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SUBGROUPS OF SIMPLE ALGEBRAIC GROUPS CONTAINING REGULAR TORI, AND IRREDUCIBLE REPRESENTATIONS WITH MULTIPLICITY 1 NON-ZERO WEIGHTS

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Abstract. Our main goal is to determine, under certain restrictions, the maximal closed connected subgroups of simple linear algebraic groups containing a regular torus. We call a torus *regular* if its centralizer is abelian. We also obtain some results of independent interest. In particular, we determine the irreducible representations of simple algebraic groups whose non-zero weights occur with multiplicity 1.

1. Introduction

Let H be a simple linear algebraic group defined over an algebraically closed field F of characteristic $p \geq 0$. The closed subgroups of H containing a maximal torus, so-called subgroups of maximal rank, play a substantial role in the structure theory of semisimple linear algebraic groups. A naturally occurring family of maximal rank subgroups of H are subsystem subgroups; a *subsystem subgroup of H* is a closed semisimple subgroup G of H, normalized by a maximal torus of H. For such a subgroup G, the root system of G is, in a natural way, a subset of the root system of H. The main results of [32, 4] imply that every subgroup of maximal rank is either contained in a parabolic subgroup of H, or lies in the normalizer of a subsystem subgroup. Moreover, as described in [10, 3], we have a precise understanding of the subsystem subgroups of a given simple group H.

Our aim here is to generalize the Dynkin-Borel-De Siebenthal classification by replacing a maximal torus by a *regular torus*, that is, a torus whose centralizer in H is a maximal torus of H. If G is a closed subgroup of H which contains a regular torus of H, then the maximal tori of G are regular in H. Hence, we would like to determine the closed connected subgroups G of H whose maximal tori are regular in H. In its most general form, the question as stated is intractable. For instance, by Proposition 1, for H of type A_m the problem involves a classification of indecomposable representations of simple algebraic groups G, with the property

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that all weight spaces are one-dimensional, a result which is not available in case $\text{char}(F) > 0$. A tractable version of the above question, and the one which we consider and solve here, is the following:

Problem 1. Determine (up to conjugacy) the maximal closed connected subgroups G of a simple algebraic group H containing a regular torus of H.

The solution to Problem 1 must include of course listing all maximal subgroups of H of maximal rank. As mentioned above, the maximal connected, maximal rank subgroups are either parabolic or subsystem. (See [18, §13] for a detailed discussion.) Therefore, we will henceforth consider reductive subgroups G with $\text{rank}(G) < \text{rank}(H)$.

For a classical type group H with natural module V, a criterion for determining whether a torus in H is regular is given by the following proposition. Recall that the *multiplicity* of a weight in a representation is the dimension of the corresponding weight space.

Proposition 1. *Let H be a classical type simple algebraic group over F. Let V be the natural FH-module and let T be a (not necessarily maximal) torus in H. Then the following statements are equivalent:*

(1) *T is a regular torus in H;*
(2) *Either all T-weights on V have multiplicity 1, or H is of type D_m, and exactly one T-weight has multiplicity 2 and all other T-weights have multiplicity 1.*

The above proposition motivates our consideration of irreducible representations of simple algebraic groups G having at most one T-weight space of dimension greater than 1 (where T is now taken to be a maximal torus of G). Since all weights in a fixed Weyl group orbit occur with equal multiplicites, if there exists a unique weight occurring with multiplicity greater than 1, the weight must be the zero weight. Our next result gives a complete classification of irreducible FG-modules, both p-restricted (Thm. 2(1)) and non p-restricted (Thm. 2(2)), all of whose non-zero weights occur with multiplicity 1.

For the statement of the result we require an additional notation and definition. Let $X(T)$ denote the group of rational characters of T; for a dominant weight $\lambda \in X(T)$, let $L_G(\lambda)$ denote the irreducible FG-module of highest weight λ.

In the following we refer to Tables 1 and 2; all tables can be found in Section 8.

Theorem 2. *Let G be a simple algebraic group over F and let $T \subset G$ be a maximal torus.*

(1) *Let $\lambda \in X(T)$ be a non-zero dominant weight. If $\text{char}(F) = p > 0$, assume in addition that λ is p-restricted. All non-zero weights of $L_G(\lambda)$ are of multiplicity 1 if and only if λ is as in Table 1 or Table 2.*
(2) *Let $\mu \in X(T)$ be a non-zero dominant weight such that all non-zero weights of $L_G(\mu)$ are of multiplicity 1. Then either all weights of $L_G(\mu)$ have multiplicity 1, or $\mu = p^k\lambda$ for some integer $k \geq 0$ and λ is as in Table 1 or Table 2, where $k = 0$ if $\text{char}(F) = 0$.*

Note that Table 2 also contains the data on the multiplicity of the zero weight in $L_G(\lambda)$.

We will apply Proposition 1 and Theorem 2 to solve Problem 1 for H a classical group with subgroup of the form $\rho(G)$, where $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation of a simple algebraic group. In order to do so, for ρ of highest weight as in Theorem 2, we must determine whether $\rho(G)$ stabilizes a non-degenerate quadratic or alternating form on the associated module. This will be carried out in Section 5 for tensor-indecomposable representations. See Section 2 for a discussion of the general case.

The main result for the classical groups is then the following:

Theorem 3. *Let G be a simple algebraic group over F and $\rho : G \to \mathrm{GL}(V)$ an irreducible rational representation with p-restricted highest weight λ. Let $H \subset \mathrm{GL}(V)$ be the smallest simple classical group on V containing $\rho(G)$ and assume $\rho(G)$ contains a regular torus of H. Then the pairs (G, λ) are those appearing in Tables* 4, 5, 6, 7, 8, 9. *Moreover, each of the groups $\rho(G)$ in the cited tables, contains a regular torus of the classical group H.*

Observe that Theorem 3 includes the case of $\mathrm{char}(F) = 0$. Moreover, we do not require $\rho(G)$ to be maximal in H. The specific case of maximal subgroups of classical type groups is treated in Theorems 25 and 26.

Our result for the exceptional groups is the following:

Theorem 4. *Let H be an exceptional simple algebraic group over F and $M \subset H$ a maximal closed connected subgroup containing a regular torus of H. Then either M contains a maximal torus of H or the pair (M, H) is as in Table* 10. *Moreover, each of the subgroups M ocurring in Table* 10 *contains a regular torus of the simple algebraic group H.*

We expect to use our results for recognition of linear groups, and more generally, for recognition of subgroups of algebraic groups that contain an element of a specific nature. This is the principal motivation for our consideration of Problem 2 below. For the statement, we recall the following:

Definition 1. An element x of a connected reductive algebraic group H is said to be *regular* if $\dim C_H(x) = \mathrm{rank}(H)$.

Note that for a semisimple element $x \in H$, this is equivalent to saying that $C_H(x)^\circ$ is a torus, or equivalently that $C_H(x)$ contains no non-identity unipotent element. (See [25, Cor. 4.4] and [18, 14.7].) This is also equivalent to saying that $C_H(x)$ has an abelian normal subgroup of finite index (still for x semisimple).

We now apply the following standard result.

Lemma 5. [12, Prop. 16.4] *Let K be a connected algebraic group. A torus S of K is regular if and only if S contains a regular element.*

As every semisimple element of G belongs to a torus in G, Problem 1 is therefore equivalent to the following.

Problem 2. Let H be a simple algebraic group over F. Determine (up to conjugacy) all maximal closed connected subgroups G of H such that G contains a regular semisimple element of H.

Problem 2 can be viewed as a "recognition" result; that is, given a single element g of H described in convenient terms, determine the closed subgroups G of H containing g. There are many such results in the literature, both for simple algebraic groups H and for finite groups of Lie type. The interested reader might want to consult some of the references [34, 35], [36], [19], [33], [13], [8, 9], [21], [14], [28], [22], [29].

It is of course natural to consider Problem 2 for arbitrary regular elements (i.e., not necessarily semisimple). The following proposition, proven in Section 2, shows that the determination of subgroups containing a regular element of H can be reduced to the case of regular semisimple elements. We would like to thank the editors for providing a new proof, which is much simpler than our original proof.

Proposition 6. *Let H be a connected reductive group over F. Let $G \subset H$ be a closed connected reductive subgroup. If G contains a regular element of H then G contains a regular semisimple element of H.*

Note that Problem 2, where we replace "semisimple" by "unipotent" was already studied and solved in [22], [29], and [28]. We indicate in Section 7 how one can use these results and the results of the current manuscript to classify pairs (G, g), G a closed subgroup of H, $g \in G$ a regular element of H.

We now describe briefly our approach to the resolution of Problem 1, which differs according to whether H is of classical or of exceptional type. In the former case our strategy is to reduce Problem 1 to the recognition of linear representations of simple algebraic groups G whose weights satisfy certain specified properties. Denote by V the natural module for a classical type simple algebraic group H. Suppose first that H is of type A_m. Then G reductive maximal implies that G acts irreducibly on V, and (as alluded to above) the condition for a torus T of G to be a regular torus in H can be expressed in geometric terms. Specifically, T is a regular torus in H if and only if all T-weight spaces of V are one-dimensional. The embedding $G \to H$ can be viewed as a representation. Therefore, having reduced to the case G simple, one needs to determine the irreducible representations of simple algebraic groups all of whose weight spaces are one-dimensional. In this form, the problem we are discussing was considered in [23]. For his purposes, Seitz only needed infinitesimally irreducible representations satisfying the condition on weight spaces; his result was later extended to general irreducible representations in [37]. (We quote these results in Proposition 8 below.)

Now suppose that H is a classical group not of type A_m, still with natural module V. (Note that when H is of type B_n and $p = 2$, the natural module is a $(2n + 1)$-dimensional vector space equipped with a nondegenerate quadratic form and H is the derived subgroup of the isometry group of this form.) Again we view the embedding $G \to H$ as a representation of G. As G is assumed to be maximal among closed connected subgroups, and not containing a maximal torus of H, a direct application of [23, Thm. 3] shows that either G acts irreducibly on V or the pair (G, H) is $(B_{m-k}B_k, D_{m+1})$, for some $0 < k \leq m$. In the latter case, it is straightforward to show that G contains a regular torus of H; see Section 2 for details. Let now G be a closed subgroup of H, acting irreducibly on V and containing a regular torus of H. It is easier to determine such subgroups when H

is of type B_m or C_m due to the fact that a regular torus of H is regular in $\mathrm{SL}(V)$. Hence, we may refer to the previously mentioned classification of representations having 1-dimensional weight spaces. If $H = D_m$, we apply Proposition 1 and Theorem 2.

Now we turn to the case where H is of exceptional Lie type. In contrast with the classical group case, the classification of maximal positive-dimensional closed subgroups of H is explicit [15, Table 1]; we analyse the maximal connected subgroups which are not of maximal rank, and decide for which of these a maximal torus is regular in H. This is done in Proposition 27. Here our method uses a different aspect of representation theory than that used in the classical group case. It is based upon the following fact.

Proposition 7. *Let H be a connected algebraic group over F, with Lie algebra* $\mathrm{Lie}(H)$. *Let $T \subset H$ be a (not necessarily maximal) torus. The torus T is regular in H if and only if* $\dim C_{\mathrm{Lie}(H)}(T) = \mathrm{rank}(H)$.

This follows from the fact that $\mathrm{Lie}(C_H(S)) = C_{\mathrm{Lie}(H)}(S)$, for any torus $S \subset H$. (See [12, Prop. A. 18.4]). Now T lies in a maximal torus T_H of H and so $\mathrm{Lie}(T_H) = C_{\mathrm{Lie}(H)}(T_H) \subset C_{\mathrm{Lie}(H)}(T)$. Hence, T is regular in H if and only if $C_{\mathrm{Lie}(H)}(T) = \mathrm{Lie}(T_H)$. If G is explicitly given as a subgroup of H, one can determine the composition factors of the restriction of $\mathrm{Lie}(H)$ as FG-module; indeed this information is available in [15]. Next, for every composition factor, we determine the multiplicity of the zero weight. Then $C_{\mathrm{Lie}(H)}(T) = \mathrm{Lie}(T_H)$ if and only if the sum of these multiplicities equals the rank of H.

Notation. We fix the notation and terminology to be used throughout the paper. We write $\mathbb{N}_0$ for the set of non-negative integers, including 0, and $\mathbb{N}$ for the set $\mathbb{N}_0 \setminus \{0\}$. Let F be an algebraically closed field, of characteristic 0 or of prime characteristic $\mathrm{char}(F) = p > 0$. For a natural number $a \geq 1$, we write $p \geq a$ (respectively $p > a$, $p \neq a$) to mean that either $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) = p \neq 0$ and $p \geq a$ (resp. $p > a$, $p \neq a$). For a linear algebraic group X defined over F, we write X° for the connected component of the identity. All groups considered will be linear algebraic groups over F, and all subgroups will be closed subgroups of the ambient algebraic group.

Let G be a reductive algebraic group over F. All FG-modules are assumed to be rational, and we will not make further reference to this fact. We fix a maximal torus and Borel subgroup $T \subset B$ of G, the root system $\Phi(G)$ with respect to T, a set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ corresponding to B, the corresponding set of positive roots Φ^+, and the corresponding fundamental dominant weights $\{\omega_1, \dots, \omega_n\}$. Write $X(T)$ for the group of rational characters on T. Given a dominant weight $\lambda \in X(T)$, we write $L_G(\lambda)$ for the irreducible FG-module with highest weight λ, $W_G(\lambda)$ for the Weyl module of highest weight λ, and $\mathrm{rad}(W_G(\lambda))$ for the unique maximal submodule of the latter. A dominant weight $\lambda \in X(T)$ is said to be *p-restricted* if either $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) = p$ and $\lambda = \sum a_i\omega_i$ with $a_i < p$ for all i. Recall that a weight $\mu \in X(T)$ is said to be *subdominant* to λ if λ and μ are dominant weights and $\mu = \lambda - \sum a_i\alpha_i$ for some $a_i \in \mathbb{N}_0$. For an FG-module M and a weight $\mu \in X(T)$, we let M_μ denote the T-weight space corresponding to the weight μ. Set $W_G := N_G(T)/T$, the Weyl group of G and write s_i for the reflection in W_G corresponding to the simple root α_i. We label Dynkin diagrams as in Bourbaki [5].

When G is a classical type simple algebraic group, by the "natural" module for G we mean $L_G(\omega_1)$, unless G is of type B_n and $p = 2$, in which case the natural module is $W_G(\omega_1)$. We assume further that $n \geq 2$ if G is of type C_n and $n \geq 3$ if G is of type B_n. It is well-known that G preserves a non-degenerate symplectic or quadratic form on the natural module.

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2. Initial reductions

In this section, we prove Proposition 6, and therefore reduce the determination of connected reductive overgroups of regular elements to the consideration of regular semisimple elements. Then as remarked earlier, we will treat this by considering connected overgroups of regular tori.

The following proof was provided by one of the editors, and greatly simplifies the orginial proof. We are very thankful to the editor for communicating this improvement.

Proof of Proposition 6. Let H_{reg} denote the set of regular elements in H. This is an open set in H ([26, 5.4]) and hence $G \cap H_{reg}$ is open in G, and non-empty by hypothesis. As well, the set of semisimple elements in G is dense in G (see [12, Thm. 22.2]) and so intersects nontrivially $H_{reg} \cap G$. Hence G contains a semisimple regular element of H.

We give one further reduction for subgroups of classical groups containing regular tori. Let H be a simple algebraic group over F and let G be a maximal closed connected subgroup of H which contains a regular torus of H. If G is not reductive, then the main results of [32] and [4, §2] imply that G is a maximal parabolic subgroup of H and hence contains a maximal torus of H. Henceforth we will restrict our attention to connected reductive subgroups G.

Consider now the case where H is a classical type simple algebraic group with natural module V. We use the general reduction theorem, [23, Thm. 3], on maximal closed connected subgroups of H; this allows us to restrict our considerations to irreducibly acting subgroups of H. For a detailed discussion of this see [18, §18]. Precisely, an application of [18, Prop. 18.4] yields the following.

Let H be a classical group with natural module V and let G be maximal among closed connected subgroups of H. Then one of:

(1) G contains a maximal torus of H.
(2) H is of type A_m and G is of type $B_{\frac{m}{2}}$, $C_{\frac{m+1}{2}}$, or $D_{\frac{m+1}{2}}$.
(3) H is of type D_m and either $V = U \oplus U^{\perp}$, where U is an odd-dimensional non-degenerate subspace with respect to the bilinear form on V, $2 \leq \dim U \leq \dim V - 2$, and $G = \mathrm{Stab}_H(U)^{\circ}$, or $\mathrm{char}(F) = 2$ and G is the stabilizer in H of a non-singular 1-space of V.

(4) $V = V_1 \otimes V_2$, each of V_1 and V_2 is equipped with either the zero form (in case V has no non-degenerate H-invariant form), or a non-degenerate bilinear or quadratic form, and the form on V is obtained as the product form. Moreover, G is the connected component of $(\mathrm{Isom}(V_1) \otimes \mathrm{Isom}(V_2)) \cap H$. Note that if V is equipped with a quadratic form and $\mathrm{char}(F) = 2$, then $G = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$.

(5) G is simple acting irreducibly and tensor indecomposably on V.

In the first four cases, it is straightforward to show that G contains a regular torus of H. Hence we are reduced to considering simple subgroups which act irreducibly and tensor indecomposably on V.

3. Irreducible representations whose non-zero weights are of multiplicity 1

In this section, we determine the irreducible representations of simple algebraic groups all of whose non-zero weights have multiplicity 1, thereby establishing Theorem 2. Throughout this section we take G to be a simply connected simple algebraic group over F. The rest of the notation will be as fixed in Section 1. We inroduce the following notation.

Definition 2. Let G be a semisimple algebraic group with maximal torus T. We denote by $\Omega_2(G)$ the set of p-restricted dominant weights $\lambda \in X(T)$ such that all non-zero weights of $L_G(\lambda)$ have multiplicity 1, and by $\Omega_1(G)$ the set of weights $\lambda \in \Omega_2(G)$ such that all weights of $L_G(\lambda)$ have multiplicity 1.

As discussed in Section 1, we will require the following classification.

Proposition 8. *Let $\lambda \in X(T)$ be a non-zero dominant weight.*

(1) *Assume in addition that λ is p-restricted. Then all weights of $L_G(\lambda)$ are of multiplicity* 1 *if and only if λ is as in the second column of Table* 1. *In other words, the set $\Omega_1(G) \setminus \{0\}$ is as given in Table* 1.

(2) *Suppose that $p > 0$ and λ is not p-restricted, so $\lambda = \sum_{i=0}^{k} p^i \lambda_i$, where λ_i is p-restricted for all i and $\lambda_i \neq 0$ for some $i > 0$. Then all weights of $L_G(\lambda)$ are of multiplicity* 1 *if and only if the following hold:*

(a) *for all $0 \le l \le k$, $\lambda_l \in \Omega_1(G)$, and*

(b) *for all $0 \le l < k$,*

$(\lambda_l, \lambda_{l+1}) \neq (\omega_n, \omega_1)$ *if $p = 2$, $G = C_n$;*

$(\lambda_l, \lambda_{l+1}) \neq (\omega_1, \omega_n)$ *if $p = 2$, $G = B_n$;*

$(\lambda_l, \lambda_{l+1}) \neq (\omega_1, \omega_1)$ *if $p = 2$, $G = G_2$;*

$(\lambda_l, \lambda_{l+1}) \neq (\omega_2, \omega_1)$ *if $p = 3$, $G = G_2$.*

Proof. The result (1) follows from [23, 6.1] and [37]. Part (2) follows from [37, Prop.2], assuming $p \neq 2$ when $G = B_n$. For this exceptional case, we apply the isogeny $B_n \to C_n$ induced by the action of B_n on $W_{B_n}(\omega_1)/\mathrm{rad}(W_{B_n}(\omega_1))$ and the result of [37, Prop.2].

We now collect some results on dimensions of certain weight spaces in infinitesimally irreducible FG-modules.

Lemma 9. *Let $\lambda \in X(T)$ be a p-restricted dominant weight.*

(1) *If G is of type A_n and $\lambda = a\omega_j + b\omega_k$, with $1 \leq j < k \leq n$ and $ab \neq 0$, then the multiplicity of the weight $\lambda - \alpha_j - \alpha_{j+1} - \cdots - \alpha_k$ in $L_G(\lambda)$ is $k-j+1$ unless $p|(a+b+k-j)$, in which case the multiplicity is $k-j$.*

(2) *If G is of type A_n and $\lambda = c\omega_i$ for some $1 < i < n$ and $c > 1$, then the multiplicity of the weight $\lambda - \alpha_{i-1} - 2\alpha_i - \alpha_{i+1}$ in $L_G(\lambda)$ is 2 unless $c = p-1$ in which case the multiplicity is 1.*

(3) *If G is of type C_2 and $\lambda = a\omega_1 + b\omega_2$, with $ab \neq 0$, the multiplicity of the weight $\lambda - \alpha_1 - \alpha_2$ in $L_G(\lambda)$ is 2 unless $a + 2b + 2 \equiv 0 \pmod p$ in which case the multiplicity is 1.*

(4) *If G is of type B_n with $\lambda = \omega_1 + \omega_n$, then the weight $\lambda - \alpha_1 - \cdots - \alpha_n$ has multiplicity n in $L_G(\lambda)$, unless $p|(2n+1)$ in which case it has multiplicity $n-1$.*

(5) *If G is of type G_2 and $\lambda = a\omega_1 + b\omega_2$, with $ab \neq 0$, the multiplicity of the weight $\lambda - \alpha_1 - \alpha_2$ in $L_G(\lambda)$ is 2 unless $3a + b + 3 \equiv 0 \pmod p$ in which case the multiplicity is 1.*

(6) *If G is of type A_n and $\lambda = a\omega_i + b\omega_{i+1} + c\omega_{i+2}$, with $abc \neq 0$ and $a + b = p - 1 = b + c$, the weight $\lambda - \alpha_i - \alpha_{i+1} - \alpha_{i+2}$ has multiplicity at least 2 in the module $L_G(\lambda)$.*

(7) *If G is of type D_4 and $\lambda = a\omega_1$, with $a > 1$, then the weight $\lambda - 2\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$ has multiplicity at least 2 in $L_G(\lambda)$.*

(8) *If G is of type G_2, $\lambda = b\omega_1$ with $b > 1$, and $p > 3$, then the weight $\lambda - 2\alpha_1 - \alpha_2$ has multiplicity 2 in $L_G(\lambda)$.*

(9) *If G is of type G_2, $\lambda = a\omega_2$, with $p > 3$ and $a = \frac{p-1}{2}$, then the weight $\lambda - 2\alpha_1 - 2\alpha_2$ has multiplicity 2 in $L_G(\lambda)$.*

Proof. Part (1) is [23, 8.6], (3) and (5) are [30, 1.35]. For (2), see the proof of [23, 6.7] and apply the main result of [20]. Part (4) is [7, 2.2.7], when $p \neq 2$. For the case $p = 2$, use [23, 1.6]. The proof of (6) is contained in the proof of [23, 6.10]. Part (7) is proved in [23, 6.13]. Finally, the proofs of (8) and (9) follow from the proof in [23, 6.18].

We now turn our attention to the determination of the set $\Omega_2(G) \setminus \Omega_1(G)$. Let λ be a p-restricted dominant weight for the group G. It will be useful to work inductively, restricting the representation $L_G(\lambda)$ to certain subgroups and applying the following analogue of [23, 6.4].

Lemma 10. *Let X be a subsystem subgroup of G normalized by T. Let $\lambda \in \Omega_2(G)$. Let $L_X(\mu)$ be an FX-composition factor of $L_G(\lambda)$, for some dominant weight μ in the character group of $X \cap T$. Then $\mu \in \Omega_2(X)$.*

Proof. The argument is completely analogous to the proof of [23, Lemma 6.4]. Set $W := L_G(\lambda)$. Write $TX = XZ$, where $Z = C_T(X)^\circ$. Let $0 \subset M_1 \subset \cdots \subset M_t = W$ be an $F(XT)$-composition series of W. Then there exists i such that $L_X(\mu) \cong M_i/M_{i-1}$. Now $M_i = M_{i-1} \oplus M'$ as FT-modules, Z acts by scalars on M' and the set of $(T \cap X)$-weights in M' (and their multiplicities) are precisely the same as in $L_X(\mu)$. Also, if ν is a non-zero weight of $L_X(\mu)$, then ν corresponds to a non-zero T-weight of M'. Therefore if ν is a $(T \cap X)$-weight occurring in $L_X(\mu)$ with multiplicity greater than 1, there exists a T-weight ν' such that $\dim(M')_{\nu'} \geq 2$. So $\nu' = 0$ and hence $\nu = 0$. The result follows.

Proposition 11. *The set of weights $\Omega_2(G) \setminus \Omega_1(G)$ is as given in Table 2. Moreover, the multiplicity of the zero weight in $L_G(\lambda)$ is as indicated in the fourth column.*

Proof. We first note that for G and λ as in Table 2, the multiplicity of the zero weight in $L_G(\lambda)$ can be deduced from [17, Table 2]. We now show that the list in Table 2 contains all weights in $\Omega_2(G) \setminus \Omega_1(G)$. Let $\lambda \in \Omega_2(G) \setminus \Omega_1(G)$; in particular, 0 must be subdominant to λ, and so λ lies in the root lattice. We will proceed as in [23, §6]. We apply Lemma 10 to various subsystem subgroups of G; all of such are taken to be normalized by the fixed maximal torus T.

Case $\mathbf{A_3}$. Consider first the case where $\lambda = b\omega_2$. By the above remarks, $b > 1$, and so we have $p > 2$. If $b = 2$, the only weights subdominant to λ are $\lambda - \alpha_2$, which has multiplicity 1 in $L_G(\lambda)$ and $\mu = \lambda - \alpha_1 - 2\alpha_2 - \alpha_3$ which is the zero weight. Hence $\lambda \in \Omega_2(G)$ and by Lemma 9(2), $\lambda \in \Omega_1(G)$ if and only if $p = 3$. If $b > 2$, the weight μ is a non-zero weight and Lemma 9(2) implies that $b = p - 1$, in which case $\lambda \in \Omega_1(G)$.

Now consider the general case $\lambda = a\omega_1 + b\omega_2 + c\omega_3$. Assume for the moment that $abc \neq 0$. Applying Lemma 9(1), we see that $a + b = p - 1 = c + d$. But then Lemma 9(6) rules out this possibility. Hence we must have $abc = 0$. If $ab \neq 0$ as above we have $a + b = p - 1$ and $\lambda \in \Omega_1(G)$. The case $bc \neq 0$ is analogous. If $b = 0$ and $ac \neq 0$, Lemma 9(1) implies that the weight $\lambda - \alpha_1 - \alpha_2 - \alpha_3$ must be the zero weight and hence $a = 1 = c$. This weight appears in Table 2. Finally, if $b = c = 0$ or $a = b = 0$, then $\lambda \in \Omega_1(G)$. This completes the case G of type A_3.

Case $\mathbf{A_n}$, $\mathbf{n \neq 3}$. If $n = 2$, Lemma 9(1) and Proposition 8 give the result. So we now assume $n > 3$, and $\lambda = \sum_{i=1}^{n} b_i\omega_i$. We apply Lemma 10 to various A_3 Levi factors of G, as well as the result of Lemma 9(1) and (2). For example, for each set of consecutive simple roots $\{\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+k}\}$ with $k > 1$, $b_i b_{i+k} \neq 0$ and $b_j = 0$ for $i < j < i + k$, we deduce that $b_i = 1 = b_{i+k}$ and $i = 1$ and $i + k = n$. Further considerations of this type allow us to reduce to configurations of the form:

a) $\lambda = \omega_i$ for some i,
b) $\lambda = (p - 1)\omega_i$, for $1 < i < n$,
c) $\lambda = a\omega_i$, for $i = 1, n$,
d) $\lambda = c\omega_i + d\omega_{i+1}$, $cd \neq 0$, $1 \leq i < n$ and $c + d = p - 1$,
e) $\lambda = \omega_1 + \omega_n$.

Each of these weights is included either in $\Omega_1(G)$ or in Table 2.

Case $\mathbf{C_2}$. Let $\lambda = d\omega_1 + c\omega_2$. The arguments of [23, 6.11] together with Lemma 9(3) show that either $\lambda \in \Omega_1(G)$ or one of the following holds:

(a) $d = 0$, $c > 1$, $c \neq \frac{p-1}{2}$, and the weight $\lambda - 2\alpha_1 - 2\alpha_2$ is the zero weight.
(b) $c = 0$, $d > 1$, and the weight $\lambda - 2\alpha_1 - \alpha_2$ is the zero weight.
(c) $cd \neq 0$, $d > 1$, $2c + d + 2 \equiv 0 \pmod{p}$ and $\lambda - \alpha_1 - \alpha_2$ is the zero weight.

Case (i) is satisfied only if $c = 2$; (ii) is satisfied only if $d = 2$; (iii) is not possible.

Case $\mathbf{C_3}$. Let $\lambda = a\omega_1 + b\omega_2 + c\omega_3$. We apply Lemma 10 to three different C_2 subsystem subgroups of G, namely X_1, the Levi factor corresponding to the set

$\{\alpha_2, \alpha_3\}$, X_2, the conjugate of this group by the reflection s_1, and $X_3 = X_1^w$, where $w = s_1 s_2$. Restricting λ to X_1 gives that

$$(b, c) \in \{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, \frac{p-3}{2}), (0, \frac{p-1}{2})\}.$$

Note also that $\lambda|_{T \cap X_2}$ has highest weight $(a + b)\mu_1 + c\mu_2$, where μ_1, μ_2 are the fundamental dominant weights corresponding to the base $\{\alpha_1 + \alpha_2, \alpha_3\}$.

Suppose first that $c \neq 0$, so $b \in \{0, 1\}$. If $a + b < p$, we can again apply the C_2 result to the group X_2 to see that $a + b = 0$ or 1. On the other hand, if $a + b \geq p$, we must have $b = 1$ and $a = p - 1$. This latter case is not possible as Lemma 9(1) implies that the non-zero weight $\lambda - \alpha_1 - \alpha_2$ has multiplicity 2. Hence when $c \neq 0$, we have $(a, b, c) = (0, 0, c)$, or $(a, b, c) = (1, 0, c)$, or $(a, b, c) = (0, 1, \frac{p-3}{2})$. As the third possibility corresponds to a weight in $\Omega_1(G)$, we consider the first two possibilities. If $(a, b, c) = (0, 0, c)$, by Proposition 8, we may assume $c \neq \frac{p-1}{2}$, and $c \neq 1$. This leaves us with the weight $2\omega_3$, and $p \neq 5$. But then [17] shows that a non-zero weight has multiplicity greater than 1. If $(a, b, c) = (1, 0, c)$, then $c \in \{1, 2, \frac{p-1}{2}\}$. The restriction of λ to the subgroup X_2 is $\mu_1 + c\mu_2$. But here Lemma 9(3) shows that the weight $\mu = \lambda - (\alpha_1 + \alpha_2) - \alpha_3$ has multiplicity 2 unless $c = \frac{p-3}{2}$. Since μ is a non-zero weight, either $c = 1$ and $p = 5$ or $c = 2$ and $p = 7$. Again, we refer to [17] to see that there is a non-zero weight with multiplicity greater than one in each case.

Suppose now $c = 0$. Then the restriction to X_1 implies that (a, b, c) is one of $(a, 1, 0)$, $(a, 2, 0)$, $(a, 0, 0)$. Suppose $(a, b, c) = (a, 1, 0)$. If $a = 0$, then the only subdominant weight in $L_G(\lambda)$ is the 0 weight and hence this gives an example. If $a \neq 0$, Lemma 9(1) implies that $a = p - 2$. Then the restriction of λ to X_3 is the weight $a\eta_1 + \eta_2$, where η_1, η_2 are the fundamental dominant weights corresponding to the base $\{\alpha_1, 2\alpha_2 + \alpha_3\}$. But then Lemma 9(3) implies that the non-zero weight $\lambda - \alpha_1 - 2\alpha_2 - \alpha_3$ occurs with multiplicity 2. If $(a, b, c) = (a, 2, 0)$, then the subdominant weight $\lambda - 2\alpha_2 - \alpha_3$ has multiplicity 2 and hence this is not an example. Finally, if $(a, b, c) = (a, 0, 0)$, we consider the restriction of λ to the subgroup X_3 and the C_2 result implies that $a = 1$ or $a = 2$. If $a = 1$, then $\lambda \in \Omega_1(G)$, while if $a = 2$, [17, Table 2] shows that $\lambda \in \Omega_2(G)$. This completes the consideration of the case $G = C_3$.

Case $\mathbf{C_n}$, $\mathbf{n \geq 4}$. If $\lambda = \sum a_i\omega_i$ with $a_i = 0$ for $i \leq n - 2$, then the C_3 result and Lemma 10 (applied to the standard C_3 Levi factor) implies that either λ is one of the weights in Table 1 or in Table 2, or $\lambda = \omega_{n-1}$ or $\lambda = \omega_n$, with $p \neq 3$ in each case. If $\lambda = \omega_{n-1}$, then we refer to [17], for the group C_4, to see that the subdominant weight ω_{n-3} occurs with multiplicity 2. This then shows that $\lambda = \omega_{n-1} \notin \Omega_2(G)$. Now if $\lambda = \omega_n$, again use [17] and find that $\lambda \in \Omega_2(G)$ when $n = 4$, while if $n > 4$, the weight ω_{n-4} occurs with multiplicity 2 and so $\lambda \notin \Omega_2(G)$.

We may now assume that there exists $i \leq n - 2$ with $a_i \neq 0$. Choose $i \leq n - 2$ maximal with $a_i \neq 0$ and consider the C_3 subsystem subgroup X with root system base $\{\alpha_i + \cdots + \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$, so that $\lambda|_{T \cap X} = a_i\eta_1 + a_{n-1}\eta_2 + a_n\eta_3$, where $\{\eta_1, \eta_2, \eta_3\}$ are the fundamental dominant weights corresponding to the given base. As $a_i \neq 0$, the C_3 case considerations imply that $a_{n-1} + a_n = 0$ and $a_i = 1$ or 2. If $i = 1$, λ occurs in the statement of the result. If $i = 2$, so $\lambda = a_1\omega_1 + a_2\omega_2$, then we may assume $a_1 \neq 0$, or $a_2 = 2$, as $\lambda = \omega_2$ occurs in the statement of the result. But then the restriction of λ to the C_3 subsystem subgroup with root system base $\{\alpha_1, \alpha_2 + \cdots + \alpha_{n-1}, \alpha_n\}$ has non-zero weights occurring with multiplicity greater than 1.

So finally, we may assume $i > 2$, and so $n \geq 5$. If $n = 5$ and so $i = 3$, we apply the result for C_4 to see that $a_2 = 0$ and $a_3 = 1$. But the non-zero weight $\lambda - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$ has multiplicity at least 2 by the C_4 result. Hence the result holds for the case $n = 5$. If $n \geq 6$, we consider the C_5 subsystem subgroup whose root system has base $\{\alpha_{i-2}, \alpha_{i-1}, \alpha_i + \cdots + \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ and obtain a contradiction.

Case $\mathbf{D_n}$. Suppose first that $n = 4$. Let $\lambda = \sum a_i\omega_i$. If $a_2 = 0$, Lemma 9(1) implies that $\lambda = a_i\omega_i$ for $i = 1$, 3 or 4; assume by symmetry that $i = 1$. Then Lemma 9(7) shows that either $a_1 = 1$ or $\lambda - 2\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$ is the zero weight. In either case, λ is as in the statement of the result. So we now assume $a_2 \neq 0$. Using the result for A_3, applied to the three standard A_3 Levi factors, we see that at most one of a_1, a_3, a_4 is non-zero. However if $a_1 + a_3 + a_4 \neq 0$, Lemma 9(1) and (2) provide a contradiction. Hence $\lambda = a_2\omega_2$, and since $\lambda = \omega_2$ is in Table 2, we may assume $a_2 > 1$. Now $\lambda - \alpha_1 - 2\alpha_2 - \alpha_3$ is a non-zero weight, so Lemma 9(2) implies that $a_2 = p - 1$. But we now apply Lemma 10 to the A_3 subsystem subgroup with root system having as base $\{\alpha_2, \alpha_1, \alpha_2 + \alpha_3 + \alpha_4\}$ to obtain a contradiction.

Now consider the general case where $n > 4$. We argue by induction on n. Apply the result for D_{n-1} to the standard D_{n-1} Levi factor of G to see that either λ is as in the statement of the result or $\lambda = a\omega_1$, $a\omega_1 + \omega_2$, $a\omega_1 + \omega_{n-1}$, $a\omega_1 + \omega_n$, $a\omega_1 + \omega_3$, or $a\omega_1 + 2\omega_2$. Now consider the D_4 subsystem subgroup whose root system has base $\{\alpha_1, \alpha_2 + \cdots + \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$. The result for D_4 then implies that either λ appears in Table 1 or Table 2, or $\lambda = \omega_3$. But then the restriction of λ to the standard D_{n-1} Levi factor X affords a composition factor which is the unique nontrivial composition factor of Lie(X). The weight corresponding to the zero weight in Lie(X) is the weight $\lambda - \alpha_1 - 2(\alpha_2 + \cdots + \alpha_{n-2}) - \alpha_{n-1} - \alpha_n$, which is a non-zero weight in $L_G(\lambda)$ with multiplicity at least $n - 3$. (See [17, Table 2].) This completes the consideration of type D_n.

Case $\mathbf{B_n}$. If $p = 2$, we may deduce the result from the case of $G = C_n$; hence we assume for the remainder of this case that $p \neq 2$. Consider first the case $n = 3$. We apply the result for C_2 to the standard C_2 Levi factor of G and the A_3 result to the subsystem subgroup X with root system base $\{\alpha_2 + 2\alpha_3, \alpha_1, \alpha_2\}$. Note that if $\lambda = a\omega_1 + b\omega_2 + c\omega_3$, then the restriction of λ to X affords a composition factor with highest weight $(b + c)\eta_1 + a\eta_2 + b\eta_3$, where $\{\eta_1, \eta_2, \eta_3\}$ is the set of fundamental dominant weights dual to the given base. We deduce that either λ appears in Table 1 or in Table 2 or $\lambda = \omega_1 + \omega_3$, $2\omega_3$, $(p - 3)\omega_1 + 2\omega_3$, or $(p - 2)\omega_1 + \omega_3$. The first case is ruled out by Lemma 9(4). For the second and third, where $p > 2$, we see that the restriction of λ to the standard C_2 Levi factor affords a composition factor isomorphic to the Lie algebra of the Levi factor, in which the non-zero weight $\lambda - \alpha_2 - 2\alpha_3$ has multiplicity 2. For the final case, when $\lambda = (p - 2)\omega_1 + \omega_3$, consider the C_2 subsystem subgroup X^{s_1}, with root system base $\{\alpha_1 + \alpha_2, \alpha_3\}$, for which λ affords a composition factor with highest weight $(p-2)\zeta_1 + \zeta_2$ (where ζ_1, ζ_2 are the fundamental dominant weights with respect to the base $\{\alpha_1 + \alpha_2, \alpha_3\}$), contradicting Lemma 10. This completes the consideration of $G = B_3$.

Consider now the general case $n \geq 4$. Let $\lambda = \sum a_i\omega_i$. By considering the restriction of λ to the standard B_3 Levi subgroup, we see that $a_{n-1} + a_n < p$. Now consider the maximal rank D_n subsystem subgroup X with root system base $\{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$. The above remarks imply that $\lambda|_{T\cap X}$ is a p-restricted weight and the result for D_n then gives the result for B_n.

Case $\mathbf{E_n}$. For $n = 6$, we apply the D_n result to the two standard D_5 Levi subgroups of G and the A_n result to the standard A_5 Levi subgroup of G to see that either λ is as in Table 2 or $\lambda = \omega_3$, ω_5, $2\omega_1$ or $2\omega_6$. In the first two cases, the restriction of λ to one of the D_5 Levi factors affords a composition factor which is isomorphic to the unique nontrivial composition factor of the Lie algebra of the Levi factor. But the zero weight in this composition factor corresponds to a non-zero weight with multiplicity at least 4. In the last two configurations, we note that the D_5 composition factor afforded by λ has zero as a subdominant weight of multiplicity at least 3 (here we use [16]). But this subdominant weight is a non-zero weight with respect to T.

Now for $n = 7$, we apply the result for E_6 as well as for D_6 and A_6 to see that either λ is as in the statement of the result or $\lambda = \omega_6$ or $2\omega_7$. These two configurations can be ruled out exactly as in the case of E_6. The case of $G = E_8$ is completely analogous.

Case $\mathbf{F_4}$. For this case, we use the standard B_3 and C_3 Levi factors and the maximal rank D_4 subsystem subgroup whose root system base is $\{\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_2, \alpha_1, \alpha_2 + 2\alpha_3\}$. This leads immediately to the result.

Case $\mathbf{G_2}$. Let $\lambda = b\omega_1 + a\omega_2$, where $3(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$. We first treat the cases where $p = 2$ or $p = 3$ by referring to [16] to see that λ is as in Table 2 or $\lambda = 2\omega_1 + 2\omega_2$ and $p = 3$. But then Lemma 9(5) shows that the non-zero weight $\lambda - \alpha_1 - \alpha_2$ has multiplicity 2. So we may now assume $p > 3$. We will consider the restriction of λ to X, the A_2 subsystem subgroup corresponding to the long roots in $\Phi(G)$; we have $\lambda|_{T\cap X} = (a+b)\eta_1 + a\eta_2$, where $\{\eta_1, \eta_2\}$ are the fundamental dominant weights for X.

Assume that λ is not as in Table 2, so $\lambda \neq \omega_i$, $i = 1, 2$. Now Lemma 9(8) implies that $a \neq 0$. Consideration of the action of X, together with Lemma 9(1), implies that either $a + b \geq p$ or $2a + b + 1 \equiv 0 \pmod p$. Now if $b = 0$, we must be in the second case, so $a = \frac{p-1}{2}$; in particular, $a > 1$. But then Lemma 9(9) gives a contradiction. Hence we must have $b \neq 0$. Then Lemma 9(5) implies that $3a + b + 3 \equiv 0 \pmod p$ and either $2a + b + 1 \equiv 0 \pmod p$ or $a + b \geq p$. In the first case we have $a = p - 2$ and $b = 3$; in the second case we must have $b \geq 2$. So in either case $\lambda - 2\alpha_1 - \alpha_2$ is a subdominant weight. Arguing as in [23, 6.18], we see that the non-zero weight $\lambda - 2\alpha_1 - \alpha_2$ has multiplicity 1 only if $6a + 4b + 8 \equiv 0 \pmod p$, which together with the previous congruence relation implies that $b = p - 1$, and hence $3a + 2 \equiv 0 \pmod p$. We are now in the situation where $a + b \geq p$; indeed, as $a = \frac{p-2}{3}$ or $\frac{2p-2}{3}$, we have that either $p = 5$, $a = 1$, $b = 4$ and [16] gives a contradiction, or $\lambda|_{T\cap X} = p\eta_1 + (c\eta_1 + d\eta_2)$, with $cd \neq 0$, $(c, d) = (\frac{p-5}{3}, \frac{p-2}{3})$ or $(\frac{2p-5}{3}, \frac{2p-2}{3})$. In each case, the result for A_2 leads to a contradiction.

It remains to verify for each weight λ in Table 2 that all non-zero weights of $L_G(\lambda)$ do indeed have multiplicity 1. This is straightfoward using [17] and [16].

The following corollary is immediate.

Corollary 12. *Let $\lambda \in X(T)$ be dominant and p-restricted. If all non-zero weights of $L_G(\lambda)$ occur with multiplicity* 1*, then the zero weight occurs with multiplicity at most* $\mathrm{rank}(G)$.

Note that bounds for the maximal weight multiplicities in irreducible representations of G are studied in [1, 2]. The above corollary does not however follow from their results.

Corollary 13. *Let $\lambda \in \Omega_2(G)$. Then one of the following holds:*

(1) *$\lambda \in \Omega_1(G)$.*
(2) *$L_G(\lambda)$ is the unique nontrivial composition factor of* $\mathrm{Lie}(G)$.
(3) *(G, λ) is one of $(A_3, 2\omega_2)$ with $p > 3$, $(B_n, 2\omega_1)$, (C_n, ω_2) with $n > 2$ and $p \neq 3$ if $n = 3$, (B_n, ω_2) with $p = 2$, $(C_2, 2\omega_2)$ with $p \neq 5$, (C_4, ω_4) with $p > 3$, $(D_n, 2\omega_1)$ with $n > 3$, or (F_4, ω_4) with $p \neq 3$.*

Proof. The statement about $\mathrm{Lie}(G)$ follows from the known structure of the FG-module $\mathrm{Lie}(G)$; see for example [23, 1.9].

We record in the following corollary the cases where the 0 weight has multiplicity 2 in $L_G(\lambda)$, for $\lambda \in \Omega_2(G)$. This will be required for the resolution of Problem 1 in case $H = D_n$.

Corollary 14. *Let $\lambda \in \Omega_2(G)$ and suppose that the zero weight has multiplicity* 2 *in $L_G(\lambda)$. Then the pair (G, λ) is as in Table* 3.

We can now complete the proof of Theorem 2.

Theorem 15. *Let $\lambda \in X(T)$ be a non-zero dominant weight. Then at most one weight space of $L_G(\lambda)$ is of dimension greater than one if and only if one of the following holds:*

(1) *the module $L_G(\lambda)$ is as described in Proposition* 8*;*
(2) *$\lambda = p^k\mu$ for some $k \in \mathbb{N}_0$, $k = 0$ if* $\mathrm{char}(F) = 0$*, and for some weight $\mu \in \Omega_2(G) \setminus \Omega_1(G)$, as given in Table* 2.

Proof. By Propositions 8 and 11, for the modules $L_G(\lambda)$ described in (1) and (2), all non-zero weights are of multiplicity at most 1. So now take $\lambda \in X(T)$ a non-zero dominant weight such that at most one weight space of $L_G(\lambda)$ has dimension greater than 1, and suppose λ is not as in (1). In particular, $L_G(\lambda)$ has a weight of multiplicity greater than 1. Note that if a non-zero weight in $L_G(\lambda)$ occurs with multiplicity greater than 1, then so do all of its conjugates under the Weyl group. Hence, the weight occurring with multiplicity greater than one in $L_G(\lambda)$ must be the zero weight.

Now, let $\lambda = \sum_{i=1}^{l} p^{k_i}\lambda_i$, where λ_i is a non-zero dominant p-restricted weight for all i; so we have $L_G(\lambda) \cong L_G(p^{k_1}\lambda_1) \otimes \cdots \otimes L_G(p^{k_l}\lambda_l)$. If $l = 1$ then (2) holds. Let $l > 1$. Then for all $1 \leq i \leq l$, the weights of $L_G(p^{k_i}\lambda_i)$ must have multiplicity 1 (else a non-zero weight has multiplicity greater than 1 in $L_G(\lambda)$). Then by Proposition 8, we see that there exists a pair (λ_i, λ_j) with $k_j = k_i + 1$, as in Proposition 8(2)(b). So we investigate the multiplicity of the zero weight in the tensor products associated to these pairs of weights.

Let $L_G(\lambda) = L_G(\lambda_1) \otimes L_G(p\lambda_2)$, where $(G, p, \lambda_1, \lambda_2)$ is one of the following: (a) $(C_n, 2, \omega_n, \omega_1)$, (b) $(B_n, 2, \omega_1, \omega_n)$, (c) $(G_2, 2, \omega_1, \omega_1)$, or (d) $(G_2, 3, \omega_2, \omega_1)$. Let ε_i be the weights defined as in [5, Planche II, III, IX].

In case (a), the weights of $L_G(\omega_n)$ are $\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n$ and the weights of $L_G(2\omega_1)$ are $\pm 2\varepsilon_1 \ldots \pm 2\varepsilon_n$; in particular, there are no common weights. It follows that the 0 weight does not occur in $L_G(\lambda_1) \otimes L_G(2\lambda_2)$.

In case (b), the weights of $L_G(\omega_1)$ are $\pm\varepsilon_i$ and 0 and the weights of $L_G(2\omega_n)$ are $\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_n$ and we conclude as in the previous case.

In case (c), the weights of $L_G(\omega_1)$ are $\{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3)\}$. So $L_G(\omega_1)$ and $L_G(2\omega_1)$ again have no common weight, and hence the 0 weight does not occur in $L_G(\omega_1) \otimes L_G(2\omega_1)$.

Finally for case (d), the weights of $L_G(3\omega_1)$ are $\{0, \{\pm 3(\varepsilon_1 - \varepsilon_2), \pm 3(\varepsilon_1 - \varepsilon_3), \pm 3(\varepsilon_2 - \varepsilon_3)\}$, and the weights of $L_G(\omega_2)$ are $0, \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3)$. Then the multiplicity of the weight 0 in $L_G(\lambda_1) \otimes L_G(3\lambda_2)$ is 1.

It follows from Proposition 8 that the weights of $L_G(\lambda_1) \otimes L_G(p\lambda_2)$ whose multiplicity is greater than 1 are non-zero. Hence there are no examples of $\lambda = \sum_{i=1}^{l} p^{k_i}\lambda_i$ with $l > 1$ and all λ_i different from 0 such that the zero weight has multiplicity greater than 1 and all non-zero weights have multiplicity 1 in the module $L_G(\lambda)$.

4. Semisimple regular elements in classical groups

In this section, we prove Proposition 1. Throughout, H is a simply connected simple algebraic group of classical type, with natural module V. Let $\tilde{H}$ be the image of H in $\mathrm{GL}(V)$. Note that $x \in H$ is regular if and only if the image of x in $\tilde{H}$ is regular in $\tilde{H}$.

Lemma 16. *Let H be of type A_m, B_m, C_m or D_m, and assume $p \neq 2$ if H has type B_m. Let $t \in \tilde{H}$ be a semisimple regular element. Then t is a regular element in $\mathrm{GL}(V)$, except for the following cases:*

(1) *H is of type B_m and -1 is an eigenvalue of t on V with multiplicity 2;*
(2) *H is of type D_m and at least one of 1 and -1 is an eigenvalue of t on V with multiplicity 2.*

Proof. The result is trivial if H is of type A_m, so we assume that V is equipped with a nondegenerate symplectic or quadratic form and that $\tilde{H} \subset \mathrm{Isom}(V)$ is the corresponding simple algebraic group. Let $d = \dim V$, and let $b_1, \ldots, b_d$ be a basis of V with respect to which the Gram matrix of the associated bilinear form is anti-diagonal, with all non-zero entries in the set $\{1, -1\}$. We may assume the matrix of t with respect to this basis is diagonal and of the form $t = \mathrm{diag}(t_1, \ldots t_m, x, t_m^{-1}, \ldots, t_1^{-1})$, where x is absent if d is even and equal to 1 otherwise. Since t is regular in H, $t_1, \ldots, t_m$ are distinct. Moreover, if H is of type C_m then all $t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}$ are distinct. Indeed, the roots of C_m take values $\{t_i t_j^{-1}, t_i t_j, t_j^2 \mid 1 \le i \ne j \le m\}$ and t lies in the kernel of no root. If H is of type B_m, $m > 2$, then the roots of H take values $\{t_i t_j^{-1}, t_i t_j, t_j \mid 1 \le i \ne j \le m\}$ on t. So no t_i is equal to 1 and therefore the eigenvalues $t_1, \ldots, t_m, x, t_m^{-1}, \ldots, t_1^{-1}$ are distinct except for the case (1). Finally, for H of type D_m, we argue as above, using the fact that the roots of H take values $\{t_i t_j^{-1}, t_i t_j \mid 1 \le i \ne j \le m\}$ on t, which leads to (2).

Lemma 17. *Let H be of type A_m, B_m, C_m or D_m, and let T' be a regular torus in $\tilde{H}$. Then T' is a regular torus in* $\mathrm{GL}(V)$*, except for the case where H is of type D_m and the fixed point subspace of T' on V is of dimension* 2.

Proof. Appying Lemma 5, we choose $t \in T'$ such that $C_H(t)^\circ = C_H(T')$. By Lemma 16, t is regular in $\mathrm{GL}(V)$ if H is of type A_m or C_m, and so the result holds. If $\dim V$ is odd, that is, if H is of type B_m, let T be a maximal torus of H. Then for all T-weights μ, ν of V, the difference $\mu - \nu$ is a multiple of a root of H (with respect to T). Since T' is a regular torus in $\tilde{H}$, $T' \not\subset \ker(\beta)$ for any root β of H. Hence T' has distinct weights on V, and so is a regular torus in $\mathrm{GL}(V)$.

Finally, consider the case $H = D_m$. Let $b_1 \ldots b_{2m}$ be a basis for V as in the previous proof. We may assume that T' consists of diagonal matrices with respect to this basis. Now, T' regular implies that the weight of T' afforded by $\langle b_j \rangle$ is distinct from the weight afforded by $\langle b_{2m-k+1} \rangle$ for all $k \notin \{j, 2m-j+1\}$. So the weights of T' on V are distinct unless there exists j, $1 \le j \le m$, such that the weight afforded by $\langle b_j \rangle$ is equal to that afforded by $\langle b_{2m-j+1} \rangle$. So assume these two weights coincide. Since the weights afforded by these two vectors differ by a sign, if they are equal, they must be 0. Hence there is a unique such j and the result holds.

Proposition 18. *Let H be of type D_m with $m \ge 4$. Let T be a torus in $\tilde{H}$. Suppose that there exists at most one T-weight of V of multiplicity greater than* 1*, and if such a weight exists, its multiplicity is* 2*. Then $C_{\tilde{H}}(T)$ is a maximal torus in $\tilde{H}$, that is, T is a regular torus in $\tilde{H}$.*

Proof. If all weights of T on V have multiplicity 1, then T is a regular torus in $\mathrm{GL}(V)$, and hence in $\tilde{H}$. Suppose that there is a weight μ of T on V of multiplicity 2, and let M be the corresponding weight space. Denote by R the set of singular vectors in M together with the zero vector.

If $R = M$ then M is totally singular, $M \subset M^\perp$, and $V|_T = M^\perp \oplus V_2$. It is well-known that the FT-module V_2 is dual to M, and hence is contained in a T-weight space. This is a contradiction as $\dim V_2 = 2$ and by hypothesis M is the unique weight space of dimension greater than 1.

Suppose that M is neither totally singular nor non-degenerate with respect to the bilinear form on V. Then R is a 1-dimensional T-invariant subspace of M and $M \subset R^\perp$. Let $M|_T = R \oplus R'$ for a T-submodule R'. Then non-zero vectors $x \in R'$ are non-singular, and if $t \in T$ then $tx = \mu(t)x$. Let Q be the quadratic form on V preserved by $\tilde{H}$. Then $Q(x) = Q(tx) = \mu(t)^2 Q(x)$, for all $t \in T$, whence $\mu = 0$. Therefore, T acts trivially on M, and hence on R. However, $V/R^\perp$ is an FT-module dual to R, so T acts trivially on $V/R^\perp$. As V is a completely reducible

FT-module, it follows that the dimension of the zero T-weight space on V is at least 3, which contradicts the hypothesis.

Finally, suppose that M is non-degenerate with respect to the bilinear form on V. The reductive group $C_{\tilde{H}}(T)$ stabilizes all T-weight spaces of V. Since $\mathrm{SO}_2(F)$ has no unipotent elements, $C_{\tilde{H}}(T)$ must be a torus, as required.

Proof of Proposition 1. The Proposition follows directly from Lemma 17 and Proposition 18.

As explained in Section 1 and Section 2, for a classical group H with natural module V, Proposition 1 reduces the resolution of Problems 1 and 2 to a question about weight multiplicities in the module $V|_G$, for a simple algebraic group G acting irreducibly and tensor-indecomposably on V. Given a simple subgroup G of H, we see that a maximal torus T of G is a regular torus of H if and only if one of the following holds:

a) all T-weight spaces of V are 1-dimensional, and so V is described by Proposition 8, or
b) $G \subset H = D_m$ and the 0 weight space of V (viewed as a T-module) is 2-dimensional, while all other T-weight spaces of V are 1-dimensional, so V is described by Theorem 15 and Proposition 11.

5. Orthogonal and symplectic representations

As in the previous sections, we take G, T and the rest of the notation to be as fixed in Section 1. In this section we partition the irreducible representations ρ of G with highest weight in $\Omega_1(G)$ into four families depending on whether $\rho(G)$ is contained in a group of type B_m, C_m, D_m or in none of them. (It is well-known that the latter holds if and only if the associated FG-module is not self-dual.) In addition, we determine which of the weights in $\Omega_2(G)$ correspond to a representation whose image contains a regular torus of $H = D_m$. This information is collected in Tables 4, 5, 6, 7, 8, 9, and completes the proof of Theorem 3. For simplicity, we will say that an irreducible representation of G (or the corresponding module V) is *symplectic*, respectively *orthogonal*, if G preserves a non-degenerate alternating form, respectively a non-degenerate quadratic form on V. As discussed in Section 2, for our application, it suffices to consider tensor-indecomposable modules.

Until the end of the section $\lambda \in X(T)$ is assumed to be a p-restricted dominant weight. Recall that the highest weight of the irreducible FG-module $L_G(\lambda)^*$ (the dual of $L_G(\lambda)$), is $-w_0\lambda$, where w_0 is the longest word in the Weyl group of G. Since $-w_0 = \mathrm{id}$ for the groups A_1, B_n, C_n, D_n, (n even), E_7, E_8, F_4, and G_2, all irreducible modules are self-dual for these groups. (See [18, 16.1].) The following result, whose entirely straightfoward proof is omitted, treats the remaining cases.

Proposition 19. *Let G be of type A_n, $n > 1$, D_n, n odd, or E_6. Let $\lambda \in \Omega_2(G)$. Then $L_G(\lambda)$ is a self-dual FG-module if and only if one of the following holds:*

(1) *$G = A_n$, $n > 1$, and either*
(i) *$\lambda = \omega_1 + \omega_n$, or*
(ii) *$\lambda = \omega_{(n+1)/2}$, for n odd, or*

(iii) $\lambda = (p-1)\omega_{(n+1)/2}$, *for n odd, or*
(iv) $\lambda = \frac{p-1}{2}(\omega_{n/2} + \omega_{(n+2)/2})$, *for n even and p odd, or*
(v) $\lambda = 2\omega_2$ *for* $n = 3$.
(2) $G \cong D_n$, *n odd, and* $\lambda \in \{\omega_1, \omega_2, 2\omega_1\}$*;*
(3) $G \cong E_6$ *and* $\lambda = \omega_2$.

Lemma 20. (1) *Let* $G = A_n$, $p > 2$ *and let* $\lambda = (p-1)\omega_{(n+1)/2}$ *for n odd, and* $\lambda = \frac{p-1}{2}(\omega_{n/2} + \omega_{(n+2)/2})$ *for n even. Then* $\dim L_G(\lambda)$ *is odd.*
(2) *Let* $G = A_n$, *n odd, and* $\lambda = \omega_{(n+1)/2}$. *Then* $\dim L_G(\lambda) = \binom{n+1}{(n+1)/2}$ *and so* $\dim L_G(\lambda)$ *is even.*
(3) *Let* $G = C_n$, $n > 1$, $p > 2$ *and let* $\lambda_1 = \frac{p-1}{2}\omega_n$, $\lambda_2 = \omega_{n-1} + \frac{p-3}{2}\omega_n$. *Then* $\dim L_G(\lambda_1) = (p^n+1)/2$ *and* $\dim L_G(\lambda_2) = (p^n-1)/2$.

Proof. (1) Let $B := \{(c,i) \mid 0 \le c \le p-1, 0 \le i \le n\}$ and for the purposes of this proof set ω_0 and ω_{n+1} to be the 0 weight. Then by [38, Prop. 1.2] (and the discussion on page 555 *loc.cit)*, the direct sum of all irreducible representations $L_G(\mu)$ with μ running over the set $\{(p-1-c)\omega_i + c\omega_{i+1} \mid (c,i) \in B\}$ has dimension p^{n+1}. Note that the trivial representation of G occurs twice among the $L_G(\mu)$ (specifically, for $(c,i) = (0,0)$ and $(p-1,n)$). We also observe that $L_G(\lambda)$ with λ as in (1) is the only nontrivial self-dual module among the $L_G(\mu)$, whereas the other $L_G(\mu)$ (with $\mu \neq \lambda, 0$) occur in the sum as dual pairs. Therefore, the parity of $\dim L_G(\lambda)$ coincides with that of p^{n+1}, which is an odd number.

(2) The irreducible FG-module $L_G(\omega_j)$ is the j-th exterior power of the natural FG-module, whence the result.

(3) The assertion is proven in [37], see the statement A of the Main Theorem.

For the proof of Proposition 22, we first recall the following result from [27].

Lemma 21. [27, Lemma 79] *Let G be a simply connected simple algebraic group over F, with root system* Φ. *Fix a maximal torus T of G and for each* $\alpha \in \Phi$, *let* U_α *denote the* 1*-dimensional root subgroup normalized by T, corresponding to the root* α, *and fix isomorphisms* $x_\alpha : \mathbf{G}_a \to U_\alpha$. *For* $c \in F^*$, *set* $h_\alpha(c) = w_\alpha(c)w_\alpha(1)^{-1}$, *where* $w_\alpha(c) = x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c)$. *Finally set* $h = \prod_{\alpha\in\Phi^+} h_\alpha(-1)$. *Then*
(1) *h is in the center of G and* $h^2 = 1$.
(2) *For a dominant weight* λ, *if* $w_0(\lambda) = -\lambda$, *then G preserves a symplectic form on V if* $\lambda(h) = -1$, *and a nondegenerate symmetric bilinear form on V if* $\lambda(h) = 1$.

Proposition 22. *Let* $\lambda \in \Omega_2(G)$. *Assume moreover that* $p > 2$. *Then* $L_G(\lambda)$ *is symplectic if and only if the pair* (G,λ) *is as in Table* 4. *In particular, if* $\lambda \in \Omega_2(G)$ *with* $L_G(\lambda)$ *symplectic, then* $\lambda \in \Omega_1(G)$.

Proof. First, by Lemma 21, if $|Z(G)|$ is odd then every self-dual irreducible representation of G is orthogonal; in particular, this is the case if $G = A_n$ with n even. So we assume that $|Z(G)|$ is even. It follows that G is classical or of type E_7. Furthermore, in the adjoint representation of G the center acts trivially, so again by Lemma 21 the representations arising from the adjoint representation are orthogonal. More generally, any representation where all weights are roots must be orthogonal. Now let $h \in Z(G)$ as defined in Lemma 21.

Using the description of the sum of the positive roots in $\Phi(E_7)$ given in [5, Planche VI], we deduce that h acts nontrivially on $L_{E_7}(\omega_7)$, and hence by Lemma 21 $L_{E_7}(\omega_7)$ is symplectic. So we are left with the classical groups.

Let $G = A_n$ with n odd. In view of Proposition 19, Lemma 20, and the above comments, we must consider the cases $\lambda = 2\omega_2$ for $n = 3$ and $\lambda = \omega_{(n+1)/2}$ with n odd. In the former case, h acts trivially on $L_G(2\omega_2)$, so the module is orthogonal. In the second case, $L_G(\lambda)$ is the $(n+1)/2$-th exterior power of the natural FG-module, and so h acts as $(-1)^{(n+1)/2} \cdot \mathrm{Id}$ on $L_G(\lambda)$. Then the above Lemma gives the result.

Let $G = C_n$, $n > 1$. Since $L_G(\omega_1)$ is the natural symplectic module for G, h acts as $-\mathrm{Id}$ on $L_G(\omega_1)$, so in particular $h \neq 1$. As $Z(G)$ is of order 2, $L_G(\lambda)$ is symplectic if and only if G acts faithfully on $L_G(\lambda)$. As the root lattice $\mathbb{Z}\Phi(G)$ has index 2 in the weight lattice, it follows that G acts faithfully on $L_G(\lambda)$ if and only if $\lambda \notin \mathbb{Z}\Phi(G)$. The weights $\lambda = \omega_2, 2\omega_1, 2\omega_2$ all lie in $\mathbb{Z}\Phi(G)$. Note that ω_n (respectively, ω_{n-1}) lies in $\mathbb{Z}\Phi(G)$ if and only if n is even (respectively, odd). (See [5, Planche III].) Therefore, $\frac{p-1}{2}\omega_n \notin \mathbb{Z}\Phi(G)$ if and only if $\frac{n(p-1)}{2}$ is odd. Observe that $\omega_n - \omega_{n-1} \notin \mathbb{Z}\Phi(G)$ and $\omega_{n-1} + \frac{p-3}{2}\omega_n = \omega_{n-1} - \omega_n + \frac{p-1}{2}\omega_n$; it follows that $\omega_{n-1} + \frac{p-3}{2}\omega_n \notin \mathbb{Z}\Phi(G)$ if and only if $\frac{n(p-1)}{2}$ is even, as stated in Table 4.

Let $G = D_n$, $n > 3$ odd; here the weights which we must consider are ω_1, $2\omega_1$ and ω_2. Since $L_G(\omega_1)$ is the natural orthogonal module for G, h acts trivially on $L_G(\omega_1)$. Since $2\omega_1$ and ω_2 each occur in $L_G(\omega_1) \otimes L_G(\omega_1)$, $L_G(2\omega_1)$ and $L_G(\omega_2)$ are also orthogonal.

Let $G = D_n$, $n > 3$ even. For the weights ω_1, $2\omega_1$ and ω_2, the argument of the previous paragraph is valid. So we must consider the weights ω_{n-1} and ω_n. A direct check using the information in [5, Planche IV] allows one to see that $\omega_{n-1}(h) = (-1)^{\frac{n(n-1)}{2}} = \omega_n(h)$, and so $L_G(\omega_{n-1})$ and $L_G(\omega_n)$ are symplectic if and only if $n \equiv 2 \pmod 4$.

Let $G = B_n$, $n > 2$. In this case ω_n is the only fundamental dominant weight which does not lie in the root lattice. In the natural embedding of B_n in D_{n+1}, B_n acts irreducibly on each of the spin modules for D_{n+1}; the restriction is the FG-module $L_{B_n}(\omega_n)$ in each case. Hence, for n odd, $L_G(\omega_n)$ is symplectic if and only if $n+1 \equiv 2 \pmod 4$. When n is even, we consider the natural embedding of D_n in B_n, where the spin module for B_n decomposes as a direct sum of the two distinct spin modules for D_n, and the summands are non self-dual, non isomorphic, and hence non-degenerate with respect to the form. Hence $L_G(\omega_n)$ is symplectic if and only if $n \equiv 2 \pmod 4$. So to summarize, for the group $G = B_n$, $L_G(\omega_n)$ is symplectic if and only if $n \equiv 1$ or $2 \pmod 4$.

This completes the proof of the proposition.

Continuing with the case where G preserves a non-degenerate form on $L_G(\lambda)$ and $p > 2$, for the weights $\lambda \in \Omega_2(G)$ not listed in Table 4 the module $L_G(\lambda)$ is orthogonal. In order to decide whether the image of G under the corresponding representation lies in a subgroup of type B_n or D_n, one has only to determine whether $\dim L_G(\lambda)$ is odd or even. Since we are interested in the solution to

Problem 2, we consider those weights $\lambda \in \Omega_2(G)$ for which the multiplicity of the 0 weight is at most 1 if $L_G(\lambda)$ is odd-dimensional, and at most 2 if $\dim L_G(\lambda)$ is even.

The following lemma can be deduced directly from [17] and the preceding results.

Lemma 23. *Let $\lambda \in \Omega_2(G)$. Assume $p > 2$, $L_G(\lambda)$ is orthogonal, and moreover the multiplicity of the 0 weight in $L_G(\lambda)$ is at most 2. Then $\dim L_G(\lambda)$ is even if and only if the pair (G, λ) is as in Table 5.*

We give the odd-dimensional orthogonal representations $L_G(\lambda)$, with $\lambda \in \Omega_1(G)$ in Table 6.

We now turn to the situation where $p = 2$, and $L_G(\lambda)$ is self-dual.

Lemma 24. *Let $\lambda \in \Omega_2(G)$ such that $L_G(\lambda)$ is self-dual and the multiplicity of the 0 weight in $L_G(\lambda)$ is at most 2. Assume in addition that $p = 2$. Then $L_G(\lambda)$ has a non-degenerate G-invariant quadratic form if and only if (G, λ) are as in Table 7.*

Proof. We first inspect the last column of Table 2, where the dimension of the 0 weight space in $L_G(\lambda)$ is given. In addition, the result of Proposition 19, and the remarks preceding the proposition, further restrict the list of pairs (G, λ) for $\lambda \in \Omega_2(G)$, which must be considered. We find that the only even-dimensional $L_G(\lambda)$ in addition to those listed in Table 7 are as follows:

a) $G = A_1$, $\lambda = \omega_1$, $\dim L_G(\lambda) = 2$;
b) $G = C_n$, $\lambda = \omega_1$, $\dim L_G(\lambda) = 2n$;
c) $G = B_n$, $\lambda = \omega_1$, $\dim L_G(\lambda) = 2n$;
d) $G = G_2$, $\lambda = \omega_1$, $\dim L_G(\lambda) = 6$.

In cases (a), (b), (c) and (d) above, it is well-known that G preserves no non-degenerate quadratic form on $L_G(\lambda)$. Hence we now turn to the list of pairs (G, λ) of Table 7. The pair (D_n, ω_1) is clear as $L_{D_n}(\omega_1)$ is the natural representation of the classical group of type D_n.

For cases where $L_G(\lambda)$ occurs as a composition factor of the adjoint representation of G, we refer to [11, §3] to conclude that G preserves a non-degenerate quadratic form on $L_G(\lambda)$. (Note that we may also apply the exceptional isogeny $F_4 \to F_4$.) The pairs (C_3, ω_2), (C_4, ω_2), (E_7, ω_7) are covered in [6, Table 2], and we can then use the isogeny $B_n \to C_n$ to settle the analagous cases for $G = B_n$. This leaves us with the pairs $(A_n, \omega_{(n+1)/2})$, (C_n, ω_n), (B_n, ω_n), (D_n, ω_j), $j = n-1, n$. For the second case, we may consider the group B_n acting on $L_G(\omega_n)$. Then in all cases the Weyl module with the given highest weight is irreducible and the existence of a G-invariant quadratic form follows from [24, 2.4].

Additionally, we provide in Table 9 the list of modules $L_G(\lambda)$, with $\lambda \in \Omega_1(G)$, $L_G(\lambda) \ncong L_G(\lambda)^*$. So in particular, the image of G under the corresponding representation contains a regular torus of $H = \mathrm{SL}(L_G(\lambda))$ and G does not lie in a proper classical subgroup of H. Finally, Table 8 records the non-orthogonal symplectic modules $L_G(\lambda)$ for $\lambda \in \Omega_1(G)$ when $p = 2$.

Note that at this point the proof of Theorem 3 is complete.

We can now give an explicit solution to Problem 1 for simple subgroups G of classical groups. In Proposition 25, we treat the case of tensor-indecomposable irreducible representations of G all of whose weight spaces are 1-dimensional. As discussed in Section 2, the image of G under a tensor-decomposable representation is not maximal in the classical group. In Proposition 26, we handle the orthogonal irreducible representations of G whose zero weight space has dimension 2 while all other weight spaces are 1-dimensional. In the following two Propositions, we determine whether the image of G under the given representation is a maximal subgroup of the minimal classical group containing it.

Proposition 25. *Let G be a simple algebraic group and let $\lambda \in \Omega_1(G)$. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G with highest weight λ. Then $\rho(G)$ is a maximal subgroup in the minimal classical group on V containing $\rho(G)$, except for the following cases:*

(1) *$G = A_1$, $\lambda = 6\omega_1$, $p \geq 7$, where there is an intermediate subgroup of type G_2;*

(2) *$G = A_2$, $\lambda = \omega_1 + \omega_2$, $p = 3$, where there is an intermediate subgroup of type G_2;*

(3) *$G = B_n$, $\lambda = \omega_n$, where there is an intermediate subgroup of type D_{n+1};*

(4) *$G = C_n$, $\lambda = \omega_n$, $p = 2$, where there is an intermediate subgroup of type D_{n+1}.*

Proof. This follows from Seitz's classification of maximal closed connected subgroups of the classical type simple algebraic groups, see [23, Thm. 3, Table 1] and our results above.

Proposition 26. *Let λ be a weight of G occurring in Tables* 5 *or* 7 *but not in Table* 1*. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with highest weight λ. In particular, $\rho(G)$ lies in the (simple) orthogonal group on V, and does not contain a regular torus of $\mathrm{SL}(L_G(\lambda))$. Then $\rho(G)$ is a maximal subgroup of the orthogonal group containing $\rho(G)$, except for the following cases:*

(1) *$G = A_3$, $\lambda = \omega_1 + \omega_3$, $p = 2$, where there is an intermediate subgroup of type C_3;*

(2) *$G = D_4$, $\lambda = \omega_2$, $p = 2$, where there are intermediate subgroups of type C_4 and F_4;*

(3) *$G = G_2$, $\lambda = \omega_2$, $p = 2$, where there is an intermediate subgroup of type C_3.*

Proof. The proof is carried out exactly as the proof of Proposition 25.

6. Maximal reductive subgroups of exceptional groups containing regular tori

In this section, we consider Problem 1 for the case where H is an exceptional type simple algebraic group over F. We will determine all maximal closed connected subgroups M of H which contain a regular torus. As discussed in Section 2, we will assume M to be reductive and $\mathrm{rank}(M) < \mathrm{rank}(H)$. The main tool is the classification of the maximal closed connected subgroups of H, as given in [15, Cor. 2(ii)]. In order to be consistent with the tables and notation used in *loc. cit.*, we will throughout this section (and only in this section) allow both B_2 and C_2, contrary to our standing convention.

For a semisimple group $M = M_1 M_2 \cdots M_t$, with M_i simple, and with respect to a fixed maximal torus T_M of M, we will write $\{\omega_{i1}, \ldots, \omega_{i\ell_i}\}$, for the set of fundamental dominant weights of $T_M \cap M_i$ (so $\mathrm{rank}(M_i) = \ell_i$). In case M is simple, we will simply write $\{\omega_1, \ldots, \omega_\ell\}$.

Proposition 27. *Let M be a maximal closed connected positive-dimensional subgroup of an exceptional type simple algebraic group H. Assume* $\mathrm{rank}(M) < \mathrm{rank}(H)$. *Then M contains a regular torus of H if and only if the pair (M, H) is as given in Table* 10. *In particular, if M contains a regular torus of H, then M is semisimple.*

Before proving the result, it is interesting to compare the above table with [15, Thm. 1], which describes the maximal closed connected positive-dimensional subgroups of the exceptional simple algebraic groups. There are precisely four pairs (M, H), M a maximal closed connected positive-dimensional subgroup of an exceptional algebraic group H with $\mathrm{rank}(M) < \mathrm{rank}(H)$, and where M does *not* contain a regular torus of H: one class of A_1 subgroups in $H = E_7$, 2 classes of A_1 subgroups in $H = E_8$ and a maximal B_2 in E_8.

Proof. Let M be as in the statement of the result and fix T_M, a maximal torus of M. (Throughout the proof we will refer to M as a *maximal* subgroup, even though M may only be maximal among connected subgroups.) Then [15, Cor. 2] implies that M is semisimple. The method of proof is quite simple. By Proposition 7, T_M is a regular torus in H if and only if $\dim(C_{\mathrm{Lie}(H)}(T_M)) = \mathrm{rank}(H)$. Hence, we need only determine the dimension of the 0 weight space for T_M acting on $\mathrm{Lie}(H)$. This can be deduced from the information in [15, Table 10.1].

If M is of type A_1, the notation $T(m_1; m_2; \ldots; m_k)$, used in [15, Table 10.1], represents an FM-module whose composition factors are the same as those of $W_M(m_1\omega_1) \oplus \cdots \oplus W_M(m_k\omega_1)$. Since the multiplicity of the 0 weight in each Weyl module for A_1 is precisely 1, we see that the only maximal A_1-subgroups containing a regular torus are those listed above. This covers the case $H = G_2$.

Consider now the two remaining cases in $H = F_4$. If M is the maximal G_2 subgroup in H (occurring only for $p = 7$), then $\mathrm{Lie}(H)|_M$ has composition factors $L_M(\omega_2)$ and $L_M(\omega_1 + \omega_2)$. Now consulting [16], we see that the 0 weight has multiplicity 2 in each of these irreducible modules and hence multiplicity 4 in $\mathrm{Lie}(H)$. This then implies that T_M is a regular torus in H. For the semisimple subgroup $M = A_1 G_2$ in F_4 (which exists when $p \geq 3$), we must explain an additional notation used in [15]. In [15, Table 10.1], the notation $\Delta(\mu_1; \mu_2)$ denotes a certain indecomposable FM-module whose composition factors are $L_M(\mu_1)$, $L_M(\mu_2)$ and two factors $L_M(\nu)$, where μ_1 and μ_2 are dominant weights such that the tilting modules $T(\mu_1)$ and $T(\mu_2)$ each have socle and irreducible quotient of highest weight ν. Now if $p > 3$, $\mathrm{Lie}(H)|_M$ has composition factors $L_M(4\omega_{11} + \omega_{21})$, $L_M(2\omega_{11})$, and $L_M(\omega_{22})$. (See the paragraph preceding Proposition 27 for an explanation of the notation used here.) The multiplicity of the 0 weight in these modules is 1, 1, 2, respectively, and hence T_M is a regular torus of H. In case $p = 3$, the composition factors are $L_M(4\omega_{11} + \omega_{21})$, $L_M(\omega_{22})$ $L_M(\omega_{21})$, $L_M(\omega_{21})$, and $L_M(2\omega_{11})$, and again the multiplicity of the 0 weight is 4. This completes the consideration of the case $H = F_4$.

Now we consider the case $H = E_6$ and $\mathrm{rank}(M) \geq 2$. There is a maximal A_2 subgroup M of H (when $p \geq 5$) whose action on $\mathrm{Lie}(H)$ is $\mathrm{Lie}(H)|_M = L_M(4\omega_1 + \omega_2) \oplus L_M(\omega_1 + 4\omega_2) \oplus L_M(\omega_1 + \omega_2)$. Consulting [16], we see that the multiplicity of the zero weight in this module is 6 and so M contains a regular torus of H. The group H also has a maximal G_2 subgroup M when $p \neq 7$, such that $\mathrm{Lie}(H)|_M$ has the same set of composition factors as $W_M(\omega_1 + \omega_2) \oplus W_M(\omega_2)$. Now we consult [16] and find that the multiplicity of the zero weight in $\mathrm{Lie}(H)|_M$ is 6 for all characteristics $p \neq 7$; hence M contains a regular torus of H.

Turn now to the maximal closed connected subgroups of $H = E_6$, of rank at least 4. The maximal subgroup M of type C_4 acts on $\mathrm{Lie}(H)$ with composition factors $L_M(2\omega_1)$ and $L_M(\omega_4)$, if $p \neq 3$, and with these same composition factors plus an additional 1-dimensional composition factor, if $p = 3$. As usual, we find that the dimension of the 0 weight space is 6 and so C_4 contains a regular torus of H. For the maximal F_4 subgroup M of H, which exists in all characteristics, $\mathrm{Lie}(H)|_M = L_M(\omega_4) \oplus L_M(\omega_1)$, if $p > 2$, and when $p = 2$, the Lie algebra is isomorphic to the tilting module of highest weight ω_1, which has a composition factor $L_M(\omega_1)$ and two factors $L_M(\omega_4)$. The usual argument shows that M contains a regular torus of H. Finally, we consider the maximal subgroup $M \subset H$ of type A_2G_2; $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the FM-module $W_M(\omega_{11} + \omega_{12} + \omega_{21}) \oplus W_M(\omega_{11} + \omega_{12}) \oplus W_M(\omega_{22})$. One checks as usual that the multiplicity of the 0 weight is indeed 6.

We now turn to the case $H = E_7$, and M is a maximal closed connected subgroup of rank 2. There exists a maximal A_2 subgroup M of H when $p \geq 5$, whose action on $\mathrm{Lie}(H)$ is given by $L_M(4\omega_1 + 4\omega_2) \oplus L_M(\omega_1 + \omega_2)$, when $p \neq 7$, and $\mathrm{Lie}(H)|_M = T(4\omega_1 + 4\omega_2)$, when $p = 7$. Again using [16] one verifies the multiplicity of the 0 weight in $\mathrm{Lie}(H)|_M$ is 7 and hence T_M is a regular torus of H. The maximal A_1A_1 subgroup M of H is such that $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the module $W_M(2\omega_{11} + 8\omega_{21}) \oplus W_M(4\omega_{11} + 6\omega_{21}) \oplus W_M(6\omega_{11} + 4\omega_{21}) \oplus W_M(2\omega_{11} + 4\omega_{21}) \oplus W_M(4\omega_{11} + 2\omega_{21}) \oplus W_M(2\omega_{11}) \oplus W_M(2\omega_{21})$. One verifies that the multiplicity of the zero weight is 7 and hence M contains a regular torus of H. This completes the consideration of the rank two reductive maximal connected subgroups.

We now handle the remaining maximal connected subgroups of $H = E_7$. The maximal A_1G_2 subgroup M of H, which exists for all $p \geq 3$, satisfies: $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the FM-module $W_M(4\omega_{11} + \omega_{21}) \oplus W_M(2\omega_{11} + 2\omega_{21}) \oplus W_M(2\omega_{11}) \oplus W_M(\omega_{22})$. As usual, we check that the multiplicity of the zero weight in $\mathrm{Lie}(H)|_M$ is 7. The maximal A_1F_4 subgroup M is such that $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the module $W_M(2\omega_{11} + \omega_{24}) \oplus W_M(2\omega_{11}) \oplus W_M(\omega_{21})$. This module has a 7-dimensional 0 weight space and so M contains a regular torus of H. Finally, we consider the maximal G_2C_3 subgroup M, which exists in all characteristics. In this case, $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the FM-module $W_M(\omega_{11} + \omega_{22}) \oplus W_M(\omega_{12}) \oplus W_M(2\omega_{21})$, which has a 7-dimensional 0 weight space and again M contains a regular torus of H.

To complete the proof, we now turn to the case $H = E_8$ and M a maximal closed connected subgroup of rank at least 2. There exists a unique (up to conjugacy) rank 2 reductive maximal subgroup of H, namely $M = B_2$, when $p \geq 5$. Here $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as $W_M(6\omega_2) \oplus W_M(3\omega_1 + 2\omega_2) \oplus W_M(2\omega_2)$; but this latter has a 12-dimensional 0 weight space and so T_M is not a regular torus of H. We now consider the group $M = A_1A_2$. Here $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the FM-module $W_M(6\omega_{11} + \omega_{21} + \omega_{22}) \oplus W_M(2\omega_{11} + 2\omega_{21} + 2\omega_{22}) \oplus W_M(4\omega_{11} + 3\omega_{21}) \oplus W_M(4\omega_{11} + 3\omega_{22}) \oplus W_M(2\omega_{11}) \oplus W_M(\omega_{21} + \omega_{22})$, which has an 8-dimensional 0 weight space and hence M contains a regular torus of H. Finally, we consider the maximal G_2F_4 subgroup M, which exists in all characteristics. Here $\mathrm{Lie}(H)|_M$ has the same set of T_M-weights (and multiplicities) as the FM-module $W_M(\omega_{11} + \omega_{24}) \oplus W_M(\omega_{12}) \oplus W_M(\omega_{21})$, which has an 8-dimensional 0 weight space and so T_M is a regular torus of H.

7. Non-semisimple regular elements

We conclude the paper with some remarks about how one can determine the connected reductive overgroups of a regular element, which is neither semisimple nor unipotent. Of course, if G is a closed connected reductive subgroup of H containing a regular element of H, then Proposition 6 implies that G contains a regular semisimple element and hence has been determined (assuming G is maximal). Nevertheless, one might be interested in determining closed connected reductive subgroups which contain a non semisimple regular element of H. As mentioned before, this has been done for the regular unipotent elements in [22] and [29]. So now suppose $g = us = su \in G \subset H$ with s semisimple and u unipotent and g regular in H. Set $Y = C_H(s)^\circ$ and $X = C_G(s)^\circ$. We first claim that u is a regular unipotent element of Y. Indeed, if u is not regular in Y then $\dim C_H(g) = \dim C_Y(u) > \mathrm{rank}(Y) = \mathrm{rank}(H)$, contradicting the regularity of g. Thus u is a regular unipotent element in the connected reductive group $Y = C_H(s)^\circ$, lying in the connected reductive group $X = C_G(s)^\circ$. We now appeal to the following classification:

Theorem 28. [29, Theorem 1.4] *Let G be a closed semisimple subgroup of the simple algebraic group H, containing a regular unipotent element of H. Then G is simple and either the pair (H, G) is as given in Table 11, or G is of type A_1 and $p = 0$ or $p \geq h$, where h is the Coxeter number for H. Moreover, for each pair of root systems $(\Phi(H), \Phi(G))$ as in the table, respectively, for $(\Phi(H), A_1, p)$, with $p = 0$ or $p \geq h$, there exists a closed simple subgroup $X \subset H$ of type $\Phi(G)$, respectively A_1, containing a regular unipotent element of H.*

The above classification applies to simple groups H, but it is straightforward to deduce from this the set of possible pairs of reductive groups $(C_G(s)^\circ, C_H(s)^\circ)$. Now we use information about the structure of centralizers of semisimple elements in H and in G. The connected components of centralizers of semisimple elements are subsystem subgroups and thus can be obtained via the Borel-De Siebenthal algorithm. Hence one can inductively determine the pairs (G, H) such that there exists a semisimple element $s \in G$ with $(C_G(s)^\circ, C_H(s)^\circ)$ one of the pairs given by Theorem 28.

Remark 1. We point out here two inaccuracies, in [22] and [29]. In the statement of [22, Thm. A], the condition given for the existence of an A_1 subgroup containing a regular unipotent element is $p > h$. But in fact, as shown in [31], such a subgroup exists for all $p \geq h$. This has been correctly stated in Theorem 28. In addition, in Table 1 of [29], we indicated that $p \neq 2$ for the example of G_2 in D_4; however, this prime restriction is not necessary and should be omitted. This has been corrected in Table 11 below. Finally we also point out that in [29], we determined all connected reductive overgroups of regular unipotent elements, and indeed established that such a group must be simple (see Proposition 2.3 *loc.cit*). Therefore, the caption of Table 11 below reflects this.

8. Tables

We recall here our convention for reading the tables in case $\mathrm{char}(F) = 0$: for a natural number a the expressions $p > a$, $p \geq a$ or $p \neq a$ are to be interpreted as the absence of any restriction, that is, a is allowed to be any natural number. Note further that when a weight λ has coefficients expressed in terms of p, we are assuming that $\mathrm{char}(F) = p > 0$.

G	$\Omega_1(G) \setminus \{0\}$
A_1	$a\omega_1, 1 \leq a < p$
$A_n, n > 1$	$a\omega_1, b\omega_n,\ 1 \leq a, b < p$ $\omega_i,\ 1 < i < n$ $c\omega_i + (p-1-c)\omega_{i+1},\ 1 \leq i < n,\ 0 \leq c < p$
$B_n, n > 2$	$\omega_1,\ \omega_n$
$C_n, n > 1, p = 2$	$\omega_1,\ \omega_n$
$C_2, p > 2$	$\omega_1,\ \omega_2,\ \omega_1 + \frac{p-3}{2}\omega_2,\ \frac{p-1}{2}\omega_2$
C_3	ω_3
$C_n, n > 2, p > 2$	$\omega_1,\ \omega_{n-1} + \frac{p-3}{2}\omega_n,\ \frac{p-1}{2}\omega_n$
$D_n, n > 3$	$\omega_1,\ \omega_{n-1},\ \omega_n$
E_6	$\omega_1,\ \omega_6$
E_7	ω_7
$F_4, p = 3$	ω_4
$G_2, p \neq 3$	ω_1
$G_2, p = 3$	$\omega_1,\ \omega_2$

Table 1: Irreducible p-restricted FG-modules with all weights of multiplicity 1

G	conditions	$\Omega_2(G) \setminus \Omega_1(G)$	weight0 multiplicity
A_n,	$n > 1, (n,p) \neq (2,3)$	$\omega_1 + \omega_n$	$\begin{cases} n-1 & \text{if } p \mid (n+1) \\ n & \text{if } p \nmid (n+1) \end{cases}$
A_3	$p > 3$	$2\omega_2$	2
B_n	$n > 2, p \neq 2$	ω_2	n
	$n > 2, p = 2$	ω_2	$n - \gcd(2,n)$
		$2\omega_1$	$\begin{cases} n & \text{if } p \mid (2n+1) \\ n+1 & \text{if } p \nmid (2n+1) \end{cases}$
C_n	$n > 1$	$2\omega_1$	n
	$n > 2, (n,p) \neq (3,3)$	ω_2	$\begin{cases} n-2 & \text{if } p \mid n \\ n-1 & \text{if } p \nmid n \end{cases}$
C_2	$p \neq 5$	$2\omega_2$	2
C_4	$p \neq 2, 3$	ω_4	2
D_n	$n > 3$	$2\omega_1$	$\begin{cases} n-2 & \text{if } p \mid n \\ n-1 & \text{if } p \nmid n \end{cases}$
	$n > 3, p \neq 2$	ω_2	n
	$n > 3, p = 2$	ω_2	$n - \gcd(2,n)$
E_6		ω_2	$\begin{cases} 5 & \text{if } p = 3 \\ 6 & \text{if } p \neq 3 \end{cases}$
E_7		ω_1	$\begin{cases} 6 & \text{if } p = 2 \\ 7 & \text{if } p \neq 2 \end{cases}$
E_8		ω_8	8
F_4		ω_1	$\begin{cases} 2 & \text{if } p = 2 \\ 4 & \text{if } p \neq 2 \end{cases}$
	$p \neq 3$	ω_4	2
G_2	$p \neq 3$	ω_2	2

Table 2: Irreducible p-restricted FG-modules with non-zero weights of multiplicity 1 and whose zero weight has multiplicity greater than 1.

G	λ	conditions	$\dim L_G(\lambda)$
A_2	$\omega_1+\omega_2$	$p \neq 3$	8
A_3	$2\omega_2$	$p > 3$	20
	$\omega_1+\omega_3$	$p = 2$	14
B_3	ω_2	$p = 2$	14
B_4	ω_2	$p = 2$	26
C_2	$2\omega_1$		10
	$2\omega_2$	$p \neq 5$	14
C_3	ω_2	$p \neq 3$	14
C_4	ω_2	$p = 2$	26
	ω_4	$p \neq 2, 3$	42
D_4	ω_2	$p = 2$	26
F_4	ω_1	$p = 2$	26
	ω_4	$p \neq 3$	26
G_2	ω_2	$p \neq 3$	14

Table 3: $\lambda \in \Omega_2(G)$, 0 weight in $L_G(\lambda)$ of multiplicity 2

G	λ	conditions	$\dim L_G(\lambda)$
A_1	$a\omega_1$	a odd	$a+1$
A_n	$\omega_{(n+1)/2}$	$n > 1$ odd, $\frac{n+1}{2}$ odd	$\binom{n+1}{\frac{n+1}{2}}$
B_n	ω_n	$n > 2, n \equiv 1$ or $2 \pmod 4$	2^n
C_3	ω_3		14
C_n	ω_1	$n > 1$	$2n$
	$\omega_{n-1}+\frac{p-3}{2}\omega_n$	$n > 1, p \geq 3, n(p-1)/2$ even	$(p^n-1)/2$
	$\frac{p-1}{2}\omega_n$	$n > 1, p \geq 3, n(p-1)/2$ odd	$(p^n+1)/2$
D_n	$\omega_{n-1},\ \omega_n$	$n > 3$ even, $n \equiv 2 \pmod 4$	2^{n-1}
E_7	ω_7		56

Table 4: $\lambda \in \Omega_2(G)$, $p \neq 2$, $L_G(\lambda)$ symplectic

G	λ	conditions	$\dim L_G(\lambda)$
A_n	$\omega_{(n+1)/2}$	$n>1$ odd, $(n+1)/2$ even	$\binom{n+1}{\frac{n+1}{2}}$
A_2	$\omega_1+\omega_2$	$p\neq 3$	8
A_3	$2\omega_2$	$p>3$	20
B_n	ω_n	$n\equiv 0$ or $3 \pmod 4$	2^n
C_2	$2\omega_1$		10
	$2\omega_2$	$p\neq 5$	14
C_3	ω_2	$p>3$	14
C_4	ω_4	$p>3$	42
D_n	ω_1	$n>3$	$2n$
	ω_{n-1},ω_n	$n\equiv 0 \pmod 4$	2^{n-1}
F_4	ω_4	$p>3$	26
G_2	ω_2	$p>3$	14

Table 5: $\lambda\in\Omega_2(G)$, $p\neq 2$, $L_G(\lambda)$ even-dimensional orthogonal, with 0 weight of multiplicity at most 2

G	λ	conditions	$\dim L_G(\lambda)$
A_1	$a\omega_1$	$a>0$ even	$a+1$
A_n	$(p-1)\omega_{(n+1)/2}$	$n>1, n$ odd	†
	$\frac{p-1}{2}(\omega_{n/2}+\omega_{(n+2)/2})$	n even	†
B_n	ω_1	$n>2$	$2n+1$
C_2	ω_2		5
C_n	$\omega_{n-1}+\frac{p-3}{2}\omega_n$	$n\geq 2$, $\frac{n(p-1)}{2}$ odd	$(p^n-1)/2$
	$\frac{p-1}{2}\omega_n$	$n\geq 2$, $\frac{n(p-1)}{2}$, even	$(p^n+1)/2$
F_4	ω_4	$p=3$	25
G_2	ω_1		7
	ω_2	$p=3$	7

Table 6: $\lambda\in\Omega_1(G)$, $p\neq 2$, $L_G(\lambda)$ odd-dimensional orthogonal

† The dimensions of these modules can be deduced from the fact that the representations can be realized in the action of SL_{n+1} on the homogeneous components of the truncated polynomial ring $F[Y_1,\dots,Y_{n+1}]/\langle Y_1^p,\dots,Y_{n+1}^p\rangle$. See [38] for example.

G	λ	conditions	$\dim L_G(\lambda)$
A_n	$\omega_{(n+1)/2}$	$n > 1$ odd	$\binom{n+1}{\frac{n+1}{2}}$
A_2	$\omega_1 + \omega_2$		8
A_3	$\omega_1 + \omega_3$		14
B_n, C_n	ω_n		2^n
B_3, C_3	ω_2		14
B_4, C_4	ω_2		26
D_n	ω_1	$n > 3$	$2n$
D_n	ω_{n-1}, ω_n	$n > 3$ even	2^{n-1}
D_4	ω_2		26
E_7	ω_7		56
F_4	ω_1, ω_4		26
G_2	ω_2		14

Table 7: $\lambda \in \Omega_2(G)$, $p = 2$, $L_G(\lambda)$ orthogonal, with 0 weight of multiplicity at most 2

G	λ	conditions	$\dim L_G(\lambda)$
A_1	ω_1		2
B_n	ω_1	$n \geq 3$	$2n$
C_n	ω_1	$n \geq 2$	$2n$
G_2	ω_1		6

Table 8: $p = 2$, $\lambda \in \Omega_1(G)$, $L_G(\lambda)$ non-orthogonal symplectic

G	λ	conditions
A_n	$a\omega_1, b\omega_n$	$n > 1, 1 \leq a, b < p$
	ω_i	$1 < i < n, i \neq (n+1)/2$ if n is odd
	$c\omega_i + (p-1-c)\omega_{i+1}$	$1 \leq i < n, 0 \leq c < p$, and $c \neq (p-1)/2$ if n is even and $i = n/2$; $c \neq 0$ if n is odd and $i = (n-1)/2$; $c \neq p-1$ if n is odd and $i = (n+1)/2$.
D_n	ω_{n-1}, ω_n	$n > 3$ odd
E_6	ω_1, ω_6	

Table 9: $\lambda \in \Omega_1(G)$, $\lambda \neq -w_0\lambda$ and $\rho(G)$ contains a regular torus of $\mathrm{SL}(L_G(\lambda))$

H	M simple	M non simple
G_2	A_1 $(p \geq 7)$	
F_4	A_1 $(p \geq 13), G_2$ $(p = 7)$	A_1G_2 $(p \geq 3)$
E_6	A_2 $(p \geq 5), G_2$ $(p \neq 7)$ C_4 $(p \geq 3), F_4$	A_2G_2
E_7	A_1 $(p \geq 19), A_2$ $(p \geq 5)$	A_1A_1 $(p \geq 5), A_1G_2$ $(p \geq 3)$ A_1F_4, G_2C_3
E_8	A_1 $(p \geq 31)$	A_1A_2 $(p \geq 5), G_2F_4$

Table 10: Maximal connected reductive subgroups $M \subset H$, H exceptional, $\mathrm{rank}(M) < \mathrm{rank}(H)$, with M containing a regular torus of H

H	G
A_6 A_5	$G_2, p \neq 2$ $G_2, p = 2$
C_3	$G_2, p = 2$
B_3	G_2
D_4	G_2 B_3
E_6	F_4
$A_{n-1}, n > 1$	$C_{n/2}, n$ even $B_{(n-1)/2}, n$ odd$, p \neq 2$
$D_n, n > 4$	B_{n-1}

Table 11: Connected reductive subgroups $G \subset H$ containing a regular unipotent element

References

[1] A. A. Baranov, A. A. Osinovskaya, and I. D. Suprunenko, *Modular representations of classical groups with small weight multiplicities*, Sovrem. Mat. Prilozh., (2008), pp. 162–174.

[2] ———, *Modular representations of the special linear groups with small weight multiplicities*, J. Algebra, **397** (2014), pp. 225–251.

[3] A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv., **23** (1949), pp. 200–221.

[4] A. Borel and J. Tits, *Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I*, Invent. Math., **12** (1971), pp. 95–104.

[5] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[6] J. Brundan, *Double coset density in classical algebraic groups*, Trans. Amer. Math. Soc., **352** (2000), pp. 1405–1436 (electronic).

[7] T. Burness, S. Ghandour, and D. Testerman, *Irreducible geometric subgroups of classical algebraic groups.* Memoirs of the Amer. Math. Soc., to appear.

[8] B. N. Cooperstein, *Subgroups of exceptional groups of Lie type generated by long root elements. I. Odd characteristic*, J. Algebra, **70** (1981), pp. 270–282.

[9] ———, *Subgroups of exceptional groups of Lie type generated by long root elements. II. Characteristic two*, J. Algebra, **70** (1981), pp. 283–298.

[10] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik N.S., **30**(72) (1952), pp. 349–462 (3 plates).

[11] R. Gow and W. Willems, *Methods to decide if simple self-dual modules over fields of characteristic* 2 *are of quadratic type*, J. Algebra, **175** (1995), pp. 1067–1081.

[12] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[13] W. M. Kantor, *Subgroups of classical groups generated by long root elements*, Trans. Amer. Math. Soc., **248** (1979), pp. 347–379.

[14] M. W. Liebeck and G. M. Seitz, *Subgroups generated by root elements in groups of Lie type*, Ann. of Math. (2), **139** (1994), pp. 293–361.

[15] ———, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc., **169** (2004), pp. vi+227.

[16] F. Lübeck, *Tables of weight multiplicities.* http://www.math.rwth-aachen.de/~Frank.Lübeck/chev/WMSmall/index.html.

[17] ———, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math., **4** (2001), pp. 135–169.

[18] G. Malle and D. Testerman, *Linear algebraic groups and finite groups of Lie type*, vol. 133 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2011.

[19] H. Pollatsek, *Irreducible groups generated by transvections over finite fields of characteristic two*, J. Algebra, **39** (1976), pp. 328–333.

[20] A. A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Mat. Sb. (N.S.), **133**(175) (1987), pp. 167–183, 271.

[21] A. A. Premet and I. D. Suprunenko, *The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights*, Comm. Algebra, **11** (1983), pp. 1309–1342.

[22] J. Saxl and G. M. Seitz, *Subgroups of algebraic groups containing regular unipotent elements*, J. London Math. Soc. (2), **55** (1997), pp. 370–386.

[23] G. M. Seitz, *The maximal subgroups of classical algebraic groups*, Mem. Amer. Math. Soc., **67** (1987), pp. iv+286.

[24] P. Sin and W. Willems, *G-invariant quadratic forms*, J. Reine Angew. Math., **420** (1991), pp. 45–59.

[25] T. A. Springer and R. Steinberg, *Conjugacy classes*, in Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266.

[26] R. Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math., (1965), pp. 49–80.

[27] ———, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[28] I. D. Suprunenko, *Irreducible representations of simple algebraic groups containing matrices with big Jordan blocks*, Proc. London Math. Soc. (3), **71** (1995), pp. 281–332.

[29] D. Testerman and A. Zalesski, *Irreducibility in algebraic groups and regular unipotent elements*, Proc. Amer. Math. Soc., **141** (2013), pp. 13–28.

[30] D. M. Testerman, *Irreducible subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc., **75** (1988), pp. iv+190.

[31] ———, *A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups*, J. Algebra, **177** (1995), pp. 34–76.

[32] B. J. Veĭsfeĭler, *A class of unipotent sub-groups of semi-simple algebraic groups*, Uspehi Mat. Nauk, **21** (1966), pp. 222–223.

[33] A. Wagner, *Collineation groups generated by homologies of order greater than* 2, Geom. Dedicata, **7** (1978), pp. 387–398.

[34] A. Wagner, *Determination of the finite primitive reflection groups over an arbitrary field of characteristic not* 2*. I*, Geom. Dedicata, **9** (1980), pp. 239–253.

[35] ———, *Determination of the finite primitive reflection groups over an arbitrary field of characteristic not two. II, III*, Geom. Dedicata, **10** (1981), pp. 191–203, 475–523.

[36] A. E. Zalesskiĭ and V. N. Serežkin, *Finite linear groups generated by reflections*, Izv. Akad. Nauk SSSR Ser. Mat., **44** (1980), pp. 1279–1307, 38.

[37] A. E. Zalesskiĭ and I. D. Suprunenko, *Representations of dimension $(p^n \pm 1)/2$ of the symplectic group of degree $2n$ over a field of characteristic p*, Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk, (1987), pp. 9–15, **123**.

[38] ———, *Truncated symmetric powers of natural realizations of the groups $\mathrm{SL}_m(P)$ and $\mathrm{Sp}_m(P)$ and their constraints on subgroups*, Sibirsk. Mat. Zh., 31 (1990), pp. 33–46, **220**.