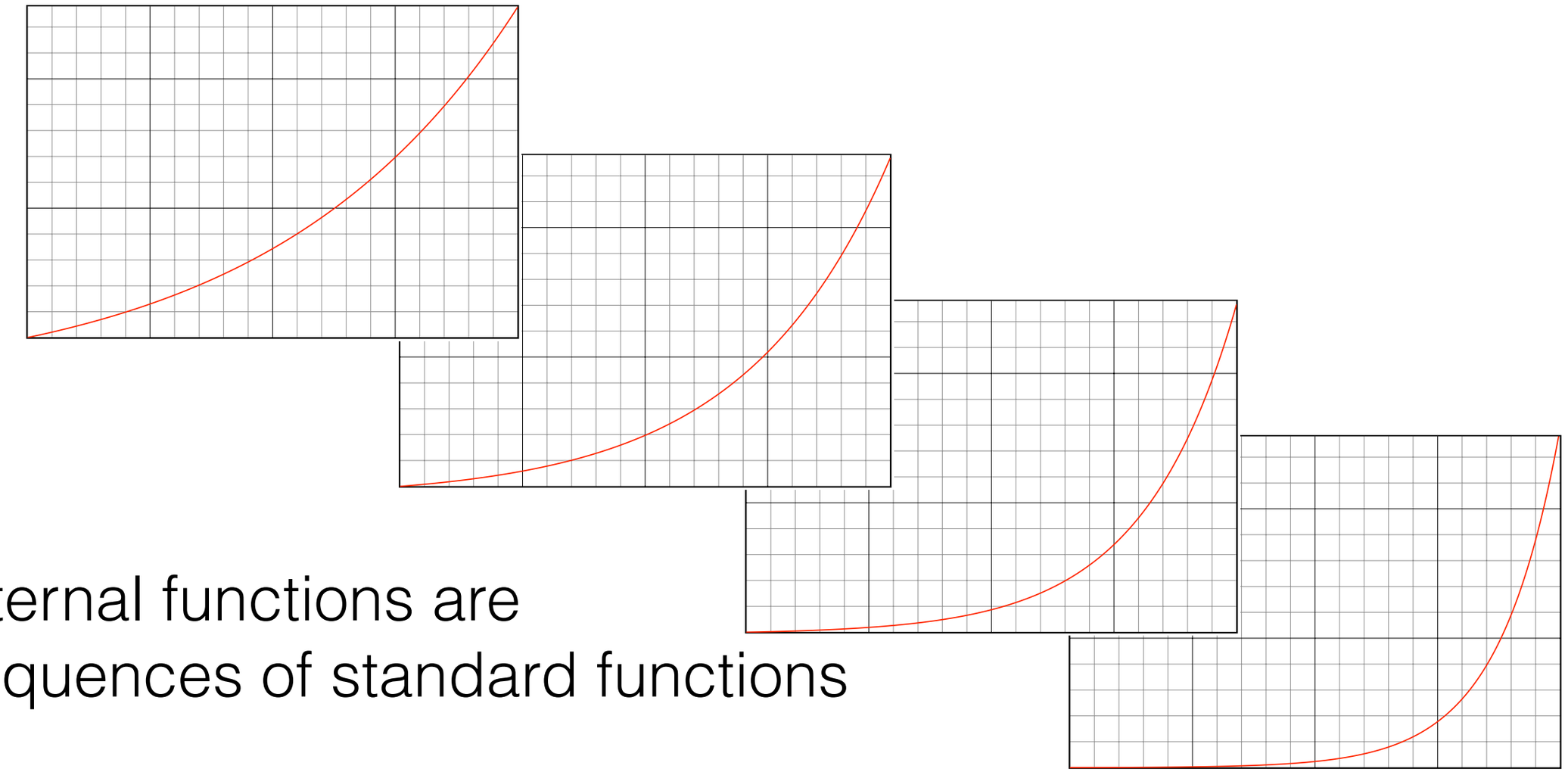
Signets



- Consider signals as sequences of additive signets
- A **signet** is a non-standard continuous **internal** function

Specifying abstraction



- Internal functions are sequences of standard functions

- As a consequence of Łoś' theorem, we can reason on standard functions to draw conclusions about the signet

- Use this to derive interval boundaries for the interval abstraction

Conclusion

- We proposed a semantic model for hybrid signals
  - Uniform (linear) and dense nature of time
  - The “physical” properties of signals (read “continuity”)
- Operational, although not directly implementable
  - Describes how to compute the exact solution of a system of dynamic equations
  - Disregarding the finiteness of computational resources
- Can serve as a basis for reasoning and implementation
  - Concrete implementations approximate the solution with **non-infinitesimal** error
  - New language features can be discussed on a sound basis
- First step towards formalising signal abstraction

Appendix

```
model BouncingBall
   Real v, x;
   constant Real g = 10;
initial equation
   v = 1.0;
   x = 0.0;
equation
   der(v) = -g;
   der(x) = v;
   when x < 0 then
      reinit(v, -0.8 * pre(v));
      reinit(x, 0.0);
   end when;
end BouncingBall;
```

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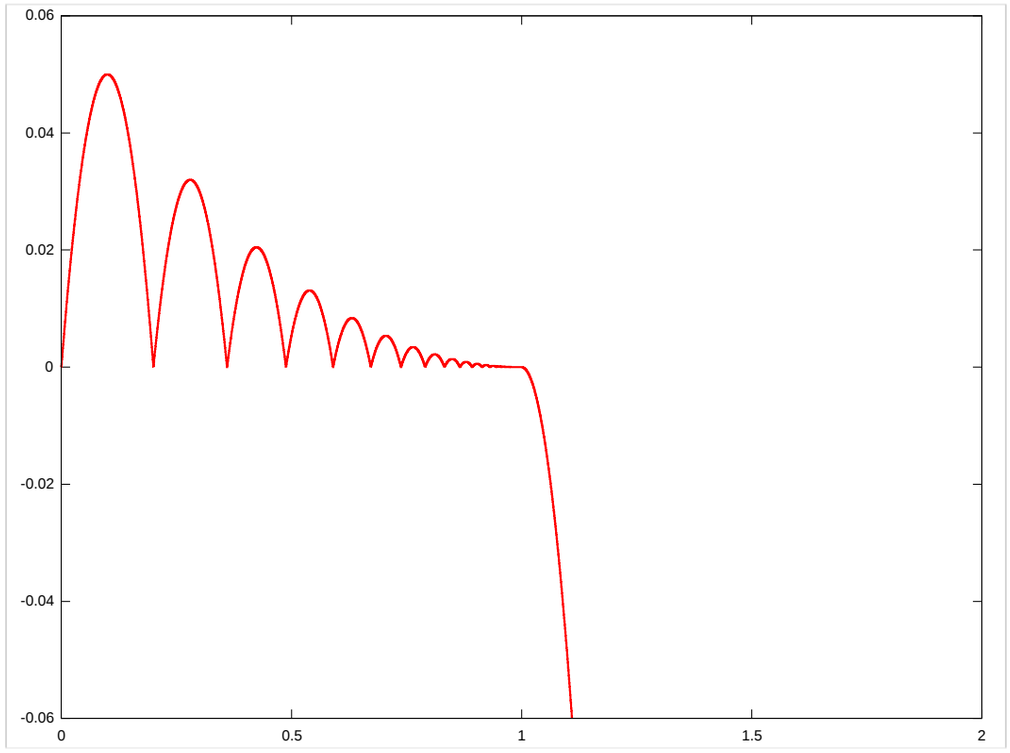
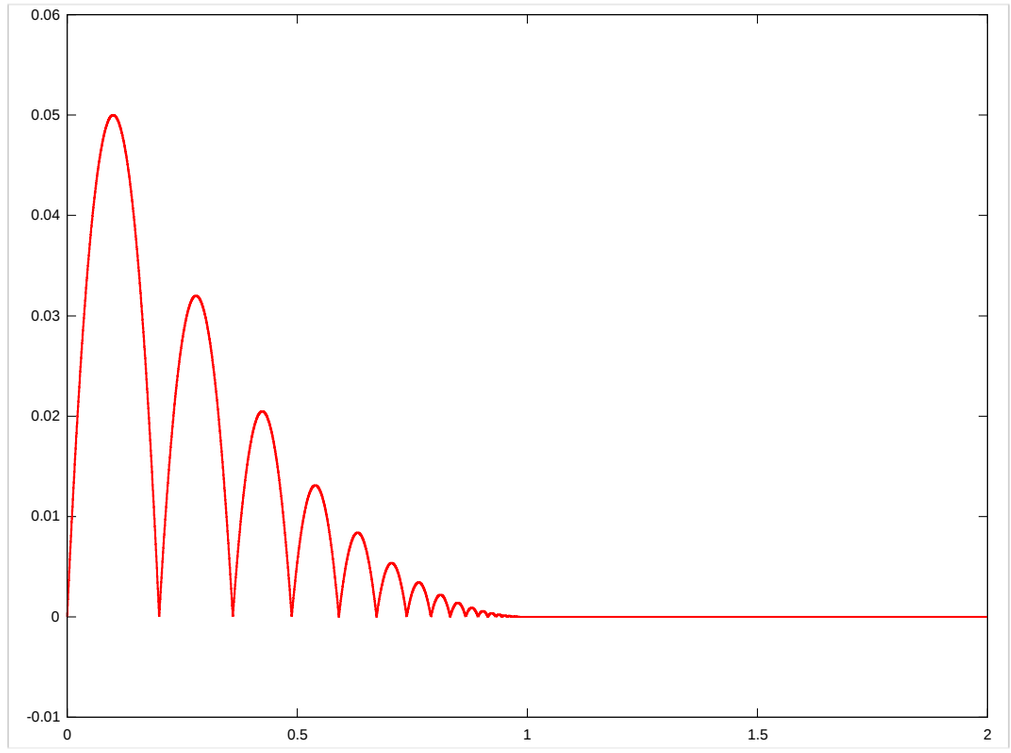
Which one is correct?

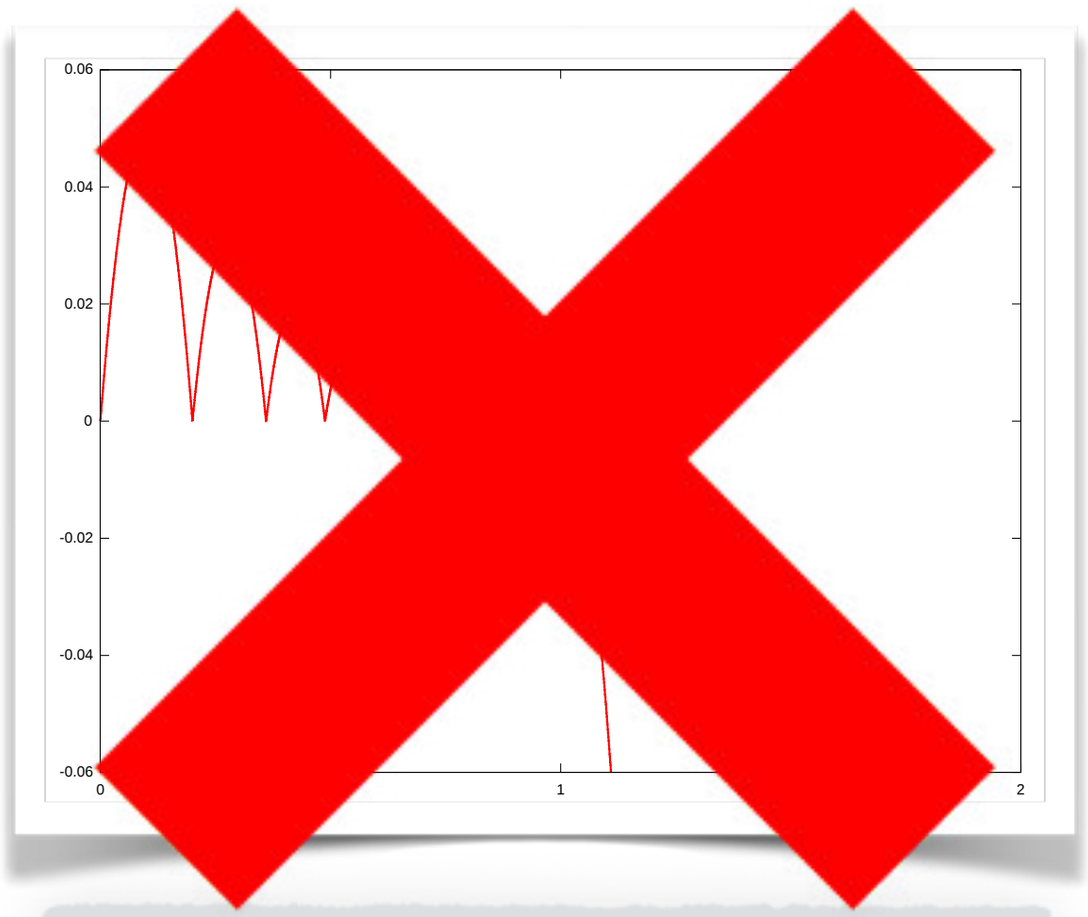


Results from simulator A



Results from simulator B

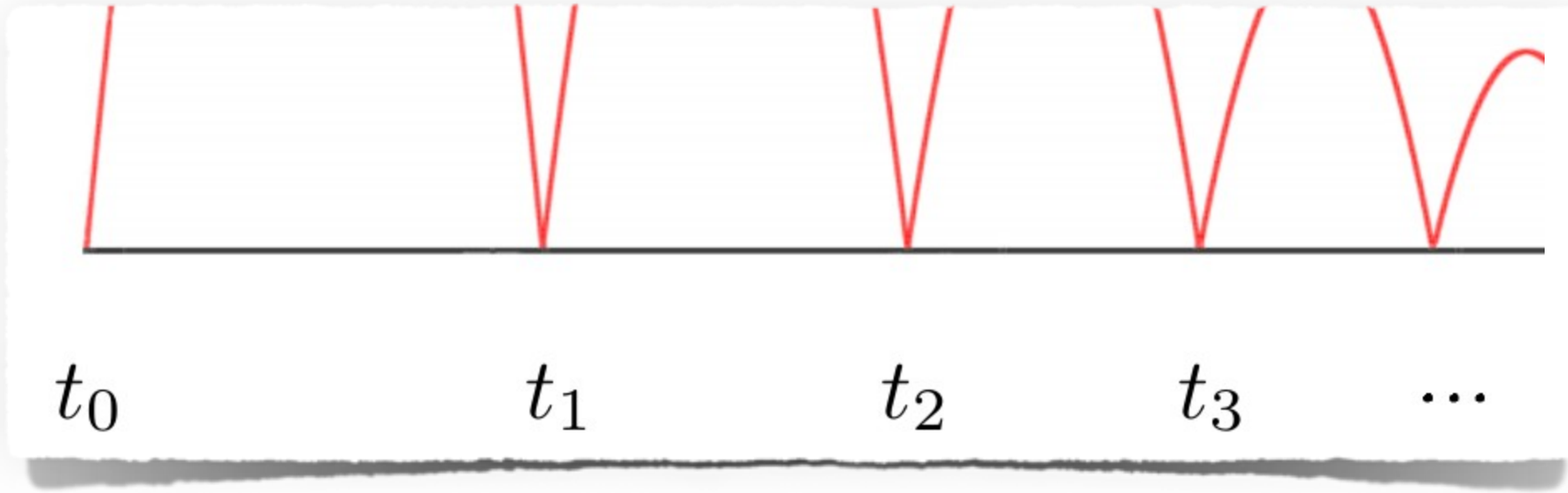
Which one is correct?



Results from simulator A

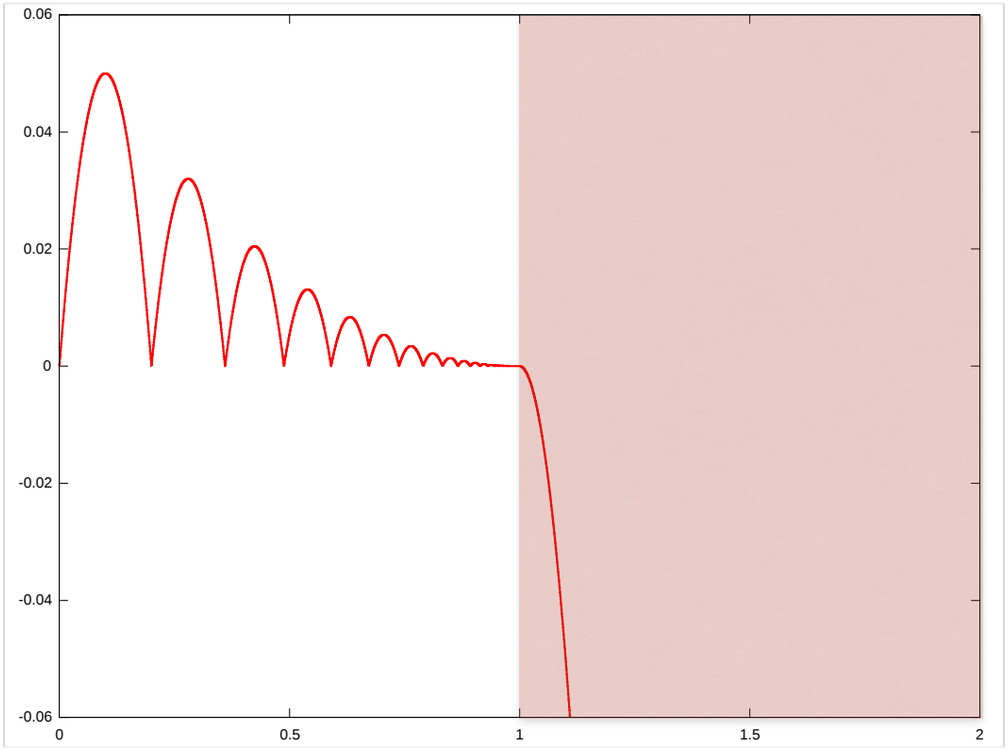
Results from simulator B

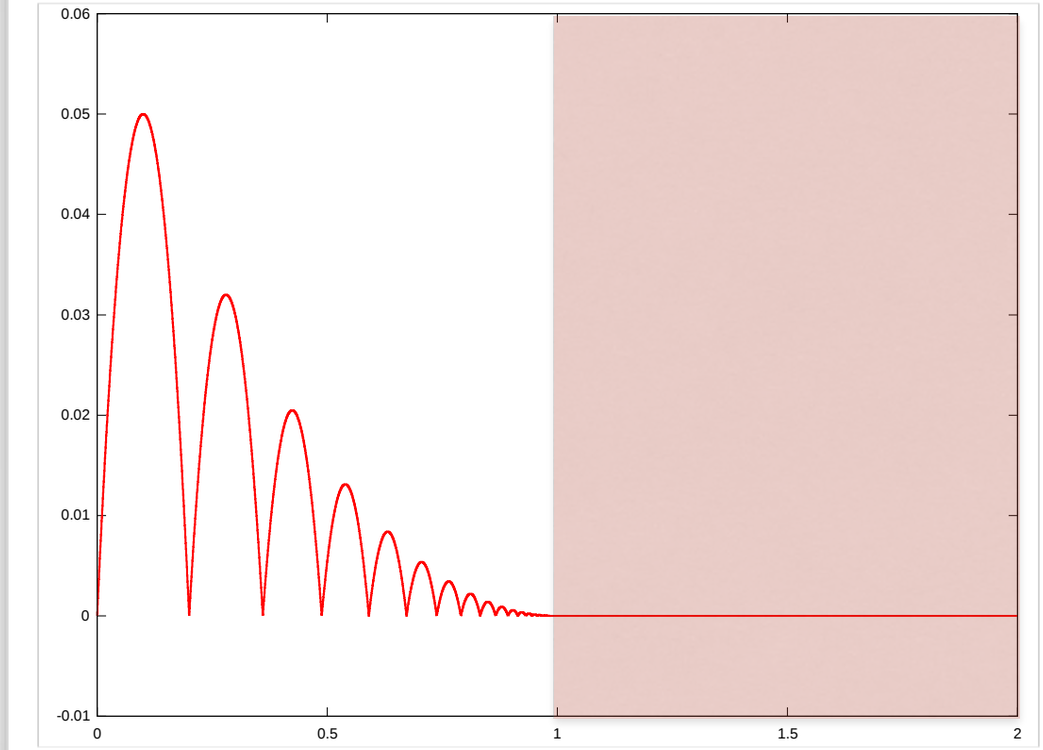
What's wrong?



$$\lim_{n\to\infty} t_n - t_0 = \frac{10v_0}{g} = 1$$

Zeno point





The model is undefined beyond the Zeno point!

Abstraction

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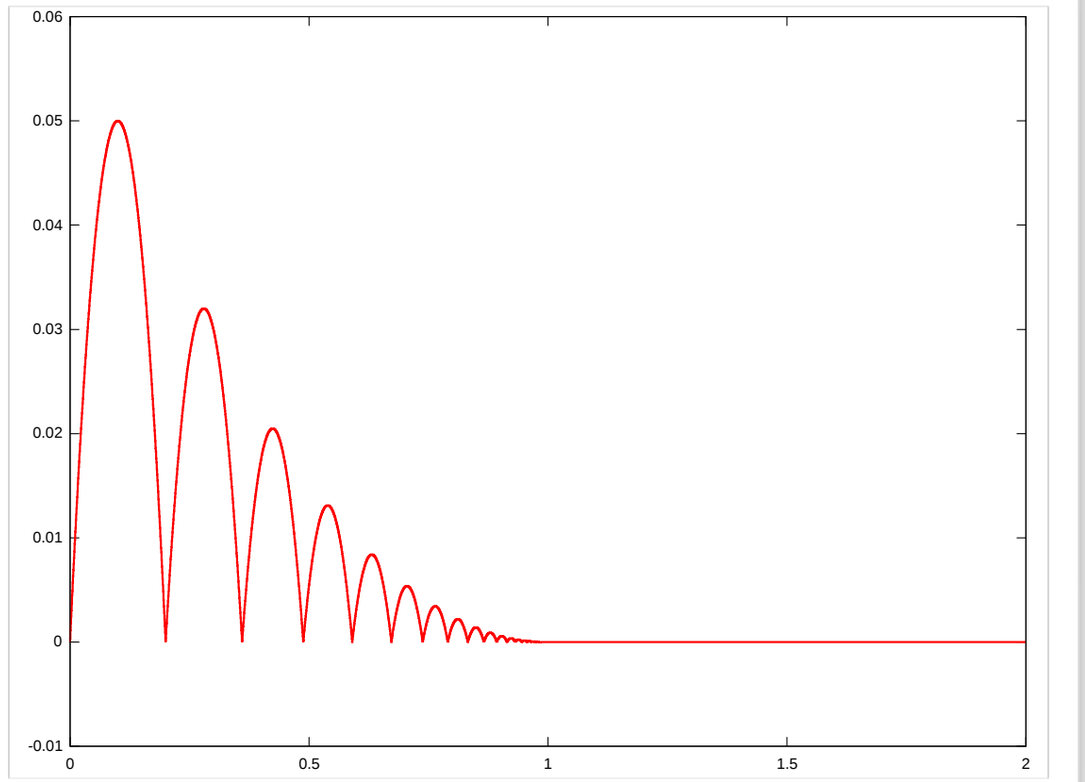
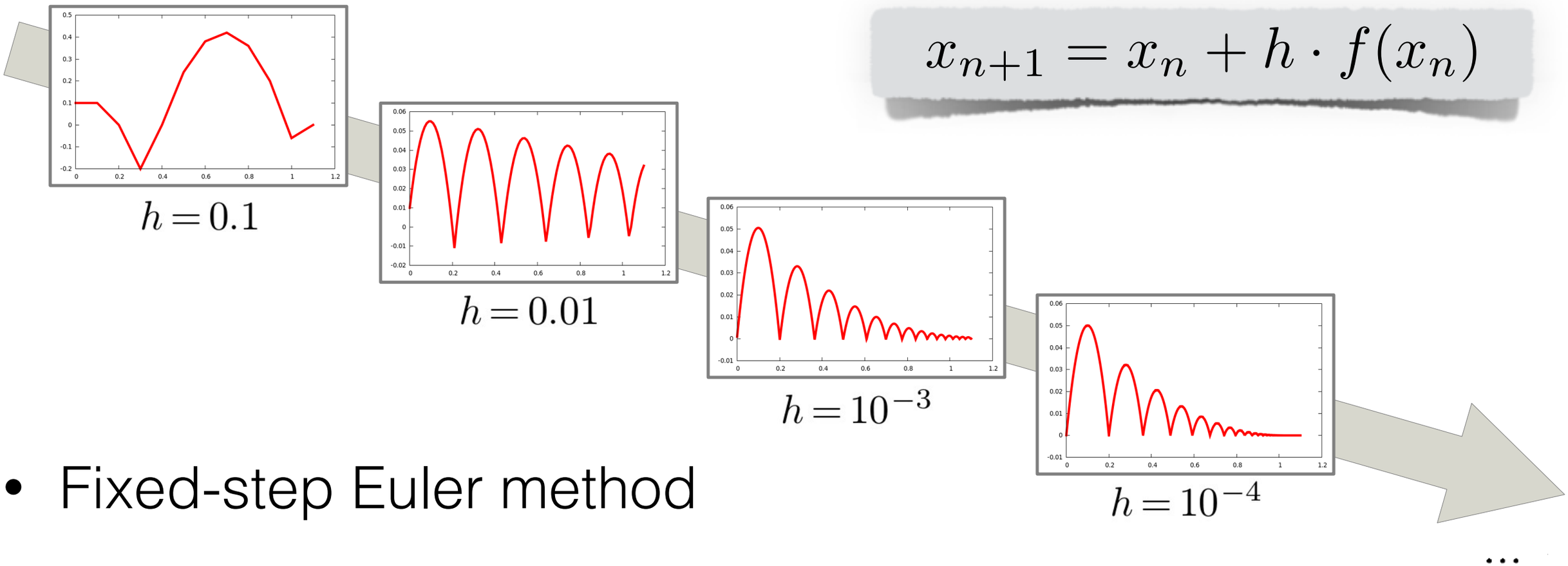


This model is an **idealised** representation of the real-world behaviour of the ball.

Approximation

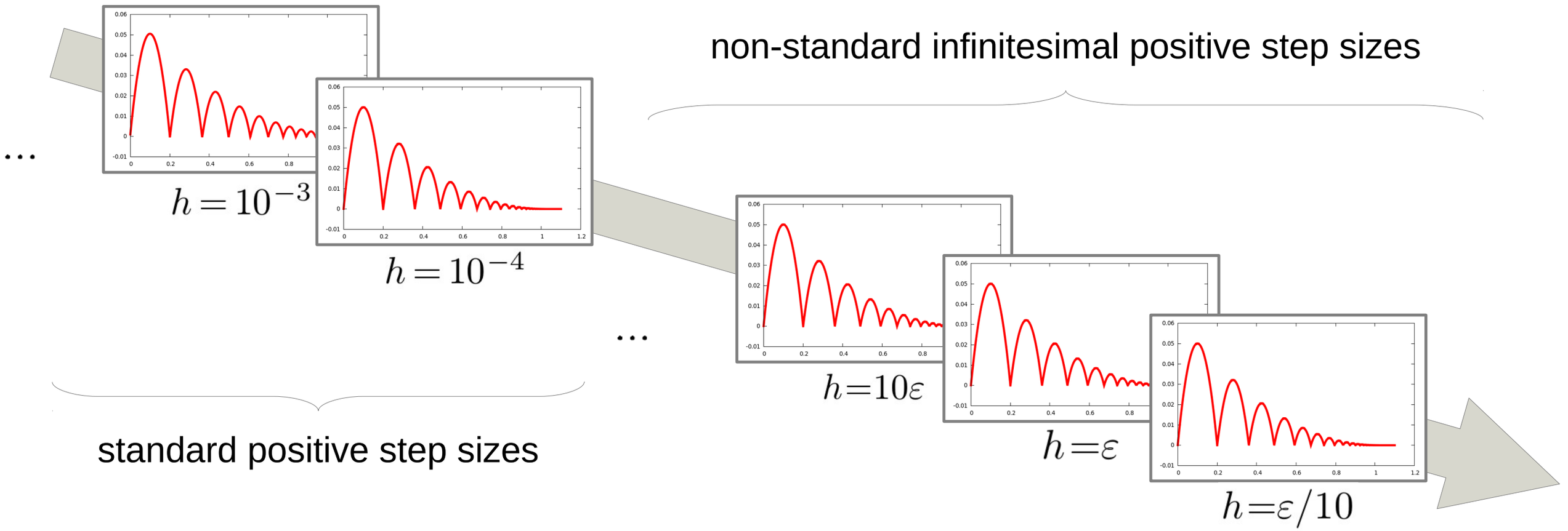
$$x_{n+1} = x_n + h \cdot f(x_n)$$



- Fixed-step Euler method
- Approximates the desired model behaviour
  - Necessarily oversteps the Zeno point
- To fit all models, we need an **infinitesimal** step.

Non-standard semantics

non-standard infinitesimal positive step sizes

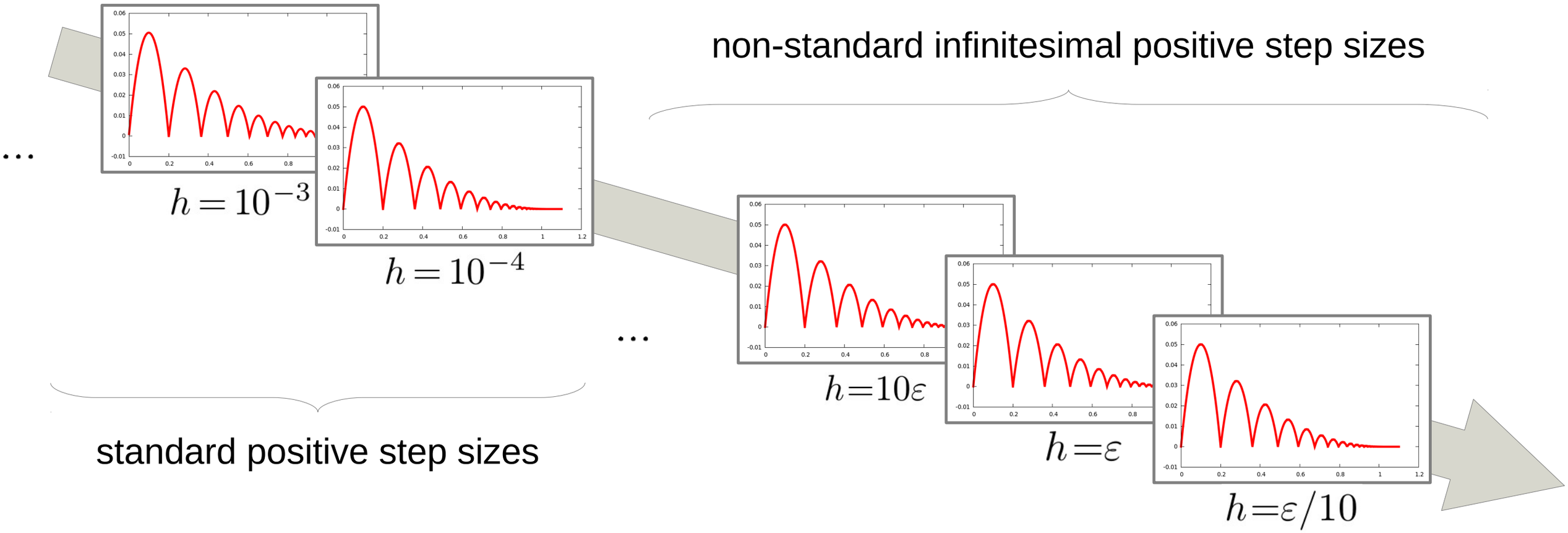


standard positive step sizes

$$^*\mathbb{T} \overset{def}{=} \{\varepsilon \cdot n \,|\, n \in {}^*\mathbb{N}_0\}$$

$$\forall \varepsilon \approx 0,\ \forall x \in {}^*\mathbb{R},\ \exists n \in {}^*\mathbb{Z} :\ n\varepsilon < x \leq (n+1)\varepsilon$$

Non-standard semantics



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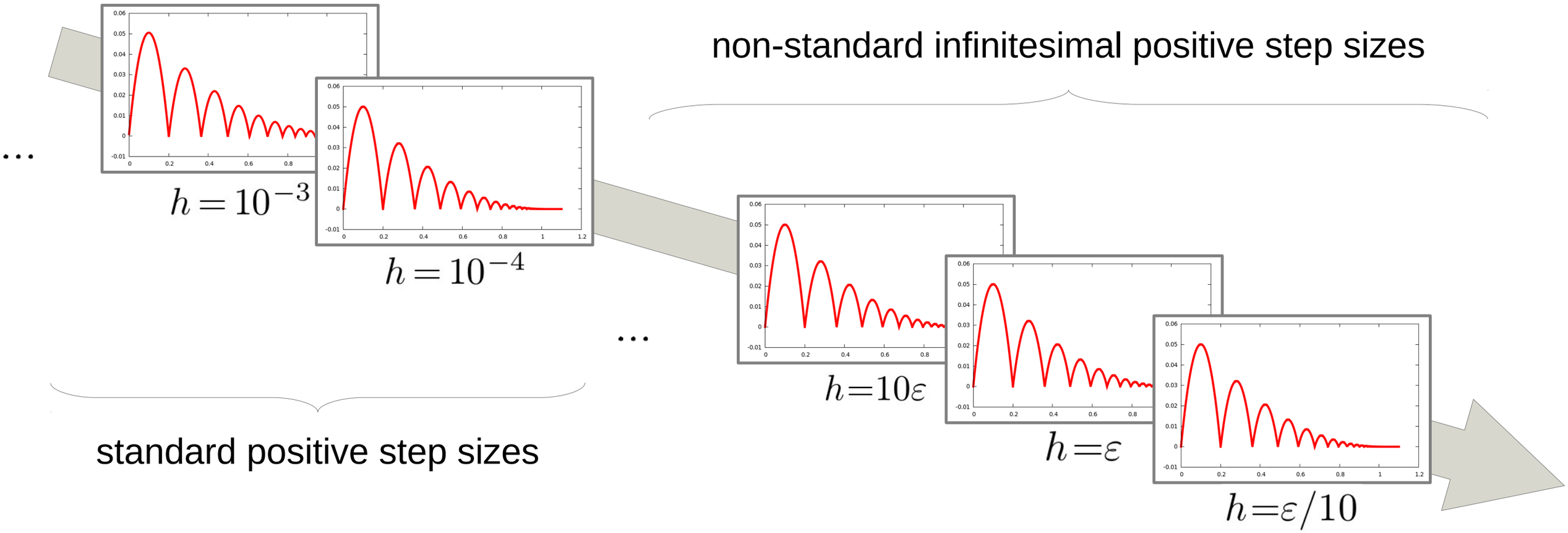

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