INT 207/02 October 2002

COMMENT ON DEPLOYMENT OF ANALYTICAL EXPRESSIONS FOR FLUX SURFACE SHAPING AND TOROIDAL EFFECTS ON THE IDEAL MHD m=n=1 INSTABILITY

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Comment on deployment of analytical expressions for flux surface shaping and toroidal effects on the ideal MHD $m = n = 1$ instability

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Abstract

The effect of shaping on the ideal internal kink mode is considered for cases where $\Delta q = |1-q_0|$ may or may not be small. For an analysis which treats the ellipticity e, triangularity δ and inverse aspect ratio (toroidicity) ϵ to be small expansion parameters, a pure triangularity δW contribution to the kink mode exists which is leading order in magnitude even when Δq is small. Where there is no elongation, only a pure triangularity term contributes to the effect of shaping on stability.

The motivation for this note is to reflect on a recent paper by Eriksson and Wahlberg [1] which considers the effect of combined triangularity δ and ellipticity e on the internal kink mode potential energy δW. The largest shaping contributions in Ref. [1] have been shown to be identical to the shaping terms of Mercier criterion [2]. The latter Mercier terms have been employed to interprete sawtoothing behaviour in shaped TCV plasmas [3]. However, recent numerical studies [4] using the KINX code [5] present results that cannot be explained by Ref. [1]. These include improved internal kink mode stabilisation for negative triangularity - a regime planned for possible future sawtoothing studies in the TCV tokamak. The present contribtion attemps to explain the origin of this stabilising effect by taking into account a pure triangularity term [6, 7]. Although this term was ignored in Ref. [1] it is shown here that it is generally leading order in magnitude.

In Ref. [1] special attention is given to small $\Delta q \equiv |1 - q_0|$. This follows because for $\Delta q \sim 1$ the shaping terms of Eriksson and Wahlberg [1] are a factor of ϵ smaller than the toroidal term [8] and the pure ellipticity and triangularity of Refs. [6, 7]. Expanding in small Δq and keeping only Δq^0 terms facilitates the previously mentioned agreement between an expression for δW and the shaping terms of the Mercier criterion [2]. However, it is important to note that while e, δ and ϵ are treated as formal expansion parameters in Ref. [1], Δq is not. In fact the ordering of $\Delta q \sim \epsilon$ must be treated carefully from the outset as was done for the toroidal case of Ref. [9]. Here it was seen that the field line bending contribution to δW

$$\frac{2\pi^2 B_0^2}{\mu_0 R_0}\int_0^a dr\, r^3 \left(\frac{d\xi_{r0}}{dr}\right)^2 \left(1-\frac{1}{q}\right)^2 \tag{1}$$

for $\Delta q \sim \epsilon$ is a factor of ϵ^2 smaller than for conventional q profiles. Hence for $\Delta q \sim \epsilon$, Eq. (1) is the same order of magnitude as the effects that arise from toroidicity and shaping and therefore does not in general identify the leading order radial eigenfunction ξ_{r0} as the top-hat. For q profiles which have $r_1/a \gg \epsilon_1$ (where 1 denotes values at the $q = 1$ surface) one finds from Eq. (14) of Ref. [9] that the internal kink mode is always unstable for conventional sized beta ($\beta \sim \epsilon^2$). Such a finding gave credence to Wesson's quasi interchange mode [10]. In particular, $\beta \sim \epsilon^3$ is required for an ideal stability threshold to exist. Hence it is noted here that one cannot have absolute confidence in Eriksson and Whalberg's [1] analytical expressions (e.g. Eq. (40) of Ref. [1]) of δW for cases where $\Delta q \lesssim \epsilon$. This questions the correctness of the statement in Ref. [1] which claims that when q_0 approaches unity the geometric factor governing the stability of the internal kink mode is identical to the corresponding factor in the Mercier criterion for shaped plasmas. More analytical work is required for the formal treatment of small Δq in shaped plasmas.

The remainder of this note is dedicated to including all the known analytical shaping contributions to δW [6, 7, 1] and comparing the sizes of these terms for both $\Delta q \sim 1$ and $\Delta q \sim \epsilon$. This therefore requires that ξ_{r0} is assumed to be top-hat throughout. In references [7, 11] the equilibrium has the form:

$$R = R_0 - r\cos\omega - \Delta(r) + \sum_n S^{(n)}(r)\cos(n-1)\omega$$

$$Z = r\sin\omega + \sum_n S^{(n)}(r)\sin(n-1)\omega,$$

where r is the minor radius and ω is a non-orthogonal (to r) angular variable. Δ is the Shafranov shift and $S^{(n)}$ the imposed shaping of the flux surfaces with n denoting the Fourier harmonic. It is noted that the equilibrium equations of Ref. [7] yield the radial dependence of the shaping coefficients:

$$S^{(n)}(r) \sim \left(\frac{r}{a}\right)^{n-1} S^{(n)}(a). \tag{2}$$

The Mercier analysis of Refs. [11] use transformations between the coefficients $S^{(2)}$ and $S^{(3)}$ and the conventional definitions of elongation κ and triangularity δ as:

$$\frac{S^{(2)}}{r} = \frac{\kappa-1}{2} \quad \text{and} \quad \frac{S^{(3)}}{r} = \frac{\delta}{4}, \tag{3}$$

and hence $S^{(2)}/r$ is identified with the ellipticity $e \equiv (\kappa-1)/2$.

Assuming at the start that $\Delta q \sim O(\epsilon^0)$ as required in the analysis of Refs. [8, 6, 7] and using ϵ as a formal expansion parameter obtains: $\delta W_{-2} + \delta W_0 + \delta W_2$ and eigenfunction $\boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$, where the subscript denotes the corresponding ordering in ϵ. Here δW_{-2} is minimised to zero by the incompressible fluid relation $\nabla \cdot \boldsymbol{\xi}_{\perp 0} = 0$. δW_0, given by Eq. (1) above, is minimised to zero by identifying ξ_{r0} with the top hat function. In a real torus the pure cylindrical contribution to δW_2 is minimised to zero thus leaving toroidal and shaping effects only. Given that $\kappa \sim \delta \sim \epsilon$ and employing the normalisation $\delta W \to \delta W/(2\pi^2 R_0 B_0^2 \xi_0^2 \epsilon_1^4/\mu_0)$ one obtains [8, 6, 7] the toroidal contribution (ϵ^2):

$$\delta W_{\epsilon^2} = \delta W_{f1} + \delta W_{f2}\beta_p + \delta W_{f3}\beta_p^2 \tag{4}$$

where

$$\begin{aligned}
\delta W_{f1} &= \frac{\sigma}{2} + \frac{\left[\frac{9}{4}(b^{(2)}-1)(1-c^{(2)}) - 6\sigma(b^{(2)}-1)(c^{(2)}+3) - 4\sigma^2(c^{(2)}+3)(b^{(2)}+3)\right]}{[16(b^{(2)}-c^{(2)})]},\\
\delta W_{f2} &= -\frac{(c^{(2)}+3)[3(b^{(2)}-1)+4\sigma(b^{(2)}+3)]}{8(b^{(2)}-c^{(2)})},\\
\delta W_{f3} &= -\frac{(b^{(2)}+3)(c^{(2)}+3)}{4(b^{(2)}-c^{(2)})},
\end{aligned} \tag{5}$$

with

$$\beta_p = -\frac{2\mu_0}{B_0^2\epsilon_1^2 r_1^2}\int_0^{r_1} r^2 \frac{dP}{dr}dr \quad \text{and} \quad \sigma = \frac{1}{r_1^4}\int_0^{r_1} r^3\left(\frac{1}{q(r)^2}-1\right)dr.$$

The pure elongation e^2 and pure triangularity δ^2 terms are given by:

$$\delta W_{e^2} = \frac{1}{12\epsilon_1^2}\left[\frac{\left[S_1^{(2)\prime} - S_1^{(2)}/r_1\right]^2 (b^{(3)}+4)(c^{(3)}+4)}{c^{(3)}-b^{(3)}} + \frac{\left[S_1^{(2)\prime} + 3S_1^{(2)}/r_1\right]^2 b^{(-1)}c^{(-1)}}{c^{(-1)}-b^{(-1)}}\right],$$

and

$$\delta W_{\delta^2} = \frac{1}{12\epsilon_1^2}\left[\frac{\left[S_1^{(3)\prime} - 2S_1^{(3)}/r_1\right]^2 (b^{(4)}+5)(c^{(4)}+5)}{c^{(4)}-b^{(4)}}\right. \tag{6}$$

$$\left. + \frac{\left[S_1^{(3)\prime} + 4S_1^{(3)}/r_1\right]^2 (b^{(-2)}-1)(c^{(-2)}-1)}{c^{(-2)}-b^{(-2)}}\right], \tag{7}$$

where subscript '1' denotes evaluation at r_1. Note that Ref. [7] also obtains the contributions from pure quadrupole d^2 and other higher harmonics. The quantities $b^{(m)}$ and $c^{(m)}$ are defined as

$$b^{(m)} = \left.\frac{r}{\xi^{(m)}}\frac{d\xi^{(m)}}{dr}\right|_{r=r_1-} \quad \text{and} \quad c^{(m)} = \left.\frac{r}{\xi^{(m)}}\frac{d\xi^{(m)}}{dr}\right|_{r=r_1+}, \tag{8}$$

with functions $\xi^{(m)}(r < r_1)$ and $\xi^{(m)}(r > r_1)$ being the solutions of the homogeneous equation for the eigenfunction ξ:

$$\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{m}\right)^2\frac{d\xi}{dr}\right] - r(m^2-1)\left(\frac{1}{q}-\frac{1}{m}\right)^2\xi = 0. \tag{9}$$

Permissible solutions are regular as $r \to 0$ and $r \to r_2$ (where $q(r_2) = 2$). The boundary conditions $[\xi(0) = 0\,,\ d\xi(0)/dr = 1]$ are used to obtain $b^{(2)}$ and $[\xi(r_2) = 1\,,\ d\xi(r_2)/dr = 0]$ to obtain $c^{(2)}$.

Analytical approximations are available for the toroidal and shaping terms if one assumes that $r_1/a \sim \epsilon$ and the q profile has have the form: $q = 1 - \Delta q(1 - (r/r_1)^\lambda)$. Choosing the parabolic case ($\lambda = 2$) and using the substitutions given in Eq. (3) together with $d = S^{(4)}/r$ gives

$$\delta W_{\epsilon^2} + \delta W_{e^2} + \delta W_{\delta^2} + \delta W_{d^2} = \frac{\Delta q}{\epsilon^2}\left[\left(\frac{13}{48} - 3\beta_p^2\right)\epsilon^2 - \Delta q^2\frac{2e^2}{3} + \frac{\delta^2}{4} + 12d^2\right], \tag{10}$$

where e, δ, ϵ and d are again evaluated at r_1. It is noted that the elongation term is proportional to Δq^3 (not Δq^2 as quoted in Ref. [12]) and for this reason is considered to be ignorable for small Δq. However, the pure triangularity term is proportional to Δq only and thus cannot be considered small enough to ignore. In particular, with an ordering $\delta \sim \epsilon$ one should not drop the pure triangular shaping term while retaining the toroidal term, which is also proportional to Δq, as has been done in Ref. [1]. Indeed δW_{δ^2} was considered to be large in Ref. [7], and for experimentally relevant parameters could give rise to an increase in the critical threshold beta to $\beta_p \sim 1$. It is also noted that the quadrupolarity term δW_{d^2} is much less important than δW_{δ^2}. This follows from Eq. (2) where it is seen that $d(r) \sim (r/a)^2 d(a)$ has only a weak penetration into the plasma.

If one wishes now to artificially assume that $\Delta q \sim \epsilon$, and ignore the non-trivial effect on the ordering of the internal kink expansion, then the terms in Eq. (10) are one order higher in ϵ. Hence the leading order components of the $\epsilon^2 e$ and $\epsilon e\delta$ contributions described in Ref. [1] should also be included:

$$\delta W(\epsilon, \delta, \kappa, \Delta q) = \Delta q\left(\frac{13}{48} - 3\beta_p^2\right) + \frac{\Delta q}{4}\left(\frac{\delta}{\epsilon}\right)^2 - \frac{3}{4}(\kappa - 1)\beta_p\left(1 - \frac{2\delta}{\epsilon}\right) + O(\epsilon^2) \tag{11}$$

where the effect of quadrupolarity and higher harmonics has been ignored. It is seen that the pure triangularity term is all that remains of the shaping effects if the flux surfaces are not elongated! The critical (threshold) beta which corresponds to vanishing Eq. (11) is given by:

$$\beta_p^c = -\frac{\kappa - 1}{8\Delta q}\left(1 - \frac{2\delta}{\epsilon}\right) \pm \left\{\left[\frac{\kappa - 1}{8\Delta q}\left(1 - \frac{2\delta}{\epsilon}\right)\right]^2 + \frac{1}{12}\left[\left(\frac{\delta}{\epsilon}\right)^2 + \frac{13}{12}\right]\right\}^{1/2}. \tag{12}$$

Hence, where there is no elongation the critical beta is given by:

$$\beta_p^c = \frac{1}{\sqrt{12}}\left[\left(\frac{\delta}{\epsilon}\right)^2 + \frac{13}{12}\right]^{1/2}.$$

Figure 1(a) plots β_p^c as a function of δ/ϵ for $\Delta q = 0.25$ and $\kappa = 1.2$. The dotted line ignores the contribution from the δ^2 contribution and is thus similar to the $\Delta q = 0.2$ curve in Fig.2 of Ref. [1]. The solid curve, which this time does include the δ^2 contribution, differs greatly from the dotted curve for $|\delta| \sim \epsilon$. Furthermore, for negative δ of around $\epsilon/2$ it is seen that there is a minimum value of β_p^c. For even more negative triangularity the quadratic δ^2 contribution dominates and thus stabilises the mode. Similar characteristics are observed in Fig. 1(b) which plots δW as a function of δ/ϵ for $\Delta q = 0.25$, $\kappa = 1.2$ and $\beta_p = 0.19$. Again, the solid line in Fig. 1(b) includes the δ^2 contribution while the dotted line does not.

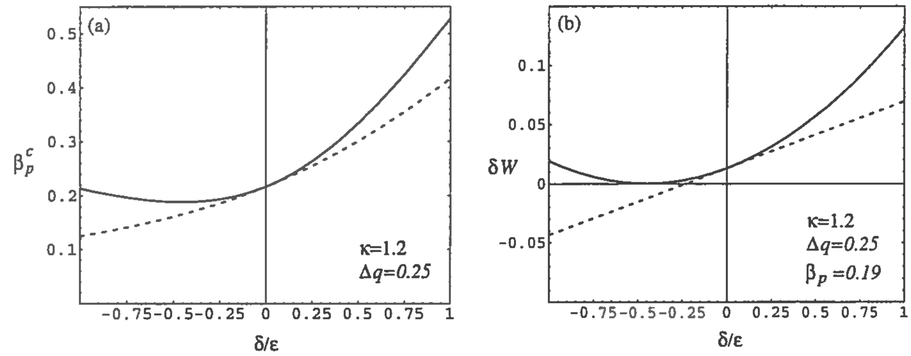

Figure 1: Showing (a) the critical poloidal beta β_p^c (for which $\delta W = 0$) and (b) the potential energy δW as a function of triangularity δ/ϵ with fixed parameters $\Delta q = 0.25$ and $\kappa = 1.2$. The solid lines include the pure triangularity contribution (δ^2), while the dotted lines do not.

Figure 2 shows three different contour plots of β_p^c as a function of δ/ϵ and κ. Figure 2(a) plots Eq. (12) without the pure triangularity term while in Fig. 2(b) the pure triangularity term is retained. For $\kappa \gtrsim 1$ the latter two plots exhibit the expected contrasting dependence on δ/ϵ. Figure 2(b) is very similar to that of Fig. 4(a) in Ref. [4]. Note that the value of δ/ϵ which gives the minimum β_p^c is increasingly large and negative as κ increases from unity. The minimum is at $\delta = 0$ when $\kappa = 1$. This can be understood from inspection of Eq. (11) or (12) where it is seen that Whalberg's $\epsilon e \delta$ contribution is zero for $\kappa = 1$ regardless of δ, while the δ^2 term is quadratic in δ and independent of κ.

For cases where $\kappa = 1$ (i.e. no elongation) the minimum of both δW and β_p^c is not in general at vanishing δ/ϵ as ϵ increases. One can see this by noting that there are higher order contributions to δW which are an odd function of δ and independent of κ. From Ref. [1] it can be seen that contributions to δW of order $\epsilon^\mu e^\nu \delta^\lambda$ exist for $\int_0^{2\pi} d\theta \cos^\mu \theta \cos^\nu 2\theta \cos^\lambda 3\theta \neq 0$. Hence the non-vanishing term $\epsilon^4(\delta/\epsilon)$ is linear in δ/ϵ and independent of κ. For $\delta \sim \epsilon$, the latter term is ϵ^2 smaller than the pure triangluarity term $\epsilon^2(\delta/\epsilon)^2$ and would therefore only be significant for ϵ larger than is appropriate in analytical studies. However, regimes with moderate ϵ could explain KINX simulations which claim to yield a minimum in β_p^c at non-zero δ when $\kappa = 1$.

Finally, the effect of the pure elongation term (e^2) is included in Fig. 2(c) for the parameters employed in Figs. 2(a) and (b). One can evaluate the effect of the pure elongation term on β_p^c by employing the following transformation in Eq. (12):

$$\frac{13}{12} \rightarrow \frac{13}{12} - \frac{4\Delta q^2}{6}\left(\frac{\kappa - 1}{\epsilon}\right)^2$$

This transformation introduces an explicit ϵ dependence in β_p^c. Upon comparing Fig. 2(b) and (c) it can be seen that the destabilisng quadratic dependence of the pure elongation term is significant for $\kappa \gtrsim 1.2$. However, for $\Delta q \approx \epsilon = 0.1$ (rather than $\Delta q = 0.25$ as in Fig. 2(c)) it is found that $\kappa \sim 3$ is required for the pure elongation term to significantly influence the stability. Such extreme elongation is out of the range of what is permitted for an analyitcal treatment because the corresponding value of the ellipticity e is unity.

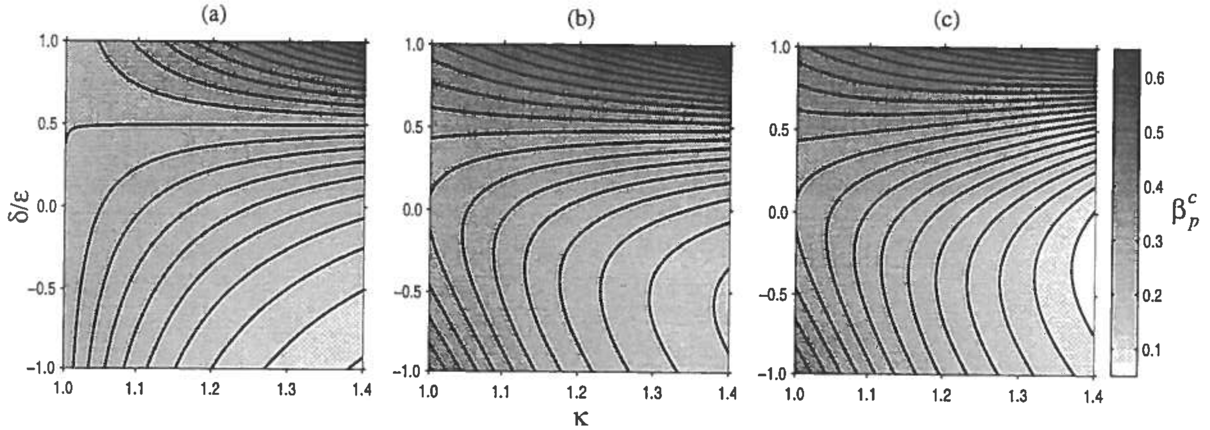

Figure 2: Contour plots of β_p^c as a function of δ/ϵ and κ with fixed parameter $\Delta q = 0.25$. (a) plots Eq. (12) without the pure triangularity term, while in (b) it is retained. (c) also includes the contribution from pure elongation (e^2) which requires an additional numerical parameter: $\epsilon = 0.1$. Darker colours correspond to larger values of β_p^c.

In conclusion, the effects of pure elongation and triangularity [6, 7] on the ideal internal kink mode potential energy δW have been included in conjunction with shaping contributions of Eriksson and Whalberg [1]. It is seen that for $\Delta q \sim 1$, the pure elongation and triangularity terms are leading order of magnitude, while those of Eriksson and Whalberg [1] are a factor of ϵ smaller. For $\Delta q \sim \epsilon$ the shaping terms of Ref. [1], the pure triangularity term of Refs. [6, 7] and the toroidal contribution [8] are all leading order of magnitude. Here it has been assumed that contrary to the analysis of Refs. [9, 10], the leading order eigenfunction is assumed to be the top-hat. Inclusion of the pure triangularity term provides agreement with KINX simulations [4] which show that negative triangularity can stabilise the internal kink mode. Such a regime is planned for future sawtoothing experiments.

This work has been supported in part by the Swiss National Science Foundation.

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