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The Hochschild complex of a twisting cochain



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ABSTRACT

Given any twisting cochain $t : C \rightarrow A$, where C is a connected, coaugmented chain coalgebra and A is an augmented chain algebra over an arbitrary commutative ring R , we construct a twisted extension of chain complexes

$$A \xrightarrow{\quad} \mathcal{H}(t) \longrightarrow C$$

of which both the well-known Hochschild complex of an augmented, associative algebra and the coHochschild complex of a coaugmented, coassociative coalgebra [13] are special cases. We therefore call $\mathcal{H}(t)$ the Hochschild complex of the twisting cochain t .

We explore the extent of the naturality of the Hochschild complex construction and apply the results of this exploration to determining conditions under which $\mathcal{H}(t)$ admits multiplicative or comultiplicative structure. In particular, we show that the Hochschild complex on a chain Hopf algebra always admits a natural comultiplication.

Furthermore, when A is a chain Hopf algebra, we determine conditions under which $\mathcal{H}(t)$ admits an r th-power map extending the usual r th-power map on A and lifting the identity on C . As special cases, we obtain that both the Hochschild complex of any cocommutative Hopf algebra and the co-Hochschild complex of the normalized chain complex of a simplicial double suspension admit power maps.

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1. Introduction

It has long been known that the Hochschild homology of a (differential graded) commutative algebra admits natural power operations [6,19], which are closely related to the power operations on free loop spaces [4,25]. These power operations in Hochschild homology can be constructed as follows.

If H is a Hopf algebra over a ring R , then the R -module $\text{End}(H)$ of endomorphisms of H admits a multiplication $*$, called the *convolution product*. For all $f, g \in \text{End}(H)$, the convolution product of f and g is the composite

$$H \xrightarrow{\delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H,$$

where δ and μ are the comultiplication and multiplication on H , respectively. The r th-power map on H is then just

$$\lambda_r = \text{Id}_H^{*r} = \mu^{(r)} \delta^{(r)}, \tag{1.1}$$

where $\delta^{(r)} : H \rightarrow H^{\otimes r}$ and $\mu^{(r)} : H^{\otimes r} \rightarrow H$ denote the iterated comultiplication and multiplication.

Recall that the Hochschild complex of an augmented algebra A , which we denote $\mathcal{H}(A)$, can be seen as a twisted extension of A by $\mathcal{B}(A)$, the bar construction, i.e., there is a twisted tensor extension

$$A \twoheadrightarrow \mathcal{H}(A) \twoheadrightarrow \mathcal{B}(A). \tag{1.2}$$

If A is commutative, then $\mathcal{B}(A)$ is naturally a commutative Hopf algebra, where the multiplication is the shuffle product. The multiplication on $\mathcal{B}(A)$ lifts to $\mathcal{H}(A)$, so that

(1.2) becomes a sequence of algebra maps. Moreover, the r th-power map on $\mathcal{B}(A)$ also lifts to a linear map $\tilde{\lambda}_r : \mathcal{H}(A) \rightarrow \mathcal{H}(A)$ such that

$$\begin{array}{ccccc}
 A & \twoheadrightarrow & \mathcal{H}(A) & \twoheadrightarrow & \mathcal{B}(A) \\
 \parallel & & \tilde{\lambda}_r \downarrow & & \lambda_r \downarrow \\
 A & \twoheadrightarrow & \mathcal{H}(A) & \twoheadrightarrow & \mathcal{B}(A)
 \end{array} \tag{1.3}$$

commutes. The map induced by $\tilde{\lambda}_r$ in Hochschild homology is the r th-power map of [6,19].

In this paper we define and study a significant generalization of the power map on Hochschild homology. In [15] it is shown that a special case of this construction provides an integral algebraic model for the topological power map on the free loop space of a double suspension.

For any twisting cochain $t : C \rightarrow A$, we define a chain complex $\mathcal{H}(t)$ that is a twisted tensor extension of A by C . The complex $\mathcal{H}(t)$ generalizes both the well-known Hochschild complex $\mathcal{H}(A)$ of a chain algebra A and the coHochschild complex $\widehat{\mathcal{H}}(C)$ of a chain coalgebra C [13].

- If $t_{\mathcal{B}} : \mathcal{B}A \rightarrow A$ is the couniversal twisting cochain associated to A , where $\mathcal{B}A$ denotes the (reduced) bar construction on A , then $\mathcal{H}(t_{\mathcal{B}}) = \mathcal{H}(A)$.
- If $t_{\Omega} : C \rightarrow \Omega C$ is the universal twisting cochain associated to C , where ΩC denotes the (reduced) cobar construction on C , then $\mathcal{H}(t_{\Omega}) = \widehat{\mathcal{H}}(C)$.

We first show that the construction $\mathcal{H}(t)$ is natural with respect to the most obvious notion of morphisms of twisting cochains: pairs (f, g) , where f is a map of coalgebras and g is a map of algebras, commuting with the twisting cochains. In particular, any twisting cochain $t : C \rightarrow A$ induces a chain map $\widehat{\mathcal{H}}(C) \rightarrow \mathcal{H}(A)$.

We then prove that the Hochschild complex construction admits an extended naturality, with respect to pairs of maps (f, g) , where either f is a map of coalgebras up to strong homotopy (Theorem 2.36) or g is a map of algebras up to strong homotopy (Theorem 2.46). We point out that the natural section $A \rightarrow \Omega\mathcal{B}A$ of the counit $\Omega\mathcal{B}A \rightarrow A$ of the bar/cobar adjunction is a map of algebras up to strong homotopy, while the natural retraction $\mathcal{B}\Omega C \rightarrow C$ of the unit map $C \rightarrow \mathcal{B}\Omega C$ is a map of coalgebras up to strong homotopy. Consequently, the natural chain map $\widehat{\mathcal{H}}(C) \rightarrow \mathcal{H}(\Omega C)$ admits a retraction, while $\mathcal{H}(\mathcal{B}A) \rightarrow \mathcal{H}(A)$ admits a section (Corollaries 2.39 and 2.48).

We are interested in determining conditions that guarantee the existence of operations and cooperations on the Hochschild complex, which motivates us to study certain types of (co)algebras with additional structure, known as Alexander–Whitney (co)algebras. A chain coalgebra C is an Alexander–Whitney coalgebra if its comultiplication map is a map of coalgebras up to strong homotopy, and the higher homotopies induce a coassociative comultiplication on ΩC . Every Alexander–Whitney coalgebra C is therefore a Hirsch

coalgebra [18], but, as we show (Example 3.19), not all Hirsch coalgebras are Alexander–Whitney coalgebras. Dually, a chain algebra A is an Alexander–Whitney algebra if its multiplication map is a map of algebras up to strong homotopy, and the higher homotopies induce an associative multiplication on $\mathcal{B}A$. Every Alexander–Whitney algebra A is therefore a Hirsch algebra. Alexander–Whitney (co)algebras are special types of B_∞ -(co)algebras [7,1].

We prove that if H is a chain Hopf algebra, then $\mathcal{B}H$ is an Alexander–Whitney coalgebra and ΩH is an Alexander–Whitney algebra (Theorem 3.12), strengthening a result of Kadeishvili [18]. In the course of the proof, we establish a result that is interesting in and of itself (Theorem A.11): for every pair of chain algebras A and A' , the natural “Alexander–Whitney” map $\mathcal{B}(A \otimes A') \rightarrow \mathcal{B}A \otimes \mathcal{B}A'$ is map of coalgebras up to strong homotopy.

As a consequence of the extended naturality of the Hochschild complex construction, we obtain that if $t : C \rightarrow H$ is a twisting cochain such that C is an Alexander–Whitney coalgebra and H is a chain Hopf algebra, then $\mathcal{H}(t)$ admits a comultiplication extending that on H and lifting that on C (Theorem 3.23). Dually, if $t : H \rightarrow A$ is a twisting cochain such that A is an Alexander–Whitney algebra and H is a chain Hopf algebra, then $\mathcal{H}(t)$ admits a multiplication extending that on A and lifting that on H (Theorem 3.26). In particular, if C is an Alexander–Whitney coalgebra, then $\widehat{\mathcal{H}}(C)$ admits a comultiplication, while if A is an Alexander–Whitney algebra, then $\mathcal{H}(A)$ admits a multiplication.

The heart of this article concerns the existence of power maps on the Hochschild complex of a twisting cochain. We show that, under certain cocommutativity conditions, if $t : C \rightarrow H$ is a twisting cochain, where C is a Hirsch coalgebra and H is a chain Hopf algebra, then $\mathcal{H}(t)$ admits an r th-power map $\tilde{\lambda}_r$ extending the usual r th-power map on H and lifting the identity on C (Theorem 4.1). In particular, if H is a cocommutative Hopf algebra, then $\mathcal{H}(H)$ admits an r th-power map extending the usual r th-power map on H and lifting the identity map on $\mathcal{B}H$ (Corollary 4.3). Dually, if C is a Hirsch coalgebra such that associated comultiplication on ΩC is cocommutative, then $\widehat{\mathcal{H}}(C)$ admits an r th-power map extending the usual r th-power map on ΩC and lifting the identity on C (Corollary 4.2). We also show that the natural map $\widehat{\mathcal{H}}(C) \rightarrow \mathcal{H}(H)$ induced by the twisting cochain t commutes with the r th-power maps.

Throughout the article we provide numerous concrete examples of our constructions, primarily topological in nature.

Remark 1.1. The r th-power maps on the Hochschild complex of a cocommutative Hopf algebra should induce a Hodge-type decomposition of its Hochschild homology, at least in characteristic zero. It would be interesting to study this decomposition.

Remark 1.2. It is probably possible to generalize the constructions here to higher-order Hochschild complexes, in the sense of Pirashvili and Ginot [23,8].

1.1. *Notation and conventions*

- Throughout this paper we are working over a commutative ring R , so that all undecorated tensor products are implicitly taken over R . We denote the category of (non-negative) chain complexes over R by Ch_R , the category of augmented, associative chain algebras over R by Alg_R , the category of coaugmented, coassociative, connected chain coalgebras by Coalg_R and the category of connected chain Hopf algebras by Hopf_R . The underlying graded modules of all chain complexes are assumed to be R -free. For any object A in Alg_R (respectively, C in Coalg_R), we denote by 1 the image of the unit element in R under the unit map (respectively, the coaugmentation).

The degree of an element v of a chain complex V is denoted $|v|$.

Given chain complexes (V, d) and (W, d) , the notation $f : (V, d) \xrightarrow{\cong} (W, d)$ indicates that f induces an isomorphism in homology. In this case we refer to f as a *quasi-isomorphism*.

Let $f, g : A \rightarrow A'$ be morphisms of chain algebras. A *derivation homotopy* from f to g consists of a chain homotopy $H : A \rightarrow A'$ from f to g such that $H(ab) = H(a)f(b) + (-1)^{|a|}g(a)H(b)$ for all $a, b \in A$.

- The *suspension* endofunctor s on the category of graded modules is defined on objects $V = \bigoplus_{i \in \mathbb{Z}} V_i$ by $(sV)_i \cong V_{i-1}$. Given a homogeneous element v in V , we write sv for the corresponding element of sV . The suspension s admits an obvious inverse, which we denote s^{-1} .
- Let T denote the endofunctor on the category of free graded R -modules given by

$$TV = \bigoplus_{n \geq 0} V^{\otimes n},$$

where $V^{\otimes 0} = R$. An element of the summand $V^{\otimes n}$ of TV is a sum of terms denoted $v_1 | \cdots | v_n$, where $v_i \in V$ for all i .

- The *bar construction* functor $\mathcal{B} : \text{Alg}_R \rightarrow \text{Coalg}_R$ is defined by

$$\mathcal{B}A = (T(s\bar{A}), d_{\mathcal{B}})$$

where \bar{A} denotes the augmentation ideal of A , and if d is the differential on A , then

$$\pi \circ d_{\mathcal{B}}(sa) = -s(da)$$

and

$$\pi \circ d_{\mathcal{B}}(sa|sb) = (-1)^{|a|}s(ab),$$

where $\pi : T(s\bar{A}) \rightarrow s\bar{A}$ is the projection. The entire differential is determined by its projection onto $s\bar{A}$, since the graded R -module underlying $\mathcal{B}A$ is naturally a cofree coassociative coalgebra, with comultiplication given by splitting of words.

- Throughout this article we apply the *Koszul rule*, a sign convention for commuting elements of a graded module or for commuting a morphism of graded modules past an element of the source module. For example, if V and W are graded algebras and $v \otimes w, v' \otimes w' \in V \otimes W$, then

$$(v \otimes w) \cdot (v' \otimes w') = (-1)^{|w| \cdot |v'|} v v' \otimes w w'.$$

Furthermore, if $f : V \rightarrow V'$ and $g : W \rightarrow W'$ are morphisms of graded modules, then for all $v \otimes w \in V \otimes W$,

$$(f \otimes g)(v \otimes w) = (-1)^{|g| \cdot |v|} f(v) \otimes g(w).$$

Moreover, starting from an element $v_1 | \cdots | v_n$, with sign $+1$, of TV for some free graded R -module V , the Koszul rule implies that for a permutation σ of $\{1, \dots, n\}$, the induced sign on the permuted word $v_{\sigma(1)} | \cdots | v_{\sigma(n)}$ is $(-1)^{\epsilon_\sigma}$, where

$$\epsilon_\sigma = \sum_{\{j < k | \sigma(k) < \sigma(j)\}} |v_j| \cdot |v_k|.$$

- The *cobar construction* functor $\Omega : \text{Coalg}_R \rightarrow \text{Alg}_R$ is defined by

$$\Omega C = (T(s^{-1}\bar{C}), d_\Omega)$$

where \bar{C} denotes the coaugmentation coideal of C , and if d denotes the differential on C and c is a homogeneous element of C , then

$$d_\Omega(s^{-1}c) = -s(dc) + (-1)^{|c_i|} s^{-1}c_i | s^{-1}c^i,$$

where the reduced comultiplication applied to c is $c_i \otimes c^i$ (using Einstein implicit summation notation). The entire differential is determined by its restriction to $s^{-1}\bar{C}$, since the graded R -module underlying ΩC is naturally a free associative algebra, with multiplication given by concatenation.

- The category of simplicial sets is denoted sSet in this article. Its full subcategory of reduced simplicial sets (i.e., simplicial sets with a unique 0-simplex) is denoted sSet_0 , while the category of pointed simplicial sets and basepoint-preserving simplicial maps is denoted sSet_* . Observe that sSet_0 can naturally be viewed as a full subcategory of sSet_* . Objects in sSet , sSet_* , and sSet_0 are usually denoted M , L and K , respectively, in this paper. The normalized chains functor from simplicial sets to chain complexes is denoted C_* .
- The *reduced simplicial suspension functor* $E : \text{sSet}_* \rightarrow \text{sSet}_0$ ([20, Definition 27.6]) is defined by $(EL)_0 = \{a_0\}$, and for $n > 0$,

$$(EL)_n = \{s_0^n a_0\} \amalg \coprod_{1 \leq k \leq n} \{k\} \times L_{n-k} / \sim,$$

where $(k, s_0^{n-k}x_0) \sim s_0^n a_0$ for all $1 \leq k \leq n$, for every pointed simplicial set (L, x_0) . The *unreduced simplicial suspension functor* $E^u : \mathbf{sSet} \rightarrow \mathbf{sSet}_*$ ([26, Example 2.16]) is defined for any simplicial set M by $(E^u M)_0 = \{b_0, c_0\}$, where b_0 is the basepoint, and c_0 is the cone point, and for $n > 0$,

$$(E^u M)_n = \{s_0^n b_0, s_0^n c_0\} \amalg \coprod_{1 \leq k \leq n} \{k\} \times M_{n-k}.$$

The *double suspension functor* for unpointed simplicial sets is then defined to be $S^2 = EE^u : \mathbf{sSet} \rightarrow \mathbf{sSet}_0$.

2. The Hochschild complex of a twisting cochain

We begin this section by recalling the definition and certain well-known examples of twisting cochains. We observe that twisting cochains are the objects of a category \mathbf{Tw} , which admits an interesting monoidal structure. We then define the Hochschild complex functor $\mathcal{H} : \mathbf{Tw} \rightarrow \mathbf{Ch}_R$, which turns out to be strongly monoidal, and consider certain important special cases.

Weakening the definition of morphisms in \mathbf{Tw} somewhat, we then consider two faithful, wide embeddings (i.e., injective on morphisms and bijective on objects) $\mathbf{Tw} \hookrightarrow \mathbf{Tw}^{\text{sh}}$ and $\mathbf{Tw} \hookrightarrow \mathbf{Tw}_{\text{sh}}$ and show that the Hochschild complex functor extends over both \mathbf{Tw}^{sh} and \mathbf{Tw}_{sh} . The extended naturality of the Hochschild construction that we obtain in this manner plays an essential role in the later sections of the paper.

2.1. Twisting cochains: definition and examples

Seen as functors from coalgebras to algebras and vice versa, the cobar and bar constructions form an adjoint pair $\Omega \dashv \mathcal{B}$. Let $\eta : \text{Id} \rightarrow \mathcal{B}\Omega$ denote the unit of this adjunction. It is well known that for all connected, coaugmented chain coalgebras C , the counit map

$$\eta_C : C \xrightarrow{\cong} \mathcal{B}\Omega C \tag{2.1}$$

is a quasi-isomorphism of chain coalgebras [22, Corollary 10.5.4].

Dually, let $\varepsilon : \Omega\mathcal{B} \rightarrow \text{Id}$ denote the counit of this adjunction. For all augmented chain algebras A , the counit map

$$\varepsilon_A : \Omega\mathcal{B}A \xrightarrow{\cong} A \tag{2.2}$$

is a quasi-isomorphism of chain algebras [22, Corollary 10.5.4].

Definition 2.1. A *twisting cochain* from a connected, coaugmented chain coalgebra (C, d) with comultiplication Δ to an augmented chain algebra (A, d) with multiplication m

consists of a linear map $t : C \rightarrow A$ of degree -1 such that

$$dt + td = m(t \otimes t)\Delta.$$

Remark 2.2. A twisting cochain $t : C \rightarrow A$ induces both a chain algebra map

$$\alpha_t : \Omega C \rightarrow A$$

specified by $\alpha_t(s^{-1}c) = t(c)$ and a chain coalgebra map

$$\beta_t : C \rightarrow \mathcal{B}A,$$

satisfying

$$\alpha_t = \varepsilon_A \circ \Omega\beta_t \quad \text{and} \quad \beta_t = \mathcal{B}\alpha_t \circ \eta_C.$$

It follows that α_t is a quasi-isomorphism if and only if β_t is a quasi-isomorphism.

Example 2.3. Let C be a connected, coaugmented chain coalgebra. The *universal twisting cochain*

$$t_\Omega : C \rightarrow \Omega C$$

is defined by $t_\Omega(c) = s^{-1}c$ for all $c \in C$, where $s^{-1}c$ is defined to be 0 if $|c| = 0$. Note that $\alpha_{t_\Omega} = \text{Id}_{\Omega C}$, so that $\beta_{t_\Omega} = \eta_C$. Moreover, t_Ω truly is universal, as all twisting cochains $t : C \rightarrow A$ factor through t_Ω , since the diagram

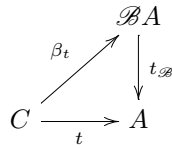
$$\begin{array}{ccc} C & \xrightarrow{t_\Omega} & \Omega C \\ & \searrow t & \downarrow \alpha_t \\ & & A \end{array}$$

always commutes.

Example 2.4. Let A be an augmented chain algebra. The *couniversal twisting cochain*

$$t_{\mathcal{B}} : \mathcal{B}A \rightarrow A$$

is defined by $t_{\mathcal{B}}(sa) = a$ for all $a \in A$ and $t_{\mathcal{B}}(sa_1 | \cdots | sa_n) = 0$ for all $n \neq 1$. Note that $\beta_{t_{\mathcal{B}}} = \text{Id}_{\mathcal{B}A}$, so that $\alpha_{t_{\mathcal{B}}} = \varepsilon_A$. Moreover, $t_{\mathcal{B}}$ truly is couniversal, as all twisting cochains $t : C \rightarrow A$ factor through $t_{\mathcal{B}}$, since the diagram



always commutes.

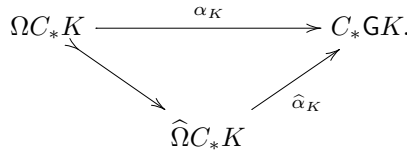
Example 2.5. Let K be a reduced simplicial set, and let $\mathbf{G}K$ denote its Kan loop group. In 1961 [24], Szczarba gave an explicit formula for a twisting cochain

$$t_K : C_*K \rightarrow C_*\mathbf{G}K,$$

natural in K that induces a chain algebra map

$$\alpha_K := \alpha_{t_K} : \Omega C_*K \rightarrow C_*\mathbf{G}K. \tag{2.3}$$

As shown in [16], α_K factors naturally through an “extended cobar construction,” $\widehat{\Omega}C_*K$, of which the usual cobar construction is a chain subalgebra, i.e., there is a commuting diagram of chain algebra maps



Moreover, $\widehat{\alpha}_K$ admits a natural retraction ρ_K such that $\widehat{\alpha}_K\rho_K$ is chain homotopic to the identity on $C_*\mathbf{G}K$. In particular, $\widehat{\alpha}_K$ is a quasi-isomorphism for all reduced simplicial sets K . Since $\widehat{\Omega}C_*K = \Omega C_*K$ if K is actually 1-reduced, it follows that α_K itself is a quasi-isomorphism if K is 1-reduced.

Remark 2.6. If $t : C \rightarrow A$ is a twisting cochain, $f : C' \rightarrow C$ is a chain coalgebra map and $g : A \rightarrow A'$ is a chain algebra map, then $gtf : C' \rightarrow A'$ is also a twisting cochain.

Notation 2.7. Let \mathbf{Tw} denote the category such that

- $\text{Ob Tw} = \{t : C \rightarrow A \mid t \text{ twisting cochain}\}$, and
- if $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ are twisting cochains, then

$$\text{Tw}(t, t') = \{(f, g) \in \text{Coalg}_R(C, C') \times \text{Alg}_R(A, A') \mid g \circ t = t' \circ f\}.$$

Composition of morphisms in \mathbf{Tw} is defined componentwise.

Remark 2.8. Note that $(f, g) \in \text{Tw}(t, t')$ if and only if $\mathcal{B}g \circ \beta_t = \beta_{t'} \circ f$, which is true if and only if $g \circ \alpha_t = \alpha_{t'} \circ \Omega f$.

Later in this paper we are led to consider the following variant of Tw . Below, and elsewhere in this paper, if (H, δ) denotes a chain Hopf algebra, then H is the underlying chain algebra, and δ is the comultiplication.

Notation 2.9. Let Tw_{Hopf} denote the category with

- $\text{Ob Tw}_{\text{Hopf}} = \{(C \xrightarrow{t} H, (H, \delta)) \mid t \in \text{Ob Tw}, (H, \delta) \in \text{Ob Hopf}_R\}$, and
- if $(t, (H, \delta))$ and $(t', (H', \delta'))$ are objects in Tw_{Hopf} , then

$$\text{Tw}_{\text{Hopf}}(t, t') = \{(f, g) \in \text{Tw}(t, t') \mid (g \otimes g)\delta = \delta'g\}.$$

The proposition below gives a categorical formulation of the universality of t_Ω and of the couniversality of $t_{\mathcal{B}}$.

Proposition 2.10. *If $S : \text{Tw} \rightarrow \text{Coalg}_R$ is the functor that projects onto the source of a twisting cochain and $U : \text{Coalg}_R \rightarrow \text{Tw}$ is the “universal twisting cochain functor,” specified by $U(C) = t_\Omega : C \rightarrow \Omega C$ and $U(f) = (f, \Omega f)$, then U is left adjoint to S .*

Similarly, if $T : \text{Tw} \rightarrow \text{Alg}_R$ is the functor that projects onto the target of a twisting cochain and $V : \text{Alg}_R \rightarrow \text{Tw}$ is the “couniversal twisting cochain functor,” specified by $V(A) = t_{\mathcal{B}} : \mathcal{B}A \rightarrow A$ and $V(g) = (\mathcal{B}g, g)$, then V is right adjoint to T .

Proof. Note that $(f, g) \in \text{Tw}(U(C'), t)$ implies that

$$\begin{array}{ccc} C' & \xrightarrow{f} & C \\ t_\Omega \downarrow & & \downarrow t \\ \Omega C' & \xrightarrow{g} & A \end{array}$$

commutes. It follows that $g = \alpha_t \circ \Omega f$, since the graded algebra underlying $\Omega C'$ is free, and g is therefore determined by its values on the generators $s^{-1}\overline{C'}$.

The natural isomorphism

$$\zeta : \text{Coalg}_R(C', S(t)) \xrightarrow{\cong} \text{Tw}(U(C'), t)$$

for $C' \in \text{Ob Coalg}_R$ and a twisting cochain $t : C \rightarrow A$ is thus defined by $\zeta(f) = (f, \alpha_t \circ \Omega f)$, with inverse ζ^{-1} defined by $\zeta^{-1}(f, g) = f$.

The proof that V is right adjoint to T is similar. \square

There is an important binary operation on the set of twisting cochains, defined as follows.

Definition 2.11. Let $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ be twisting cochains. Let $\varepsilon : C \rightarrow R$ and $\varepsilon' : C' \rightarrow R$ be the counits (augmentations), and let $\eta : R \rightarrow A$ and $\eta' : R \rightarrow A'$ be the units (coaugmentations). Set

$$t * t' = t \otimes \eta' \varepsilon' + \eta \varepsilon \otimes t' : C \otimes C' \rightarrow A \otimes A'$$

Then $t * t'$ is a twisting cochain, called the *cartesian product* of t and t' .

Example 2.12. An important special case of the cartesian product of twisting cochains is

$$t_\Omega * t_\Omega : C \otimes C' \rightarrow \Omega C \otimes \Omega C',$$

for $C, C' \in \text{Coalg}_R$. To simplify notation, we write

$$q = \alpha_{t_\Omega * t_\Omega} : \Omega(C \otimes C') \rightarrow \Omega C \otimes \Omega C'. \tag{2.4}$$

Milgram proved in [21] that the chain algebra map q was a quasi-isomorphism if C and C' were simply connected, i.e., connected and $C_1 = C'_1 = 0$. In [12] it was shown that q is in fact a chain homotopy equivalence, for all $C, C' \in \text{Coalg}_R$.

Dually, for all $A, A' \in \text{Alg}_R$, there is a map of chain coalgebras that is a chain homotopy equivalence

$$\nabla = \beta_{t * t'} : \mathcal{B}A \otimes \mathcal{B}A' \rightarrow \mathcal{B}(A \otimes A'). \tag{2.5}$$

The equivalence ∇ is often called the “Eilenberg–Zilber” equivalence, by analogy with the Eilenberg–Zilber equivalence of algebraic topology. We refer the reader to [Appendix A](#) for further details of this equivalence.

Remark 2.13. Let $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ be twisting cochains. Note that

$$\alpha_{t * t'} = (\alpha_t \otimes \alpha_{t'}) \circ q : \Omega(C \otimes C') \rightarrow A \otimes A'$$

and that

$$\beta_{t * t'} = \nabla \circ (\beta_t \otimes \beta_{t'}) : C \otimes C' \rightarrow \mathcal{B}(A \otimes A').$$

Remark 2.14. Endowed with the cartesian product of twisting cochains, the category Tw is clearly monoidal, where the unit object is the zero map $0 : R \rightarrow R$.

2.2. Definition of the Hochschild complex

We can now define the Hochschild construction functor $\mathcal{H} : \text{Tw} \rightarrow \text{Ch}_R$, which we study and apply throughout the remainder of this paper. We begin by introducing a somewhat more general construction, which also englobes the other familiar constructions of chain complexes built from twisting cochains.

Definition 2.15. Let $t : C \rightarrow A$ be a twisting cochain. Let N be a C -bicomodule with left C -coaction λ_N and right C -coaction ρ_N , and let M be an A -bimodule with left A -action λ_M and right A -action ρ_M . Let d_M and d_N denote the differentials on M and N , respectively. The *Hochschild complex* of t with coefficients in N and M , denoted $\mathcal{H}_t(N, M)$, is the chain complex with underlying graded R -module $N \otimes M$ and with differential d_t , defined by

$$\begin{aligned} d_t &= d_N \otimes M + N \otimes d_M \\ &\quad - (N \otimes \lambda_M)(N \otimes t \otimes M)(\rho_N \otimes M) \\ &\quad - (N \otimes \rho_M)(213)(t \otimes N \otimes M)(\lambda_N \otimes M), \end{aligned}$$

where (213) denotes the cyclic permutation of the three tensor factors.

It is a matter of straightforward, but somewhat tedious, computation, using the definition of twisting cochains, to show that $d_t^2 = 0$, i.e., that $\mathcal{H}_t(N, M)$ really is a chain complex. Furthermore, if M is augmented over R and N is coaugmented over R , there is a twisted extension of chain complexes

$$M \twoheadrightarrow \mathcal{H}_t(N, M) \twoheadrightarrow N.$$

Remark 2.16. If we apply d_t to $y \otimes x \in N \otimes M$, we obtain

$$\begin{aligned} d_t(y \otimes x) &= d_N y \otimes x + (-1)^{|y|} y \otimes d_M x \\ &\quad - (-1)^{|y_j|} y_j \otimes t(c^j) \cdot x - (-1)^{(|c_i|-1)(|y^i|+|x|)} y^i \otimes x \cdot t(c_i), \end{aligned}$$

where \cdot denotes the left and right actions of A on M , $\lambda_N(y) = c_i \otimes y^i$ and $\rho_N(y) = y_j \otimes c^j$ (using Einstein summation notation).

Notation 2.17. When $N = C$, seen as a bicomodule over itself via its comultiplication, and $M = A$, seen as a bimodule over itself via its multiplication, then we write

$$\mathcal{H}(t) := \mathcal{H}_t(C, A) = (C \otimes A, d_t).$$

Note that in this case

$$\begin{aligned} d_t &= d_C \otimes A + C \otimes d_A \\ &\quad - (C \otimes \mu)((C \otimes t \otimes A) + (213)(t \otimes C \otimes A))(\Delta \otimes A). \end{aligned}$$

Example 2.18. If $A \in \text{Alg}_R$ and M is an A -bimodule, then $\mathcal{H}_{t_\emptyset}(\mathcal{B}A, M)$ is exactly the usual Hochschild complex on A with coefficients in M . In particular,

$$\mathcal{H}(t_{\mathcal{B}}) = \mathcal{H}(A)$$

is the usual Hochschild complex on A .

Example 2.19. If $C \in \text{Coalg}_R$ and N is a C -bicomodule, then $\mathcal{H}_{t_\Omega}(N, \Omega C)$ is exactly the coHochschild complex on C with coefficients in N , as defined in [13]. In particular,

$$\mathcal{H}(t_\Omega) = \widehat{\mathcal{H}}(C)$$

is the coHochschild complex on C .

Example 2.20. The usual twisted extension over a twisting cochain $t : C \rightarrow A$ of a right C -module N by a left A -module M is a special case of the Hochschild complex construction defined above. It suffices to consider N as a C -bicomodule with trivial left C -coaction and M as an A -bimodule with trivial right A -action.

Proposition 2.21. *The Hochschild complex construction extends to a functor*

$$\mathcal{H} : \text{Tw} \rightarrow \text{Ch}_R.$$

Proof. Let $(f, g) : t \rightarrow t'$ be a morphism in Tw from $t : C \rightarrow A$ to $t' : C' \rightarrow A'$. Define

$$\mathcal{H}(f, g) : \mathcal{H}(t) \rightarrow \mathcal{H}(t')$$

by $\mathcal{H}(f, g)(c \otimes a) = f(c) \otimes g(a)$. An easy calculation shows that $\mathcal{H}(f, g)$ is then a chain map. Moreover, it is obvious that $\mathcal{H}(\text{Id}_C, \text{Id}_A) = \text{Id}_{\mathcal{H}(t)}$ and that \mathcal{H} respects composition. \square

Thanks to this proposition, we obtain a new proof of Theorem 4.1 in [13].

Corollary 2.22. *A twisting cochain $t : C \rightarrow A$ induces a chain map*

$$\beta_t \otimes \alpha_t : \widehat{\mathcal{H}}(C) \rightarrow \mathcal{H}(A),$$

which is a quasi-isomorphism if α_t and β_t are quasi-isomorphisms.

Proof. Applying Proposition 2.21 to the morphism of twisting cochains

$$\begin{array}{ccc} C & \xrightarrow{\beta_t} & \mathcal{B}A \\ t_\Omega \downarrow & & \downarrow t_{\mathcal{B}} \\ \Omega C & \xrightarrow{\alpha_t} & A, \end{array}$$

we obtain a morphism of chain complexes

$$\mathcal{H}(\beta_t, \alpha_t) = \beta_t \otimes \alpha_t : \mathcal{H}(t_\Omega) \rightarrow \mathcal{H}(t_{\mathcal{B}}).$$

Since (cf. [Examples 2.18 and 2.19](#)) $\widehat{\mathcal{H}}(C) = \mathcal{H}(t_\Omega)$ and $\mathcal{H}(A) = \mathcal{H}(t_{\mathcal{B}})$, we can conclude.

The quasi-isomorphism result follows from an easy argument based on the Zeeman comparison theorem. \square

Remark 2.23. Applied to the universal twisting cochain $t_\Omega : C \rightarrow \Omega C$, [Corollary 2.22](#) implies the existence of a quasi-isomorphism

$$\eta_C \otimes \text{Id}_{\Omega C} : \widehat{\mathcal{H}}(C) \xrightarrow{\cong} \mathcal{H}(\Omega C). \tag{2.6}$$

Dually, for the couniversal twisting cochain $t_{\mathcal{B}} : \mathcal{B}A \rightarrow A$, we obtain a quasi-isomorphism

$$\text{Id}_{\mathcal{B}A} \otimes \varepsilon_A : \widehat{\mathcal{H}}(\mathcal{B}A) \xrightarrow{\cong} \mathcal{H}(A). \tag{2.7}$$

The following alternative description of the Hochschild construction of a twisting cochain turns out to be quite useful, as we see in [sections 2.3 and 4.2](#). First we define yet another complex that can be constructed from a twisting cochain.

Definition 2.24. Let $t : C \rightarrow A$ be a twisting cochain. The *twisted double extension* of C by A is the differential graded A -bimodule

$$\mathcal{D}(t) = (A \otimes C \otimes A, D_t),$$

where D_t is the derivation of A -bimodules specified by

$$D_t(c) = 1 \otimes dc \otimes 1 - t(c_i) \otimes c^i \otimes 1 - (-1)^{c_i} 1 \otimes c_i \otimes t(c^i),$$

for all $c \in C$, where the reduced comultiplication applied to c is $c_i \otimes c^i$.

It is a straightforward exercise, using the definition of twisting cochains, to show that $D_t^2 = 0$.

Remark 2.25. Note that for any twisting cochain $t : C \rightarrow A$,

$$\mathcal{D}(t) \cong A \otimes_{\Omega C} \mathcal{D}(t_\Omega) \otimes_{\Omega C} A.$$

The relationship between the twisted double extension and the Hochschild complex can be expressed as follows.

Lemma 2.26. *If $t : C \rightarrow A$ is a twisting cochain, then there is an isomorphism of chain complexes*

$$\mathcal{H}(t) \cong \mathcal{D}(t) \otimes_{A \otimes A^{op}} A,$$

where the left $A \otimes A^{op}$ -action on A arises, as usual, from the left and right actions of A on itself, while the right $A \otimes A^{op}$ -action on $\mathcal{H}(t)$ is that given by

$$(a \otimes c \otimes a') \cdot (b' \otimes b) = (-1)^{|b|(|a|+|c|+|a'|+|b'|)}(ba) \otimes c \otimes (a'b').$$

The proof of this lemma is an easy computation, once one observes that there is an isomorphism of right $A \otimes A^{op}$ -modules

$$A \otimes C \otimes A \xrightarrow{\cong} C \otimes A \otimes A^{op} : a \otimes c \otimes a' \mapsto (-1)^{|a|(|c|+|a'|)} c \otimes a' \otimes a.$$

The derivation defined below plays a particularly important role in our discussion in the next section of the extended naturality of the Hochschild complex functor. Note that for any twisting cochain $t : C \rightarrow A$, the twisted double extension $\mathcal{D}(t)$ is an ΩC -bimodule, since it is an A -bimodule.

Definition 2.27. Let $t : C \rightarrow A$ be a twisting cochain. Let

$$\sigma_t : (\Omega C)_{>0} \rightarrow \mathcal{D}(t)$$

denote the derivation of ΩC -bimodules determined by

$$\sigma_t(s^{-1}c) = 1 \otimes c \otimes 1.$$

That σ_t is a derivation of ΩC -bimodules implies, for example, that

$$\sigma_t(s^{-1}c_1|s^{-1}c_2) = 1 \otimes c_1 \otimes t(c_2) - (-1)^{|c_1|}t(c_1) \otimes c_2 \otimes 1.$$

Direct application of the definitions implies the following useful identity.

Lemma 2.28. *For any twisting cochain $t : C \rightarrow A$,*

$$D_t \sigma_t = -\sigma_t d_\Omega : (\Omega C)_{>0} \rightarrow \mathcal{D}(t).$$

2.3. Extended naturality of the Hochschild construction

We show in this section that the Hochschild construction on a twisting cochain is actually natural with respect to a weaker notion of morphism than that adopted in the definition of Tw .

Definition 2.29. (See [9].) Given $C, C' \in \text{Ob Coalg}_R$, a chain map $f : C \rightarrow C'$ is called a *DCSH (Differential Coalgebra with Strong Homotopy) map* or a *strongly homotopy-comultiplicative map* if there is a chain algebra map $\varphi : \Omega C \rightarrow \Omega C'$ such that

$$\begin{array}{ccc}
 \bar{C} & \xrightarrow{f} & \bar{C}' \\
 t_\Omega \downarrow & & \downarrow s^{-1} \\
 \Omega C & \xrightarrow{\varphi} \Omega C' \xrightarrow{\text{proj}} & s^{-1}\bar{C}'
 \end{array}$$

commutes. The chain algebra map φ is said to *realize the strong homotopy structure* of the DCSH map f .

Notation 2.30. For any chain algebra map $\varphi : \Omega C \rightarrow \Omega C'$, let $f_\varphi : C \rightarrow C'$ denote the R -linear map of degree zero given by the composite

$$\bar{C} \xrightarrow{t_\Omega} \Omega C \xrightarrow{\varphi} \Omega C' \xrightarrow{\text{proj}} s^{-1}\bar{C}' \xrightarrow{s} C'$$

and by $f_\varphi(1)$. Note that since φ is a chain map, f_φ is as well. Indeed, it is a DCSH map, with φ realizing its strong homotopy structure.

Example 2.31. If K and L are 1-reduced simplicial sets, then the natural Alexander–Whitney map

$$C_*(K \times L) \xrightarrow{\cong} C_*K \otimes C_*L$$

is a DCSH map [9].

Example 2.32. As we prove in Appendix A (Theorem A.11), if A and A' are augmented chain algebras, then the natural Alexander–Whitney map

$$\mathcal{B}(A \otimes A') \xrightarrow{\cong} \mathcal{B}A \otimes \mathcal{B}A'$$

defined by Eilenberg and Mac Lane in [5] is a DCSH map.

Example 2.33. Let C be a connected, coaugmented chain coalgebra. Let $\rho_C : \mathcal{B}\Omega C \rightarrow C$ denote the composite

$$\mathcal{B}\Omega C \xrightarrow{t_{\mathcal{B}}} \Omega C \xrightarrow{\text{proj}} s^{-1}C \xrightarrow{s} C,$$

i.e., $\rho_C(s(s^{-1}c)) = c$ for all $c \in \bar{C}$ and $\rho_C(sa_1 | \cdots | sa_n) = 0$ otherwise. It is obvious that ρ_C is a retraction of η_C , i.e.,

$$\begin{array}{ccc}
 C & \xrightarrow{\eta_C} & \mathcal{B}\Omega C \\
 & \searrow \text{Id}_C & \downarrow \rho_C \\
 & & C
 \end{array}$$

commutes, which implies that ρ_C is a quasi-isomorphism. Moreover, ρ_C is a DCSH map, since

$$\begin{array}{ccc}
 \mathcal{B}_+\Omega C & \xrightarrow{\rho_C} & \overline{C} \\
 t_\Omega \downarrow & & \downarrow s^{-1} \\
 \Omega\mathcal{B}\Omega C & \xrightarrow{\varepsilon_{\Omega C}} \Omega C \xrightarrow{\text{proj}} & s^{-1}\overline{C}
 \end{array}$$

commutes, where $\mathcal{B}_+\Omega C$ denotes the R -submodule of positively graded elements. In particular, $\varepsilon_{\Omega C}$ realizes the strong homotopy structure of ρ_C .

Notation 2.34. Let Tw^{sh} denote the category such that

- $\text{Ob Tw}^{\text{sh}} = \{t : C \rightarrow A \mid t \text{ twisting cochain}\}$, and
- if $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ are twisting cochains, then

$$\text{Tw}^{\text{sh}}(t, t') = \{(\varphi, g) \in \text{Alg}_R(\Omega C, \Omega C') \times \text{Alg}_R(A, A') \mid g \circ \alpha_t = \alpha_{t'} \circ \varphi\}.$$

Composition of morphisms in Tw^{sh} is defined componentwise.

Remark 2.35. There is an obvious faithful functor Tw to Tw^{sh} , which is the identity on objects and which sends a morphism $(f, g) : t \rightarrow t'$ to the morphism $(\Omega f, g) : t \rightarrow t'$.

Theorem 2.36. *The Hochschild construction functor extends to a functor*

$$\mathcal{H}^{\text{sh}} : \text{Tw}^{\text{sh}} \rightarrow \text{Ch}_R.$$

In particular, given twisting cochains $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ and a commutative diagram in Alg_R

$$\begin{array}{ccc}
 \Omega C & \xrightarrow{\alpha_t} & A \\
 \varphi \downarrow & & \downarrow g \\
 \Omega C' & \xrightarrow{\alpha_{t'}} & A',
 \end{array}$$

there is a chain map $\mathcal{H}^{\text{sh}}(\varphi, g) : \mathcal{H}(t) \rightarrow \mathcal{H}(t')$ such that

$$\begin{array}{ccccccc}
 \Omega C & \xrightarrow{\alpha_t} & A & \twoheadrightarrow & \mathcal{H}^{\rho}(t) & \twoheadrightarrow & C \\
 \varphi \downarrow & & g \downarrow & & \mathcal{H}^{sh}(\varphi, g) \downarrow & & f_{\varphi} \downarrow \\
 \Omega C' & \xrightarrow{\alpha_{t'}} & A' & \twoheadrightarrow & \mathcal{H}^{\rho}(t') & \twoheadrightarrow & C'
 \end{array} \tag{2.8}$$

commutes (cf. [Notation 2.30](#)). Furthermore, if φ and g are quasi-isomorphisms, then so is $\mathcal{H}^{sh}(\varphi, g)$.

Proof. We begin by proving extended naturality for twisted double extensions ([Definition 2.24](#)), from which that of Hochschild complexes then follows easily. Let $(\varphi, g) : t \rightarrow t'$ be a morphism in Tw^{sh} from $t : C \rightarrow A$ to $t' : C' \rightarrow A'$. The pair (φ, g) induces a morphism of A -bimodules

$$\mathcal{D}(\varphi, g) : \mathcal{D}(t) \rightarrow \mathcal{D}(t')$$

between the twisted double extensions associated to the t and t' , which is specified by

$$\mathcal{D}(\varphi, g)(1 \otimes c \otimes 1) = \sigma_{t'} \varphi(s^{-1}c) \quad \forall c \in C_{>0} \quad \text{and} \quad \mathcal{D}(\varphi, g)(1 \otimes 1 \otimes 1) = 1,$$

where $\sigma_{t'} : (\Omega C')_{>0} \rightarrow \mathcal{D}(t')$ is the derivation from [Definition 2.27](#). Thus,

$$\mathcal{D}(\varphi, g)(a \otimes c \otimes b) = g(a)\mathcal{D}(\varphi, g)(1 \otimes c \otimes 1)g(b)$$

for all $a, b \in A$ and $c \in C$, whence in particular

$$\mathcal{D}(\varphi, g)(1 \otimes 1 \otimes a) = g(a) \tag{2.9}$$

for all $a \in A$. Moreover, for all $c \in C_{>0}$,

$$\mathcal{D}(\varphi, g)(1 \otimes c \otimes 1) = 1 \otimes f_{\varphi}(c) \otimes 1 + \text{terms in } (A_+ \otimes C \otimes A + A \otimes C \otimes A_+). \tag{2.10}$$

Straightforward computations using [Lemma 2.28](#) show that

$$D_{t'} \mathcal{D}(\varphi, g) = \mathcal{D}(\varphi, g) D_t,$$

i.e., $\mathcal{D}(\varphi, g)$ is a morphism of differential graded A -bimodules or, equivalently, right $A \otimes A^{op}$ -modules. Applying [Lemma 2.26](#), we then obtain a morphism of chain complexes

$$\mathcal{H}^{sh}(\varphi, g) = \mathcal{D}(\varphi, g) \otimes_{A \otimes A^{op}} A : \mathcal{H}(t) \rightarrow \mathcal{H}(t')$$

that makes diagram (2.8) commute, thanks to (2.9) and (2.10).

An easy spectral sequence argument shows that $\mathcal{H}^{sh}(\varphi, g)$ is a quasi-isomorphism if φ and g are quasi-isomorphisms. \square

For use later in this paper, we single out the following consequence of the proof above.

Scholium 2.37. For every morphism $(\varphi, g) : t \rightarrow t'$ in $\mathbb{T}\mathbf{W}^{\text{sh}}$ from $t : C \rightarrow A$ to $t' : C' \rightarrow A'$, there is a morphism of differential graded A -bimodules

$$\mathcal{D}(\varphi, g) : \mathcal{D}(t) \rightarrow \mathcal{D}(t')$$

such that

$$\begin{array}{ccc} (\Omega C)_{>0} & \xrightarrow{\sigma_t} & \mathcal{D}(t) \\ \varphi \downarrow & & \downarrow \mathcal{D}(\varphi, g) \\ (\Omega C')_{>0} & \xrightarrow{\sigma_{t'}} & \mathcal{D}(t') \end{array}$$

commutes.

Proof. It suffices to check commutativity of the diagram on elements of the form $s^{-1}c$, which follows immediately from the definition of $\mathcal{D}(\varphi, g)$. \square

As a consequence of [Theorem 2.36](#), we obtain a new proof of [Theorem 3.3](#) in [\[13\]](#).

Corollary 2.38. A DCSH map $f : C \rightarrow C'$ with a fixed choice of chain algebra map $\varphi : \Omega C \rightarrow \Omega C'$ realizing its strong homotopy structure naturally induces a chain map

$$\widehat{\varphi} : \widehat{\mathcal{H}}(C) \rightarrow \widehat{\mathcal{H}}(C')$$

such that

$$\begin{array}{ccccc} \Omega C & \twoheadrightarrow & \widehat{\mathcal{H}}(C) & \twoheadrightarrow & C \\ \downarrow \varphi & & \downarrow \widehat{\varphi} & & \downarrow f \\ \Omega C' & \twoheadrightarrow & \widehat{\mathcal{H}}(C') & \twoheadrightarrow & C' \end{array}$$

commutes.

Proof. Recall that $\widehat{\mathcal{H}}(C) = \mathcal{H}(t_\Omega)$ ([Example 2.19](#)). Let $\widehat{\varphi} = \mathcal{H}^{\text{sh}}(\varphi, \varphi)$. \square

[Theorem 2.36](#) also provides us with a new, more conceptual proof of [Theorem B](#) from [\[17\]](#), which was reformulated as [Theorem 4.3](#) in [\[13\]](#) essentially as follows. Recall the DCSH retraction $\rho_C : \mathcal{B}\Omega C \rightarrow C$ from [Example 2.33](#) and the chain map $\eta_C \otimes \text{Id}_{\Omega C} : \widehat{\mathcal{H}}(C) \xrightarrow{\cong} \mathcal{H}(\Omega C)$ ([2.6](#)).

Corollary 2.39. Let C be a connected, coaugmented chain coalgebra. There is a quasi-isomorphism $\widehat{\rho}_C : \mathcal{H}(\Omega C) \xrightarrow{\cong} \widehat{\mathcal{H}}(C)$, natural in C , such that

$$\begin{array}{ccccc}
 \Omega C & \twoheadrightarrow & \mathcal{H}(\Omega C) & \twoheadrightarrow & \mathcal{B}\Omega C \\
 \text{Id}_{\Omega C} \downarrow & & \hat{\rho}_C \downarrow & & \rho_C \downarrow \\
 \Omega C & \twoheadrightarrow & \widehat{\mathcal{H}}(C) & \twoheadrightarrow & C
 \end{array}$$

commutes and such that $\hat{\rho}_C \circ (\eta_C \otimes \text{Id}_{\Omega C}) = \text{Id}_{\widehat{\mathcal{H}}(C)}$.

Proof. Let $\hat{\rho}_C = \mathcal{H}^{sh}(\varepsilon_{\Omega C}, \text{Id}_{\Omega C})$. The naturality of \mathcal{H}^{sh} then implies that

$$\hat{\rho}_C \circ (\eta_{\Omega C} \otimes \text{Id}_{\Omega C}) = \text{Id}_{\widehat{\mathcal{H}}(C)}. \quad \square$$

The definitions and results above can be dualized as follows.

Definition 2.40. (See [9].) Given $A, A' \in \text{Ob Alg}_R$, a chain map $g : A \rightarrow A'$ is called a *DASH (Differential Algebra with Strong Homotopy) map* or a *strongly homotopy-multiplicative map* if there is a chain coalgebra map $\gamma : \mathcal{B}A \rightarrow \mathcal{B}A'$ such that

$$\begin{array}{ccccc}
 s\bar{A} & \xrightarrow{\text{incl}} & \mathcal{B}A & \xrightarrow{\gamma} & \mathcal{B}A' \\
 s^{-1} \downarrow & & & & \downarrow t_{\mathcal{B}} \\
 \bar{A} & \xrightarrow{g} & & & \bar{A}'
 \end{array}$$

commutes. The chain coalgebra map γ is said to *realize the strong homotopy structure* of the DASH map g .

Example 2.41. Dualizing Example 2.31, we see that if K and L are 1-reduced simplicial sets of finite type, then the dual of the Alexander–Whitney map (i.e., the cross product)

$$C^*K \otimes C^*L \xrightarrow{\cong} C^*(K \times L)$$

is a DASH map.

Example 2.42. Dualizing Example 2.32, we obtain that if C and C' are connected, coaugmented chain coalgebras of finite type, then the dual

$$\Omega C \otimes \Omega C' \xrightarrow{\cong} \Omega(C \otimes C')$$

of the Alexander–Whitney map for the bar construction is a DASH map.

Example 2.43. Let A be an augmented chain algebra. Let $\sigma_A : A \rightarrow \Omega \mathcal{B}A$ denote the chain map defined by $\sigma_A(a) = s^{-1}(sa)$. It is obvious that σ_A is a section of ε_A , i.e.,

$$\begin{array}{ccc}
 A & \xrightarrow{\sigma_A} & \Omega\mathcal{B}A \\
 & \searrow \text{Id}_A & \downarrow \varepsilon_A \\
 & & A
 \end{array}$$

commutes, which implies that σ_A is a quasi-isomorphism. Moreover, σ_A is a DASH map, since

$$\begin{array}{ccccc}
 s\bar{A} & \xrightarrow{\text{incl}} & \mathcal{B}A & \xrightarrow{\eta_{\mathcal{B}A}} & \mathcal{B}\Omega\mathcal{B}A \\
 s^{-1} \downarrow & & & & \downarrow t_{\mathcal{B}} \\
 \bar{A} & \xrightarrow{\sigma_A} & \overline{\Omega\mathcal{B}A} & &
 \end{array}$$

commutes, where $\overline{\Omega\mathcal{B}A}$ denotes the augmentation ideal of $\Omega\mathcal{B}A$. In particular, $\eta_{\mathcal{B}A}$ realizes the strong homotopy structure of σ_A .

Notation 2.44. Let Tw_{sh} denote the category such that

- $\text{Ob Tw}_{\text{sh}} = \{t : C \rightarrow A \mid t \text{ twisting cochain}\}$, and
- if $t : C \rightarrow A$ and $t' : C' \rightarrow A'$ are twisting cochains, then

$$\text{Tw}_{\text{sh}}(t, t') = \{(f, \gamma) \in \text{Coalg}_R(C, C') \times \text{Coalg}_R(\mathcal{B}A, \mathcal{B}A') \mid \gamma \circ \beta_t = \beta_{t'} \circ f\}.$$

Composition of morphisms in Tw_{sh} is defined componentwise.

Remark 2.45. There is an obvious faithful functor Tw to Tw_{sh} , which is the identity on objects and which sends a morphism $(f, g) : t \rightarrow t'$ to the morphism $(f, \mathcal{B}g) : t \rightarrow t'$.

To avoid truly nasty explicit formulas, we permit ourselves a slight restriction in dualizing [Theorem 2.36](#). Let $\text{Twist}_{\text{sh},f}$ denote the full subcategory of twisting cochains $t : C \rightarrow A$ such that both C and A are connected and of finite type, i.e., are finitely generated free R -modules in each degree.

Theorem 2.46. *The Hochschild construction functor extends to a functor*

$$\mathcal{H}_{\text{sh}} : \text{Twist}_{\text{sh},f} \rightarrow \text{Ch}_R.$$

In particular, given twisting cochains $t : C \rightarrow A$ and $t' : C' \rightarrow A'$, where C, A, C' and A' are connected and of finite type, and a commutative diagram in Coalg_R

$$\begin{array}{ccc}
 C & \xrightarrow{\beta_t} & \mathcal{B}A \\
 f \downarrow & & \downarrow \gamma \\
 C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A',
 \end{array}$$

there is a chain map $\mathcal{H}_{sh}(f, \gamma) : \mathcal{H}(t) \rightarrow \mathcal{H}(t')$ such that

$$\begin{array}{ccccccc}
 A & \twoheadrightarrow & \mathcal{H}(t) & \twoheadrightarrow & C & \xrightarrow{\beta_t} & \mathcal{B}A \\
 g \downarrow & & \mathcal{H}_{sh}(f, \gamma) \downarrow & & f \downarrow & & \downarrow \gamma \\
 A' & \twoheadrightarrow & \mathcal{H}(t') & \twoheadrightarrow & C' & \xrightarrow{\beta_{t'}} & \mathcal{B}A'
 \end{array}$$

commutes, where $g(a) = t_{\mathcal{B}}\gamma(sa)$ for all $a \in A$.

Proof. Recall that the R -dual of any R -coalgebra is an R -algebra, while the R -dual of any finite-type R -algebra is an R -coalgebra. It follows that in order to prove this theorem, we can dualize, then apply [Theorem 2.36](#) and finally dualize again to obtain the desired map. \square

We can also dualize [Corollary 2.38](#), obtaining a result not explicitly stated in [\[13\]](#).

Corollary 2.47. A DASH map $g : A \rightarrow A'$ between finite-type, connected chain algebras, with a fixed choice of chain coalgebra map $\gamma : \mathcal{B}A \rightarrow \mathcal{B}A'$ realizing its strong homotopy structure, naturally induces a chain map

$$\widehat{\gamma} : \mathcal{H}(A) \rightarrow \mathcal{H}(A')$$

such that

$$\begin{array}{ccccccc}
 A & \twoheadrightarrow & \mathcal{H}(A) & \twoheadrightarrow & \mathcal{B}A \\
 g \downarrow & & \widehat{\gamma} \downarrow & & \downarrow \gamma \\
 A' & \twoheadrightarrow & \mathcal{H}(A') & \twoheadrightarrow & \mathcal{B}A'
 \end{array}$$

commutes.

Proof. Recall that $\mathcal{H}(A) = \mathcal{H}(t_{\mathcal{B}})$ ([Example 2.18](#)). Let $\widehat{\gamma} = \mathcal{H}_{sh}(\gamma, \gamma)$. \square

There is also a result dual to [Corollary 2.39](#) that holds. Recall the DASH map $\sigma_A : A \rightarrow \Omega\mathcal{B}A$ from [Example 2.43](#) and the chain map $\text{Id}_{\mathcal{B}A} \otimes \varepsilon_A : \widehat{\mathcal{H}}(\mathcal{B}A) \xrightarrow{\cong} \mathcal{H}(A)$ (2.7).

Corollary 2.48. *Let A be a connected, augmented chain algebra of finite type. There is a quasi-isomorphism $\widehat{\sigma}_A : \mathcal{H}(A) \xrightarrow{\cong} \widehat{\mathcal{H}}(\mathcal{B}A)$, natural in A , such that*

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & \mathcal{H}(A) & \twoheadrightarrow & \mathcal{B}A \\
 \sigma_A \downarrow & & \widehat{\sigma}_A \downarrow & & \text{Id}_{\mathcal{B}A} \downarrow \\
 \Omega \mathcal{B}A & \xrightarrow{\quad} & \widehat{\mathcal{H}}(\mathcal{B}A) & \twoheadrightarrow & \mathcal{B}A
 \end{array}$$

commutes and such that $(\text{Id}_{\mathcal{B}A} \otimes \varepsilon_A) \circ \widehat{\sigma}_A = \text{Id}_{\mathcal{H}(A)}$.

Proof. Let $\widehat{\sigma}_A = \mathcal{H}_{sh}(\text{Id}_{\mathcal{B}A}, \eta_{\mathcal{B}A})$. The naturality of \mathcal{H}_{sh} implies then that

$$(\text{Id}_{\mathcal{B}A} \otimes \varepsilon_A) \circ \widehat{\sigma}_A = \text{Id}_{\mathcal{H}(A)}. \quad \square$$

3. Operations and cooperations on the Hochschild complex

As mentioned in the Introduction, it is well known that the Hochschild complex of a commutative chain algebra is naturally a commutative chain algebra. Indeed, if A is commutative, then $\mathcal{B}A$ admits a commutative multiplication

$$\mathcal{B}A \otimes \mathcal{B}A \xrightarrow{\nabla} \mathcal{B}(A \otimes A) \xrightarrow{\mathcal{B}\mu} \mathcal{B}A,$$

where ∇ is the Eilenberg–Zilber equivalence (2.5), and μ is the multiplication map of A , which is an algebra map since A is commutative. It is easy to check that

$$\mathcal{H}A \otimes \mathcal{H}A \rightarrow \mathcal{H}A : (w \otimes a) \otimes (w' \otimes a') \mapsto w \cdot w' \otimes aa',$$

where \cdot denotes the multiplication on $\mathcal{B}A$ defined above, is a chain map, as well as commutative, associative and unital. Along similar lines, in [13], the authors specified conditions on a coalgebra C under which $\widehat{\mathcal{H}}(C)$ admits a comultiplication, a special case of the following result.

Proposition 3.1. *Let H be a Hopf algebra, and let C be a connected, coaugmented chain coalgebra. Let $\delta : H \rightarrow H \otimes H$ and $\Delta : C \rightarrow C \otimes C$ denote the comultiplications on H and C . If $t : C \rightarrow H$ is a twisting cochain such that (Δ, δ) is a morphism in Tw from t to $t * t$, then $\mathcal{H}(t)$ admits a natural coassociative comultiplication.*

Proposition 3.1 is a consequence of the following, almost obvious result.

Lemma 3.2. *The Hochschild construction functor \mathcal{H} is strongly monoidal, i.e., for all $t, t' \in \text{Ob Tw}$, there are natural isomorphisms*

$$v_{t,t'} : \mathcal{H}(t) \otimes \mathcal{H}(t') \xrightarrow{\cong} \mathcal{H}(t * t')$$

satisfying the usual coherency diagrams [3], where the necessary unit map is the isomorphism $R \rightarrow R \otimes R = \mathcal{H}(0)$.

The proof of Lemma 3.2 consists a simple calculation, based on the natural symmetry of the tensor product of chain complexes. Recall (Remark 2.14) that Tw is a monoidal category with respect to $*$, and that all undecorated tensor products are implicitly taken over R .

Note that (Δ, δ) is a morphism in Tw only if Δ is a morphism of coalgebras, which is true if and only if C is cocommutative.

Proof of Proposition 3.1. The desired comultiplication is given by the composite

$$\mathcal{H}(t) \xrightarrow{\mathcal{H}(\Delta, \delta)} \mathcal{H}(t * t) \xrightarrow{v_{t,t}} \mathcal{H}(t) \otimes \mathcal{H}(t).$$

The coassociativity of Δ and of δ implies that of the composite above, since \mathcal{H} is strongly monoidal. \square

Example 3.3. Let C be a connected, coaugmented chain coalgebra with cocommutative comultiplication Δ . Since Δ is cocommutative, it induces a chain algebra map $\Omega\Delta : \Omega C \rightarrow \Omega(C \otimes C)$ and therefore a coassociative comultiplication $\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$, defined to be the composite

$$\Omega C \xrightarrow{\Omega\Delta} \Omega(C \otimes C) \xrightarrow{q} \Omega C \otimes \Omega C,$$

where q is Milgram’s equivalence (Example 2.12). (In fact, we have simply endowed ΩC with the comultiplication such that all generators are primitive, but we emphasize this construction in terms of q , since we generalize it in section 3.) Moreover, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ t_\Omega \downarrow & & \downarrow t_{\Omega * t_\Omega} \\ \Omega C & \xrightarrow{\psi} & \Omega C \otimes \Omega C \end{array}$$

commutes, i.e., (Δ, ψ) is a morphism of twisting cochains. Applying Proposition 3.1, we obtain a coassociative comultiplication on $\mathcal{H}(t_\Omega)$.

Motivated by these special cases, we provide below conditions on a twisting cochain $t : C \rightarrow A$, in the spirit of those in [13], under which $\mathcal{H}(t)$ admits a (co)multiplicative structure.

3.1. Alexander–Whitney (co)algebras

In this section we define Alexander–Whitney coalgebras and algebras, which are the type of highly structured coalgebras and algebras for which the Hochschild complex

admits a natural comultiplication or a natural multiplication. We recall well-known examples of Alexander–Whitney coalgebras and algebras coming from topology and introduce new classes of examples, including the bar construction on any chain Hopf algebra.

Definition 3.4. (See [14].) A *weak Alexander–Whitney coalgebra* consists of a connected, coaugmented chain coalgebra C such that the comultiplication $\Delta : C \rightarrow C \otimes C$ is a DCSH map, together with a choice of chain algebra map $\omega : \Omega C \rightarrow \Omega(C \otimes C)$ that realizes the DCSH structure of Δ . If the composite

$$\Omega C \xrightarrow{\omega} \Omega(C \otimes C) \xrightarrow{q} \Omega C \otimes \Omega C$$

is a coassociative comultiplication on ΩC , where q denotes the Milgram equivalence (Example 2.12), then (C, ω) is an *Alexander–Whitney coalgebra*. We call $q\omega$ the *associated loop comultiplication*. Note that $(\Omega C, q\omega)$ is a chain Hopf algebra.

An Alexander–Whitney coalgebra (C, ω) is *balanced* if the associated loop comultiplication is cocommutative.

Alexander–Whitney algebras, which we usually denote (A, ν) —and their weak or balanced variants—are defined dually. If (A, ν) is an Alexander–Whitney algebra, then the composite

$$\mathcal{B}A \otimes \mathcal{B}A \xrightarrow{\nabla} \mathcal{B}(A \otimes A) \xrightarrow{\nu} \mathcal{B}A$$

is an associative multiplication on $\mathcal{B}A$, where ∇ denotes the Eilenberg–Zilber map (Example 2.12). We call $\nu\nabla$ the *associated bar multiplication*. Note that $(\mathcal{B}A, \nu\nabla)$ is a chain Hopf algebra.

Remark 3.5. An Alexander–Whitney (co)algebra is a special type of B_∞ -(co)algebra [1,7].

If $\Delta : C \rightarrow C \otimes C$ is a DCSH map and $\omega : \Omega C \rightarrow \Omega(C \otimes C)$ realizes its DCSH structure, then $\text{Id}_C \otimes \Delta$ and $\Delta \otimes \text{Id}_C$ are both DCSH maps as well. In particular, there are chain algebra maps

$$\text{Id}_C \wedge \omega, \omega \wedge \text{Id}_C : \Omega(C \otimes C) \rightarrow \Omega(C \otimes C \otimes C)$$

realizing their DCSH structure. For further details of this construction, we refer the reader to section 1 in [10] and section 2 in [11].

Definition 3.6. A *strict Alexander–Whitney coalgebra* is a weak Alexander–Whitney coalgebra (C, ω) such that

$$(\text{Id}_C \wedge \omega)\omega = (\omega \wedge \text{Id}_C)\omega.$$

A *quasistrict Alexander–Whitney coalgebra* is a weak Alexander–Whitney coalgebra (C, ω) such that there is a derivation homotopy from $(\text{Id}_C \wedge \omega)\omega$ to $(\omega \wedge \text{Id}_C)\omega$.

Strict Alexander–Whitney algebras and *quasistrict Alexander–Whitney algebras* are defined dually.

Remark 3.7. The naturality of the Milgram equivalence q implies that any strict Alexander–Whitney coalgebra is an Alexander–Whitney coalgebra. A similar statement holds for strict Alexander–Whitney algebras, due to the naturality of the Eilenberg–Zilber equivalence ∇ .

Example 3.8. If C is a connected, coaugmented, cocommutative coalgebra with comultiplication Δ , then $(C, \Omega\Delta)$ is a strict, balanced Alexander–Whitney coalgebra. Dually, if A is an augmented, commutative algebra with multiplication m , then $(A, \mathcal{B}m)$ is a strict, balanced Alexander–Whitney algebra.

Example 3.9. (See [14].) For any reduced simplicial set K , there is a natural choice of chain algebra map $\omega_K : \Omega C_*K \rightarrow \Omega(C_*K \otimes C_*K)$ such that (C_*K, ω_K) is an Alexander–Whitney coalgebra.

In general C_*K is not a strict Alexander–Whitney coalgebra. On the other hand, as explained in Example 3.8 of [13], C_*K is always a quasistrict Alexander–Whitney coalgebra.

Along the same lines, Theorem 4.9 in [12] implies that if L is a pointed simplicial set such that C_*L is a cocommutative coalgebra, then there is a natural choice of chain algebra map $\omega_{EL} : \Omega C_*EL \rightarrow \Omega(C_*EL \otimes C_*EL)$ such that (C_*EL, ω_{EL}) is a balanced Alexander–Whitney coalgebra, where E denotes the simplicial suspension functor [20]. More precisely, $\Omega C_*EL \cong T(\overline{C}_*L)$, the tensor algebra generated by coaugmentation coideal of C_*L , endowed with the strictly linear differential induced by the differential on C_*L . Moreover, the comultiplication $\psi_{EL} = q\omega_{EL}$ satisfies

$$\psi_{EL}(x) = x_i \otimes x^i \in T(\overline{C}_*L) \otimes T(\overline{C}_*L),$$

for all $x \in \overline{C}_*L$, where $\Delta(x) = x_i \otimes x^i$, and Δ is the usual comultiplication on C_*L . In particular, if S^2 is the simplicial double suspension functor, then (C_*S^2M, ω_{S^2K}) is a balanced Alexander–Whitney coalgebra for all simplicial sets M , since all positive-degree elements of C_*E^uM are primitive.

Inspired by Example 3.9, we formulate the following definition.

Definition 3.10. A reduced simplicial set K is *symmetric* if $C_*K \otimes \mathbb{F}_2$ is a strictly cocommutative coalgebra, where \mathbb{F}_2 denotes the field of 2 elements.

In these terms, the result from [12] cited above in Example 3.9 says that if K is a symmetric simplicial set, then $C_*EK \otimes \mathbb{F}_2$ is a balanced Alexander–Whitney coalgebra.

Remark 3.11. The class of symmetric simplicial sets includes all simplicial suspensions, both reduced and unreduced, since the natural comultiplication on the normalized chain complex of a simplicial suspension is trivial in positive degrees. Moreover an easy calculation shows that the nerve of $\mathbb{Z}/2\mathbb{Z}$, which is a simplicial model of $\mathbb{R}P^\infty$, is also symmetric (cf. [Example 4.11](#)).

From these examples of symmetric simplicial sets, one can construct many others, including all wedges of truncated real projective spaces of arbitrary dimension. It is clear that all subsimplicial sets and all quotients of symmetric simplicial sets are also symmetric, as is any wedge sum of symmetric simplicial sets. We intend to study symmetric simplicial sets in greater detail in future work.

The next theorem provides us with another important class of Alexander–Whitney (co)algebras.

Theorem 3.12. *If H is a chain Hopf algebra, then $\mathcal{B}H$ is naturally an Alexander–Whitney coalgebra. If H is connected and of finite type, then ΩH is naturally an Alexander–Whitney algebra. Moreover, the associated loop comultiplication on $\Omega\mathcal{B}H$ and associated bar multiplication on $\mathcal{B}\Omega H$ are such that both of the natural quasi-isomorphisms $\varepsilon_H : \Omega\mathcal{B}H \rightarrow H$ and $\eta_H : H \rightarrow \mathcal{B}\Omega H$ are maps of Hopf algebras.*

We refer the reader to [Appendix A](#) for the proof of this theorem.

When we define power maps on Hochschild complexes in section 4, we can actually relax slightly the conditions on the (co)algebras we consider and study the Hirsch (co)algebras of Kadeishvili [\[18\]](#).

Definition 3.13. A *Hirsch coalgebra* consists of a connected, coaugmented chain coalgebra C , together with a coassociative comultiplication $\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$, called its *loop comultiplication*, that is a morphism of chain algebras. A Hirsch coalgebra (C, ψ) is *balanced* if ψ is cocommutative.

Hirsch algebras are defined dually.

Notation 3.14. Let Hirsch_R denote the category of which the objects are Hirsch coalgebras (C, ψ) and where

$$\text{Hirsch}_R((C, \psi), (C', \psi')) = \{f \in \text{Coalg}_R(C, C') \mid (\Omega f \otimes \Omega f)\psi = \psi'\Omega f\}.$$

Remark 3.15. If (C, ω) is an Alexander–Whitney coalgebra, then $(C, q\omega)$ is a Hirsch coalgebra.

Remark 3.16. As explained in [\[18\]](#), the homology of a Hirsch algebra is naturally a *Gerstenhaber algebra*, i.e., a commutative graded algebra endowed with a Lie bracket of degree -1 that is a biderivation with respect to the multiplication. In particular, the homology of any Alexander–Whitney algebra is a Gerstenhaber algebra.

Example 3.17. (See [18].) The Hochschild cochain complex of a chain algebra A with coefficients in A , usually denoted $C^*(A, A)$, is a Hirsch algebra.

Remark 3.18. In [2] Baues provided combinatorial formulas for a natural Hirsch coalgebra structure on C_*K , for all reduced simplicial sets K . The Alexander–Whitney structure defined in [14] lifts Baues’ Hirsch structure.

Example 3.19. Not all Hirsch coalgebras are induced from Alexander–Whitney coalgebras, as in Remark 3.15. Let C denote the free graded abelian group with five generators: u in degree 0 (which plays the role of 1), x in degree 3, y and y' in degree 4 and z in degree 7. Endow C with a differential d specified by

$$dy = 2x \quad \text{and} \quad dy' = 3x,$$

while u , x , and z are cycles for degree reasons. Define a comultiplication Δ on C by setting x , y and y' to be primitive, while

$$\Delta(z) = u \otimes z + 3x \otimes y - 2x \otimes y' + z \otimes u.$$

It is easy to check that Δ is a chain map. Moreover, Δ is cocommutative up to chain homotopy, where the chain homotopy F is given by $F(x) = F(y) = F(y') = 0$ and

$$F(z) = y' \otimes y - y \otimes y'.$$

There is a cocommutative, coassociative comultiplication $\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$, defined to be primitive on all generators, except $s^{-1}z$, where

$$\psi(s^{-1}z) = s^{-1}z \otimes 1 + s^{-1}y' \otimes s^{-1}y - s^{-1}y \otimes s^{-1}y' + 1 \otimes s^{-1}z.$$

Thus, (C, ψ) is a balanced Hirsch coalgebra. Easy computations show that there is no algebra map $\omega : \Omega C \rightarrow \Omega(C \otimes C)$ such that $q\omega = \psi$. It follows that (C, ψ) is not realizable as the Hirsch coalgebra of a simplicial set.

Remark 3.20. Any Hirsch coalgebra (C, ψ) is weakly equivalent to an Alexander–Whitney coalgebra, since $\mathcal{B}(\Omega C, \psi)$ is an Alexander–Whitney coalgebra, by Theorem 3.12, and $\eta_C : C \rightarrow \mathcal{B}\Omega C$ is a quasi-isomorphism of chain coalgebras.

When we construct power maps on the Hochschild complex of a twisting cochain later in this paper, we are led to consider variants of the category Tw (cf. Notation 2.7) that involve Hirsch coalgebra structure.

Notation 3.21. Let $\mathbf{Tw}_{\text{Hirsch}}$ denote the category with

- $\text{Ob } \mathbf{Tw}_{\text{Hirsch}} = \{((C, \psi), C \xrightarrow{t} A) \mid (C, \psi) \in \text{Ob Hirsch}_R, t \in \text{Ob Tw}\}$, and
- if $((C, \psi), t)$ and $((C', \psi'), t')$ are objects in $\mathbf{Tw}_{\text{Hirsch}}$, then

$$\mathbf{Tw}_{\text{Hirsch}}\left(\left((C, \psi), t\right), \left((C', \psi'), t'\right)\right) = \{(f, g) \in \text{Tw}(t, t') \mid (\Omega f \otimes \Omega f)\psi = \psi' \Omega f\}.$$

We sometimes need an even more highly structured category. Recall [Notation 2.9](#).

Notation 3.22. Let \mathbf{Tw}_{HH} denote the category the objects of which are triples

$$\left((C, \psi), C \xrightarrow{t} H, (H, \delta)\right),$$

where $(C, \psi) \in \text{Ob Hirsch}_R$, $t \in \text{Ob Tw}$, and $(H, \delta) \in \text{Ob Hopf}_R$. If $((C, \psi), t, (H, \delta))$ and $((C', \psi'), t', (H', \delta'))$ are objects in \mathbf{Tw}_{HH} , then

$$\begin{aligned} \mathbf{Tw}_{\text{HH}}\left(\left((C, \psi), t, (H, \delta)\right), \left((C', \psi'), t', (H', \delta')\right)\right) \\ = \mathbf{Tw}_{\text{Hirsch}}\left(\left((C, \psi), t\right), \left((C', \psi'), t'\right)\right) \cap \mathbf{Tw}_{\text{Hopf}}\left(\left(t, (H, \delta)\right), \left(t', (H', \delta')\right)\right) \end{aligned}$$

3.2. Existence of (co)multiplication on the Hochschild complex

We are now ready to generalize [Proposition 3.1](#), as well as Theorem 3.9 in [\[13\]](#), which says that the coHochschild complex of an Alexander–Whitney coalgebra admits a natural comultiplication.

Theorem 3.23. *Let (C, ω) be an Alexander–Whitney coalgebra with underlying comultiplication $\Delta : C \rightarrow C \otimes C$. Let H be a chain Hopf algebra, with comultiplication $\delta : H \rightarrow H \otimes H$. Let $t : C \rightarrow H$ be a twisting cochain.*

*If (ω, δ) is a morphism in \mathbf{Tw}^{sh} from t to $t * t$, then the Hochschild complex of t admits a comultiplication $\widehat{\delta} : \mathcal{H}(t) \rightarrow \mathcal{H}(t) \otimes \mathcal{H}(t)$ such that*

$$\begin{array}{ccccc} H & \xrightarrow{\quad} & \mathcal{H}(t) & \xrightarrow{\quad} & C \\ \delta \downarrow & & \widehat{\delta} \downarrow & & \Delta \downarrow \\ H \otimes H & \xrightarrow{\quad} & \mathcal{H}(t) \otimes \mathcal{H}(t) & \xrightarrow{\quad} & C \otimes C \end{array}$$

commutes. Moreover, $\widehat{\delta}$ is coassociative (respectively, coassociative up to chain homotopy) if (C, ω) is a strict (respectively, quasistrict) Alexander–Whitney coalgebra.

Note that, according to [Remark 2.13](#), asking for (ω, δ) to be a morphism in \mathbf{Tw}^{sh} from t to $t * t$ is equivalent to requiring the diagram

$$\begin{array}{ccc}
 \Omega C & \xrightarrow{\omega} & \Omega(C \otimes C) \\
 \alpha_t \downarrow & & \downarrow q \\
 & & \Omega C \otimes \Omega C \\
 & & \downarrow \alpha_t \otimes \alpha_t \\
 H & \xrightarrow{\delta} & H \otimes H
 \end{array}$$

to commute, i.e., to requiring α_t to be a map of chain coalgebras with respect to the associated loop comultiplication

$$\Omega C \xrightarrow{\omega} \Omega(C \otimes C) \xrightarrow{q} \Omega C \otimes \Omega C.$$

Proof. Apply [Theorem 2.36](#) to $(\omega, \delta) : t \rightarrow t * t$, obtaining a chain map

$$\mathcal{H}^{sh}(\omega, \delta) : \mathcal{H}(t) \rightarrow \mathcal{H}(t * t).$$

The desired comultiplication $\widehat{\delta}$ is then given by the composite

$$\mathcal{H}(t) \xrightarrow{\mathcal{H}^{sh}(\omega, \delta)} \mathcal{H}(t * t) \xrightarrow{v_{t,t}} \mathcal{H}(t) \otimes \mathcal{H}(t)$$

(cf. [Lemma 3.2](#)). Together with the formulas in the proof of [Theorem 2.36](#), naturality of v implies that $\widehat{\delta}$ is coassociative (respectively, coassociative up to chain homotopy) if and only if

$$\begin{array}{ccc}
 \mathcal{H}(t) & \xrightarrow{\mathcal{H}^{sh}(\omega, \delta)} & \mathcal{H}(t * t) \\
 \mathcal{H}^{sh}(\omega, \delta) \downarrow & & \downarrow \mathcal{H}^{sh}(\omega \wedge \text{Id}_C, \delta \otimes \text{Id}_C) \\
 \mathcal{H}(t * t) & \xrightarrow{\mathcal{H}^{sh}(\text{Id}_C \wedge \omega, \text{Id}_C \otimes \delta)} & \mathcal{H}(t * t * t)
 \end{array}$$

commutes (respectively, commutes up to chain homotopy), which is true if

$$\begin{array}{ccc}
 \mathcal{D}(t) & \xrightarrow{\mathcal{D}(\omega, \delta)} & \mathcal{D}(t * t) \\
 \mathcal{D}(\omega, \delta) \downarrow & & \downarrow \mathcal{D}(\omega \wedge \text{Id}_C, \delta \otimes \text{Id}_C) \\
 \mathcal{D}(t * t) & \xrightarrow{\mathcal{D}(\text{Id}_C \wedge \omega, \text{Id}_C \otimes \delta)} & \mathcal{D}(t * t * t)
 \end{array}$$

commutes (respectively, commutes up to chain homotopy), where $\mathcal{D}(-)$ denotes the twisted double extension of [Definition 2.24](#). Finally, the naturality of the derivation $\sigma_{(-)}$ ([Scholium 2.37](#)) implies that the last diagram commutes (respectively, commutes up to chain homotopy) if $(\text{Id}_C \wedge \omega)\omega = (\omega \wedge \text{Id}_C)\omega$ (respectively, if there is a derivation homotopy from $(\text{Id}_C \wedge \omega)\omega$ to $(\omega \wedge \text{Id}_C)\omega$). \square

Example 3.24. Let C be a quasistrict Alexander–Whitney coalgebra. Applying [Theorem 3.23](#) to the universal twisting cochain $t_\Omega : C \rightarrow \Omega C$, we obtain a homotopy coassociative comultiplication on $\widehat{\mathcal{H}}(C) = \mathcal{H}(t_\Omega)$. [Theorem 3.9](#) in [\[13\]](#) is therefore a special case of our [Theorem 3.23](#).

Example 3.25. Since the bar construction on a chain Hopf algebra is an Alexander–Whitney coalgebra, the Hochschild complex of H , which is equal to $\mathcal{H}(t_{\mathcal{B}})$, admits a comultiplication. We conjecture that if H is cocommutative, then $\mathcal{B}H$ is quasistrict and therefore the comultiplication on $\mathcal{H}(t_{\mathcal{B}})$ is coassociative up to chain homotopy.

Dualizing both the statement and the proof of the theorem above, we obtain the following result.

Theorem 3.26. *Let (A, ν) be an Alexander–Whitney algebra of finite type, with underlying multiplication $m : A \otimes A \rightarrow A$. Let H be a chain Hopf algebra of finite type, with multiplication $\mu : H \otimes H \rightarrow H$. Let $t : H \rightarrow A$ be a twisting cochain.*

*If (μ, ν) is a morphism in Tw_{sh} from $t * t$ to t , then the Hochschild complex of t admits a multiplication $\widehat{\mu} : \mathcal{H}(t) \otimes \mathcal{H}(t) \rightarrow \mathcal{H}(t)$ such that*

$$\begin{array}{ccccc}
 A \otimes A & \twoheadrightarrow & \mathcal{H}(t) \otimes \mathcal{H}(t) & \twoheadrightarrow & H \otimes H \\
 m \downarrow & & \widehat{\mu} \downarrow & & \mu \downarrow \\
 A & \twoheadrightarrow & \mathcal{H}(t) & \twoheadrightarrow & H
 \end{array}$$

commutes. Moreover, $\widehat{\mu}$ is associative (respectively, associative up to chain homotopy) if (A, ν) is a strict (respectively, quasistrict) Alexander–Whitney algebra.

Note that, according to [Remark 2.13](#), asking for (μ, ν) to be a morphism in Tw_{sh} from $t * t$ to t is equivalent to requiring the diagram

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\mu} & H \\
 \beta_t \otimes \beta_t \downarrow & & \downarrow \beta_t \\
 \mathcal{B}A \otimes \mathcal{B}A & & \mathcal{B}A \\
 \nabla \downarrow & & \downarrow \nu \\
 \mathcal{B}(A \otimes A) & \xrightarrow{\nu} & \mathcal{B}A
 \end{array}$$

to commute, i.e., to requiring β_t to be a map of chain algebras with respect to the multiplication

$$\mathcal{B}A \otimes \mathcal{B}A \xrightarrow{\nabla} \mathcal{B}(A \otimes A) \xrightarrow{\nu} \mathcal{B}A.$$

Example 3.27. Since the cobar construction on a finite-type, connected chain Hopf algebra is an Alexander–Whitney algebra, the coHochschild complex of H , which is equal to $\mathcal{H}(t_\Omega)$, admits a multiplication.

4. Power maps on the Hochschild complex of a twisting cochain

Let $t : C \rightarrow H$ be a twisting cochain, where H is a chain Hopf algebra. The goal of this section is to prove the existence, under certain cocommutativity conditions, of an r th-power map $\tilde{\lambda}_r$ on the Hochschild complex of t , extending the usual r th-power map λ_r on H (cf. equation (1.1)). In [15] we show that if C is the chain complex on a simplicial double suspension K , and $H = \Omega C$, then the algebraic r th-power map $\tilde{\lambda}_r$ is topologically meaningful, in the sense that it models the topological r th-power map on $\mathcal{L}|K|$.

4.1. The existence theorem for power maps

Theorem 4.1. *Let C be a Hirsch coalgebra, with loop comultiplication $\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$. Let H be a chain Hopf algebra, with comultiplication $\delta : H \rightarrow H \otimes H$. Let $t : C \rightarrow H$ be a twisting cochain.*

If

- (1) *the induced chain algebra map $\alpha_t : \Omega C \rightarrow H$ is also a map of coalgebras, and*
- (2) *$\tau\delta t = \delta t$, where $\tau : H \otimes H \xrightarrow{\cong} H \otimes H$ is the symmetry isomorphism,*

then for any positive integer r , there is an endomorphism of chain complexes

$$\tilde{\lambda}_r : \mathcal{H}(t) \rightarrow \mathcal{H}(t),$$

natural with respect to morphisms in Tw_{HH} (cf. Notation 3.22), such that

$$\begin{array}{ccccc} H & \twoheadrightarrow & \mathcal{H}(t) & \twoheadrightarrow & C \\ \lambda_r \downarrow & & \tilde{\lambda}_r \downarrow & & \parallel \\ H & \twoheadrightarrow & \mathcal{H}(t) & \twoheadrightarrow & C \end{array}$$

commutes, where λ_r denotes the r th-power map on H . In particular, if $s^{-1}c$ is a primitive of $(\Omega C, \psi)$ for all $c \in C$, then $\tilde{\lambda}_r = \text{Id}_C \otimes \lambda_r$.

There are two special cases of Theorem 4.1 that are particularly worthy of note.

Corollary 4.2. *If (C, ψ) is a balanced Hirsch coalgebra, then the coHochschild complex $\widehat{\mathcal{H}}(C)$ of C admits an r th-power map $\tilde{\lambda}_r$, for all positive integers r , that is natural with respect to morphisms in Hirsch_R (cf. Notation 3.14). In particular,*

$$\begin{array}{ccccc}
 \Omega C & \twoheadrightarrow & \widehat{\mathcal{H}}(C) & \twoheadrightarrow & C \\
 \lambda_r \downarrow & & \tilde{\lambda}_r \downarrow & & \parallel \\
 \Omega C & \twoheadrightarrow & \widehat{\mathcal{H}}(C) & \twoheadrightarrow & C
 \end{array}$$

commutes, where λ_r denotes the r th-power map on ΩC .

Proof. Apply [Theorem 4.1](#) to the twisting cochain $t_\Omega : C \rightarrow \Omega C$. Hypotheses (1) and (2) are satisfied because $\alpha_{t_\Omega} = \text{Id}_{\Omega C}$ and because (C, ψ) is balanced.

With respect to the naturality of $\tilde{\lambda}_r$, note that if $f \in \text{Hirsch}((C, \psi), (C', \psi'))$, then $(f, \Omega f) \in \text{Tw}_{\text{HH}}(t_\Omega, t_\Omega)$. \square

Corollary 4.3. *If H is a cocommutative chain Hopf algebra, then the Hochschild complex $\mathcal{H}(H)$ of H admits an r th-power map $\tilde{\lambda}_r$, for all positive integers r , that is natural with respect to chain Hopf algebra maps. In particular,*

$$\begin{array}{ccccc}
 H & \twoheadrightarrow & \mathcal{H}(H) & \twoheadrightarrow & \mathcal{B}H \\
 \lambda_r \downarrow & & \tilde{\lambda}_r \downarrow & & \parallel \\
 H & \twoheadrightarrow & \mathcal{H}(H) & \twoheadrightarrow & \mathcal{B}H
 \end{array}$$

commutes, where λ_r denotes the r th-power map on H .

Proof. Apply [Theorem 4.1](#) to the twisting cochain $t_{\mathcal{B}} : \mathcal{B}H \rightarrow H$. Since $\varepsilon_H : \Omega \mathcal{B}H \rightarrow H$ is a map of coalgebras ([Theorem 3.12](#)), hypothesis (1) holds, while hypothesis (2) follows from the cocommutativity of H .

With respect to the naturality of $\tilde{\lambda}_r$, note that [Theorem 3.12](#) implies that the Alexander–Whitney coalgebra structure on $\mathcal{B}H$ is natural in H . Any chain Hopf algebra map $g : H \rightarrow H'$ therefore induces a morphism $(\mathcal{B}g, g) : t_{\mathcal{B}} \rightarrow t_{\mathcal{B}}$ in Tw_{HH} . \square

The naturality of the power map enables us to compare the constructions of the two corollaries above, via a twisting cochain.

Corollary 4.4. *Let (C, ψ) be a balanced Hirsch coalgebra, and let H be a cocommutative chain Hopf algebra with comultiplication δ . Let ψ_H denote the natural comultiplication on $\Omega \mathcal{B}H$ with respect to which $\varepsilon_H : (\Omega \mathcal{B}H, \psi_H) \rightarrow (H, \delta)$ is a morphism of chain Hopf algebras (cf. [Theorem 3.12](#)).*

If $t : C \rightarrow H$ is a twisting cochain such that $\Omega \beta_t : (\Omega C, \psi) \rightarrow (\Omega \mathcal{B}H, \psi_H)$ is a morphism of chain Hopf algebras, then

$$\begin{array}{ccc}
 \widehat{\mathcal{H}}(C) & \xrightarrow{\mathcal{H}(\beta_t, \alpha_t)} & \mathcal{H}(H) \\
 \tilde{\lambda}_r \downarrow & & \tilde{\lambda}_r \downarrow \\
 \widehat{\mathcal{H}}(C) & \xrightarrow{\mathcal{H}(\beta_t, \alpha_t)} & \mathcal{H}(H)
 \end{array}$$

commutes.

Proof. Observe that

$$\begin{array}{ccc}
 C & \xrightarrow{\beta_t} & \mathcal{B}H \\
 t_\Omega \downarrow & & \downarrow t_{\mathcal{B}} \\
 \Omega C & \xrightarrow{\alpha_t} & H
 \end{array}$$

always commutes, i.e., that $(\beta_t, \alpha_t) : t_\Omega \rightarrow t_{\mathcal{B}}$ is always a morphism in \mathbf{Tw} . Since $\Omega\beta_t : (\Omega C, \psi) \rightarrow (\Omega\mathcal{B}H, \psi_H)$ is a morphism of chain Hopf algebras by hypothesis, and $\alpha_t = \varepsilon_H \circ \Omega\beta_t$ (Remark 2.2), $\alpha_t : (\Omega C, \psi) \rightarrow (H, \delta)$ is also a morphism of chain Hopf algebras, whence (β_t, α_t) is actually a morphism in $\mathbf{Tw}_{\mathbf{HH}}$. \square

Example 4.5. Recall from Example 3.9 that if L is a pointed simplicial set such that C_*L is cocommutative, e.g., if L is a simplicial suspension (reduced or unreduced), then C_*EL admits a natural, balanced Alexander–Whitney coalgebra structure. Corollary 4.2 therefore implies that if C_*L is cocommutative, then $\widehat{\mathcal{H}}(C_*EL)$ admits an r th-power map, for all positive integers r . If L is itself a simplicial suspension, then $s^{-1}c$ is a primitive of $(\Omega C_*EL, \psi_{EL})$ for all $c \in C_*EL$, and thus $\tilde{\lambda}_r = \text{Id}_{C_*EL} \otimes \lambda_r$.

Moreover, if C_*L is cocommutative, then C_*GEL is a cocommutative chain Hopf algebra, as easily follows from an examination of the formulas for the simplicial suspension functor E and for the Kan loop group functor G (cf., e.g., sections 2.1 (a) and (b) in [12]), which imply the existence of a simplicial map

$$L \rightarrow GEL : x \mapsto \overline{(1, x)}.$$

It therefore follows from Corollary 4.3 that if C_*L is cocommutative, then $\mathcal{H}(C_*GEL)$ also admits an r th-power map, for all positive integers r .

Let $t_{EL} : C_*EL \rightarrow C_*GEL$ denote the Szczarba twisting cochain for EL (Example 2.5), with associated chain coalgebra map $\beta_{EL} : C_*EL \rightarrow \mathcal{B}C_*GEL$ and chain algebra map $\alpha_{EL} : \Omega C_*EL \rightarrow C_*GEL$, which together induce a chain map

$$\mathcal{H}(\beta_{EL}, \alpha_{EL}) : \widehat{\mathcal{H}}(C_*EL) \rightarrow \mathcal{H}(C_*GEL).$$

It is natural to wonder under what conditions this map commutes with the r th-power maps. Recall from Example 2.5 that α_{EL} , and thus β_{EL} and $\mathcal{H}(\beta_{EL}, \alpha_{EL})$, are quasi-isomorphisms if L is actually reduced, since EL is then 1-reduced.

It follows from the proof of Theorem 4.11 in [12] that for any pointed simplicial set L ,

$$\alpha_{EL} : (\Omega C_*EL, \psi_{EL}) \rightarrow (C_*GEL, \Delta)$$

is a chain Hopf algebra map, where Δ is the usual comultiplication on C_*GEL . On the other hand, since C_*EL is a trivial coalgebra,

$$\beta_{EL}(\overline{(1, x)}) = s(t_{EL}\overline{(1, x)}),$$

which implies that

$$\Omega\beta_{EL}(s^{-1}\overline{(1, x)}) = s^{-1}\left(s(t_{EL}\overline{(1, x)})\right).$$

The formulas in the proof of Theorem A.11 for the DCSH structure of the Alexander–Whitney map $f : \mathcal{B}(H \otimes H) \rightarrow \mathcal{B}H \otimes \mathcal{B}H$ imply that

$$\psi_H(s^{-1}(sa)) = s^{-1}(sa) \otimes 1 + 1 \otimes s^{-1}(sa)$$

for all $a \in H$ and for all connected chain Hopf algebras H . In particular, therefore,

$$\psi_{C_*GEL} \circ \Omega\beta_{EL}(s^{-1}\overline{(1, x)}) = s^{-1}\left(s(t_{EL}\overline{(1, x)})\right) \otimes 1 + 1 \otimes s^{-1}\left(s(t_{EL}\overline{(1, x)})\right)$$

for all $x \in C_*L$.

On the other hand, for all $x \in C_*L$

$$\begin{aligned} (\Omega\beta_{EL} \otimes \Omega\beta_{EL}) \circ \psi_{EL}(s^{-1}\overline{(1, x)}) &= \Omega\beta_{EL}(s^{-1}\overline{(1, x_i)}) \otimes \Omega\beta_{EL}(s^{-1}\overline{(1, x^i)}) \\ &= s^{-1}\left(s(t_{EL}\overline{(1, x_i)})\right) \otimes s^{-1}\left(s(t_{EL}\overline{(1, x^i)})\right), \end{aligned}$$

where $\Delta(x) = x_i \otimes x^i$. Since $t_{EL}\overline{(1, y)} \neq 0$ for all $y \in C_*L \setminus \{0\}$ (cf. the explicit formula for t_{EL} given in [12] just before Theorem 4.11), we conclude that $\Omega\beta_{EL}$ is a chain Hopf algebra map if and only if C_*L is a trivial coalgebra. In particular, if L itself is a simplicial suspension (reduced or unreduced), then $\Omega\beta_{EL}$ is a chain Hopf algebra map, and Corollary 4.4 therefore implies that

$$\begin{array}{ccc} \widehat{\mathcal{H}}(C_*EL) & \xrightarrow{\mathcal{H}(\beta_{EL}, \alpha_{EL})} & \mathcal{H}(C_*GEL) \\ \tilde{\lambda}_r \downarrow & & \tilde{\lambda}_r \downarrow \\ \widehat{\mathcal{H}}(C_*EL) & \xrightarrow{\mathcal{H}(\beta_{EL}, \alpha_{EL})} & \mathcal{H}(C_*GEL) \end{array}$$

commutes.

4.2. Proof of the existence of power maps via “loop concatenation”

The key to the proof of [Theorem 4.1](#) is the following result, which is analogous to the existence of topological loop concatenation. In the statement below, for a Hopf algebra H with multiplication μ and comultiplication δ , we let $\mu^{(r)} : H^{\otimes r} \rightarrow H$ and $\delta^{(r)} : H \rightarrow H^{\otimes r}$ denote the iterated multiplication and comultiplication maps.

Theorem 4.6. *Let C be a Hirsch coalgebra, with loop comultiplication $\psi : \Omega C \rightarrow \Omega C \otimes \Omega C$. Let H be a chain Hopf algebra, with multiplication μ and comultiplication $\delta : H \rightarrow H \otimes H$. Let $t : C \rightarrow H$ be a twisting cochain.*

If

- (1) *the induced chain algebra map $\alpha_t : \Omega C \rightarrow H$ is also a map of coalgebras, and*
- (2) *$\tau\delta t = \delta t$, where $\tau : H \otimes H \xrightarrow{\cong} H \otimes H$ is the symmetry isomorphism,*

then there is a chain map $\tilde{\mu}_r : \mathcal{H}(\delta^{(r)}t) \rightarrow \mathcal{H}(t)$, natural with respect to morphisms in $\mathbf{Tw}_{\text{Hirsch}}$ (cf. [Notation 3.21](#)), such that

$$\begin{array}{ccccc}
 H^{\otimes r} & \twoheadrightarrow & \mathcal{H}(\delta^{(r)}t) & \twoheadrightarrow & C \\
 \mu^{(r)} \downarrow & & \tilde{\mu}_r \downarrow & & \parallel \\
 H & \twoheadrightarrow & \mathcal{H}(t) & \twoheadrightarrow & C
 \end{array}$$

commutes. In particular, if $s^{-1}c$ is a primitive of $(\Omega C, \psi)$ for all $c \in C$, then $\tilde{\mu}_r = \text{Id}_C \otimes \mu^{(r)}$.

Remark 4.7. Existence of the map $\tilde{\mu}_r$ does not follow immediately from the naturality—even extended—of the Hochschild construction, since $\mu^{(r)}$ is in general not a map of algebras.

Proof. We define $\tilde{\mu}_r : \mathcal{H}(\delta^{(r)}t) \rightarrow \mathcal{H}(t)$ by

$$\tilde{\mu}_r(1 \otimes (w_1 \otimes \cdots \otimes w_r)) = 1 \otimes w_1 \cdots w_r$$

and on elements of the form $c \otimes (w_1 \otimes \cdots \otimes w_r)$ for $c \in C_{>0}$ to be the composite

$$\begin{array}{ccc}
 C \otimes H^{\otimes r} & \xrightarrow{t_\Omega \otimes H^{\otimes r}} & \Omega C \otimes H^{\otimes r} \xrightarrow{\psi^{(r)} \otimes H^{\otimes r}} (\Omega C)^{\otimes r} \otimes H^{\otimes r} \xrightarrow{\text{sh}_r} (\Omega C \otimes H)^{\otimes r} \\
 & & \downarrow \sigma_t \otimes H \otimes (\alpha_t \otimes H)^{\otimes r-1} \\
 & & H \otimes C \otimes H \otimes H^{\otimes 2r-1} \\
 & & \downarrow \xi_r \\
 & & C \otimes H^{\otimes 2r+1} \\
 & & \downarrow C \otimes \mu^{(2r+1)} \\
 & & C \otimes H,
 \end{array}$$

where $\sigma_t : (\Omega C)_{>0} \rightarrow H \otimes C \otimes H$ is the derivation from [Definition 2.27](#), while

$$\text{sh}_r(u_1 \otimes \cdots \otimes u_r \otimes w_1 \otimes \cdots \otimes w_r) = (-1)^{\epsilon_{\text{sh}}} u_1 \otimes w_1 \otimes \cdots \otimes u_r \otimes w_r,$$

with $\epsilon_{\text{sh}} = \sum_{i=1}^r |w_i|(|u_{i+1}| + \cdots + |u_r|)$, and ξ_r is the cyclic permutation, i.e.,

$$\xi_r(w_0 \otimes c \otimes w_1 \otimes \cdots \otimes w_{2r}) = (-1)^{\epsilon_\xi} c \otimes w_1 \otimes \cdots \otimes w_{2r} \otimes w_0,$$

with $\epsilon_\xi = |w_0|(|c| + \sum_{i=1}^{2r} |w_i|)$. Note that naturality of $\tilde{\mu}_r$ with respect to morphisms in $\text{Tw}_{\text{Hirsch}}$ follows immediately from its definition.

In the composite above the differentials on $\Omega C \otimes H^{\otimes r}$, $(\Omega C)^{\otimes r} \otimes H^{\otimes r}$ and $(\Omega C)^{\otimes r} \otimes H^{\otimes r}$ are the usual, unperturbed differentials on tensor products of chain complexes. The differential on $H \otimes C \otimes H \otimes H^{\otimes 2r-1}$ is that of $\mathcal{D}(t) \otimes H^{\otimes 2r-1}$, while the differential on $C \otimes H^{\otimes 2r+1}$ is that of $\mathcal{H}_t(C, H^{\otimes 2r+1})$, where we consider $H^{\otimes 2r+1}$ as an H -bimodule simply via multiplication by H on the first and last factors. Finally, the differentials on $C \otimes H^{\otimes r}$ and $C \otimes H$ are, of course, those of $\mathcal{H}(\delta^{(r)}t)$ and $\mathcal{H}(t)$, respectively.

To see that $\tilde{\mu}_r$ is a chain map, observe first that $\psi^{(r)} \otimes H^{\otimes r}$, sh_r , ξ_r and $C \otimes \mu^{(2r+1)}$ are all clearly chain maps, while $\sigma_t \otimes H \otimes (\alpha_t \otimes H)^{\otimes r-1}$ commutes with the differentials up to a sign ([Lemma 2.28](#)). The crucial factor in the composite to consider is therefore $t_\Omega \otimes H^{\otimes r}$.

Observe that

$$\begin{aligned}
 & (t_\Omega \otimes H^{\otimes r})d_{\delta^{(r)}t} \\
 &= (t_\Omega \otimes H^{\otimes r})(d_C \otimes H^{\otimes r} + C \otimes d_{H^{\otimes r}} - (C \otimes \mu_{H^{\otimes r}})(C \otimes \delta^{(r)}t \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \\
 &\quad - (C \otimes \mu_{H^{\otimes r}})\xi_r'(\delta^{(r)}t \otimes C \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r})) \\
 &= t_\Omega d_C \otimes H^{\otimes r} + t_\Omega \otimes d_{H^{\otimes r}} - (C \otimes \mu_{H^{\otimes r}})(t_\Omega \otimes \delta^{(r)}t \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \\
 &\quad - (C \otimes \mu_{H^{\otimes r}})\xi_r'(\delta^{(r)}t \otimes t_\Omega \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}),
 \end{aligned}$$

where

$$\xi'(w_1 \otimes \cdots \otimes w_r \otimes c \otimes w'_1 \otimes \cdots \otimes w'_r) = (-1)^{\epsilon_{\xi'}} c \otimes w'_1 \otimes \cdots \otimes w'_r \otimes w_1 \otimes \cdots \otimes w_r,$$

with

$$\epsilon_{\xi'} = \left(\sum_{i=1}^r |w_i| \right) (|c| + \sum_{i=1}^r |w'_i|),$$

while

$$(d_\Omega \otimes H^{\otimes r} + \Omega C \otimes d_{H^{\otimes r}})(t_\Omega \otimes H^{\otimes r}) = d_\Omega t_\Omega \otimes H^{\otimes r} - t_\Omega \otimes d_{H^{\otimes r}}.$$

It follows that if Γ denotes the composite of the last five factors in $\tilde{\mu}_r$, then

$$\begin{aligned} \tilde{\mu}_r d_{\delta^{(r)}t} &= \Gamma(t_\Omega \otimes H^{\otimes r}) d_{\delta^{(r)}t} \\ &= \Gamma(t_\Omega d_C \otimes H^{\otimes r} + t_\Omega \otimes d_{H^{\otimes r}} - (C \otimes \mu_{H^{\otimes r}})(t_\Omega \otimes \delta^{(r)}t \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \\ &\quad - (C \otimes \mu_{H^{\otimes r}})\xi'_r(\delta^{(r)}t \otimes t_\Omega \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r})), \end{aligned}$$

and

$$\begin{aligned} d_t \tilde{\mu}_r &= -\Gamma(d_\Omega \otimes H^{\otimes r} + \Omega C \otimes d_{H^{\otimes r}})(t_\Omega \otimes H^{\otimes r}) \\ &= \Gamma(-d_\Omega t_\Omega \otimes H^{\otimes r} + t_\Omega \otimes d_{H^{\otimes r}}). \end{aligned}$$

Thus, since t_Ω is a twisting cochain,

$$\begin{aligned} \tilde{\mu}_r d_{\delta^{(r)}t} - d_t \tilde{\mu}_r &= \Gamma\left((\mu_{\Omega C}(t_\Omega \otimes t_\Omega)\Delta) \otimes H^{\otimes r} - (C \otimes \mu_{H^{\otimes r}})(t_\Omega \otimes \delta^{(r)}t \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \right. \\ &\quad \left. - (C \otimes \mu_{H^{\otimes r}})\xi'_r(\delta^{(r)}t \otimes t_\Omega \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \right). \end{aligned}$$

It remains therefore to show that

$$\begin{aligned} \Gamma\left((\mu_{\Omega C}(t_\Omega \otimes t_\Omega)\Delta) \otimes H^{\otimes r} \right) &= \Gamma\left((C \otimes \mu_{H^{\otimes r}})(t_\Omega \otimes \delta^{(r)}t \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \right. \\ &\quad \left. + (C \otimes \mu_{H^{\otimes r}})\xi'_r(\delta^{(r)}t \otimes t_\Omega \otimes H^{\otimes r})(\Delta \otimes H^{\otimes r}) \right), \end{aligned}$$

for which it suffices to establish that

$$\begin{aligned} \Gamma(\mu_{\Omega C} \otimes H^{\otimes r}) &= \Gamma\left((C \otimes \mu_{H^{\otimes r}})(\Omega C \otimes \alpha_t^{\otimes r} \psi^{(r)} \otimes H^{\otimes r}) \right. \\ &\quad \left. + (C \otimes \mu_{H^{\otimes r}})\xi'_r(\alpha_t^{\otimes r} \psi^{(r)} \otimes \Omega C \otimes H^{\otimes r}) \right), \end{aligned} \tag{4.1}$$

as morphisms from $(\Omega C)^{\otimes 2} \otimes H^{\otimes r}$ to $C \otimes H$, since $\delta^{(r)}t = \alpha_t^{\otimes r} \psi^{(r)} t_\Omega$, by hypothesis.

As we show below, equation (4.1) is a consequence of the fact that σ_t is a derivation and of the hypothesis that $\tau\delta t = \delta t$. In the computations below, if $u \in \Omega C$, we use Sweedler-type notation and write

$$\psi^{(r)}(u) = u_{(1)} \otimes \cdots \otimes u_{(r)}.$$

Moreover, to ease the notation, we suppress one summation and write

$$u = s^{-1}c_1(u) | \cdots | s^{-1}c_m(u)$$

and

$$u^{k,l} = s^{-1}c_k(u) | \cdots | s^{-1}c_l(u), \quad \forall 1 \leq k \leq l \leq m.$$

Finally, also in the interest of simplifying notation, we do not give the signs explicitly, as they are complicated to write down but very easy to derive: all of the signs arise in a straightforward manner from the Koszul rule (cf. section 1.1), as we explain more precisely below.

For all $u, v \in \Omega C$ and $w_1 \otimes \cdots \otimes w_r \in H^{\otimes r}$,

$$\begin{aligned} &\Gamma(\mu_{\Omega C} \otimes H^{\otimes r})(u \otimes v \otimes w_1 \otimes \cdots \otimes w_r) \\ &= \pm(C \otimes \mu^{(2r+1)})\xi_r(\sigma_t(u_{(1)}v_{(1)}) \otimes w_1 \otimes \alpha_t(u_{(2)}v_{(2)}) \otimes \cdots \otimes \alpha_t(u_{(r)}v_{(r)}) \otimes w_r) \\ &= \pm(C \otimes \mu^{(2r+1)})\xi_r(\sigma_t(u_{(1)})\alpha_t(v_{(1)}) \otimes w_1 \otimes \alpha_t(u_{(2)}v_{(2)}) \otimes \cdots \otimes \alpha_t(u_{(r)}v_{(r)}) \otimes w_r) \\ &\quad \pm(C \otimes \mu^{(2r+1)})\xi_r(\alpha_t(u_{(1)})\sigma_t(v_{(1)}) \otimes w_1 \otimes \alpha_t(u_{(2)}v_{(2)}) \otimes \cdots \otimes \alpha_t(u_{(r)}v_{(r)}) \otimes w_r) \\ &= \sum_i \pm c_i(u_{(1)}) \otimes \alpha_t(u_{(1)}^{i+1,m}v_{(1)})w_1\alpha_t(u_{(2)}v_{(2)}) \cdots \alpha_t(u_{(r)}v_{(r)})w_r\alpha_t(u_{(1)}^{1,i-1}) \\ &\quad + \sum_i \pm c_i(v_{(1)}) \otimes \alpha_t(v_{(1)}^{i+1,m})w_1\alpha_t(u_{(2)}v_{(2)}) \cdots \alpha_t(u_{(r)}v_{(r)})w_r\alpha_t(u_{(1)}v_{(1)}^{1,i-1}), \end{aligned} \tag{4.2}$$

where we used the fact that σ_t is a derivation. Moreover,

$$\begin{aligned} &\Gamma\left((C \otimes \mu_{H^{\otimes r}})(\Omega C \otimes \alpha_t^{\otimes r}\psi^{(r)} \otimes H^{\otimes r})\right)(u \otimes v \otimes w_1 \otimes \cdots \otimes w_r) \\ &= \pm(C \otimes \mu^{(2r+1)})\xi_r(\sigma_t(u_{(1)}) \otimes \alpha_t(v_{(1)})w_1 \otimes \alpha_t(u_{(2)}) \otimes \cdots \otimes \alpha_t(u_{(r)}) \otimes \alpha_t(v_{(r)})w_r) \\ &= \sum_i \pm c_i(u_{(1)}) \otimes \alpha_t(u_{(1)}^{i+1,m}v_{(1)})w_1\alpha_t(u_{(2)}v_{(2)}) \cdots \alpha_t(u_{(r)}v_{(r)})w_r\alpha_t(u_{(1)}^{1,i-1}) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\Gamma\left((C \otimes \mu_{H^{\otimes r}})\xi'_r(\alpha_t^{\otimes r}\psi^{(r)} \otimes \Omega C \otimes H^{\otimes r})\right)(u \otimes v \otimes w_1 \otimes \cdots \otimes w_r) \\ &= \pm(C \otimes \mu^{(2r+1)})\xi_r(\sigma_t(v_{(1)}) \otimes w_1\alpha_t(u_{(1)}) \otimes \alpha_t(v_{(2)}) \otimes \cdots \otimes \alpha_t(v_{(r)}) \otimes w_r\alpha_t(u_r)) \end{aligned}$$

$$= \sum_i \pm c_i(v_{(1)}) \otimes \alpha_t(v_1^{i+1,m}) w_1 \alpha_t(u_{(1)}v_{(2)}) \cdots \alpha_t(u_{(r-1)}v_{(r)}) w_r \alpha_t(u_{(r)}v_1^{1,i-1}). \tag{4.4}$$

Since $\tau\delta t = \delta t$, and therefore $\tau(\alpha_t \otimes \alpha_t)\psi = (\alpha_t \otimes \alpha_t)\psi$,

$$\alpha_t(u_{(1)}) \otimes \cdots \otimes \alpha_t(u_r) = (-1)^{|u_{(1)}| \cdot \sum_{i>1} |u_{(i)}|} \alpha_t(u_{(2)}) \otimes \cdots \otimes \alpha_t(u_r) \otimes \alpha_t(u_{(1)}).$$

The computations above therefore establish that equation (4.1) holds as desired, since all signs are determined uniquely by the Koszul rule, i.e., depend only what permutation has been applied to

$$u_{(1)}^{1,i-1} \otimes c_i(u_1) \otimes u_{(1)}^{i+1,m} \otimes u_{(2)} \otimes \cdots \otimes u_{(r)} \otimes v_{(1)} \otimes \cdots \otimes v_{(r)} \otimes w_{(1)} \otimes \cdots \otimes w_{(r)},$$

where the original sign is +1, in the case of (4.3) and of the first summand of (4.2), and applied to

$$u_{(1)} \otimes \cdots \otimes u_{(r)} \otimes v_{(1)}^{1,i-1} \otimes c_i(v_1) \otimes v_{(1)}^{i+1,m} \otimes v_{(2)} \cdots \otimes v_{(r)} \otimes w_{(1)} \otimes \cdots \otimes w_{(r)},$$

where the original sign is +1, in the case of (4.4) and of the second summand of (4.2). □

Remark 4.8. An explicit formula for $\tilde{\mu}_r$ can be given as follows. For all $c \in \bar{C}$, we suppress one summation and write

$$\psi^{(r)}(s^{-1}c) = u_1(c) \otimes u_2(c) \otimes \cdots \otimes u_r(c)$$

and

$$u_1(c) = s^{-1}c_1 | \cdots | s^{-1}c_k.$$

Using this notation,

$$\begin{aligned} &\tilde{\mu}_r(c \otimes (w_1 \otimes \cdots \otimes w_r)) \\ &= \sum_{1 \leq j \leq k} \pm c_j \otimes t(c_{j+1}) \cdots t(c_k) \cdot w_1 \cdot \alpha_t(u_2(c)) \cdots \\ &\quad \cdot \alpha_t(u_r(c)) \cdot w_r \cdot t(c_1) \cdots t(c_{j-1}), \end{aligned}$$

where the signs are determined by the Koszul rule, i.e., depend only on the permutation applied to

$$s^{-1}c_1 | \cdots | s^{-1}c_k \otimes u_2(c) \otimes \cdots \otimes u_r(c) \otimes w_{(1)} \otimes \cdots \otimes w_{(r)},$$

where the original sign is +1 (cf. section 1.1). Note that if $s^{-1}c$ is a primitive element of $(\Omega C, \psi)$ for all $c \in C$, then this formula reduces to $\tilde{\mu}_r = \text{Id}_C \otimes \mu^{(r)}$.

Motivated by the definition of topological r th-power map on a free loop space as the composite of iterated loop concatenation and of the diagonal map, we now complete the proof of the existence of the algebraic r th-power map.

Proof of Theorem 4.1. Since the diagram

$$\begin{array}{ccc}
 C & \xlongequal{\quad} & C \\
 \downarrow t & & \downarrow \delta^{(r)}t \\
 H & \xrightarrow{\delta^{(r)}} & H^{\otimes r}
 \end{array}$$

commutes, the pair $(\text{Id}_C, \delta^{(r)})$ is a morphism in Tw from t to $\delta^{(r)}t$ and therefore induces a chain map $\tilde{\delta}_r := \mathcal{H}(\text{Id}_C, \delta^{(r)}) : \mathcal{H}(t) \rightarrow \mathcal{H}(\delta^{(r)}t)$ such that

$$\begin{array}{ccccc}
 H & \xrightarrow{\quad} & \mathcal{H}(t) & \xrightarrow{\quad} & C \\
 \delta^{(r)} \downarrow & & \tilde{\delta}_r \downarrow & & \parallel \\
 H^{\otimes r} & \xrightarrow{\quad} & \mathcal{H}(\delta^{(r)}t) & \xrightarrow{\quad} & C
 \end{array}$$

commutes. Note that $\tilde{\delta}_r$ is natural with respect to morphisms in Tw_{Hopf} , as chain Hopf algebra maps commute with iterated comultiplications.

Let $\tilde{\lambda}_r = \tilde{\mu}_r \circ \tilde{\delta}_r$, which is natural with respect to morphisms in Tw_{HH} , since $\tilde{\delta}_r$ and $\tilde{\mu}_r$ are natural with respect to morphisms in Tw_{Hopf} and $\text{Tw}_{\text{Hirsch}}$, respectively. \square

Remark 4.9. Combining the formula developed in the proof of Theorem 4.6 for $\tilde{\mu}_r$ with the identity $\tilde{\delta}_r = \text{Id}_C \otimes \delta^{(r)}$, we obtain the following formula for $\tilde{\lambda}_r$. If $c \in C$ and $w \in H$, then

$$\begin{aligned}
 \tilde{\lambda}_r(c \otimes w) &= \sum_{1 \leq j \leq k} \pm c_j \otimes t(c_{j+1}) \cdot \dots \cdot t(c_k) \cdot w_1 \cdot \alpha_t(u_2(c)) \cdot \dots \\
 &\quad \cdot \alpha_t(u_r(c)) \cdot w_r \cdot t(c_1) \cdot \dots \cdot t(c_{j-1}),
 \end{aligned}$$

where signs are determined by the Koszul rule precisely as in Remark 4.8 and (suppressing obvious summations)

$$\begin{aligned}
 \delta^{(r)}(w) &= w_1 \otimes \dots \otimes w_r, \\
 \psi^{(r)}(s^{-1}c) &= u_1(c) \otimes u_2(c) \otimes \dots \otimes u_r(c),
 \end{aligned}$$

and

$$u_1(c) = s^{-1}c_1 | \dots | s^{-1}c_k.$$

Example 4.10. Recall [Example 3.9](#). Let K be a simplicial double suspension, either E^2L for some pointed simplicial set L or S^2M for some simplicial set M . If $C = C_*K$, then every element of C is primitive, as is every element $s^{-1}c$ of ΩC , which implies that

$$\widehat{\mathcal{H}}(C) = (C \otimes T(s^{-1}\overline{C}), d_{\widehat{\mathcal{H}}}),$$

where, if $[-, -]$ denotes the graded commutator, then

$$d_{\widehat{\mathcal{H}}}(c \otimes w) = dc \otimes w + (-1)^{|c|}c \otimes d_{\Omega}w - 1 \otimes [s^{-1}c, w],$$

and

$$\widetilde{\lambda}_r(x \otimes w) = x \otimes \lambda_r(w)$$

for all $x \in C$ and for all $w \in \Omega C$.

Example 4.11. Let K be the nerve of the cyclic group of order two, which is a reduced simplicial model of $\mathbb{R}P^\infty$ (cf. [Remark 3.11](#)). An easy calculation shows that C_kK is free abelian on one generator z_k for each k . Moreover,

$$\Delta(z_k) = \sum_{i=0}^k z_i \otimes z_{k-i}$$

for all k , which implies that $C_*K \otimes \mathbb{F}_2$ is cocommutative, i.e., that K is a symmetric simplicial set. Note that the differential in $C_*K \otimes \mathbb{F}_2$ is exactly 0.

Let $C = C_*EK \otimes \mathbb{F}_2$, and let y_k denote the suspension of z_k . Consider $\widehat{\mathcal{H}}(C) = (C \otimes T(C_+K), d_{\widehat{\mathcal{H}}})$, where

$$d_{\widehat{\mathcal{H}}}(y_l \otimes z_{k_1} | \cdots | z_{k_m}) = -1 \otimes z_l | z_{k_1} | \cdots | z_{k_m} + (-1)^{lk_1} 1 \otimes z_{k_1} | \cdots | z_{k_m} | z_l,$$

where $k = k_1 + \cdots + k_m$. Moreover, for all k ,

$$\psi_{EK}(z_k) = \sum_{i=0}^k z_i \otimes z_{k-i} \in TC_+K \otimes TC_+K$$

and so

$$\psi_{EK}^{(r)}(z_k) = \sum_{k_1 + \cdots + k_r = k} z_{k_1} \otimes \cdots \otimes z_{k_r} \in (TC_+K)^{\otimes r}$$

for all r . The formula in [Remark 4.9](#) therefore implies that

$$\widetilde{\lambda}_r(y_l \otimes z_{k_1} | \cdots | z_{k_m}) = \sum y_{l_1} \otimes z_{k_{1,1}} | \cdots | z_{k_{m,1}} | z_{l_2} | \cdots | z_{l_r} | z_{k_{1,r}} | \cdots | z_{k_{m,r}},$$

where the sum is taken over

- all (l_1, \dots, l_r) such that $\sum_j l_j = l$, and
- all $(k_{i,1}, \dots, k_{i,r})$ such that $\sum_j k_{i,j} = k_i$, for all $1 \leq i \leq m$.

Remark 4.12. All of the results in this section can be dualized, at least in the finite-type case. We leave the straightforward task of dualizing the statements to the interested reader.

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Appendix A. The proof of Theorem 3.12

We begin by recalling those elements of homological perturbation theory that we need in order to prove that applying the bar construction to a chain Hopf algebra gives rise to an Alexander–Whitney coalgebra.

Definition A.1. Suppose that $\nabla : (X, \partial) \rightarrow (Y, d)$ and $f : (Y, d) \rightarrow (X, \partial)$ are morphisms in Ch_R . If $f\nabla = \text{Id}_X$ and there exists a chain homotopy $h : (Y, d) \rightarrow (Y, d)$ such that

- (1) $dh + hd = \nabla f - \text{Id}_Y$,
- (2) $h\nabla = 0$,
- (3) $fh = 0$, and
- (4) $h^2 = 0$,

then $(X, d) \xrightarrow[\nabla]{f} (Y, d) \circlearrowleft h$ is a *strong deformation retract (SDR) of chain complexes*.

If, moreover, (Y, d, Δ_Y) and (X, d, Δ_X) are chain coalgebras and ∇ is a morphism of coalgebras, then the SDR $(X, d) \xrightarrow[\nabla]{f} (Y, d) \circlearrowleft h$ is called *Eilenberg–Zilber data* [9].

Remark A.2. If $(X, d) \xrightarrow[\nabla]{f} (Y, d) \circlearrowleft h$ is Eilenberg–Zilber data, then

$$(d \otimes \text{Id}_X + \text{Id}_X \otimes d)((f \otimes f)\Delta_Y h) + ((f \otimes f)\Delta_Y h)d = \Delta_X f - (f \otimes f)\Delta_Y,$$

i.e., f is a map of coalgebras up to chain homotopy. In fact, as stated precisely in the next theorem (due to Gugenheim and Munkholm and slightly strengthened in section 2.3 of [12]), f is a DCSH map, under reasonable local finiteness conditions.

Recall that if V is a non-negatively graded R -module with $V_0 = R$, then \bar{V} denotes $V_{>0}$.

Theorem A.3. (See [9,12].) *Let $(X, d) \xrightarrow[\bar{f}]{\nabla} (Y, d) \circlearrowleft h$ be Eilenberg–Zilber data such that X and Y are connected. Let $\bar{\Delta}_Y : \bar{Y} \rightarrow \bar{Y}^{\otimes 2}$ denote the reduced comultiplication on Y . Let $F_0 = 0$, and let F_1 be the composite*

$$\bar{Y} \xrightarrow{f} \bar{X} \xrightarrow{s^{-1}} s^{-1}\bar{X}.$$

For $k \geq 2$, let

$$F_k = - \sum_{i+j=k} (F_i \otimes F_j) \bar{\Delta}_Y h : \bar{Y} \rightarrow T^k(s^{-1}\bar{X}).$$

If for all $y \in Y$, there exists $N(y) \in \mathbb{N}$ such that $F_k(y) = 0$ for all $k > N(y)$, then

$$F = \prod_{k \geq 1} F_k = \bigoplus_{k \geq 1} F_k : Y \rightarrow \Omega X$$

is a twisting cochain. In particular, $f : Y \rightarrow X$ is a DCSH map, and $\alpha_F : \Omega Y \rightarrow \Omega X$ realizes its strong homotopy structure.

Remark A.4. Given Eilenberg–Zilber data $(X, d) \xrightarrow[\bar{f}]{\nabla} (Y, d) \circlearrowleft h$, there is a closed formula for each of the F_k ’s above. For any $k \geq 2$, let

$$h_k = \sum_{0 \leq i \leq k-2} \text{Id}_{\bar{Y}}^{\otimes i} \otimes \bar{\Delta}_Y h \otimes \text{Id}_{\bar{Y}}^{\otimes k-i-2} : \bar{Y}^{\otimes k-1} \rightarrow \bar{Y}^{\otimes k}$$

and let

$$H_k = h_k \circ h_{k-1} \circ \cdots \circ h_2 : \bar{Y} \rightarrow \bar{Y}^{\otimes k}. \tag{A.1}$$

Then

$$F_k = (-1)^{k+1} (s^{-1}f)^{\otimes k} \circ H_k.$$

We prove [Theorem 3.12](#) in this section by applying [Theorem A.3](#) to appropriately chosen Eilenberg–Zilber data. We now set up the desired SDR.

In the development below, we use the following helpful notation for simplicial expressions.

Notation A.5. If J is any set of non-negative integers $j_1 < j_2 < \dots < j_r$, let

$$s_J = s_{j_r} \cdots s_{j_1},$$

and let $|J| = r$.

For non-negative integers $m \leq n$, let $[m, n] = \{j \in \mathbb{Z} \mid m \leq j \leq n\}$. Let Δ denote the category with objects

$$\text{Ob } \Delta = \{[0, n] \mid n \geq 0\}$$

and

$$\Delta([0, m], [0, n]) = \{f : [0, m] \rightarrow [0, n] \mid f \text{ order-preserving set map}\}.$$

Viewing a simplicial R -module M_\bullet as a contravariant functor from Δ to the category of R -modules, given $x \in M_n := M([0, n])$ and $0 \leq a_1 < a_2 < \dots < a_m \leq n$, let

$$x_{a_1 \dots a_m} := M(\mathbf{a})(x) \in M_m$$

where $\mathbf{a} : [0, m] \rightarrow [0, n] : j \mapsto a_j$. Note that in particular

$$x_{0 \dots r} = d_{r+1} \cdots d_n x,$$

while for all $m < r$,

$$x_{0 \dots m r \dots n} = d_{m+1} \cdots d_{r-1} x.$$

Example A.6. Let \mathcal{A} denote the usual functor from simplicial R -modules to Ch_R , i.e., for any simplicial R -module M_\bullet , the graded R -module underlying $\mathcal{A}(M_\bullet)$ is $\{M_n\}_{n \geq 0}$, and the differential in degree n is given by the alternating sum of the face maps from M_n to M_{n-1} . Let \mathcal{A}_N denote its normalized variant.

In Theorem 2.1a) of [5] Eilenberg and Mac Lane gave explicit formulas for a natural SDR of chain complexes

$$\mathcal{A}_N(M_\bullet) \otimes \mathcal{A}_N(M'_\bullet) \xrightarrow[f]{\nabla} \mathcal{A}_N(M_\bullet \boxtimes M'_\bullet) \circlearrowleft h, \tag{A.2}$$

where \boxtimes denotes the levelwise tensor product of simplicial R -modules. In particular, if $x \in M_m$ and $x' \in M'_n$, then

$$f(x \boxtimes x') = \sum_{0 \leq \ell \leq n} x_{0 \dots \ell} \otimes x'_{\ell \dots n} \tag{A.3}$$

and

$$\nabla(x \otimes x') = \sum_{0 \leq \ell \leq n} \sum_{\substack{A \cup B = [0, n-1] \\ |A| = n-\ell, |B| = \ell}} \pm s_A x \boxtimes s_B x', \tag{A.4}$$

where the sign of a summand is the sign of the shuffle permutation corresponding to the pair (A, B) .

If $R[K]$ denotes the free simplicial R -module generated by a simplicial set K , then $C_*K \otimes R \cong \mathcal{A}_N(R[K])$. It follows that, when applied to $M_\bullet = R[K]$ and $M'_\bullet = R[K']$, for simplicial sets K and K' , Eilenberg and Mac Lane’s strong deformation retract becomes the usual Eilenberg–Zilber/Alexander–Whitney equivalence

$$C_*K \otimes C_*K' \underset{f}{\overset{\nabla}{\cong}} C_*(K \times K') \circlearrowleft h,$$

which is in fact Eilenberg–Zilber data. In the case $K = K'$, these Eilenberg–Zilber data give rise to the Alexander–Whitney coalgebra structure on C_*K [14].

In order to prove [Theorem 3.12](#), we consider another special case of the Eilenberg–Mac Lane SDR. Recall that if A is an augmented chain algebra, then $\mathcal{B}A$ is the normalized chain complex associated to the simplicial chain algebra $\mathcal{B}_\bullet A$, where $\mathcal{B}_n A = A^{\otimes n}$. The degeneracy maps are given in terms of the unit map $R \rightarrow A$ by

$$s_i : A^{\otimes n} \rightarrow A^{\otimes n+1} : a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n,$$

while the face maps are given in terms of the multiplication or the augmentation ε by

$$d_i : A^{\otimes n} \rightarrow A^{\otimes n-1} : a_1 \otimes \cdots \otimes a_n \mapsto \begin{cases} \varepsilon(a_1) \cdot (a_2 \otimes a_3 \otimes \cdots \otimes a_n) & : i = 0 \\ a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_n & : 0 < i < n \\ (a_1 \otimes \cdots \otimes a_{n-1}) \cdot \varepsilon(a_n) & : i = n. \end{cases}$$

If $M_\bullet = \mathcal{B}_\bullet A$ and $M'_\bullet = \mathcal{B}_\bullet A'$, then Eilenberg and Mac Lane’s strong deformation retract becomes

$$\mathcal{B}A \otimes \mathcal{B}A' \underset{f}{\overset{\nabla}{\cong}} \mathcal{B}(A \otimes A') \circlearrowleft h, \tag{A.5}$$

after identifying $\mathcal{B}(A \otimes A')$ with $\mathcal{A}_N(M_\bullet \boxtimes M'_\bullet)$ via a levelwise isomorphism

$$(A \otimes A')^{\otimes n} \cong A^{\otimes n} \otimes A'^{\otimes n}. \tag{A.6}$$

The map ∇ is exactly the equivalence defined in [Example 2.12](#) via the twisting cochain $t_{\mathcal{B}} * t_{\mathcal{B}}$. In particular, ∇ is a map of coalgebras, which implies that [\(A.5\)](#) is Eilenberg–Zilber data.

Note that equation [\(A.4\)](#) implies that $\nabla(sa_1 | \cdots | sa_m \otimes sa'_1 | \cdots | sa'_n)$ is equal to the signed sum of all possible (m, n) -shuffles of

$$s(a_1 \otimes 1) | \cdots | s(a_m \otimes 1) | s(1 \otimes a'_1) | \cdots | s(1 \otimes a'_n). \tag{A.7}$$

Moreover, equation (A.3) implies that

$$f(s(a_1 \otimes a'_1) | \cdots | s(a_n \otimes a'_n)) = \sum_{0 \leq \ell \leq n} \varepsilon(a_{\ell+1}) \cdots \varepsilon(a_n) \varepsilon(a'_1) \cdots \varepsilon(a'_\ell) \cdot sa_1 | \cdots | sa_\ell \otimes sa'_{\ell+1} | \cdots | sa'_n \tag{A.8}$$

Equation (A.8) implies that

$$f(\mathcal{B}_n(A \otimes A')) \subset \bigoplus_{n'+n''=n} \mathcal{B}_{n'}A \otimes \mathcal{B}_{n''}A'.$$

To prove [Theorem 3.12](#), we apply [Theorem A.3](#) to the bar construction SDR of Eilenberg and Mac Lane (A.5). We must therefore prove local finiteness of the associated F_k 's, which follows from a technical result proved in [14] (Lemma 5.3), expressed below in terms of simplicial R -modules instead of simplicial sets.

Lemma A.7. (See [14].) *Let M_\bullet and M'_\bullet be simplicial R -modules. Let $m < r \leq n$ be non-negative integers, and let A and B be disjoint sets of non-negative integers such that $A \cup B = [m + 1, n]$ and $|B| = r - m$.*

Let $h^{A,B} : (M \boxtimes M')_n \rightarrow (M \boxtimes M')_{n+1}$ be the R -linear map given by

$$h^{A,B}(x \boxtimes x') = s_{A \cup \{m\}} x_{0\dots r} \boxtimes s_B x'_{0\dots mr\dots n}$$

for all $x \in M_n$ and $x' \in M'_n$. Then the Eilenberg–Mac Lane homotopy in level n

$$h : \mathcal{A}_n(M_\bullet \boxtimes M'_\bullet) = M_n \otimes M'_n \rightarrow M_{n+1} \otimes M'_{n+1} = \mathcal{A}_{n+1}(M_\bullet \boxtimes M'_\bullet)$$

is given by

$$h(x \boxtimes x') = \sum_{\substack{m < r, A \cup B = [m+1, n] \\ |A| = n-r, |B| = r-m}} \pm h^{A,B}(x \boxtimes x'),$$

where the sign is that of the shuffle permutation associated to the couple (A, B) .

Example A.8. We are particularly interested in the case where $M_\bullet = \mathcal{B}_\bullet A$ and $M'_\bullet = \mathcal{B}_\bullet A'$, and we apply the identification (A.6) above. We compute here one term of $h(s(a_1 \otimes a'_1) | s(a_2 \otimes a'_2) | s(a_3 \otimes a'_3))$, to give some indication of the form of this homotopy, before providing general formulas below.

Observe that if $x = a_1 \otimes \cdots \otimes a_n \in \mathcal{B}_n A$, then

$$x_{0\dots r} = \varepsilon(a_{r+1} \cdots a_n) \cdot (a_1 \otimes \cdots \otimes a_r) \in \mathcal{B}_r A,$$

while if $x' = a'_1 \otimes \cdots \otimes a'_n \in \mathcal{B}_n A'$, then

$$x'_{0\dots mr\dots n} = a'_1 \otimes \cdots \otimes a'_m \otimes a'_{m+1} \cdots \cdots a'_r \otimes a'_{r+1} \otimes \cdots \otimes a'_n.$$

When $n = 3, r = 2, m = 1, A = \{3\}$ and $B = \{2\}$,

$$\begin{aligned} h^{A,B}(s(a_1 \otimes a'_1) | s(a_2 \otimes a'_2) | s(a_3 \otimes a'_3)) \\ = \varepsilon(a_3) \cdot (s(a_1 \otimes a'_1) | s(1 \otimes a'_2) | s(a_2 \otimes 1) | s(1 \otimes a'_3)), \end{aligned}$$

because

$$s^{A \cup \{1\}}(a_1 \otimes a_2 \otimes a_3)_{012} = \varepsilon(a_3) \cdot s_3 s_1(a_1 \otimes a_2) = \varepsilon(a_3) \cdot (a_1 \otimes 1 \otimes a_2 \otimes 1)$$

and

$$s^B(a'_1 \otimes a'_2 \otimes a'_3)_{0123} = s_2(a'_1 \otimes a'_2 \otimes a'_3) = a'_1 \otimes a'_2 \otimes 1 \otimes a'_3.$$

In the case of the bar construction SDR, we obtain the following general, explicit formulas, where we use the notational shortcut

$$v = v_1 | \cdots | v_m \quad \text{and} \quad w = w_1 | \cdots | w_n \implies v | w := v_1 | \cdots | v_m | w_1 | \cdots | w_n.$$

Corollary A.9. *Let $A, A' \in \text{Ob Alg}_R$. If $M_\bullet = \mathcal{B}_\bullet A$ and $M'_\bullet = \mathcal{B}_\bullet A'$, then the Eilenberg–Mac Lane homotopy $h : \mathcal{B}_*(A \otimes A') \rightarrow \mathcal{B}_{*+1}(A \otimes A')$ satisfies the following equations.*

- (1) $h(s(1 \otimes a'_1) | \cdots | s(1 \otimes a'_n)) = 0$ for all $a'_1, \dots, a'_n \in A$.
- (2) If $|a_r| \neq 0$, then

$$\begin{aligned} & h(s(a_1 \otimes a'_1) | \cdots | s(a_r \otimes a'_r) | s(1 \otimes a'_{r+1}) | \cdots | s(1 \otimes a'_n)) \\ &= \sum_{0 \leq m < r} \pm s(a_1 \otimes a'_1) | \cdots | s(a_m \otimes a'_m) | s(1 \otimes a'_{m+1} \cdots a'_r) | \\ & \quad \times \nabla(sa_{m+1} | \cdots | sa_r \otimes sa'_{r+1} | \cdots | sa'_n) \end{aligned}$$

for all $a_1, \dots, a_{r-1} \in A$ and $a'_1, \dots, a'_n \in A'$.

Remark A.10. Note that the formulas above imply that

$$h(s(a_1 \otimes 1) | \cdots | s(a_r \otimes 1) | s(1 \otimes a'_{r+1}) | \cdots | s(1 \otimes a'_n)) = 0,$$

for all $a_1, \dots, a_r \in A$ and $a'_{r+1}, \dots, a'_n \in A'$ and for all $r \geq 0$, since in this case $a'_{m+1} = \cdots = a'_r = 1$ for all $m < r$.

Proof. Let

$$w = s(a_1 \otimes a'_1) | \cdots | s(a_M \otimes a'_M) | s(1 \otimes a'_{M+1}) | \cdots | s(1 \otimes a'_n) \in \mathcal{B}_n(A \otimes A'),$$

where $|a_M| > 0$. Note that [Lemma A.7](#) implies that $h(w) \in \mathcal{B}_{n+1}(A \otimes A')$.

It is clear that

$$r < M \implies h^{A,B}(w) = 0$$

for all $0 \leq m < r$ and A and B disjoint sets of non-negative integers such that $A \cup B = [m + 1, n]$ and $|B| = r - m$, since $\varepsilon(a_M) = 0$.

We can also show that

$$r > M \implies h^{A,B}(w) = 0$$

for all $0 \leq m < r$ and A and B disjoint sets of non-negative integers such that $A \cup B = [m + 1, n]$ and $|B| = r - m$. To establish this implication, we consider the following two cases. Suppressing summation, write

$$h^{A,B}(w) = s(b_1 \otimes b'_1) | \cdots | s(b_{n+1} \otimes b'_{n+1}).$$

- (1) If $r > m \geq M$, then $|b_{m+1}| = \cdots = |b_n| = 0$, while the list b'_{m+1}, \dots, b'_n includes at least $r - m$ elements of degree 0. There exists therefore $k \in [m + 1, n]$ such that both b_k and b'_k are of degree zero and therefore $s(b_k \otimes b'_k)$ is degenerate in $\mathcal{B}_\bullet(A \otimes A')$, i.e., $s(b_k \otimes b'_k) = 0$ in the normalized complex.
- (2) If $r > M > m$, then the list b_{m+1}, \dots, b_n includes at most $M - m$ elements of positive degree, i.e., at least $n - M$ elements of degree 0. On the other hand, the list b'_{m+1}, \dots, b'_n includes at least $r - m$ elements of degree 0. Since

$$(r - m) + (n - M) = n - (m + M - r) > n - (m + 1),$$

there exists $k \in [m + 1, n]$ such that both b_k and b'_k are of degree zero and therefore $s(b_k \otimes b'_k)$ is degenerate in $\mathcal{B}_\bullet(A \otimes A')$, i.e., $s(b_k \otimes b'_k) = 0$ in the normalized complex.

We conclude that the only nonzero summands of $h(w)$ are those for which $r = M$, in which case the formula given in the corollary follows by straightforward application of the formula in [Lemma A.7](#). \square

Theorem A.11. *For all $A, A' \in \text{Ob Alg}_R$, the Alexander–Whitney map*

$$f : \mathcal{B}(A \otimes A') \rightarrow \mathcal{B}A \otimes \mathcal{B}A'$$

is a DCSH map.

Proof. We prove this proposition by applying [Theorem A.3](#) to the Eilenberg–Zilber data [\(A.5\)](#). Note first that $\mathcal{B}A \otimes \mathcal{B}A'$ and $\mathcal{B}(A \otimes A')$ are both connected, by definition of the bar construction.

Given a nonzero element $w = s(a_1 \otimes a_1) | \cdots | s(a_n \otimes a'_n) \in \mathcal{B}_n(A \otimes A')$, let

$$\zeta(w) = \#\{i \mid |a_i| = 0\} + \#\{j \mid |a'_j| = 0\}.$$

Let $\zeta(0) = +\infty$.

Let $w = s(a_1 \otimes a_1) | \cdots | s(a_n \otimes a'_n) \in \mathcal{B}_n(A \otimes A')$. If $\zeta(w) > n$, then there exists $j \in [1, n]$ such that $|a_j| = 0 = |a'_j|$ and therefore w corresponds to a degenerate element in $\mathcal{B}_\bullet(A \otimes A')$. Since $\mathcal{B}(A \otimes A')$ is the normalized complex associated to $\mathcal{B}_\bullet(A \otimes A')$, it follows that $w = 0$. We therefore conclude that

$$0 \neq w \in \mathcal{B}_n(A \otimes A') \implies \zeta(w) \leq n. \tag{A.9}$$

Define a bifiltration of $\mathcal{B}(A \otimes A')$ by

$$\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A')) = \{w \in \mathcal{B}_{\leq n}(A \otimes A') \mid \zeta(w) \geq p\},$$

and consider the induced bifiltration

$$\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A')^{\otimes k}) = \bigoplus_{\substack{p_1 + \cdots + p_k = p \\ n_1 + \cdots + n_k = n}} \mathcal{F}^{p_1, n_1}(\mathcal{B}(A \otimes A')) \otimes \cdots \otimes \mathcal{F}^{p_k, n_k}(\mathcal{B}(A \otimes A')).$$

It is easy to check that the comultiplication $\Delta : \mathcal{B}(A \otimes A') \rightarrow \mathcal{B}(A \otimes A') \otimes \mathcal{B}(A \otimes A')$ is a bifiltered map. Moreover, it follows from implication [\(A.9\)](#) that

$$\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A')^{\otimes k}) = 0 \text{ for all } p > n \text{ and } k \geq 1. \tag{A.10}$$

To prove local finiteness of the F_k 's associated to the SDR [\(A.5\)](#), we show that

$$\zeta(h(w)) \geq \zeta(w) + 2 \tag{A.11}$$

for all $w \in \mathcal{B}(A \otimes A')$. It follows that

$$h(\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A'))) \subset \mathcal{F}^{p+2, n+1}(\mathcal{B}(A \otimes A'))$$

for all p and n , since the formulas in [Corollary A.9](#) imply that

$$h(\mathcal{B}_n(A \otimes A)) \subset \mathcal{B}_{n+1}(A \otimes A).$$

Consequently, if $\bar{\Delta}$ denotes the reduced comultiplication on $\mathcal{B}(A \otimes A')$, then

$$\bar{\Delta}h(\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A'))) \subset \mathcal{F}^{p+2, n+1}(\mathcal{B}(A \otimes A') \otimes \mathcal{B}(A \otimes A')),$$

which is the base step in an easy recursive argument showing that

$$H_{k+1} \left(\mathcal{F}^{p,n}(\mathcal{B}(A \otimes A')) \right) \subset \mathcal{F}^{p+2k,n+k}(\mathcal{B}(A \otimes A')^{\otimes k}),$$

for all $k \geq 1$, where the map H_{k+1} is defined as in (A.1).

Equation (A.10) therefore implies that for all $w \in \mathcal{F}^{p,n}(\mathcal{B}(A \otimes A'))$ and for all $k > n - p + 1$,

$$F_k(w) = (s^{-1}f)^{\otimes k} \circ H_k(w) = 0.$$

Local finiteness of the F_k 's, which allows us to apply Theorem A.3 and therefore conclude that $f : \mathcal{B}(A \otimes A') \rightarrow \mathcal{B}(A) \otimes \mathcal{B}(A')$ is a DCSH map, is thus a consequence of inequality (A.11).

To complete the proof, we must prove that inequality (A.11) holds. It follows immediately from inspection of the formulas in Corollary A.9 and for ∇ (A.4) that, in comparison with $w = s(a_1 \otimes a_1) | \cdots | s(a_n \otimes a'_n)$, each summand in the expression

$$s(a_1 \otimes a'_1) | \cdots | s(a_m \otimes a'_m) | s(1 \otimes a'_{m+1} \cdots a'_r) | \nabla(sa_{m+1} | \cdots | sa_r \otimes sa'_{r+1} | \cdots | sa'_n),$$

if nonzero,

- contains $(n - r) + (r - m) + 1$ new 1's, inserted in the last $n - m$ terms, and
- has lost at most $(r - m - 1)$ 1's, in the process of multiplying $a'_{m+1} \cdots a'_r$.

We see thus that

$$\begin{aligned} &\zeta(s(a_1 \otimes a'_1) | \cdots | s(a_m \otimes a'_m) | s(1 \otimes a'_{m+1} \cdots a'_r) | \nabla(sa_{m+1} | \cdots | sa_r \otimes sa'_{r+1} | \cdots | sa'_n)) \\ &\geq \zeta(w) + (n - m + 1) - (r - m - 1) \\ &= \zeta(w) + n - r + 2 \\ &\geq \zeta(w) + 2. \quad \square \end{aligned}$$

Before proving Theorem 3.12, we establish a technical lemma that plays an important role in showing that $\varepsilon_H : \Omega\mathcal{B}H \rightarrow H$ is a coalgebra map. Recall the cartesian product of twisting cochains from Definition 2.11.

Lemma A.12. *Let A and A' be augmented chain algebras. For all $n > 1$, the composite*

$$\begin{array}{ccc}
 \mathcal{B}_n(A \otimes A') \xrightarrow{h} \mathcal{B}_{n+1}(A \otimes A') & \xrightarrow{\Delta} & \bigoplus_{\ell+m=n+1} \mathcal{B}_\ell(A \otimes A') \otimes \mathcal{B}_m(A \otimes A') \\
 & & \downarrow f \otimes f \\
 & & \bigoplus_{\substack{\ell+m=n+1 \\ \ell'+\ell''=\ell \\ m'+m''=m}} \left(\mathcal{B}_{\ell'} A \otimes \mathcal{B}_{\ell''} A' \right) \otimes \left(\mathcal{B}_{m'} A \otimes \mathcal{B}_{m''} A' \right) \\
 & & \downarrow (t_{\mathcal{B}} * t_{\mathcal{B}})^{\otimes 2} \circ (t_{\mathcal{B}} * t_{\mathcal{B}}) \\
 & & (A \otimes A') \otimes (A \otimes A')
 \end{array}$$

is equal to zero. Moreover,

$$(t_{\mathcal{B}} * t_{\mathcal{B}})^{\otimes 2} \circ f^{\otimes 2} \circ \Delta \circ h(s(a \otimes a')) = \begin{cases} (-1)^{|a| \cdot |a'|} (1 \otimes a') \otimes (a \otimes 1) & : |a| \cdot |a'| > 0 \\ 0 & : \text{else.} \end{cases}$$

Proof. Let $\delta = (t_{\mathcal{B}} * t_{\mathcal{B}})^{\otimes 2} \circ f^{\otimes 2} \circ \Delta \circ h$. Recall that $t_{\mathcal{B}}(sa) = a$ for all a in A or A' , while $t_{\mathcal{B}}(sa_1 | \dots | sa_n) = 0$ for all $n > 1$.

If $n > 1$ and thus $\ell' + \ell'' + m' + m'' = n + 1 > 2$, then

- at least one of ℓ' , ℓ'' , m' and m'' is greater than 1, or
- $\ell' + \ell'' = 2$ and $m' + m'' \leq 2$, or
- $\ell' + \ell'' \leq 2$ and $m' + m'' = 2$.

In the first case, the corresponding summand of δ is zero, since $\mathcal{B}_{\geq 2} A \subset \ker t_{\mathcal{B}}$ and similarly for A' . In the second and third cases, the corresponding summand of δ is also zero, since

$$\mathcal{B}_1 A \otimes \mathcal{B}_1 A' = s\bar{A} \otimes s\bar{A}' \subset \ker(t_{\mathcal{B}} * t_{\mathcal{B}}).$$

The case $n = 1$ is established by a straightforward calculation. \square

Corollary A.13. *Let A and A' be augmented chain algebras. Consider the composite*

$$\Omega \mathcal{B}(A \otimes A') \xrightarrow{\alpha_F} \Omega(\mathcal{B}A \otimes \mathcal{B}A') \xrightarrow{q} \Omega \mathcal{B}A \otimes \Omega \mathcal{B}A' \xrightarrow{\varepsilon_A \otimes \varepsilon_{A'}} A \otimes A',$$

where $F : \mathcal{B}(A \otimes A') \rightarrow \Omega(\mathcal{B}A \otimes \mathcal{B}A')$ is the twisting cochain of [Theorem A.11](#). For all $n > 1$,

$$s^{-1} \mathcal{B}_n(A \otimes A') \subset \ker \left((\varepsilon_A \otimes \varepsilon_{A'}) q \alpha_F \right).$$

Proof. [Remark 2.13](#) implies that

$$\alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}} = (\varepsilon_A \otimes \varepsilon_{A'})q : \Omega(\mathcal{B}A \otimes \mathcal{B}A') \rightarrow A \otimes A'.$$

Furthermore, for all $w_1, \dots, w_n \in \mathcal{B}A$ and $w'_1, \dots, w'_n \in \mathcal{B}A'$,

$$\alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}}(s^{-1}(w_1 \otimes w'_1) | \cdots | s^{-1}(w_n \otimes w'_n)) = \pm a_1 \cdots a_n \otimes a'_1 \cdots a'_n,$$

where (suppressing summation) $\alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}}(s^{-1}(w_j \otimes w'_j)) = a_j \otimes a'_j$ for all $1 \leq j \leq n$, and the sign is determined by the Koszul rule, i.e., determined entirely by the permutation applied to

$$(a_1 \otimes a'_1) \otimes \cdots \otimes (a_n \otimes a'_n),$$

where the original sign is +1 (cf. section 1.1).

Recall equation (A.1), the definition of the operators H_k associated to Eilenberg–Zilber data. A straightforward inductive argument, of which Lemma A.12 is the base step, shows that for all $n > 1$

$$f^{\otimes k} \circ H_k(\mathcal{B}_n(A \otimes A')) \subset \ker(t_{\mathcal{B}} * t_{\mathcal{B}})^{\otimes k}.$$

It then follows from the second half of Remark A.4 that

$$\alpha_F(\mathcal{B}_n(A \otimes A')) \subset \ker \alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}},$$

and we can conclude. \square

Proof of Theorem 3.12. Let H be a chain Hopf algebra. From Theorem A.11 it follows that $\mathcal{B}H$ is a weak Alexander–Whitney coalgebra, where the chain algebra map realizing the DCSH structure of $\delta : H \rightarrow H \otimes H$ is the composite

$$\Omega\mathcal{B}H \xrightarrow{\Omega\mathcal{B}\delta} \Omega\mathcal{B}(H \otimes H) \xrightarrow{\alpha_F} \Omega(\mathcal{B}H \otimes \mathcal{B}H).$$

It remains to show that $(\mathcal{B}H, \alpha_F \circ \Omega\mathcal{B}\delta)$ is actually an Alexander–Whitney coalgebra, i.e., that the comultiplication

$$\Omega\mathcal{B}H \xrightarrow{\alpha_F \circ \Omega\mathcal{B}\delta} \Omega(\mathcal{B}H \otimes \mathcal{B}H) \xrightarrow{q} \Omega\mathcal{B}H \otimes \Omega\mathcal{B}H$$

is coassociative. Essentially the same argument as in the proof of coassociativity of the canonical diagonal on ΩC_*K (Theorem 4.2 in [14]) works here, since the comultiplication on the cobar constructions comes in both cases from the Alexander–Whitney map in the original Eilenberg–Mac Lane SDR (A.2).

Let $\psi = q \circ \alpha_F \circ \Omega\mathcal{B}\delta : \Omega\mathcal{B}H \rightarrow \Omega\mathcal{B}H \otimes \Omega\mathcal{B}H$. To prove that $\varepsilon_H : \Omega\mathcal{B}H \rightarrow H$ is a coalgebra map, we must verify the following two claims.

- (1) $(\varepsilon_H \otimes \varepsilon_H) \circ \psi(s^{-1}(sa)) = \delta(a)$ for all $a \in H$.
- (2) $(\varepsilon_H \otimes \varepsilon_H) \circ \psi(s^{-1}(sa_1 | \cdots | sa_n)) = 0$ for all $n > 1$.

Claim (2) is an immediate consequence of [Corollary A.13](#).

On the other hand, [Lemma A.12](#) implies that if $\delta(a) = a \otimes 1 + 1 \otimes a + a_i \otimes a^i$, then

$$\begin{aligned} (\varepsilon_H \otimes \varepsilon_H) \psi(s^{-1}(sa)) &= \alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}} \alpha_F(s^{-1}(s(a \otimes 1) + s(1 \otimes a) + s(a_i \otimes a^i))) \\ &= \alpha_{t_{\mathcal{B}} * t_{\mathcal{B}}}(s^{-1}(sa \otimes 1) + s^{-1}(1 \otimes sa) + s^{-1}(1 \otimes sa^i) | s^{-1}(sa_i \otimes 1)) \\ &= \delta(a), \end{aligned}$$

and so Claim (1) holds as well.

To prove the dual result, concerning the cobar construction on H , note that if H is connected and of finite type, then $\text{hom}_R(\Omega H, R)$ is isomorphic to $\mathcal{B} \text{hom}_R(H, R)$, which is an Alexander–Whitney coalgebra by the argument above. It follows that ΩH is an Alexander–Whitney algebra and that $\eta_H : H \rightarrow \mathcal{B}\Omega H$ is an algebra map. \square

Remark A.14. In [\[18\]](#), Kadeishvili showed that if H is a chain Hopf algebra, then ΩH is a Hirsch algebra. In proving above that ΩH is an Alexander–Whitney algebra, we have established a stronger result, one that is necessary to proving the existence of multiplicative structure on $\widehat{\mathcal{H}}(H)$.

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