Introducing Geometry in Active Learning for Image Segmentation (Supplementary Material)

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Algorithm for the Search for the Best Plane

In this document we describe in more details the algorithm of Sec. 5 that searches for the most uncertain planar patch in the image stack. It uses a Branch-and-Bound approach to quickly find the optimal plane in the search space. We start by reformulating the problem and introducing some definitions and assumptions. Then, we define a bounding function, which we introduced in Sec. 5, a search procedure and a termination condition.

A. Problem formulation and definitions

Problem formulation. Let us consider the most uncertain supervoxel s_i for which we would like to find a circular projection of maximum uncertainty with it in the center. Such a patch can be seen as the intersection of a sphere of radius r with a plane of arbitrary orientation going through s_i as shown in Fig. 1.

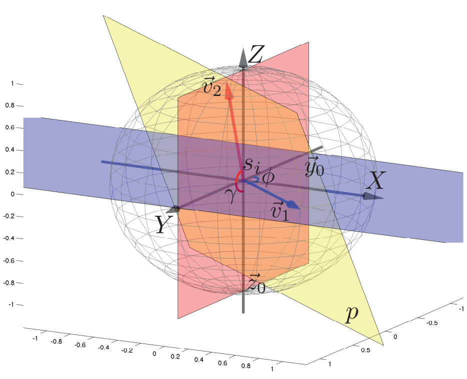

Figure 1. Coordinate system for planes. We are looking for a circular patch that is defined as the intersection of a plane with a sphere. Plane p_i (yellow) is defined by two angles, ϕ – the intersection between plane p and plane Xs_iY (blue) and γ – the intersection between plane p and plane Ys_iZ (red). Best seen in colour.

Supervoxel approximation. We assume that any supervoxel s_j can be well approximated by a spherical object of radius κ (that is set to a constant for a particular dataset) and its center w_j (Fig. 2).

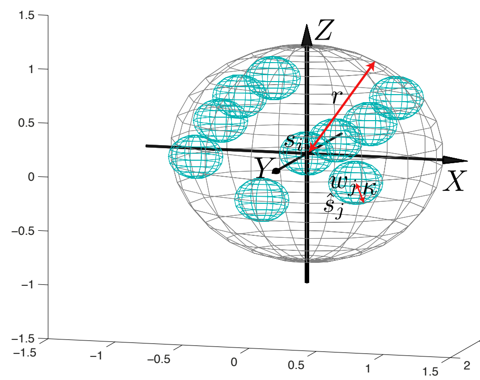

Figure 2. Supervoxel approximation. Each supervoxel can be considered as a sphere of radius κ and center w_j. We are interested in the neighbourhood of supervoxel s_i defined by a sphere of radius r.

We will refer to such an approximation as $\hat{s}_j$. Then, every $\hat{s}_j$ is characterised by its center w_j and the common radius κ: $\hat{s}_j = (w_j, \kappa)$.

Sphere. Let $\hat{S}_i^r$ be the set of supervoxels within radius r from s_i. It defines a sphere of interest, as shown in Fig. 2:

$$\hat{S}_i^r = \{\hat{s}_j = (w_j, \kappa) \mid \|w_j - w_i\| \leq r\}. \tag{1}$$

We will operate exclusively on the objects from this set.

Plane. The Cartesian coordinate system does not serve our purposes because it is cumbersome to find coordinates of planes that satisfy certain properties. The Spherical coordinate system, due to its non-symmetric coordinates, does not allow to divide regions into equal parts easily. We therefore introduce the coordinate system depicted by Fig. 1 that makes it is easy to define planes and split the space into regions of equal size, a requirement of the proposed Branch-and-Bound search procedure.

Let P_i be the set of all planes bisecting the image volume at the center of $\hat{s}_i$. Let us consider some $p \in P_i$ (Fig. 1). It intersects plane $X s_i Y$ along some line that can be characterised by a vector $\vec{v}_1 = [x_1, y_1, 0]$, that we can choose such that $x_1 > 0$ and $||\vec{v}_1|| = 1$. Similarly, let us consider the intersection of p with plane $Y s_i Z$ and characterise it by the vector $\vec{v}_2 = [0, y_2, z_2]$ with $y_2 > 0$ and $||\vec{v}_2|| = 1$. Let $\vec{y}_0 = [0, -1, 0]$, $\vec{z}_0 = [0, 0, -1]$. With this, we can define the plane p by two angles: $\phi = \sphericalangle(\vec{v}_1, \vec{y}_0) \in [0, \pi)$ and $\gamma = \sphericalangle(\vec{v}_2, \vec{z}_0) \in [0, \pi)$ (see Fig. 1). We will refer to the plane's angular coordinates as (ϕ, γ).

Sector. Suppose now that we have two planes: $p_{\min} = (\phi_{\min}, \gamma_{\min})$ and $p_{\max} = (\phi_{\max}, \gamma_{\max})$, where $\phi_{\min} < \phi_{\max}$ and $\gamma_{\min} < \gamma_{\max}$. We will call the area between them a sector and refer to it as $[p_{\min}, p_{\max}]$. For example, in Fig. 3 the sector is the area that includes green points, but not black ones. The sector can be seen as the convex hull of $\{p_{\min}, p_{\max}\}$ and any plane $p_0 = (\alpha_1\phi_{\min} + \beta_1\phi_{\max}, \alpha_2\gamma_{\min} + \beta_2\gamma_{\max})$, where $\alpha_1 + \beta_1 = 1, \alpha_2 + \beta_2 = 1$, is included in the sector $[p_{\min}, p_{\max}]$.

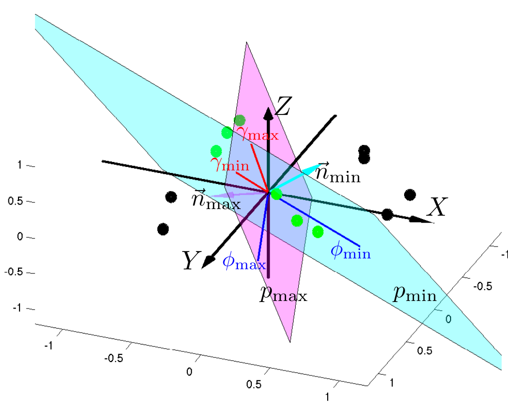

Figure 3. Sector. Sector is the area between $p_{\min}$ and $p_{\max}$. Points correspond to supervoxel centres. Green points are included into sector $[p_{\min}, p_{\max}]$ and black points are not included. $\vec{n}_{\min}$ and $\vec{n}_{\max}$ are normals to planes $p_{\min}$ and $p_{\max}$. Best seen in colour.

Uncertainty of a plane and a sector. Let us define by $C_i^r(p)$ the set of supervoxels $\hat{s}_j \in \hat{S}_i^r$ lying on p. Lying on p means that there exists a voxel q that belongs to the plane p and to $\hat{s}_j$.

$$C_i^r(p) = \{\hat{s}_j \mid \exists q : q \in \hat{s}_j, \hat{s}_j \in \hat{S}_i^r \text{ and } q \in p\}. \tag{2}$$

We associate to each $\hat{s}_j$ an uncertainty value $U(\hat{s}_j) \geq 0$ according to Sec. 4. Then, we define the uncertainty of a plane p as

$$U(p) = \sum_{\hat{s}_j \in C_i^r(p)} U(\hat{s}_j). \tag{3}$$

We take the uncertainty score of a sector to be the sum of uncertainty scores of the supervoxels lying between the two planes that define the sector. For a supervoxel to be included in the sector, it is enough for its center to lie between the planes or to be no further than 2κ away from any of them. To find the supervoxels enclosed between planes $p_{\min}$ and $p_{\max}$, we first find normal vectors to these planes $\vec{n}_{\min}$ and $\vec{n}_{\max}$ chosen such that they are pointing both inside or both outside of the sector as shown in Fig. 3. To ensure the orientation of the normals we examine their inner product $d = \vec{n}_{\min} \cdot \vec{n}_{\max}$. With acute angles of the sectors, to guarantee the orientation, it is enough to ensure that d is negative. Finally, the points enclosed between the planes should have the inner products of their centres with the two normal vectors either both positive or both negative (this will correspond to a set of points on the other side of the planes intersection) with offset 2κ:

$$\begin{aligned} &C_i^r([p_{\min}, p_{\max}]) = \\ &= \{\hat{s}_j \mid \hat{s}_j \in \hat{S}_i^r : \{\vec{w}_j \cdot \vec{n}_{\min} > -2\kappa \text{ and } \vec{w}_j \cdot \vec{n}_{\max} > -2\kappa\} \\ &\qquad \text{or } \{\vec{w}_j \cdot \vec{n}_{\min} < 2\kappa \text{ and } \vec{w}_j \cdot \vec{n}_{\max} < 2\kappa\}\}. \end{aligned} \tag{4}$$

Then the uncertainty of a sector is defined as sum of the uncertainties of supervoxels in $C_i^r([p_{\min}, p_{\max}])$

$$U([p_{\min}, p_{\max}]) = \sum_{\hat{s}_j \in C_i^r([p_{\min}, p_{\max}])} U(\hat{s}_j). \tag{5}$$

Optimisation problem. Recall that our target is to find a circular patch p^* of maximum uncertainty. Finally, given the above notation, this can be formulated as

$$p^* = (\phi^*, \gamma^*) = \arg\max_{p \in P_i} U(p). \tag{6}$$

B. Implementation

To find the optimal circular patch and solve Eq. (6), we use a Branch-and-Bound optimization approach. It involves evaluating entire subsets of the parameter space, *i.e.* ϕ and γ, using a bounding function and progressively reducing the search space. The optimal parameters are then attained when the evaluated subset is a singleton. As such, we now define a bounding function, a search procedure and then a termination condition.

Bounding function. Sectors can be treated as subsets of the parameter space and their properties allow us to define

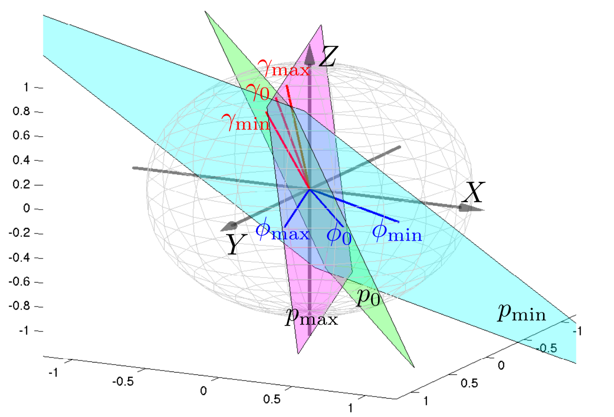

Figure 4. Sector splitting procedure. $U(p_0) < U([p_{\min}, p_{\max}])$. We split the sector $[p_{\min}, p_{\max}]$ into $[p_{\min}, p_0]$ and $[p_0, p_{\max}]$ and evaluate their uncertainty values. Among all available sectors we select a sector with the highest value to be split next. Best seen in colour.

bounding function. Consider any plane $p_0 = (\phi_0, \gamma_0) \in [p_{\min}, p_{\max}]$ as shown in Fig. 4. Given that $U(\hat{s}_j) \geq 0$ and that Eq. (6) is linear in $U(\hat{s}_j)$, the score of this plane will certainly be less or equal to the score of all the points included between two planes $p_{\min}$ and $p_{\max}$

$$U(p_0) \leq U([p_{\min}, p_{\max}]). \quad (7)$$

This observation allows us to bound the score of any plane on the top and to search for planes in the most promising parameter intervals.

Search procedure. We keep a priority queue L of sectors. At each step of the algorithm we remove the sector $[p^j_{\min}, p^j_{\max}]$ with the highest uncertainty $U([p^j_{\min}, p^j_{\max}])$ according to Eq. (5) and process it as follows. We divide each of the angles $\phi^j_{\min} s_i \phi^j_{\max}$ and $\gamma^j_{\min} s_i \gamma^j_{\max}$ into two by a bisector plane $p^j_0 = (\phi^j_0, \gamma^j_0)$, where $\phi^j_0 = (\phi^j_{\min} + \phi^j_{\max})/2$ and $\gamma^j_0 = (\gamma^j_{\min} + \gamma^j_{\max})/2$ as shown in Fig. 4. We compute the uncertainty of sectors $[p^j_{\min}, p^j_0]$ and $[p^j_0, p^j_{\max}]$ and add them to the priority queue L. Note, that we always operate on acute angles after the first iteration with initialisation $[0; \pi)$, that allows us to compute uncertainty scores of sectors as shown in Eq. (4) and Eq. (5).

Termination condition. The procedure of splitting areas is continued while the range of angles of interest: $\phi_{\max} - \phi_{\min}$ or $\gamma_{\max} - \gamma_{\min}$ is bigger than the angle that allows to fit a spherical voxel at the end of a segment as depicted by Fig. 5. This minimal angle is defined as

$$\alpha_{\min} = 2 \arctan \frac{\kappa}{r}. \quad (8)$$

It is possible to find a single plane that includes voxels

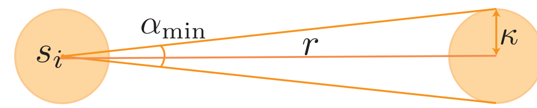

Figure 5. Minimal angle of interest. The search procedure is terminated when the angle of a sector becomes less than the angle allowing to fit a single supervoxel at the end of the sector.

from each of supervoxels in this sector $C^r_i([p_{\min}, p_{\max}])$.

C. Global optimization

Finally, recall that we performed all the operations described above for the most uncertain supervoxel s_i. If we consider the whole image stack, we would need to perform the search procedure for every possible supervoxel s_j, which would be prohibitively expensive. Thus, we restrict our search to t top supervoxels in the volume. We assume that the uncertainty scores are often consistent in small neighbourhoods, which is especially true for the Geometric uncertainty described in Sec. 4.2. It enables us to find a solution that is close to the optimal one with a low value of t. So, the final algorithm is the following: we take all supervoxels S with uncertainty U and fix t. Then, we find the best plane for each of the t top supervoxels and choose the best plane among them.