

Restricting representations of classical algebraic groups to maximal subgroups

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Résumé

Soient K un corps algébriquement clos de caractéristique $p \geq 0$ et Y un groupe classique sur K . Aussi, soit $X \subset Y$ un sous-groupe fermé connexe, maximal parmi les sous-groupes fermés connexes de Y , et considérons un KY -module rationnel et irréductible V . Dans cette thèse, nous nous intéressons aux triplets (Y, X, V) tels que la restriction $V|_X$ de V au sous-groupe X ait exactement deux facteurs de composition pour X , observant qu'il s'agit d'une généralisation d'un problème introduit par Dynkin dans les années 1950, depuis étudié par de nombreux mathématiciens. En particulier, nous étudions le plongement naturel du groupe $\text{Spin}_{2n}(K)$ à l'intérieur de $\text{Spin}_{2n+1}(K)$ ainsi que celui de $\text{SO}_{2n}(K)$ dans $\text{SL}_{2n}(K)$ et en déduisons des informations sur certains modules de Weyl.

Mots-clefs: Groupes algébriques, groupes classiques, théorie des représentations, multiplicités de poids, modules irréductibles, facteurs de composition, règles de restrictions.

Abstract

Fix an algebraically closed field K having characteristic $p \geq 0$ and let Y be a simple algebraic group of classical type over K . Also let X be maximal among closed connected subgroups of Y and consider a p -restricted irreducible rational KY -module V . In this thesis, we investigate the triples (Y, X, V) such that X acts with exactly two composition factors on V and see how it generalizes a question initially investigated by Dynkin in the 1950s and then further studied by numerous mathematicians. In particular, we study the natural embeddings of $\text{Spin}_{2n}(K)$ in $\text{Spin}_{2n+1}(K)$ as well as $\text{SO}_{2n}(K)$ in $\text{SL}_{2n}(K)$ and obtain results on the structure of certain Weyl modules.

Key words: Algebraic groups, classical groups, representation theory, weight multiplicities, irreducible modules, composition factors, restriction rules.

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CHAPTER 1

Introduction

In the 1950s, Dynkin [Dyn52] determined the maximal closed connected subgroups of the classical algebraic groups over \mathbb{C} . The difficult part of the investigation concerned irreducible closed simple subgroups X of $\mathrm{SL}(V)$. Indeed, in the course of his analysis, Dynkin observed that if X is a simple algebraic group over \mathbb{C} and if $\phi : X \rightarrow \mathrm{SL}(V)$ is an irreducible rational representation, then with specified exceptions the image of X is maximal among closed connected subgroups in one of the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$. Here Dynkin determined the triples (Y, X, V) where Y is a closed connected subgroup of $\mathrm{SL}(V)$, V is an irreducible KY -module different from the natural module for Y or its dual, and X is a closed connected subgroup of Y such that the restriction of V to X , written $V|_X$, is also irreducible. Such triples shall be referred to as *irreducible triples* in the remainder of the thesis.

In the 1980s, Seitz [Sei87] extended the problem to the situation of fields of arbitrary characteristic. By introducing new techniques, he determined all irreducible triples (Y, X, V) where Y is a simply connected simple algebraic group of classical type over an algebraically closed field K having characteristic $p \geq 0$, X is a closed connected proper subgroup of Y and V is an irreducible, tensor indecomposable KY -module. His investigation was then extended by Testerman [Tes88] to exceptional algebraic groups Y , again for X a closed connected subgroup.

The work of Dynkin, Seitz and Testerman provides a complete classification of irreducible triples (Y, X, V) where Y is a simple algebraic group, X is a closed connected proper subgroup of Y and V is an irreducible, tensor indecomposable KY -module. In the 1990s, Ford [For96],[For99] investigated irreducible triples (Y, X, V) in the special case where Y is of classical type, the connected component $X^\circ \subsetneq X$ of X is simple and the restriction $V|_{X^\circ}$ has p -restricted composition factors.

More recently, Ghandour [Gha10] gave a complete classification of irreducible triples (Y, X, V) in the case where Y is a simple algebraic group of exceptional type, X is a closed disconnected positive-dimensional subgroup of Y and V is an irreducible p -restricted rational KY -module. Finally, in [BGT15], [BGMT15] and [BMT], Burness, Marion, Ghandour and Testerman treat the case of triples (Y, X, V) where Y is of classical type, X is a closed positive-dimensional subgroups of Y and V is an irreducible tensor indecomposable KY -module, removing the previously mentioned assumption of Ford.

Now notice that if (Y, X, V) is an irreducible triple with X° simple such that $[X : X^\circ] = 2$ and $V|_{X^\circ}$ is reducible, then $V|_{X^\circ}$ has exactly two direct summands. Knowing such direct sum decompositions can yield information about the structure of V and $V|_X$, e.g. their dimension or composition factors. In [For95], Ford even applied his work (more precisely, the methods used in the proof of [For96, Proposition 3.1]) to study representations of the symmetric group. It thus seems worthwhile to relax the hypothesis, when considering the action of simple subgroups.

In this thesis, we investigate triples (Y, X, V) where Y is a simply connected (so that the weight lattice of the underlying root system for Y coincides with the character group of a maximal torus of Y) simple algebraic group of classical type over an algebraically closed field K of characteristic $p \geq 0$, X is a closed connected subgroup of Y and V is an irreducible, tensor indecomposable, p -restricted KY -module such that X has exactly two composition factors on V . Now if G is a closed subgroup of Y such that $X \subsetneq G \subsetneq Y$, then X has exactly two composition factors on V if and only if one of the following holds.

1. The restriction $V|_G$ is irreducible and X has exactly two composition factors on the KG -module $V|_G$.
2. The subgroup G acts with exactly two composition factors on V , say V_1, V_2 , and both $V_1|_X, V_2|_X$ are irreducible.

Therefore it is only natural to start the investigation by assuming X is maximal among closed connected subgroups of Y . Also, let $F : G \rightarrow G$ be a standard Frobenius morphism on G and denote by U^F the Frobenius twist of a given KY -module U . If V is an irreducible KY -module, then the *Steinberg Tensor Product Theorem* (see Theorem 2.3.2) yields the existence of irreducible p -restricted KY -modules V_1, \dots, V_k such that

$$V \cong V_1^{Fr_1} \otimes \cdots \otimes V_k^{Fr_k}.$$

Hence if X has exactly two composition factors on V , then there exists a unique $j \in \mathbb{Z}_{>0}$ such that $1 \leq j \leq k$ and $V_j|_X$ has exactly two composition factors. Consequently, we shall consider the situation in which X is maximal among connected subgroups of Y and V is an irreducible p -restricted KY -module. Finally, in the case where K has characteristic zero, we refer the reader to [KT87, Proposition 2.5.1], in which the embeddings $\mathrm{Sp}_{2n}(K) \subset \mathrm{SL}_{2n}(K)$, $\mathrm{SO}_{2n}(K) \subset \mathrm{SL}_{2n}(K)$, and $\mathrm{SO}_{2n+1}(K) \subset \mathrm{SL}_{2n+1}(K)$ are investigated.

Statements of results

In this section, we record the main results of this thesis and comment briefly on the methods. For starters, we fix an algebraically closed field K having characteristic $p \geq 0$ and refer the reader to Chapter 2 for some background material, such as the construction and classification of the irreducible p -restricted KG -modules for a semisimple algebraic group G over K , as well as ways of computing weight multiplicities in these irreducibles. For such a group G , we fix a maximal torus T_G and write $L_G(\lambda)$ to denote the irreducible KG -module having highest weight $\lambda \in X^+(T_G)$, where $X^+(T_G)$ denotes the character group of T_G . Also, we adopt Bourbaki notation [Bou68, Chapter VI, Section 4] concerning the labelling of the corresponding Dynkin diagram of G . Finally, if V is a KG -module on which G acts with exactly two composition factors having highest weights $\mu, \nu \in X^+(T_G)$, we write $V = \mu/\nu$ for simplicity. We refer the reader to page 191 for a complete list of notations.

In Chapter 3, we let Y be a simple algebraic group of classical type over K having rank n and let X be a maximal proper parabolic subgroup of Y . Here we may and will assume that $X = P_r$ is the parabolic subgroup of Y obtained by removing the r^{th} node in the corresponding Dynkin diagram of Y , for some $1 \leq r \leq n$. We then consider a non-trivial irreducible p -restricted KY -module V and observe that in this situation, $V|_X$ is reducible (see Lemma 3.1). Writing $X = QL$, where L is a Levi subgroup of X and Q the unipotent radical of X , we consider the well-known filtration of KL' -submodules of V

$$V \supset [V, Q] \supseteq [V, Q^2] \supseteq \dots \supseteq [V, Q^k] \supseteq [V, Q^{k+1}] = 0, \quad (1.1)$$

called the Q -commutator series of V (see Section 2.3.2 for more details). As $Q \trianglelefteq X$, the filtration (1.1) is a series of KX -submodules of V and hence if we suppose that X acts with exactly two composition factors on the latter, we immediately get $k = 1$. A result of Seitz [Sei87, Proposition 2.3] on the structure of the successive quotients of (1.1) then leads to structural information on $V|_X$ and allows us to narrow down the possible candidates for V . Finally, arguing by dimension on each of the aforementioned candidates leads to a complete classification of triples (Y, X, V) satisfying the desired condition (see Theorem 3.2).

Next let P be an arbitrary proper parabolic subgroup of Y and suppose that V is an irreducible p -restricted KY -module such that P has exactly two composition factors on V . Then P is contained in a maximal proper parabolic subgroup $X = QL$ acting with exactly two composition factors on V as well, say V_1, V_2 , such that $V_1|_P$ and $V_2|_P$ are irreducible. By Lemma 3.1 again, one deduces that either X is maximal among parabolic subgroups of Y , or L must be semisimple, which can only happen in one specific situation by Theorem 3.2. An argument on the Q -commutator series of V then shows the necessity for P to be maximal, yielding the following result. (Here T_Y and $T_{L'}$ are such that $T_{L'} \subset T_Y$ and we let $\{\lambda_1, \dots, \lambda_n\}$, $\{\omega_1, \dots, \omega_{n-1}\}$ respectively denote the corresponding sets of fundamental weights.)

Theorem 1

Let Y be a simple algebraic group of classical type over K and let X be a proper parabolic subgroup of Y . Also consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if $X = P_r$ is maximal among parabolic subgroups of Y and Y, X, λ are as in Table 1.1, where we give λ up to graph automorphisms.

Y	X	λ	$V _{L'}$	Dimensions
A_1	B_Y	λ_1	0/0	1, 1
$A_n(n \geq 2)$	P_1	λ_1	$0/\omega_1$	1, n
	P_r ($1 < r < n$)	λ_1	$\omega_{1,1}/\omega_{2,1}$	$r, n - r + 1$
	P_n	λ_1	$\omega_1/0$	$n, 1$
	P_1	λ_i ($1 < i < d$)	ω_{i-1}/ω_i	$\binom{n}{i-1}, \binom{n}{i}$
$B_n(n \geq 2)$	P_1	λ_n	$\omega_{n-1}/\omega_{n-1}$	$2^{n-1}, 2^{n-1}$
$C_n(n \geq 3)$	P_n	λ_1	ω_1/ω_{n-1}	n, n
$D_n(n \geq 4)$	P_n	λ_1	ω_1/ω_{n-1}	n, n
	P_1	λ_n	$\omega_{n-1}/\omega_{n-2}$	$2^{n-2}, 2^{n-2}$

Table 1.1: Triples (Y, X, V) where X is a maximal parabolic subgroup of Y . Here L' denotes the derived subgroup of a Levi subgroup of X and $d = \lfloor \frac{n+1}{2} \rfloor$ the integer part of $\frac{n+1}{2}$.

Remarks

In the fifth column of Table 1.1, we record the dimension of each composition factor of $V|_{L'}$ for completeness. Also, observe that $A_1 = B_1 = C_1 = D_1$, $B_2 = C_2$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$, thus justifying the conditions on n in the first column of Table 1.1. Finally, notice that the results in Theorem 1 are independent of p .

We next focus our attention on the embedding of $X = \text{Spin}_{2n}(K)$ in $Y = \text{Spin}_{2n+1}(K)$, where we view X as the derived subgroup of the stabilizer of a non-singular one-dimensional subspace of the natural module for Y . Fix T_Y a maximal torus of Y and T_X a maximal torus of X such that $T_X \subset T_Y$ and consider an irreducible KY -module V having p -restricted highest weight $\lambda \in X^+(T_Y)$. If $p \neq 2$, then it is easy to show (see Section 4.3.1, for example) that X has at least two composition factors on V , while on the other hand if $p = 2$, then $V|_X$ is almost always irreducible by [Sei87, Theorem 1, Table 1 (MR₄)]. In other words, $V|_X$ is reducible in general and we aim at determining whether or not X has exactly two composition factors on V . It turns out that this question is related to the aforementioned work of Ford [For96].

More precisely, Ford [For96, Section 3] considers an irreducible KY -module V having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r \in X^+(T_Y)$, where $\{\lambda_1, \dots, \lambda_n\}$ denotes a set of fundamental weights for T_Y . Assuming that the standard graph automorphism θ of order 2 of X does not act trivially on V , he immediately gets $a_n \neq 0$. Furthermore, since interested in pairs (λ, p) such that $X \langle \theta \rangle$ acts irreducibly on V , Ford easily deduces that we may as well assume $a_n = 1$, in which case X acts with exactly two composition factors on V , interchanged by θ . Working with the Lie algebras associated to Y and X , he then argues on the possible elements generating certain weight spaces in V and finally concludes relying on the fact that X can be seen as the subgroup of Y generated by the root subgroups corresponding to the long roots for T_Y .

Surprisingly, the argument of Ford can be generalized to fit the situation in which θ acts trivially on V , that is, $V = L_Y(\lambda)$ for some p -restricted T_Y -weight $\lambda = \sum_{r=1}^n a_r \lambda_r$ such that $a_n = 0$. In Chapter 4, we determine every pair (λ, p) such that X acts with exactly two composition factors on $V = L_Y(\lambda)$, thus extending [For96, Theorem 3.3]. For more details, we refer the reader to the preamble of Chapter 4, in which a brief outline of the proof is given. Here $\{\omega_1, \dots, \omega_n\}$ denotes the set of fundamental weights for T_X .

Theorem 2

Let Y be a simply connected simple algebraic group of type B_n over K and let X be the subgroup of type D_n , embedded in Y in the usual way. Also consider an irreducible non-trivial KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r \in X^+(T_Y)$, and if $\lambda \neq a_n \lambda_n$, let $1 \leq k < n$ be maximal such that $a_k \neq 0$. Then X has exactly two composition factors on V if and only if $a_n \leq 1$ and one of the following holds.

1. $\lambda = \lambda_k$ and $p \neq 2$.
2. $\lambda = \lambda_n$.
3. λ is neither as in 1 nor 2, $p \neq 2$ and the following divisibility conditions are satisfied.
 - (a) $p \mid a_i + a_j + j - i$ for every $1 \leq i < j < n$ such that $a_i a_j \neq 0$ and $a_r = 0$ for $i < r < j$.
 - (b) $p \mid 2(a_n + a_k + n - k) - 1$.

Furthermore, if (λ, p) is as in 1, 2 or 3, then $L_Y(\lambda)|_X$ is completely reducible.

Remark

Let (λ, p) be as in 2 or 3, with $a_n = 1$ in the latter case. Then the KX -composition factors of V have respective highest weights $\omega = \sum_{r=1}^{n-1} a_r \omega_r + (a_{n-1} + 1)\omega_n$ and $\omega' = \omega^\theta$. If on the other hand (λ, p) is as in 1 or 3, with $a_n = 0$ in the latter case, then the KX -composition factors of V have respective highest weights $\omega = \sum_{r=1}^k a_r \omega_r + \delta_{k,n-1} \omega_n$ and $\omega' = \sum_{r=1}^{k-2} a_r \omega_r + (a_{k-1} + 1)\omega_{k-1}$.

In the last three chapters of the thesis, we let $n \geq 3$ and consider the natural embedding of $X = \mathrm{SO}_{2n}(K)$ in $Y = \mathrm{SL}_{2n}(K)$. Also, let $\{\lambda_1, \dots, \lambda_{2n-1}\}$ be a set of fundamental weights for T_Y and fixing a maximal torus T_X of X such that $T_X \subset T_Y$, we get a set of fundamental T_X -weights $\{\omega_1, \dots, \omega_n\}$. Finally, let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight $\lambda \in X^+(T_Y)$.

In Chapter 5, we consider the case where $n = 3$ and start by observing that the restriction of λ to T_X , say ω , always affords the highest weight of a composition factor of V for X . In order to show the existence of a second KX -composition factor of V , it thus suffices to find a dominant T_X -weight $\mu \in X(T)$ such that $\dim(V|_X)_\mu > \dim L_X(\omega)_\mu$. Also, as soon as the highest weight, say ω' , of a second KX -composition factor of V is known, then finding a dominant T_X -weight $\nu \in X(T_X)$ such that $\dim(V|_X)_\mu > \dim L_X(\omega)_\mu + \dim L_X(\omega')_\mu$ yields the existence of a third composition factor of V for X . Applying this method, we thus get a smaller list of candidates for (λ, p) . We then conclude by comparing dimensions (using [Lüb01]), thus getting a complete classification of pairs (λ, p) satisfying the desired property (see Theorem 5.1).

In Chapter 6, we assume $n = 4$ and consider a D_3 -parabolic subgroup $P_X = Q_X L_X$ of X . Following the idea of Seitz [Sei87, Proposition 2.8], we first construct a canonical parabolic $P = QL$ of Y as the stabilizer of the filtration

$$W \supset [W, Q_X] \supset [W, Q_X^2] \supset \dots \supset [W, Q_X^k] \supset 0$$

of the natural KY -module W . It turns out that L' is simple of type A_5 and if X has exactly two composition factors on V , then L' acts with at most two composition factors on $V/[V, Q]$ (see Lemma 2.3.10). Consequently, a small list of candidates for (λ, p) can be deduced inductively thanks to the list obtained in the case where $n = 3$ and [Sei87, Theorem 1]. Again, arguing on weight multiplicities and dimensions then yields the desired result (see Theorem 6.1).

Finally, let $n > 4$ and assume a complete classification is known for every $N < n$. By considering a D_{n-1} -parabolic subgroup of $P_X = Q_X L_X$ of X and constructing a suitable parabolic $P = QL$ of Y as above, a shorter list of possible candidates can be obtained. However, the method described above to show the existence of a third KX -composition factor of V requires a very good knowledge of certain weight multiplicities in V and in the general case, even the use of the Jantzen p -sum formula fails to give us enough information to proceed further. Therefore, a complete classification was not obtained in this situation. Nevertheless, following the ideas of McNinch [McN98, Lemma 4.9.1], we were able to determine the structure of certain Weyl modules for X (see Theorem 5 below, for example), by embedding them in suitable tensor products. This led to a partial answer to the question, recorded in Theorem 3. Furthermore, we record a conjecture on what a complete classification should look like, based on various examples. Notice that the conjecture holds in the cases where $n = 3, 4$.

Theorem 3

Let Y be a simply connected simple algebraic group of type A_{2n-1} over K , with $n \geq 3$, and let X be the subgroup of type D_n , embedded in Y in the usual way. Also let λ and p be as in Table 1.2, with $p \nmid n+1$ in the case where $\lambda = 2\lambda_1 + \lambda_j$ for some $j \neq n-1$. Then X has exactly two composition factors on the irreducible KY -module $V = L_Y(\lambda)$. Furthermore, if $n \leq 4$ and X has exactly two composition factors on an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$, then λ and p are as in Table 1.2.

λ	Conditions	$V _X$
$2\lambda_1$	$p \nmid n$	$2\omega_1/0$
$3\lambda_1$	$p \nmid n+1$	$3\omega_1/\omega_1$
λ_2 (n odd)	$p = 2$	$\omega_2/0$
λ_3 (n even)	$p = 2$	ω_3/ω_1
λ_n	$p \neq 2$	$2\omega_{n-1}/2\omega_n$
$\lambda_1 + \lambda_j$ $1 < j < n-1$	$p \nmid 2n-j+1$	$\omega_1 + \omega_j/\omega_{j-1}$
$\lambda_1 + \lambda_{n-1}$	$p \nmid n+2$	$\omega_1 + \omega_{n-1} + \omega_n/\omega_{n-2}$
$\lambda_1 + \lambda_{n+2}$	$p \nmid n-1$	$\omega_1 + \omega_{n-2}/\omega_{n-1} + \omega_n$
$\lambda_1 + \lambda_j$ $n+2 < j < 2n$	$p \nmid 2n-j+1$	$\omega_1 + \omega_{2n-j}/\omega_{2n-j+1}$
$2\lambda_1 + \lambda_j$ $1 < j < n-1$	$p \mid j+2, p \nmid n+2$	$2\omega_1 + \omega_j/\omega_1 + \omega_{j-1}$
$2\lambda_1 + \lambda_{n-1}$	$p \mid n+1$	$2\omega_1 + \omega_{n-1} + \omega_n/\omega_1 + \omega_{n-2}$
$2\lambda_1 + \lambda_{n+2}$	$p \mid n+4$	$2\omega_1 + \omega_{n-2}/\omega_1 + \omega_{n-1} + \omega_n$
$2\lambda_1 + \lambda_j$ $n+1 < j < 2n$	$p \mid j+2, p \nmid n+2$	$2\omega_1 + \omega_{2n-j}/\omega_1 + \omega_{2n-j+1}$

Table 1.2: The case $\mathrm{SO}_{2n}(K) \subset \mathrm{SL}_{2n}(K)$.

Conjecture 4

Let Y be a simply connected simple algebraic group of type A_{2n-1} over K and let X be the subgroup of type D_n , embedded in Y in the usual way. Also consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if λ and p are as in Table 1.2.

Consequences and additional results

Applying the method introduced in Chapter 2, Section 2.3 to efficiently use *Freudenthal's formula* together with the *Jantzen p -sum formula* and other techniques, the investigation of the aforementioned embeddings lead to various results on weight multiplicities. If interested in such multiplicities in the case where G is of type A_3 over K , we refer the reader to Section 5.1. References to results on other weight multiplicities are recorded in Table 1.3.

G	λ	Conditions	μ	Reference
A_n ($n > 3$)	$a\lambda_1 + \lambda_{n-1}$	$a > 1$	$\lambda - 2 \dots 21$	Lemma 6.1.3
	$a\lambda_1 + \lambda_{n-2}$	$a > 2$	$\lambda - 3 \dots 321$	Lemma 6.1.4
	$a\lambda_1 + b\lambda_2 + c\lambda_n$	$abc > 0$	$\lambda - 1 \dots 1$	Proposition 6.1.10
	$a\lambda_2 + \lambda_{n-1}$	$a > 1$	$\lambda - 12 \dots 21$	Proposition 7.5.5
B_n ($n > 2$)	$a\lambda_1$	a	$\lambda - 2 \dots 2$	Proposition 4.2.4
	$a\lambda_1 + \lambda_2$	$a > 1$	$\lambda - 12 \dots 2$	Proposition 4.2.12
	$a\lambda_1 + \lambda_j$	$a > 1, 2 < j < n$	$\lambda - 1 \dots 12 \dots 2$	Proposition 4.2.18
D_n ($n > 3$)	$a\lambda_1$	$a > 1$	$\lambda - 2 \dots 211$	Lemma 7.2.2
	$2\lambda_2$		$\lambda - 12 \dots 211$	Lemma 7.4.11
	$\lambda_2 + \lambda_j$	$2 < j < n - 1$	$\lambda - 12 \dots 211$	Lemma 7.5.12

Table 1.3: Some weight multiplicities.

In the course of the investigation of the embedding $X \subset Y$, where $X = \mathrm{SO}_{2n}(K)$ and $Y = \mathrm{SL}_{2n}(K)$ are as above, we were able to generalize the idea of [McN98, Lemma 4.8.2] in order to determine the structure of the Weyl module $V_X(\omega_1 + \omega_j + \delta_{j,n-1}\omega_n)$ having highest weight $\omega_1 + \omega_j + \delta_{j,n-1}\omega_n$ for $p \neq 2$ and $1 < j < n$. The result, which was already known for $j = 2$ (see [McN98, Lemma 4.9.2]) is recorded in the following theorem, whose proof can be found in Section 7.3. We also record a direct consequence on the dimension of the corresponding irreducible KX -modules.

Theorem 5

Let X be a simple algebraic group of type D_n over K and assume $p \neq 2$. Also fix $1 < j < n$ and consider the dominant T_X -weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. Then the following assertions hold.

1. If $1 < j < n - 2$, we have $V_X(\omega) = \omega/\omega_{j+1}^{\epsilon_p(j+1)}/\omega_{j-1}^{\epsilon_p(2n-j+1)}$. Furthermore, if p divides both $j + 1$ and $2n - j + 1$, then $V_X(\omega) \supset L_X(\omega_{j+1}) \oplus L_X(\omega_{j-1}) \supset L_X(\omega_{j+1}) \supset 0$ is a composition series of $V_X(\omega)$.
2. If $j = n - 2$, we have $V_X(\omega) = \omega/(\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)}/\omega_{n-3}^{\epsilon_p(n+3)}$. Moreover, if p divides $(n-1)(n+3)$, then $V_X(\omega) \supset L_X(\omega_{n-3})^{\epsilon_p(n+3)} \oplus L_X(\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)} \supset 0$ is a composition series of $V_X(\omega)$.
3. If $\omega = \omega_1 + \omega_{n-1} + \omega_n$, we have $V_X(\omega) = \omega/2\omega_{n-1}^{\epsilon_p(n)}/2\omega_n^{\epsilon_p(n)}/\omega_{n-2}^{\epsilon_p(n+2)}$. Moreover, if p divides n , then $V_X(\omega) \supset L_X(2\omega_{n-1}) \oplus L_X(2\omega_n) \supset L_X(2\omega_{n-1}) \supset 0$ is a composition series of $V_X(\omega)$.

Corollary 6

Let X be a simple algebraic group of type D_n over K and assume $p \neq 2$. Also fix $1 < j < n$ and consider the dominant T_X -weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. Then

$$\dim L_X(\omega) = \binom{2n+2}{j+1} \frac{j(2n-j)}{2n+1} - \epsilon_p(j+1) \binom{2n}{j+1} - \epsilon_p(2n-j+1) \binom{2n}{j-1}.$$

2.1 Notation

We first fix some notation that will be used throughout the thesis. Let G be a semisimple algebraic group of classical type defined over an algebraically closed field K of characteristic $p \geq 0$. Also fix a Borel subgroup $B_G = U_G T_G$ of G , where T_G is a maximal torus of G and U_G denotes the unipotent radical of B_G . Let $n = \text{rank } G = \dim T_G$ and let $\Pi(G) = \{\alpha_1, \dots, \alpha_n\}$ be a corresponding base of the root system $\Phi(G) = \Phi^+(G) \cup \Phi^-(G)$ of G , where $\Phi^+(G)$ and $\Phi^-(G)$ denote the set of positive and negative roots of G , respectively. Let

$$X(T_G) = \text{Hom}(T_G, K^*)$$

denote the character group of T_G and write $(-, -)$ for the usual inner product on the vector space $X(T_G)_{\mathbb{R}} = X(T_G) \otimes_{\mathbb{Z}} \mathbb{R}$. Also let $\{\lambda_1, \dots, \lambda_n\}$ be the set of fundamental weights for T_G corresponding to our choice of base $\Pi(G)$, that is $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for every $1 \leq i, j \leq n$, where

$$\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)},$$

for every $\lambda, \alpha \in X(T_G)$. Set

$$X^+(T_G) = \{\lambda \in X(T_G) : \langle \lambda, \alpha \rangle \geq 0 \text{ for every } \alpha \in \Pi(G)\}$$

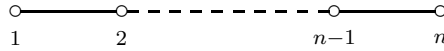
and call a character $\lambda \in X^+(T_G)$ a *dominant character*. Every such character can be written in the form $\lambda = \sum_{r=1}^n a_r \lambda_r$, where $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$. Finally, for $\alpha \in \Phi(G)$, define the reflection $s_\alpha : X(T_G)_{\mathbb{R}} \rightarrow X(T_G)_{\mathbb{R}}$ relative to α by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$, this for every $\lambda \in X(T_G)_{\mathbb{R}}$, and denote by $\mathscr{W} = \mathscr{W}_G$ the finite group $\langle s_{\alpha_i} : 1 \leq i \leq n \rangle$, called the *Weyl group of G* .

2.2 Bourbaki's construction of irreducible root systems

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a base of an irreducible root system of classical type Φ . In this section, we give a description of Π and the corresponding fundamental weights $\lambda_1, \dots, \lambda_n$ in terms of an orthonormal basis of a Euclidean space $(E, (-, -))$, as well as the description of the action of the corresponding Weyl group \mathscr{W} on Φ . We also record the value of $\rho = \sum_{r=1}^n \lambda_r$ in terms of the aforementioned basis, as it is needed in Chapters 5, 6 and 7. We refer the reader to [Bou68, Chapter VI] for more details.

2.2.1 $\Phi = A_n$ ($n \geq 2$)

Let $\Phi = A_n$ ($n \geq 2$) and let $\{\varepsilon_1, \dots, \varepsilon_{n+1}\}$ be an orthonormal basis of the Euclidean space $E = \mathbb{R}^{n+1}$, with standard inner product $(-, -)$. We choose the labelling of the associated Dynkin diagram as follows



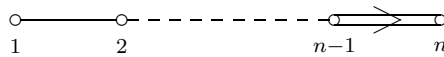
and denote by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a corresponding base of Φ . Here the set Φ^+ of positive roots in Φ corresponding to Π is given by $\Phi^+ = \{\alpha_i + \dots + \alpha_j\}_{1 \leq i \leq j \leq n}$ and Φ can be realized in E by setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq n$. One then easily checks that this yields $\alpha_i + \dots + \alpha_j = \varepsilon_i - \varepsilon_{j+1}$ for every $1 \leq i < j \leq n$. Also using [Hum78, Table 1, p.69] gives

$$\lambda_i = \sum_{r=1}^i \varepsilon_r - \frac{i}{n+1} \sum_{r=1}^{n+1} \varepsilon_r,$$

from which one deduces that $\rho = \frac{1}{2} \sum_{r=0}^n (n-2r)\varepsilon_{r+1}$. For $1 \leq i < j \leq n$, the element $s_{\varepsilon_i - \varepsilon_j}$ exchanges ε_i and ε_j , leaving ε_k ($k \neq i, j$) unchanged. Thus the Weyl group $\mathscr{W} \cong \mathfrak{S}_{n+1}$ acts by permuting the indices of the ε_i , $1 \leq i \leq n+1$. (Throughout this thesis, \mathfrak{S}_l denotes the symmetric group on $\{1, \dots, l\}$.)

2.2.2 $\Phi = B_n$ ($n \geq 2$)

Let $\Phi = B_n$ and let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be an orthonormal basis of the Euclidean space $E = \mathbb{R}^n$, with standard inner product $(-, -)$. We choose the labelling of the associated Dynkin diagram as follows



and denote by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a corresponding base of Φ . Here the set Φ^+ of positive roots in Φ is given by $\Phi^+ = \{\alpha_i + \dots + \alpha_j\}_{1 \leq i \leq j \leq n} \cup \{\alpha_i + \dots + \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_n\}_{1 \leq i \leq j \leq n-1}$.

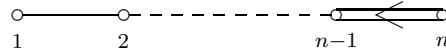
Also Φ can be realized in E by setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n - 1$, and $\alpha_n = \varepsilon_n$. Adopting the convention $\varepsilon_{n+1} = 0$, one checks that this yields

$$\begin{aligned}\varepsilon_i - \varepsilon_{j+1} &= \alpha_i + \cdots + \alpha_j, \\ \varepsilon_i + \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n,\end{aligned}$$

for $1 \leq i < j \leq n$. Also using [Hum78, Table 1, p.69] gives $\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $1 \leq i \leq n - 1$, and $\lambda_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$, from which one deduces that $\rho = \frac{1}{2} \sum_{r=0}^{n-1} (2(n-r) - 1)\varepsilon_{r+1}$. Here for $1 \leq i < j \leq n$, the element $s_{\varepsilon_i - \varepsilon_j}$ exchanges ε_i and ε_j , leaving ε_k ($k \neq i, j$) invariant, while for $1 \leq i \leq n$, the element s_{ε_i} sends ε_i to $-\varepsilon_i$, leaving ε_k ($k \neq i$) unchanged. Thus the Weyl group $\mathscr{W} \cong \mathfrak{S}_2 \wr \mathfrak{S}_n = (\mathfrak{S}_2)^n \cdot \mathfrak{S}_n$ acts by all permutations and sign changes of the ε_i , $1 \leq i \leq n$.

2.2.3 $\Phi = C_n$ ($n \geq 3$)

Let $\Phi = C_n$ and let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be an orthonormal basis of the Euclidean space $E = \mathbb{R}^n$, with standard inner product $(-, -)$. We choose the labelling of the associated Dynkin diagram as follows



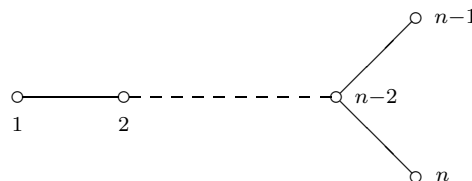
and denote by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a corresponding base of Φ . Here the set Φ^+ of positive roots in Φ corresponding to Π is given by

$$\begin{aligned}\Phi^+ &= \{\alpha_i + \cdots + \alpha_j\}_{1 \leq i \leq j \leq n} \cup \{\alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \alpha_n\}_{1 \leq i \leq j \leq n-2} \\ &\quad \cup \{2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n\}_{1 \leq i \leq n-1}\end{aligned}$$

and Φ can be realized in E by setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n - 1$, and $\alpha_n = 2\varepsilon_n$. Again one checks that this gives $\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $1 \leq i \leq n$, from which one deduces that $\rho = \sum_{r=0}^{n-1} (n-r)\varepsilon_{r+1}$. Here the Weyl group $\mathscr{W} \cong \mathfrak{S}_2 \wr \mathfrak{S}_n = (\mathfrak{S}_2)^n \cdot \mathfrak{S}_n$ acts on the basis of E exactly as in the case $\Phi = B_n$.

2.2.4 $\Phi = D_n$ ($n \geq 4$)

Finally, let $\Phi = D_n$ and let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be an orthonormal basis of the Euclidean space $E = \mathbb{R}^n$, with standard inner product $(-, -)$. We choose the labelling of the associated Dynkin diagram as follows



and denote by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a corresponding base of Φ . Here the set Φ^+ of positive roots in Φ corresponding to Π is given by

$$\begin{aligned} \Phi^+ = & \{\alpha_i + \dots + \alpha_n\}_{1 \leq i \leq n-2} \cup \{\alpha_i + \dots + \alpha_{n-2} + \alpha_n\}_{1 \leq i \leq n-2} \cup \{\alpha_i + \dots + \alpha_j\}_{1 \leq i < j \leq n-1} \\ & \cup \{\alpha_i + \dots + \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\}_{1 \leq i < j \leq n-3} \end{aligned}$$

and Φ can be realized in E by setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. One checks that this gives

$$\begin{aligned} \varepsilon_i - \varepsilon_j &= \alpha_i + \dots + \alpha_{j-1}, \\ \varepsilon_r + \varepsilon_s &= \alpha_r + \dots + \alpha_{s-1} + 2\alpha_s + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \\ \varepsilon_s + \varepsilon_{n-1} &= \alpha_s + \dots + \alpha_n, \\ \varepsilon_s + \varepsilon_n &= \alpha_s + \dots + \alpha_{n-2} + \alpha_n, \end{aligned}$$

for every $1 \leq i < j \leq n$ and $1 \leq r < s \leq n-2$. Also using [Hum78, Table 1, p.69] yields $\lambda_i = \varepsilon_1 + \dots + \varepsilon_i$ for $1 \leq i \leq n-2$, while $\lambda_{n-1} = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{n-1} - \varepsilon_n)$, and $\lambda_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$, from which one deduces that $\rho = \sum_{r=1}^n (n-r)\varepsilon_r$. Here for $1 \leq i < j \leq n$, the element $s_{\varepsilon_i - \varepsilon_j}$ exchanges ε_i and ε_j , leaving ε_k ($k \neq i, j$) unchanged. Similarly, for $1 \leq i < j \leq n$, the element $s_{\varepsilon_i - \varepsilon_j} s_{\varepsilon_i + \varepsilon_j}$ sends ε_i to $-\varepsilon_i$, ε_j to $-\varepsilon_j$, and leaves ε_k ($k \neq i, j$) unchanged. Thus the Weyl group $\mathscr{W} \cong (\mathfrak{S}_2)^{n-1} \cdot \mathfrak{S}_n$ acts as the group of all permutations and even number of sign changes of the ε_i , $1 \leq i \leq n$.

2.3 Weights and multiplicities

Let G be a semisimple algebraic group defined over an algebraically closed field K of characteristic $p \geq 0$, $B = B_G = UT$ a Borel subgroup of G , where $T = T_G$ is a maximal torus of G and $U = U_G$ is the unipotent radical of B , and let V denote a finite-dimensional rational KG -module. (Throughout this thesis, we shall only consider finite-dimensional, rational modules.) Unless specified otherwise, the results recorded in this section can be found in [Hum75, Chapter XI, Section 31]. Recall first that V can be decomposed into a direct sum of KT -modules

$$V = \bigoplus_{\mu \in X(T)} V_\mu,$$

where for every $\mu \in X(T)$,

$$V_\mu = \{v \in V : t \cdot v = \mu(t)v \text{ for every } t \in T\}.$$

A character $\mu \in X(T)$ with $V_\mu \neq 0$ is called a T -weight of V , and V_μ is said to be its corresponding *weight space*. The dimension of V_μ is called the *multiplicity of μ in V* and is denoted by $m_V(\mu)$. Write $\Lambda(V)$ to denote the set of T -weights of V , and define a partial order on the latter by saying that $\mu \in \Lambda(V)$ is *under* $\lambda \in \Lambda(V)$ (written $\mu \preceq \lambda$) if and only if there exist non-negative integers c_α ($\alpha \in \Pi$) such that $\mu = \lambda - \sum_{\alpha \in \Pi} c_\alpha \alpha$. We also write $\mu \prec \lambda$ to indicate that μ is strictly under λ and set $\Lambda^+(V) = \Lambda(V) \cap X^+(T)$. Any weight in $\Lambda^+(V)$ is called *dominant*.

The natural action of the Weyl group \mathscr{W} of G on $X(T)$ induces an action on $\Lambda(V)$ and we say that $\lambda, \mu \in X(T)$ are *conjugate under the action of \mathscr{W}* (or *\mathscr{W} -conjugate*) if there exists $w \in \mathscr{W}$ such that $w\lambda = \mu$. It is well-known (see [Hum78, Section 13.2, Lemma A], for example) that each weight in $X(T)$ is \mathscr{W} -conjugate to a unique dominant weight. Also, if $\lambda \in X^+(T)$, then $w\lambda \preceq \lambda$ for every $w \in \mathscr{W}$. Finally, $\Lambda(V)$ is a union of \mathscr{W} -orbits and all weights in a \mathscr{W} -orbit have the same multiplicity.

Definition 2.3.1

Let $G, B = UT$ and V be as above. A dominant T -weight $\lambda \in \Lambda^+(V)$ is called a *highest weight of V* if $\{\mu \in \Lambda^+(V) : \lambda \prec \mu\} = \emptyset$.

Now by the Lie-Kolchin Theorem ([Hum75, Theorem 17.6]), there exists $0 \neq v^+ \in V$ such that $\langle v^+ \rangle_K$ is invariant under the action of B . We call such a vector v^+ a *maximal vector in V for B* . Note that since $\langle v^+ \rangle_K$ is stabilized by any maximal torus of B , there exists $\lambda \in X(T)$ such that $v^+ \in V_\lambda$. In fact, one can show that $\lambda \in \Lambda^+(V)$.

2.3.1 Irreducible modules

In general, an arbitrary finite-dimensional KG -module V can have many distinct highest weights. However if V is irreducible and $v^+ \in V_\lambda$ is a maximal vector in V for B , then $V = Gv^+$, $m_V(\lambda) = 1$, and every weight $\mu \in \Lambda(V)$ can be obtained from λ by subtracting positive roots, so that λ is the unique highest weight of V . Reciprocally, given a dominant weight $\lambda \in X^+(T)$, one can construct a finite-dimensional irreducible KG -module with highest weight λ . This correspondence defines a bijection

$$X^+(T) \longleftrightarrow \{\text{isomorphism classes of irreducible } KG\text{-modules}\}.$$

From now on, for $\lambda \in X^+(T)$, we let $L_G(\lambda)$ denote the irreducible KG -module having highest weight λ . In addition, we say that λ is *p -restricted* if $p = 0$ or $0 \leq \langle \lambda, \alpha \rangle < p$, for every $\alpha \in \Pi$. It is only natural to wonder whether a given irreducible KG -module is tensor indecomposable or not (in characteristic zero, all irreducible modules are tensor indecomposable) and a partial answer to this question is given by the following well-known result, due to Steinberg (see [Ste63, Theorem 1] for a proof). Here $F : G \rightarrow G$ is a standard Frobenius morphism and for V a KG -module, V^{F^i} is the KG -module on which G acts via $g \cdot v = F^i(g) \cdot v$, for every $g \in G, v \in V$.

Theorem 2.3.2 (The Steinberg Tensor Product Theorem)

Assume $p > 0$ and G is simply connected. Let $\lambda \in X^+(T)$ be a dominant T -weight. Then there exist $k \in \mathbb{Z}_{\geq 0}$ and p -restricted dominant T -weights $\mu_0, \mu_1, \dots, \mu_k \in X^+(T)$ such that $\lambda = \mu_0 + \mu_1 p + \dots + \mu_k p^k$ and

$$L_G(\lambda) \cong L_G(\mu_0) \otimes L_G(\mu_1)^F \otimes \dots \otimes L_G(\mu_k)^{F^k}.$$

In view of Theorem 2.3.2, in positive characteristic, the investigation of all irreducible KG -modules is reduced to the study of the finitely many ones having p -restricted highest weights, on which we focus our attention in the remainder of this section. Let then $\lambda \in X^+(T)$ be such a weight and denote by $V_{\mathcal{L}_{\mathbb{C}}(G)}(\lambda)$ the $\mathcal{L}_{\mathbb{C}}(G)$ -module over \mathbb{C} having highest weight λ . (Here $\mathcal{L}_{\mathbb{C}}(G)$ denotes the Lie algebra over \mathbb{C} having same type as G .) Choosing a minimal admissible lattice in $V_{\mathcal{L}_{\mathbb{C}}(G)}(\lambda)$ allows one to “reduce modulo p ”, providing the latter with a structure of KG -module, denoted $V_G(\lambda)$. We refer to [Hum78, 26.4] for a proof of the existence of such a lattice. In the literature, $V_G(\lambda)$ is referred to as the *Weyl module of G with highest weight λ* and we recall that $V_G(\lambda)$ is generated by a maximal vector for B of weight λ . It is indecomposable and has a unique maximal submodule $\text{rad}(\lambda)$ (called the *radical of $V_G(\lambda)$*) such that $L_G(\lambda) \cong V_G(\lambda)/\text{rad}(\lambda)$.

Definition 2.3.3

A pair (G, p) is called *special* if G is simple and $(\Phi(G), p) \in \{(B_n, 2), (C_n, 2), (F_4, 2), (G_2, 3)\}$.

It is well-known (see [Hum78, 21.3]) that the set of weights of $V_G(\lambda)$, written $\Lambda(\lambda)$, is *saturated* (i.e. $\mu - i\alpha \in \Lambda(\lambda)$ for every $\mu \in \Lambda(\lambda)$, $\alpha \in \Phi$ and $0 \leq i \leq \langle \mu, \alpha \rangle$), containing all dominant weights under λ (such weights are said to be *subdominant to λ*) together with all their \mathscr{W} -conjugates. Obviously $\Lambda(L_G(\lambda)) \subseteq \Lambda(\lambda)$ and it turns out that the converse also holds if (G, p) is not special.

Theorem 2.3.4 (Premet, [Pre87])

Let $\lambda \in X^+(T)$ be a p -restricted dominant weight for T , and assume (G, p) is simple but not special. Then $\Lambda(L_G(\lambda)) = \Lambda(\lambda)$.

2.3.2 Parabolic embeddings

For $\alpha \in \Phi$, set $U_\alpha = \{u_\alpha(c) : c \in K\}$, where $u_\alpha : K \rightarrow G$ is an injective morphism of algebraic groups such that $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$ for every $t \in T$ and $c \in K$. Also for $J \subset \Pi$, denote by Φ_J^+ the subset of Φ^+ generated by the simple roots in J and define *the opposite of the standard parabolic subgroup of G corresponding to J* to be $P_J = Q_J L_J$, where

$$L_J = \langle T, U_{\pm\alpha} : \alpha \in J \rangle, \quad Q_J = \langle U_{-\beta} : \beta \in \Phi^+ - \Phi_J^+ \rangle$$

respectively, denote a Levi factor of P_J with root system Φ_J , respectively the unipotent radical of P_J . Finally, let $V = L_G(\lambda)$ be an irreducible KG -module having p -restricted highest weight $\lambda \in X^+(T)$.

Definition 2.3.5

Let μ be a T -weight of V , so $\mu = \lambda - \sum_{\alpha \in \Pi} c_\alpha \alpha$, with $c_\alpha \in \mathbb{Z}_{\geq 0}$ for every $\alpha \in \Pi$. Then the Q_J -level of μ is $\sum_{\alpha \in \Pi - J} c_\alpha$.

Following the ideas of [Sei87, Section 2], we define a series of KL_J -modules by setting $[V, Q_J^0] = V$ and $[V, Q_J^i] = \langle qv - v : v \in [V, Q_J^{i-1}], q \in Q_J \rangle$, for every $i \in \mathbb{Z}_{>0}$. The flag

$$V \supset [V, Q_J] \supset [V, Q_J^2] \supset \dots \supset 0 \quad (2.1)$$

is called the Q_J -commutator series of V . Observe that for every $i \in \mathbb{Z}_{\geq 0}$, the KL_J -module $[V, Q_J^i]$ is Q_J -stable as well, making (2.1) a series of KP_J -modules. We now record a few results on this filtration, starting with a description of its first quotient $V/[V, Q_J]$. In the remainder of this section, we let $T_{L'_J} = T \cap L'_J$.

Lemma 2.3.6

The KL'_J -module $V/[V, Q_J]$ is irreducible with highest $T_{L'_J}$ -weight $\lambda|_{T_{L'_J}}$.

Proof. See [Smi82] or [Sei87, Proposition 2.1]. □

The following consequence of Lemma 2.3.6 makes it easier to compute weight multiplicities in certain situations. We leave the easy proof to the reader.

Lemma 2.3.7

Let J and V be as above, and consider $\mu = \lambda - \sum_{j \in J} c_j \alpha_j \in \Lambda(V)$. Then $m_V(\mu) = m_{V'}(\mu')$, where $\mu' = \mu|_{T_{L'_J}}$ and $V' = V_{L'_J}(\lambda|_{T_{L'_J}})$.

From Lemma 2.3.6, we know that the first quotient of (2.1) is irreducible as a KL'_J -module. The next result gives a description of the remaining terms and successive quotients of the Q_J -commutator series of V using the notion of Q_J -levels introduced above. We recall that the pair (G, p) is special if $(\Phi(G), p) \in \{(B_n, 2), (C_n, 2), (F_4, 2), (G_2, 3)\}$.

Proposition 2.3.8

Let $i \in \mathbb{Z}_{\geq 0}$ be a non-negative integer and suppose that (G, p) is not special. Then the following assertions hold.

1. $[V, Q_J^i] = \bigoplus V_\mu$, the sum ranging over the weights $\mu \in X(T)$ having Q_J -level at least i .
2. $[V, Q_J^i]/[V, Q_J^{i+1}] \cong \bigoplus V_\mu$, the sum ranging over the weights $\mu \in X(T)$ having Q_J -level exactly i .

Proof. See [Sei87, Proposition 2.3]. □

Now let Y be a simply connected simple algebraic group of classical type over K and let X be a closed semisimple subgroup of Y acting irreducibly on $W = V_Y(\lambda_1)$. Assume (Y, p) non-special and let $P_X = Q_X L_X$ be a parabolic subgroup of X . One can use the Q_X -commutator series of the natural KY -module W to construct a parabolic subgroup of Y with some nice properties. This construction was initially introduced by Seitz.

Lemma 2.3.9

The stabilizer in Y of the Q_X -commutator series of W is a parabolic subgroup $P_Y = Q_Y L_Y$ of Y which satisfies the following properties.

1. $P_X \leq P_Y$ and $Q_X \leq Q_Y$.
2. $L_Y = C_Y(Z)$ is a Levi factor of P_Y containing L_X , where $Z = Z(L_X)^\circ$.
3. If T_Y is a maximal torus of Y containing T_X , then $T_Y \leq L_Y$.

Proof. See [Sei87, Proposition 2.8] or [For96, 2.7]. □

Finally, let Y , X and P_X be as above, with $P_Y = Q_Y L_Y$ the parabolic subgroup of Y given by Lemma 2.3.9, and let V be an irreducible KY -module having p -restricted highest weight. Recall that by Lemma 2.3.6, the KL_Y -module $V/[V, Q_Y]$ is irreducible.

Lemma 2.3.10

If X has exactly two composition factors on V , then either L_X acts irreducibly on $V/[V, Q_Y]$ or has exactly two composition factors on it.

Proof. By assumption, there exists an irreducible maximal KX -submodule M of V . We have $[V/M, Q_X] = ([V, Q_X] + M)/M$, which gives the isomorphism

$$V / ([V, Q_X] + M) \cong V/M / [V/M, Q_X]. \quad (2.2)$$

The latter being irreducible for L_X by Lemma 2.3.6, we get that $[V, Q_X] + M$ is a maximal KL_X -submodule of V . Hence considering the series $V \supset [V, Q_Y] + M \supset [V, Q_X] + M \supset 0$ gives either

$$[V, Q_Y] + M = V \text{ or } [V, Q_Y] \subset [V, Q_X] + M.$$

In the former case, observe that $M \not\subset [V, Q_Y]$ (since $M \subsetneq V$), so that we immediately get $[M, Q_X] \subseteq M \cap [V, Q_X] \subseteq M \cap [V, Q_Y] \subsetneq M$, and as M is irreducible as a KX -module, Lemma 2.3.6 applies, yielding $[M, Q_X] = M \cap [V, Q_Y]$. Therefore since $[V, Q_Y] + M = V$, we have

$$V/[V, Q_Y] \cong M/(M \cap [V, Q_Y]) = M/[M, Q_X],$$

hence the irreducibility of $V/[V, Q_Y]$ for L_X .

In the case where $[V, Q_Y] \subset [V, Q_X] + M$, first observe that if $M \subset [V, Q_Y]$ (so that $[V, Q_Y] = [V, Q_X] + M$), then $[V, Q_Y]$ is a maximal KL_X -module of V by (2.2), so the result holds in this situation. If on the other hand $M \not\subset [V, Q_Y]$, then consider the filtration of KL_X -modules

$$V/[V, Q_Y] \supset ([V, Q_X] + M)/[V, Q_Y] \supsetneq 0.$$

Using (2.2) shows that $([V, Q_X] + M)/[V, Q_Y]$ is a maximal KL_X -submodule of $V/[V, Q_Y]$, thus in order to complete the proof, we only need to show that $([V, Q_X] + M)/[V, Q_Y]$ is irreducible as a KL_X -module. As above $[M, Q_X] = M \cap [V, Q_Y]$, so that

$$([V, Q_X] + M)/[V, Q_Y] = ([V, Q_Y] + M)/[V, Q_Y] \cong M/[M, Q_X].$$

An application of Lemma 2.3.6 then yields the desired result. \square

2.3.3 Weight multiplicities

Since knowing the multiplicity of a given T -weight in $V_G(\lambda)$ is a first step in computing its multiplicity in $L_G(\lambda)$, we introduce a way of calculating $m_{V_G(\lambda)}(\mu)$ for a weight μ subdominant to a given $\lambda \in X^+(T)$, using the well-known Freudenthal's formula. Set

$$d(\lambda, \mu) = 2(\lambda + \rho, \lambda - \mu) - (\lambda - \mu, \lambda - \mu), \quad (2.3)$$

where ρ denotes the half-sum of all positive roots in Φ , or equivalently, the sum of all fundamental weights, as defined in Section 2.2. The following formula gives a recursive way to compute the multiplicity of μ in $V_G(\lambda)$. We refer the reader to [Hum78, Theorem 22.3] for more details.

Theorem 2.3.11 (Freudenthal's Formula)

Let λ be as above and let $\mu \in X(T)$ be such that $\mu \prec \lambda$. Then the multiplicity of μ in $V_G(\lambda)$ is given recursively by

$$m_{V_G(\lambda)}(\mu) = \frac{2}{d(\lambda, \mu)} \sum_{i>0} \sum_{\alpha>0} m_{V_G(\lambda)}(\mu + i\alpha)(\mu + i\alpha, \alpha).$$

Assume $\text{rank } G = n$ and consider $\lambda = \sum_{r=1}^n a_r \lambda_r \in X^+(T)$. Write $\Lambda^+(\lambda) = \Lambda(\lambda) \cap X^+(T)$ and let $\mu \in X(T)$ be such that $\mu = \lambda - \sum_{r=1}^n c_r \alpha_r \in \Lambda^+(\lambda)$ for some $c_1, \dots, c_n \in \mathbb{Z}_{\geq 0}$. Adopting the notation $\lambda_0 = \lambda_{n+1} = 0$, we then define

$$\mu_{i,x} = \sum_{r=0}^{i-1} a_r \lambda_r + x \lambda_i + \sum_{r=i+1}^{n+1} a_r \lambda_r - \sum_{r=1}^n c_r \alpha_r,$$

for every $1 \leq i \leq n$ and $x \in \mathbb{Z}_{>0}$. (Observe that $\lambda_{i,a_i} = \lambda$ and $\mu_{i,a_i} = \mu$ for every $1 \leq i \leq n$.) Finally, for $1 \leq i \leq n$, write $S_i = \{x \in \mathbb{Z} : x \geq a_i\}$. Using Theorem 2.3.11, we study the value of $m_{V_G(\lambda_{i,x})}(\mu_{i,x})$ for $1 \leq i \leq n$ and $x \in S_i$ satisfying a certain condition.

Proposition 2.3.12

Let K, G be as above and let $\lambda = \sum_{r=1}^n a_r \lambda_r \in X^+(T)$. Also let $\mu \prec \lambda$ be a dominant T -weight and assume the existence of $1 \leq i \leq n$ such that

$$m_{V_G(\lambda_{i,x})}(\mu_{i,x} + j\alpha) = m_{V_G(\lambda_{i,y})}(\mu_{i,y} + j\alpha) \quad (2.4)$$

for every $\alpha \in \Phi^+$, $x, y \in S_i$, and $j \in \mathbb{Z}_{>0}$. Then $m_{V_G(\lambda_{i,x})}(\mu_{i,x}) = m_{V_G(\lambda)}(\mu)$ for every $x \in S_i$.

Proof. Write $\mu = \lambda - \sum_{r=1}^n c_r \alpha_r$. If $c_i = 0$, then set $J = \Pi - \{\alpha_i\}$ and adopting the notation introduced in Section 2.3.2, consider the parabolic subgroup $P_J = Q_J L_J$ of G . Denote by

$$H = L'_J = \langle U_{\pm \alpha_r} : 1 \leq r \leq n, r \neq i \rangle$$

the derived subgroup of the Levi subgroup of P_J , so H is semisimple and J is a base of the root system of H . Hence $(\lambda_{i,x})|_{T_H} = \lambda|_{T_H}$ and $(\mu_{i,x})|_{T_H} = \mu|_{T_H}$ for every $x \in \mathbb{Z}_{\geq 0}$, so that an application of Lemma 2.3.7 yields

$$m_{V_G(\lambda_{i,x})}(\mu_{i,x}) = m_{V_H(\lambda|_{T_H})}(\mu|_{T_H})$$

for every $x \in \mathbb{Z}_{\geq 0}$. Therefore $m_{V_G(\lambda_{i,x})}(\mu_{i,x})$ is independent of x if $c_i = 0$ and the result holds in this situation.

Assume $c_i \neq 0$ for the remainder of the proof and denote by $\mathbb{Z}[X]_r$ (respectively, $\mathbb{Z}[X]_{\leq r}$) the set of all polynomials in the indeterminate X with coefficients in \mathbb{Z} and having degree r (respectively, at most r). Writing $\nu = \lambda_{i,x} - \mu_{i,x}$, we get

$$\begin{aligned} d(\lambda_{i,x}, \mu_{i,x}) &= 2(\lambda_{i,x} + \rho, \nu) - (\nu, \nu) \\ &= 2c_i(\lambda_i, \alpha_i)x + 2 \sum_{r=1}^{i-1} (a_r \lambda_r, \nu) + \sum_{r=i+1}^n (a_r \lambda_r, \nu) - (\nu, \nu) \end{aligned}$$

for every $x \in \mathbb{Z}_{>0}$, and since $\nu = \sum_{r=1}^n c_r \alpha_r$ is independent of x , there exists $f \in \mathbb{Z}[X]_1$ such that $d(\lambda_{i,x}, \mu_{i,x}) = f(x)$ for every $x \in \mathbb{Z}_{>0}$ (and hence for every $x \in S_i$ as well). Also, one easily checks that by (2.4), there exists $g \in \mathbb{Z}[X]_{\leq 1}$ such that

$$m_{V_G(\lambda_{i,x})}(\mu_{i,x}) = \frac{g(x)}{f(x)}$$

for every $x \in S_i$. Now by Theorem 2.3.11, $\frac{g(x)}{f(x)} \in \mathbb{Z}_{>0}$ for every $x \in S_i$, showing the existence of $h \in \mathbb{Z}[X]$ such that $g = hf$. Therefore $\deg(f) = \deg(g)$ and we get that $h \in \mathbb{Z}$, from which the result follows. \square

We next use Theorem 2.3.11 together with Proposition 2.3.12 in order to determine weight multiplicities in various Weyl modules for a simple algebraic group of type A_n over K , starting with the following well-known result.

Lemma 2.3.13

Let G be a simple algebraic group of type A_n over K and fix $a, b \in \mathbb{Z}_{>0}$. Also consider the T -weight $\lambda = a\lambda_1 + b\lambda_n$ and let $\mu = \lambda - (\alpha_1 + \cdots + \alpha_n)$. Then

$$m_{V_G(\lambda)}(\mu) = n.$$

Proof. Since $(\alpha, \alpha) = 1$ for every $\alpha \in \Phi$ (as all roots in Φ are long in this situation), we have $\langle \alpha, \beta \rangle = \frac{1}{2}(\alpha, \beta)$ for every $\alpha, \beta \in \Phi$. Here $\Phi^+ = \{\alpha_r + \cdots + \alpha_s : 1 \leq r \leq s \leq n\}$ and one easily shows that

$$m_{V_G(\lambda_{1,x})}(\mu_{1,x} + j\alpha) = \delta_{j,1} \quad (2.5)$$

for every $x, j \in \mathbb{Z}_{>0}$ and $\alpha \in \Phi^+$. On the other hand, a straightforward computation yields the existence of $k \in \mathbb{Z}$ such that $d(\lambda_{1,x}, \mu_{1,x}) = x + k$ for every $x \in \mathbb{Z}_{>0}$. Now using (2.5), we successively get

$$\begin{aligned} \sum_{j>0} \sum_{\alpha>0} m_{V_G(\lambda_{1,x})}(\mu_{1,x} + j\alpha)(\mu_{1,x} + j\alpha, \alpha) &= \sum_{\alpha>0} (\mu_{1,x} + \alpha, \alpha) \\ &= \sum_{r=1}^n (\mu_{1,x}, \alpha_r) + l \\ &= \frac{1}{2}nx + l, \end{aligned}$$

where $l = |\Phi^+| + \sum_{r=2}^n \sum_{s=r}^n (\mu_{1,x}, \alpha_r + \cdots + \alpha_s) \in \mathbb{Z}$ is independent of x . Therefore an application of Theorem 2.3.11 yields

$$m_{V_G(\lambda_{1,x})}(\mu_{1,x}) = \frac{nx+2l}{x+k}, \quad (2.6)$$

for every $x \in \mathbb{Z}_{>0}$. Finally, notice that (2.5) holds for any $x \in S_1$ (as $S_1 \subset \mathbb{Z}_{>0}$) and hence Proposition 2.3.12 applies, so that (2.6) translates to $m_{V_G(\lambda)}(\mu) = \frac{nx+2l}{x+k}$ for every $x \in S_1$. As $m_{V_G(\lambda)}(\mu)$ is independent of x , the result follows. \square

Lemma 2.3.14

Let G be a simple algebraic group of type A_n over K and fix $a \in \mathbb{Z}_{>1}$. Also consider the T -weight $\lambda = a\lambda_1 + \lambda_{n-1} \in X^+(T)$ and let $\mu = \lambda - (2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n)$. Then

$$m_{V_G(\lambda)}(\mu) = \frac{1}{2}(n-1)n.$$

Proof. Proceed exactly as in the proof of Lemma 2.3.13, first observing that thanks to the latter, we have

$$m_{V_G(\lambda_{1,x})}(\mu_{1,x} + j\alpha) = \begin{cases} 1 & \text{if } \alpha = \alpha_n; \\ \delta_{j,1}(n-1) & \text{otherwise} \end{cases}$$

for every $x > 1$, $j \in \mathbb{Z}_{>0}$ and $\alpha \in \Phi^+$, so that Proposition 2.3.12 applies (with $S_1 = \mathbb{Z}_{>1}$), yielding $m_{V_G(\lambda)}(\mu) = m_{V_G(\lambda_{1,x})}(\mu_{1,x})$ for every $x \in S_1$. Again, one easily shows the existence of $k, l \in \mathbb{Z}$ such that $d(\lambda_{1,x}, \mu_{1,x}) = 2x + k$ for every $x \in S_1$ and

$$\sum_{j>0} \sum_{\alpha>0} m_{V_G(\lambda_{1,x})}(\mu_{1,x} + j\alpha)(\mu_{1,x} + j\alpha, \alpha) = \frac{1}{2}(n-1)nx + l.$$

Therefore an application of Theorem 2.3.11 yields $m_{V_G(\lambda)}(\mu) = \frac{(n-1)nx+2l}{2x+k}$ for every $x \in S_1$, and arguing exactly as in the proof of Lemma 2.3.13 completes the proof. \square

Finally, assume G is of type A_n and let $\lambda = \sum_{r=1}^n a_r \lambda_r$, where $a_1 a_n \neq 0$. Also let $I_\lambda = \{r_1, \dots, r_{N_\lambda}\}$ be maximal in $\{1, \dots, n\}$ such that $r_1 < \dots < r_{N_\lambda}$ and $\prod_{r \in I_\lambda} a_r \neq 0$.

Proposition 2.3.15

Let G be a simple algebraic group of type A_n over K and consider the dominant T -weight $\lambda = \sum_{r=1}^n a_r \lambda_r$, where $a_1 a_n \neq 0$. Also let $I_\lambda = \{r_1, \dots, r_{N_\lambda}\}$ be as above and consider $\mu = \lambda - (\alpha_1 + \dots + \alpha_n)$. Then

$$m_{V_G(\lambda)}(\mu) = \prod_{i=2}^{N_\lambda} (r_i - r_{i-1} + 1).$$

Proof. First assume $N_\lambda = 2$, that is, $I_\lambda = \{1, n\}$ and $\lambda = a_1 \lambda_1 + a_n \lambda_n$ for some $a_1, a_n \in \mathbb{Z}_{>0}$. Using Lemma 2.3.13, one gets $m_{V_G(\lambda)}(\mu) = n$ and hence the result holds in this situation. Moreover, since $m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_r) = 1$ for every $1 \leq r \leq n$ one gets $\sum_{r=1}^n m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_r) = n = m_{V_G(\lambda)}(\mu)$. Proceeding by induction on N_λ , we will show that

$$m_{V_G(\lambda)}(\mu) = \sum_{r=1}^n m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_r) = \prod_{i=2}^{N_\lambda} (r_i - r_{i-1} + 1). \quad (2.7)$$

Assume the existence of $N_0 \in \mathbb{Z}_{>0}$ such that (2.7) holds for every $\lambda' = \sum_{r=1}^n a'_r \lambda_r$ with $a'_1 a'_n \neq 0$ and $2 \leq N_{\lambda'} < N_0$, and let $\lambda \in X^+(T)$ be such that $N_\lambda = N_0$. An appropriate use of Lemma 2.3.7 and our induction hypothesis shows that $m_{V_G(\lambda)}(\mu + \alpha) = m_{V_G(\lambda_{1,x})}(\mu_{1,x} + \alpha)$ for every $x \in \mathbb{Z}_{>0}$ and $\alpha \in \Phi^+$. Therefore

$$m_{V_G(\lambda)}(\mu) = m_{V_G(\lambda_{1,x})}(\mu_{1,x}) \quad (2.8)$$

for every $x \in \mathbb{Z}_{>0}$ by Proposition 2.3.12. Now a straightforward computation yields the existence of $k \in \mathbb{Z}$ such that $d(\lambda_{1,x}, \mu_{1,x}) = x + k$ for every $x \in \mathbb{Z}_{>0}$, as well as $l \in \mathbb{Z}$ such that

$$\sum_{\alpha \in \Phi^+} m_{V_G(\lambda_{1,x})}(\mu_{1,x} + \alpha)(\mu_{1,x} + \alpha, \alpha) = \frac{1}{2} \sum_{r=1}^n m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_r) x + l.$$

Arguing as in the proof of Lemma 2.3.13 (using Theorem 2.3.11 and (2.8)), one shows that $m_{V_G(\lambda)}(\mu) = \sum_{r=1}^n m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_r)$. Moreover, since $N_{\mu + \alpha_{r_2} + \dots + \alpha_n} < N_\lambda$ and $\mu + \alpha_1 + \dots + \alpha_r$ is \mathcal{W} -conjugate to $\mu + \alpha_1 + \dots + \alpha_{r_2-1}$ for every $1 \leq r \leq r_2 - 1$, our induction assumption applies and we have $m_{V_G(\lambda)}(\mu) = r_2 m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_{r_2-1})$. Finally, another application of the induction hypotheses yields

$$m_{V_G(\lambda)}(\mu + \alpha_1 + \dots + \alpha_{r_2-1}) = \prod_{i=3}^{N_\lambda} (r_i - r_{i-1} + 1)$$

and thus $m_{V_G(\lambda)}(\mu) = r_2 \prod_{i=3}^{N_\lambda} (r_i - r_{i-1} + 1) = \prod_{i=2}^{N_\lambda} (r_i - r_{i-1} + 1)$ as desired, completing the proof. \square

Assuming G simple, we now record some preliminary results on weight multiplicities in irreducible KG -modules. Let then $V = L_G(\lambda)$ be an irreducible KG -module having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r$. We refer the reader to [Tes88, Proposition 1.30] for a proof of the next result, used implicitly in the remainder of the thesis.

Lemma 2.3.16

Let λ and $V = L_G(\lambda)$ be as above, with $a_i > 0$. Then $\mu = \lambda - d\alpha_i \in \Lambda(V)$ for every $1 \leq d \leq a_i$. Moreover $m_V(\mu) = 1$.

We saw above (see Theorem 2.3.4) that if (G, p) is not special, then the set of weights $\Lambda(V)$ is saturated, yielding the following result. Again, we shall apply it without explicit reference.

Lemma 2.3.17

Suppose that (G, p) is not special and let $\mu \in \Lambda(V)$. Then $\mu - r\alpha \in \Lambda(V)$ for every $\alpha \in \Phi^+$ and $0 \leq r \leq \langle \mu, \alpha \rangle$.

Suppose that G is a simple algebraic group of classical type over K . In [Sei87, Section 6], Seitz proved that if $m_V(\mu) \leq 1$ for every $\mu \in X(T)$, then G, λ and p are as in Table 2.1.

G	p	λ
$A_n(n \geq 1)$	any	$\lambda_i (1 \leq i \leq n)$ $a\lambda_1, a\lambda_n (a > 1)$
	$p \mid a + b + 1$	$a\lambda_i + b\lambda_{i+1} (1 \leq i \leq n - 1)$
$B_n(n \geq 2)$	any	λ_1, λ_n
$C_n(n \geq 3)$	any	λ_1, λ_n
	$p \mid 2a + 1$	$a\lambda_n$
	$p \mid 2a + 3$	$\lambda_{n-1} + a\lambda_n$
$D_n(n \geq 4)$	any	$\lambda_1, \lambda_{n-1}, \lambda_n$

Table 2.1: Modules with 1-dimensional weight spaces.

Now it turns out that weight spaces of KG -modules as in Table 2.1 are indeed 1-dimensional. We refer the reader to [ZS87], [ZS90], and [BOS14] for a proof of this result.

Theorem 2.3.18

Let G, λ , and p be as in Table 2.1. Then $m_V(\mu) \leq 1$ for every $\mu \in X(T)$.

To conclude this section, we assume G is simple of type A_n ($n \geq 2$) over K and consider the T -weight $\lambda = a\lambda_i + b\lambda_j$, with $ab \neq 0$, and $1 \leq i < j \leq n$. By Lemma 2.3.17, the character

$$\mu = \lambda - (c\alpha_i + \alpha_{i+1} + \cdots + \alpha_j)$$

is a T -weight of $V = L_G(\lambda)$ for every $1 \leq c \leq a+1$. Furthermore, its multiplicity is given by the following lemma. We refer the reader to [Sei87, Proposition 8.6] or Proposition 4.1.3 for a proof in the case $c = 1$. The proof for the general case is entirely similar, hence the details are left to the reader.

Lemma 2.3.19

Let G , λ and μ be as above, with $2c \leq a+1$. Then the T -weight μ is dominant and its multiplicity in $V = L_G(\lambda)$ is given by

$$m_V(\mu) = \begin{cases} j-i & \text{if } p \mid a+b+j-i; \\ j-i+1 & \text{otherwise.} \end{cases}$$

2.4 Some dimension calculations

In this section, G denotes a simply connected simple algebraic group of rank n over K and $V = L_G(\lambda)$ an irreducible KG -module having p -restricted highest weight $\lambda \in X^+(T)$. In general, the dimension of V is unknown, or at least there is no known formula holding for λ arbitrary. Nevertheless, the dimension of $V_G(\lambda)$ is given by the well-known *Weyl's dimension formula* (or *Weyl's degree formula*), whose proof can be found in [Hum78, Section 24.3].

Theorem 2.4.1 (Weyl's Degree Formula)

The dimension of the Weyl module $V_G(\lambda)$ corresponding to $\lambda \in X^+(T)$ is given by

$$\dim V_G(\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Let $G = \text{CL}_n(K) \in \{A_n, B_n, C_n, D_n\}$ be a classical algebraic group over K having rank n , and for $0 \leq i \leq 2$, set $\Phi_i^+ = \{\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n \in \Phi^+ : a_i = i\}$. Clearly $\Phi^+ = \Phi_0^+ \sqcup \Phi_1^+ \sqcup \Phi_2^+$ thanks to the description of Φ^+ given in Section 2.2, and using Theorem 2.4.1, one easily sees that

$$\dim V_G(\lambda) = \left(\prod_{\alpha \in \Phi_1^+ \sqcup \Phi_2^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \right) \dim V_{\tilde{G}}(\tilde{\lambda}), \quad (2.9)$$

where \tilde{G} is a simple algebraic group of type $\text{CL}_{n-1}(K)$ over K and $\tilde{\lambda} = \sum_{i=1}^{n-1} \langle \lambda, \alpha_{i+1} \rangle \lambda_{i+1}$. In [Hum78, Section 24.3], explicit formulas are given for $\dim V_G(\lambda)$ for G of type A_2 , B_2 , G_2 , and λ arbitrary, and using (2.9), one checks that the following result holds. (Note that similar expressions can be found for every type of irreducible root system.)

Proposition 2.4.2

Assume G has type A_n over K and let $\lambda = \sum_{r=1}^n a_r \lambda_r$ be a dominant T -weight. Then the dimension of $V_G(\lambda)$ is given by

$$\dim V_G(\lambda) = \left(\prod_{l=1}^n \frac{1}{l!} \right) \prod_{1 \leq i \leq j \leq n} \left(\sum_{k=i}^j (a_k + 1) \right).$$

The following result gives a way to efficiently compute the dimension of a given irreducible KG -module V , provided that the multiplicity of every weight in $\Lambda^+(V)$ is known. Its proof directly follows from [Sei87, Theorem 1.10].

Proposition 2.4.3

Let V be as above and for a dominant weight $\mu \in X^+(T)$, consider the subgroup \mathscr{W}_μ of \mathscr{W} defined by $\mathscr{W}_\mu = \langle s_\alpha : \alpha \in \Pi \text{ with } \langle \mu, \alpha \rangle = 0 \rangle$. Then

$$\dim V = \sum_{\mu \in X^+(T)} [\mathscr{W} : \mathscr{W}_\mu] m_V(\mu).$$

We now record some information on the dimension of various irreducible KG -modules, starting with the following result on the symmetric powers of the natural KG -module for G of type A_n over K . We refer the reader to [Sei87, Lemma 1.14] for a proof.

Lemma 2.4.4

Let G be a simple algebraic group of type A_n over K and consider an irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1$, where $a \in \mathbb{Z}_{\geq 0}$. Then $V \cong \text{Sym}^a W$, where $\text{Sym}^a W$ denotes the a^{th} symmetric power of the natural KG -module W . In particular, we have

$$\dim V = \binom{a+n}{a}.$$

The dimension of the exterior powers of the natural KG -module for G of type A_n can easily be determined as well, using Proposition 2.4.3 and the fact that $\Lambda^+(V_G(\lambda_i)) = \{\lambda_i\}$, this for every $1 \leq i \leq n$. The details are left to the reader.

Lemma 2.4.5

Let G be a simple algebraic group of type A_n over K and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda = \lambda_i$, where $1 \leq i \leq n$. Then $V = V_G(\lambda) \cong \bigwedge^i W$, where $\bigwedge^i W$ denotes the i^{th} exterior power of the natural KG -module W . In particular, we have

$$\dim V = \binom{n+1}{i}.$$

Assume $p \neq 2$, fix $1 \leq i \leq n-1$ and let G be a simple algebraic group of type D_n over K . Then by [Sei87, Theorem 1, Table 1 (I₄, I₅)], we get that $L_G(\lambda_i + \delta_{i,n-1}\lambda_n)$ is isomorphic to the restriction to G of the i^{th} exterior power of the natural module for G' of type A_{2n-1} over K . Using this observation together with Lemma 2.4.5, one shows the following result.

Lemma 2.4.6

Assume $p \neq 2$, let G be a simple algebraic group of type D_n over K and consider an irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda = \lambda_i + \delta_{i,n-1}\lambda_n$, where $1 \leq i < n$. Then $V = V_G(\lambda)$ and

$$\dim V = \binom{2n}{i}.$$

A proof of the next Lemma, concerning the dimension of the irreducible KG -module $L_G(2\lambda_{n-1})$ for G of type D_n , can be found in [BGT15, Lemma 2.3.6].

Lemma 2.4.7

Let G be a simple algebraic group of type D_n over K and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda = 2\lambda_{n-1}$. Then

$$\dim L_G(\lambda) = \begin{cases} 2^{n-1} & \text{if } p = 2; \\ \frac{1}{2} \binom{2n}{n} & \text{otherwise.} \end{cases}$$

Using Lemma 2.3.19 together with Proposition 2.4.3, we now determine the dimension of $L_G(\lambda_1 + \lambda_j)$, where $2 \leq j \leq n$. We introduce the following notation: for $l \in \mathbb{Z}_{\geq 0}$ a prime, let $\epsilon_l : \mathbb{Z}_{\geq 0} \rightarrow \{0, 1\}$ be the map defined by

$$\epsilon_l(z) = \begin{cases} 1 & \text{if } l \mid z; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.4.8

Let G be a simple algebraic group of type A_n over K , fix $1 < j \leq n$ and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda = \lambda_1 + \lambda_j$. Then

$$\dim V = j \binom{n+2}{j+1} - \epsilon_p(j+1) \binom{n+1}{j+1}.$$

Proof. First observe that $\Lambda^+(\lambda) = \{\lambda, \lambda_{j+1}\}$, where we adopt the notation $\lambda_{n+1} = 0$. An application of Proposition 2.4.3 and Lemma 2.3.19 thus yields

$$\dim V = [\mathfrak{S}_{n+1} : \mathfrak{S}_{j-1} \times \mathfrak{S}_{n-j+1}] + [\mathfrak{S}_{n+1} : \mathfrak{S}_{j+1} \times \mathfrak{S}_{n-j}] m_V(\lambda_{j+1}),$$

where $m_V(\lambda_{j+1}) = j - \epsilon_p(j+1)$. An elementary computation then completes the proof. \square

Finally, let G be a simple algebraic group of type A_n over K and consider the dominant T -weight $\lambda = 2\lambda_1 + \lambda_n$. Again, one easily checks that $\Lambda^+(\lambda) = \{\lambda, \lambda_2 + \lambda_n, \lambda_1\}$, and proceeding exactly as in the proof of Lemma 2.4.8 yields the following result. We leave the details to the reader.

Lemma 2.4.9

Assume $p \neq 2$, let G be a simple algebraic group of type A_n over K and consider an irreducible KG -module $V = L_G(\lambda)$ having highest weight $\lambda = 2\lambda_1 + \lambda_n$. Then

$$\dim V = \frac{1}{2}(n+1)(n(n+3) - 2\epsilon_p(n+2)).$$

2.5 Lie algebras

In this section, we recall some elementary facts on Lie algebras, their representations, as well as their relation with algebraic groups. Most of the results presented here can be found in [Hum78, Chapter VII] or [Car89, Chapter 4].

2.5.1 Structure constants and Chevalley basis

Let K be an algebraically closed field having characteristic zero and let \mathcal{L} be a finite-dimensional simple Lie algebra over K . Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\mathcal{L}}$ of \mathcal{L} containing a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\mathcal{L}}$ of \mathcal{L} , and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote a corresponding base of the root system Φ of \mathcal{L} . Recall the existence of a *standard Chevalley basis*

$$\mathcal{B} = \{e_\alpha, f_\alpha = e_{-\alpha}, h_{\alpha_i} : \alpha \in \Phi^+, 1 \leq i \leq n\}$$

of \mathcal{L} , whose elements satisfy the usual relations (see [Car89, Theorem 4.2.1]). For all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, we have

$$[e_\alpha, e_\beta] = N_{(\alpha,\beta)}e_{\alpha+\beta} = \pm(q+1)e_{\alpha+\beta}, \tag{2.10}$$

where q is the greatest integer for which $\alpha - q\beta \in \Phi$. The $N_{(\alpha,\beta)}$ are called the *structure constants*. Now one can easily check that for any pair of roots (α, β) , we have

$$N_{(\beta,\alpha)} = -N_{(\alpha,\beta)} = N_{(-\alpha,-\beta)}, \tag{2.11}$$

and using the Jacobi identity of \mathcal{L} , one can prove that for every $\alpha, \beta, \gamma \in \Phi$ satisfying $\alpha + \beta + \gamma = 0$, we have

$$\frac{N_{(\alpha,\beta)}}{(\gamma, \gamma)} = \frac{N_{(\beta,\gamma)}}{(\alpha, \alpha)} = \frac{N_{(\gamma,\alpha)}}{(\beta, \beta)}. \tag{2.12}$$

Finally, one can show (see the proof of [Car89, Theorem 4.1.2]) that for every $\alpha, \beta, \gamma, \delta \in \Phi$ such that $\alpha + \beta + \gamma + \delta = 0$ and no pair are opposites, we have

$$\frac{N_{(\alpha,\beta)}N_{(\gamma,\delta)}}{(\alpha + \beta, \alpha + \beta)} + \frac{N_{(\gamma,\alpha)}N_{(\beta,\delta)}}{(\gamma + \alpha, \gamma + \alpha)} + \frac{N_{(\beta,\gamma)}N_{(\alpha,\delta)}}{(\beta + \gamma, \beta + \gamma)} = 0. \tag{2.13}$$

Following the ideas in [Car89, Section 2.1], we fix an ordering on Φ^+ by saying that $\alpha \preceq \beta$ if either $\alpha = \beta$ or $\beta - \alpha = \sum_{i=1}^n c_i \alpha_i$ with the last non-zero coefficient c_i positive. We shall also write $\alpha \prec \beta$ if $\alpha \preceq \beta$ but $\alpha \neq \beta$.

Definition 2.5.1

An ordered pair of roots (α, β) is *special* if $\alpha + \beta \in \Phi$ and $0 \prec \alpha \prec \beta$. Also, such a pair is *extraspecial* if for all special pairs (γ, δ) satisfying $\gamma + \delta = \alpha + \beta$, we have $\alpha \preceq \gamma$.

Remark 2.5.2

In view of Definition 2.5.1, one immediately notices that if $\gamma \in \Phi^+$, then either $\gamma \in \Pi$ or there exist unique $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta = \gamma$ and (α, β) is extraspecial.

Now by [Car89, Proposition 4.2.2], the structure constants of a simple Lie algebra \mathcal{L} are uniquely determined by their values on the set of extraspecial pairs, for which we can arbitrarily choose the sign in (2.10). Throughout this thesis, we shall always assume that $N_{(\alpha, \beta)} > 0$ for any extraspecial pair (α, β) .

Lemma 2.5.3

Let \mathcal{L} be a simple Lie algebra of type A_n over K , Φ the corresponding root system and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a base of Φ . Then the extraspecial pairs are $(\alpha_i, \alpha_{i+1} + \dots + \alpha_j)$, where $1 \leq i < j \leq n$. Moreover $N_{(\alpha_i + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)} = 1$ for every $1 \leq i \leq r < j \leq n$.

Proof. We first show that the extraspecial pairs are as mentioned, starting by considering $\gamma \in \Phi^+$ such that $\gamma \notin \Pi$. By the description of Φ^+ recorded in Section 2.2.1, there exist unique $1 \leq i < j \leq n$ such that $\gamma = \alpha_i + \dots + \alpha_j$ and one easily sees that the special pairs (α, β) satisfying $\alpha + \beta = \gamma$ are $(\alpha_i + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)$, where $i \leq r \leq j - 1$. The assertion on extraspecial pairs then immediately follows from remark 2.5.2 and hence in the remaining of the proof, we shall assume $N_{(\alpha_i, \alpha_{i+1} + \dots + \alpha_j)} = 1$ for every $1 \leq i < j \leq n$ thanks to (2.10) and our assumption on the positivity of structure constants.

We next suppose that $1 \leq i < r < j$, in which case applying (2.13) to the roots $\alpha = \alpha_i$, $\beta = -(\alpha_i + \dots + \alpha_r)$, $\gamma = -(\alpha_{r+1} + \dots + \alpha_j)$ and $\delta = \alpha_{i+1} + \dots + \alpha_j$ yields

$$0 = N_{(\alpha, \beta)} N_{(\gamma, \delta)} + N_{(\beta, \gamma)} N_{(\alpha, \delta)}.$$

Now by (2.11), $N_{(\alpha_i + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)} = -N_{(\beta, \gamma)}$. Also, by the $r = i$ case, we know that $N_{(\alpha, \delta)} = 1$. Finally, by (2.11), (2.12) and the $r = i$ case again, we get $N_{(\alpha, \beta)} = -1$ and $N_{(\gamma, \delta)} = N_{(\alpha_{r+1} + \dots + \alpha_j, \alpha_{i+1} + \dots + \alpha_r)}$, so that

$$N_{(\alpha_i + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)} = N_{(\alpha_{i+1} + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)}.$$

The result then follows by induction. □

We next deal with the case of a simple Lie algebra of type B_n over K . Here we leave to the reader to check that the extraspecial pairs (α, β) are as in Table 2.2.

α	β	Conditions
α_i	$\alpha_{i+1} + \cdots + \alpha_j$	$1 \leq i < j \leq n$
α_i	$\alpha_{i+1} + \cdots + \alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_n$	$i < k < n$
α_{i+1}	$\alpha_i + \alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_n$	$i < k < n$

Table 2.2: Extraspecial pairs (α, β) for Φ of type B_n over K .

Lemma 2.5.4

Let \mathcal{L} be a simple Lie algebra of type B_n over K , Φ the corresponding root system and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a base of Φ . Then

1. $N_{(\alpha_i + \cdots + \alpha_r, \alpha_{r+1} + \cdots + \alpha_j)} = 1$, $1 \leq i \leq r < j \leq n$.
2. $N_{(\alpha_j, \alpha_r + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n)} = 1$, $1 \leq r < j < n$.
3. $N_{(\alpha_i + \cdots + \alpha_r, \alpha_{r+1} + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n)} = 1$, $1 \leq i \leq r < j < n$.
4. $N_{(\alpha_j + \cdots + \alpha_n, \alpha_r + \cdots + \alpha_n)} = 2$, $1 \leq r < j \leq n$.
5. $N_{(\alpha_s + \cdots + \alpha_j, \alpha_r + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n)} = 1$, $1 \leq r < s \leq j < n$.
6. $N_{(\alpha_i + \cdots + \alpha_j, \alpha_{r+1} + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n)} = -1$, $1 \leq i \leq r < j < n$.

Proof. Proceed exactly as in the proof of Lemma 2.5.3. The details are left to the reader. \square

Finally, we consider the case of a simple Lie algebra of type D_n ($n \geq 4$) over K . Again, we leave to the reader to check that the extraspecial pairs (α, β) are as in Table 2.3.

α	β	Conditions
α_i	$\alpha_{i+1} + \cdots + \alpha_j$	$1 \leq i < n - 1$, $i < j \leq n - 1$
α_{n-2}	α_n	
α_i	$\alpha_{i+1} + \cdots + \alpha_{n-2} + \alpha_n$	$1 \leq i < n - 2$
α_i	$\alpha_{i+1} + \cdots + \alpha_n$	$1 \leq i < n - 2$
α_{n-1}	$\alpha_{n-2} + \alpha_n$	
α_i	$\alpha_{i+1} + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$1 \leq i < j < n - 2$
α_{i+1}	$\alpha_i + \alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$1 \leq i < n - 3$
α_{n-2}	$\alpha_{n-1} + \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1} + \alpha_n$	

Table 2.3: Extraspecial pairs for Φ of type D_n over K .

Lemma 2.5.5

Let \mathcal{L} be a simple Lie algebra of type D_n over K , Φ the corresponding root system and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ a base of Φ . Also let $1 \leq i \leq r < s \leq j \leq n$. Then

1. $N_{(\alpha_i + \dots + \alpha_r, \alpha_{r+1} + \dots + \alpha_j)} = 1$, for every $1 \leq i \leq r < j \leq n - 1$.
2. $N_{(\alpha_i + \dots + \alpha_j, \alpha_{j+1} + \dots + \alpha_{n-2} + \alpha_n)} = 1$ for every $1 \leq i \leq j \leq n - 3$.
3. $N_{(\alpha_i + \dots + \alpha_j, \alpha_{j+1} + \dots + \alpha_n)} = 1$ for every $1 \leq i \leq j \leq n - 3$.
4. $N_{(\alpha_i + \dots + \alpha_{n-1}, \alpha_n)} = -1$ for every $1 \leq i \leq n - 2$.
5. $N_{(\alpha_i + \dots + \alpha_{n-1}, \alpha_{n-2} + \alpha_n)} = -1$ for every $1 \leq i \leq n - 3$.
6. $N_{(\alpha_i + \dots + \alpha_{n-1}, \alpha_n)} = -1$.
7. $N_{(\alpha_j + \dots + \alpha_{n-1}, \alpha_i + \dots + \alpha_{n-2} + \alpha_n)} = 1$ for every $1 \leq i < j \leq n - 1$.
8. $N_{(\alpha_j + \dots + \alpha_{n-2}, \alpha_i + \dots + \alpha_n)} = 1$ for every $1 \leq i < j \leq n - 3$.
9. $N_{(\alpha_j + \dots + \alpha_k, \alpha_i + \dots + \alpha_k + 2\alpha_{k+1} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)} = 1$, $1 \leq i < j \leq k \leq n - 3$.

2.5.2 Relations with algebraic groups

Let K be an algebraically closed field having characteristic $p \geq 0$ and let G be a simple algebraic group of classical type over K . Following the ideas in [Hum75, III.9], one sees that the space of left-invariant derivations of $K[G]$ is a Lie algebra over K (having same type as G), which we denote by $\mathcal{L}(G)$, and that $\mathcal{L}(G)$ is isomorphic to the tangent space $\mathcal{T}(G)_{1_G}$. Given a morphism of algebraic groups $f : G \rightarrow \tilde{G}$, we obtain a morphism of Lie algebras $df : \mathcal{L}(G) \rightarrow \mathcal{L}(\tilde{G})$ by differentiating at 1_G . Finally, if $\phi : G \rightarrow \mathrm{GL}(V)$ is a rational representation of G , then $d\phi_{1_G} : \mathcal{L}(G) \rightarrow \mathfrak{gl}(V)$ is a representation of $\mathcal{L}(G)$. We recall the following well-known result, recorded here without proof.

Lemma 2.5.6

Let $\phi : G \rightarrow \mathrm{GL}(V)$ be as above, and suppose that $U \subset V$ is invariant under the action of G . Then U is $\mathcal{L}(G)$ -invariant as well.

Let $V_G(\lambda)$ be the Weyl module corresponding to a p -restricted weight $\lambda \in X^+(T)$ and consider $\mu = \lambda - \sum_{r=1}^n c_r \alpha_r \prec \lambda$. If $\mathcal{B} = \{e_\alpha, f_\alpha, h_{\alpha_i} : \alpha \in \Phi^+, 1 \leq i \leq n\}$ is a standard Chevalley basis of $\mathcal{L}(G)$, then it is well-known (see [BCC70, Lemma 6.2]) that

$$V_G(\lambda)_\mu = \left\langle \frac{f_{\beta_1}^{k_1}}{k_1!} \cdots \frac{f_{\beta_r}^{k_r}}{k_r!} v^\lambda : \beta_1 \prec \dots \prec \beta_r \in \Phi^+, \quad \mu + \sum_{i=1}^r k_i \beta_i = \lambda \right\rangle_K, \quad (2.14)$$

where $v^\lambda \in V_G(\lambda)$ denotes a maximal vector for B and \prec is the ordering on Φ^+ introduced before Definition 2.5.1.

Lemma 2.5.7

Let λ, μ be as above and let $\beta_1 \prec \beta_2 \preceq \dots \preceq \beta_r \in \Phi^+$ be such that $\lambda - \beta_1 \notin \Lambda(\lambda)$. Then $f_{\beta_1}^k f_{\beta_2} \cdots f_{\beta_r} v^\lambda \in \langle f_{\gamma_1} f_{\gamma_2} \cdots f_{\gamma_s} v^\lambda : \beta_1 \prec \gamma_1 \preceq \gamma_2 \preceq \dots \preceq \gamma_s \in \Phi^+ \rangle_K$, for every $k \in \mathbb{Z}_{>0}$.

Proof. We first show the result for $k = 1$, proceeding by induction on r . Starting with the case where $r = 2$, we have $f_{\beta_1} f_{\beta_2} v^\lambda = -N_{(\beta_1, \beta_2)} f_{\beta_1 + \beta_2} v^\lambda + f_{\beta_2} f_{\beta_1} v^\lambda$, where $N_{(\beta_1, \beta_2)} = 0$ if $\beta_1 + \beta_2 \notin \Phi$. Also, since $\lambda - \beta_1 \notin \Lambda(\lambda)$, we get $f_{\beta_2} f_{\beta_1} v^\lambda = 0$ and hence $f_{\beta_1} f_{\beta_2} v^\lambda \in \langle f_{\beta_1 + \beta_2} v^\lambda \rangle_K$ as desired. Next let $r_0 > 2$ be such that the Lemma holds for every $2 \leq r < r_0$ and consider $\beta_1 \prec \beta_2 \preceq \dots \preceq \beta_{r_0} \in \Phi^+$, where $\lambda - \beta_1 \notin \Lambda(\lambda)$. Then

$$f_{\beta_1} \cdots f_{\beta_{r_0}} v^\lambda = -N_{(\beta_1, \beta_2)} f_{\beta_1 + \beta_2} f_{\beta_3} \cdots f_{\beta_{r_0}} v^\lambda + f_{\beta_2} f_{\beta_1} f_{\beta_3} \cdots + f_{\beta_{r_0}} v^\lambda,$$

where again $N_{(\beta_1, \beta_2)} = 0$ if $\beta_1 + \beta_2 \notin \Phi^+$, and the result for $k = 1$ then easily follows from our inductive hypothesis. Now in the case of $k \in \mathbb{Z}_{>0}$ arbitrary, proceeding by induction again completes the proof. \square

Thanks to Lemma 2.5.7, we are now able to determine a smaller set of generating elements for $V_G(\lambda)_\mu$ in certain situations. We shall even see that in some cases, the newly obtained set consists of a basis of $V_G(\lambda)_\mu$ (see Proposition 4.1.1, for example).

Proposition 2.5.8

Let λ, μ be as above and set $\Phi_\lambda^+ = \{\beta \in \Phi^+ : \lambda - \beta \in \Lambda(\lambda)\}$. Then

$$V_G(\lambda)_\mu = \left\langle \frac{f_{\beta_1}^{k_1}}{k_1!} \cdots \frac{f_{\beta_r}^{k_r}}{k_r!} v^\lambda : \beta_1 \prec \dots \prec \beta_r \in \Phi_\lambda^+, \quad \mu + \sum_{i=1}^r k_i \beta_i = \lambda \right\rangle_K.$$

Proof. We proceed by induction on $s = |\Phi^+| - |\Phi_\lambda^+|$. If $s = 0$, then there is nothing to do, while by Lemma 2.5.7, the result holds in the case where $s = 1$. Hence let $s_0 > 1$ be such that the proposition holds for every $0 \leq s < s_0$ and assume $\Phi^+ - \Phi_\lambda^+ = \{\gamma_1, \dots, \gamma_{s_0}\}$, where $\gamma_1 \prec \gamma_2 \prec \dots \prec \gamma_{s_0}$. Also let $\delta_1 \prec \dots \prec \delta_t \in \Phi^+$ be such that

$$\frac{f_{\delta_1}^{l_1}}{l_1!} \cdots \frac{f_{\delta_t}^{l_t}}{l_t!} v^\lambda \in V_G(\lambda)_\mu$$

and assume the existence of $1 \leq t_1 < \dots < t_{s_0} \leq t$ with $\delta_{t_i} = \gamma_i$ for every $1 \leq i \leq s_0$. Without any loss of generality, we can suppose $t_1 = 1$, in which case an inductive argument (considering the weight $\mu + l_1 \delta_1$) completes the proof. \square

Clearly, if v^+ denotes the image of v^λ in $V_G(\lambda)/\text{rad}(\lambda)$, then replacing v^λ by v^+ in (2.14) gives a generating set for $L_G(\lambda)_\mu$. Concretely, we get

$$L_G(\lambda)_\mu = \left\langle f_{\beta_1} \cdots f_{\beta_r} v^+ : \beta_1 \preceq \dots \preceq \beta_r \in \Phi^+, \quad \mu + \sum_{i=1}^r \beta_i = \lambda \right\rangle_K. \quad (2.15)$$

We conclude this section with a result showing that irreducible KG -modules with p -restricted highest weights behave well with respect to the differential. We refer the reader to [Cur60, Theorem 1] for a proof of the following.

Theorem 2.5.9 (Curtis)

Let G , $\mathcal{L}(G)$ be as above and consider an irreducible KG -module $L_G(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T)$. Then $L_G(\lambda)$ is irreducible as a module for $\mathcal{L}(G)$ as well.

2.6 Filtrations and extensions of modules

Let G be a simple algebraic group over an algebraically closed field K having characteristic $p \geq 0$. In this section, we introduce some notation and recall a few basic results concerning filtrations and extensions of KG -modules.

2.6.1 Filtrations of modules

Let V be a KG -module and recall that a *filtration* of V is a sequence of KG -submodules $V = V^0 \supset V^1 \supset \dots \supset V^r \supset V^{r+1} = 0$, with $r \in \mathbb{Z}_{\geq 0}$. Such a filtration is called a *composition series* of V if for every $0 \leq i \leq r$, the quotient $S^i = V^i/V^{i+1}$ is irreducible. Let then $\{\mu_1, \dots, \mu_s\} \subset X^+(T)$ be of minimal cardinality such that for every $0 \leq i \leq r$, there exists $1 \leq j_i \leq s$ with $S^i \cong L_G(\mu_{j_i})$. The irreducibles $L_G(\mu_1), \dots, L_G(\mu_s)$ are the *KG -composition factors* of V and we say that an irreducible KG -module $L_G(\mu)$ occurs with multiplicity m_μ in V if $|\{1 \leq i \leq r : S^i \cong L_G(\mu)\}| = m_\mu$, in which case we write $[V, L_G(\mu)] = m_\mu$. We then adopt the notation

$$V = \mu_1^{m_1} / \mu_2^{m_2} / \dots / \mu_s^{m_s} \tag{2.16}$$

to indicate that V is a KG -module with composition factors $L_G(\mu_1), \dots, L_G(\mu_s)$, $L_G(\mu_i)$ occurring with multiplicity m_i , $1 \leq i \leq s$. The following well-known result guarantees that the notion of composition factors of V is independent of the choice of a composition series for V .

Theorem 2.6.1 (The Jordan-Hölder Theorem)

Consider two distinct composition series $V = U^0 \supsetneq U^1 \supsetneq \dots \supsetneq U^r \supsetneq U^{r+1} = 0$ and $V = V^0 \supsetneq V^1 \supsetneq \dots \supsetneq V^s \supsetneq V^{s+1} = 0$ of V . Then $r = s$ and the list $\{U^i/U^{i+1}\}_{i=0}^r$ is a rearrangement of the list $\{V^i/V^{i+1}\}_{i=0}^r$, up to isomorphisms.

Fix $B = UT$ a Borel subgroup of G containing a maximal torus T and let $\lambda \in X(T)$. Clearly λ determines a 1-dimensional KT -module K_λ on which $t \in T$ acts as multiplication by $\lambda(t)$ and one observes that we get a KB -module structure on K_λ , given by $(ut)x = \lambda(t)x$, for every $ut \in B$ and $x \in K_\lambda$.

For $r \geq 0$, we let $H^r(-) = H^r(G/B, -)$ denote the r^{th} derived functor of the left exact functor $\text{ind}_B^G(-)$ and write $H^r(\lambda) = H^r(K_\lambda)$. It turns out (see [Jan03, II, 2.13]) that if $\lambda \in X^+(T)$, then $H^0(\lambda) \cong V_G(-w_0\lambda)^*$, where w_0 denotes the longest element in the Weyl group \mathscr{W} of G . Consequently $L_G(\lambda) \cong L_G(-w_0)^*$ is the unique irreducible submodule of $H^0(\lambda)$ and hence is the *socle* of $H^0(\lambda)$, written $\text{soc}(\lambda)$. Since we only work with $H^0(\lambda)$ in this thesis, we omit the details here and refer the reader to [Jan03, Section 2.1].

Definition 2.6.2

A filtration $V = V^0 \supseteq V^1 \supseteq \dots \supseteq V^r \supseteq V^{r+1} = 0$ of V is called a *Weyl filtration* if for every $0 \leq i \leq r$, there exists a weight $\mu_i \in X^+(T)$ with $V^i/V^{i+1} \cong V_G(\mu_i)$. Similarly, such a filtration is called a *good filtration* if for every $0 \leq i \leq r$, there exists a weight $\mu_i \in X^+(T)$ with $V^i/V^{i+1} \cong H^0(\mu_i)$. Finally, we call a KG -module *tilting* if it admits both a good and a Weyl filtration.

It turns out that modules with filtrations as above behave nicely with respect to tensor products and exterior (respectively, symmetric) powers, as recorded in the following.

Proposition 2.6.3

If U, V are two KG -modules admitting a good (respectively, Weyl) filtration then $U \otimes V$ also admits a good (respectively, Weyl) filtration. In addition, W is a KG -module affording a good (respectively, Weyl) filtration, then each of $\text{Sym}^r W$ and $\bigwedge^r W$ admits a good (respectively, Weyl) filtration as well, for $r \in \mathbb{Z}_{>0}$.

Proof. The first general proof of the result on the tensor product was given in [Mat90], but it had already been proven in most cases in [Don85]. We refer to [HM13, Proposition 2.2.5] for a proof of the second assertion. □

Let $\mu, \nu \in X^+(T)$ be two dominant weights. By Proposition 2.6.3, the tensor product $V_G(\mu) \otimes V_G(\nu)$ admits a Weyl filtration and the following result gives further useful properties.

Proposition 2.6.4

Let μ, ν be as above, set $\lambda = \mu + \nu$, and consider $V = V_G(\mu) \otimes V_G(\nu)$. Then the following assertions hold.

1. *Any dominant weight $\sigma \in \Lambda^+(V)$ satisfies $\sigma \preceq \lambda$, and $m_V(\lambda) = 1$. In other words, λ is the unique highest weight of V .*
2. *There is an injective morphism of KG -modules $\iota : V_G(\lambda) \hookrightarrow V$.*
3. *If in addition $V_G(\mu)$ and $V_G(\nu)$ are irreducible, then V is tilting and there is a surjective morphism of KG -modules $\phi : V \twoheadrightarrow H^0(\lambda)$, with $\iota(\text{rad}(\lambda)) \subset \ker(\phi)$.*

Proof. We refer the reader to [McN98, Proposition 4.6.2] for a proof of 1, 2 and the existence of a surjective morphism of KG -modules $V \twoheadrightarrow H^0(\lambda)$ under the hypotheses of 3. Now let $\phi \in \text{Hom}_G(V, H^0(\lambda))$ and identify $V_G(\lambda)$ with $\iota(V_G(\lambda))$, where ι is as in 2. Also write $N = \ker(\phi) \cap V_G(\lambda)$, and denote by $\bar{\phi} : V_G(\lambda)/N \hookrightarrow H^0(\lambda)$ the injective morphism of KG -modules induced by $\phi \circ \iota$. As $\text{rad}(\lambda)$ is the unique maximal submodule of $V_G(\lambda)$, we have $N \subset \text{rad}(\lambda)$, and if $N \subsetneq \text{rad}(\lambda)$, we get $0 \subsetneq \bar{\phi}(\text{rad}(\lambda)/N) \subset \text{Im}(\bar{\phi}) \subset H^0(\lambda)$, a contradiction with $\text{soc}(H^0(\lambda)) = L_G(\lambda)$, as $\lambda \notin \Lambda(\text{rad}(\lambda))$. Therefore $N = \text{rad}(\lambda)$ and so 3 holds. \square

2.6.2 Extensions of modules

Following [Jan03, II, 2.12-2.14], we now record some information on extensions of KG -modules. Let V_1, V_2 be two KG -modules and identify $\text{Ext}_G^1(V_2, V_1)$ with the set of equivalence classes of all short exact sequences $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ of KG -modules. One can then show that for any dominant T -weight $\lambda \in X^+(T)$, we have

$$\text{Ext}_G^1(L_G(\lambda), L_G(\lambda)) = 0.$$

In other words, any short exact sequence $0 \rightarrow L_G(\lambda) \rightarrow V \rightarrow L_G(\lambda) \rightarrow 0$ splits. Also, one can prove that

$$\text{Ext}_G^1(L_G(\lambda), L_G(\mu)) \cong \text{Ext}_G^1(L_G(\mu), L_G(\lambda)),$$

this for any dominant T -weights $\lambda, \mu \in X^+(T)$. Finally, the following result shall prove useful later on.

Proposition 2.6.5

Let $\lambda, \mu \in X^+(T)$, with $\mu \prec \lambda$, and suppose that $[V_G(\lambda), L_G(\mu)] = 0$. Then

$$\text{Ext}_G^1(L_G(\lambda), L_G(\mu)) = 0.$$

Proof. This follows from [Jan03, II, 2.14]. \square

2.7 On the structure of Weyl modules

Let G be a simple algebraic group over K , with $B = UT$ a Borel subgroup of G , Π a corresponding base of the root system $\Phi = \Phi^+ \cup \Phi^-$ of G , and \mathscr{W} the Weyl group of G , as usual. In this section, we introduce a few tools which shall be of use in order to better understand the composition factors of a given Weyl module for G . So far we know that $[V_G(\lambda), L_G(\lambda)] = 1$ for any $\lambda \in X^+(T)$ and that if $\mu \in X^+(T)$, then $[V_G(\lambda), L_G(\mu)] \neq 0$ implies $\mu \preceq \lambda$. Most of the results presented here can be found in [Jan03, II, Sections 4,5,8], to which we refer the reader for more details.

2.7.1 The dot action and linkage principle

Let ρ denote the half-sum of all positive roots in Φ , or equivalently, the sum of all fundamental weights, as in Section 2.2. The *dot action* of \mathscr{W} on $X(T)$ is given by the formula

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \text{ for } w \in \mathscr{W} \text{ and } \lambda \in X(T).$$

Also define the *support* of an element $z \in \mathbb{Z}\Phi$ to be the subset $I \subset \Pi$ consisting of those simple roots α such that $c_\alpha \neq 0$ in the decomposition $z = \sum c_\alpha \alpha$. We refer the reader to [McN98, Lemma 4.5.6] for a proof of the following technical result, originally due to Jantzen.

Lemma 2.7.1

Let μ be a T -weight subdominant to $\lambda \in X^+(T)$. If $\alpha \in \Phi^+$ is such that $\lambda - r\alpha \in \mathscr{W} \cdot \mu$ for some $1 < r < \langle \lambda + \rho, \alpha \rangle$, then α and $\lambda - \mu$ have the same support.

For $r \in \mathbb{Z}$ and $\alpha \in \Phi$, we denote by $s_{\alpha,r} : X(T) \rightarrow X(T)$ the affine reflection on $X(T)$ defined by

$$s_{\alpha,r}(\lambda) = s_\alpha(\lambda) + r\alpha, \lambda \in X(T).$$

Also for l a prime, set \mathscr{W}_l equal to the subgroup of $\text{Aff}(X(T))$ generated by all $s_{\alpha,nl}$, with $\alpha \in \Phi$, $n \in \mathbb{Z}$, and call \mathscr{W}_l the *affine Weyl group associated to G and l* . The dot action introduced above can be extended to an action of \mathscr{W}_l on $X(T)$ and $X(T)_\mathbb{R}$ in the obvious way, setting $w \cdot \lambda = w(\lambda + \rho) - \rho$, $w \in \mathscr{W}_l$, $\lambda \in X(T)$. The following result gives us some information on possible non-trivial extensions between two irreducible KG -modules.

Proposition 2.7.2 (The Linkage Principle)

Let G be as above and suppose that $\lambda, \mu \in X^+(T)$ are such that $\text{Ext}_G^1(L_G(\lambda), L_G(\mu)) \neq 0$. Then $\lambda \in \mathscr{W}_p \cdot \mu$.

Proof. See [And80]. □

Finally, let $\lambda, \mu \in X^+(T)$ be such that $\mu \prec \lambda$ and let $d(\lambda, \mu) = 2(\lambda + \rho, \lambda - \mu) - (\lambda - \mu, \lambda - \mu)$ be as in (2.3). The following corollary to Proposition 2.7.2 gives a necessary condition for μ to afford the highest weight of a KG -composition factor of $V_G(\lambda)$, in the case where G is of classical type and $p > 2$. We refer the reader to [Sei87, Proposition 6.2] for a proof.

Corollary 2.7.3

Let G , λ and μ be as above, with G classical and $p > 2$. Also assume the inner product on $\mathbb{Z}\Phi$ is normalized so that long roots have length 1 and let $d(\lambda, \mu)$ be as above. If $\mu \prec \lambda$ affords the highest weight of a composition factor of $V_G(\lambda)$, then

$$2d(\lambda, \mu) \in p\mathbb{Z}.$$

2.7.2 The Jantzen p -sum formula

Let V be a KG -module and let $\{e^\mu\}_{\mu \in X(T)}$ denote the standard basis of the group ring $\mathbb{Z}[X(T)]$ over \mathbb{Z} . The Weyl group \mathscr{W} of G acts on $\mathbb{Z}[X(T)]$ by $we^\mu = e^{w\mu}$, $w \in \mathscr{W}$, $\mu \in X(T)$, and we write $\mathbb{Z}[X(T)]^\mathscr{W}$ to denote the set of fixed points. The *formal character* of V is the linear polynomial $\text{ch } V \in \mathbb{Z}[X(T)]^\mathscr{W}$ defined by

$$\text{ch } V = \sum_{\mu \in X(T)} m_V(\mu) e^\mu.$$

Following the ideas in [Jan03, II, 5.5], we also associate to every T -weight $\lambda \in X(T)$ the linear polynomial

$$\chi(\lambda) = \sum_{r \geq 0} (-1)^r \text{ch } H^r(\lambda).$$

If $\lambda \in X^+(T)$, Kempf's vanishing Theorem [Jan03, II, 4.5] shows that $H^r(\lambda) = 0$ for $r > 0$ and hence $\chi(\lambda) = \text{ch } H^0(\lambda)$ in this situation. Recall from [Jan03, II, 2.13] that if $\lambda \in X^+(T)$, then $\chi(\lambda) = \text{ch } V_G(\lambda)$ as well. One shows (see [Jan03, II, 5.8]) that each of $\{\chi(\lambda)\}_{\lambda \in X^+(T)}$ and $\{\text{ch } L_G(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$. In addition for $\mu \in X^+(T)$, we denote by $\chi^\mu(\lambda)$ the truncated sum

$$\chi^\mu(\lambda) = \sum_{\mu \preceq \nu \preceq \lambda} [V_G(\lambda), L_G(\nu)] \text{ch } L_G(\nu). \quad (2.17)$$

Finally, for l a prime number and $n \in \mathbb{Z}$, we write $\nu_l(n)$ to denote the greatest integer r such that l^r divides n (adopting the notation $\nu_0(n) = 0$ for every $n \in \mathbb{Z}$ as well). The following result, known as the *Jantzen p -sum formula*, provides a powerful tool for understanding Weyl modules.

Proposition 2.7.4 (The Jantzen p -sum Formula)

Let K, G be as above and let $\lambda \in X^+(T)$ be a dominant weight. Then there exists a filtration of KG -modules $V_G(\lambda) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\lambda)$ such that $V^0/V^1 \cong L_G(\lambda)$ and

$$\sum_{i=1}^k \text{ch } V^i = \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \chi(s_{\alpha, r} \cdot \lambda). \quad (2.18)$$

Proof. See [Jan03, II, 8.19]. □

We adopt the notation of [Jan03, II, 8.14], writing $\nu_c(T_\lambda)$ to denote the expression (2.18). Since $\nu_c(T_\lambda)$ is the character of the KG -module $V^1 \oplus \dots \oplus V^k$ and $\{\text{ch } L_G(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$, there exist unique $a_\nu \in \mathbb{Z}_{\geq 0}$ ($\nu \prec \lambda$) such that

$$\nu_c(T_\lambda) = \sum_{\nu \prec \lambda} a_\nu \text{ch } L_G(\nu). \quad (2.19)$$

Proposition 2.7.5

Let $\lambda \in X^+(T)$ and consider a dominant T -weight $\mu \prec \lambda$. Then μ affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $a_\mu \neq 0$ in (2.19).

Proof. Let $\mu \prec \lambda$ be such that $a_\mu \neq 0$ in (2.19). Then there exists $1 \leq i \leq k$ such that $[V^i, L_G(\mu)] \neq 0$, hence $[V_G(\lambda), L_G(\mu)] \neq 0$ as well. Reciprocally, if $\mu \prec \lambda$ affords the highest weight of a composition factor of $V_G(\lambda)$, there exists $1 \leq i \leq k$ such that $[V^i, L_G(\mu)] \neq 0$, since $V_G(\lambda)/V^1 \cong L_G(\lambda)$. The result then follows. \square

Although (2.18) can be evaluated (for fixed and small ranks) using a computer implementation of an algorithm (see [McN98, Remark 4.5.8] for a description of the latter), it is not convenient for large ranks. Hence we aim at finding an alternative expression to (2.18).

First let

$$\mathcal{D} = \{\lambda \in X(T) : \langle \lambda + \rho, \alpha \rangle \geq 0 \text{ for every } \alpha \in \Phi^+\}.$$

Then one easily sees that \mathcal{D} is a fundamental domain for the dot action of \mathcal{W} on $X(T)$, that is, for every $\mu \in X(T)$, there exist $w \in \mathcal{W}$ and a unique $\lambda \in \mathcal{D}$ such that $w \cdot \mu = \lambda$. This observation, together with the next result, provide the necessary tools to compute $\chi(\lambda)$ for any given $\lambda \in X(T)$. For $w \in \mathcal{W}$, we write $\det(w)$ for the determinant of w as an invertible linear transformation of $X(T)_{\mathbb{R}}$.

Lemma 2.7.6

Let $\lambda \in X(T)$ and $w \in \mathcal{W}$. Then $\chi(w \cdot \lambda) = \det(w)\chi(\lambda)$. Moreover, if $\lambda \in \mathcal{D}$ is not in $X^+(T)$, then $\chi(\lambda) = 0$.

Proof. The first assertion immediately follows from [Jan03, II, 5.9 (1)] and we refer the reader to [Jan03, II, 5.5] for a proof of the second. \square

We now give another formulation for (2.18) in Proposition 2.7.4, using Lemma 2.7.6 and the fact that \mathcal{D} is a fundamental domain for the dot action.

Corollary 2.7.7

Let $\lambda \in X^+(T)$ and let $V_G(\lambda) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\lambda)$ given by Proposition 2.7.4. Then

$$\nu_c(T_\lambda) = - \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \det(w_{\alpha,r}) \chi(\mu_{\alpha,r}), \quad (2.20)$$

where for $\alpha \in \Phi^+$ and $1 < r < \langle \lambda + \rho, \alpha \rangle$, $\mu_{\alpha,r}$ denotes the unique weight in $\mathcal{W} \cdot (\lambda - r\alpha) \cap \mathcal{D}$ and $w_{\alpha,r}$ an element in \mathcal{W} satisfying $w_{\alpha,r} \cdot \mu_{\alpha,r} = \lambda - r\alpha$.

Proof. For any $\alpha \in \Phi^+$ and $r \in \mathbb{Z}$, we have (by definition of the dot action)

$$\begin{aligned} s_{\alpha,r} \cdot \lambda &= s_{\alpha,r}(\lambda + \rho) - \rho \\ &= s_{\alpha}(\lambda + \rho) + r\alpha - \rho \\ &= s_{\alpha}(\lambda - r\alpha + \rho) - \rho \\ &= s_{\alpha} \cdot (\lambda - r\alpha), \end{aligned}$$

and thus $\chi(s_{\alpha,r} \cdot \lambda) = -\chi(\lambda - r\alpha)$ by Lemma 2.7.6. The Jantzen p -sum formula (2.18) can then be rewritten as

$$\nu_c(T_\lambda) = - \sum_{\alpha \in \Phi^+} \sum_{r=2}^{\langle \lambda + \rho, \alpha \rangle - 1} \nu_p(r) \chi(\lambda - r\alpha).$$

Now by the previous remark, for every $r \in \mathbb{Z}$, $\alpha \in \Phi^+$, there exist $w_{\alpha,r} \in \mathscr{W}$ and a unique $\mu_{\alpha,r} \in \mathscr{D}$ such that $w_{\alpha,r} \cdot \mu_{\alpha,r} = \lambda - r\alpha$. An application of Lemma 2.7.6 then completes the proof. \square

To conclude this section, we give a “truncated version” of Proposition 2.7.5 and an immediate consequence. Let $\lambda \in X^+(T)$ and consider the series $V_G(\lambda) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\lambda)$ given by Proposition 2.7.4. Also let $\mu \in X^+(T)$ be a dominant T -weight and write $\nu_c^\mu(T_\lambda)$ to designate the truncated sum

$$\nu_c^\mu(T_\lambda) = - \sum_{(\alpha,r) \in I_\mu} \nu_p(r) \det(w_{\alpha,r}) \chi^\mu(\mu_{\alpha,r}), \quad (2.21)$$

where

$$I_\mu = \{(\alpha, r) \in \Phi^+ \times [2, \langle \lambda + \rho, \alpha \rangle - 1] : \mu_{\alpha,r} \in X^+(T), \mu \preceq \mu_{\alpha,r} \prec \lambda\}.$$

(Here $[i, j] = \{i, i+1, \dots, j\}$ for $i < j \in \mathbb{Z}_{\geq 0}$.) Since $\{\text{ch } L_G(\lambda)\}_{\lambda \in X^+(T)}$ forms a \mathbb{Z} -basis of $\mathbb{Z}[X(T)]^\mathscr{W}$, there exist $b_\nu \in \mathbb{Z}$ ($\mu \preceq \nu \prec \lambda$) such that

$$\nu_c^\mu(T_\lambda) = \sum_{\mu \preceq \nu \prec \lambda} b_\nu \text{ch } L_G(\nu), \quad (2.22)$$

and one easily sees (using the fact that $[V_G(\nu_1), L_G(\nu_2)] = 0$ if $\nu_1, \nu_2 \in X^+(T)$ are such that $\nu_2 \not\prec \nu_1$) that $b_\nu = a_\nu$ for every $\mu \preceq \nu \prec \lambda$, with $\{a_\nu\}_{\nu \prec \lambda}$ as in (2.19). Therefore $b_\nu \in \mathbb{Z}_{\geq 0}$ and the following Proposition holds.

Proposition 2.7.8

Let λ, μ be as above, and let $\mu \preceq \nu \prec \lambda$ be a dominant T -weight. Then ν affords the highest weight of a composition factor of $V_G(\lambda)$ if and only if $b_\nu \neq 0$ in (2.22).

Since not all the coefficients in $\nu_c^\mu(T_\lambda)$ need be non-negative, there can exist $\nu \in X^+(T)$ such that $\chi^\mu(\nu)$ appears in $\nu_c^\mu(T_\lambda)$, but $[V_G(\lambda), L_G(\nu)] = 0$. However, this cannot happen if ν is “maximal”, as recorded in the following result. The details are left to the reader.

Corollary 2.7.9

Let λ, μ and ν be as in Proposition 2.7.8, with ν maximal (with respect to the partial order on $X(T)$ introduced in Section 2.3) such that $\chi^\mu(\nu)$ appears as a summand of $\nu_c^\mu(T_\lambda)$ in (2.20). Then ν affords the highest weight of a composition factor of $V_G(\lambda)$. In particular if (G, p) is not special and $\nu \prec \lambda$ is such that $m_{V_G(\lambda)}(\nu) = 1$, then $\chi^\mu(\nu)$ cannot appear in $\nu_c(T_\lambda)$.

2.7.3 Weight multiplicities using Corollary 2.7.7

Let G be a classical algebraic group of rank n defined over K , $B = UT$ a Borel subgroup of G , with T a maximal torus of G and U the unipotent radical of B . Also set

$$d_G = \begin{cases} n+1 & \text{if } G = A_n; \\ n & \text{otherwise.} \end{cases}$$

and for $l \in \mathbb{Z}_{>0}$ and $A = (a_j)_{j=1}^l \in \mathbb{Q}^l$, we write $|A| = (|a_j|)_{j=1}^l$ and consider the usual action of \mathfrak{S}_l on \mathbb{Q}^l given by $(\sigma \cdot A) = (a_{\sigma(j)})_{j=1}^l$.

Definition 2.7.10

Let $A = (a_j)_{j=1}^{d_G}, B = (b_j)_{j=1}^{d_G} \in \mathbb{Q}^{d_G}$. We say that A and B are G -conjugate (and write $A \sim_G B$) if one of the following holds.

1. $G = A_n$ and there exists $\sigma \in \mathfrak{S}_{n+1}$ such that $\sigma \cdot A = B$.
2. $G = B_n$ or C_n and there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot |A| = |B|$.
3. $G = D_n$, there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot |A| = |B|$ and

$$|\{1 \leq j \leq n : a_j < 0\}| + |\{1 \leq j \leq n : b_j < 0\}| \in 2\mathbb{Z}.$$

In other words, $A \sim_G B$ if and only if there exists $\sigma \in \mathfrak{S}_n$ such that $\sigma \cdot |A| = |B|$ and the minimal number of necessary sign changes to get $\{b_j\}_{j=1}^n$ from $\{a_j\}_{j=1}^n$ is even.

Let $\lambda \in X^+(T)$ be a dominant T -weight and recall from Section 2.2 the description of the simple roots and fundamental weights for T in terms of a basis $\{\varepsilon_1, \dots, \varepsilon_{d_G}\}$ for a Euclidean space E . Following the ideas of [McN98, Lemma 4.5.7], for $\alpha \in \Phi^+$ and $r \in \mathbb{Z}_{\geq 0}$ such that $1 < r < \langle \lambda + \rho, \alpha \rangle$, we write $\lambda + \rho - r\alpha = a_1\varepsilon_1 + \dots + a_{d_G}\varepsilon_{d_G}$ and set $A_{\alpha,r} = (a_j)_{j=1}^{d_G} \in \mathbb{Q}^{d_G}$. Also for $\mu \in X^+(T)$, write $\mu + \rho = b_1\varepsilon_1 + \dots + b_{d_G}\varepsilon_{d_G}$ and set $B_\mu = (b_j)_{j=1}^{d_G} \in \mathbb{Q}^{d_G}$. The proof of the next result directly follows from the description of the action of \mathscr{W} on Φ in terms of the ε_i given in Section 2.2, together with Lemma 2.7.1.

Lemma 2.7.11

Let λ be as above and let $\mu \in X^+(T)$, $\alpha \in \Phi^+(G)$, and $r \in \mathbb{Z}_{\geq 0}$ be such that $1 < r < \langle \lambda + \rho, \alpha \rangle$. Then $\mu \in \mathscr{W} \cdot (\lambda - r\alpha)$ if and only if $A_{\alpha,r} \sim_G B_\mu$, in which case α and $\lambda - \mu$ have equal support.

Proof. We show the result in the case where G is of type D_n over K and leave the other cases to the reader, as they can be dealt with in a similar fashion. First assume the existence of $w \in \mathscr{W}$ such that $w \cdot (\lambda - r\alpha) = \mu$ and let $A_{r,\alpha}$ and B_μ be as above. As seen in Section 2.2.3, the Weyl group \mathscr{W} of G acts as the group of all permutations and even number of sign changes of the ε_i for $1 \leq i \leq n+1$, which by Definition 2.7.10 (Part 3) translates to $A_{\alpha,r} \sim_G B_\mu$ as desired. Conversely, if $A_{\alpha,r} \sim_G B_\mu$, then using the description of the action of \mathscr{W} in terms of the ε_i again, one easily finds $w \in \mathscr{W}$ such that $w \cdot (\lambda - r\alpha) = \mu$. Finally, observe that the assertion on the support immediately follows from Lemma 2.7.1, thus completing the proof. \square

Let $V_G(\lambda) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\lambda)$ given by Proposition 2.7.4 and let $\mu \in X^+(T)$ be a dominant T -weight. We can now describe an algorithm for finding an upper bound for $m_{L_G(\lambda)}(\mu)$, provided that the decomposition (2.17) of $\chi^\mu(\nu)$ is known for every $\mu \preceq \nu \prec \lambda$.

1. For every $\mu \preceq \nu \prec \lambda$, we first find every $\alpha \in \Phi^+$ and $1 < r < \langle \lambda + \rho, \alpha \rangle$ for which there exists $w_{\alpha,r} \in \mathscr{W}$ such that $\nu = \mu_{\alpha,r} = w_{\alpha,r} \cdot (\lambda - r\alpha)$, using Lemma 2.7.11. We then determine such $w_{\alpha,r}$ and an application of Lemma 2.7.6 yields the coefficients in the truncated sum (2.21).
2. Substituting $\chi^\mu(\nu)$ by its decomposition in terms of characters of irreducibles (known by assumption) for every $\mu \preceq \nu \prec \lambda$ then gives $\{b_\nu\}_{\mu \preceq \nu \prec \lambda} \subset \mathbb{Z}_{\geq 0}$ as in (2.22).
3. By Proposition 2.7.8, every $\mu \preceq \nu \prec \lambda$ such that $b_\nu \neq 0$ in (2.22) affords the highest weight of a composition factor of $V_G(\lambda)$. Therefore since $L_G(\lambda) \cong V_G(\lambda)/\text{rad}(\lambda)$, we have

$$\begin{aligned} m_{L_G(\lambda)}(\mu) &= m_{V_G(\lambda)}(\mu) - \sum_{\mu \preceq \nu \prec \lambda} [V_G(\lambda), L_G(\nu)] m_{L_G(\nu)}(\mu) \\ &\leq m_{V_G(\lambda)}(\mu) - \sum_{\substack{\mu \preceq \nu \prec \lambda \\ b_\nu \neq 0}} m_{L_G(\nu)}(\mu). \end{aligned}$$

To conclude this chapter, we refer the reader to Lemmas 5.1.2 and 7.2.6 for detailed applications of the above algorithm.

CHAPTER 3

Parabolic Subgroups

Let K be an algebraically closed field having characteristic $p \geq 0$ and let $Y = \mathrm{CL}_n(K)$ be a simple algebraic group of classical type over K having rank n . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_n\}$ denote a corresponding base of the root system $\Phi(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_n\}$ be the set of fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. The following well-known result gives constitutes the key to the proof of Theorem 1, to which this chapter is devoted.

Lemma 3.1

Let G be a simple algebraic group of classical type over K and consider a non-trivial irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_G)$. If P is a proper parabolic subgroup of G , then $V|_P$ is reducible.

Proof. Write $P = QL$, where L is a Levi subgroup of P and $Q \neq 1$ is the unipotent radical of P . Then the fixed point space $V^Q \subsetneq V$ of Q is a proper non-zero KP -submodule of V , from which the result follows. \square

Keeping the notation introduced in Section 2.3.2, set $P_r = P_{\Pi(Y) - \{\alpha_r\}}$ for $1 \leq r \leq n$. In other words, P_r is the opposite of the standard parabolic subgroup of Y obtained by removing the r^{th} node in the corresponding Dynkin diagram of Y . Write X to denote the derived subgroup of $L_r = \langle T_Y, U_{\pm\alpha_i} : 1 \leq i \leq n, i \neq r \rangle$ and fix $B_X = T_X U_X$, where $T_X = T_Y \cap X$ is a maximal torus of X and $U_X = U_Y \cap X$ the unipotent radical of B_X . Clearly X is semisimple (unless $Y = \mathrm{SL}_2(K)$) and before going further, we describe the restriction to T_X of the fundamental weights for T_Y . Three situations may occur.

If $r = 1$, then X is simple of type $\text{CL}_{n-1}(K)$ and $\Pi(X) = \{\beta_1, \dots, \beta_{n-1}\}$, where $\beta_i = \alpha_{i+1}$, for every $1 \leq i \leq n-1$. Here if $\{\omega_1, \dots, \omega_{n-1}\}$ denotes the set of fundamental weights for T_X corresponding to our choice of base, then one easily checks that

$$\lambda_1|_{T_X} = 0, \lambda_i|_{T_X} = \omega_{i-1}, \text{ for every } 1 < i \leq n. \quad (3.1)$$

If $1 < r < n$ (without loss of generality, we may and will assume $Y \neq D_n$ if $r = n-1$, thanks to the graph automorphism of D_n), then X is a semisimple subgroup of Y of type $A_{r-1} \times \text{CL}_{n-r}(K)$ and $\Pi(X) = \{\beta_1, \dots, \beta_{r-1}\} \cup \{\gamma_1, \dots, \gamma_{n-r}\}$, where $\beta_i = \alpha_i$ for every $1 \leq i < r$, and $\gamma_j = \alpha_{j+r}$, for every $1 \leq j \leq n-r$. Here again, if

$$\{\omega_{1,i} : 1 \leq i \leq r-1\} \cup \{\omega_{2,j} : 1 \leq j \leq n-r\}$$

denotes the set of fundamental weights for T_X corresponding to our choice of base $\Pi(X)$, one easily sees that for every $1 \leq i < r$ and $r < j \leq n$, we have

$$\lambda_i|_{T_X} = \omega_{1,i}, \lambda_r|_{T_X} = 0, \text{ and } \lambda_j|_{T_X} = \omega_{2,j-r}. \quad (3.2)$$

Finally if $r = n$, then X is simple of type A_{n-1} and $\Pi(X) = \{\beta_1, \dots, \beta_{n-1}\}$, where $\beta_i = \alpha_i$, for every $1 \leq i < n$. As above, if $\omega_1, \dots, \omega_{n-1}$ are the fundamental weights for T_X corresponding to our choice of base, then we leave to the reader to check that

$$\lambda_i|_{T_X} = \omega_i, \lambda_n|_{T_X} = 0, \text{ for every } 1 \leq i < n. \quad (3.3)$$

In Section 3.1, we let X be as above (that is, $X = L'_r$, where $P_r = Q_r L_r$ for some $1 \leq r \leq n$), and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. By studying the Q_r -commutator series of V

$$V \supset [V, Q_r] \supset [V, Q_r^2] \supset \dots \supset [V, Q_r^k] \supsetneq 0$$

introduced in Section 2.3.2, we then show that in general, L_r acts with more than two composition factors on V , unless $\lambda = \lambda_i$ for some $1 \leq i \leq n$. We then determine every triple (Y, X, V) such that the Q_r -commutator series of V is of the form

$$V \supset [V, Q_r] \supset 0.$$

Since $V/[V, Q_r]$ is irreducible as a KX -module (see Lemma 2.3.6), it remains to determine whether the KX -module $[V, Q_r]$ is irreducible or not, which can be done by a dimension argument, yielding the following result. Notice that since L_r normalizes Q_r , each triple (Y, X, λ) recorded in Table 3.1 is such that P_r acts with exactly two composition factors on $V = L_Y(\lambda)$ as well. Also observe that in general, B_Y is not maximal among parabolic subgroups of Y , except if $Y = \text{SL}_2(K)$, in which case B_Y acts with exactly two composition factors on V if and only if $\dim V = 2$.

Theorem 3.2

Let X, Y be as above and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if Y, X , and λ are as in Table 3.1, where we give λ up to graph automorphisms. Furthermore, if (Y, X, λ) is recorded in Table 3.1, then $V|_X$ is completely reducible.

Y	X	λ	$V _X$	Dimensions
$A_n(n \geq 2)$	L'_1	λ_1	$0/\omega_1$	$1, n$
	L'_r ($1 < r < n$)	λ_1	$\omega_{1,1}/\omega_{2,1}$	$r, n - r + 1$
	L'_n	λ_1	$\omega_1/0$	$n, 1$
	L'_1	λ_i ($1 < i < d$)	ω_{i-1}/ω_i	$\binom{n}{i-1}, \binom{n}{i}$
$B_n(n \geq 2)$	L'_1	λ_n	$\omega_{n-1}/\omega_{n-1}$	$2^{n-1}, 2^{n-1}$
$C_n(n \geq 3)$	L'_n	λ_1	ω_1/ω_n	n, n
$D_n(n \geq 4)$	L'_n	λ_1	ω_1/ω_{n-1}	n, n
	L'_1	λ_n	$\omega_{n-1}/\omega_{n-2}$	$2^{n-2}, 2^{n-2}$

Table 3.1: The parabolic case. Here $d = \lceil \frac{n+1}{2} \rceil$ denotes the integer part of $(n+1)/2$.

Finally, we establish as well the following Proposition, which together with Theorem 3.2 constitute a proof of Theorem 1.

Proposition 3.3

Let Y be as above and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. If P is a proper parabolic of Y acting with exactly two composition factors on Y , then P is maximal among all proper parabolic subgroups of Y .

3.1 A first reduction

Let K, Y, X be as in the statement of Theorem 3.2 and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Also for $1 \leq r \leq n$, denote by Q_r the unipotent radical of P_r and recall the existence of the Q_r -commutator series of V

$$V \supset [V, Q_r] \supset [V, Q_r^2] \supset \dots \supset [V, Q_r^k] \supsetneq 0, \quad (3.4)$$

where $k \in \mathbb{Z}_{\geq 0}$. (Properties of this series were discussed in Section 2.3.2.) Notice that in such a situation X has at least $k+1$ composition factors on V (including multiplicities) and using this observation, we first tackle the case where (Y, p) is not special (see Definition 2.3.3).

Lemma 3.1.1

Assume (Y, p) is not special and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. If X has exactly two composition factors on V , then $\lambda = \lambda_i$ for some $1 \leq i \leq n$.

Proof. The remark made above forces $k = 1$ in (3.4), that is, the Q_r -commutator series of V is of the form $V \supset [V, Q_r] \supset 0$, which by Proposition 2.3.8 means that every T_Y -weight $\mu \in \Lambda(V)$ has Q_r -level at most 1. We first claim that $a_i \leq 1$ for every $1 \leq i \leq n$, since otherwise $\lambda - 2(\alpha_1 + \dots + \alpha_n)$ is a T_Y -weight having Q_r -level 2 for every $1 \leq r \leq n$, a contradiction. Also, if $a_i a_j \neq 0$ for some $1 \leq i < j \leq n$, then again $\lambda - 2(\alpha_1 + \dots + \alpha_n)$ is a T_Y -weight having Q_r -level 2 for every $1 \leq r \leq n$, thus completing the proof. \square

In the next result, we extend Lemma 3.1.1 to the case where (Y, p) is special, which in our situation forces Y to be of type B_n or C_n and $p = 2$. The existence of isogenies between B_n and C_n allows us to only consider the case where Y is of type C_n over K .

Lemma 3.1.2

Assume $p = 2$ and Y is of type C_n . Also consider an irreducible KY -module $V = L_Y(\lambda)$ having 2-restricted highest weight $\lambda \in X^+(T_Y)$. If X has exactly two composition factors on V , then $\lambda = \lambda_i$ for some $1 \leq i \leq n$.

Proof. Here again, the strategy is to argue on the existence of T_Y -weights in V having certain Q_r -levels. Indeed for every $i \in \mathbb{Z}_{>0}$, the subspace $V^i = \bigoplus V_\mu$ (the sum ranging over those weights $\mu \in X(T_Y)$ having Q_r -level at least i) is a KX -module. However, Proposition 2.3.8 cannot be applied in this situation (as $p = 2$) and hence showing the existence of a T_Y -weight having Q_r -level 2 is not enough, since it does not imply the existence of a T_Y -weight having Q_r -level 1.

First consider the case where $\langle \lambda, \alpha_n \rangle = 0$ and assume the existence of $1 \leq i < j < n$ such that $a_i a_j \neq 0$. Here $\lambda - (\alpha_1 + \dots + \alpha_n) \in \Lambda(V)$, since it is \mathscr{W}_Y -conjugate to $\lambda - (\alpha_i + \dots + \alpha_j)$ and the latter is obviously a T_Y -weight in V . (In fact, we even have $\lambda - (\alpha_i + \dots + \alpha_j) \in \Lambda^+(V)$.) Similarly $\lambda - 2(\alpha_1 + \dots + \alpha_n) \in \Lambda(V)$, contradicting our initial assumption.

Finally, consider the case where $\langle \lambda, \alpha_n \rangle = 1$ and suppose that there exists $1 \leq i < n$ such that $a_i \neq 0$. By [Sei87, Proposition 1.6], we have

$$V \cong L_G(\lambda_i) \otimes L_G(\lambda_n),$$

from which one easily sees that $V_{\lambda - (\alpha_i + \dots + \alpha_n)} \neq 0$. Therefore both $\lambda - (\alpha_1 + \dots + \alpha_n)$ and $\lambda - 2(\alpha_1 + \dots + \alpha_n)$ (the latter being \mathscr{W}_Y -conjugate to the former) are T_Y -weights of V , a contradiction. The proof is thus complete. \square

3.2 Conclusion

Let K, Y, X be as in the statement of Theorem 3.2 and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. In this section, we give a complete proof of Theorem 3.2, starting with the case where Y is of type A_n over K .

Lemma 3.2.1

Let Y be a simple algebraic group of type A_n over K and suppose that X has exactly two composition factors on V . Then Y, X and λ appear in Table 3.1, up to graph automorphisms.

Proof. By Lemma 3.1.1, we can assume $\lambda = \lambda_i$ for some $1 \leq i \leq n$. If $i = 1$, observe that all T_Y -weights of V have Q_r -levels smaller or equal to 1, this for every $1 \leq r \leq n$. The same holds in the case where $i = n$, while if $1 < i < n$, then $\lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n)$ is a T_Y -weight having Q_r -level 2, for every $1 < r < n$, so that $X = L'_1$ or L'_n as desired. \square

We now prove that the candidates for Y, X and λ obtained in Lemma 3.2.1 indeed satisfy the desired property.

Lemma 3.2.2

Let Y be a simple algebraic group of type A_n over K and suppose that X and $V = L_Y(\lambda)$ are such that (Y, X, λ) appears in Table 3.1. Then X has exactly two composition factors on V .

Proof. First consider the case where $\lambda = \lambda_1$ and $X = L'_1$. By Lemma 2.3.6 the restriction of λ to T_X affords the highest weight of a first composition factor of V , isomorphic to $V/[V, Q_1]$. Also, every T_X -weight of V having Q_1 -level equal to 1 is under the restriction of $\lambda - \alpha_1$, which thus affords the highest weight of a second KX -composition factor of V . Applying (3.1) then yields $V/[V, Q_1] \cong K$ as well as $(\lambda - \alpha_1)|_{T_X} = \omega_1$. Since

$$\dim K + \dim L_X(\omega_1) = 1 + n - 1 = n = \dim V,$$

the result holds in this situation. The case where $\lambda = \lambda_1$ and $X = L'_n$ can be dealt with in a similar fashion and hence is left to the reader.

Next consider the situation where $\lambda = \lambda_1$ and $X = L'_r$ for some $1 < r < n$. Arguing as above, one gets that each of $\lambda|_{T_X}$ and $(\lambda - (\alpha_1 + \cdots + \alpha_r))|_{T_X}$ affords the highest weight of a composition factor of V for X . An application of (3.2) then yields $\lambda|_{T_X} = \omega_{1,1}$ as well as $(\lambda - (\alpha_1 + \cdots + \alpha_r))|_{T_X} = \omega_{2,1}$. Again, since $\dim L_X(\omega_{1,1}) + \dim L_X(\omega_{2,1}) = \dim V$, the result follows.

Finally, assume $\lambda = \lambda_i$ for some $1 < i < d$ (where d denotes the integer part of $(n+1)/2$) and $X = L'_1$. Arguing as in the previous cases shows that each of $\omega = \lambda|_{T_X}$ and $\omega' = (\lambda - (\alpha_1 + \cdots + \alpha_i))|_{T_X}$ affords the highest weight of a composition factor of V for X . Lemma 2.4.5 yields $\dim V = \dim V/[V, Q_1] + \dim L_X(\omega')$, thus completing the proof. \square

Let Y be a simple algebraic group of type A_n over K and let X denote the derived subgroup of a Levi subgroup of a maximal proper parabolic subgroup of Y . Also consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$ and assume X acts with exactly two composition factors on V . Then X and λ are as in Table 3.1 by Lemma 3.2.1. Conversely, if X and λ are recorded in Table 3.1, then X has exactly two composition factors on V by Lemma 3.2.2. In other words, both lemmas provide a proof of Theorem 3.2 in the case where Y is of type A_n over K .

Next assume $p \neq 2$ and suppose that Y is of type B_n over K . We proceed as in the previous case, starting by showing a first direction of Theorem 3.2.

Lemma 3.2.3

Assume $p \neq 2$ and let Y be a simple algebraic group of type B_n over K . Also suppose that X acts with at most two composition factors on V . Then Y , X and λ appear in Table 3.1.

Proof. By Lemma 3.1.1 again, we can assume $\lambda = \lambda_i$ for some $1 \leq i \leq n$. Now if $1 \leq i < n$, then $\lambda - (2\alpha_1 + \cdots + 2\alpha_n)$ is a T_Y -weight having Q_r -level 2, this for every $1 \leq r \leq n$, forcing $\lambda = \lambda_n$. Therefore $\lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$ is a T_Y -weight of V , yielding $X = L'_1$ as desired. \square

It remains to show that $X = L'_1$ has exactly two composition factors on $V = L_Y(\lambda_n)$, which can be done exactly as in the first part of Lemma 3.2.2 (replacing Lemma 2.4.5 by [BGT15, Lemma 2.3.2], for example). Notice that here both λ and $\lambda - (\alpha_1 + \cdots + \alpha_n)$ restrict to ω_{n-1} by (3.1).

Lemma 3.2.4

Let Y be a simple algebraic group of type C_n over K and suppose that X acts with at most two composition factors on V . Then Y , X and λ appear in Table 3.1.

Proof. First assume $p \neq 2$, in which case Lemma 3.1.1 yields $\lambda = \lambda_i$ for some $1 \leq i \leq n$. Now if $i = n$, then $\lambda - 2(\alpha_1 + \cdots + \alpha_n)$ is a T_Y -weight of V having Q_r -level 2, for any $1 \leq r \leq n$. Similarly, if $1 < i < n$, then $\lambda - (2\alpha_1 + \cdots + 2\alpha_{i-1} + 3\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_n)$ is a T_Y -weight of V , forcing $\lambda = \lambda_1$, in which case $\lambda - (2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n)$ is a T_Y -weight and thus $X = L'_n$ as desired. Now if $p = 2$, Lemma 3.1.2 also yields $\lambda = \lambda_i$ for some $1 \leq i \leq n$. In addition, observe that if $i \neq n$, then $\lambda - (\alpha_1 + \cdots + \alpha_n), \lambda - (2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n) \in \Lambda(V)$, since both are \mathscr{W}_Y -conjugate to λ , forcing $i = 1$ and $X = L'_n$ as desired. \square

Proceeding as in the proof of Lemma 3.2.2 shows that $X = L'_n$ acts with exactly two composition factors on $L_Y(\lambda_1)$. Here λ_1 restricts to ω_1 , while $\lambda_1 - (\alpha_1 + \cdots + \alpha_n)$ restricts to ω_{n-1} by (3.3). Again the details are left to the reader. Finally, suppose that Y has type D_n over K ($n \geq 4$), in which case we shall assume $r \neq n - 1$.

Lemma 3.2.5

Let Y be a simple algebraic group of type D_n over K and suppose that X acts with at most two composition factors on V . Then Y , X and λ appear in Table 3.1, up to graph automorphisms.

Proof. By Lemma 3.1.1 we have $\lambda = \lambda_i$ for some $1 \leq i \leq n$. If $i = 1$, observe that $\lambda - (2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$ is a T_Y -weight having Q_r -level greater than or equal to 2 for every $1 \leq r \leq n - 2$, forcing $X = L'_{n-1}$ or L'_n as desired. If on the other hand $1 < i < n - 1$, then $\lambda - (2\alpha_1 + 3\alpha_2 + \cdots + 3\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n)$ is a T_Y -weight having Q_r level greater than or equal to 2 for every $1 \leq r \leq n$, a contradiction. Finally, assume $i = n$ (or $n - 1$) and $X \neq L'_1$. Then the T_Y -weight $\lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-3} + 3\alpha_{n-2} + 2\alpha_{n-1} + 2\alpha_n)$ has Q_r -level greater than or equal to 2, from which the result follows. \square

In order to complete the proof of Theorem 3.2, it remains to first show that $X = L'_n$, $\lambda = \lambda_1$ and $X = L'_1$, $\lambda = \lambda_n$ are indeed examples, which can be done exactly as in the proof of Lemma 3.2.2. Using (3.1) and (3.3), one also checks that we have the desired restrictions, and then concludes using the fact that in each case, the KX -composition factors are KP_r -modules as well by construction. Finally, we give a proof of Proposition 3.3, which together with Theorem 3.2 yield Theorem 1.

Proof of Proposition 3.3: Let $P = Q_J L_J$, Y and $V = L_Y(\lambda)$ be as in the statement of the proposition. Also let $1 \leq r \leq n$ such that $P \subseteq P_r = Q_r L_r$ and observe that P_r must act with at most two composition factors on V . By Lemma 3.1, P_r has exactly two composition factors on V and hence an application of Theorem 3.2 shows that (Y, λ) appears in Table 3.1. Consequently, we may assume $p \neq 2$, in which case one easily concludes that P has to be maximal using Proposition 2.3.8.

The case $\mathrm{Spin}_{2n}(K) \subset \mathrm{Spin}_{2n+1}(K)$

Let $Y = \mathrm{Spin}_{2n+1}(K)$ be a simply connected simple algebraic group of type B_n ($n \geq 3$) over K and consider the subgroup X of type D_n , embedded in Y in the usual way, as the derived subgroup of the stabilizer of a non-singular one-dimensional subspace of the natural module for Y . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_n\}$ denote a corresponding base of the root system $\Phi(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_n\}$ be the set of fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. Here we have

$$X = \langle U_\alpha : \alpha \in \Phi(Y) \text{ is a long root} \rangle.$$

Let $B_X = U_X T_X$ be a Borel subgroup of X , where $T_X = T_Y \cap X$ is a maximal torus of X and $U_X = U_Y \cap X$ the unipotent radical of B_X , and denote by $\Pi(X) = \{\beta_1, \dots, \beta_n\}$ the corresponding base of the root system $\Phi(X)$ of X . Here $\beta_i = \alpha_i$ for every $1 \leq i < n$, $\beta_n = \alpha_{n-1} + 2\alpha_n$, while the corresponding fundamental dominant T_X -weights $\omega_1, \dots, \omega_n$ satisfy the restrictions

$$\lambda_i|_{T_X} = \omega_i, \text{ for } 1 \leq i < n - 1, \lambda_{n-1}|_{T_X} = \omega_{n-1} + \omega_n, \text{ and } \lambda_n|_{T_X} = \omega_n. \quad (4.1)$$

Finally, let θ denote the graph automorphism of X and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. In [For96, Section 3], Ford determined the pairs (λ, p) such that $V|_{X^{(\theta)}}$ is irreducible and $V|_X$ is reducible. He observed that in this situation, X acts with exactly two composition factors on V , interchanged by θ . He proceeded by first finding two maximal vectors in V for B_X , say v^+, w^+ , and then determined under which conditions V could be decomposed as a direct sum of $V_1 = \mathcal{L}(X)v^+$ and $V_2 = \mathcal{L}(X)w^+$. (Here $\mathcal{L}(X)$ denotes the Lie algebra of X .) Finally, he showed that both V_1, V_2 are irreducible as KX -modules and that $V_2 = V_1^\theta$. The aim of this chapter is to extend [For96, Theorem 3.3] to the following more general result.

Theorem 4.1

Let K , Y and X be as above, and consider an irreducible non-trivial KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = \sum_{r=1}^n a_r \lambda_r \in X^+(T_Y)$. Also if $\lambda \neq a_n \lambda_n$, let $1 \leq k < n$ be maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$. Then X has exactly two composition factors on V if and only if $\langle \lambda, \alpha_n \rangle \leq 1$ and one of the following holds.

1. $\lambda = \lambda_k$ and $p \neq 2$.
2. $\lambda = \lambda_n$.
3. λ is neither as in 1 nor 2, $p \neq 2$ and the following divisibility conditions are satisfied.
 - (a) $p \mid a_i + a_j + j - i$ for every $1 \leq i < j < n$ such that $a_i a_j \neq 0$ and $a_r = 0$ for $i < r < j$.
 - (b) $p \mid 2(a_n + a_k + n - k) - 1$.

Furthermore, if (λ, p) is as in 1, 2 or 3, then $L_Y(\lambda)|_X$ is completely reducible.

In Sections 4.1 and 4.2, we investigate various weight spaces that shall play a role in the proof of Theorem 4.1. Both sections being very technical, we advise the reader to skip them in the first place and then come back to them when needed. Even though we are only interested in a Lie algebra of type B_n over K , we start by investigating certain weight multiplicities for a Lie algebra of type A_n . Indeed, this helps us in determining bases for weight spaces for $\mathcal{L}(Y)$ by considering a suitable Levi subalgebra of the latter.

Let v^+ denote a maximal vector in V for B_Y and observe that since $B_X \subset B_Y$, then v^+ is a maximal vector for B_X as well. As in the proof of [For96, Theorem 3.3], we find another maximal vector w^+ in V for B_X and first aim at showing (see Section 4.3.2) that if X has exactly two composition factors on V , then $\langle \lambda, \alpha_n \rangle \leq 1$ and one of 1, 2 or 3 is satisfied. We start by observing that both $\langle Xv^+ \rangle$ and $\langle Xw^+ \rangle$ are irreducible p -restricted KX -modules and hence are irreducible as modules for the Lie algebra $\mathcal{L}(X)$ of X . Therefore $V = \langle Xv^+ \rangle \oplus \langle Xw^+ \rangle = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)w^+$ and thus we obviously get

$$f_{\alpha_r + \dots + \alpha_n} v^+, f_{\alpha_r + \dots + \alpha_n} w^+ \in \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)w^+ \quad (4.2)$$

for every $1 \leq r \leq n$. Finally, a generalization of [For96, Proposition 3.1] (namely, Proposition 4.3.7) leads us to carefully investigate certain weight spaces of V .

Reciprocally, in Section 4.3.3, we suppose that one of 1, 2 or 3 holds and aim at showing that $V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)w^+$. Now we know that V can be written as a direct sum of T_Y -weight spaces, which by (2.15) are spanned by vectors of the form $f_{\gamma_1} \cdots f_{\gamma_r} v^+$, where $\gamma_1, \dots, \gamma_r \in \Phi^+(Y)$ are such that $\gamma_1 \preceq \gamma_2 \preceq \dots \preceq \gamma_r$. Therefore, it suffices to show that any such element belongs to $\mathcal{L}(X)v^+ \oplus \mathcal{L}(X)w^+$, or equivalently (thanks to an analogue of [For96, Lemma 3.4], namely Proposition 4.3.2), that (4.2) holds. Again, a study of certain weight spaces of V then allows us to conclude.

4.1 Weight spaces for G of type A_n

Let G be a simple algebraic group of type A_n ($n \geq 2$) over K , fix a Borel subgroup $B = UT$ of G as usual, and let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ be a corresponding base of the root system Φ of G . Let $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental weights corresponding to our choice of base Π and denote by \mathcal{L} the Lie algebra of G . Also let \mathfrak{h} be the Lie algebra of T and let \mathfrak{b} be the Borel subalgebra of \mathcal{L} corresponding to Π , so that

$$\mathcal{L} = \mathcal{L}(T) \oplus \left(\bigoplus_{\gamma \in \Phi} \mathcal{L}(U_\gamma) \right).$$

Consider a standard Chevalley basis

$$\mathcal{B} = \{e_\gamma, f_\gamma, h_{\gamma_r} : \gamma \in \Phi^+, 1 \leq r \leq n\}$$

of \mathcal{L} , as in Section 2.5.1, and for $\sigma \in X^+(T)$, simply write $V(\sigma)$ (respectively, $L(\sigma)$) to denote the Weyl module for G corresponding to σ (respectively, the irreducible KG -module having highest weight σ). In this section, we consider the p -restricted dominant T -weight $\sigma = a\sigma_1 + b\sigma_n$, where $a, b \in \mathbb{Z}_{>0}$, and set

$$\mu = \sigma - (\gamma_1 + \dots + \gamma_n).$$

Also for $1 \leq r \leq s \leq n$, we adopt the notation

$$f_{r,s} = f_{\gamma_r + \dots + \gamma_s}.$$

By (2.14) and our choice of ordering \preceq on Φ^+ , the weight space $V(\sigma)_\mu$ is spanned by $f_{1,n}v^\sigma$ and elements of the form $f_{1,r_1}f_{r_1+1,r_2} \cdots f_{r_m+1,n}v^\sigma$, where $v^\sigma \in V(\sigma)_\sigma$ denotes a maximal vector in $V(\sigma)$ for B , $1 \leq m \leq n$ and $1 \leq r_1 < r_2 < \dots < r_m < n$. Now observe that $\sigma - \alpha_r + \dots + \alpha_s$ is a T -weight of $V(\sigma)$ if and only if either $r = 1$ or $s = n$. Therefore the list

$$\{f_{1,r}f_{r+1,n}v^\sigma\}_{1 \leq r \leq n-1} \cup \{f_{1,n}v^\sigma\} \quad (4.3)$$

forms a generating set for $V(\sigma)$ by Proposition 2.5.8. Furthermore, an application of Lemma 2.3.13 yields $\dim V(\sigma)_\mu = n$, forcing the generating elements of (4.3) to be linearly independent, so that the following holds.

Proposition 4.1.1

Let $\sigma = a\sigma_1 + b\sigma_n$, where $a, b \in \mathbb{Z}_{>0}$, and set $\mu = \sigma - (\gamma_1 + \dots + \gamma_n)$. Then μ is dominant and the set (4.3) forms a basis of the weight space $V(\sigma)_\mu$.

We now study the relation between the quadruple (a, b, n, p) and the existence of a maximal vector in $V(\sigma)_\mu$ for \mathfrak{b} . For $A = (A_r)_{1 \leq r \leq n} \in K^n$, we set

$$u(A) = \sum_{r=1}^{n-1} A_r f_{1,r} f_{r+1,n} v^\sigma + A_n f_{1,n} v^\sigma \in V(\sigma)_\mu. \quad (4.4)$$

Lemma 4.1.2

Let σ, μ be as above, and adopt the notation of (4.4). Then the following assertions are equivalent.

1. There exists $0 \neq A \in K^n$ such that $e_\gamma u(A) = 0$ for every $\gamma \in \Pi$.
2. There exists $A \in K^{n-1} \times K^*$ such that $e_\gamma u(A) = 0$ for every $\gamma \in \Pi$.
3. The divisibility condition $p \mid a + b + n - 1$ is satisfied.

Proof. Let $A = (A_r)_{1 \leq r \leq n} \in K^n$ and set $u = u(A)$. Then applying Lemma 2.5.3 successively yields

$$\begin{aligned}
e_{\gamma_1} u &= \sum_{r=1}^{n-1} A_r e_{\gamma_1} f_{1,r} f_{r+1,n} v^\sigma + A_n e_{\gamma_1} f_{1,n} v^\sigma \\
&= (a+1)A_1 f_{2,n} v^\sigma - \sum_{r=2}^{n-1} A_r f_{2,r} f_{r+1,n} v^\sigma - A_n f_{2,n} v^\sigma \\
&= \left((a+1)A_1 + \sum_{r=2}^{n-1} A_r - A_n \right) f_{2,n} v^\sigma, \\
e_{\gamma_r} u &= (A_r - A_{r-1}) f_{1,r-1} f_{r+1,n} v^\sigma, \\
e_{\gamma_n} u &= (A_n + bA_{n-1}) f_{1,n-1} v^\sigma,
\end{aligned}$$

where $1 < r < n$. Now $e_{\gamma_{n-1}} \cdots e_{\gamma_2} f_{2,n} v^\sigma = \pm f_{\gamma_n} v^\sigma \neq 0$, showing that $f_{2,n} v^\sigma \neq 0$. Similarly, one checks that each of the vectors $f_{\gamma_1} f_{3,n} v^\sigma, \dots, f_{2,n-2} f_{\gamma_n} v^\sigma, f_{2,n} v^\sigma$ is non-zero, so that $e_\gamma u(A) = 0$ for every $\gamma \in \Pi$ if and only if A is a solution to the system of equations

$$\begin{cases} A_n = (a+1)A_1 + \sum_{r=2}^{n-1} A_r \\ A_{r-1} = A_r \text{ for every } 1 < r < n \\ A_n = -bA_{n-1}. \end{cases} \quad (4.5)$$

Now one easily sees that (4.5) admits a non-trivial solution A if and only if $p \mid a + b + n - 1$ (showing that 1 and 3 are equivalent), in which case $A \in \langle (1, \dots, 1, -b) \rangle_K$ (so that 1 and 2 are equivalent), completing the proof. \square

Let σ, μ be as above and consider an irreducible KG -module $V = L(\sigma)$ having highest weight σ . Take $V = V(\sigma)/\text{rad}(\sigma)$ and write v^+ to denote the image of v^σ in V , that is, v^+ is a maximal vector in V for B . By Proposition 4.1.1, the weight space V_μ is spanned by

$$\{f_{1,r} f_{r+1,n} v^+\}_{1 \leq r \leq n-1} \cup \{f_{1,n} v^+\}. \quad (4.6)$$

We write $V_{1,n}$ to denote the span of all the generators in (4.6) except for $f_{1,n} v^+$. The following result gives a precise description of the weight space V_μ , as well as a characterization for $[V(\sigma), L(\mu)]$ to be non-zero.

Proposition 4.1.3

Let G be a simple algebraic group of type A_n and fix $a, b \in \mathbb{Z}_{>0}$. Also consider an irreducible KG -module $V = L(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_n \in X^+(T)$ and let $\mu = \sigma - (\gamma_1 + \cdots + \gamma_n) \in \Lambda^+(\sigma)$. Then the following assertions are equivalent.

1. The weight μ affords the highest weight of a composition factor of $V(\sigma)$.
2. The generators in (4.6) are linearly dependent.
3. The element $f_{1,n}v^+$ lies inside $V_{1,n}$.
4. The divisibility condition $p \mid a + b + n - 1$ is satisfied.

Proof. Clearly both 1 and 3 imply 2. Also if 2 holds, then $\text{rad}(\sigma) \cap V(\sigma)_\mu \neq 0$, so $L(\nu)$ occurs as a composition factor of $V(\sigma)$ for some $\nu \in \Lambda^+(\sigma)$ such that $\mu \preceq \nu \prec \sigma$. Now one easily sees that $m_{V(\sigma)}(\nu) = 1$ for every $\mu \prec \nu \prec \sigma$, hence 1 holds by Theorem 2.3.4. Still assuming 2, this also shows that there exists $0 \neq A \in K^n$ such that $u(A) \in \text{rad}(\sigma) \cap V(\sigma)_\mu$ is a maximal vector in $V(\sigma)$ for \mathcal{L} , where we adopt the notation of (4.4). Therefore 2 implies 4 as well by Lemma 4.1.2. Finally, suppose that 4 holds. By Lemma 4.1.2, there exists $A \in K^{n-1} \times K^*$ such that $e_\gamma u(A) = 0$ for every $\gamma \in \Pi$. Consequently, we also get $e_\gamma(u(A) + \text{rad}(\sigma)) = 0$ for every $\gamma \in \Pi$, that is, $u(A) + \text{rad}(\sigma) \in \langle v^+ \rangle_K \cap V_\mu = 0$ and so 3 holds. \square

To conclude this section, let σ, μ be as above and assume $p \mid a + b + n - 1$. By Proposition 4.1.3, μ affords the highest weight of a composition factor of $V(\sigma)$, and $f_{1,n}v^+ \in V_{1,n}$. Moreover, the proof of Lemma 4.1.2 showed that

$$u^+ = f_{1,n}v^\sigma - b^{-1} \sum_{r=1}^{n-1} f_{1,r}f_{r+1,n}v^\sigma \tag{4.7}$$

is a maximal vector in $V(\sigma)_\mu$ for \mathfrak{b} , leading to a precise description of $f_{1,n}v^+$ in terms of a basis of $V_{1,n}$.

4.2 Weight spaces for G of type B_n

Let G be a simple algebraic group of type B_n ($n \geq 2$) over K , fix a Borel subgroup $B = UT$ of G as usual, and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a corresponding base of the root system Φ of G . Let $\{\lambda_1, \dots, \lambda_n\}$ be the set of fundamental weights corresponding to our choice of base Π and denote by \mathcal{L} the Lie algebra of G . Also let \mathfrak{h} be the Lie algebra of T and let \mathfrak{b} be the Borel subalgebra of \mathcal{L} corresponding to Π , so that

$$\mathcal{L} = \mathcal{L}(T) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathcal{L}(U_\alpha) \right).$$

Consider a standard Chevalley basis

$$\mathcal{B} = \{e_\alpha, f_\alpha, h_{\alpha_r} : \alpha \in \Phi^+, 1 \leq r \leq n\}$$

of \mathcal{L} , as in Section 2.5.1, and for $\sigma \in X^+(T)$, simply write $V(\sigma)$ (respectively, $L(\sigma)$) to denote the Weyl module for G corresponding to σ (respectively, the irreducible KG -module having highest weight σ). Although most of the results presented here hold for K having arbitrary characteristic, we shall assume $p \neq 2$ throughout this section for simplicity. Indeed, Theorem 4.1 is an immediate consequence of [Sei87, Theorem 1, Table 1 (MR₄)] together with [For96, Theorem 3.3] in the case where $p = 2$, hence there is no harm in ruling out this possibility here. Finally, adopt the notation

$$f_{i,j} = f_{\alpha_i + \dots + \alpha_j},$$

for every $1 \leq i \leq j \leq n$, as well as

$$F_{r,s} = f_{\alpha_r + \dots + \alpha_{s-1} + 2\alpha_s + \dots + 2\alpha_n},$$

for every $1 \leq r < s \leq n$.

4.2.1 Study of $L(a\lambda_1)$ ($a \in \mathbb{Z}_{>0}$)

Let $a \in \mathbb{Z}_{>0}$ and consider the p -restricted dominant weight $\lambda = a\lambda_1 \in X^+(T)$. Also write $\mu = \lambda - 2(\alpha_1 + \dots + \alpha_n)$. By Proposition 2.5.8 and our choice of ordering \preceq on Φ^+ , one sees that

$$V(\sigma)_\mu = \langle \frac{1}{2}(f_{1,n})^2 v^\lambda, f_{1,j} F_{1,j+1} v^\lambda : 1 \leq j < n \rangle_K,$$

where $v^\lambda \in V(\lambda)_\lambda$ denotes a maximal vector in $V(\lambda)$ for B . Since we are assuming $p \neq 2$, we get that $\frac{1}{2}(f_{1,n})^2 v^\lambda \in V(\lambda)_\mu$ if and only if $(f_{1,n})^2 v^\lambda \in V(\lambda)_\mu$, so that the weight space $V(\lambda)_\mu$ is spanned by

$$\{f_{1,j} F_{1,j+1} v^\lambda\}_{1 \leq j < n} \cup \{(f_{1,n})^2 v^\lambda\}, \quad (4.8)$$

Now if $a = 1$, then μ is \mathcal{W} -conjugate to λ , which has multiplicity 1 in $V(\lambda)$. Furthermore, successively applying $e_{\alpha_1}, \dots, e_{\alpha_n}$ to the element $f_{\alpha_1} F_{1,2} v^\lambda$ shows that it is non-zero, hence $V(\lambda)_\mu = \langle f_{\alpha_1} F_{1,2} v^\lambda \rangle_K$. Finally, we leave to the reader to check (using Lemma 2.5.4 together with the fact that $V(\lambda)_{\lambda - (2\alpha_1 + \alpha_2 + \dots + \alpha_n)} = 0$) that the following result holds.

Proposition 4.2.1

Let $\lambda = \lambda_1$ and consider $\mu = \lambda - 2(\alpha_1 + \dots + \alpha_n) \in \Lambda(\lambda)$. Then $V(\lambda)_\mu = \langle f_{\alpha_1} F_{1,2} v^\lambda \rangle_K$ and the following assertions hold.

1. $f_{1,j} F_{1,j+1} v^\lambda = f_{\alpha_1} F_{1,2} v^\lambda$ for every $1 \leq j < n$.
2. $(f_{1,n})^2 v^\lambda = 2f_{\alpha_1} F_{1,2} v^\lambda$.

For the remainder of this section, we assume $a > 1$, in which case the weight μ is dominant. An application of Theorem 2.3.11 gives $\dim V(\lambda)_\mu = n$, so that the generating elements of (4.8) are linearly independent, leading to the following result.

Proposition 4.2.2

Let $a \in \mathbb{Z}_{>1}$, set $\lambda = a\lambda_1$, and consider $\mu = \lambda - 2(\alpha_1 + \cdots + \alpha_n) \in \Lambda^+(\lambda)$. Then the set (4.8) forms a basis of the weight space $V(\lambda)_\mu$.

We now study the relation between the triple (a, n, p) and the existence of a maximal vector in $V(\lambda)_\mu$ for \mathfrak{b} . For $A = (A_r)_{1 \leq r \leq n} \in K^n$, we set

$$w(A) = \sum_{j=1}^{n-1} A_j f_{1,j} F_{1,j+1} v^\lambda + A_n (f_{1,n})^2 v^\lambda. \quad (4.9)$$

Lemma 4.2.3

Let λ, μ be as above and adopt the notation of (4.9). Then the following assertions are equivalent.

1. There exists $0 \neq A \in K^n$ such that $e_\alpha w(A) = 0$ for every $\alpha \in \Pi$.
2. There exists $A \in K^{n-1} \times K^*$ such that $e_\alpha w(A) = 0$ for every $\alpha \in \Pi$.
3. The divisibility condition $p \mid 2(a+n) - 3$ is satisfied.

Proof. Let $A = (A_r)_{1 \leq r \leq n} \in K^n$ and set $w = w(A)$. Then Lemma 2.5.4 yields

$$\begin{aligned} e_{\alpha_1} w &= aA_1 F_{1,2} v^\lambda - \sum_{j=2}^{n-1} A_j f_{2,j} F_{1,j+1} v^\lambda - A_n f_{2,n} f_{1,n} v^\lambda \\ &= \left(aA_1 + \sum_{j=2}^{n-1} A_j + 2A_n \right) F_{1,2} v^\lambda, \end{aligned}$$

as well as $e_{\alpha_r} w = (A_r - A_{r-1}) f_{1,r-1} F_{1,r+1} v^\lambda$, for every $1 < r < n$. Finally, one checks that $e_{\alpha_n} w = (4A_n - A_{n-1}) f_{1,n-1} f_{1,n} v^\lambda$. As in the proof of Lemma 4.1.2, one checks that each of the vectors $F_{1,2} v^\lambda, f_{\alpha_1} F_{1,3} v^\lambda, \dots, f_{1,n-2} F_{1,n} v^\lambda, f_{1,n-1} f_{1,n} v^\lambda$ is non-zero, so that $e_\alpha w(A) = 0$ for every $\alpha \in \Pi$ if and only if $A \in K^n$ is a solution to the system of equations

$$\begin{cases} 2A_n + aA_1 &= -\sum_{r=2}^{n-1} A_r \\ A_{r-1} &= A_r \text{ for every } 1 < r < n \\ A_{n-1} &= 4A_n. \end{cases} \quad (4.10)$$

Now one easily sees that (4.10) admits a non-trivial solution $A \in K^n$ if and only if $p \mid 2(a+n) - 3$ (showing that 1 and 3 are equivalent), in which case $A \in \langle (4, \dots, 4, 1) \rangle_K$ (so that 1 and 2 are equivalent), completing the proof. \square

Let λ, μ be as above and consider an irreducible KG -module $V = L(\lambda)$ having highest weight λ . Assume $V = L(\lambda)/\text{rad}(\lambda)$ and write v^+ to denote the image of v^λ in $V = L(\lambda)$, that is, v^+ is a maximal vector in V for B . By (4.8) and our choice of ordering \preceq on Φ^+ , the weight space V_μ is spanned by

$$\{f_{1,j}F_{1,j+1}v^+\}_{1 \leq j < n} \cup \{(f_{1,n})^2v^+\}. \quad (4.11)$$

We write $V_{1,n}^2$ to denote the span of all the generators in (4.11) except for $(f_{1,n})^2v^+$. The following result gives a precise description of the weight space V_μ , as well as a characterization for $[V(\lambda), L(\mu)]$ to be non-zero.

Proposition 4.2.4

Let G be a simple algebraic group of type B_n over K and fix $a \in \mathbb{Z}_{>1}$. Also consider an irreducible KG -module $V = L(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 \in X^+(T)$ and let $\mu = \lambda - 2(\alpha_1 + \cdots + \alpha_n) \in \Lambda^+(\lambda)$. Then the following assertions are equivalent.

1. The weight μ affords the highest weight of a composition factor of $V(\lambda)$.
2. The generators in (4.11) are linearly dependent.
3. The element $(f_{1,n})^2v^+$ lies inside $V_{1,n}^2$.
4. The divisibility condition $p \mid 2(a+n) - 3$ is satisfied.

Proof. First observe that the weights $\nu \in \Lambda^+(\lambda)$ such that $\mu \prec \nu \prec \lambda$ are $\lambda - \alpha_1$, $\lambda - 2\alpha_1 - \alpha_2$ (if $a > 2$), $\lambda - (\alpha_1 + \cdots + \alpha_n)$ and $\lambda - (2\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ (if $a > 2$), which all satisfy $m_{V(\lambda)}(\nu) = 1$. Proceeding exactly as in the proof of Proposition 4.1.3, using Lemma 4.2.3 instead of Lemma 4.1.2 then yields the desired result. We leave the details to the reader. \square

4.2.2 Study of $L(\lambda_i)$ ($1 < i < n$)

Next let $\lambda = \lambda_2$ and write $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$. (Observe that μ is the zero weight.) By (2.14), our choice of ordering \preceq on Φ^+ , and Proposition 4.2.1, one checks that the weight space $V(\lambda)_\mu$ is spanned by

$$\begin{aligned} & \{F_{1,2}v^\lambda\} \cup \{f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda\} \\ & \cup \{f_{1,j}F_{2,j+1}v^\lambda\}_{2 \leq j < n} \\ & \cup \{f_{2,j}F_{1,j+1}v^\lambda\}_{2 \leq j < n} \\ & \cup \{f_{2,n}f_{1,n}v^\lambda\}. \end{aligned}$$

Proposition 4.2.5

Let $\lambda = \lambda_2$ and set $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n) \in \Lambda^+(\lambda)$. Then $\mathfrak{m}_{V(\lambda)}(\mu) = n$ and a basis of $V(\lambda)_\mu$ is given by

$$\begin{aligned} & \{F_{1,2}v^\lambda\} \cup \{f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda\} \\ & \cup \{f_{2,j}F_{1,j+1}v^\lambda\}_{2 \leq j < n}. \end{aligned} \quad (4.12)$$

Proof. By Theorem 2.3.11, the assertion on the dimension holds, so it remains to show that $f_{1,j}F_{2,j+1}v^\lambda$ ($2 \leq j < n$) and $f_{2,n}f_{1,n}v^\lambda$ can be expressed as linear combinations of elements of (4.12). Let then $2 \leq j < n$ be fixed. By Lemma 2.5.4 and Proposition 4.2.1 (part 1) applied to the B_{n-1} -Levi subalgebra corresponding to the simple roots $\alpha_2, \dots, \alpha_n$ (noticing that the constant structures have been chosen in a compatible way in Section 2.5.1), we have

$$\begin{aligned} f_{1,j}F_{2,j+1}v^\lambda &= f_{2,j}f_{\alpha_1}F_{2,j+1}v^\lambda - f_{\alpha_1}f_{2,j}F_{2,j+1}v^\lambda \\ &= -f_{2,j}F_{1,j+1}v^\lambda - f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda, \end{aligned}$$

that is, $f_{1,j}F_{2,j+1}v^\lambda \in \langle f_{2,j}F_{1,j+1}v^\lambda, f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda \rangle_K$. On the other hand, Lemma 2.5.4 and Proposition 4.2.1 (part 2) applied to the B_{n-1} -Levi subalgebra corresponding to the simple roots $\alpha_2, \dots, \alpha_n$ yield

$$\begin{aligned} f_{2,n}f_{1,n}v^\lambda &= -2F_{1,2}v^\lambda + f_{1,n}f_{2,n}v^\lambda \\ &= -2F_{1,2}v^\lambda - f_{\alpha_1}(f_{2,n})^2v^\lambda + f_{2,n}f_{\alpha_1}f_{2,n}v^\lambda \\ &= -2F_{1,2}v^\lambda - 2f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda - f_{2,n}f_{1,n}v^\lambda, \end{aligned}$$

so that $f_{2,n}f_{1,n}v^\lambda = -F_{1,2}v^\lambda - f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda$. Therefore $f_{2,n}f_{1,n}v^\lambda$ lies in the subspace of $V(\lambda)_\mu$ generated by $F_{1,2}v^\lambda$ and $f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda$, as desired. \square

Let λ, μ be as above and consider an irreducible KG -module $V = L(\lambda)$ having highest weight λ . As usual, take $V = V(\lambda)/\text{rad}(\lambda)$ and write v^+ to denote the image of v^λ in V , that is, v^+ is a maximal vector in V for B . By Proposition 4.2.5, the weight space V_μ is spanned by

$$\begin{aligned} & \{F_{1,2}v^+\} \cup \{f_{\alpha_1}f_{\alpha_2}F_{2,3}v^+\} \\ & \cup \{f_{2,j}F_{1,j+1}v^+\}_{2 \leq j < n}. \end{aligned} \quad (4.13)$$

Now by [Lüb01, Theorems 4.4, 5.1], the KG -module $V(\lambda)$ is irreducible (since $p \neq 2$), which in particular yields the following result.

Proposition 4.2.6

Consider an irreducible KG -module $V = L(\lambda)$ having highest weight $\lambda = \lambda_2$. Then $V = V(\lambda)$ and the T -weight $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$ is dominant. Also $\mathfrak{m}_V(\mu) = n$ and the set (4.13) forms a basis of V_μ .

Finally, consider an irreducible KG -module $V = L(\lambda)$ having highest weight $\lambda = \lambda_i$, where $1 < i < n$. Also set $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_n)$. Proceeding as in the proof of Proposition 4.2.5, one easily deduces that the weight space V_μ is spanned by

$$\begin{aligned} & \{F_{1,i}v^+\} \cup \{f_{1,i-1}f_{\alpha_i}F_{i,i+1}v^+\} \\ & \cup \{f_{i,j}F_{1,j+1}v^+\}_{i \leq j < n}. \end{aligned} \quad (4.14)$$

Hence applying Lemma 2.3.7 to the B_{n-i+2} -Levi subgroup of G corresponding to the simple roots $\alpha_{i-1}, \dots, \alpha_n$ together with Proposition 4.2.6 yields the following result. The details are left to the reader.

Proposition 4.2.7

Consider an irreducible KG -module $V = L(\lambda)$ having highest weight $\lambda = \lambda_i$, where $1 < i < n$. Then the T -weight $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_n)$ is dominant, $m_V(\mu) = n - i + 2$ and the set (4.14) forms a basis of V_μ .

4.2.3 Study of $L(a\lambda_1 + \lambda_2)$ ($a \in \mathbb{Z}_{>0}$)

Assume $p \neq 2$ and consider the p -restricted dominant weight $\lambda = a\lambda_1 + \lambda_2$, where $a \in \mathbb{Z}_{>0}$. Also write $\mu_{1,2} = \lambda - \alpha_1 - \alpha_2$ and $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$. By Proposition 2.5.8, our choice of ordering \preceq on Φ^+ and Proposition 4.2.1, one sees that the weight space $V(\lambda)_\mu$ is spanned by

$$\begin{aligned} & \{F_{1,2}v^\lambda\} \cup \{f_{\alpha_1}f_{\alpha_2}F_{2,3}v^\lambda\} \\ & \cup \{f_{1,j}F_{2,j+1}v^\lambda\}_{1 < j < n} \\ & \cup \{f_{2,j}F_{1,j+1}v^\lambda\}_{1 < j < n} \\ & \cup \{f_{2,n}f_{1,n}v^\lambda\}, \end{aligned} \quad (4.15)$$

where $v^\lambda \in V(\lambda)_\lambda$ denotes a maximal vector in $V(\lambda)$ for B . As usual, an application of Theorem 2.3.11 gives $\dim V(\lambda)_\mu = 2n - 1$, so that the generating elements of (4.15) are linearly independent. The following assertion then holds.

Proposition 4.2.8

Let $\lambda = a\lambda_1 + \lambda_2$, where $a \in \mathbb{Z}_{>0}$, and set $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n) \in \Lambda^+(\lambda)$. Then the set (4.15) forms a basis of the weight space $V(\lambda)_\mu$.

Suppose for the remainder of this section that $p \mid a + 2$, so that $\mu_{1,2}$ affords the highest weight of a composition factor of $V(\lambda)$ for B by Proposition 4.1.3. Also denote by u^+ the corresponding maximal vector in $V(\lambda)_{\mu_{1,2}}$ for B given by (4.7), and set $\overline{V(\lambda)} = V(\lambda)/\langle Gu^+ \rangle_K$.

Lemma 4.2.9

Let λ , μ , u^+ , and $\overline{V(\lambda)}$ be as above, with $p \mid a + 2$. Then $[\langle Gu^+ \rangle_K, L(\mu)] = 0$. In particular $[V(\lambda), L(\mu)] = [\overline{V(\lambda)}, L(\mu)]$.

Proof. The result follows from the fact that $\langle Gu^+ \rangle_K$ is an image of $V(\mu_{1,2})$, in which $L(\mu)$ cannot occur as a composition factor by Proposition 4.2.6. \square

In view of Lemma 4.2.9, we are led to investigate the structure of the quotient $\overline{V(\lambda)}$. Write \bar{v}^λ for the image of v^λ in $\overline{V(\lambda)}$. By Lemma 2.5.4 and (4.7), we successively get

$$f_{1,r}\bar{v}^\lambda = f_{3,r}f_{1,2}\bar{v}^\lambda = f_{3,r}f_{\alpha_1}f_{\alpha_2}\bar{v}^\lambda = f_{\alpha_1}f_{3,r}f_{\alpha_2}\bar{v}^\lambda = f_{\alpha_1}f_{2,r}\bar{v}^\lambda, \quad (4.16)$$

for every $2 < r \leq n$. Also, since $\langle Gu^+ \rangle_K$ is an image of $V(\mu_{1,2})$ and $m_{L(\mu_{1,2})}(\mu) = n - 1$ by Proposition 4.2.6, we have $\dim \overline{V(\lambda)}_\mu = n$. Those observations can be used to determine a basis of the weight space $\overline{V(\lambda)}_\mu$, as the following result shows.

Proposition 4.2.10

Let $a \in \mathbb{Z}_{>0}$ be such that $p \mid a + 2$ and let $\lambda = a\lambda_1 + \lambda_2$. Also write $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n)$ and let u^+ be the maximal vector in $V(\lambda)_{\mu_{1,2}}$ for B given by (4.7). Finally, write \bar{v}^λ for the image of v^λ in $\overline{V(\lambda)} = V(\lambda)/\langle Gu^+ \rangle_K$. Then a basis of the weight space $\overline{V(\lambda)}_\mu$ is given by

$$\{F_{1,2}\bar{v}^\lambda\} \cup \{f_{2,j}F_{1,j+1}\bar{v}^\lambda\}_{1 < j < n} \cup \{f_{2,n}f_{1,n}\bar{v}^\lambda\}. \quad (4.17)$$

Proof. We start by showing that each of $f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda$, $f_{1,j}F_{2,j+1}\bar{v}^\lambda$ ($1 < j < n$) can be written as a linear combination of elements of (4.17). Fix $1 < j < n$. By Lemmas 2.5.4, 4.2.1 (part 1), and (4.16), we successively get

$$\begin{aligned} f_{1,j}F_{2,j+1}\bar{v}^\lambda &= F_{1,2}\bar{v}^\lambda + F_{2,j+1}f_{\alpha_1}f_{2,j}\bar{v}^\lambda \\ &= F_{1,2}\bar{v}^\lambda + F_{1,j+1}f_{2,j}\bar{v}^\lambda + f_{\alpha_1}f_{2,j}F_{2,j+1}\bar{v}^\lambda \\ &= 2F_{1,2}\bar{v}^\lambda + f_{2,j}F_{1,j+1}\bar{v}^\lambda + f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda, \end{aligned}$$

so that $f_{1,j}F_{2,j+1}\bar{v}^\lambda \in \langle F_{1,2}\bar{v}^\lambda, f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda, f_{2,j}F_{1,j+1}\bar{v}^\lambda \rangle_K$. It then remains to show that $f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda$ is a linear combination of elements of (4.17). By Lemmas 2.5.4, 4.2.1 (part 2), and (4.16), we have

$$\begin{aligned} f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda &= \frac{1}{2}f_{\alpha_1}(f_{2,n})^2\bar{v}^\lambda \\ &= \frac{1}{2}(f_{2,n}f_{\alpha_1}f_{2,n}\bar{v}^\lambda - f_{1,n}f_{2,n}\bar{v}^\lambda) \\ &= \frac{1}{2}(f_{2,n}f_{1,n}\bar{v}^\lambda - f_{1,n}f_{2,n}\bar{v}^\lambda) \\ &= -F_{1,2}\bar{v}^\lambda, \end{aligned}$$

hence $f_{\alpha_1}f_{\alpha_2}F_{2,3}\bar{v}^\lambda \in \langle F_{1,2}\bar{v}^\lambda \rangle_K$, showing that $\overline{V(\lambda)}_\mu$ is spanned by the set of vectors in (4.17). Therefore the assertion on $\dim \overline{V(\lambda)}_\mu$ given above allows us to conclude. \square

We now study the relation between the pair (n, p) and the existence of a maximal vector in $\overline{V(\lambda)}_\mu$ for \mathfrak{b} . For $A = (A_r)_{1 \leq r \leq n} \in K^n$, we set

$$\bar{w}(A) = A_1 F_{1,2} \bar{v}^\lambda + \sum_{j=2}^{n-1} A_j f_{2,j} F_{1,j+1} \bar{v}^\lambda + A_n f_{2,n} f_{1,n} \bar{v}^\lambda. \quad (4.18)$$

Lemma 4.2.11

Let λ, μ be as above, with $p \mid a + 2$, and adopt the notation of (4.18). Then the following assertions are equivalent.

1. There exists $0 \neq A \in K^n$ such that $e_\alpha \bar{w}(A) = 0$ for every $\alpha \in \Pi$.
2. There exist $A \in K^{n-1} \times K^*$ such that $e_\alpha \bar{w}(A) = 0$ for every $\alpha \in \Pi$.
3. The divisibility condition $p \mid 2n - 3$ is satisfied.

Proof. Let $A = (A_r)_{1 \leq r \leq n} \in K^n$ and set $\bar{w} = \bar{w}(A)$. Starting by using Lemmas 2.5.4 and 4.2.1 (parts 1 and 2), we get

$$\begin{aligned} e_{\alpha_1} \bar{w} &= - \sum_{i=2}^{n-1} A_i f_{2,i} F_{2,i+1} \bar{v}^\lambda - A_n (f_{2,n})^2 \bar{v}^\lambda \\ &= - \left(\sum_{i=2}^{n-1} A_i + 2A_n \right) f_{\alpha_2} F_{2,3} \bar{v}^\lambda, \end{aligned}$$

while Lemma 2.5.4 yields

$$\begin{aligned} e_{\alpha_2} \bar{w} &= -A_1 F_{1,3} \bar{v}^\lambda + A_2 h_{\alpha_2} F_{1,3} \bar{v}^\lambda - \sum_{i=3}^{n-1} A_i f_{3,i} F_{1,i+1} \bar{v}^\lambda - A_n f_{3,n} f_{1,n} \bar{v}^\lambda \\ &= \left(-A_1 + 2A_2 + \sum_{r=3}^{n-1} A_r + 2A_n \right) F_{1,3} \bar{v}^\lambda. \end{aligned}$$

Similarly, one easily checks that $e_{\alpha_r} \bar{w} = (A_r - A_{r-1}) f_{2,r-1} F_{1,r+1} \bar{v}^\lambda$, for every $2 < r < n$, and finally, we have

$$\begin{aligned} e_{\alpha_n} \bar{w} &= 2A_n (f_{2,n-1} f_{1,n} \bar{v}^\lambda + f_{2,n} f_{1,n-1} \bar{v}^\lambda) - A_{n-1} f_{2,n-1} f_{1,n} \bar{v}^\lambda \\ &= (4A_n - A_{n-1}) f_{2,n-1} f_{1,n} \bar{v}^\lambda, \end{aligned}$$

where the last equality comes from the fact that $V(\lambda)_{\lambda - (2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n)} = 0$ and (4.16) applied to $f_{2,n} f_{1,n-1} \bar{v}^\lambda$.

As usual, one checks that each of the vectors $f_{\alpha_2}F_{2,3}\bar{v}^\lambda, F_{1,3}\bar{v}^\lambda, f_{\alpha_2}F_{1,4}\bar{v}^\lambda, \dots, f_{2,n-2}F_{1,n}\bar{v}^\lambda$, and $f_{2,n-1}f_{1,n}\bar{v}^\lambda$ is non-zero. Consequently, $e_\alpha\bar{w}(A) = 0$ for every $\alpha \in \Pi$ if and only if $A \in K^n$ is a solution to the system of equations

$$\begin{cases} 2A_n &= -\sum_{j=2}^{n-1} A_j \\ A_1 &= 2(A_2 + A_n) + \sum_{j=3}^{n-1} A_j \\ A_{r-1} &= A_r \text{ for every } 2 < r \leq n-1 \\ A_{n-1} &= 4A_n. \end{cases} \quad (4.19)$$

Now one easily sees that (4.19) admits a non-trivial solution A if and only if $p \mid 2n-3$ (showing that 1 and 3 are equivalent), in which case $A \in \langle (4, \dots, 4, 1) \rangle_K$ (so that 1 and 2 are equivalent), completing the proof. \square

Let λ and μ be as above, with $p \mid a+2$, and consider an irreducible \mathcal{L} -module $V = L(\lambda)$ having highest weight λ . As usual, take $V = V(\lambda)/\text{rad}(\lambda)$, so that

$$V \cong \overline{V(\lambda)} / \overline{\text{rad}(\lambda)},$$

where $\overline{\text{rad}(\lambda)} = \text{rad}(\lambda)/\langle Gu^+ \rangle_K$. Also write v^+ to denote the image of \bar{v}^λ in V , that is, v^+ is a maximal vector in V for B . By Proposition 4.2.10, the weight space V_μ is spanned by

$$\begin{aligned} &\{F_{1,2}v^+\} \cup \{f_{2,j}F_{1,j+1}v^+\}_{1 < j < n} \\ &\cup \{f_{2,n}f_{1,n}v^+\}. \end{aligned} \quad (4.20)$$

We write $V_{1,2,n}$ to denote the span of all the generators in (4.20) except for $f_{2,n}f_{1,n}v^+$. As usual, the following result consists of a precise description of the weight space V_μ , as well as a characterization for $[V(\lambda), L(\mu)]$ to be non-zero.

Proposition 4.2.12

Let G be a simple algebraic group of type B_n over K and consider an irreducible KG -module $V = L(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + \lambda_2$, where $a \in \mathbb{Z}_{>0}$ is such that $p \mid a+2$. Also write $\mu = \lambda - (\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n) \in \Lambda^+(\lambda)$. Then the following assertions are equivalent.

1. The weight μ affords the highest weight of a composition factor of $V(\lambda)$.
2. The generators in (4.20) are linearly dependent.
3. The element $f_{2,n}f_{1,n}v^+$ lies inside $V_{1,2,n}$.
4. The divisibility condition $p \mid 2n-3$ is satisfied.

Proof. Clearly 3 implies 2, while if 1 holds, then Lemma 4.2.9 yields $[\overline{V(\lambda)}, L(\mu)] \neq 0$, so that 2 holds. Now if 2 is satisfied, then $L(\nu)$ occurs as a composition factor of $V(\lambda)$ for some $\nu \in \Lambda^+(\lambda)$ such that $\mu \preceq \nu \prec \lambda$ by Proposition 4.2.10. Since only μ can afford the highest weight of such a composition factor, 1 holds by Lemma 4.2.9 again. This also shows the existence of $0 \neq A \in K^n$ such that $\bar{w}(A)$ is a maximal vector in $\overline{V(\lambda)}$ for \mathfrak{b} , where we adopt the notation of (4.18). Therefore 2 implies 4 by Lemma 4.2.11. Finally suppose that 4 holds. By Lemma 4.2.11, there exists $A \in K^{n-1} \times K^*$ such that $e_\alpha \bar{w}(A) = 0$ for every $\alpha \in \Pi$. Consequently, we also get $e_\alpha(\bar{w}(A) + \text{rad}(\lambda)) = 0$ for every $\alpha \in \Pi$, that is, $\bar{w}(A) + \text{rad}(\lambda) \in \langle v^+ \rangle_K \cap V(\lambda)_\mu = 0$ by Theorem 2.5.9. Therefore 3 holds and the proof is complete. \square

4.2.4 Study of $L(a\lambda_1 + \lambda_k)$ ($2 < k < n$ and $p \neq 2$)

Let $\lambda = a\lambda_1 + \lambda_k$, where $a \in \mathbb{Z}_{>0}$, and $2 < k < n$. Also write $\mu_{1,k} = \lambda - (\alpha_1 + \cdots + \alpha_k)$ and $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_n)$. By Proposition 2.5.8, our choice of ordering \preceq on Φ^+ , and Proposition 4.2.5, one checks that the weight space $V(\lambda)_\mu$ is spanned by

$$\begin{aligned}
& \{F_{1,k}v^\lambda\} \cup \{f_{1,k-1}f_{\alpha_k}F_{k,k+1}v^\lambda\} \\
& \quad \cup \{f_{1,i}F_{i+1,k}v^\lambda\}_{1 \leq i \leq k-2} \\
& \quad \cup \{f_{1,j}F_{k,j+1}v^\lambda\}_{k \leq j < n} \\
& \quad \cup \{f_{1,i}f_{i+1,k-1}f_{\alpha_k}F_{k,k+1}v^\lambda\}_{1 \leq i \leq k-2} \\
& \quad \cup \{f_{1,i}f_{k,j}F_{i+1,j+1}v^\lambda\}_{1 \leq i \leq k-2, k \leq j < n} \\
& \quad \cup \{f_{k,j}F_{1,j+1}v^\lambda\}_{k \leq j < n} \\
& \quad \cup \{f_{k,n}f_{1,n}v^\lambda\}, \tag{4.21}
\end{aligned}$$

where $v^\lambda \in V(\lambda)_\lambda$ is a maximal vector in $V(\lambda)$ for B . As usual, an application of Theorem 2.3.11 yields $m_{V(\lambda)}(\mu) = k(n - k + 2) - 1$, forcing the generating elements of (4.21) to be linearly independent. The following result thus holds.

Proposition 4.2.13

Fix $a \in \mathbb{Z}_{>0}$ and $2 < k < n$. Also let $\lambda = a\lambda_1 + \lambda_k \in X^+(T)$ and consider the dominant T -weight $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_n) \in \Lambda^+(\lambda)$. Then the set (4.21) forms a basis of $V(\lambda)_\mu$.

Suppose for the remainder of this section that $p \mid a + k$, so that $\mu_{1,k}$ affords the highest weight of a composition factor of $V(\lambda)$ for B by Proposition 4.1.3. Also denote by u^+ the corresponding maximal vector in $V(\lambda)_{\mu_{1,k}}$ for \mathfrak{b} given by (4.7), and set $\overline{V(\lambda)} = V(\lambda)/\langle Gu^+ \rangle_K$.

Lemma 4.2.14

Let λ , μ , u^+ , and $\overline{V(\lambda)}$ be as above, with $p \mid a + k$. Then $[\langle Gu^+ \rangle_K, L(\mu)] = 0$. In particular $[V(\lambda), L(\mu)] = [\overline{V(\lambda)}, L(\mu)]$.

Proof. The result follows from the fact that $\langle Gu^+ \rangle_K$ is an image of $V(\mu_{1,k})$, in which $L(\mu)$ cannot occur as a composition factor by Proposition 4.2.6. \square

In view of Lemma 4.2.14, it is only natural to investigate the structure of the quotient $\overline{V(\lambda)}$. Write \bar{v}^λ for the class of v^λ in $\overline{V(\lambda)}$. By Lemma 2.5.4 and (4.7), we successively get

$$f_{1,r}\bar{v}^\lambda = f_{k+1,r}f_{1,k}\bar{v}^\lambda = \sum_{s=1}^{k-1} f_{k+1,r}f_{1,s}f_{s+1,k}\bar{v}^\lambda = \sum_{s=1}^{k-1} f_{1,s}f_{s+1,r}\bar{v}^\lambda, \quad (4.22)$$

for every $k < r \leq n$. Also, since $\langle Gu^+ \rangle_K$ is an image of $V(\mu_{1,k})$ and $m_{L(\mu_{1,k})}(\mu) = n - k + 1$ by Proposition 4.2.7, we have $\dim \overline{V(\lambda)}_\mu = (k - 1)(n - k + 2)$. Those observations can be used to determine a basis of the weight space $\overline{V(\lambda)}_\mu$, as the following result shows.

Proposition 4.2.15

Let $a \in \mathbb{Z}_{>0}$ and $2 < k < n$ be such that $p \mid a + k$ and consider the dominant T -weight $\lambda = a\lambda_1 + \lambda_k$. Also set $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_n) \in \Lambda^+(\lambda)$ and let u^+ be the maximal vector in $V(\lambda)_{\mu_{1,k}}$ for B given by (4.7). Finally, set $\overline{V(\lambda)} = V(\lambda)/\langle Gu^+ \rangle_K$, and write \bar{v}^λ for the class of v^λ in $\overline{V(\lambda)}$. Then a basis of the weight space $\overline{V(\lambda)}_\mu$ is given by

$$\begin{aligned} & \{F_{1,k}\bar{v}^\lambda\} \cup \{f_{1,i}F_{i+1,k}\bar{v}^\lambda\}_{1 \leq i \leq k-2} \\ & \cup \{f_{1,i}f_{i+1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda\}_{1 \leq i \leq k-2} \\ & \cup \{f_{1,i}f_{k,j}F_{i+1,j+1}\bar{v}^\lambda\}_{1 \leq i \leq k-2, k \leq j < n} \\ & \cup \{f_{k,j}F_{1,j+1}\bar{v}^\lambda\}_{k \leq j < n} \\ & \cup \{f_{k,n}f_{1,n}\bar{v}^\lambda\}. \end{aligned} \quad (4.23)$$

Proof. We first show that $f_{1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda$ lies inside the subspace of $\overline{V(\lambda)}$ generated by the elements $F_{1,k}\bar{v}^\lambda$, $f_{1,i}F_{i+1,k}\bar{v}^\lambda$, $f_{1,i}f_{i+1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda$, and $f_{1,i}f_{k,j}F_{i+1,j+1}\bar{v}^\lambda$, where $1 \leq i \leq k - 2$ and $k \leq j < n$. By Lemma 2.5.4 and Proposition 4.2.1 (part 2), we have

$$\begin{aligned} f_{1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda &= \frac{1}{2}f_{1,k-1}(f_{k,n})^2\bar{v}^\lambda \\ &= \frac{1}{2}(f_{k,n}f_{1,k-1}f_{k,n}\bar{v}^\lambda - f_{1,n}f_{k,n}\bar{v}^\lambda) \\ &= \frac{1}{2}(f_{k,n}f_{1,k-1}f_{k,n}\bar{v}^\lambda - 2F_{1,k}\bar{v}^\lambda - f_{k,n}f_{1,n}\bar{v}^\lambda), \end{aligned}$$

and by (4.22), we get $f_{k,n}f_{1,k-1}f_{k,n}\bar{v}^\lambda = f_{k,n}f_{1,n}\bar{v}^\lambda - \sum_{r=1}^{k-2} f_{1,r}f_{k,n}f_{r+1,n}\bar{v}^\lambda$. An application of Proposition 4.2.7 then yields the desired result in this case.

Finally, let $k \leq j < n$, and first observe that by Lemma 2.5.4 and (4.22), we have

$$\begin{aligned} f_{1,j}F_{k,j+1}\bar{v}^\lambda &= F_{1,k}\bar{v}^\lambda + F_{k,j+1}f_{1,j}\bar{v}^\lambda \\ &= F_{1,k}\bar{v}^\lambda + \sum_{r=1}^{k-2} f_{1,r}F_{k,j+1}f_{r+1,j}\bar{v}^\lambda + F_{k,j+1}f_{1,k-1}f_{k,j}\bar{v}^\lambda. \end{aligned}$$

Applying Proposition 4.2.7 shows that $f_{1,r}F_{k,j+1}f_{r+1,j}\bar{v}^\lambda$ lies inside the subspace of $\overline{V(\lambda)}$ generated by the elements of (4.23) as desired, while an application of Proposition 4.2.1 (part 1) yields

$$\begin{aligned} F_{k,j+1}f_{1,k-1}f_{k,j}\bar{v}^\lambda &= F_{1,j+1}f_{k,j}\bar{v}^\lambda + f_{1,k-1}F_{k,j+1}f_{k,j}\bar{v}^\lambda \\ &= F_{1,k}\bar{v}^\lambda + f_{k,j}F_{1,j+1}\bar{v}^\lambda + f_{1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda. \end{aligned}$$

Therefore $\overline{V(\lambda)}_\mu$ is spanned by the elements of (4.23) and the assertion on $\dim \overline{V(\lambda)}_\mu$ given above allows us to conclude. \square

In order to investigate the existence of a maximal vector in $\overline{V(\lambda)}_\mu$ for \mathfrak{b} as in Lemma 4.2.11, we require the following technical result.

Lemma 4.2.16

Let λ , and μ be as above, with $p \mid a + k$. Then the following assertions hold.

1. $f_{k,n}f_{2,n}\bar{v}^\lambda = -F_{2,k}\bar{v}^\lambda - f_{2,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda$.
2. $f_{k,n-1}f_{1,n}\bar{v}^\lambda = -\sum_{r=1}^{k-2} f_{1,r}f_{r+1,n-1}f_{k,n}\bar{v}^\lambda + f_{1,n-1}f_{k,n}\bar{v}^\lambda$.
3. $f_{k,n-1}f_{r+1,n}\bar{v}^\lambda = -f_{r+1,n-1}f_{k,n}\bar{v}^\lambda$, for every $1 \leq r < k - 1$.

Proof. By Lemma 2.5.4 and Proposition 4.2.1 (part 2), we get

$$\begin{aligned} f_{k,n}f_{2,n}\bar{v}^\lambda &= -f_{2,n}f_{k,n}\bar{v}^\lambda - f_{2,k-1}(f_{k,n})^2\bar{v}^\lambda \\ &= -2F_{2,k}\bar{v}^\lambda - f_{k,n}f_{2,n}\bar{v}^\lambda - f_{2,k-1}(f_{k,n})^2\bar{v}^\lambda \\ &= -2F_{2,k}\bar{v}^\lambda - f_{k,n}f_{2,n}\bar{v}^\lambda - 2f_{2,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda, \end{aligned}$$

from which 1 immediately follows. Finally, by Lemma 2.5.4 and (4.22), we have

$$\begin{aligned} f_{k,n-1}f_{1,n}\bar{v}^\lambda &= \sum_{r=1}^{k-2} f_{1,r}f_{k,n-1}f_{r+1,n}\bar{v}^\lambda + f_{k,n-1}f_{1,k-1}f_{k,n}\bar{v}^\lambda \\ &= \sum_{r=1}^{k-2} f_{1,r}f_{k,n-1}f_{r+1,n}\bar{v}^\lambda + f_{1,n-1}f_{k,n}\bar{v}^\lambda. \end{aligned}$$

Noticing that $f_{k,n-1}f_{r+1,n}\bar{v}^\lambda = -f_{k,n-1}f_{r+1,k-1}f_{k,n}\bar{v}^\lambda = -f_{r+1,n-1}f_{k,n}\bar{v}^\lambda$ for $1 \leq r \leq k - 1$ then yields 2 and 3, completing the proof. \square

We now study the relation between the triple (n, k, p) and the existence of a maximal vector in $\overline{V(\lambda)}_\mu$ for \mathfrak{b} , assuming $p \mid a + k$. For $X = (A, B_i, C_i, D_{ij}, E_j, F) \in K^{(k-1)(n-k+2)}$ ($1 \leq i \leq k-2$ and $k \leq j \leq n-1$), we set

$$\begin{aligned} \bar{w}(X) = & AF_{1,k}\bar{v}^\lambda + \sum_{i=1}^{k-2} B_i f_{1,i} F_{i+1,k} \bar{v}^\lambda + \sum_{i=1}^{k-2} C_i f_{1,i} f_{i+1,k-1} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda \\ & + \sum_{i=1}^{k-2} \sum_{j=k}^{n-1} D_{ij} f_{1,i} f_{k,j} F_{i+1,j+1} \bar{v}^\lambda + \sum_{j=k}^{n-1} E_j f_{k,j} F_{1,j+1} \bar{v}^\lambda \\ & + F f_{k,n} f_{1,n} \bar{v}^\lambda. \end{aligned} \quad (4.24)$$

Lemma 4.2.17

Let λ, μ be as above, with $p \mid a + k$, and adopt the notation of (4.24). Then the following assertions are equivalent.

1. There exists $0 \neq X = (A, B_i, C_i, D_{ij}, E_j, F) \in K^{(k-1)(n-k+2)}$ ($1 \leq i \leq k-2$ and $k \leq j \leq n-1$) such that $e_\alpha \bar{w}(X) = 0$ for every $\alpha \in \Pi$.
2. There exists $X = (A, B_i, C_i, D_{ij}, E_j, F) \in K^{(k-1)(n-k+2)-1} \times K^*$ ($1 \leq i \leq k-2$ and $k \leq j \leq n-1$) such that $e_\alpha \bar{w}(X) = 0$ for every $\alpha \in \Pi$.
3. The divisibility condition $p \mid 2(n-k) + 1$ is satisfied.

Proof. We start by assuming $k = 3$, respectively write B, C and D_j for B_1, C_1 and $D_{1,j}$ ($3 \leq j \leq n-1$), and let $X = (A, B, C, D_j, E_j, F) \in K^{2(n-1)}$. With these simplifications, $\bar{w} = \bar{w}(X)$ can be rewritten as

$$\begin{aligned} \bar{w} = & AF_{1,3}\bar{v}^\lambda + B f_{\alpha_1} F_{2,3} \bar{v}^\lambda + C f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} F_{3,4} \bar{v}^\lambda \\ & + \sum_{j=3}^{n-1} D_j f_{\alpha_1} f_{3,j} F_{2,j+1} \bar{v}^\lambda + \sum_{j=3}^{n-1} E_j f_{3,j} F_{1,j+1} \bar{v}^\lambda \\ & + F f_{3,n} f_{1,n} \bar{v}^\lambda. \end{aligned}$$

Lemma 2.5.4 then yields

$$\begin{aligned} e_{\alpha_1} \bar{w} = & (-A + (a+1)B) F_{2,3} \bar{v}^\lambda + (a+1) C f_{\alpha_2} f_{\alpha_3} F_{3,4} \bar{v}^\lambda \\ & + \sum_{j=3}^{n-1} ((a+1)D_j - E_j) f_{3,j} F_{2,j+1} \bar{v}^\lambda - F f_{3,n} f_{2,n} \bar{v}^\lambda \\ = & (-A + (a+1)B + F) F_{2,3} \bar{v}^\lambda + ((a+1)C + F) f_{\alpha_2} f_{\alpha_3} F_{3,4} \bar{v}^\lambda \\ & + \sum_{j=3}^{n-1} ((a+1)D_j - E_j) f_{3,j} F_{2,j+1} \bar{v}^\lambda, \end{aligned}$$

where the last equality follows from Lemma 4.2.16 (part 1). Similarly, Lemma 2.5.4 yields

$$\begin{aligned} e_{\alpha_2} \bar{w} &= (2C - D_3) f_{\alpha_1} f_{\alpha_3} F_{3,4} \bar{v}^\lambda - \sum_{j=4}^{n-1} D_j f_{\alpha_1} f_{3,j} F_{3,j+1} \bar{v}^\lambda \\ &= \left(2C - \sum_{j=3}^{n-1} D_j \right) f_{\alpha_1} f_{\alpha_3} F_{3,4} \bar{v}^\lambda, \end{aligned}$$

where the last equality follows from Proposition 4.2.1 (part 1). Again, applying Lemma 2.5.4 gives

$$\begin{aligned} e_{\alpha_3} \bar{w} &= (2E_3 - A) F_{1,4} \bar{v}^\lambda - \sum_{j=4}^{n-1} D_j f_{\alpha_1} f_{4,j} F_{2,j+1} \bar{v}^\lambda \\ &\quad + (2D_3 - B - C) f_{\alpha_1} F_{2,4} \bar{v}^\lambda - \sum_{j=4}^{n-1} E_j f_{4,j} F_{1,j+1} \bar{v}^\lambda \\ &\quad - F f_{4,n} f_{1,n} \bar{v}^\lambda \\ &= \left(2E_3 + \sum_{j=4}^{n-1} E_j - A + 2F \right) F_{1,4} \bar{v}^\lambda \\ &\quad + \left(2D_3 + \sum_{j=4}^{n-1} D_j - B - C \right) f_{\alpha_1} F_{2,4} \bar{v}^\lambda, \end{aligned}$$

while for every $4 \leq r \leq n-1$ one checks that

$$e_{\alpha_r} \bar{w} = (D_r - D_{r-1}) f_{\alpha_1} f_{3,r-1} F_{2,r+1} \bar{v}^\lambda + (E_r - E_{r-1}) f_{3,r-1} F_{1,r+1} \bar{v}^\lambda.$$

Finally, we leave to the reader to check (using Lemma 4.2.16 (part 2) and (4.22)) that

$$\begin{aligned} f_{\alpha_1} f_{3,n-1} f_{2,n} \bar{v}^\lambda &= f_{3,n-1} f_{1,n} \bar{v}^\lambda - f_{3,n-1} f_{1,2} f_{3,n} \bar{v}^\lambda \\ &= f_{3,n-1} f_{1,n} \bar{v}^\lambda - f_{1,n-1} f_{3,n} \bar{v}^\lambda \\ &= -f_{\alpha_1} f_{2,n-1} f_{3,n} \bar{v}^\lambda \end{aligned}$$

and hence

$$\begin{aligned} e_{\alpha_n} \bar{w} &= -D_{n-1} f_{\alpha_1} f_{3,n-1} f_{2,n} \bar{v}^\lambda + (2F - E_{n-1}) f_{3,n-1} f_{1,n} \bar{v}^\lambda + 2F f_{1,n-1} f_{3,n} \bar{v}^\lambda \\ &= (2F - D_{n-1} - E_{n-1}) f_{\alpha_1} f_{2,n-1} f_{3,n} \bar{v}^\lambda + (4F - E_{n-1}) f_{1,n-1} f_{3,n} \bar{v}^\lambda. \end{aligned}$$

As usual, one then checks that the vector $f_{\alpha_1} f_{\alpha_3} F_{3,4} \bar{v}^\lambda$ is non-zero. Also by Lemma 4.2.7, the list $\{F_{2,3} \bar{v}^\lambda, f_{\alpha_2} f_{\alpha_3} F_{3,4} \bar{v}^\lambda, f_{3,j} F_{2,j+1} \bar{v}^\lambda : 3 \leq j < n\}$ is linearly independent. Similarly, one sees that each of the lists $\{F_{1,4} \bar{v}^\lambda, f_{\alpha_1} F_{2,4} \bar{v}^\lambda\}$, $\{f_{\alpha_1} f_{3,j} F_{2,j+2} \bar{v}^\lambda, f_{3,j} F_{1,j+2} \bar{v}^\lambda\}$ (for every $3 \leq j \leq n-2$), $\{f_{\alpha_1} f_{2,n-1} f_{3,n} \bar{v}^\lambda, f_{1,n-1} f_{3,n} \bar{v}^\lambda\}$ is linearly independent as well. Consequently, $e_\alpha \bar{w}(X) = 0$ for every $\alpha \in \Pi$ if and only if X is a solution to the system of equations

$$\left\{ \begin{array}{l} A = (a+1)B + F \\ F = -(a+1)C \\ E_j = (a+1)D_j \text{ for every } 3 \leq j < n \\ 2C = \sum_{j=3}^{n-1} D_j \\ 2E_3 = A - \sum_{j=4}^{n-1} E_j - 2F \\ 2D_3 = B + C - \sum_{j=4}^{n-1} D_j \\ D_{r-1} = D_r \text{ for every } 3 < r < n \\ E_{r-1} = E_r \text{ for every } 3 < r < n \\ D_{n-1} = 2F - E_{n-1} \\ E_{n-1} = 4F. \end{array} \right. \quad (4.25)$$

Now one easily sees that (4.25) admits a non-trivial solution X if and only if $p \mid 2n-5$ (showing that 1 and 3 are equivalent), in which case

$$X \in \langle (4, 1-n, 3-n, \underbrace{-2, \dots, -2}_{n-3}, \underbrace{4, \dots, 4}_{n-3}, 1) \rangle_K$$

(so that 1 and 2 are equivalent). The result follows in this situation and assume $3 < k < n$ for the remainder of the proof. Let then $X = (A, B_i, C_i, D_{ij}, E_j, F) \in K^{(k-1)(n-k+2)}$, where $1 \leq i \leq k-2$ and $k \leq j \leq n-1$. By Lemma 2.5.4 and Lemma 4.2.16, we have

$$\begin{aligned} e_{\alpha_1} \bar{w} &= \left((a+1)B_1 + \sum_{i=2}^{k-2} B_i - A \right) F_{2,k} \bar{v}^\lambda \\ &+ \left((a+1)C_1 + \sum_{i=2}^{k-2} C_i \right) f_{2,k-1} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda \\ &+ \sum_{j=k}^{n-1} \left((a+1)D_{1j} + \sum_{i=2}^{k-2} D_{ij} - E_j \right) f_{k,j} F_{2,j+1} \bar{v}^\lambda \\ &- F f_{k,n} f_{2,n} \bar{v}^\lambda \\ &= \left((a+1)B_1 + \sum_{i=2}^{k-2} B_i - A + F \right) F_{2,k} \bar{v}^\lambda \\ &+ \left((a+1)C_1 + \sum_{i=2}^{k-2} C_i + F \right) f_{2,k-1} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda \\ &+ \sum_{j=k}^{n-1} \left((a+1)D_{1j} + \sum_{i=2}^{k-2} D_{ij} - E_j \right) f_{k,j} F_{2,j+1} \bar{v}^\lambda, \end{aligned}$$

where the last equality can be deduced from Lemma 4.2.16 (part 1) Also, for $1 < r < k - 1$, we get

$$e_{\alpha_r} \bar{w} = (B_r - B_{r-1}) f_{1,r-1} F_{r+1,k} \bar{v}^\lambda + (C_r - C_{r-1}) f_{1,r-1} f_{r+1,k-1} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda \\ + \sum_{j=k}^{n-1} (D_{rj} - D_{r-1,j}) f_{1,r-1} f_{k,j} F_{r+1,j+1} \bar{v}^\lambda,$$

while Proposition 4.2.1 (part 1) yields

$$e_{\alpha_{k-1}} \bar{w} = (2C_{k-2} - D_{k-2,k}) f_{1,k-2} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda - \sum_{j=k+1}^{n-1} D_{k-2,j} f_{1,k-2} f_{k,j} F_{k,j+1} \bar{v}^\lambda \\ = \left(2C_{k-2} - \sum_{j=k}^{n-1} D_{k-2,j} \right) f_{1,k-2} f_{\alpha_k} F_{k,k+1} \bar{v}^\lambda.$$

Also

$$e_{\alpha_k} \bar{w} = \left(-A + 2E_k + \sum_{j=k+1}^{n-1} E_j + 2F \right) F_{1,k+1} \bar{v}^\lambda \\ - \sum_{i=1}^{k-2} \left(B_i + C_i - 2D_{i,k} - \sum_{j=k+1}^{n-1} D_{i,j} \right) f_{1,i} F_{i+1,k+1} \bar{v}^\lambda,$$

while for $k < s < n$, we have

$$e_{\alpha_s} \bar{w} = \sum_{i=1}^{k-2} (D_{is} - D_{i,s-1}) f_{1,i} f_{k,s-1} F_{i+1,s+1} \bar{v}^\lambda + (E_s - E_{s-1}) f_{k,s-1} F_{1,s+1} \bar{v}^\lambda.$$

Finally, thanks to Lemma 4.2.16 (part 3), we see that

$$e_{\alpha_n} \bar{w} = \sum_{i=1}^{k-2} D_{i,n-1} f_{1,i} f_{i+1,n-1} f_{k,n} \bar{v}^\lambda + (2F - E_{n-1}) f_{k,n-1} f_{1,n} \bar{v}^\lambda \\ + 2F f_{1,n-1} f_{k,n} \bar{v}^\lambda \\ = \sum_{i=1}^{k-2} (D_{i,n-1} + E_{n-1} - 2F) f_{1,i} f_{i+1,n-1} f_{k,n} \bar{v}^\lambda \\ + (4F - E_{n-1}) f_{1,n-1} f_{k,n} \bar{v}^\lambda,$$

where the last equality follows from Lemma 4.2.16 (part 2). One checks that $f_{1,k-2}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda$ is non-zero and that each of the lists $\{F_{2,k}\bar{v}^\lambda, f_{2,k-1}f_{\alpha_k}F_{k,k+1}, f_{k,j}F_{2,j+1}\bar{v}^\lambda : k \leq j \leq n-1\}$, $\{f_{1,r-1}F_{r+1,k}\bar{v}^\lambda, f_{1,r-1}f_{r+1,k-1}f_{\alpha_k}F_{k,k+1}\bar{v}^\lambda, f_{1,r-1}f_{k,j}F_{r+1,j+1}\bar{v}^\lambda\}$ (for every $1 < r < k-1$), $\{f_{1,i}f_{k,s-1}F_{i+1,s+1}\bar{v}^\lambda, f_{k,s-1}F_{1,s+1}\bar{v}^\lambda\}$ (for $k < s < n$), $\{F_{1,k+1}\bar{v}^\lambda, f_{1,i}F_{i+1,k+1}\bar{v}^\lambda\}$ as well as $\{f_{1,i}f_{i+1,n-1}f_{k,n}\bar{v}^\lambda, f_{1,n-1}f_{k,n}\bar{v}^\lambda\}$ is linearly independent. Consequently, $e_\alpha\bar{w}(X) = 0$ for every $\alpha \in \Pi$ if and only if X is a solution to the system of equations

$$\left\{ \begin{array}{l} F = A - (a+1)B_1 - \sum_{i=2}^{k-2} B_i \\ F = -(a+1)C_1 - \sum_{i=2}^{k-2} C_i \\ E_j = (a+1)D_{1j} + \sum_{i=2}^{k-2} D_{ij} \text{ for every } k \leq j \leq n-1 \\ B_{r-1} = B_r \text{ for every } 1 < r < k-1 \\ C_{r-1} = C_r \text{ for every } 1 < r < k-1 \\ D_{r-1,j} = D_{rj} \text{ for every } 1 < r < k-1, k \leq j \leq n-1 \\ 2C_{k-2} = \sum_{j=k}^{n-1} D_{k-2,j} \\ 2F = A - 2E_k - \sum_{j=k+1}^{n-1} E_j \\ B_i + C_i = 2D_{ik} + \sum_{j=k+1}^{n-1} D_{ij} \text{ for every } 1 \leq i \leq k-2 \\ D_{i,r-1} = D_{ir} \text{ for every } 1 \leq i \leq k-2, k < r < n \\ E_{r-1} = E_r \text{ for every } k < r < n \\ E_{n-1} = 4F \\ D_{i,n-1} = 2F - E_{n-1} \text{ for every } 1 \leq i \leq k-2. \end{array} \right. \quad (4.26)$$

Now one easily sees that (4.26) admits a non-trivial solution if and only if $p \mid 2(n-k) + 1$ (showing that 1 and 3 are equivalent), in which case

$$X \in \langle (4, \underbrace{n-k-1, \dots, n-k-1}_{k-2}, \underbrace{k-n, \dots, k-n}_{k-2}, \underbrace{-2, \dots, -2}_{(n-k)(k-2)}, \underbrace{4, \dots, 4}_{n-k}, 1) \rangle_K,$$

thus completing the proof. \square

Let λ , and μ be as above, with $p \mid a+k$, and consider an irreducible KG -module $V = L(\lambda)$ having highest weight λ . As in the case where $k = 2$, set $V = V(\lambda)/\text{rad}(\lambda)$, so that

$$V \cong \overline{V(\lambda)} / \overline{\text{rad}(\lambda)},$$

where $\overline{\text{rad}(\lambda)} = \text{rad}(\lambda)/\langle Gu^+ \rangle_K$. Also write v^+ to denote the image of \bar{v}^λ in V , that is, v^+ is a maximal vector in V for B . By Proposition 4.2.15, the weight space V_μ is spanned by

$$\begin{aligned} & \{F_{1,k}v^+\} \cup \{f_{1,i}F_{i+1,k}v^+\}_{1 \leq i \leq k-2} \\ & \cup \{f_{1,i}f_{i+1,k-1}f_{\alpha_k}F_{k,k+1}v^+\}_{1 \leq i \leq k-2} \\ & \cup \{f_{1,i}f_{k,j}F_{i+1,j+1}v^+\}_{1 \leq i \leq k-2, k \leq j < n} \\ & \cup \{f_{k,j}F_{1,j+1}v^+\}_{k \leq j < n} \\ & \cup \{f_{k,n}f_{1,n}v^+\}. \end{aligned} \quad (4.27)$$

We write $V_{1,k,n}$ to denote the span of all the generators in (4.27) except for $f_{k,n}f_{1,n}v^+$. As usual, the following result consists of a precise description of the weight space V_μ , as well as a characterization for $[V(\lambda), L(\mu)]$ to be non-zero.

Proposition 4.2.18

Let G be a simple algebraic group of type B_n over K and consider an irreducible KG -module $V = L(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + \lambda_k$, where $a \in \mathbb{Z}_{>0}$, and $2 < k < n$. Also let $\mu = \lambda - (\alpha_1 + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_n) \in \Lambda^+(\lambda)$ and assume $p \mid a + k$. Then the following assertions are equivalent.

1. The weight μ affords the highest weight of a composition factor of $V(\lambda)$.
2. The generators in (4.27) are linearly dependent.
3. The element $f_{k,n}f_{1,n}v^+$ lies inside $V_{1,k,n}$.
4. The divisibility condition $p \mid 2(n - k) + 1$ is satisfied.

Proof. Proceed exactly as in the proof of Proposition 4.2.12, replacing $\mu_{1,2}$ by $\mu_{1,k}$, Lemma 4.2.9 by Lemma 4.2.14, and Lemma 4.2.11 by Lemma 4.2.17. \square

4.3 Proof of Theorem 4.1

Let K be an algebraically closed field of characteristic $p \geq 0$, Y a simply connected simple algebraic group of type B_n ($n \geq 2$) over K , and X the subgroup of Y of type D_n generated by the root subgroups of Y corresponding to long roots, as in the introduction of this chapter. Also consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. In this section, we finally give a proof of Theorem 4.1, starting by a reduction to the case where $\langle \lambda, \alpha_n \rangle = 0$ (relying on a result of Ford [For96, Theorem 3.3]), as well as a few technical results.

4.3.1 Preliminary considerations

Let $V = L_Y(\lambda)$ be as above and suppose first that $\langle \lambda, \alpha_n \rangle \neq 0$. Write $\omega = \lambda|_{T_X}$ and let $v^+ \in V_\lambda$ denote a maximal vector in V for B_Y . Since $B_X \subset B_Y$, the latter is a maximal vector for B_X as well, showing that ω affords the highest weight of a KX -composition factor of V . Also observe that the element $f_{\alpha_n}v^+$ is non-zero and satisfies $u_\beta(c)f_{\alpha_n}v^+ = f_{\alpha_n}v^+$ for every $\beta \in \Pi(X)$, $c \in K$, that is, $f_{\alpha_n}v^+$ is a maximal vector in V for B_X . Therefore the T_X -weight $\omega' = (\lambda - \alpha_n)|_{T_X}$ affords the highest weight of a second KX -composition factor of V , and one easily sees that ω and ω' are p -restricted T_X -weights interchanged by a graph automorphism of the Dynkin diagram corresponding to X , and that

$$\Lambda(\omega) \cap \Lambda(\omega') = \emptyset. \tag{4.28}$$

Theorem 4.3.1 (The case $\langle \lambda, \alpha_n \rangle \neq 0$)

Let λ, V be as above, with $\langle \lambda, \alpha_n \rangle \neq 0$ and let $1 \leq k < n$ be maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$. Then X has exactly two composition factors on V if and only if one of the following holds

1. $\lambda = \lambda_n$.
2. $\langle \lambda, \alpha_n \rangle = 1$, $p \mid 2(a_k + n - k) + 1$ and $p \mid a_i + a_j + j - i$ for every $1 \leq i < j < n$ such that $a_i a_j \neq 0$, but $a_r = 0$ for every $i < r < j$.

Moreover, if X has exactly two composition factors on V , then $V|_X$ is completely reducible.

Proof. First suppose that X has exactly two composition factors on V and assume $a_n > 1$. Here $(\lambda - 2\alpha_n)|_{T_X} \in \Lambda(V|_X)$ is neither in $\Lambda(\omega)$ nor $\Lambda(\omega')$, giving the existence of a third KX -composition factor of V , a contradiction. Therefore $a_n = 1$, in which case $L_X(\omega)$ and $L_X(\omega')$ are interchanged by θ and [For96, Theorem 3.3] applies, yielding the first assertion. Finally, the complete reducibility of $V|_X$ immediately follows from (4.28). \square

From now on, assume $\langle \lambda, \alpha_n \rangle = 0$ and $p \neq 2$, since otherwise X acts irreducibly on V by [Sei87, Theorem 1, Table 1 (MR₄)]. As above, write $\omega = \lambda|_{T_X}$ and let v^+ denote a maximal vector in V for B_Y . Since $B_X \subset B_Y$, the latter is a maximal vector for B_X as well, showing that ω affords the highest weight of a KX -composition factor of V . Also define

$$k = \max\{1 \leq r < n : \langle \lambda, \alpha_r \rangle \neq 0\}.$$

Since $p \neq 2$, the element $f_{k,n}v^+$ is non-zero and one easily sees that $u_\beta(c)f_{k,n}v^+ = f_{k,n}v^+$ for every $\beta \in \Pi(X)$, $c \in K$, that is, $f_{k,n}v^+$ is a maximal vector in V for B_X . Therefore the T_X -weight $\omega' = (\lambda - (\alpha_k + \cdots + \alpha_n))|_{T_X}$ affords the highest weight of a second KX -composition factor of V and as above, we observe that

$$\Lambda(\omega) \cap \Lambda(\omega') = \emptyset. \tag{4.29}$$

Now by Theorem 2.5.9, the KY -module V is irreducible as an $\mathcal{L}(Y)$ -module as well, where $\mathcal{L}(Y)$ denotes the Lie algebra of Y . Let then

$$\mathcal{B} = \{e_\alpha, f_\alpha, h_{\alpha_r} : \alpha \in \Phi^+(Y), 1 \leq r \leq n\}$$

be a Chevalley basis of $\mathcal{L}(Y)$ as in Section 2.5.1, and for $1 \leq r \leq s \leq n$, adopt the notation

$$\mu_{r,s} = \lambda - (\alpha_r + \cdots + \alpha_s).$$

The key to the proof of Theorem 4.1 lies in the following result, whose proof is similar to that of [For99, Lemma 3.4]. Here $\mathcal{L}(X)$ denotes the Lie algebra of X .

Proposition 4.3.2

Let $\mathcal{L}(X)$, V and v^+ be as above. Then $V = \mathcal{L}(X)f_{k,n}v^+ \oplus \mathcal{L}(X)v^+$ if and only if $f_{i,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ and $f_{i,n}f_{k,n}v^+ \in \mathcal{L}(X)v^+$ for every $1 \leq i \leq k$.

Proof. Assume first that $V = \mathcal{L}(X)f_{k,n}v^+ \oplus \mathcal{L}(X)v^+$. Also fix $1 \leq i \leq k$ and denote by $u \in \mathcal{L}(X)f_{k,n}v^+$ and $w \in \mathcal{L}(X)v^+$ the unique elements in V such that $f_{i,n}v^+ = u + w$. Moreover, as $f_{i,n}v^+$ lies in the weight space $V_{\mu_{i,n}}$, so do u and w . Observe however that $\mathcal{L}(X)v^+ \cap V_{\mu_{i,n}} = 0$ by (4.29), forcing $w = 0$ and thus $f_{i,n}v^+ = u \in \mathcal{L}(X)f_{k,n}v^+$. A similar argument shows that $f_{i,n}f_{k,n}v^+ \in \mathcal{L}(X)v^+$, thus the desired result.

Conversely, suppose that $f_{i,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ and $f_{i,n}f_{k,n}v^+ \in \mathcal{L}(X)v^+$ for every $1 \leq i \leq k$, and write $U = \mathcal{L}(X)f_{k,n}v^+ \oplus \mathcal{L}(X)v^+ \subseteq V$. We first show that

$$f_{\gamma_1} \cdots f_{\gamma_s}(f_{k,n})^\epsilon v^+ \in U,$$

for every $\epsilon \in \{0, 1\}$, $\gamma_1 \in \Phi^+(Y)$ short and $\gamma_2, \dots, \gamma_s \in \Phi^+(Y)$ long. Ab absurdo, suppose that it is not the case and let $2 \leq m$ be the smallest integer for which there exist $\gamma_1, \dots, \gamma_m$ as above such that $f_{\gamma_1} \cdots f_{\gamma_m} v^+ \notin U$ (the case where $\epsilon = 1$ can be dealt with in an identical fashion). Then

$$f_{\gamma_1} \cdots f_{\gamma_m} v^+ = f_{\gamma_2} f_{\gamma_1} f_{\gamma_3} \cdots f_{\gamma_m} v^+ + N_{(\gamma_1, \gamma_2)} f_{\gamma_1 + \gamma_2} f_{\gamma_3} \cdots f_{\gamma_m} v^+,$$

and by minimality of m , both $f_{\gamma_1} f_{\gamma_3} \cdots f_{\gamma_m} v^+$ and $f_{\gamma_1 + \gamma_2} f_{\gamma_3} \cdots f_{\gamma_m} v^+$ lie inside U . However, since γ_2 is long, we get $f_{\gamma_1} \cdots f_{\gamma_m} v^+ \in U$, contradicting our initial assumption.

Finally, let $r_1, \dots, r_m \in \Phi^+(Y)$ be such that $f_{r_1} \cdots f_{r_m} v^+ \notin U$, with m minimal with respect to this property. By minimality, we can rewrite $f_{r_1} \cdots f_{r_m} v^+$ as

$$f_{r_1} \cdots f_{r_m} v^+ = f_{r_1} \left(\sum a_{\gamma_1, \dots, \gamma_s} f_{\gamma_1} \cdots f_{\gamma_s} v^+ + \sum b_{\delta_1, \dots, \delta_t} f_{\delta_1} \cdots f_{\delta_t} f_{k,n} v^+ \right),$$

where each sum ranges over n -tuples ($n \in \mathbb{Z}_{>0}$) of long roots in $\Phi(Y)$. Two situations can occur: either r_1 is long or short. If the former holds, then $f_{r_1} \cdots f_{r_m} v^+ \in U$, a contradiction. Therefore r_1 is short, which by above also yields $f_{r_1} \cdots f_{r_m} v^+ \in U$. Consequently $U = V$ as desired. \square

For $1 \leq i < j \leq n$, set $P(i, j) = \{(m_r)_{r=1}^s : s > 0, i \leq m_1 < \dots < m_s < j\}$ and for $(m) = (m_r)_{r=1}^s \in P(i, j)$, write

$$f_{(m)} = f_{i, m_1} f_{m_1 + 1, m_2} \cdots f_{m_s + 1, j}.$$

By (2.15) and our choice of ordering \preceq on $\Phi^+(Y)$, we get that the weight space $V_{\mu_{i,j}}$ is spanned by

$$\{f_{(m)} v^+ : (m) \in P(i, j)\}. \quad (4.30)$$

We let $V_{i,j}$ denote the span of all the above terms except for $f_{i,j} v^+$. In order to apply Proposition 4.3.2, it is convenient to relate the subspace $\mathcal{L}(X)f_{k,n}v^+$ to the family of subspaces $\{V_{r,n}\}_{1 \leq r < k}$. The following result provides us with an alternative to [For96, Lemma 3.5], slightly modified to fit our situation.

Lemma 4.3.3

Let λ, V be as above, and let $1 \leq i < k < n$. Then $f_{r,n}v^+ \in V_{r,n}$ for every $i \leq r < k$ if and only if $f_{r,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for every $i \leq r < k$.

Proof. First observe that $\mathcal{L}(X)f_{k,n}v^+ \cap V_{\mu_{r,n}} \subset V_{r,n}$ for every $i \leq r < k$. Therefore, if $f_{r,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for some $i \leq r < k$, then clearly $f_{r,n}v^+ \in V_{r,n}$. Conversely, assume $f_{r,n}v^+ \in V_{r,n}$ for every $i \leq r < k$. We shall proceed by induction on $k - i$. If $k - i = 1$, then $r = k - 1$ and $V_{k-1,n}$ is at most 1-dimensional, thus the result is immediate. Let then $1 \leq i_0 < k - 1$ be such that $f_{r,n}v^+ \in V_{r,n}$ for every $i_0 \leq r < k$ and suppose that the assertion holds for every $i_0 < i < k$. By assumption $f_{i_0,n}v^+ \in V_{i_0,n}$, so there exist $\eta_{i_0}, \dots, \eta_{k-1} \in K$ such that

$$f_{i_0,n}v^+ = \sum_{s=i_0}^{k-1} \eta_s f_{i_0,s} f_{s+1,n}v^+.$$

By the inductive hypothesis, $f_{s+1,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for every $i_0 \leq s < k - 1$ (obviously $f_{k,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ as well) and since $\alpha_{i_0} + \dots + \alpha_s$ is long for every $i_0 \leq s < k$, we get the desired result. \square

Next assume $\langle \lambda, \alpha_k \rangle > 1$ and let $\mu = \lambda - 2(\alpha_k + \dots + \alpha_n) \in \Lambda^+(\lambda)$. Using (4.11) together with Lemma 2.3.7 (applied to the B_{n-k+1} -parabolic corresponding to the simple roots $\alpha_k, \dots, \alpha_n$) shows that the weight space V_μ is spanned by

$$\{(f_{k,n})^2v^+\} \cup \{f_{k,j}F_{k,j+1}v^+\}_{k \leq j < n}. \quad (4.31)$$

As in Section 4.2, we write $V_{k,n}^2$ to designate the span of all the generators in (4.31) except for $(f_{k,n})^2v^+$. Clearly, we have $V_{k,n}^2 \subset \mathcal{L}(X)v^+$, since the elements of $V_{k,n}^2$ are of the form $f_{\gamma_1}f_{\gamma_2}v^+$, with both γ_1 and γ_2 long roots in $\Phi^+(Y)$. Conversely, one easily sees that $\mathcal{L}(X)v^+ \cap V_\mu \subset V_{k,n}^2$, leading to the following result.

Lemma 4.3.4

Let λ and V be as above and assume $\langle \lambda, \alpha_k \rangle > 1$. Then $(f_{k,n})^2v^+ \in V_{k,n}^2$ if and only if $(f_{k,n})^2v^+ \in \mathcal{L}(X)v^+$.

Finally, assume $\langle \lambda, \alpha_k \rangle = 1$ and the existence of $1 \leq l < k$ such that $\langle \lambda, \alpha_l \rangle \neq 0$, but $\langle \lambda, \alpha_r \rangle = 0$ for every $l < r < k$. Also suppose that $p \mid a_l + k - l + 1$ and write $\mu = \lambda - (\alpha_l + \dots + \alpha_{k-1} + 2\alpha_k + \dots + 2\alpha_n)$. Using Proposition 4.2.15 together with Lemma 2.3.7 (applied to the B_{n-l+1} -parabolic corresponding to the simple roots $\alpha_l, \dots, \alpha_n$) shows that the weight space V_μ is spanned by

$$\begin{aligned} & \{F_{l,k}v^+\} \cup \{f_{l,i}F_{i+1,k}v^+\}_{l \leq i \leq k-2} \cup \{f_{l,i}f_{i+1,k-1}f_{\alpha_k}F_{k,k+1}v^+\}_{l \leq i \leq k-2} \\ & \cup \{f_{l,i}f_{k,j}F_{i+1,j+1}v^+\}_{l \leq i \leq k-2, k \leq j < n} \cup \{f_{k,j}F_{l,j+1}v^+\}_{k \leq j < n} \\ & \cup \{f_{k,n}f_{l,n}v^+\}. \end{aligned} \quad (4.32)$$

As in Section 4.2, we write $V_{l,k,n}$ to designate the span of all the generators in (4.32) except for $f_{k,n}f_{l,n}v^+$. Clearly, we have $V_{l,k,n} \subset \mathcal{L}(X)v^+$, since the elements of $V_{l,k,n}$ are of the form $f_{\gamma_1} \cdots f_{\gamma_r}v^+$, with $\gamma_1, \dots, \gamma_r$ long roots in $\Phi^+(Y)$. Conversely, one easily sees that $\mathcal{L}(X)v^+ \cap V_\mu \subset V_{l,k,n}$, leading to the following result.

Lemma 4.3.5

Let λ and V be as above and assume $\langle \lambda, \alpha_k \rangle = 1$. Then $f_{k,n}f_{l,n}v^+ \in V_{l,k,n}$ if and only if $f_{k,n}f_{l,n}v^+ \in \mathcal{L}(X)v^+$.

4.3.2 Tackling a first direction

Assume $p \neq 2$ and let Y and X be as in the statement of Theorem 4.1. Also consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$ such that $\langle \lambda, \alpha_n \rangle = 0$. In this section we show that if X acts with exactly two composition factors on V , then one of 1, 2 or 3 in Theorem 4.1 holds.

Suppose then that X has exactly two composition factors on V , write v^+ to denote a maximal vector in V for B_Y and let $1 \leq k < n$ be maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$. Recall then that both v^+ and $f_{k,n}v^+$ are maximal vectors in V for B_X , so that each of $\omega = \lambda|_{T_X}$ and $\omega' = (\lambda - (\alpha_k + \cdots + \alpha_n))|_{T_X}$ affords the highest weight of a KX -composition factor of V . Since we are assuming that X has exactly two composition factors on V , one immediately deduces that the KX -submodules $\langle Xv^+ \rangle$ and $\langle Xf_{k,n}v^+ \rangle$ of V are isomorphic to $L_X(\omega)$ and $L_X(\omega')$ respectively, and (4.29) yields

$$V = \langle Xv^+ \rangle \oplus \langle Xf_{k,n}v^+ \rangle \cong L_X(\omega) \oplus L_X(\omega'). \quad (4.33)$$

Now ω is p -restricted by (4.1), and in order to be able to apply Theorem 2.5.9, we need ω' to be p -restricted as well, which in fact follows from our assumption that X has exactly two composition factors on V , as the next result shows.

Lemma 4.3.6

Let λ be as above and suppose that X has exactly two composition factors on $V = L_Y(\lambda)$, having highest weights ω and ω' respectively. Then ω and ω' are p -restricted and

$$V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)f_{k,n}v^+.$$

Proof. Ab absurdo, assume ω' is not p -restricted. Then (4.1) yields $p \mid a_{k-1} + 1$, so that $\mu_{k-1,n}|_{T_X} \notin \Lambda(\omega) \cup \Lambda(\omega')$ by Theorem 2.3.2. Consequently $\mu_{k-1,n}|_{T_X}$ occurs in a third KX -composition factor of V , a contradiction. We then get that $L_X(\omega)$ and $L_X(\omega')$ are irreducible as $\mathcal{L}(X)$ -modules by Theorem 2.5.9 and so (4.33) completes the proof. \square

Before being able to apply Proposition 4.3.2 to its full potential, we need the following technical result, inspired by [For96, Proposition 3.1].

Proposition 4.3.7

Let λ be as above and let $1 \leq i < m \leq k$ be such that $a_i a_m \neq 0$. If $f_{r,m} v^+ \in V_{r,m}$ for all $i \leq r < m$, then $f_{i,j} v^+ \in V_{i,j}$, where $i < j \leq m$ is minimal such that $a_j \neq 0$. Also, if $f_{r,n} v^+ \in V_{r,n}$ for some $1 \leq r < k$, then $f_{r,k} v^+ \in V_{r,k}$.

Proof. We refer the reader to [For96, Proposition 3.1] for a proof of the first assertion and then consider $1 \leq r < k$ be such that $f_{r,n} v^+ \in V_{r,n}$. By (2.15), for every $r \leq s < k$, there exists $\{a_{(m)}\}_{(m) \in P(r,s)} \subset K$ such that

$$f_{r,n} v^+ = \sum_{s=r}^{k-1} \sum_{(m) \in P(r,s)} a_{(m)} f_{(m)} f_{s+1,n} v^+.$$

Now applying successively $e_{\alpha_n}, e_{\alpha_{n-1}}, \dots, e_{\alpha_{k+1}}$ gives a non-zero multiple of $f_{r,k} v^+$ on the left-hand side and elements lying inside $V_{r,k}$ on the right-hand side, yielding the desired result. \square

As a consequence of Lemma 4.3.6 and Proposition 4.3.7, we now show that if X has exactly two composition factors on V , then the divisibility conditions 3a in Theorem 4.1 are satisfied.

Corollary 4.3.8

Let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight $\lambda \in X^+(T_Y)$, with $\langle \lambda, \alpha_n \rangle = 0$. Assume in addition that X has exactly two composition factors on V . Then $p \mid a_i + a_j + j - i$ for every $1 \leq i < j < n$ such that $a_i a_j \neq 0$, and $a_r = 0$ for $i < r < j$.

Proof. Let $1 \leq k < n$ be maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$, $1 \leq i < j < n$ be such that $a_i a_j \neq 0$ and $a_r = 0$ for every $i < r < j$. By Lemma 4.3.6, we have $V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)f_{k,n}v^+$, where v^+ denotes a maximal vector in V for B_Y , and so we may apply Proposition 4.3.2, from which we get that $f_{r,n} v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for every $i \leq r < k$. Therefore $f_{r,n} v^+ \in V_{r,n}$ for every $i \leq r < k$ by Lemma 4.3.3, and thus Proposition 4.3.7 gives $f_{i,j} v^+ \in V_{i,j}$. Applying Proposition 4.1.3 then yields $p \mid a_i + a_j + j - i$, completing the proof. \square

Finally, assume that $\lambda \neq \lambda_k$, and suppose that X has exactly two composition factors on V . By Corollary 4.3.8, the divisibility conditions 3a in Theorem 4.1 are satisfied, and the following result shows that the remaining divisibility condition 3b in Theorem 4.1 holds as well.

Corollary 4.3.9

Let $\lambda = \sum_{r=1}^l a_r \lambda_r + a_k \lambda_k$, with $a_l \neq 0$ if $a_k = 1$, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight λ . Assume in addition that X has exactly two composition factors on V . Then $p \mid 2(a_k + n - k) - 1$.

Proof. First assume $a_k > 1$. Then $(f_{k,n})^2 v^+ \in \mathcal{L}(X)v^+$ by Proposition 4.3.2, which by Lemma 4.3.4 translates to $(f_{k,n})v^+ \in V_{k,n}^2$. Therefore $p \mid 2(a_k + n - k) - 1$ by Proposition 4.2.4, yielding the result in this situation. Finally, assume $\langle \lambda, \alpha_k \rangle = 1$ (i.e. $a_k = 1$) and let $1 \leq l < k$ be as above. By Corollary 4.3.8, the divisibility condition $p \mid a_l + l - k + 1$ is satisfied, hence Proposition 4.3.2 yields $f_{k,n} f_{l,n} v^+ \in \mathcal{L}(X)v^+$, which by Lemma 4.3.5 translates to $f_{k,n} f_{l,n} v^+ \in V_{l,k,n}$. Therefore $p \mid 2(n - k) + 1$ by Propositions 4.2.12 (if $l = k - 1$), or 4.2.18 (if $l < k - 1$), completing the proof. \square

4.3.3 Other direction and conclusion

Assume $p \neq 2$ and let Y and X be as in the statement of Theorem 4.1. Also adopt the notation introduced in the previous section and let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight λ , with (λ, p) as in 1 or 3 of Theorem 4.1. By Theorem 2.5.9, the KY -module V is irreducible when viewed as an $\mathcal{L}(Y)$ -module as well, where $\mathcal{L}(Y)$ denotes the Lie algebra of Y . We first aim at showing that $V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)f_{k,n}v^+$, using Proposition 4.3.2, starting by investigating whether or not $f_{r,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$, for $1 \leq r \leq k$.

Lemma 4.3.10

Let (λ, p) be as in 1 or 3 in Theorem 4.1. Then $f_{i,j}v^+ \in V_{i,j}$ for every $1 \leq i < j \leq k$ such that $a_i a_j \neq 0$.

Proof. For every $1 \leq i < j \leq k$ such that $a_i a_j \neq 0$, set $N(i, j) = |\{i \leq r \leq j : a_r \neq 0\}|$. We proceed by induction on $N(i, j)$, observing that the case $N(i, j) = 2$ directly follows from Proposition 4.1.3. Suppose the result proven for every $1 \leq i < j \leq k$ such that $a_i a_j \neq 0$ and $2 \leq N(i, j) < N_0$, and let $1 \leq i_0 < j_0 \leq k$ be such that $a_{i_0} a_{j_0} \neq 0$ and $N(i_0, j_0) = N_0$. If $i_0 < s < j_0$ is maximal such that $a_s \neq 0$, then Lemma 2.5.4 yields

$$f_{i_0, j_0} v^+ = f_{s+1, j_0} f_{i_0, s} v^+ - f_{i_0, s} f_{s+1, j_0} v^+,$$

and thus $f_{i_0, j_0} v^+ \in V_{i_0, j_0}$ if and only if $f_{s+1, j_0} f_{i_0, s} v^+ \in V_{i_0, j_0}$. Now $f_{i_0, s} v^+ \in V_{i_0, s}$ by our inductive assumption and thus is a sum of terms of type (4.30) with more than one f_α . It is clear that f_{s+1, j_0} commutes with all but the last f_α , from which the result follows. \square

We can now show that under the divisibility conditions 3a of Theorem 4.1, we have $f_{r,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for every $1 \leq r \leq k$.

Proposition 4.3.11

Assume $p \neq 2$ and consider a non-trivial irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight λ , with $\langle \lambda, \alpha_n \rangle = 0$. Also let $1 \leq k < n$ be maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$, and assume either $\lambda = \lambda_k$ or the divisibility conditions 3a of Theorem 4.1 hold. Then $f_{i,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ for every $1 \leq i \leq k$.

Proof. The result obviously holds for $i = k$, so we may assume $1 \leq i < k$, in which case it suffices to show that $f_{i,n}v^+ \in V_{i,n}$ for every $1 \leq i < k$ by Lemma 4.3.3. Observe that if $a_i = 0$, then $f_{i,n}v^+ \in V_{i,n}$ if and only if $f_{i+1,n} \in V_{i+1,n}$, since $f_{i,n}v^+ = -f_{\alpha_i}f_{i+1,n}v^+$ (which in particular yields the result in the case where $\lambda = \lambda_k$ for some $1 \leq k < n$). If on the other hand $a_i \neq 0$, observe that $f_{i,k}v^+ \in V_{i,k}$ by Lemma 4.3.10, so that

$$f_{i,n}v^+ = f_{k+1,n}f_{i,k}v^+ = \sum_{s=i}^{k-1} \sum_{(m) \in P(i,s)} a_{(m)} f_{(m)} f_{k+1,n} f_{s+1,k} v^+$$

for some $\{a_{(m)}\}_{(m) \in P(i,s)} \subset K$ ($i \leq s \leq k-1$). As $f_{k+1,n}f_{s+1,k}v^+ = f_{s+1,n}v^+$ for every $i \leq s \leq k-1$, the result follows by induction on s . \square

In order to apply Proposition 4.3.2, we still need to show that $f_{k,n}f_{r,n}v^+ \in \mathcal{L}(X)v^+$ for every $1 \leq r \leq k$. For the remainder of this chapter, we define

$$l = \max\{1 \leq r < k : \langle \lambda, \alpha_r \rangle \neq 0\}$$

and start by proving that $f_{k,n}f_{r,n}v^+ \in \mathcal{L}(X)v^+$ for every $l < r \leq k$, in which case no fundamental distinction needs to be made between the situations $a_k > 1$ and $a_k = 1$.

Lemma 4.3.12

Let λ be as above and assume $p \mid 2(a_k + n - k) - 1$ if $a_k > 1$. Then $f_{k,n}f_{r,n}v^+ \in \mathcal{L}(X)v^+$ for every $l < r \leq k$.

Proof. We first show that the assertion holds in the case where $r = k$. If $a_k = 1$, then this is immediate by Proposition 4.2.1, while if $a_k > 1$, then $(f_{k,n})^2v^+ \in V_{k,n}^2$ by Proposition 4.2.4, and the result then follows from Proposition 4.3.4. Now consider $l < r < k$. By Lemma 2.5.4, we have

$$\begin{aligned} f_{k,n}f_{r,n}v^+ &= -f_{k,n}f_{r,k-1}f_{k,n}v^+ \\ &= -f_{r,n}f_{k,n}v^+ - f_{r,k-1}(f_{k,n})^2v^+ \\ &= -2F_{r,k}v^+ - f_{k,n}f_{r,n}v^+ - f_{r,k-1}(f_{k,n})^2v^+, \end{aligned}$$

so that

$$f_{k,n}f_{r,n}v^+ = -F_{r,k}v^+ - \frac{1}{2}f_{r,k-1}(f_{k,n})^2v^+.$$

Clearly $F_{r,k}v^+ \in \mathcal{L}(X)v^+$ for every $l < r < k$ and the same holds for $(f_{k,n})^2v^+$ by what we saw above, completing the proof. \square

Now if $a_k = 1$, then $f_{k,n}f_{l,n}v^+ \in V_{l,k,n}$ by Proposition 4.2.12 or 4.2.18 (depending on whether $l = k-1$ or not), which by Proposition 4.3.5 implies that $f_{k,n}f_{l,n}v^+ \in \mathcal{L}(X)v^+$. This assertion also holds in the case where $a_k > 1$, but is not that immediate.

Lemma 4.3.13

Let λ be as above and assume $p \mid 2(a_k + n - k) - 1$ if $a_k > 1$. Then $f_{k,n}f_{l,n}v^+ \in \mathcal{L}(X)v^+$.

Proof. We refer to the remark above in the case where $a_k = 1$ and assume $a_k > 1$ for the remainder of the proof. By Lemma 2.5.4 and (4.22), one easily shows that

$$\begin{aligned} a_k f_{k,n} f_{l,n} v^+ &= \sum_{i=l}^{k-2} f_{l,i} f_{k,n} f_{i+1,n} v^+ + f_{k,n} f_{l,k-1} f_{k,n} v^+ \\ &= \sum_{i=l}^{k-2} f_{l,i} f_{k,n} f_{i+1,n} v^+ + f_{l,n} f_{k,n} v^+ + f_{l,k-1} (f_{k,n})^2 v^+ \\ &= \sum_{i=l}^{k-2} f_{l,i} f_{k,n} f_{i+1,n} v^+ + 2F_{l,k} v^+ + f_{k,n} f_{l,n} v^+ + f_{l,k-1} (f_{k,n})^2 v^+, \end{aligned}$$

and since $a_k > 1$, we finally obtain

$$f_{k,n} f_{l,n} v^+ = (a_k - 1)^{-1} \left(\sum_{i=l}^{k-2} f_{l,i} f_{k,n} f_{i+1,n} v^+ + 2F_{l,k} v^+ + f_{l,k-1} (f_{k,n})^2 v^+ \right).$$

Now by Lemma 4.3.12, we have $f_{l,k-1} (f_{k,n})^2 v^+ \in \mathcal{L}(X)v^+$, as well as $f_{l,i} f_{k,n} f_{i+1,n} v^+$, for every $l \leq i \leq k-2$, so that $f_{k,n} f_{l,n} v^+ \in \mathcal{L}(X)v^+$ as desired. \square

We are finally ready to show that under the divisibility conditions of Theorem 4.1, the element $f_{k,n} f_{r,n} v^+$ lies in $\mathcal{L}(X)v^+$ for every $1 \leq r \leq k$.

Proposition 4.3.14

Assume $p \neq 2$ and let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight $\lambda = \sum_{r=1}^l a_r \lambda_r + a_k \lambda_k$, where $1 \leq l \leq k < n$, and $a_k \neq 0$. Assume either $\lambda = \lambda_k$ or the divisibility conditions 3a and 3b of Theorem 4.1 hold. Then $f_{k,n} f_{r,n} v^+ \in \mathcal{L}(X)v^+$ for every $1 \leq r \leq k$.

Proof. We proceed by induction on $k - r$, the cases $l \leq r \leq k$ following from Lemmas 4.3.12 and 4.3.13. Let then $1 \leq r < l$. By Proposition 4.3.11 and Lemma 4.3.3, we immediately get that $f_{r,n} v^+ \in V_{r,n}$, hence the existence of $\{a_{(m)}\}_{(m) \in P(r,s)} \subset K$ for every $r \leq s \leq k-1$ such that

$$f_{k,n} f_{r,n} = \sum_{s=r}^{k-1} \sum_{(m) \in P(r,s)} a_{(m)} f_{(m)} f_{k,n} f_{s+1,n} v^+.$$

The result then easily follows by induction, so we leave the details to the reader. \square

Corollary 4.3.15

Assume $p \neq 2$ and let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight $\lambda \in X^+(T_Y)$, where $1 \leq k < n$ is maximal such that $\langle \lambda, \alpha_k \rangle \neq 0$. Assume either $\lambda = \lambda_k$ or the divisibility conditions 3a and 3b of Theorem 4.1 hold. Then

$$V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)f_{k,n}v^+.$$

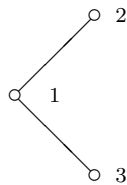
Proof. By Propositions 4.3.11 and 4.3.14, we immediately get that $f_{r,n}v^+ \in \mathcal{L}(X)f_{k,n}v^+$ and $f_{r,n}f_{k,n}v^+ \in \mathcal{L}(X)v^+$, for every $1 \leq r \leq k$. An application of Proposition 4.3.2 then completes the proof. \square

Proof of Theorem 4.1: Adopt the notation of Theorem 4.1 and start by supposing that X has exactly two composition factors on $V = L_Y(\lambda)$. If $\langle \lambda, \alpha_n \rangle \neq 0$, then Theorem 4.3.1 yields the result, so we may assume $\langle \lambda, \alpha_n \rangle = 0$ for the remainder of the proof. Also if $p = 2$, then V is irreducible as a KX -module by [Sei87, Theorem 1, Table 1 (MR₄)], a contradiction. We may thus assume $p \neq 2$ as well, in which case Corollaries 4.3.8 and 4.3.9 then yield the desired divisibility conditions.

Conversely, assume (λ, p) as in 1 or 3 of Theorem 4.1, in which case an application of Corollary 4.3.15 yields $V = \mathcal{L}(X)v^+ \oplus \mathcal{L}(X)f_{k,n}v^+$. Therefore V has a quotient isomorphic to $L_X(\omega)$ and since $V \cong V^*$ as a KY -module (and thus as a KX -module as well), we can assume the existence of a submodule U of V , isomorphic to $L_X(\omega)$. Since $V_\lambda = \langle v^+ \rangle_K$, we get $v^+ \in U$ and thus $\mathcal{L}(X)v^+ \subset \langle Xv^+ \rangle_K \subset U$, so that $\mathcal{L}(X)v^+ \cong L_X(\omega)$. A similar argument shows that $\mathcal{L}(X)f_{k,n}v^+ \cong L_X(\omega')$, hence $V \cong L_X(\omega) \oplus L_X(\omega')$ as KX -modules, completing the proof.

The case $SO_6(K) \subset SL_6(K)$

Let Y be a simply connected simple algebraic group of type A_5 over K and consider the subgroup X of type D_3 , embedded in Y in the usual way, as the stabilizer of a non-degenerate quadratic form on the natural module for Y . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ denote a corresponding base of the root system $\Phi(Y)$ of Y , and let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be the set of fundamental dominant T_Y -weights corresponding to $\Pi(Y)$. Also let $B_X = U_X T_X$, where $U_X = U_Y \cap X$, $T_X = T_Y \cap X$, let $\Pi(X) = \{\beta_1, \beta_2, \beta_3\}$ be a corresponding set of simple T_X -roots and let $\{\omega_1, \omega_2, \omega_3\}$ be the corresponding set of fundamental dominant T_X -weights.



The A_2 -parabolic subgroup of X corresponding to the simple roots $\{\beta_1, \beta_2\}$ embeds in an $A_2 \times A_2$ -parabolic subgroup of Y , and up to conjugacy, we may assume that this gives $\alpha_1|_{T_X} = \alpha_5|_{T_X} = \beta_1$, and $\alpha_2|_{T_X} = \alpha_4|_{T_X} = \beta_2$. By considering the action of the Levi factors of these parabolics on the natural KY -module $L_Y(\lambda_1)$, we can deduce that $\alpha_3|_{T_X} = \beta_3 - \beta_2$. Finally, using [Hum78, Table 1, p.69] and the fact that $\lambda_1|_{T_X} = \omega_1$ yields

$$\lambda_5|_{T_X} = \omega_1, \quad \lambda_2|_{T_X} = \lambda_4|_{T_X} = \omega_2 + \omega_3, \quad \lambda_3|_{T_X} = 2\omega_3. \tag{5.1}$$

In [Sei87], Seitz showed that if V is an irreducible KY -module having p -restricted highest weight, then $V|_X$ is reducible except when $V = L_Y(\lambda_i)$ for some $1 \leq i \leq 2n - 1$ such that $i \neq n$ (see [Sei87, Theorem 1, Table 1 (I_4, I_5)]). In this chapter, we determine the pairs (V, p) such that X has exactly two composition factors on V . In other words, we give a proof that Conjecture 4 (recorded here as Theorem 5.1) holds in the case where $n = 3$.

Theorem 5.1

Let K, Y, X be as above and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if λ and p are as in Table 5.1, where we give λ up to graph automorphisms. Moreover, if (λ, p) is recorded in Table 5.1, then $V|_X$ is completely reducible if and only if $(\lambda, p) \neq (\lambda_2, 2)$.

λ	p	$V _X$	Dimensions
$\lambda_1 + \lambda_2$	$\neq 5$	$\omega_1 + \omega_2 + \omega_3/\omega_1$	$64 - 20\delta_{p,3}, 6$
$\lambda_1 + \lambda_5$	$\neq 2$	$2\omega_1/\omega_2 + \omega_3$	$20 - \delta_{p,3}, 15$
$2\lambda_1$	$\neq 2, 3$	$2\omega_1/0$	$20, 1$
$2\lambda_1 + \lambda_5$	$= 7$	$3\omega_1/\omega_1 + \omega_2 + \omega_3$	$50, 64$
$3\lambda_1$	$\neq 2, 3$	$3\omega_1/\omega_1$	$50, 6$
λ_2	$= 2$	$\omega_2 + \omega_3/0$	$15, 1$
λ_3	$\neq 2$	$2\omega_2/2\omega_3$	$10, 10$

Table 5.1: The case $\text{SO}_6(K) \subset \text{SL}_6(K)$.

Here we say a few words about the method of the proof. Let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight λ and let $v^+ \in V_\lambda$ denote a maximal vector in V for B_Y . Since $B_X \subset B_Y$, v^+ is a maximal vector in V for B_X as well and the T_X -weight $\omega = \lambda|_{T_X}$ affords the highest weight of a KX -composition factor of V . Furthermore, it turns out (see Lemma 3.1.1) that in general, every T_X -weight $\nu \in \Lambda(V|_X)$ satisfies $\nu \preceq \omega$ and thus if $\omega' \in \Lambda(\omega)$ is maximal such that $m_{V|_X}(\omega') > m_{L_X(\omega)}(\omega')$, then ω' affords the highest weight of a second KX -composition factor of V . Finally, finding $\omega'' \in \Lambda(\omega)$ such that $m_{V|_X}(\omega'') > m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'')$ translates to the existence of a third composition factor of V for X . Therefore determining the pairs (λ, p) such that X acts with exactly two composition factors on V requires a good knowledge of weight multiplicities in V , $L_X(\omega)$ and $L_X(\omega')$.

In section 5.1, we investigate such weight multiplicities in certain irreducible modules for a simple group of type D_3 over K . To do so, we proceed as explained in Section 2.7.3, using the Jantzen p -sum formula to obtain information on the KX -composition factors of carefully chosen Weyl modules for X .

In Section 5.2, we first assume that X has exactly two composition factors on $V = L_Y(\lambda)$ and proceeding by a case-by-case analysis, we apply the method introduced above and use the previously calculated weight multiplicities to obtain a small list of possible candidates for (λ, p) . Finally, arguing on dimensions allows us to show that the aforementioned candidates satisfy the desired property.

5.1 Preliminary considerations

Let K be an algebraically closed field of characteristic $p \geq 0$ and G a simple algebraic group of type D_3 over K . Fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U is the unipotent radical of B , let $\Pi = \{\gamma_1, \gamma_2, \gamma_3\}$ denote a corresponding base of the root system Φ of G and let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the set of fundamental dominant weights for T corresponding to our choice of base Π . Also let $V = L_G(\sigma)$ be an irreducible KG -module having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_3$, where $a, b, c \in \mathbb{Z}_{\geq 0}$. In this section, we record some useful results on certain T -weights of V and their multiplicities, starting with the case where $a \neq b = c = 0$. Here for $c_1, c_2, c_3 \in \mathbb{Z}$, we adopt the notation $\sigma - c_1c_2c_3$ to designate $\sigma - c_1\gamma_1 - c_2\gamma_2 - c_3\gamma_3$.

Lemma 5.1.1

Let V be as above, with $a > 1$, $b = c = 0$, and consider $\mu = \sigma - 211 \in X(T)$. Then μ is dominant and its multiplicity in V is given by

$$m_V(\mu) = \begin{cases} 1 & \text{if } p \mid a + 1; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. An application of Theorem 2.3.11 gives $m_{V_G(\sigma)}(\mu) = 2$, while every weight $\nu \in \Lambda^+(\sigma)$ such that $\mu \prec \nu \preceq \sigma$ has multiplicity 1 in $V_G(\sigma)$, thus cannot afford the highest weight of a composition factor of $V_G(\sigma)$ by Theorem 2.3.4. Finally, an application of Corollary 2.7.3 shows that if μ affords the highest weight of a composition factor of $V_G(\sigma)$, then $p \mid a + 1$, in which case Theorem 2.3.18 yields the desired result. \square

We next apply the method introduced in Section 2.7 in order to determine the multiplicity of the T -weight $\sigma - 422 \in X(T)$ in an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1$, where $a > 3$.

Lemma 5.1.2

Let V be as above, with $a > 3$, $b = c = 0$, and consider $\mu = \sigma - 422 \in X(T)$. Then μ is dominant, and its multiplicity in V is given by

$$m_V(\mu) = \begin{cases} 1 & \text{if } p \mid a + 1; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. We assume $a > 5$ and refer the reader to [Lüb15] for the cases where $a = 4$ or 5 . Let then $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. As explained in Section 2.7, we start by finding the expression of $\nu_c^\mu(T_\sigma)$ in terms of $\chi(\nu)$ ($\nu \in X^+(T)$). By Corollary 2.7.9, we only need to consider those weights $\mu \preceq \nu \prec \sigma$ appearing in Table 5.2.

ν	$m_{V_G(\sigma)}(\nu)$	Contribution to $\nu_c^\mu(T_\sigma)$
$\sigma - 211$	2	$-\nu_p(2) + \nu_p(a + 1)$
$\sigma - 311$	2	$-\nu_p(3) + \nu_p(a)$
$\sigma - 411$	2	$-\nu_p(4) + \nu_p(a - 1)$
$\sigma - 412$	2	none
$\sigma - 421$	2	none
μ	3	none

Table 5.2: Dominant T -weights $\mu \preceq \nu \prec \sigma$ with $m_{V_G(\sigma)}(\nu) > 1$.

For such a weight ν , applying Lemma 2.7.11 requires us to find every root $\gamma \in \Phi^+$ and every integer $1 < r < \langle \sigma + \rho, \gamma \rangle$ such that $A_{\gamma,r} \sim_G B_\nu$. Also notice that $\sigma - \nu$ has support Π , so $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ and we only need to look for those $1 < r < \langle \sigma + \rho, \gamma \rangle$ such that

$$B_\nu \sim_G (a + 2 - r, 1 - r, 0).$$

First consider the T -weight $\nu = \sigma - 211$. Here $B_\nu = (a, 1, 0)$ and one easily checks that $B_\nu \sim_G A_{\gamma,r}$ if and only if $r = 2$ or $a + 1$. In the former case, we have $A_{\gamma,2} = (a, -1, 0)$, so that $\nu = (s_{\gamma_2} s_{\gamma_3}) \cdot (\sigma - 2\gamma)$, while in the latter case, we have $A_{\gamma,a+1} = (1, -a, 0)$, thus $\nu = (s_{\gamma_1} s_{\gamma_2} s_{\gamma_3}) \cdot (\sigma - (a + 1)\gamma)$. Corollary 2.7.7 then yields the contribution to $\nu_c^\mu(T_\sigma)$ stated in Table 5.2 and we leave the reader to check the remaining contributions, as they can be dealt with in a similar fashion. In the end, since σ is p -restricted (and so $p > 5$ and $a < p$), we get

$$\nu_c^\mu(T_\sigma) = \nu_p(a + 1) \chi^\mu(\sigma - 211). \quad (5.2)$$

Therefore $m_V(\mu) = m_{V_G(\sigma)}(\mu)$ if $p \nmid a + 1$ by Proposition 2.7.8 and one easily sees (using Theorem 2.3.11) that $m_{V_G(\sigma)}(\mu) = 3$, so that the assertion holds in this situation. Assume $p \mid a + 1$ for the remainder of the proof and write $\tau = \sigma - 211 = (a - 2)\sigma_1$, so that $\mu = \tau - 211$. An application of Lemma 5.1.1 then yields $\chi^\mu(\tau) = \text{ch } L_G(\tau)$, in which case (5.2) can be rewritten as

$$\nu_c^\mu(T_\sigma) = \nu_p(a + 1) \text{ch } L_G(\tau).$$

Consequently $\chi^\mu(\sigma) = \text{ch } L_G(\sigma) + \text{ch } L_G(\tau)$ by Proposition 2.7.8 and Lemma 5.1.1, so that $m_V(\mu) = m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. As seen above, we have $m_{V_G(\sigma)}(\mu) = 3$, while $m_{L_G(\tau)}(\mu) = 2$ by Lemma 5.1.1 and thus the proof is complete. \square

For the next two lemmas, we consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_2$, where $a, b > 0$. In order to show the first result, we again apply the method introduced in Section 2.7.

Lemma 5.1.3

Let V be as above, with $a > 1$, $b > 0$, $c = 0$, and consider $\mu = \sigma - 211 \in X(T)$. Then μ is

dominant and its multiplicity in V is given by

$$m_V(\mu) = \begin{cases} 1 & \text{if } p \mid a + b + 1; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. If $p \mid a + b + 1$, then the result follows from Theorem 2.3.18, so for the remainder of the proof, we may assume $p \nmid a + b + 1$ and consider the series $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ given by Proposition 2.7.4. Here the T -weight $\sigma - 110$ does not afford the highest weight of a composition factor of $V_G(\sigma)$ by Lemma 2.3.19. Proceeding exactly as in the proof of Lemma 5.1.2, one first checks that the only dominant T -weights $\nu \in \Lambda^+(\sigma)$ such that $\mu \preceq \nu \prec \sigma$ and $m_{V_G(\sigma)}(\nu) > 1$ are $\sigma - 110$, $\sigma - 210$ (if $a > 2$) and μ itself, and that neither $\sigma - 210$ nor μ contributes to $\nu_c^\mu(T_\sigma)$. Therefore $m_V(\mu) = m_{V_G(\sigma)}(\mu)$ by Proposition 2.7.8 and the result then follows from Theorem 2.3.11. \square

Lemma 5.1.4

Let V be as above, with $a = 1$, $b > 1$, $c = 0$, and consider $\mu = \sigma - 221 \in X(T)$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } L_G(\sigma) + \epsilon_p(b + 2) \text{ch } L_G(\sigma - 110)$ and

$$m_V(\mu) = \begin{cases} 1 & \text{if } p \mid b + 2; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. First observe that the weights $\nu \in \Lambda^+(\sigma)$ such that $\mu \preceq \nu \prec \sigma$ and $m_{V_G(\sigma)}(\nu) > 1$ are $\tau = \sigma - 110 = (b - 1)\sigma_2 + \sigma_3$, $\sigma - 120 = \tau - 010$ (if $b > 2$) and μ . Now if $p \nmid b + 2$, then $[V_G(\sigma), L_G(\tau)] = [V_G(\sigma), L_G(\tau - 010)] = 0$ by Lemma 2.3.19, while Corollary 2.7.3 yields $[V_G(\sigma), L_G(\mu)] = 0$ as well. The assertion then holds in this situation and we may assume $p \mid b + 2$ for the remainder of the proof, in which case $[V_G(\sigma), L_G(\tau)] = 1$ by Lemma 2.3.19. One finally checks (using Lemma 2.3.19 again) that $m_{L_G(\tau)}(\mu) = 2$ and an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 3$, allowing us to conclude. \square

In the following lemma, we study the multiplicity of the T -weight $\mu = \sigma - 222$ in a given irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = b\sigma_2 + b\sigma_3$ for some $b > 1$.

Lemma 5.1.5

Let V be as above, with $a = 0$, $b = c > 1$, and consider $\mu = \sigma - 222 \in X(T)$. Then μ is dominant and its multiplicity in V satisfies

$$m_V(\mu) \leq \begin{cases} 3 & \text{if } p \mid b + 1; \\ 5 & \text{if } p \mid 2b + 1; \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We shall assume $b > 2$ and refer the reader to [Lüb15] for the other cases. Write $\tau = \sigma - 111$ and consider the filtration $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\sigma)$ given by Proposition 2.7.4. One first checks that the T -weights $\nu \in \Lambda^+(\sigma)$ such that $\mu \preceq \nu \prec \sigma$ and $m_{V_G(\sigma)}(\nu) > 1$ together with their contribution to $\nu_c^\mu(T_\sigma)$ are as in Table 5.3, so that $\nu_c^\mu(T_\sigma) = \nu_p(2b+2)\chi^\mu(\tau) + \nu_p(2b+1)\chi^\mu(\mu)$.

ν	$m_{V_G(\sigma)}(\nu)$	Contribution to $\nu_c^\mu(T_\sigma)$
$\sigma - 111$	3	$\nu_p(2b+2)$
$\sigma - 121$	3	none
$\sigma - 112$	3	none
$\sigma - 212$	3	none
$\sigma - 222$	6	$\nu_p(2b+1)$

Table 5.3: Dominant T -weights $\mu \preceq \nu \prec \sigma$ with $m_{V_G(\sigma)}(\nu) > 1$.

Now if $p \nmid (b+1)(2b+1)$, then $\chi^\mu(\sigma) = \text{ch } L_G(\sigma)$ by Proposition 2.7.8, while an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 6$, thus showing the assertion in this case. If on the other hand $p \mid b+1$, then $\nu_c^\mu(T_\sigma) = \nu_p(2b+2)\chi^\mu(\tau)$, while $\chi^\mu(\tau) = \text{ch } L_G(\tau)$ by Lemma 2.3.19. Therefore $\chi^\mu(\sigma) = \text{ch } L_G(\sigma) + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8. Now $m_{L_G(\tau)}(\mu) = 3$ by Lemma 2.3.19, from which the result follows in this situation. Finally, if $p \mid 2b+1$, then $\nu_c^\mu(T_\sigma) = \nu_p(2b+1)\text{ch } L_G(\mu)$, and one deduces that μ affords the highest weight of a composition factor of $V_G(\sigma)$ using Proposition 2.7.8. Hence $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - 1$, completing the proof. \square

The result given by Lemma 5.1.5 could easily be improved. Indeed, the proof of the latter showed that if $p \nmid 2b+1$, then $m_V(\mu) = 6 - 3\epsilon_p(b+1)$. We next investigate the multiplicity of various T -weights in a given irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_3$, where $abc \neq 0$.

Lemma 5.1.6

Let V be as above, with $abc \neq 0$ such that p divides both $a+b+1$ and $a+c+3$. Also let $\mu_1 = \sigma - 111 \in X(T)$ as well as $\mu_2 = \sigma - 121 \in X(T)$ and $\mu_3 = \sigma - 112 \in X(T)$. Then for $1 \leq i \leq 3$, we have $\chi^{\mu_i}(\sigma) = \text{ch } L_G(\sigma) + \text{ch } L_G(\sigma - 110)$, $m_{V_G(\sigma)}(\mu_i) = 4$ and $m_V(\mu_i) = 3$.

Proof. First observe that the weights $\nu \in X^+(T)$ such that $\mu_1 \preceq \nu \prec \sigma$ and $m_{V_G(\sigma)}(\nu) > 1$ are $\sigma - 110$, $\sigma - 101$ and μ_1 . By Corollary 2.7.3, neither $\sigma - 101$ nor μ_1 can afford the highest weight of a composition factor of $V_G(\sigma)$, while by Lemma 2.3.19, we know that $[V_G(\sigma), L_G(\sigma - 110)] = 1$. Now Theorem 2.3.11 gives $m_{V_G(\sigma)}(\mu_1) = 4$, while $m_{V(\sigma - 110)}(\mu_1) = 1$ by Lemma 2.3.16, hence the assertion on $m_V(\mu_1)$. Proceeding in a similar fashion (notice that $[V_G(\sigma), L_G(\mu_2)] = [V_G(\sigma), L_G(\mu_3)] = 0$ by Corollary 2.7.3) then yields the assertions on μ_2 and μ_3 . The details are left to the reader. \square

Lemma 5.1.7

Let V be as above, with $abc \neq 0$ such that p divides both $a + b + 1$ and $a + c + 1$. Also let $\mu_1 = \sigma - 111 \in X(T)$ and $\mu_2 = \sigma - 122 \in X(T)$. Then μ_1 is dominant, $m_{V_G(\sigma)}(\mu_1) = 4$ and $m_V(\mu_1) = 2$. If in addition $b, c > 2$, then μ_2 is also dominant, $m_{V_G(\sigma)}(\mu_2) = 4$ and $m_V(\mu_2) \leq 2$.

Proof. By Corollary 2.7.3, the T -weight μ_1 cannot afford the highest weight of a composition factor of $V_G(\sigma)$, while by Lemma 2.3.19, we know that each of $\sigma - 110$ and $\sigma - 101$ does. Now Proposition 2.3.15 gives $m_{V_G(\sigma)}(\mu_1) = 4$, while $m_{V(\sigma-110)}(\mu_1) = m_{V(\sigma-101)}(\mu_1) = 1$ by Lemma 2.3.16, hence the assertion on $m_V(\mu_1)$. Assume $b, c > 2$ for the remainder of the proof. An application of Proposition 2.3.15 yields $m_{V_G(\sigma)}(\mu_2) = 4$ and thus the assertion on $m_V(\mu_2)$ easily follows. \square

Lemma 5.1.8

Let V be as above, with $a > 1$ and $bc \neq 0$ such that p divides both $a + b + 1$ and $a + c + 1$. Also let $\mu = \sigma - 211 \in X(T)$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 5$ and $m_V(\mu) \leq 2$.

Proof. Write $\tau_1 = \sigma - 110 \in X(T)$ as well as $\tau_2 = \sigma - 101 \in X(T)$ and consider the filtration $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\sigma)$ given by Proposition 2.7.4. As usual, we leave to the reader to check that

$$\nu_c^\mu(T_\sigma) = \nu_p(a + b + 1)\chi^\mu(\tau_1) + \nu_p(a + c + 1)\chi^\mu(\tau_2),$$

and since $\chi^\mu(\tau_i) = \text{ch } L_G(\tau_i) + \text{ch } L_G(\mu)$ for $i = 1, 2$ by Lemma 2.3.19, an application of Proposition 2.7.8 shows that each of τ_1, τ_2 , and μ affords the highest weight of a composition factor of $V_G(\sigma)$. Now $m_{V_G(\sigma)}(\mu) = 5$ by Theorem 2.3.11, while applying Lemma 2.3.19 yields $m_{L_G(\tau_1)}(\mu) = m_{L_G(\tau_2)}(\mu) = 1$, thus completing the proof. \square

Lemma 5.1.9

Let V be as above, with $a > 1$ and $b = c = 1$ such that $p \mid a + 2$. Also let $\mu_1 = \sigma - 322 \in X(T)$ and $\mu_2 = \sigma - 311 \in X(T)$. Then μ_1 is dominant, $m_{V_G(\sigma)}(\mu_1) = 8$ and $m_V(\mu_1) \leq 3$. Similarly, if $a > 3$, then μ_2 is dominant, $m_{V_G(\sigma)}(\mu_2) = 5$ and $m_V(\mu_2) \leq 2$.

Proof. We shall assume $a > 3$ throughout the proof and refer the reader to [Lüb15] for the case where $a = 3$. Write $\tau_1 = \sigma - 110, \tau_2 = \sigma - 101, \tau = \sigma - 211$, and consider the filtration $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\sigma)$ given by Proposition 2.7.4. Again we proceed as in the proof of Lemma 5.1.2 (starting with the T -weight μ_2) and leave to the reader to check that we have $\nu_c^{\mu_2}(T_\sigma) = \nu_p(a + 2)(\chi^{\mu_2}(\tau_1) + \chi^{\mu_2}(\tau_2))$. Now Lemma 2.3.19 yields $\chi^{\mu_2}(\tau_i) = \text{ch } L_G(\tau_i) + \text{ch } L_G(\tau)$ for $i = 1, 2$, so that

$$\nu_c^{\mu_2}(T_\sigma) = \nu_p(a + 2)(\text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + 2 \text{ch } L_G(\tau)).$$

Therefore each of τ_1 , τ_2 and τ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8 and $m_V(\mu_2) \leq m_{V_G(\sigma)}(\mu_2) - m_{L_G(\tau_1)}(\mu_2) - m_{L_G(\tau_2)}(\mu_2) - m_{L_G(\tau)}(\mu_2)$. An application of Theorem 2.3.11 then yields $m_{V_G(\sigma)}(\mu_2) = 5$, from which the assertion on $m_V(\mu_2)$ easily follows. Next we consider the T -weight μ_1 and again leave to the reader to check that $\nu_c^{\mu_1}(T_\sigma) = \nu_p(a+2)(\chi^{\mu_1}(\tau_1) + \chi^{\mu_1}(\tau_2) + \chi^{\mu_1}(\mu_1))$. Now $[V_G(\tau_i), L_G(\tau)] = 1$ for $i = 1, 2$ by Lemma 2.3.19, so that each of τ_1 , τ_2 , τ and μ_1 affords the highest weight of a composition factor of $V_G(\sigma)$ and hence

$$m_V(\mu_1) \leq m_{V_G(\sigma)}(\mu_1) - m_{L_G(\tau_1)}(\mu_1) - m_{L_G(\tau_2)}(\mu_1) - m_{L_G(\tau)}(\mu_1) - m_{L_G(\mu_1)}(\mu_1).$$

Finally, Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu_1) = 8$, while $m_{L_G(\tau_i)}(\mu_1) = 1$ for $i = 1, 2$ by Theorem 2.3.18 and $m_{L_G(\tau)}(\mu_1) \geq 2$ by Theorem 2.3.18, completing the proof. \square

Lemma 5.1.10

Let V be as above, with $a > 0$ and $b, c > 1$ such that p divides both $a + b + 1$ and $a + c + 1$. Also let $\mu = \sigma - 222 \in X(T)$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 10 - \delta_{a,1}$ and $m_V(\mu) \leq 3$.

Proof. Write $\tau_1 = \sigma - 110$, $\tau_2 = \sigma - 101$ and first assume $a = 1$. By Lemma 2.3.19, each of the weights τ_1 and τ_2 affords the highest weight of a composition factor of $V_G(\sigma)$. Also $m_{L_G(\tau_i)}(\mu) = 3$ for $i = 1, 2$ by Lemma 2.3.19 again and since $m_{V_G(\sigma)}(\mu) = 9$ by Theorem 2.3.11, the assertion holds in the case where $a = 1$. For the remainder of the proof, assume $a > 1$, write $\tau = \sigma - 211$ and consider the filtration $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_G(\sigma)$ given by Proposition 2.7.4. We leave to the reader to check that

$$\nu_c^\mu(T_\sigma) = \nu_p(a+b+1)\chi^\mu(\tau_1) + \nu_p(a+c+1)\chi^\mu(\tau_2)$$

and as $\chi^\mu(\tau_i) = \text{ch } L_G(\tau_i) + \text{ch } L_G(\tau)$ for $i = 1, 2$ by Lemma 5.1.6, we get that each of τ_1 , τ_2 , and τ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8. Finally, $m_{V_G(\sigma)}(\mu) = 10$ by Theorem 2.3.11, while $m_{L_G(\tau_1)}(\mu) = m_{L_G(\tau_2)}(\mu) = 3$ by Lemma 5.1.6, completing the proof. \square

Finally, suppose that G is a simple algebraic group of type A_4 over K , and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 \in X^+(T)$, where $a, b, c, d \in \mathbb{Z}_{\geq 0}$. Here for $c_1, c_2, c_3, c_4 \in \mathbb{Z}$, we adopt the notation $\sigma - c_1c_2c_3c_4$ to designate the T -weight $\sigma - c_1\alpha_1 - c_2\alpha_2 - c_3\alpha_3 - c_4\alpha_4$.

Remark 5.1.11

In the proof of the following result, it is more convenient to view G as an A_4 -Levi subgroup of a simple algebraic group of type D_7 over K , the reason being the complexity of the description of fundamental weights in terms of an orthonormal basis of a Euclidean space E for G of type A_n over K (see Section 2.2). Indeed, in general it is convenient to work in a simple algebraic group of type D_{n+3} instead of A_n when applying the method introduced in Section 2.7 .

Lemma 5.1.12

Let V be as above, with $a = c = d = 0$, $b > 2$, and let $\mu = \sigma - 1321 \in X(T)$. Then μ is dominant and its multiplicity in V is given by

$$m_V(\mu) = \begin{cases} 1 & \text{if } p \mid b + 1; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Consider the filtration $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ of $V_G(\sigma)$ given by Proposition 2.7.4. The only T -weights $\nu \in \Lambda^+(\sigma)$ such that $\mu \preceq \nu \prec \sigma$ and $m_{V_G(\sigma)}(\nu) > 1$ are $\tau = \sigma - 1210$, $\sigma - 1310$ (if $b > 3$) and μ itself, and one then easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(b + 1)\chi^\mu(\tau)$. Therefore $\chi^\mu(\sigma) = \text{ch } L_G(\sigma)$ if $p \nmid b + 1$ by Proposition 2.7.8 and thus Theorem 2.3.11 yields the assertion in this situation. Finally, an application of Theorem 2.3.18 completes the proof in the case where $p \mid b + 1$. \square

5.2 Proof of Theorem 5.1

Let Y be a simply connected simple algebraic group of type A_5 over K and let X be a simple algebraic group of type D_3 , embedded in Y in the usual way. Also let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight

$$\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 + e\lambda_5 \in X^+(T_Y),$$

and denote by ω the restriction of the T_Y -weight λ to T_X , so that by (5.1), we have

$$\omega = (a + e)\omega_1 + (b + d)\omega_2 + (b + 2c + d)\omega_3.$$

Notice that if $v^+ \in V_\lambda$ is a maximal vector in V for B_Y , then v^+ is a maximal vector for B_X as well, since $B_X \subset B_Y$, showing that the T_X -weight ω affords the highest weight of a KX -composition factor of V . Every T_Y -weight of V is of the form $\lambda - \sum_{i=1}^5 c_i \alpha_i$, where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{Z}_{\geq 0}$. Throughout this section, such a weight shall be written $\lambda - c_1 c_2 c_3 c_4 c_5$ and simply called a T_Y -weight. On the other hand, a T_X -weight of $V|_X$ does not necessarily have to be under ω : for example, if $\langle \lambda, \alpha_3 \rangle \neq 0$, then $\lambda - \alpha_3|_{T_X} = \omega + \beta_2 - \beta_3 \not\prec \omega$.

Lemma 5.2.1

Let λ, ω be as above, and suppose that $\langle \lambda, \alpha_3 \rangle = 0$. Then every T_Y -weight μ of $V = L_Y(\lambda)$ satisfies $\mu|_{T_X} \preceq \omega$.

Proof. Assume for a contradiction the existence of a T_Y -weight $\mu = \lambda - c_1 c_2 c_3 c_4 c_5 \in \Lambda(V)$ such that $\mu|_{T_X} \not\prec \omega$. Recalling the restriction to T_X of the simple roots for T_Y stated in the beginning of the chapter, we have $\mu|_{T_X} = \omega - (c_1 + c_5)\beta_1 - (c_2 - c_3 + c_4)\beta_2 - c_3\beta_3$ and hence $c_3 > c_2 + c_4$. In particular, we get $\langle \mu, \alpha_3 \rangle < -c_3$, showing that $s_{\alpha_3}(\mu) \in \Lambda(V)$ is not under λ , a contradiction. \square

5.2.1 The case $\langle \lambda, \alpha_3 \rangle > 0$

Keep the notation introduced above and suppose that $c > 0$. Here the T_Y -weight $\lambda - 00100$ restricts to $\omega' = \omega + \beta_2 - \beta_3$, which is neither under nor above ω . In fact, one easily checks that ω' is a highest weight of V for T_X and hence affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a + e)\omega_1 + (b + d + 2)\omega_2 + (b + 2c + d - 2)\omega_3).$$

Lemma 5.2.2

Let λ, ω, ω' be as above and suppose that X acts with exactly two composition factors on $V = L_Y(\lambda)$. Then $c = 1$ and $\omega \in X^+(T_X)$ is p -restricted.

Proof. First suppose that $c > 1$ and observe that in this case the T_Y -weight $\lambda - 00200$ restricts to $\omega + 2\beta_2 - 2\beta_3$, which is neither under nor above ω, ω' , giving the existence of a third KX -composition factor of V , a contradiction. Consequently $c = 1$, in which case we have

$$\omega = (a + e)\omega_1 + (b + d)\omega_2 + (b + d + 2)\omega_3$$

and $\omega' = \omega^\theta$, where θ denotes the graph automorphism of X . If $\langle \omega, \beta_1 \rangle \geq p > 0$, then $ae \neq 0$, so that both $\lambda - \alpha_1$ and $\lambda - \alpha_5$ are T_Y -weights restricting to $\mu = \omega - \beta_1 \neq \omega'$. Therefore $m_{V|_X}(\mu) \geq 2$, while $m_{L_X(\omega)}(\mu) = 1$, so that μ occurs in a third KX -composition factor of V , contradicting our initial assumption. A similar argument shows that $0 \leq \langle \omega, \beta_2 \rangle < p$ if $p > 0$. Finally, suppose that $\langle \omega, \beta_3 \rangle \geq p > 0$. If $bd \neq 0$, the T_Y -weights $\lambda - 01000, \lambda - 00010$ both restrict to $\omega - \beta_2 \neq \omega'$, whose multiplicity in $L_X(\omega)$ equals 1. Without loss of generality, we may thus assume $d = 0$, so that $p \mid b + 2$ or $b + 1$. Here the T_Y -weights $\lambda - 01100, \lambda - 00110$ both restrict to $\nu = \omega - \beta_3$ and Lemma 2.3.19 yields

$$m_{V|_X}(\nu) = \begin{cases} 2 & \text{if } p \mid b + 2; \\ 3 & \text{if } p \mid b + 1, \end{cases}$$

while on the other hand $m_{L_X(\omega)}(\nu) = 1 - \epsilon_p(b + 2)$ by Theorem 2.3.2 and $m_{L_X(\omega')}(\nu) \leq 1$, yielding the existence of a third KX -composition factor of V as desired. \square

We are now able to prove a first direction of Theorem 5.1 in the case where $V = L_Y(\lambda)$ is an irreducible KG -module having p -restricted weight $\lambda \in X^+(T_Y)$ satisfying $\langle \lambda, \alpha_3 \rangle \neq 0$.

Proposition 5.2.3

Let $a, b, c, d, e \in \mathbb{Z}_{\geq 0}$, with $c > 0$, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 + e\lambda_5$. Suppose in addition that X has exactly two composition factors on V . Then (λ, p) is as in Table 5.1.

Proof. Let ω, ω' be as above and first observe that Lemma 5.2.2 implies $c = 1$. Also if $bd \neq 0$, then the T_Y -weights $\lambda - 01000, \lambda - 00010$ restrict to $\omega - \beta_2 \notin \omega'$, whose multiplicity in $L_X(\omega)$ equals 1, a contradiction. Without loss of generality, we shall then assume $d = 0$, that is, $\lambda = a\lambda_1 + b\lambda_2 + \lambda_3 + e\lambda_4$. Now the T_Y -weights $\lambda - 00110$ and $\lambda - 01100$ restrict to $\omega - \beta_3 \in \Lambda^+(\omega)$, so that

$$m_{V|X}(\omega - \beta_3) \geq \begin{cases} 2 & \text{if } b = 0 \text{ or } p \mid b + 2; \\ 3 & \text{otherwise.} \end{cases}$$

As $m_{L_X(\omega)}(\omega - \beta_3) = m_{L_X(\omega')}(\omega - \beta_3) = 1$, either $b = 0$ or $p \mid b + 2$, and since the latter cannot occur by Lemma 5.2.2, we get $b = 0$. Also, arguing as above shows that either $a = 0$ or $e = 0$ and without any loss of generality, we may assume $e = 0$ for the remainder of the proof, so that $\lambda = a\lambda_1 + \lambda_3$. Finally, if $a \neq 0$, then the T_Y -weights $\lambda - 11100, \lambda - 10110$ and $\lambda - 00111$ restrict to $\omega'' = \omega - \beta_1 - \beta_3$, so that Lemma 2.3.19 yields

$$m_{V|X}(\omega'') \geq \begin{cases} 4 & \text{if } p \mid a + 3; \\ 5 & \text{otherwise} \end{cases}$$

as well as $m_{L_X(\omega')}(\omega'') = m_{L_X(\omega)}(\omega'') = 2 - \epsilon_p(a + 3)$. Consequently ω'' occurs in a third KX -composition factor of V , forcing $\lambda = \lambda_3$ and an application of Lemma 5.2.2 then yields $p \neq 2$, thus completing the proof. \square

5.2.2 The case $\langle \lambda, \alpha_2 \rangle \neq \langle \lambda, \alpha_3 \rangle = 0$

Keep the notation introduced above and suppose that $b > c = 0$, i.e. $\lambda = a\lambda_1 + b\lambda_2 + d\lambda_4 + e\lambda_5$ and $\omega = (a + e)\omega_1 + (b + d)(\omega_2 + \omega_3)$. We start by considering the situation where $bd \neq 0$, in which case the T_Y -weights $\lambda - 01000, \lambda - 00010$ both restrict to $\omega' = \omega - \beta_2$. Therefore $m_{V|X}(\omega') \geq 2$, while $m_{L_X(\omega)}(\omega') = 1$ and since the only T_X -weight $\nu \in \Lambda(V|X)$ such that $\omega' \preceq \nu \prec \omega$ is ω' itself, we get that ω' affords the highest weight of a second KX -composition factor of V by Lemma 5.2.1, namely

$$L_X(\omega') = (a + e + 1)\omega_1 + (b + d - 2)\omega_2 + (b + d)\omega_3.$$

Lemma 5.2.4

Let λ, ω, ω' be as above, with $b \neq c = 0$, and suppose that X has exactly two composition factors on $V = L_Y(\lambda)$. Then $d = 0$.

Proof. Seeking a contradiction, assume $d > 0$. Here the T_Y -weights $\lambda - 01100, \lambda - 00110$ both restrict to $\omega'' = \omega - \beta_3 \notin \omega'$, so that $m_{V|X}(\omega'') \geq 2$, while $m_{L_X(\omega)}(\omega'') = 1$. Therefore ω'' occurs in a third KX -composition factor of V and the result follows. \square

We now consider the case where $\lambda = a\lambda_1 + b\lambda_2 + e\lambda_5$, with $b > 1$ (so that $p \neq 2$). Here the T_Y -weights $\lambda - 02100, \lambda - 01110$ both restrict to $\omega' = \omega - \beta_2 - \beta_3$, so that $m_{V|X}(\omega') \geq 2$, while $m_{L_X(\omega)}(\omega') = 1$ and thus ω' occurs in a second KX -composition factor of V . As above, an application of Lemma 5.2.1 then shows that ω' affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a + e + 2)\omega_1 + (b - 2)\omega_2 + (b - 2)\omega_3).$$

Lemma 5.2.5

Let λ, ω, ω' be as above, with $b > 1$ and $c = d = 0$. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. First observe that if $ae \neq 0$, then the T_Y -weights $\lambda - 10000, \lambda - 00001$ restrict to $\omega'' = \omega - \beta_1 \not\sim \omega'$ whose multiplicity in $L_X(\omega)$ equals 1. Therefore ω'' occurs in a third KX -composition factor of V and thus we may and will assume $ae = 0$ for the remainder of the proof.

1. We start by considering the case where $a \neq e = 0$. Here $\lambda = a\lambda_1 + b\lambda_2, \omega = a\omega_1 + b\omega_2 + b\omega_3$ and $\omega' = (a + 2)\omega_1 + (b - 2)\omega_2 + (b - 2)\omega_3$. One then checks that the T_Y -weights $\lambda - 12100, \lambda - 11110$, and $\lambda - 01111$ restrict to $\mu = \omega - \beta_1 - \beta_2 - \beta_3$. By Lemmas 2.3.19 and 5.1.3, we have

$$m_{V|X}(\mu) \geq \begin{cases} 3 & \text{if } p \mid a + b + 1; \\ 6 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\mu) \leq 4$ and $m_{L_X(\omega')}(\mu) \leq 1$ by Theorem 2.3.11. We may thus assume $p \mid a + b + 1$ (forcing ω' to be p -restricted). If $b > 2$, the T_Y -weights $\lambda - 14200, \lambda - 13210, \lambda - 12220, \lambda - 03211$ and $\lambda - 02221$ restrict to $\omega'' = \omega - \beta_1 - 2\beta_2 - 2\beta_3$, so that $m_{V|X}(\omega'') \geq 5$, while $m_{L_X(\omega)}(\omega'') \leq 2$ and $m_{L_X(\omega')}(\omega'') = 2$ by Lemma 5.1.7. Therefore ω'' occurs in a third KX -composition factor of V in this case; we deduce that $b = 2$, so that

$$\lambda = a\lambda_1 + 2\lambda_2, \omega = a\omega_1 + 2\omega_2 + 2\omega_3, \text{ and } \omega' = (a + 2)\omega_1,$$

with $p \mid a + 3$ (in particular $a > 1$). Here the T_Y -weights $\lambda - 24200, \lambda - 23210, \lambda - 22220, \lambda - 13211$, and $\lambda - 12221$ all restrict to $\omega'' = \omega - 2\beta_1 - 2\beta_2 - 2\beta_3$, while $m_{L_X(\omega')}(\omega'') = 1$ and $m_{L_X(\omega)}(\omega'') \leq 3$ by Lemmas 5.1.1 and 5.1.10 respectively. Consequently in the case where $a \neq e = 0$, X has more than two composition factors on V as desired.

2. Next assume $a = 0 \neq e$, so that $\lambda = b\lambda_2 + e\lambda_5, \omega = e\omega_1 + b\omega_2 + b\omega_3$ and also $\omega' = (e + 2)\omega_1 + (b - 2)\omega_2 + (b - 2)\omega_3$. Here the T_Y -weights $\lambda - 11000, \lambda - 01001$ and $\lambda - 00011$ restrict to $\omega'' = \omega - \beta_1 - \beta_2 \not\sim \omega'$, whose multiplicity in $L_X(\omega)$ is smaller than or equal to 2 by Theorem 2.3.11. Hence ω'' gives the existence of a third KX -composition factor of V and the assertion holds in this situation as well.

3. Finally, we suppose that $a = e = 0$, $b > 4$ and leave to the reader to check that $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ using [Lüb01, Appendices A.7, A.9] in the cases where $b = 2, 3$ or 4 . Here $\lambda = b\lambda_2$, $\omega = b\omega_2 + b\omega_3$, $\omega' = 2\omega_1 + (b-2)\omega_2 + (b-2)\omega_3$ and the T_Y -weights $\lambda - 24200$, $\lambda - 23210$, $\lambda - 22220$, $\lambda - 13211$, $\lambda - 12221$ and $\lambda - 02222$ restrict to $\omega'' = \omega - 2\beta_1 - 2\beta_2 - 2\beta_3$. Now Lemmas 5.1.1, 5.1.2 and 5.1.12 yield

$$m_{V|X}(\omega'') \geq \begin{cases} 6 & \text{if } p \mid b+1; \\ 13 & \text{otherwise,} \end{cases}$$

while on the other hand $m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 11 - 6\epsilon_p(b+1)$ by Lemmas 5.1.5 and 5.1.8, completing the proof. \square

We now assume $b = 1$ as well as $c = d = 0$, i.e. $\lambda = a\lambda_1 + \lambda_2 + e\lambda_5$, $\omega = (a+e)\omega_1 + \omega_2 + \omega_3$, and first consider the situation where $ae \neq 0$. Here the T_Y -weights $\lambda - 10000$ and $\lambda - 00001$ both restrict to $\omega' = \omega - \beta_1$, whose multiplicity inside $L_X(\omega)$ is smaller than or equal to 1. Therefore ω' occurs in a second KX -composition factor of V and since there is no weight $\nu \in \Lambda(V|X)$ such that $\omega' \prec \nu \prec \omega$, we get that ω' affords the highest weight of a second KX -composition factor of V by Lemma 5.2.1, namely

$$L_X(\omega') = L_X((a+e-2)\omega_1 + 2\omega_2 + 2\omega_3).$$

Lemma 5.2.6

Let λ , ω , ω' be as above, with $ae \neq 0$. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. First observe that the T_Y -weights $\lambda - 11000$, $\lambda - 01001$, $\lambda - 00011$ all restrict to $\mu_1 = \omega - \beta_1 - \beta_2 \in \Lambda^+(\omega)$, so that Lemma 2.3.19 yields

$$m_{V|X}(\mu_1) \geq \begin{cases} 3 & \text{if } p \mid a+2; \\ 4 & \text{otherwise,} \end{cases}$$

while an application of Theorem 2.3.11 gives $m_{L_X(\omega)}(\mu_1) + m_{L_X(\omega')}(\mu_1) \leq 3$. Hence we may and will assume $p \mid a+2$ for the remainder of the proof. Also if $e > 1$, the T_Y -weights $\lambda - 21000$, $\lambda - 11001$, $\lambda - 10011$, $\lambda - 01002$, and $\lambda - 00012$ all restrict to $\omega - 2\beta_1 - \beta_2$, whose multiplicity in both $L_X(\omega)$ and $L_X(\omega')$ is smaller than or equal to 2 by Theorem 2.3.11, giving the existence of a third KX -composition factor of V . Consequently, we may assume $e = 1$ from now on, so that $\lambda = a\lambda_1 + \lambda_2 + \lambda_5$, $\omega = (a+1)\omega_1 + \omega_2 + \omega_3$, and $\omega' = (a-1)\omega_1 + 2\omega_2 + 2\omega_3$. (Observe that in this situation both ω and ω' are p -restricted.) Here the T_Y -weights $\lambda - 12100$, $\lambda - 11110$, and $\lambda - 01111$ restrict to the T_X -weight $\mu_2 = \omega - \beta_1 - \beta_2 - \beta_3$, and by Lemma 2.3.19, we have

$$m_{V|X}(\mu_2) \geq \begin{cases} 5 & \text{if } p = 5; \\ 6 & \text{otherwise,} \end{cases}$$

while on the other hand Theorem 2.3.11 yields $m_{L_X(\omega)}(\mu_2) \leq 4$, $m_{L_X(\omega')}(\mu_2) \leq 1$. Therefore it remains to consider the case where $a = 3$ and $p = 5$. By [Lüb01, Appendices A.7, A.9], we have $\dim V = 1224$ and $\dim L_X(\omega') = 299$, while an application of Proposition 2.4.2 yields $\dim L_X(\omega) \leq 735$, thus completing the proof. \square

We now tackle the case where $ae = c = d = 0$ and $b = 1$, starting with the situation in which $a = 0 \neq e$. Here $\lambda = \lambda_2 + e\lambda_5$, $\omega = e\omega_1 + \omega_2 + \omega_3$, and one easily sees that the T_Y -weights $\lambda - 11000$, $\lambda - 01001$, and $\lambda - 00011$ restrict to $\omega' = \omega - \beta_1 - \beta_2$, whose multiplicity in $L_X(\omega)$ is smaller than or equal to 2, while $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in \Lambda(V|_X)$ such that $\omega' \prec \nu \prec \omega$. Therefore ω' affords the highest weight of a second composition factor of V by Lemma 5.2.1, namely

$$L_X(\omega') = L_X((e-1)\omega_1 + 2\omega_3).$$

Lemma 5.2.7

Let λ, ω, ω' be as above, with $a = c = d = 0$, $b = 1$ and $e \neq 0$. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. Notice that the T_Y -weights $\lambda - 11100$, $\lambda - 01101$, and $\lambda - 00111$ restrict to the T_X -weight $\omega'' = \omega - \beta_1 - \beta_3 \neq \omega'$, whose multiplicity in $L_X(\omega)$ is smaller than or equal to 2 by Theorem 2.3.11. Consequently ω'' occurs in a third composition factor of V , thus yielding the desired result. \square

Next we suppose that $a > 0$ and $c = d = e = 0$, that is, $\lambda = a\lambda_1 + \lambda_2$ and $\omega = a\omega_1 + \omega_2 + \omega_3$. Here the T_Y -weights $\lambda - 12100$, $\lambda - 11110$, $\lambda - 01111$ restrict to $\omega' = \omega - \beta_1 - \beta_2 - \beta_3$ and Lemma 2.3.19 yields

$$m_{V|X}(\omega') \geq \begin{cases} 3 & \text{if } p \mid a + 2; \\ 5 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\omega') \leq 4 - 2\epsilon_p(a + 2)$ by Lemma 5.1.7, showing that ω' occurs in a second KX -composition factor of V . One then easily checks that $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in \Lambda^+(V|_X)$ such that $\omega' \prec \nu \prec \omega$, so that ω' affords the highest weight of a KX -composition factor of V by Lemma 5.2.1, namely

$$L_X(\omega') = L_X(a\omega_1).$$

Lemma 5.2.8

Let λ, ω, ω' be as above, with $a > b = 1$ and $c = d = e = 0$. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. First observe that the T_Y -weights $\lambda - 22100$, $\lambda - 21110$, $\lambda - 11111$ restrict to the T_X -weight $\mu = \omega - 2\beta_1 - \beta_2 - \beta_3$. By Lemmas 2.3.19 and 5.1.4, we then have

$$m_{V|X}(\mu) \geq \begin{cases} 3 & \text{if } p \mid a + 2; \\ 7 & \text{otherwise,} \end{cases}$$

while Theorem 2.3.11 yields $m_{L_X(\omega)}(\mu) \leq 5$ as well as $m_{L_X(\omega')}(\mu) = 1$. We hence assume $p \mid a + 2$ for the remainder of the proof (in which case $a > 2$). Now the T_Y -weights $\lambda - 34200$, $\lambda - 33210$, $\lambda - 32220$, $\lambda - 23211$, $\lambda - 22221$, and $\lambda - 12222$ all restrict to $\omega'' = \omega - 3\beta_1 - 2\beta_2 - 2\beta_3$. Hence $m_{V|X}(\omega'') \geq 6$, while Lemmas 5.1.1 and 5.1.9 yield $m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 5$, thus completing the proof. \square

Thanks to Lemmas 5.2.4, 5.2.5, 5.2.6, 5.2.7 and 5.2.8, we are now able to prove a first direction of Theorem 5.1 in the case where $V = L_Y(\lambda)$ is an irreducible KY -module having p -restricted weight $\lambda \in X^+(T_Y)$ satisfying $\langle \lambda, \alpha_2 \rangle \neq \langle \lambda, \alpha_3 \rangle = 0$.

Proposition 5.2.9

Let $a, b, c, d, e \in \mathbb{Z}_{\geq 0}$, with $b \neq c = 0$, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 + e\lambda_5$. In addition, suppose that X has exactly two composition factors on V . Then λ and p are as in Table 5.1.

Proof. First observe that $d = 0$ by Lemma 5.2.4. Also, an application of Lemma 5.2.5 yields $b = 1$ as well. Moreover, by Lemmas 5.2.6, 5.2.7 and 5.2.8, we get that either $\lambda = \lambda_2$ or $\lambda = \lambda_1 + \lambda_2$. Assume the former case and observe that if $p \neq 2$, then X acts irreducibly on V by [Sei87, Theorem 1, Table 1 (I_4, I_5)], a contradiction. If on the other hand $\lambda = \lambda_1 + \lambda_2$ and $p = 5$, then one can check (using [Lüb01, Appendix A.7] and [Lüb01, Appendix A.9], for example) that $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, showing the existence of a third KX -composition factor of V . \square

5.2.3 Remaining cases and conclusion

We first consider the situation where $\lambda = a\lambda_1 + e\lambda_5$ for some $ae \neq 0$. Here $\omega = (a + e)\omega_1$ and one easily checks that $\omega' = \omega - \beta_1$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a + e - 2)\omega_1 + \omega_2 + \omega_3).$$

Proposition 5.2.10

Let $a, e \in \mathbb{Z}_{>0}$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + e\lambda_5 \in \Lambda^+(T_Y)$. In addition, suppose that X has exactly two composition factors on V . Then λ and p are as in Table 5.1.

Proof. Let ω, ω' be as above and first observe that if $a, e > 1$, then the T_Y -weights $\lambda - 20000$, $\lambda - 10001$, $\lambda - 00002$ restrict to $\omega - 2\beta_1$, whose multiplicity in both $L_X(\omega)$ and $L_X(\omega')$ is smaller than or equal to 1. Without any loss of generality, we thus assume $e = 1$, that is $\lambda = a\lambda_1 + \lambda_5$, $\omega = (a+1)\omega_1$, $\omega' = (a-1)\omega_1 + \omega_2 + \omega_3$. Now if $a > 2$, the T_Y -weights $\lambda - 32100$, $\lambda - 31110$, $\lambda - 22101$, $\lambda - 21111$, $\lambda - 11112$ restrict to $\omega'' = \omega - 3\beta_1 - \beta_2 - \beta_3$. By Lemma 2.3.19, $m_{V|_X}(\omega'') \geq 8$, while Theorem 2.3.11 yields $m_{L_X(\omega)}(\omega'') \leq 2$, $m_{L_X(\omega')}(\omega'') \leq 5$, a contradiction. Consequently, either $\lambda = \lambda_1 + \lambda_5$ or $\lambda = 2\lambda_1 + \lambda_5$, and we leave to the reader to complete the proof using [Lüb01, Appendix A.7] and [Lüb01, Appendix A.9], for example. \square

Finally, assume $a > 1$ and $e = 0$, so that $\lambda = a\lambda_1$, $\omega = a\omega_1$. Here the T_Y -weights $\lambda - 22100$, $\lambda - 21110$, $\lambda - 11111$ restrict to $\omega' = \omega - 2\beta_1 - \beta_2 - \beta_3$. Hence $m_{V|_X}(\omega') \geq 3$, while by Theorem 2.3.11, we have $m_{L_X(\omega)} \leq 2$. Also, since there is no weight $\nu \in \Lambda^+(V|_X)$ such that $\omega' \prec \nu \prec \omega$ and $m_{L_X(\omega)}(\nu) > 1$, we get that ω' affords the highest weight of a second KX -composition factor of V by Lemma 5.2.1, namely

$$L_X(\omega') = L_X((a-2)\omega_1).$$

Proposition 5.2.11

Let $a \in \mathbb{Z}_{>1}$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 \in \Lambda^+(T_Y)$. In addition, suppose that X has exactly two composition factors on V . Then λ and p are as in Table 5.1.

Proof. Let ω, ω' be as above and observe that if $a > 3$, then the T_Y -weights $\lambda - 44200$, $\lambda - 43210$, $\lambda - 42220$, $\lambda - 33211$, $\lambda - 32221$, and $\lambda - 22222$ all restrict to $\omega'' = \omega - 4\beta_1 - 2\beta_2 - 2\beta_3$, while Theorem 2.3.11 yields $m_{L_X(\omega)}(\omega'') \leq 3$ as well as $m_{L_X(\omega')}(\omega'') \leq 2$. Hence $a \leq 3$ and using [Lüb01, Appendices A.7, A.9], one checks that $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ if $a = 2$ and $p = 3$, thus completing the proof. \square

Thanks to Propositions 5.2.3, 5.2.9, 5.2.10 and 5.2.11, we are now able to give a proof of the main result of this chapter.

Proof of Theorem 5.1: Let K, Y, X be as in the statement of Theorem 5.1 and first suppose that X acts with exactly two composition factors on the irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight

$$\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3 + d\lambda_4 + e\lambda_5.$$

If $c > 0$, then Proposition 5.2.3 yields the desired result, so we may assume $c = 0$. Now if $b > 0$, an application of Proposition 5.2.9 shows that the assertion holds in this case as well, thus allowing us to assume $b = 0$. Finally Propositions 5.2.10 and 5.2.11 together with the fact that X acts irreducibly on $L_Y(\lambda_1)$ allow us to conclude.

In order to complete the proof, it remains to show that for every pair (λ, p) appearing in Table 5.1, X has exactly two composition factors on $V = L_Y(\lambda)$ and that $V|_X$ is completely reducible if and only if $(\lambda, p) \neq (\lambda_2, 2)$.

The first assertion can easily be proved using [Lüb01, Appendices A.7, A.9] together with Proposition 2.6.5 (in the case where $(\lambda, p) \neq (\lambda_2, 2)$). Finally, consider the irreducible KY -module $V = L_Y(\lambda_2)$ and observe that $L_Y(\lambda_2) \cong \bigwedge^2 V_Y(\lambda_1)$, so that $L_Y(\lambda_2)|_X \cong \bigwedge^2 V_X(\omega_1)$. By Proposition 2.6.3, the latter admits a Weyl filtration, yielding

$$L_Y(\lambda_2)|_X \cong V_X(\omega_2).$$

The latter being indecomposable, the proof is complete.

The case $SO_8(K) \subset SL_8(K)$

Let Y be a simply connected simple algebraic group of type A_7 over K and consider a subgroup X of type D_4 . Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_7\}$ denote a corresponding base of the root system $\Phi(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_7\}$ be the fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. Also let $\Pi(X) = \{\beta_1, \dots, \beta_4\}$ be a set of simple roots for X and let $\{\omega_1, \dots, \omega_4\}$ be the corresponding set of fundamental dominant weights for X . The A_3 -parabolic subgroup of X corresponding to the simple roots $\{\beta_1, \beta_2, \beta_3\}$ embeds in an $A_3 \times A_3$ -parabolic subgroup of Y , and up to conjugacy, we may assume that this gives $\alpha_1|_{T_X} = \alpha_7|_{T_X} = \beta_1$, $\alpha_2|_{T_X} = \alpha_6|_{T_X} = \beta_2$, and $\alpha_3|_{T_X} = \alpha_5|_{T_X} = \beta_3$. By considering the action of the Levi factors of these parabolics on the natural KY -module $L_Y(\lambda_1)$, we can deduce that $\alpha_4|_{T_X} = \beta_4 - \beta_3$. Finally, using [Hum78, Table 1, p.69] and the fact that $\lambda_1|_{T_X} = \omega_1$ yields

$$\lambda_7|_{T_X} = \omega_1, \quad \lambda_2|_{T_X} = \lambda_6|_{T_X} = \omega_2, \quad \lambda_3|_{T_X} = \lambda_5|_{T_X} = \omega_3 + \omega_4, \quad \lambda_4|_{T_X} = 2\omega_4. \quad (6.1)$$

As in the previous chapter, our goal here is to give a proof of Conjecture 4 in the case where $n = 4$. In order to do so, we first consider a suitable parabolic subgroup of X and use an inductive argument, based on Lemma 2.3.10 and Theorem 5.1, to reduce the number of possibilities for λ and p to be such that X has exactly two composition factors on the irreducible $V = L_Y(\lambda)$. We then proceed by a careful study of certain weight multiplicities in various irreducibles and assuming X has exactly two composition factors on an irreducible KY -module V , we get a relatively short list of possible candidates for V and p . Finally, we conclude by comparing dimensions as usual. For completeness, we record here the aforementioned conjecture for $n = 4$, restated as a Theorem.

Theorem 6.1

Let K, Y, X be as above, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if λ and p appear in Table 6.1, where we give λ up to graph automorphisms. Moreover, if (λ, p) is recorded in Table 6.1, then $V|_X$ is completely reducible if and only if $(\lambda, p) \neq (\lambda_3, 2)$.

λ	p	$V _X$	Dimensions
$\lambda_1 + \lambda_2$	$\neq 7$	$\omega_1 + \omega_2/\omega_1$	$160 - 56\delta_{p,3}, 8$
$\lambda_1 + \lambda_3$	$\neq 2, 3$	$\omega_1 + \omega_3 + \omega_4/\omega_2$	$350, 28$
$\lambda_1 + \lambda_6$	$\neq 2, 3$	$\omega_1 + \omega_2/\omega_3 + \omega_4$	$160 - 8\delta_{p,7}, 56$
$\lambda_1 + \lambda_7$	$\neq 2$	$2\omega_1/\omega_2$	$35, 28$
$2\lambda_1$	$\neq 2$	$2\omega_1/0$	$35, 1$
$2\lambda_1 + \lambda_3$	$= 5$	$2\omega_1 + \omega_3 + \omega_4/\omega_1 + \omega_2$	$904, 160$
$3\lambda_1$	$\neq 2, 3, 5$	$3\omega_1/\omega_1$	$112, 8$
λ_3	$= 2$	$\omega_3 + \omega_4/\omega_1$	$48, 8$
λ_4	$\neq 2$	$2\omega_3/2\omega_4$	$35, 35$

Table 6.1: The case $\mathrm{SO}_8(K) \subset \mathrm{SL}_8(K)$.

6.1 Preliminary considerations

Let G be a simple algebraic group over K with B, T, Φ, Π as usual, and consider an irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T)$. In this section we record some information on weight multiplicities in V for G of type A_n ($n \geq 3$) or D_4 over K , necessary to prove Theorem 6.1. We start by introducing a method of determining lower bounds for such multiplicities in the case where G is an arbitrary simple algebraic group over K .

Lemma 6.1.1

Let G be a simple algebraic group over K with $B, T, \Phi = \Phi^+ \sqcup \Phi^-, \Pi$ as usual, and consider an irreducible KG -module $V = L_G(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T)$. Also let $J = \{\gamma_1, \dots, \gamma_i\} \subset \Phi^+$ be such that $H = \langle U_{\pm\gamma_r} : 1 \leq r \leq i \rangle$ is a semisimple subgroup of G . Finally, let $\mu \in \Lambda^+(V)$ and write $\lambda' = \lambda|_{T_H}$, as well as $\mu' = \mu|_{T_H}$. Then $m_V(\mu) \geq m_{L_H(\lambda')}(\mu')$.

Proof. Let v^+ denote a maximal vector in V for B and set $U = \langle Hv^+ \rangle$. Clearly U is stable under the action of T and hence $m_V(\mu) \geq m_U(\mu)$. Since U is a homomorphic image of $V_G(\lambda)$ by [Jan03, II, 2.13 b)], the result follows. \square

6.1.1 Weight multiplicities for G of type A_n over K

Let K be an algebraically closed field of characteristic $p \geq 0$ and G a simple algebraic group of type A_n ($n \geq 3$) over K . Fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U is the unipotent radical of B , let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ denote a corresponding base of the root system Φ of G and let $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental dominant weights for T corresponding to our choice of base Π . Since our primary goal is to give a proof of Theorem 6.1, we could focus our attention on the cases where $2 \leq n \leq 7$. However, most of the results can easily be generalized and prove useful in the next chapter, hence we shall assume $n \geq 3$ arbitrary for the remainder of the section, unless specified otherwise. We start by considering an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_n$, where $n \geq 3$, $a, b \in \mathbb{Z}_{>0}$, and record a result similar to Lemma 2.3.19. We advise the reader to use the embedding $A_n \subset D_{n+3}$ as stated in Remark 5.1.11 in order to simplify the computations.

Lemma 6.1.2

Let V be as above, with $a > 3$, and let $\mu = \sigma - (3\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \epsilon_p(a + b + n - 1) \text{ch } L_G(\tau)$, where $\tau = \sigma - (\gamma_1 + \dots + \gamma_n)$, and

$$m_V(\mu) = \begin{cases} n - 1 & \text{if } p \mid a + b + n - 1; \\ n & \text{otherwise.} \end{cases}$$

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - (\gamma_1 + \dots + \gamma_n)$. We leave to the reader to check that

$$\nu_c^\mu(T_\sigma) = \nu_p(a + b + n - 1)\chi^\mu(\tau),$$

so that $m_V(\mu) = m_{V_G(\sigma)}(\mu)$ if $p \nmid a + b + n - 1$ by Proposition 2.7.8 and an application of Theorem 2.3.11 shows that the assertion holds in this situation. For the remainder of the proof, we assume $p \mid a + b + n - 1$. By Theorems 2.3.4 and 2.3.18, we have $\chi^\mu(\tau) = \text{ch } L_G(\tau)$ and thus $\nu_c^\mu(T_\sigma) = \nu_p(a + b + n - 1) \text{ch } L_G(\tau)$. Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8 and applying Theorem 2.3.11 completes the proof. \square

We next consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_j$, where $a \in \mathbb{Z}_{>1}$, $2 \leq j < n$, and prove the following generalization of Lemma 5.1.4.

Lemma 6.1.3

Let V be as above and let $\mu_1 = \sigma - (2\gamma_1 + \dots + 2\gamma_j + \gamma_{j+1})$, $\mu_2 = \mu_1 - \gamma_1$. Then μ_1 (respectively, μ_2 if $a > 3$) is dominant, $\chi^{\mu_1}(\sigma) = \chi^{\mu_2}(\sigma) = \text{ch } V + \epsilon_p(a + j) \text{ch } L_G(\tau)$, where $\tau = \sigma - (\gamma_1 + \dots + \gamma_j)$, and

$$m_V(\mu_1) = m_V(\mu_2) = \begin{cases} j(j-1)/2 & \text{if } p \mid a + j; \\ j(j+1)/2 & \text{otherwise.} \end{cases}$$

Proof. Fix $i = 1$ or 2 and let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. Also write $\tau = \lambda - (\gamma_1 + \dots + \gamma_j) = (a-1)\sigma_1 + \sigma_{j+1} \in X^+(T)$, and first check that

$$\nu_c^{\mu_i}(T_\sigma) = \nu_p(a+j)(\chi^{\mu_i}(\tau) - \chi^{\mu_i}(\mu_1)).$$

Therefore $m_V(\mu_i) = m_{V_G(\sigma)}(\mu_i)$ if $p \nmid a+j$ by Proposition 2.7.8 and an application of Theorem 2.3.11 shows that the assertion holds in this situation. We thus assume $p \mid a+j$ for the remainder of the proof, in which case Lemma 2.3.19 yields

$$\chi^{\mu_i}(\tau) = \text{ch } L_G(\tau) + \text{ch } L_G(\mu_1),$$

so that $\nu_c^{\mu_i}(T_\sigma) = \nu_p(a+j) \text{ch } L_G(\tau)$. Consequently $\chi^{\mu_i}(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8, so $m_V(\mu_i) = m_{V_G(\sigma)}(\mu_i) - m_{L_G(\tau)}(\mu_i)$. Finally $m_{L_G(\tau)}(\mu_i) = j$ by Lemma 2.3.19 and the result then follows from Theorem 2.3.11. \square

In the following statement, we assume $n \geq 4$ and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_j$, where $a \in \mathbb{Z}_{>2}$ and $2 \leq j < n-1$.

Lemma 6.1.4

Let V be as above and let $\mu = \sigma - (3\gamma_1 + \dots + 3\gamma_j + 2\gamma_{j+1} + \gamma_{j+2})$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \epsilon_p(a+j) \text{ch } L_G(\tau_1)$, where $\tau_1 = \sigma - (\gamma_1 + \dots + \gamma_j)$, and

$$m_V(\mu) = \begin{cases} j(j-1)(j+1)/6 & \text{if } p \mid a+j; \\ j(j+1)(j+2)/6 & \text{otherwise.} \end{cases}$$

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ be the series given by Proposition 2.7.4 and write $\tau_1 = \lambda - (\gamma_1 + \dots + \gamma_j) = (a-1)\sigma_1 + \sigma_{j+1}$, $\tau_2 = \tau_1 - (\gamma_1 + \dots + \gamma_{j+1}) = (a-2)\sigma_1 + \sigma_{j+2}$. One first checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+j)(\chi^\mu(\tau_1) - \chi^\mu(\tau_2) + \chi^\mu(\mu))$, so that $m_V(\mu) = m_{V_G(\sigma)}(\mu)$ if $p \nmid a+j$ by Proposition 2.7.8, in which case Theorem 2.3.11 yields the desired assertion. For the remainder of the proof, we thus assume $p \mid a+j$ and first observe that Lemmas 2.3.19 and 6.1.3 respectively yield

$$\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2) + \text{ch } L_G(\mu), \quad \chi^\mu(\tau_1) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2).$$

Therefore $\nu_c^\mu(T_\sigma) = \nu_p(a+j) \text{ch } L_G(\tau_1)$ and $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1)$ by Lemma 2.3.19 and Proposition 2.7.8. An application of Theorem 2.3.11 then yields $m_{V_G(\sigma)}(\mu) = \frac{1}{6}j(j+1)(j+2)$, while $m_{L_G(\tau_1)}(\mu) = \frac{1}{2}j(j+1)$ by Lemma 6.1.3, completing the proof. \square

In the remainder of this section, we focus our attention on an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_j + c\sigma_n \in X^+(T)$ where $n \geq 3$, $a, b, c \in \mathbb{Z}_{>0}$ and $2 \leq j < n$.

Lemma 6.1.5

Let V be as above and assume p divides both $a + b + j - 1$ and $b + c + n - j$. Also let $\mu = \sigma - (\gamma_1 + \cdots + \gamma_n)$ and write $\tau_1 = \sigma - (\gamma_1 + \cdots + \gamma_j)$ as well as $\tau_2 = \sigma - (\gamma_j + \cdots + \gamma_n)$. Then $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2)$, $\text{m}_{V_G(\sigma)}(\mu) = j(n - j + 1)$ and $\text{m}_V(\mu) = (n - j)(j - 1) + 1$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \cdots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and let $\tau_1 = (a - 1)\sigma_1 + (b - 1)\sigma_j + \sigma_{j+1} + c\sigma_n$, $\tau_2 = a\sigma_1 + \sigma_{j-1} + (b - 1)\sigma_j + (c - 1)\sigma_n$ be as above. One then checks that $\nu_c^\mu(T_\sigma) = \nu_p(a + b + j - 1)\chi^\mu(\tau_1) + \nu_p(b + c + n - j)\chi^\mu(\tau_2)$, while Lemma 2.3.19 yields $\chi^\mu(\tau_1) = \text{ch } L_G(\tau_1)$ and $\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2)$, so that

$$\nu_c^\mu(T_\sigma) = \nu_p(a + b + j - 1) \text{ch } L_G(\tau_1) + \nu_p(b + c + n - j) \text{ch } L_G(\tau_2).$$

Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2)$ by Lemma 2.3.19 and Proposition 2.7.8 and thus $\text{m}_V(\mu) = \text{m}_{V_G(\sigma)}(\mu) - \text{m}_{L_G(\tau_1)}(\mu) - \text{m}_{L_G(\tau_2)}(\mu)$. An application of Theorem 2.3.11 then yields $\text{m}_{V_G(\sigma)}(\mu) = j(n - j + 1)$, while $\text{m}_{L_G(\tau_1)}(\mu) = n - j$ and $\text{m}_{L_G(\tau_2)}(\mu) = j - 1$ by Lemma 2.3.19, from which the result follows. \square

Let K, G and V be as above and assume $j = 2$, that is, $V = L_G(\sigma)$ is an irreducible KG -module with p -restricted weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_n$, where $a, b, c \in \mathbb{Z}_{>0}$. In the next results, we investigate the multiplicity of the dominant T -weight $\mu = \sigma - (\gamma_1 + \cdots + \gamma_n)$ in V without the congruence conditions of the previous Lemma, using information on the structure of $V_G(\sigma)$ as an \mathcal{L} -module, where $\mathcal{L} = \mathcal{L}(G)$ denotes the Lie algebra of G . Let then $\mathcal{B} = \{e_\gamma, f_\gamma, h_{\gamma_i} : \gamma \in \Phi^+, 1 \leq i \leq n\}$ be a standard Chevalley basis of \mathcal{L} as in Section 2.5.1. By (2.14) and our choice of ordering on Φ^+ , the weight space $V_G(\sigma)_\mu$ is spanned by

$$\begin{aligned} & \{f_{1,n}v^\sigma\} \cup \{f_{\gamma_1}f_{2,r}f_{r+1,n}v^\sigma\}_{2 \leq r \leq n-1} \\ & \cup \{f_{1,s}f_{s+1,n}v^\sigma\}_{1 \leq s \leq n-1}, \end{aligned} \tag{6.2}$$

where $v^\sigma \in V_G(\sigma)_\sigma$ is a maximal vector in $V_G(\sigma)$ for B (and thus for the corresponding Borel subalgebra \mathfrak{b} of \mathcal{L} as well). Applying Theorem 2.3.11 then yields $\text{m}_{V_G(\sigma)}(\mu) = 2(n - 1)$, forcing the generating elements of (6.2) to be linearly independent, so that the following holds.

Proposition 6.1.6

Let G be a simple algebraic group of type A_n over K and consider the dominant T -weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_n$, where $a, b, c \in \mathbb{Z}_{>0}$. Also let $\mu = \sigma - (\gamma_1 + \cdots + \gamma_n)$. Then μ is dominant and the set (6.2) forms a basis of the weight space $V_G(\sigma)_\mu$.

We now study the relation between the quintuple (a, b, c, n, p) and the existence of a maximal vector in $V_G(\sigma)_\mu$ for \mathfrak{b} . For $A = (A_r)_{1 \leq r \leq 2(n-1)} \in K^{2(n-1)}$, we set

$$w(A) = A_1 f_{1,n}v^\sigma + \sum_{r=2}^{n-1} A_r f_{\gamma_1} f_{2,r} f_{r+1,n}v^\sigma + \sum_{s=1}^{n-1} A_{n+s-1} f_{1,s} f_{s+1,n}v^\sigma. \tag{6.3}$$

Lemma 6.1.7

Let G , σ , μ be as in the statement of Proposition 6.1.6 and adopt the notation of (6.3). Then the following assertions are equivalent.

1. There exists $0 \neq A \in K^{2(n-1)}$ such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$.
2. There exists $A \in K^{2n-3} \times K^*$ such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$.
3. The divisibility condition $p \mid a + b + c + n - 1$ is satisfied.

Furthermore, if $0 \neq A \in K^{2(n-1)}$ is such that $e_\gamma w(A) = 0$ for every simple root $\gamma \in \Pi$, then $A \in \langle (-a+1)c, 1, \dots, 1, -c, a+1, \dots, a+1 \rangle_K$. In particular, the subspace of $V_G(\sigma)$ spanned by all maximal vectors in $V_G(\sigma)_\mu$ for \mathbf{b} is at most 1-dimensional.

Proof. Let $A = (A_r)_{1 \leq r \leq 2(n-1)} \in K^{2(n-1)}$ and set $w = w(A)$. Then applying Lemma 2.5.3 yields

$$e_{\gamma_1} w = (-A_1 + (a+1)A_n) f_{2,n} v^\sigma + \sum_{r=2}^{n-1} ((a+1)A_r - A_{n+r-1}) f_{2,r} f_{r+1,n} v^\sigma,$$

$$e_{\gamma_2} w = \left((b+1)A_2 + \sum_{r=3}^{n-1} A_r - A_n + A_{n+1} \right) f_{\gamma_1} f_{3,n} v^\sigma,$$

while for every $3 \leq r \leq n-1$, we get

$$e_{\gamma_r} w = (-A_{r-1} + A_r) f_{\gamma_1} f_{2,r-1} f_{r+1,n} v^\sigma + (-A_{n+r-2} + A_{n+r-1}) f_{1,r-1} f_{r+1,n} v^\sigma,$$

as well as

$$e_{\gamma_n} w = (A_1 + cA_{2(n-1)}) f_{1,n-1} v^\sigma + (cA_{n-1} + A_n) f_{\gamma_1} f_{2,n-1} v^\sigma.$$

One checks that $f_{\gamma_1} f_{3,n} v^\sigma \neq 0$ and that each of the lists $\{f_{2,n} v^\sigma, f_{2,r} f_{r+1,n} v^\sigma : 2 \leq r < n\}$, $\{f_{\gamma_1} f_{2,r-1} f_{r+1,n} v^\sigma, f_{1,r-1} f_{r+1,n} v^\sigma\}$ ($2 \leq r \leq n-1$) and $\{f_{1,n-1} v^\sigma, f_{\gamma_1} f_{2,n-1} v^\sigma\}$ is linearly independent. Therefore $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$ if and only if A is a solution to the system of equations

$$\left\{ \begin{array}{l} A_1 = (a+1)A_n \\ A_{n+r-1} = (a+1)A_r \text{ for every } 2 \leq r \leq n-1 \\ (b+1)A_2 = -\sum_{r=3}^{n-1} A_r + A_n - A_{n+1} \\ A_{r-1} = A_r \text{ for every } 3 \leq r \leq n-1 \\ A_{s-1} = A_s \text{ for every } n+2 \leq s \leq 2(n-1) \\ A_1 = -cA_{2(n-1)} \\ A_n = -cA_{n-1}. \end{array} \right. \quad (6.4)$$

Now one easily sees that (6.4) admits a non-trivial solution $A \in K^{2(n-1)}$ if and only if $p \mid a + b + c + n - 1$ (showing that 1 and 3 are equivalent), in which case

$$A \in \langle \langle -(a+1)c, \underbrace{1, \dots, 1}_{n-2}, -c, \underbrace{a+1, \dots, a+1}_{n-2} \rangle \rangle_K$$

(so that 1 and 2 are equivalent), completing the proof. \square

Let $\tau = \sigma - \gamma_1 - \gamma_2 \in X^+(T)$ and assume $p \mid a + b + 1$. Then by (4.7), the element $u^\tau = f_{1,2}v^\sigma - b^{-1}f_{\gamma_1}f_{\gamma_2}v^\sigma$ is a maximal vector in $V_G(\sigma)_\tau$ for B (hence for \mathfrak{b} as well).

Lemma 6.1.8

Set $U = \langle Gu^\tau \rangle \subset \text{rad}(\sigma)$, where σ and u^τ are as above, and let $\mu = \sigma - (\gamma_1 + \dots + \gamma_n)$. Then $m_U(\mu) = n - 2$.

Proof. Write $v_1 = f_{1,n}v^\sigma$, $v_r = f_{\gamma_1}f_{2,r}f_{r+1,n}v^\sigma$ for $2 \leq r \leq n - 1$, and $v_{n+s-1} = f_{1,s}f_{s+1,n}v^\sigma$ for every $1 \leq s \leq n - 1$. Using Lemma 2.5.3, one easily checks that we have

$$\begin{aligned} f_{3,n}u^\tau &= v_1 - b^{-1}v_2 - b^{-1}v_n + v_{n+1}, \\ f_{3,r}f_{r+1,n}u^\tau &= b^{-1}v_2 - b^{-1}v_r - v_{n+1} + v_{n+r-1}, \end{aligned} \quad (6.5)$$

for every $3 \leq r \leq n - 1$. Those elements are clearly independent by Proposition 6.1.6 and thus $m_{\langle \mathcal{L}u^\tau \rangle}(\mu) = n - 2$. Since U is an image of $V_G(\tau)$ containing $\langle \mathcal{L}u^\tau \rangle$ and $m_{V_G(\tau)}(\mu) = n - 2$ by Theorem 2.3.11, the proof is complete. \square

Let U be as in the statement of Lemma 6.1.8, write $\overline{V_G(\sigma)} = V_G(\sigma)/U$ and for $v \in V_G(\sigma)$, denote by \bar{v} the class of v in $\overline{V_G(\sigma)}$. Using (6.5), one easily checks that $\bar{v}_{n+r-1} = -\bar{v}_1 + b^{-1}\bar{v}_r + b^{-1}\bar{v}_n$ for every $2 \leq r \leq n - 1$ and thus $\overline{V_G(\sigma)}_\mu = \langle \bar{v}_1, \dots, \bar{v}_n \rangle_K$. For $A = (A_r)_{r=1}^n \in K^n$, we write

$$\bar{w}(A) = A_1 f_{1,n} \bar{v}^\sigma + \sum_{r=2}^{n-1} A_r f_{\gamma_1} f_{2,r} f_{r+1,n} \bar{v}^\sigma + A_n f_{\gamma_1} f_{2,n} \bar{v}^\sigma.$$

Lemma 6.1.9

Let $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_n$ be such that $abc \neq 0$, $p \mid a + b + c + n - 1$, but $p \nmid b + c + n - 2$. Also consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight σ and let $\mu_1 = \sigma - (\gamma_1 + \dots + \gamma_n)$, $\mu_2 = \mu_1 - \gamma_1$. Then $\chi^{\mu_i}(\sigma) = \text{ch } V + \epsilon_p(a + b + 1) \text{ch } L_G(\tau) + \text{ch } L_G(\mu_1)$, where $\tau = \sigma - \gamma_1 - \gamma_2$. Moreover $m_{V_G(\sigma)}(\mu_1) = m_{V_G(\sigma)}(\mu_2) = 2(n - 2)$ and

$$m_V(\mu_1) = m_V(\mu_2) = \begin{cases} n & \text{if } p \mid a + b + 1; \\ 2n - 3 & \text{otherwise.} \end{cases}$$

Proof. Fix $i = 1$ or 2 , let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - \gamma_1 - \gamma_2$. We leave to the reader to check that

$$\nu_c^{\mu_i}(T_\sigma) = \nu_p(a+b+1)\chi^{\mu_i}(\tau) + \nu_p(a+b+c+n-1)\chi^{\mu_i}(\mu_1) \quad (6.6)$$

and that by Lemma 2.3.19, we have $\chi^{\mu_i}(\tau) = \text{ch } L_G(\tau) + \epsilon_p(c+n-2) \text{ch } L_G(\mu_1)$. Now if $p \nmid a+b+1$, then Proposition 2.7.8 shows that μ_1 affords the highest weight of a composition factor of $V_G(\sigma)$, while $[V_G(\sigma), L_G(\nu)] = 0$ for every $\mu_1 \neq \nu \in X^+(T)$ such that $\mu_2 \preceq \nu \prec \sigma$. Consequently, there exists a maximal vector in $V_G(\sigma)_{\mu_1}$ for B and an application of Lemma 6.1.7 yields $[V_G(\sigma), L_G(\mu_1)] = 1$, so that $m_V(\mu_i) = m_{V_G(\sigma)}(\mu_i) - 1$. (Indeed, $m_{V(\mu_1)}(\mu_2) = 1$.) Proposition 2.3.15 then yields the result in this situation and we may assume $p \mid a+b+1$ for the remainder of the proof. Here (6.6) can be rewritten as

$$\nu_c^{\mu_i}(T_\sigma) = \nu_p(a+b+1) \text{ch } L_G(\tau) + \nu_p(a+b+c+n-1) \text{ch } L_G(\mu_1),$$

in which case each of τ and μ_1 affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8, while $[V_G(\sigma), L_G(\nu)] = 0$ (and hence $[\overline{V_G(\sigma)}, L_G(\nu)] = 0$ as well) for every T -weight $\tau, \mu_1 \neq \nu \in X^+(T)$ such that $\mu_2 \preceq \nu \prec \sigma$. Now if μ_1 affords the highest weight of a composition factor of $\overline{V_G(\sigma)}$, then there exists $A = (A_r)_{r=1}^n \in K^n$ such that $\bar{w}(A)$ is a maximal vector in $\overline{V_G(\sigma)}$ for B_G . Now applying Lemma 2.5.3 yields

$$e_{\gamma_1} \bar{w}(A) = (-A_1 + (a+1)A_n) f_{2,n} \bar{v}^\sigma + (a+1) \sum_{r=2}^{n-1} A_r f_{2,r} f_{r+1,n} \bar{v}^\sigma$$

and as $p \nmid (a+1)(b+c+n-2)$, one gets that the elements $f_{2,n} \bar{v}^\sigma, f_{2,r} f_{r+1,n} \bar{v}^\sigma$ ($2 \leq r < n$) are linearly independent, so that $A_r = 0$ for every $2 \leq r < n$ as well as $A_1 = (a+1)A_n$. Finally, one checks that $e_{\gamma_2} \bar{w}(A) = -A_n f_{\gamma_1} f_{3,n} \bar{v}^\sigma$ and hence $A_1 = A_n = 0$. Consequently $[\overline{V_G(\sigma)}, L_G(\mu_1)] = 0$ and thus

$$[V_G(\sigma), L_G(\mu_1)] = [U, L_G(\mu_1)]. \quad (6.7)$$

Finally, notice that $m_{L_G(\tau)}(\mu_1) = n-3$ by Lemma 2.3.19, and an application of Lemma 6.1.8 yields $[U, L_G(\mu_1)] = 1$. The result then follows from (6.7). \square

Proposition 6.1.10

Let G be a simple algebraic group of type A_n ($n \geq 3$) over K and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_n \in X^+(T)$, where $a, b, c \in \mathbb{Z}_{>0}$. Also set $\tau_1 = \sigma - \gamma_1 - \gamma_2$, $\tau_2 = \sigma - (\gamma_2 + \dots + \gamma_n)$ and write $z_1 = a+b+1$, $z_2 = b+c+n-2$, $z_3 = a+b+c+n-1$. Finally, consider $\mu = \sigma - (\gamma_1 + \dots + \gamma_n) \in X(T)$. Then μ is dominant, $m_V(\mu) = 2(n-1) - \epsilon_p(z_1)(n-2) - \epsilon_p(z_2) - \epsilon_p(z_3) + \epsilon_p(z_1)\epsilon_p(z_3)$ and

$$\chi^\mu(\sigma) = \text{ch } V + \epsilon_p(z_1) \text{ch } L_G(\tau_1) + \epsilon_p(z_2) \text{ch } L_G(\tau_2) + \epsilon_p(z_3) \text{ch } L_G(\mu).$$

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. One first checks that

$$\nu_c^\mu(T_\sigma) = \nu_p(z_1)\chi^\mu(\tau_1) + \nu_p(z_2)\chi^\mu(\tau_2) + \nu_p(z_3)\chi^\mu(\mu)$$

and observes that if $p \nmid z_1 z_2 z_3$, then $\chi^\mu(\sigma) = \text{ch } V$ by Proposition 2.7.8. For the remainder of the proof, we may and will assume the existence of $1 \leq i \leq 3$ such that $p \mid z_i$.

1. We first consider the case where $p \mid z_1$. If $p \nmid z_2 z_3$, then we have $\nu_c^\mu(T_\sigma) = \nu_p(z_1)\chi^\mu(\tau_1)$ and applying Lemma 2.3.19 yields $\chi^\mu(\tau_1) = \text{ch } L_G(\tau_1)$, so that $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1)$ by Lemma 2.3.19 and Proposition 2.7.8. If on the other hand $p \mid z_2$ (and so $p \nmid z_3$), then the assertion on $\chi^\mu(\sigma)$ immediately follows from Lemma 6.1.5. Finally, if $p \mid z_3$ (and so $p \nmid z_2$), then $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\mu)$ by Lemma 6.1.9.
2. Next assume $p \nmid z_1$ and first suppose that $p \mid z_2$ (in which case one easily sees that $p \nmid z_3$). Then $\nu_c^\mu(T_\sigma) = \nu_p(z_2)\chi^\mu(\tau_2)$ and Lemma 2.3.19 yields $\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2)$, so that $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_2)$ in this case by Lemma 2.3.19 and Proposition 2.7.8. If on the other hand $p \nmid z_2$ and $p \mid z_3$, then the assertion on $\chi^\mu(\sigma)$ follows from Lemma 6.1.9.

The result on $m_V(\mu)$ is a direct consequence of the assertion on the decomposition of $\chi^\mu(\sigma)$ in terms of characters of irreducibles. We leave the details to the reader. \square

We next consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n \in X^+(T)$, where $n > 3$, $a, b \in \mathbb{Z}_{>0}$, and investigate the multiplicity of the dominant T -weight $\mu = \sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$.

Lemma 6.1.11

Let V be as above, with $n > 3$, and suppose that p divides both $a + 3$ and $b + n + 1$. Also let $\mu = \sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$ and set $\tau = \sigma - \gamma_1 - \gamma_2$. Then $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$, $m_{V_G(\sigma)}(\mu) = 3n - 4$ and $m_V(\mu) = n$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. We leave to the reader to check that $\nu_c^\mu(T_\sigma) = \nu_p(a + 3)\chi^\mu(\tau)$ and $\chi^\mu(\tau) = \text{ch } L_G(\tau)$. (Use Lemma 2.3.19 together with Corollary 2.7.3 to prove the latter assertion.) Therefore $\nu_c^\mu(T_\sigma) = \nu_p(a + 3) \text{ch } L_G(\tau)$ and $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8. Consequently $m_V(\mu) = m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$ and an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 3n - 4$, while $m_{L_G(\tau)}(\mu) = 2(n - 2)$ by Proposition 6.1.10, completing the proof. \square

Using Lemma 6.1.1 together with Lemma 6.1.11, we now give a lower bound for the multiplicity of $\sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$ in a given irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n$, where $a, b \in \mathbb{Z}_{>0}$, under certain divisibility conditions.

Lemma 6.1.12

Let V as above, with $n > 3$ and assume p divides both $a + 3$ and $b + n$. Also consider the T -weight $\mu = \sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \cdots + \gamma_n)$. Then $m_V(\mu) \geq n - 1$.

Proof. Let $J = \{\eta_1, \dots, \eta_{n-1}\}$, where $\eta_r = \gamma_r$ for every $1 \leq r < n - 1$, $\eta_{n-1} = \gamma_{n-1} + \gamma_n$, so that $H = \langle U_{\pm\eta_r} : 1 \leq r \leq n - 1 \rangle$ is simple of type A_{n-1} over K , and denote by $\{\sigma'_1, \dots, \sigma'_{n-1}\}$ the set of fundamental weights corresponding to our choice of base. Adopting the latter notation, we get $\sigma' = \sigma|_{T_H} = a\sigma'_1 + 2\sigma'_2 + b\sigma'_{n-1}$, $\mu' = \mu|_{T_H} = \sigma' - (\eta_1 + 2\eta_2 + \eta_3 + \cdots + \eta_{n-1})$, and as $p \mid b + (n - 1) + 1$, Lemma 6.1.11 applies, yielding $m_{L_H(\sigma')}(\mu') = n - 1$. The result then follows from Lemma 6.1.1. \square

We are now able to determine the exact multiplicity of $\mu = \sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \cdots + \gamma_n)$ in an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n$, where $a, b \in \mathbb{Z}_{>0}$, under the divisibility conditions of Lemma 6.1.12.

Proposition 6.1.13

Let G be a simple algebraic group of type A_n ($n \geq 3$) over K and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n \in X^+(T)$, where $a, b \in \mathbb{Z}_{>0}$. Also assume p divides both $a + 3$ and $b + n$ and let $\mu = \sigma - (\gamma_1 + 2\gamma_2 + \gamma_3 + \cdots + \gamma_n)$. Then μ is dominant and $m_V(\mu) = n - 1$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the series given by Proposition 2.7.4 and write $\tau_1 = \sigma - (\gamma_1 + \gamma_2) =$ as well as $\tau_2 = \sigma - (\gamma_2 + \cdots + \gamma_n)$. One first checks that $\nu_c^\mu(T_\sigma) = \nu_p(a + 3)\chi^\mu(\tau_1) + \nu_p(b + n)\chi^\mu(\tau_2)$ and that Lemmas 2.3.19 and 6.1.9 respectively yield

$$\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2) + \text{ch } L_G(\mu), \quad \chi^\mu(\tau_1) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\mu),$$

so that $[V_G(\sigma), L_G(\mu)] \neq 0$ by Proposition 2.7.8. Applying Theorem 2.3.11, one then gets $m_{V_G(\sigma)}(\mu) = 3n - 4$, while $m_{L_G(\tau_1)}(\mu) = 2n - 5$ by Lemma 6.1.9 and $m_{L_G(\tau_2)}(\mu) = 1$ by Lemma 2.3.19. Therefore $m_{L_G(\sigma)}(\mu) \leq n - 1$ and an application of Lemma 6.1.12 then completes the proof. \square

We now aim at proving a result similar to Proposition 6.1.13 in the situation where $\mu = \sigma - (2\gamma_1 + 2\gamma_2 + \gamma_3 + \cdots + \gamma_n)$. We start our investigation by the following two preliminary results.

Lemma 6.1.14

Assume $n \geq 4$ and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_2 + \sigma_3 + b\sigma_n$, where $a, b \in \mathbb{Z}_{>0}$. Also assume $p \neq 2, 3$ divides both $a + 4$ and $b + n + 1$, and let $\mu = \sigma - (\gamma_1 + \cdots + \gamma_n)$. Then $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$, where $\tau = \sigma - \gamma_1 - \gamma_2 - \gamma_3$, $m_{V_G(\sigma)}(\mu) = 4(n - 2)$ and $m_V(\mu) = 3n - 5$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. The reader first checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+4)\chi^\mu(\tau)$, while an application of Lemma 2.3.19 yields $\chi^\mu(\tau) = \text{ch } L_G(\tau)$, hence

$$\nu_c^\mu(T_\sigma) = \nu_p(a+4) \text{ch } L_G(\tau).$$

Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 6.1.9 and Proposition 2.7.8, so that $m_V(\mu) = m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Applying Theorem 2.3.11 gives $m_{V_G(\sigma)}(\mu) = 4(n-2)$, while $m_{L_G(\tau)}(\mu) = n-3$ by Lemma 2.3.19, completing the proof. \square

Lemma 6.1.15

Assume $n > 3$ and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n$, where $a \in \mathbb{Z}_{>1}$, $b \in \mathbb{Z}_{>0}$. Also assume p divides both $a+3$ and $b+n+1$, and let $\mu = \sigma - (2\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$. Then $m_{V_G(\sigma)}(\mu) = 4n-5$ and $m_V(\mu) = n$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - \gamma_1 - \gamma_2$, $\tau_2 = \tau_1 - \gamma_1 - \gamma_2 - \gamma_3$. One starts by checking that $\nu_c^\mu(T_\sigma) = \nu_p(a+3)(\chi^\mu(\tau_1) - \chi^\mu(\tau_2))$ while Lemmas 2.3.19 and 6.1.14 respectively yield

$$\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2), \quad \chi^\mu(\tau_1) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2),$$

so that $\nu_c^\mu(T_\sigma) = \nu_p(a+3) \text{ch } L_G(\tau_1)$. Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1)$ by Lemma 2.3.19 and Proposition 2.7.8. Finally, an application of Theorem 2.3.11 gives $m_{V_G(\sigma)}(\mu) = 4n-5$, while $m_{L_G(\tau_1)}(\mu) = 3n-5$ by Lemma 6.1.14, completing the proof. \square

Using Lemma 6.1.1 together with Lemma 6.1.15, we now give a lower bound for the multiplicity of $\sigma - (2\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$ in a given irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n$, where $a > 1$, $b > 0$, under the assumption that p divides both $a+3$ and $b+n$.

Proposition 6.1.16

Let G be a simple algebraic group of type A_n ($n > 4$) over K and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + 2\sigma_2 + b\sigma_n \in X^+(T)$, where $a, b \in \mathbb{Z}_{>0}$. Also assume p divides both $a+3$ and $b+n$ and let $\mu = \sigma - (2\gamma_1 + 2\gamma_2 + \gamma_3 + \dots + \gamma_n)$. Then μ is dominant and $m_V(\mu) \geq n-1$.

Proof. Let $J = \{\eta_1, \dots, \eta_{n-1}\}$, where $\eta_r = \gamma_r$ for every $1 \leq r < n-1$, $\eta_{n-1} = \gamma_{n-1} + \gamma_n$, so that $H = \langle U_{\pm\eta_r} : 1 \leq r \leq n-1 \rangle$ is simple of type A_{n-1} over K , and denote by $\{\sigma'_1, \dots, \sigma'_{n-1}\}$ the set of fundamental weights corresponding to our choice of base. Adopting the latter notation, we get $\sigma' = \sigma|_{T_H} = a\sigma'_1 + 2\sigma'_2 + b\sigma_{n-1}$, $\mu' = \mu|_{T_H} = \sigma' - (2\eta_1 + 2\eta_2 + \eta_3 + \dots + \eta_{n-1})$, and as $p \mid b + (n-1) + 1$, Lemma 6.1.15 applies, yielding $m_{L_H(\sigma')}(\mu') = n-1$. The result then follows from Lemma 6.1.1. \square

Finally, fix $n = 4$ and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + \sigma_2 + \sigma_3$, where $a \in \mathbb{Z}_{>0}$. In the next result, we investigate the multiplicity of $\mu = \sigma - \gamma_1 - 2\gamma_2 - 2\gamma_3 - \gamma_4$ in V , under the divisibility condition $p \mid a + 2$ ($p \neq 3$).

Lemma 6.1.17

Let V be as above and set $\mu = \sigma - \gamma_1 - 2\gamma_2 - 2\gamma_3 - \gamma_4$. Also assume $p \neq 3$ and $p \mid a + 2$. Then $m_{V_G(\sigma)}(\mu) = 8$ and $m_V(\mu) = 6$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - \gamma_1 - \gamma_2$. Then one easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+2)\chi^\mu(\tau)$, and since $p \neq 3$, we have $\chi^\mu(\tau) = \text{ch } L_G(\tau)$ by Lemma 5.1.1, so that $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8. Finally $m_{V_G(\sigma)} = 8$ by Theorem 2.3.11, while $m_{L_G(\tau)}(\mu) = 2$ by Lemma 5.1.1, completing the proof. \square

6.1.2 Weight multiplicities for G of type D_4 over K

Let K be an algebraically closed field of characteristic $p \geq 0$ and G a simple algebraic group of type D_4 over K . Fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U is the unipotent radical of B , let $\Pi = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ denote a corresponding base of the root system Φ of G and let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the set of fundamental dominant weights for T corresponding to our choice of base Π . In this section, we record some useful information on weight multiplicities and for $c_1, c_2, c_3, c_4 \in \mathbb{Z}$, we adopt the notation $\omega - c_1c_2c_3c_4$ to designate $\sigma - c_1\gamma_1 - c_2\gamma_2 - c_3\gamma_3 - c_4\gamma_4$. We start by the following three very specific situations, in which K has characteristic $p = 7$.

Lemma 6.1.18

Assume $p = 7$ and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = 4\sigma_1 + 2\sigma_2 \in X^+(T)$. Also let $\mu = \sigma - 1211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 7$ and $m_V(\mu) \leq 3$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - \gamma_1 - \gamma_2 = 3\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ affords the highest weight of a composition factor of $V_G(\sigma)$, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Now an application of 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 7$, while $m_{L_G(\tau)}(\mu) = 4$ by [Lüb15], completing the proof. \square

Lemma 6.1.19

Assume $p = 7$ and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = 4\sigma_1 + 3\sigma_2 \in X^+(T)$. Also let $\mu = \sigma - 2311$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 14$ and $m_V(\mu) \leq 9$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - 2200 = 2\sigma_1 + \sigma_2 + 2\sigma_3 + 2\sigma_4$ as well as $\tau_2 = \sigma - 1211 = 4\sigma_1 + 2\sigma_2$. One first checks that $\nu_c^\mu(T_\sigma) = \chi^\mu(\tau_1) + \chi^\mu(\tau_2)$, while Lemma 2.3.19 and Proposition 6.1.10 respectively yield

$$\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2) + \text{ch } L_G(\mu), \quad \chi^\mu(\tau_1) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\mu).$$

Therefore $\nu_c^\mu(T_\sigma) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + 2 \text{ch } L_G(\mu)$ and each of τ_1 , τ_2 and μ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8, yielding

$$m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau_1)}(\mu) - m_{L_G(\tau_2)}(\mu) - m_{L_G(\mu)}(\mu).$$

Using Proposition 6.1.10, one checks that $m_{L_G(\tau_1)}(\mu) = 3$, while $m_{L_G(\tau_2)}(\mu) = 1$ by Lemma 2.3.19 and obviously $m_{L_G(\mu)}(\mu) = 1$. Finally, applying Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 14$, completing the proof. \square

Lemma 6.1.20

Assume $p = 7$ and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = 5\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 \in X^+(T)$. Also let $\mu = \sigma - 2211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 19$ and $m_V(\mu) \leq 12$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - 1100 = 4\sigma_1 + 2\sigma_3 + 2\sigma_4$ affords the highest weight of a composition factor of $V_G(\sigma)$, and one easily checks using Lemma 2.3.19 and Corollary 2.7.3 that $[V_G(\tau), L_G(\nu)] = 0$ for every $\nu \in X^+(T)$ such that $\mu \preceq \nu \prec \tau$. Therefore $m_{L_G(\tau)}(\mu) = m_{V_G(\tau)}(\mu)$ and Theorem 2.3.11 yields $m_{V_G(\tau)}(\mu) = 7$, as well as $m_{V_G(\sigma)}(\mu) = 19$, from which the result follows. \square

We next drop the assumption $p = 7$ and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4$, where $a, b, c, d \in \mathbb{Z}_{\geq 0}$. We start by the case where $ab \neq c = d = 0$.

Lemma 6.1.21

Let V be as above, with $b = 1$, $c = d = 0$ and $a \in \mathbb{Z}_{>0}$ such that $p \mid a+2$. Also let $\mu = \sigma - 1211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 6$ and $m_V(\mu) \leq 3$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - 1100 = (a-1)\sigma_1 + \sigma_3 + \sigma_4$ affords the highest weight of a composition factor of $V_G(\sigma)$, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Also, since $p \mid a+2$ and σ is p -restricted, we get that $p \neq 2$ and Lemma 2.3.19 yields $m_{L_G(\tau)}(\mu) = 3$. An application of Theorem 2.3.11 shows that $m_{V_G(\sigma)}(\mu) = 6$ and thus allows us to conclude. \square

Lemma 6.1.22

Let V be as above, with $b = 3$, $c = d = 0$, and $a \in \mathbb{Z}_{>0}$ such that $p \mid a+4$. Also let $\mu = \sigma - 1422$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 12$ and $m_V(\mu) \leq 5$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - 1100 = (a - 1)\sigma_1 + 2\sigma_2 + \sigma_3 + \sigma_4$ affords the highest weight of a composition factor of $V_G(\sigma)$, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Also, one checks (using [Lüb15]) that $m_{L_G(\tau)}(\mu) = 7$, while an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 12$, completing the proof. \square

Lemma 6.1.23

Let V be as above, with $b = 2$, $c = d = 0$, and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 2$. Also let $\mu = \sigma - 2211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 9$ and $m_V(\mu) \leq 8$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - 2200 = (a - 2)\sigma_1 + 2\sigma_3 + 2\sigma_4$. Then one easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(a + 2)\chi^\mu(\tau)$ and since $m_{L_G(\tau)}(\mu) = 1$, we immediately get $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - 1$. Finally, an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 9$, completing the proof. \square

Lemma 6.1.24

Assume $p \neq 3$ and let V be as above, with $b = c = d = 1$ and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 2$. Also let $\mu = \sigma - 1211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 14$ and $m_V(\mu) \leq 11$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - 1100 = (a - 1)\sigma_1 + 2\sigma_3 + 2\sigma_4 \in X^+(T)$ affords the highest weight of a composition factor of $V_G(\sigma)$, and $m_{L_G(\tau)}(\mu) = 3$ by Lemma 2.3.19. Finally, an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 14$, from which the result follows. \square

Lemma 6.1.25

Let V be as above, with $b = 0$, $c = d = 1$, and $a \in \mathbb{Z}_{>0}$ such that $p \mid a + 3$. Also let $\mu = \sigma - 1111$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 7$ and $m_V(\mu) \leq 5$.

Proof. By Lemma 2.3.19, each of the T -weights $\tau_1 = \sigma - 1110$ and $\tau_2 = \sigma - 1101$ affords the highest weight of a composition factor of $V_G(\sigma)$, and since $m_{L_G(\tau_1)}(\mu) = m_{L_G(\tau_2)}(\mu) = 1$, we immediately get $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - 2$. An application of Theorem 2.3.11 then yields the desired result. \square

Lemma 6.1.26

Let V be as above, with $b = c = d = 1$ and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 4$. Also let $\mu = \sigma - 1111$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 8$ and $m_V(\mu) \leq 6$.

Proof. Proceeding exactly as in the proof of Lemma 6.1.25 (setting $\tau_1 = \sigma - 1110$ and $\tau_2 = \sigma - 1101$ and replacing Lemma 2.3.19 by Proposition 6.1.10), one easily obtains the desired result. We omit the details here. \square

Lemma 6.1.27

Assume $p \neq 3$ and let V be as above, with $b = c = d = 1$ and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 4$. Also let $\mu = \sigma - 1322$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 24$ and $m_V(\mu) = 18$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - 1110 = (a - 1)\sigma_1 + \sigma_2 + 2\sigma_4$, $\tau_2 = \sigma - 1101 = (a - 1)\sigma_1 + \sigma_2 + 2\sigma_3$. One then easily checks that

$$\nu_c^\mu(T_\sigma) = \nu_p(a + 4)\chi^\mu(\tau_1) + \nu_p(a + 4)\chi^\mu(\tau_2)$$

and using [Lüb15], we get that $\chi^\mu(\tau_1) = \text{ch } L_G(\tau_1)$ as well as $\chi^\mu(\tau_2) = \text{ch } L_G(\tau_2)$. Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2)$ by Propositions 6.1.10 and 2.7.8. Finally, Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 24$, while $m_{L_G(\tau_i)}(\mu) = 3$ for $i = 1, 2$ by [Lüb15] again, completing the proof. \square

Proposition 6.1.28

Let V be as above, with $b = 2, c = d = 0$ and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 3$. Also let $\mu = \sigma - 2422$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 24$ and $m_V(\mu) \leq 6$.

Proof. By Lemma 2.3.19, the T -weight $\tau = \sigma - 1100 = (a - 1)\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ affords the highest weight of a composition factor, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Now an application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 24$, while $m_{L_G(\tau)}(\mu) = 18$ by Lemma 6.1.27, so the result follows. \square

Lemma 6.1.29

Let V be as above, with $b = 1, c = d = 0$ and $a \in \mathbb{Z}_{>0}$ such that $p \mid a + 6$. Also let $\mu = \sigma - 1211$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 6$ and $m_V(\mu) \leq 5$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and observe that since $p \neq 2$, we have $\nu_c^\mu(T_\sigma) = \nu_p(a + 6)\chi^\mu(\mu) = \nu_p(a + 6)\text{ch } L_G(\mu)$. Therefore μ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8 and since $m_{V_G(\sigma)}(\mu) = 6$ by Theorem 2.3.11, the desired result holds. \square

Corollary 6.1.30

Let V be as above, with $b = 1, c = d = 0$, and $a \in \mathbb{Z}_{>0}$ such that $p \mid a + 6$. Also let $\mu = \sigma - 3311$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 10$ and $m_V(\mu) \leq 9$.

Proof. By Lemma 6.1.29, we know that the T -weight $\tau = \sigma - 1211$ affords the highest weight of a composition factor of $V_G(\sigma)$, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. An application of Theorem 2.3.11 then yields $m_{V_G(\sigma)}(\mu) = 10$, while $m_{L_G(\tau)}(\mu) = 1$, completing the proof. \square

We now consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$, for some $a \in \mathbb{Z}_{>0}$, and investigate the multiplicity of the T -weight $\mu = \sigma - 1111 \in X^+(T)$ in V , using information on the structure of $V_G(\sigma)$ as an \mathcal{L} -module, where \mathcal{L} denotes the Lie algebra of G . Let then $\mathcal{B} = \{e_\gamma, f_\gamma, h_{\gamma_i} : \gamma \in \Phi^+, 1 \leq i \leq 4\}$ be a standard Chevalley basis of \mathcal{L} as in Section 2.5.1. By (2.14) and our choice of ordering on Φ^+ , the weight space $V_G(\sigma)_\mu$ is spanned by

$$\begin{aligned} & \{f_{1,4}v^\sigma, f_{\gamma_1}f_{2,4}v^\sigma\} \cup \{f_{\gamma_3}f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma, f_{\gamma_1}f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma\} \\ & \cup \{f_{1,3}f_{\gamma_4}v^\sigma, f_{\gamma_1}f_{2,3}f_{\gamma_4}v^\sigma, f_{1,2}f_{\gamma_3}f_{\gamma_4}v^\sigma\}, \end{aligned} \quad (6.8)$$

where $v^\sigma \in V_G(\sigma)_\sigma$ is a maximal vector in $V_G(\sigma)$ for the Borel subgroup B of G (and thus for the corresponding Borel subalgebra \mathfrak{b} of \mathcal{L} as well). An application of Theorem 2.3.11 then yields $\dim_{V_G(\sigma)}(\mu) = 7$, forcing the generating elements of (6.8) to be linearly independent, so that the following holds.

Proposition 6.1.31

Let G be a simple algebraic group of type D_4 over K and consider the dominant T -weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$, where $a \in \mathbb{Z}_{>0}$. Also let $\mu = \sigma - 1111$. Then μ is dominant and the set (6.8) forms a basis of the weight space $V_G(\sigma)_\mu$.

As usual, we then study the relation between the pair (a, p) and the existence of a maximal vector in $V_G(\sigma)_\mu$ for \mathfrak{b} . For $X = (A_1, A_2, B_1, B_2, C_1, C_2, C_3) \in K^7$, we set

$$\begin{aligned} u(A) = & A_1f_{1,4}v^\sigma + A_2f_{\gamma_1}f_{2,4}v^\sigma + B_1f_{\gamma_3}f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma + B_2f_{\gamma_1}f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma \\ & + C_1f_{1,3}f_{\gamma_4}v^\sigma + C_2f_{\gamma_1}f_{2,3}f_{\gamma_4}v^\sigma + C_3f_{1,2}f_{\gamma_3}f_{\gamma_4}v^\sigma. \end{aligned} \quad (6.9)$$

Lemma 6.1.32

Let σ, μ be as in the statement of Proposition 6.1.31 and adopt the notation of (6.9). Then the following assertions are equivalent.

1. There exists $0 \neq X = (A_1, A_2, B_1, B_2, C_1, C_2, C_3) \in K^7$ such that $e_\gamma u(X) = 0$ for every $\gamma \in \Pi$.
2. There exists $X = (A_1, A_2, B_1, B_2, C_1, C_2, C_3) \in K^6 \times K^*$ such that $e_\gamma u(X) = 0$ for every $\gamma \in \Pi$.
3. The divisibility condition $p \mid a + 5$ is satisfied.

Furthermore, if $0 \neq A \in K^7$ is such that $e_\gamma w(A) = 0$ for every simple root $\gamma \in \Pi$, then $A \in \langle (a+1, 1, -2, 1, -2, 1, 2) \rangle_K$. In particular, the subspace of $V_G(\sigma)$ spanned by all maximal vectors in $V_G(\sigma)_\mu$ for \mathfrak{b} is at most 1-dimensional.

Proof. Let $X = (A_1, A_2, B_1, B_2, C_1, C_2, C_3) \in K^7$ and set $u = u(X)$. Then successively applying Lemma 2.5.3 yields

$$\begin{aligned}
 e_{\gamma_1}u &= (-A_1 + (a+1)A_2)f_{2,4}v^\sigma + (-B_1 + (a+1)B_2 + C_3)f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma \\
 &\quad + (-C_1 + (a+1)C_2 + C_3)f_{2,3}f_{\gamma_4}v^\sigma, \\
 e_{\gamma_2}u &= (-B_2 - C_2 + C_3)f_{\gamma_1}f_{\gamma_3}f_{\gamma_4}v^\sigma, \\
 e_{\gamma_3}u &= (-A_1 + 2B_1)f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma + (-A_2 + 2B_2 - C_2)f_{\gamma_1}f_{\gamma_2+\gamma_4}v^\sigma \\
 &\quad + (C_1 + C_3)f_{1,2}f_{\gamma_4}v^\sigma, \\
 e_{\gamma_4}u &= (-A_1 + B_1 + C_1)f_{1,3}v^\sigma + (-A_2 + C_2)f_{\gamma_1}f_{2,3}v^\sigma + (B_1 + C_3)f_{1,2}f_{\gamma_3}v^\sigma.
 \end{aligned}$$

As usual, one checks that $f_{\gamma_1}f_{\gamma_3}f_{\gamma_4}v^\sigma \neq 0$ and that the lists $\{f_{2,4}v^\sigma, f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma, f_{2,3}f_{\gamma_4}v^\sigma\}$, $\{f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma, f_{\gamma_1}f_{\gamma_2+\gamma_4}v^\sigma, f_{1,2}f_{\gamma_4}v^\sigma\}$, and $\{f_{1,3}v^\sigma, f_{\gamma_1}f_{2,3}v^\sigma, f_{1,2}f_{\gamma_3}v^\sigma\}$ are linearly independent, so that $e_\gamma u(X) = 0$ for every $\gamma \in \Pi$ if and only if X is solution to the system of equations

$$\left\{ \begin{array}{l}
 A_1 = (a+1)A_2 \\
 B_1 = (a+1)B_2 + C_3 \\
 C_1 = (a+1)C_2 + C_3 \\
 C_3 = B_2 + C_2 \\
 A_1 = 2B_1 \\
 A_2 = 2B_2 - C_2 \\
 C_1 = -C_3 \\
 A_1 = B_1 + C_1 \\
 A_2 = C_2 \\
 B_1 = -C_3.
 \end{array} \right. \quad (6.10)$$

Now one easily sees that (6.10) admits a non-trivial solution $X \in K^7$ if and only if $p \mid a+5$ (showing that 1 and 3 are equivalent), in which case $X \in \langle (a+1, 1, -2, 1, -2, 1, 2) \rangle_K$ (so that 1 and 2 are equivalent), completing the proof. \square

Proposition 6.1.33

Let G be a simple algebraic group of type D_4 over K and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4 \in X^+(T)$, where $a \in \mathbb{Z}_{>0}$. Also assume $p \neq 2$, $p \mid a+5$ and let $\mu = \sigma - 1111$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\mu)$, $\text{m}_{V_G(\sigma)}(\mu) = 7$ and $\text{m}_V(\mu) = 6$.

Proof. One first easily checks (using Lemma 2.3.19 together with the fact that $p \neq 2$ and $p \mid a+5$) that $[V_G(\sigma), L_G(\nu)] = 0$ for every $\nu \in X^+(T)$ such that $\mu \prec \nu \prec \sigma$. Therefore $[V_G(\sigma), L_G(\mu)]$ equals the dimension of the subspace of $V_G(\sigma)$ spanned by all maximal vectors in $V_G(\sigma)_\mu$ for B . An application of Lemma 6.1.32 then completes the proof. \square

Lemma 6.1.34

Let V be as above, with $b = c = 0$, $d = 2$ and $a \in \mathbb{Z}_{>0}$ such that $p \mid a+4$. Also let $\mu = \sigma - 2212$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\sigma - 1101)$, $\text{m}_{V_G(\sigma)}(\mu) = 10$ and $\text{m}_V(\mu) = 4$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - 1101$. One easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+4)(\chi^\mu(\tau) - \chi^\mu(\mu))$, and since $\chi^\mu(\tau) = \text{ch } L_G(\tau) + \text{ch } L_G(\mu)$ by Proposition 6.1.33, we get $\nu_c^\mu(T_\sigma) = \nu_p(a+4) \text{ch } L_G(\tau)$. Therefore $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau)$ by Lemma 2.3.19 and Proposition 2.7.8, thus yielding $\text{m}_V(\mu) = \text{m}_{V_G(\sigma)}(\mu) - \text{m}_{L_G(\tau)}(\mu)$. Finally, $\text{m}_{V_G(\sigma)}(\mu) = 10$ by Theorem 2.3.11 and an application of Proposition 6.1.33 then completes the proof. \square

Using Proposition 6.1.33 and Lemma 6.1.34, we now determine an upper bound for the multiplicity of $\sigma - 3322 \in X^+(T)$ in the irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$.

Proposition 6.1.35

Let V be as above, with $b = 0$, $c = d = 1$ and $a \in \mathbb{Z}_{>3}$ such that $p \mid a+3$. Also let $\mu = \sigma - 3322$. Then μ is dominant, $\text{m}_{V_G(\sigma)}(\mu) = 29$ and $\text{m}_V(\mu) \leq 14$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - 1110$, $\tau_2 = \sigma - 1101$, $\tau_3 = \sigma - 2211$. One first easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+3)(\chi^\mu(\tau_1) + \chi^\mu(\tau_2) + \chi^\mu(\mu))$, and by Lemma 6.1.34, we get

$$\chi^\mu(\tau_1) = \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_3), \quad \chi^\mu(\tau_2) = \text{ch } L_G(\tau_2) + \text{ch } L_G(\tau_3),$$

so that $\nu_c^\mu(T_\sigma) = \nu_p(a+3)(\text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + 2 \text{ch } L_G(\tau_3) + \chi^\mu(\mu))$. Therefore each of τ_1 , τ_2 , τ_3 and μ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8 and hence $\text{m}_V(\mu) \leq \text{m}_{V_G(\sigma)}(\mu) - \text{m}_{L_G(\tau_1)}(\mu) - \text{m}_{L_G(\tau_2)}(\mu) - \text{m}_{L_G(\tau_3)}(\mu) - 1$. An application of Theorem 2.3.11 yields $\text{m}_{V_G(\sigma)}(\mu) = 29$, while $\text{m}_{L_G(\tau_1)}(\mu) = \text{m}_{L_G(\tau_2)}(\mu) = 4$ by Lemma 6.1.34 and $\text{m}_{L_G(\tau_3)}(\mu) = 6$ by Proposition 6.1.33, leading to the desired result. \square

We next consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4 \in X^+(T)$, where $a \in \mathbb{Z}_{>0}$, and aim to determine the multiplicity of $\mu = \sigma - 2211$ in V . As above, we let $\mathcal{L} = \mathcal{L}(G)$ and let $\mathcal{B} = \{e_\gamma, f_\gamma, h_{\gamma_i} : \gamma \in \Phi^+, 1 \leq i \leq 4\}$ be a standard Chevalley basis of \mathcal{L} as in Section 2.5.1. By (2.14) and our choice of ordering on Φ^+ , the weight space $V_G(\sigma)_\mu$ is spanned by

$$\begin{aligned} & \{f_{\gamma_1} F_{1,2} v^\sigma\} \cup \{f_{1,2} f_{1,4} v^\sigma, f_{1,3} f_{\gamma_1 + \gamma_2 + \gamma_4} v^\sigma\} \cup \{f_{\gamma_1} f_{2,3} f_{\gamma_1 + \gamma_2 + \gamma_4} v^\sigma, f_{1,2} f_{\gamma_3} f_{\gamma_1 + \gamma_2 + \gamma_4} v^\sigma\} \\ & \cup \{f_{\gamma_1} f_{1,2} f_{2,4} v^\sigma, f_{\gamma_1} f_{1,3} f_{\gamma_2 + \gamma_4} v^\sigma\} \cup \{f_{\gamma_1}^2 f_{2,3} f_{\gamma_2 + \gamma_4} v^\sigma, f_{\gamma_1} f_{1,2} f_{\gamma_3} f_{\gamma_2 + \gamma_4} v^\sigma\} \\ & \cup \{f_{1,2} f_{1,3} f_{\gamma_4} v^\sigma\} \cup \{f_{\gamma_1} f_{1,2} f_{2,3} f_{\gamma_4} v^\sigma, (f_{1,2})^2 f_{\gamma_3} f_{\gamma_4} v^\sigma\}, \end{aligned} \quad (6.11)$$

where $v^\sigma \in V_G(\sigma)_\sigma$ is a maximal vector in $V_G(\sigma)$ for B (and thus for the corresponding Borel subalgebra \mathfrak{b} of \mathcal{L} as well). An application of Theorem 2.3.11 then yields $\text{m}_{V_G(\sigma)}(\mu) = 12$, forcing the generating elements of (6.11) to be linearly independent.

Proposition 6.1.36

Let G be a simple algebraic group of type D_4 over K and consider the dominant T -weight $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$, where $a \in \mathbb{Z}_{>1}$. Also let $\mu = \sigma - 2211$. Then μ is dominant and the set (6.11) forms a basis of $V_G(\sigma)_\mu$.

We now study the relation between the pair (a, p) and the existence of a maximal vector in $V_G(\sigma)_\mu$ for \mathfrak{b} . In order to simplify the notation, we respectively designate the elements of (6.11) by v_1, \dots, v_{12} , and for $A = (A_r)_{r=1}^{12} \in K^{12}$, we set

$$w(A) = \sum_{r=1}^{12} A_r v_r. \quad (6.12)$$

Lemma 6.1.37

Let σ, μ be as in the statement of Proposition 6.1.36 and adopt the notation of (6.12). Then the following assertions are equivalent.

1. There exists $0 \neq A \in K^{12}$ such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$.
2. There exists $A \in K^{11} \times K^*$ such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$.
3. The divisibility condition $p \mid a + 3$ is satisfied.

Furthermore, if $0 \neq A \in K^{12}$ is such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$, then $A \in \langle (0, 0, 2, -1, -1, 0, -1, 1, 1, -1, 1, 1) \rangle_K$. In particular, the subspace of $V_G(\sigma)$ spanned by all maximal vectors in $V_G(\sigma)_\mu$ for \mathfrak{b} is at most 1-dimensional.

Proof. Let $A = (A_r)_{1 \leq r \leq 12} \in K^{12}$ and set $w = w(A)$. Then successively applying Lemma 2.5.3 yields

$$\begin{aligned} e_{\gamma_1} w &= (aA_1 + A_2)F_{1,2}v^\sigma + (-A_2 + aA_6)f_{1,2}f_{2,4}v^\sigma \\ &\quad + (-A_3 + aA_4 + A_5)f_{2,3}f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma \\ &\quad + (-A_3 + aA_7 + A_{10})f_{1,3}f_{\gamma_2+\gamma_4}v^\sigma \\ &\quad + (-A_4 - A_7 + 2(a+1)A_8 + A_9 + A_{11})f_{\gamma_1}f_{2,3}f_{\gamma_2+\gamma_4}v^\sigma \\ &\quad + (-A_5 + aA_9 + 2A_{12})f_{1,2}f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma \\ &\quad + (-A_{10} + aA_{11} + 2A_{12})f_{1,2}f_{2,3}f_{\gamma_4}v^\sigma, \end{aligned}$$

$$\begin{aligned} e_{\gamma_2} w &= (-A_1 + A_2)f_{\gamma_1}f_{1,4}v^\sigma + (-A_4 + A_5)f_{\gamma_1}f_{\gamma_3}f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma \\ &\quad + A_6f_{\gamma_1}^2f_{2,4}v^\sigma + (-A_7 + A_{10})f_{\gamma_1}f_{1,3}f_{\gamma_4}v^\sigma \\ &\quad + (-A_8 + A_9)f_{\gamma_1}^2f_{\gamma_3}f_{\gamma_2+\gamma_4}v^\sigma + (-A_8 + A_{11})f_{\gamma_1}^2f_{2,3}f_{\gamma_4}v^\sigma \\ &\quad + (-A_9 - A_{11} + 2A_{12})f_{\gamma_1}f_{1,2}f_{\gamma_3}f_{\gamma_4}v^\sigma, \end{aligned}$$

$$e_{\gamma_3}w = (-A_2 + A_3 + 2A_5)f_{1,2}f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma + (A_{10} + A_{12})(f_{1,2})^2f_{\gamma_4}v^\sigma \\ + (-A_6 + A_7 + 2A_9 - A_{11})f_{\gamma_1}f_{1,2}f_{\gamma_2+\gamma_4}v^\sigma,$$

$$e_{\gamma_4}w = (-A_2 + A_3 + A_5 + A_{10})f_{1,2}f_{1,3}v^\sigma + (A_4 - A_6 + A_{11})f_{\gamma_1}f_{1,2}f_{2,3}v^\sigma \\ + (A_5 + A_{12})(f_{1,2})^2f_{\gamma_3}v^\sigma.$$

As usual, one then checks that there exists $0 \neq A \in K^{12}$ such that $e_\gamma w(A) = 0$ for every $\gamma \in \Pi$ if and only if $p \mid a + 3$ (showing that 1 and 3 are equivalent), in which case $A \in \langle (0, 0, 2, -1, -1, 0, -1, 1, 1, -1, 1, 1) \rangle_K$ (so that 1 and 2 are equivalent), completing the proof. \square

Let $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$, with $a > 1$ such that $p \mid a + 3$ and write $\tau_1 = \sigma - \gamma_1 - \gamma_2 - \gamma_3 \in X^+(T)$ as well as $\tau_2 = \sigma - \gamma_1 - \gamma_2 - \gamma_4 \in X^+(T)$. Then by (4.7), both $u^{\tau_1} = f_{1,3}v^\sigma - f_{\gamma_1}f_{2,3}v^\sigma - f_{1,2}f_{\gamma_3}v^\sigma$, $u^{\tau_2} = f_{\gamma_1+\gamma_2+\gamma_4}v^\sigma - f_{\gamma_1}f_{\gamma_2+\gamma_4}v^\sigma - f_{1,2}f_{\gamma_4}v^\sigma$ are maximal vectors in $V_G(\sigma)$ for B (hence for \mathfrak{b} as well).

Lemma 6.1.38

Set $U = \langle Gu^{\tau_1} \rangle + \langle Gu^{\tau_2} \rangle \subset \text{rad}(\sigma)$, where σ , u^{τ_1} and u^{τ_2} are as above, and let $\mu = \sigma - 2211$. Then $m_U(\mu) = 5$.

Proof. Let $\{v_r\}_{r=1}^{12}$ be the basis of $V_G(\sigma)_\mu$ introduced above. Using Lemma 2.5.5, one easily checks that we have

$$f_{\gamma_1+\gamma_2+\gamma_4}u^{\tau_1} = -v_1 - v_2 + v_3 - v_4 - v_5, \\ f_{\gamma_1}f_{\gamma_2+\gamma_4}u^{\tau_1} = -2v_1 - v_4 - v_6 + v_7 - v_8 - v_9, \\ f_{1,2}f_{\gamma_4}u^{\tau_1} = -2v_2 - v_5 + v_6 + v_{10} - v_{11} - v_{12}. \quad (6.13)$$

These elements are linearly independent by Proposition 6.1.36 and thus $m_{\langle \mathcal{L}u^{\tau_1} \rangle}(\mu) \geq 3$. Now since $\langle Gu^{\tau_1} \rangle$ is an image of $V_G(\tau_1)$ containing $\langle \mathcal{L}u^{\tau_1} \rangle$ and $m_{V_G(\tau_1)}(\mu) = 3$, we get that $m_{\langle Gu^{\tau_1} \rangle}(\mu) = 3$ as well as $\langle Gu^{\tau_1} \rangle_\mu = \langle f_{\gamma_1+\gamma_2+\gamma_4}u^{\tau_1}, f_{\gamma_1}f_{\gamma_2+\gamma_4}u^{\tau_1}, f_{1,2}f_{\gamma_4}u^{\tau_1} \rangle_K$. Similarly, one checks that

$$f_{1,3}u^{\tau_2} = v_3 - v_7 - v_{10}, \\ f_{\gamma_1}f_{2,3}u^{\tau_2} = v_4 - v_7 - v_8 - v_{11}, \\ f_{1,2}f_{\gamma_3}u^{\tau_2} = v_5 - v_9 - v_{10} - v_{12}, \quad (6.14)$$

and arguing as above yields $m_{\langle Gu^{\tau_2} \rangle}(\mu) = 3$ and $\langle Gu^{\tau_2} \rangle_\mu = \langle f_{1,3}u^{\tau_2}, f_{\gamma_1}f_{2,3}u^{\tau_2}, f_{1,2}f_{\gamma_3}u^{\tau_2} \rangle_K$. Also, an easy computation shows that $\dim \langle Gu^{\tau_1} \rangle_\mu \cap \langle Gu^{\tau_2} \rangle_\mu = 1$, so that $m_U(\mu) = 5$ as desired. \square

Let U be as in the statement of Lemma 6.1.38 and write $\overline{V_G(\sigma)} = V_G(\sigma)/U$. Also, for $v \in V_G(\sigma)$, denote by \bar{v} the class of v in $\overline{V_G(\sigma)}$. We then leave to the reader to check (using (6.13) and (6.14)) that we have $\bar{v}_{12} = 2\bar{v}_1 - \bar{v}_3 + \bar{v}_4 + \bar{v}_5 + \bar{v}_6 + \bar{v}_8$, $\bar{v}_{11} = \bar{v}_4 - \bar{v}_7 - \bar{v}_8$, $\bar{v}_{10} = \bar{v}_3 - \bar{v}_7$, $\bar{v}_9 = -2\bar{v}_1 - \bar{v}_4 - \bar{v}_6 + \bar{v}_7 - \bar{v}_8$, and $\bar{v}_5 = -\bar{v}_1 - \bar{v}_2 + \bar{v}_3 - \bar{v}_4$ and thus $\overline{V_G(\sigma)}_\mu = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_6, \bar{v}_7, \bar{v}_8 \rangle_K$. For $X = (A, B, C, D, E, G, H) \in K^7$, we write

$$\bar{w}(X) = A\bar{v}_1 + B\bar{v}_2 + C\bar{v}_3 + D\bar{v}_4 + E\bar{v}_6 + G\bar{v}_7 + H\bar{v}_8.$$

Proposition 6.1.39

Let V be as above, with $b = 0$, $c = d = 1$ and $a \in \mathbb{Z}_{>1}$ such that $p \mid a + 3$. Also set $\tau_1 = \sigma - 1110$, and $\tau_2 = \sigma - 1101$. Then the T -weight $\mu = \sigma - 2211 \in X(T)$ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + \text{ch } L_G(\mu)$, $\text{m}_{V_G(\sigma)}(\mu) = 12$ and $\text{m}_V(\mu) = 7$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - 1110 = (a - 1)\sigma_1 + 2\sigma_4$, $\tau_2 = \sigma - 1101 = (a - 1)\sigma_1 + 2\sigma_3$. One then checks that $\nu_c^\mu(T_\sigma) = \nu_p(a + 3)(\chi^\mu(\tau_1) + \chi^\mu(\tau_2))$, while applying Lemma 2.3.19 yields $\chi^\mu(\tau_i) = \text{ch } L_G(\tau_i) + \text{ch } L_G(\mu)$ for $i = 1, 2$. Therefore

$$\nu_c^\mu(T_\sigma) = \nu_p(a + 3)(\text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + 2 \text{ch } L_G(\mu)),$$

which by Proposition 2.7.8 shows that each of τ_1 , τ_2 and μ affords the highest weight of a composition factor of $V_G(\sigma)$, while $[V_G(\sigma), L_G(\nu)] = 0$ (and hence $[\overline{V_G(\sigma)}, L_G(\nu)] = 0$ as well) for every other T -weight $\tau_1, \tau_2 \neq \nu$ of $\overline{V_G(\sigma)}$ such that $\mu \prec \nu \prec \sigma$. Now if μ affords the highest weight of a composition factor of $\overline{V_G(\sigma)}$, then there exists $X \in K^7$ as above such that $\bar{w}(X)$ is a maximal vector in $\overline{V_G(\sigma)}$ for B . Arguing as in the proof of Lemma 6.1.9 then yields $X = 0$ and hence $[\overline{V_G(\sigma)}, L_G(\mu)] = 0$, so that

$$[V_G(\sigma), L_G(\mu)] = [U, L_G(\mu)]. \quad (6.15)$$

Finally, notice that $\text{m}_{L_G(\tau_1)}(\mu) = \text{m}_{L_G(\tau_2)}(\mu) = 2$ by Lemma 2.3.19 and an application of Lemma 6.1.38 yields $[U, L_G(\mu)] = 1$. The result then follows from (6.15). \square

To conclude this section, we study the multiplicity of $\sigma - 3422 \in X^+(T)$ in a given irreducible KG -module having p -restricted highest weight $\sigma = a\sigma_1 + \sigma_2$, where $a \in \mathbb{Z}_{>1}$, starting by recording the following two preliminary results. The proof of the first one being identical to that of Lemma 6.1.34 we omit the details here.

Lemma 6.1.40

Let $a \in \mathbb{Z}_{>0}$ be such that $p \mid a + 4$ and let $\sigma = a\sigma_1 + 2\sigma_4$. Also consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight σ and write $\mu = \sigma - 1212$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\sigma - 1101)$, $\text{m}_{V_G(\sigma)}(\mu) = 6$ and $\text{m}_V(\mu) = 3$.

Lemma 6.1.41

Let $a \in \mathbb{Z}_{>1}$ be such that $p \mid a + 3$ and let $\sigma = a\sigma_1 + \sigma_3 + \sigma_4$. Also consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight σ and write $\mu = \sigma - 2322$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 21$ and $m_V(\mu) = 12$.

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau_1 = \sigma - 1110 = (a-1)\sigma_1 + 2\sigma_4$, $\tau_2 = \sigma - 1101 = (a-1)\sigma_1 + 2\sigma_3$, $\tau = \sigma - 2211 = (a-2)\sigma_1 + \sigma_3 + \sigma_4$. One then checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+3)(\chi^\mu(\tau_1) + \chi^\mu(\tau_2))$. Also, applying Lemma 6.1.40 yields $\chi^\mu(\tau_i) = \text{ch } L_G(\tau_i) + \text{ch } L_G(\tau)$ for $i = 1, 2$, so that $\nu_c^\mu(T_\sigma) = \nu_p(a+3)(\text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + 2 \text{ch } L_G(\tau))$. Therefore

$$\chi^\mu(\sigma) = \text{ch } V + \text{ch } L_G(\tau_1) + \text{ch } L_G(\tau_2) + \text{ch } L_G(\tau)$$

by Lemma 2.3.19, Proposition 2.7.8 and Proposition 6.1.39. Finally $m_{V_G(\sigma)}(\mu) = 21$ by Theorem 2.3.11, while an application of Lemma 6.1.40 yields $m_{L_G(\tau_1)}(\mu) = m_{L_G(\tau_2)}(\mu) = 3$ and $m_{L_G(\tau)}(\mu) = 3$ by Lemma 2.3.19, from which the result follows. \square

Proposition 6.1.42

Let G be a simple algebraic group of type D_4 over K and consider an irreducible KG -module $V = L_G(\sigma)$ having highest weight $\sigma = a\sigma_1 + \sigma_2 \in X^+(T)$, where $a \in \mathbb{Z}_{>1}$ is such that $p \mid a + 2$. Also let $\mu = \sigma - 3422$. Then μ is dominant, $m_{V_G(\sigma)}(\mu) = 18$ and $m_V(\mu) \leq 6$.

Proof. By Lemma 2.3.19, the weight $\tau = \sigma - 1100 \in X^+(T)$ affords the highest weight of a composition factor of $V_G(\sigma)$, so that $m_V(\mu) \leq m_{V_G(\sigma)}(\mu) - m_{L_G(\tau)}(\mu)$. Now Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = 18$, while $m_{L_G(\tau)}(\mu) = 12$ by Lemma 6.1.41, completing the proof. \square

6.2 Proof of Theorem 6.1

Let Y, X be as in the statement of Theorem 6.1 and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = \sum_{i=1}^7 a_i \lambda_i \in X^+(T_Y)$. Also denote by ω the restriction of λ to T_X , so that by (6.1), we have

$$\omega = (a_1 + a_7)\omega_1 + (a_2 + a_6)\omega_2 + (a_3 + a_5)\omega_3 + (a_3 + 2a_4 + a_5)\omega_4.$$

Notice that if $v^+ \in V_\lambda$ is a maximal vector for B_Y in V , then v^+ is a maximal vector for B_X as well, since $B_X \subset B_Y$, showing that the T_X -weight ω affords the highest weight of a KX -composition factor of V . Every T_Y -weight of V is of the form $\lambda - \sum_{r=1}^7 c_r \alpha_r$, where $c_1, c_2, c_3, c_4, c_5, c_6, c_7 \in \mathbb{Z}_{\geq 0}$. Throughout this section, such a weight shall be written $\lambda - c_1 c_2 c_3 c_4 c_5 c_6 c_7$ and simply called a T_Y -weight. On the other hand, a T_X -weight of $V|_X$ does not necessarily have to be under ω : for example, if $\langle \lambda, \alpha_4 \rangle \neq 0$, then the T -weight $\lambda - \alpha_4$ restricts to $\omega + \beta_3 - \beta_4 \not\leq \omega$. The following generalization of Lemma 5.2.1 gives a condition on λ under which all T_X -weights occurring in V are under ω . Its proof being identical to that of Lemma 5.2.1, we omit the details here.

Lemma 6.2.1

Let λ, ω be as above, and suppose that $\langle \lambda, \alpha_4 \rangle = 0$. Then every T_Y -weight μ of $V = L_Y(\lambda)$ satisfies $\mu|_{T_X} \preceq \omega$.

Remark 6.2.2

Set $J = \{\beta_2, \beta_3, \beta_4\} \subset \Pi(X)$ and adopting the notation introduced in Section 2.3.2, consider the D_3 -parabolic subgroup $P_J = Q_J L_J$ of X . Also denote by $P_Y = Q_Y L_Y$ the parabolic subgroup of Y given by Lemma 2.3.9 and notice that L'_Y has type A_5 , where we thus have $\Pi(L'_Y) = \{\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5\} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Write $\tilde{X} = L'_J, \tilde{Y} = L'_Y$ and $\tilde{\lambda} = \lambda|_{T_Y \cap \tilde{Y}}$. An application of Lemma 2.3.10 and Theorem 5.1 shows that if X has exactly two composition factors on V , then either \tilde{X} acts irreducibly on $L_{\tilde{Y}}(\tilde{\lambda})$ or $(\tilde{\lambda}, p)$ appears in Table 5.1. We thus investigate each situation separately, starting with the former.

6.2.1 The irreducible case

Keep the notation introduced in Remark 6.2.2 and suppose that X' acts irreducibly on $L_{Y'}(\lambda')$. By [Sei87, Theorem 1, Table 1 (I₄, I₅)] we thus get that $\lambda' = 0, \lambda'_1$ or λ'_2 , with $p \neq 2$ in the latter situation. We first consider the case where $\lambda' = 0$, that is, $\lambda = a\lambda_1 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$.

Proposition 6.2.3

Consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + b\lambda_7$, where $a, b \in \mathbb{Z}_{\geq 0}$. Suppose in addition that X has exactly two composition factors on V . Then (λ, p) appears in Table 6.1.

Proof. First consider the case where $b = 0$, so that $\lambda = a\lambda_1, \omega = a\omega_1$. Obviously $a > 1$, in which case the T_X -weight $\omega' = \omega - 2\beta_1 - 2\beta_2 - \beta_3 - \beta_4$ is dominant. The T_Y -weights $\lambda - 2221000, \lambda - 2211100, \lambda - 2111110$, and $\lambda - 1111111$ all restrict to ω' , hence $m_{V|_X}(\omega') \geq 4$, while on the other hand, an application of Theorem 2.3.11 yields $m_{L_X(\omega)}(\omega') \leq 3$, thus showing that ω' occurs in a second KX -composition factor of V . Now one easily sees that $m_{V|_X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\omega' \prec \nu \prec \omega$, so that ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X((a-2)\omega_1).$$

Now if $a > 3$, consider $\omega'' = \omega - 4\beta_1 - 4\beta_2 - 2\beta_3 - 2\beta_4 \in \Lambda^+(T_X)$ and observe that the T_Y -weights $\lambda - 4442000, \lambda - 4432100, \lambda - 4422200, \lambda - 4332110, \lambda - 4322210, \lambda - 4222220, \lambda - 3332111, \lambda - 3322211, \lambda - 3222221$, and $\lambda - 2222222$ restrict to ω'' , hence $m_{V|_X}(\omega'') \geq 10$. On the other hand, Theorem 2.3.11 gives $m_{V_X(\omega)}(\omega'') = 6$ and $m_{V_X(\omega')}(\omega'') = 3$, so that $m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 9$. Therefore ω'' occurs in a third KX -composition factor of V , contradicting our initial assumption. Hence $a = 2$ or 3 , in which case [Lüb01, Appendices A.11, A.41] allows us to conclude.

Next consider the case where $ab \neq 0$, in which case $\omega = (a + b)\omega_1$. Here the T_Y -weights $\lambda - 1000000$ and $\lambda - 0000001$ both restrict to $\omega' = \omega - \beta_1 \in X^+(T_X)$, whose multiplicity in $L_X(\omega)$ is equal to 1. Consequently ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X((a + b - 2)\omega_1 + \omega_2).$$

If $a, b > 1$, then the T_Y -weights $\lambda - 2000000$, $\lambda - 1000001$, and $\lambda - 0000002$ restrict to $\omega - 2\beta_1$, whose multiplicity in both $L_X(\omega)$ and $L_X(\omega')$ is smaller than or equal to 1, giving the existence of a third KX -composition factor of V , a contradiction. Without any loss of generality, we may then suppose that $\lambda = a\lambda_1 + \lambda_7$, so that $\omega = (a + 1)\omega_1$ and $\omega' = (a - 1)\omega_1 + \omega_2$. The cases where $a = 1$ or 2 can be dealt with using [Lüb01, Appendices A.11, A.41], so we may assume $a \geq 3$ as well. In this situation, notice that the T_Y -weights $\lambda - 3221000$, $\lambda - 3211100$, $\lambda - 3111110$, $\lambda - 2221001$, $\lambda - 2211101$, $\lambda - 2111111$, $\lambda - 1111112$ all restrict to $\omega'' = \omega - 3\beta_1 - 2\beta_2 - \beta_3 - \beta_4 \in X^+(T_X)$. An application of Lemma 2.3.19 then yields $m_{V|X}(\omega'') \geq 12$, while by Theorem 2.3.11, we have $m_{L_X(\omega)}(\omega'') \leq 3$ and $m_{L_X(\omega')}(\omega'') \leq 8$, giving the existence of a third KX -composition factor of V . This completes the proof of the Proposition. \square

Next we tackle the situation where $\lambda' = \lambda'_1$, so that $\lambda = a\lambda_1 + \lambda_2 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, and first consider the case where $ab \neq 0$. Observe that in this situation, the T_Y -weights $\lambda - 1000000$, $\lambda - 0000001$ both restrict to $\omega' = \omega - \beta_1$, whose multiplicity in $L_X(\omega)$ equals 1, so that ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X((a + b - 2)\omega_1 + 2\omega_2).$$

Lemma 6.2.4

Let λ, ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. First consider the T_X -weight $\mu_1 = \omega - \beta_1 - \beta_2 \in X^+(T_X)$ and notice that the T_Y -weights $\lambda - 1100000$, $\lambda - 0100001$, $\lambda - 0000011$ all restrict to μ_1 . Applying Lemma 2.3.19 then yields

$$m_{V|X}(\mu_1) \geq \begin{cases} 3 & \text{if } p \mid a + 2; \\ 4 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\mu_1) + m_{L_X(\omega')}(\mu_1) \leq 3$ by Theorem 2.3.11, so we may suppose that $p \mid a + 2$ for the remainder of the proof. Also if $b > 1$, then the T_Y -weights $\lambda - 2100000$, $\lambda - 1100001$, $\lambda - 1000011$, $\lambda - 0100002$, and $\lambda - 0000012$ restrict to the T_X -weight $\mu_2 = \omega - 2\beta_1 - \beta_2$, hence $m_{V|X}(\mu_2) \geq 5$. On the other hand, Theorem 2.3.11 gives $m_{V_X(\omega)}(\mu_2) = m_{V_X(\omega')}(\mu_2) = 2$, so that $m_{L_X(\omega)}(\mu_2) + m_{L_X(\omega')}(\mu_2) \leq 4$, showing that μ_2 occurs in a third KX -composition factor of V . So assume $b = 1$ and observe that the T_Y -weights $\lambda - 1221000$, $\lambda - 1211100$, $\lambda - 1111110$, $\lambda - 0111111$ restrict to $\mu_3 = \omega - \beta_1 - 2\beta_2 - \beta_3 - \beta_4$. Applying Lemma 2.3.19 then gives

$$m_{V|X}(\mu_3) \geq \begin{cases} 8 & \text{if } p = 7; \\ 9 & \text{otherwise,} \end{cases}$$

while $m_{V_X(\omega)}(\mu_3) = 6$ and $m_{V_X(\omega')}(\mu_3) = 2$ by Theorem 2.3.11. Hence we may assume $p = 7$ (and so $a = 5$) and consider the T_X -weight $\omega'' = \omega - 2\beta_1 - 2\beta_2 - \beta_3 - \beta_4 \in X^+(T_X)$. One then checks that the T_Y -weights $\lambda - 2221000$, $\lambda - 2211100$, $\lambda - 2111110$, $\lambda - 1221001$, $\lambda - 1211101$, $\lambda - 1111111$, $\lambda - 0111112$ all restrict to ω'' . Lemma 6.1.5 then yields $m_{V|X}(\omega'') \geq 12$, while $m_{L_X(\omega)}(\omega'') \leq 8$ and $m_{L_X(\omega')}(\omega'') \leq 3$ by Theorem 2.3.11 and Lemma 6.1.18 respectively, hence showing the existence of a third KX -composition factor of V as desired. \square

We are now able to complete the study of the case where $\lambda' = \lambda'_1$, that is, $\lambda = a\lambda_1 + \lambda_2 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$.

Proposition 6.2.5

Let $\lambda = a\lambda_1 + \lambda_2 + b\lambda_7$, where $a, b \in \mathbb{Z}_{\geq 0}$, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight λ . Suppose in addition that X has exactly two composition factors on V . Then (λ, p) appears in Table 6.1, where we give λ' up to graph automorphisms.

Proof. First consider the case where $a = b = 0$, that is $\lambda = \lambda_2$ and $\omega = \omega_2$. If $p = 2$, then X acts irreducibly on $V = L_Y(\lambda)$ by [Sei87, Theorem 1, Table 1 (I_4, I_5)], so we may suppose that $p \neq 2$. An application of [Lüb01, Appendix A.41] then yields $\dim L_X(\omega) = 26$, while $\dim V = 28$ by Lemma 2.4.5. Therefore $V|_X = \omega/0^2$, that is, X has three composition factors on V . Also Lemma 6.2.4 shows that X has more than two composition factors on $L_Y(\lambda)$ if $ab \neq 0$, so for the remainder of the proof, we may suppose that either $a \neq 0 = b$ or $a = 0 \neq b$. In the former case, observe that the T_Y -weights $\lambda - 1221000$, $\lambda - 1211100$, $\lambda - 1111110$, $\lambda - 0111111$ restrict to $\omega' = \omega - \beta_1 - 2\beta_2 - \beta_3 - \beta_4 \in X^+(T_X)$. Applying Lemma 2.3.19 then yields

$$m_{V|X}(\omega') \geq \begin{cases} 4 & \text{if } p \mid a + 2; \\ 7 & \text{otherwise,} \end{cases}$$

while Theorem 2.3.11 together with Lemma 6.1.21 show that $m_{L_X(\omega)}(\omega') \leq 6 - 3\epsilon_p(a + 2)$. Hence ω' occurs in a second KX -composition factor of V and since $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \prec \omega$ (easy verification), we get that ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X(a\omega_1).$$

Seeking a contradiction, suppose that $a > 1$. The T_X -weight $\mu = \omega - 2\beta_1 - 2\beta_2 - \beta_3 - \beta_4$ is dominant in this situation and one checks that the T_Y -weights $\lambda - 2221000$, $\lambda - 2211100$, $\lambda - 2111110$, $\lambda - 1111111$ restrict to μ . Lemmas 2.3.19 and 6.1.3 then yield

$$m_{V|X}(\mu) \geq \begin{cases} 4 & \text{if } p \mid a + 2; \\ 10 & \text{otherwise,} \end{cases}$$

while on the other hand $m_{L_X(\omega)}(\mu) \leq 8$ and $m_{L_X(\omega')}(\mu) = 1$ by Theorem 2.3.11. We may thus assume $p \mid a + 2$, which forces $a > 2$. Let then $\omega'' = \omega - 3\beta_1 - 4\beta_2 - 2\beta_3 - 2\beta_4$ and observe that the T_Y -weights $\lambda - 3442000$, $\lambda - 3432100$, $\lambda - 3422200$, $\lambda - 3332110$, $\lambda - 3322210$, $\lambda - 3222220$, $\lambda - 2332111$, $\lambda - 2322211$, $\lambda - 2222221$, $\lambda - 1222222$ all restrict to ω'' , so that $m_{V|X}(\omega'') \geq 10$, while by Proposition 6.1.42 and Theorem 2.3.11, we respectively have $m_{L_X(\omega)}(\omega'') \leq 6$ and $m_{L_X(\omega')}(\omega'') \leq 3$. Consequently ω'' occurs in a third KX -composition factor of V , a contradiction, so $\lambda = \lambda_1 + \lambda_2$. Looking at [Lüb01, Appendices A.11, A.41] yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ if $p = 7$, so that the desired result holds in this case.

Finally, consider the situation where $\lambda = \lambda_2 + b\lambda_7$, for some $b \in \mathbb{Z}_{>0}$ and observe that up to graph automorphisms, we may assume $\lambda = a\lambda_1 + \lambda_6$ for some $a \in \mathbb{Z}_{>0}$. Here the T_Y -weights $\lambda - 1100000$, $\lambda - 1000010$, and $\lambda - 0000011$ restrict to $\omega' = \omega - \beta_1 - \beta_2$, hence $m_{V|X}(\omega') \geq 3$, while on the other hand, an application of Lemma 2.3.19 yields $m_{L_X(\omega)}(\omega') \leq 2$. As usual, one easily checks that $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \prec \omega$, showing that ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X((a-1)\omega_1 + \omega_3 + \omega_4).$$

Now suppose for a contradiction that $a > 1$ and let $\mu = \omega - 2\beta_1 - 2\beta_2 - \beta_3 - \beta_4$. Here the T_Y -weights $\lambda - 2221000$, $\lambda - 2211100$, $\lambda - 2111110$, $\lambda - 1111111$, $\lambda - 1011121$, $\lambda - 1001221$ restrict to μ , and Lemma 2.3.19 yields

$$m_{V|X}(\mu) \geq \begin{cases} 14 & \text{if } p \mid a + 6; \\ 16 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\mu) \leq 8$ and $m_{L_X(\omega')}(\mu) \leq 7$ by Theorem 2.3.11. Hence we may assume $p \mid a + 6$ for the rest of the proof, which in particular forces $a > 2$. Let then $\omega'' = \omega - 3\beta_1 - 3\beta_2 - \beta_3 - \beta_4$ and observe that the T_Y -weights $\lambda - 3321000$, $\lambda - 3311100$, $\lambda - 3221010$, $\lambda - 3211110$, $\lambda - 3111120$, $\lambda - 2221011$, $\lambda - 2211111$, $\lambda - 2111121$, $\lambda - 2101221$, $\lambda - 1111122$ all restrict to ω'' . By Lemmas 2.3.19 and 6.1.2, we have $m_{V|X}(\omega'') \geq 22$, while Theorem 2.3.11 and Corollary 6.1.30 respectively yield $m_{L_X(\omega')}(\omega'') \leq 12$ and $m_{L_X(\omega)}(\omega'') \leq 9$, yielding the desired contradiction. Therefore $\lambda = \lambda_1 + \lambda_6$ and one easily concludes using [Lüb01, Appendices A.11, A.41] in the case where $p = 3$. \square

Finally, it remains to treat the case where $p \neq 2$ and $\lambda = a\lambda_1 + \lambda_3 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, so $\omega = (a+b)\omega_1 + \omega_3 + \omega_4$. First suppose that $ab \neq 0$, in which case $\lambda - 1000000$, $\lambda - 0000001$ restrict to $\omega' = \omega - \beta_1$. As usual, an application of Lemma 6.2.1 shows that ω' affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a+b-2)\omega_1 + \omega_2 + \omega_3 + \omega_4).$$

Lemma 6.2.6

Let λ, ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. First observe that the T_Y -weights $\lambda - 1110000$, $\lambda - 0110001$, $\lambda - 0010011$, $\lambda - 0000111$ restrict to $\mu_1 = \omega - \beta_1 - \beta_2 - \beta_3$, and a simple application of Lemma 2.3.19 yields

$$m_{V|X}(\mu_1) \geq \begin{cases} 5 & \text{if } p \mid a + 3; \\ 6 & \text{otherwise,} \end{cases}$$

while on the other hand $m_{L_X(\omega)}(\mu_1) \leq 3$ and $m_{L_X(\omega')}(\mu_1) \leq 2$ by Theorem 2.3.11. We shall thus assume $p \mid a + 3$ for the remainder of the proof (in particular $p \neq 3$). Also if $b > 1$, then $a > 1$ as well since λ is p -restricted, and the T_Y -weights $\lambda - 2000000$, $\lambda - 1000001$, $\lambda - 0000002$ restrict to $\mu_2 = \omega - 2\beta_1 \in X^+(T_X)$, so that $m_{V|X}(\mu_2) \geq 3$. On the other hand, $m_{L_X(\omega)}(\mu_2) = m_{L_X(\omega')}(\mu_2) = 1$, hence the existence of a third KX -composition factor in V . Finally, assume $b = 1$ and observe that the T_Y -weights $\lambda - 1121000$, $\lambda - 1111100$, $\lambda - 1011110$, $\lambda - 0121001$, $\lambda - 0111101$, and $\lambda - 0111111$ restrict to $\omega'' = \omega - \beta_1 - \beta_2 - \beta_3 - \beta_4$. Hence $m_{V|X}(\omega'') \geq 12$ by Lemma 2.3.19, while on the other hand Theorem 2.3.11 gives $m_{L_X(\omega)}(\omega'') \leq 7$ as well as $m_{L_X(\omega)}(\omega'') \leq 4$, thus completing the proof. \square

We are now able to deal with the case where $\lambda = a\lambda_1 + \lambda_3 + b\lambda_7$ in its entirety and thus with the situation in which X' acts irreducibly on $L_{Y'}(\lambda')$.

Proposition 6.2.7

Let $\lambda = a\lambda_1 + \lambda_3 + b\lambda_7$, where $a, b \in \mathbb{Z}_{\geq 0}$, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight λ . Suppose in addition that X has exactly two composition factors on V . Then (λ, p) appears in Table 6.1, where we give λ up to graph automorphisms.

Proof. Suppose that X has exactly two composition factors on V , and observe that by Lemma 6.2.6, we have $ab = 0$. Also, if $\lambda = \lambda_3$, so $p \neq 2$, then X acts irreducibly on V by [Sei87, Theorem 1, Table 1 (I_4, I_5)], contradicting our initial assumption. Assume then $a \neq b = 0$ and consider the T_X -weight $\omega' = \omega - \beta_1 - \beta_2 - \beta_3 - \beta_4$. Then the T_Y -weights $\lambda - 1121000$, $\lambda - 1111100$, $\lambda - 1011110$ and $\lambda - 0011111$ restrict to ω' , so that Lemma 2.3.19 yields

$$m_{V|X}(\omega') \geq \begin{cases} 6 & \text{if } p \mid a + 3; \\ 8 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\omega') \leq 7 - 2\epsilon_p(a + 3)$ thanks to Lemma 6.1.25. One then easily checks that $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \prec \omega$, so that ω' affords the highest weight of a KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega') = L_X((a - 1)\omega_1 + \omega_2).$$

Next assume $a > 1$, consider $\mu = \omega - 2\beta_1 - 2\beta_2 - \beta_3 - \beta_4$ and observe that the T_Y -weights $\lambda - 2221000$, $\lambda - 2211100$, $\lambda - 2111110$, and $\lambda - 1111111$ all restrict to μ . Applying Lemmas 2.3.19 and 6.1.3 yields

$$m_{V|X}(\mu) \geq \begin{cases} 9 & \text{if } p \mid a + 3; \\ 15 & \text{otherwise,} \end{cases}$$

while on the other hand $m_{L_X(\omega)}(\mu) \leq 12$ and $m_{L_X(\omega')}(\mu) \leq 2$ by Theorem 2.3.11, forcing $p \mid a+3$. Also if $a > 2$, then the T_Y -weights $\lambda - 3342000$, $\lambda - 3332100$, $\lambda - 3322200$, $\lambda - 3232110$, $\lambda - 3222210$, $\lambda - 3122220$, $\lambda - 2232111$, $\lambda - 2222211$, $\lambda - 2122221$ and $\lambda - 1122222$ restrict to $\omega' = \omega - 3\omega_1 - 3\omega_2 - 2\omega_3 - 2\omega_4 \in \Lambda^+(T_X)$. Therefore $m_{V|_X}(\omega'') \geq 24$ by Lemmas 2.3.19, 6.1.2, 6.1.3, and 6.1.4, while $m_{L_X(\omega)}(\omega'') \leq 14$ and $m_{L_X(\omega')}(\omega'') \leq 8$, by Lemma 6.1.35 and Theorem 2.3.11 respectively, giving the existence of a third KX -composition factor of V , a contradiction. Consequently either $\lambda = \lambda_1 + \lambda_3$ or $2\lambda_1 + \lambda_3$ ($p = 5$), and [Lüb01, Appendices A.11, A.41] allows us to conclude in each case.

Finally, suppose that $\lambda = \lambda_3 + b\lambda_7$ for some $b \in \mathbb{Z}_{>0}$ and consider the T_X -weights $\omega' = \omega - \beta_1 - \beta_2 - \beta_3$, $\omega'' = \omega - \beta_1 - \beta_2 - \beta_4$. One checks that $\lambda - 1110000$, $\lambda - 0110001$, $\lambda - 0010011$, $\lambda - 0000111$ restrict to ω' , whose multiplicity in $L_X(\omega)$ is smaller than or equal to 3, showing the existence of a second KX -composition factor of V . A similar argument yields $[V|_X, L_X(\omega'')] \neq 0$, so that X has more than two composition factors on V . \square

6.2.2 The reducible case and conclusion

Keeping the notation introduced above, we now suppose that X' has exactly two composition factors on $L_{Y'}(\lambda')$. By Theorem 5.1, we thus get that λ' and p are as in Table 5.1, where we give λ up to graph automorphisms. We start by investigating the case where $\lambda' = 2\lambda'_1$ and $p \neq 3$, that is, $\lambda = a\lambda_1 + 2\lambda_2 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$. In the paragraph preceding the statement of Proposition 5.2.11, we showed that $m_{L_{Y'}(\lambda')|_{X'}}(\nu') = m_{L_X(\lambda|_{T_{X'}})}(\nu')$ for every $\nu' \in X^+(T_{X'})$ such that $\lambda'|_{T_{X'}} - 2\beta'_1 - \beta'_2 - \beta'_3 \prec \nu' \preceq \lambda'|_{T_{X'}}$, while on the other hand

$$m_{L_{Y'}(\lambda')|_{X'}}(\lambda'|_{T_{X'}} - 2\beta'_1 - \beta'_2 - \beta'_3) = m_{L_X(\lambda|_{T_{X'}})}(\lambda'|_{T_{X'}} - 2\beta'_1 - \beta'_2 - \beta'_3) + 1.$$

Writing $\omega = \lambda|_{T_X}$ and $\omega' = \omega - 2\beta_2 - \beta_3 - \beta_4$, an application of Lemma 2.3.7 then yields $m_{V|_X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \preceq \omega$ as well as

$$m_{V|_X}(\omega') = m_{L_X(\omega)}(\omega') + 1.$$

Therefore each of ω and ω' affords the highest weight of a second KX -composition factor of V by Lemma 6.2.1, namely

$$L_X(\omega) = L_X((a+b)\omega_1 + 2\omega_2) \text{ and } L_X(\omega') = L_X((a+b+2)\omega_1).$$

Lemma 6.2.8

Let λ , ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. We first leave to the reader to check (using [Lüb01, Appendices A.11, A.41]) that if $a = b = 0$, then $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, so that X has more than two composition factors on V . Also, if $ab \neq 0$, then one easily sees that the T_X -weight $\omega - \beta_1$ affords the highest weight of a third KX -composition factor of V . Similarly, if $a = 0 \neq b$, then one checks that the T_X -weight $\omega - \beta_1 - \beta_2$ occurs in a third KX -composition factor of V .

Hence for the remainder of the proof, we may assume $\lambda = a\lambda_1 + 2\lambda_2$, with $a \in \mathbb{Z}_{>0}$. Let then $\mu = \omega - \beta_1 - 2\beta_2 - \beta_3 - \beta_4 \in X^+(T_X)$ and observe that the T_Y -weights $\lambda - 1221000$, $\lambda - 1211100$, $\lambda - 1111110$, and $\lambda - 0111111$ restrict to μ . Lemmas 2.3.19 and 5.1.3 then yield

$$m_{V|X}(\mu) \geq \begin{cases} 4 & \text{if } p \mid a + 3; \\ 9 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\mu) \leq 7$ and $m_{L_X(\omega')}(\mu) = 1$ by Theorem 2.3.11. We thus assume $p \mid a + 3$ for the remainder of the proof, so $a > 1$, and set $\omega'' = \omega - 2\beta_1 - 4\beta_2 - 2\beta_3 - 2\beta_4 \in X^+(T_X)$. Here the T_Y -weights $\lambda - 2442000$, $\lambda - 2432100$, $\lambda - 2422200$, $\lambda - 2332110$, $\lambda - 2322210$, $\lambda - 2222220$, $\lambda - 1332111$, $\lambda - 1322211$, $\lambda - 1222221$, and $\lambda - 0222222$ restrict to ω'' , hence $m_{V|X}(\omega'') \geq 10$, while on the other hand Proposition 6.1.28 and Theorem 2.3.11 respectively yield $m_{L_X(\omega)}(\omega'') \leq 6$ and $m_{V_X(\omega')}(\omega'') = 3$, giving the existence of a third KX -composition factor of V as desired. \square

Next suppose that $\lambda = a\lambda_1 + 3\lambda_2 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$. As in the previous case, one shows that each of the T_X -weights ω and $\omega' = \omega - 2\beta_2 - \beta_3 - \beta_4$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega) = L_X((a+b)\omega_1 + 3\omega_2) \text{ and } L_X(\omega') = L_X((a+b+2)\omega_1 + \omega_2).$$

Lemma 6.2.9

Let λ , ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. Proceeding exactly as in the proof of Lemma 6.2.8 yields the desired result in the cases where $a = 0$ or $ab \neq 0$ and gives $p \mid a + 4$ if $a \neq 0 = b$. (The details are left to the reader.) In the latter situation, one checks that the T_Y -weights $\lambda - 1442000$, $\lambda - 1432100$, $\lambda - 1422200$, $\lambda - 1332110$, $\lambda - 1322210$, $\lambda - 1222220$, $\lambda - 0332111$, $\lambda - 0322211$, $\lambda - 0222221$ all restrict to $\omega'' = \omega - \beta_1 - 4\beta_2 - 2\beta_3 - 2\beta_4$, so that $m_{V|X}(\omega'') \geq 9$. By Lemmas 6.1.21 and 6.1.22 on the other hand, we get $m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 8$, showing that ω'' occurs in a third KX -composition factor of V as desired. \square

We next assume $p = 7$ and consider the situation where $\lambda = a\lambda_1 + 2\lambda_2 + \lambda_6 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, so $\omega = (a+b)\omega_1 + 3\omega_2$. Arguing as in the paragraph preceding Lemma 6.2.8 (replacing Proposition 5.2.11 by Proposition 5.2.10), one checks that the T_Y -weight $\omega' = \omega - \beta_2$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a+b+1)\omega_1 + \omega_2 + \omega_3 + \omega_4).$$

Lemma 6.2.10

Let λ , ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. As usual, if $ab \neq 0$, one easily checks that the T_X -weight $\omega - \beta_1$ occurs in a third KX -composition factor of V , so we may assume $ab = 0$ for the remainder of the proof. If $a = b = 0$, then the T_Y -weights $\lambda - 1210000$, $\lambda - 1110010$, $\lambda - 1100110$, $\lambda - 0110011$, $\lambda - 0100111$, and $\lambda - 0000121$ all restrict to $\omega'' = \omega - \beta_1 - 2\beta_2 - \beta_3 \in X^+(T_X)$. By Lemma 5.1.1, we get $m_{V|X}(\omega'') \geq 7$, while on the other hand Theorem 2.3.11 gives

$$m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 6,$$

giving the existence of a third KX -composition factor of V . Next if $a = 0 \neq b$, then the T_Y -weights $\lambda - 1200000$, $\lambda - 1100010$, $\lambda - 0200001$, $\lambda - 0100011$, and $\lambda - 0000021$ restrict to $\omega'' = \omega - \beta_1 - 2\beta_2$, so that $m_{V|X}(\omega'') \geq 5$, while by Lemma 2.3.19, we have

$$m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 4,$$

which again shows that ω'' occurs in a third KX -composition factor of V . Finally if $a \neq 0 = b$, the T_Y -weights $\lambda - 1100000$, $\lambda - 1000010$, and $\lambda - 0000011$ restrict to $\mu_1 = \omega - \beta_1 - \beta_2$. Lemma 2.3.19 thus yields

$$m_{V|X}(\mu_1) \geq \begin{cases} 3 & \text{if } a = 4; \\ 4 & \text{otherwise,} \end{cases}$$

while $m_{L_X(\omega)}(\mu_1) + m_{L_X(\mu_1)}(\mu_1) \leq 3$ by Lemma 2.3.19. We thus assume $a = 4$ and check that the T_Y -weights $\lambda - 2321000$, $\lambda - 2311100$, $\lambda - 2221010$, $\lambda - 2211110$, $\lambda - 2111120$, $\lambda - 1221011$, and $\lambda - 1211111$, $\lambda - 1111121$, $\lambda - 21012210$, and $\lambda - 0111122$ all restrict to $\omega'' = \omega - 2\beta_1 - 3\beta_2 - \beta_3 - \beta_4$. By Theorem 2.3.18, Lemma 6.1.5 and Propositions 6.1.13, 6.1.16, we have $m_{V|X}(\omega'') \geq 22$, while Lemmas 6.1.19 and 6.1.20 yield $m_{L_X(\omega)}(\omega'') + m_{L_X(\omega')}(\omega'') \leq 21$, thus completing the proof. \square

We now consider the case where $p \neq 2$ and $\lambda = a\lambda_1 + \lambda_2 + \lambda_6 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, so $\omega = (a + b)\omega_1 + 2\omega_2$. Here again, one sees that the T_X -weight $\omega' = \omega - \beta_2 \in X^+(T_X)$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a + b + 1)\omega_1 + \omega_3 + \omega_4).$$

Lemma 6.2.11

Let λ, ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. As usual, if $ab \neq 0$, one easily checks that the T_X -weight $\omega - \beta_1$ occurs in a third KX -composition factor of V , so we may assume $ab = 0$ for the remainder of the proof. Also, if both $a = 0$ and $b = 0$, then $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ by [Lüb01, Appendices A.11, A.41], so that a third KX -composition factor occurs in V . Finally, suppose that $a \neq 0 = b$, and let $\mu = \omega - \beta_1 - \beta_2 \in \Lambda^+(\omega)$. Then one shows using Lemma 2.3.19 that

$$m_{V|X}(\mu) \geq \begin{cases} 3 & \text{if } p \mid a + 2; \\ 4 & \text{otherwise,} \end{cases}$$

while Theorem 2.3.11 yields $m_{L_X(\omega)}(\mu) + m_{L_X(\omega')}(\mu) \leq 3$, so we may assume $p \mid a + 2$ and $p \neq 3$ for the remainder of the proof. (We refer the reader to [Lüb01, Appendices A.11, A.41] for the case $a = 1$, $p = 3$.) Let then $\omega'' = \omega - \beta_1 - 2\beta_2 - \beta_3 - \beta_4$, and observe that the T_Y -weights $\lambda - 1221000$, $\lambda - 1211100$, $\lambda - 1111110$, $\lambda - 0111111$, $\lambda - 0011121$, and $\lambda - 0001221$ restrict to ω'' , so that $m_{V|X}(\omega'') \geq 15$ by Theorem 2.3.18 and Lemmas 2.3.19, 6.1.9, while on the other hand Theorem 2.3.11 yields $m_{L_X(\omega)}(\omega''), m_{L_X(\omega')}(\omega'') \leq 7$, completing the proof. \square

We next assume $p \neq 5$ and $\lambda = a\lambda_1 + \lambda_2 + \lambda_3 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, so that $\omega = (a+b)\omega_1 + \omega_2 + \omega_3 + \omega_4$. Arguing as in the paragraph preceding Lemma 6.2.8 (replacing Proposition 5.2.11 by Lemma 5.2.8), one shows that the T_X -weight $\omega' = \omega - \beta_2 - \beta_3 - \beta_4$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a+b+1)\omega_1 + \omega_2).$$

Lemma 6.2.12

Let λ , ω and ω' be as above. Then X has more than two composition factors on $V = L_Y(\lambda)$.

Proof. We leave to the reader to check that if $b \neq 0$, then X has more than two composition factors on V (consider the weight $\omega - \beta_1 - \beta_2$) and thus assume $b = 0$ for the remainder of the proof. Also, if $a = 0$ as well, one checks using [Lüb01, Appendices A.11, A.41] that $\dim L_Y(\lambda) > \dim L_X(\omega) + \dim L_X(\omega')$, so that X has more than two composition factors on V .

Finally, suppose that $\lambda = a\lambda_1 + \lambda_2 + \lambda_3$ for some $a \in \mathbb{Z}_{>0}$ and consider the T_Y -weight $\mu = \omega - \beta_1 - \beta_2 - \beta_3 - \beta_4$. If $a = 1$ and $p = 3$, then [Lüb01, Appendices A.11, A.41] yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, so that X has more than two composition factors on V . Also, if $a = 2$ and $p = 3$, then ω' is not p -restricted and μ does not occur in $L_X(\omega')$. One then easily sees that μ occurs in a third composition factor of V in this situation as well. From now on, we thus assume $p \neq 3$ and observe that the T_Y -weights $\lambda - 1121000$, $\lambda - 1111100$, $\lambda - 1011110$, $\lambda - 0011111$ restrict to μ . Applying Proposition 6.1.10 then yields

$$m_{V|X}(\mu) \geq \begin{cases} 8 & \text{if } p \mid a + 2 \text{ or } a + 4; \\ 10 & \text{otherwise,} \end{cases}$$

while on the other hand, an application of Lemma 6.1.26 (recall that $p \neq 5$, so $a > 1$) and Theorem 2.3.11 gives $m_{L_X(\omega)}(\mu) \leq 8 - 2\epsilon_p(a + 4)$, $m_{V_X(\omega')}(\mu) = 1$. For the remainder of the proof, we may thus assume $p \mid a + 2$. Here the T_Y -weights $\lambda - 1221000$, $\lambda - 1211100$, $\lambda - 1111110$, $\lambda - 0111111$ restrict to $\omega'' = \omega - \beta_1 - 2\beta_2 - \beta_3 - \beta_4 \in X^+(T_X)$, so that $m_{V|X}(\omega'') \geq 14$ by Lemmas 2.3.19, 6.1.17 and Proposition 6.1.10. On the other hand, $m_{L_X(\omega)}(\omega'') \leq 11$ by Lemma 6.1.24, while $m_{L_X(\omega')}(\omega'') = 2$ by Lemma 2.3.19, giving the existence of a third KX -composition factor of V as desired. \square

Finally, consider the situation where $\lambda = a\lambda_1 + \lambda_4 + b\lambda_7$ for some $a, b \in \mathbb{Z}_{\geq 0}$, in which case $\omega = (a+b)\omega_1 + \omega_3 + \omega_4$. Arguing as in the paragraph preceding Lemma 6.2.8 (replacing Proposition 5.2.11 by Proposition 5.2.3), one shows that the T_X -weight $\omega' = \omega + \beta_3 - \beta_4$ affords the highest weight of a second KX -composition factor of V , namely

$$L_X(\omega') = L_X((a+b)\omega_1 + 2\omega_3).$$

Proposition 6.2.13

Assume $p \neq 2$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + \lambda_4 + b\lambda_7$, where $a, b \in \mathbb{Z}_{\geq 0}$. Also suppose that X has exactly two composition factors on V . Then (λ, p) appears in Table 6.1, where we give λ up to graph automorphisms.

Proof. If $ab \neq 0$, then one easily sees that the T_X -weight $\omega - \beta_1$ occurs in a third KX -composition factor of V . Without loss of generality, we thus assume $\lambda = a\lambda_1 + \lambda_4$ for the remainder of the proof. Here the T_Y -weights $\lambda - 1111000$, $\lambda - 1101100$, $\lambda - 1001110$ and $\lambda - 0001111$ restrict to $\omega'' = \omega - \beta_1 - \beta_2 - \beta_4$, so that Lemma 2.3.19 yields

$$m_{V|X}(\omega'') \geq \begin{cases} 6 & \text{if } p \mid a+4; \\ 7 & \text{otherwise} \end{cases}$$

as well as $m_{L_X(\omega')}(\omega'') = m_{L_X(\omega)}(\omega'') = 3 - \epsilon_p(a+4)$. Consequently ω'' occurs in a third KX -composition factor of V , forcing $\lambda = \lambda_4$ as desired. \square

Proof of Theorem 6.1: Let K, Y, X be as in the statement of Theorem 6.1 and first suppose that X acts with exactly two composition factors on V . By Remark 6.2.2, either X' acts irreducibly on $L_Y(\lambda')$ or (λ', p) appears in Table 5.1. In the former case, Propositions 6.2.3, 6.2.4 and 6.2.6 force λ and p to be as in Table 6.1, while if (λ', p) appears in Table 5.1, Lemmas 6.2.12, 6.2.11, 6.2.10, 6.2.8, 6.2.9 together with Propositions 6.2.7, 6.2.13 yield $(\lambda', p) = (\lambda'_2, 2)$ or $(\lambda'_3, \neq 2)$. Using [Lüb01, Appendices A.11, A.41], one can easily check that if $(\lambda, p) \in \{(\lambda_1 + \lambda_3, 2), (\lambda_3 + \lambda_7, 2), (\lambda_1 + \lambda_3 + \lambda_7, 2)\}$, then X has more than two composition factors on V .

In order to complete the proof, it remains to show that for every pair (λ, p) appearing in Table 6.1, X has exactly two composition factors on $V = L_Y(\lambda)$ and that $V|_X$ is completely reducible if and only if $(\lambda, p) \neq (\lambda_3, 2)$. This can be done using [Lüb01, Appendices A.11, A.41] and Proposition 2.6.5 (in the case where $(\lambda, p) \neq (\lambda_3, 2)$). Finally, proceeding exactly as in the proof of Theorem 5.1 completes the proof. The details are left to the reader.

CHAPTER 7

The case $SO_{2n}(K) \subset SL_{2n}(K)$

Let Y be a simply connected simple algebraic group of type A_{2n-1} ($n \geq 5$) over an algebraically closed field K and consider the subgroup X of type D_n , embedded in Y in the usual way. Fix a Borel subgroup $B_Y = U_Y T_Y$ of Y , where T_Y is a maximal torus of Y and U_Y is the unipotent radical of B_Y , let $\Pi(Y) = \{\alpha_1, \dots, \alpha_{2n-1}\}$ denote a corresponding base of the root system $\Phi(Y)$ of Y , and let $\{\lambda_1, \dots, \lambda_{2n-1}\}$ be the set of fundamental dominant weights for T_Y corresponding to our choice of base $\Pi(Y)$. Also let $\Pi(X) = \{\beta_1, \dots, \beta_n\}$ be a set of simple roots for X and let $\{\omega_1, \dots, \omega_n\}$ be the corresponding set of fundamental dominant T_X -weights. The A_{n-1} -parabolic subgroup of X corresponding to the simple roots $\{\beta_1, \dots, \beta_{n-1}\}$ embeds in an $A_{n-1} \times A_{n-1}$ -parabolic subgroup of Y , and up to conjugacy, we may assume that this gives $\alpha_i|_{T_X} = \alpha_{2n-1-i}|_{T_X} = \beta_i$, this for every $1 \leq i \leq n-1$. By considering the action of the Levi factors of these parabolics on the natural KY -module $L_Y(\lambda_1)$, we can deduce that $\alpha_n|_{T_X} = \beta_n - \beta_{n-1}$. Finally, using [Hum78, Table 1, p.69] and the fact that $\lambda_1|_{T_X} = \omega_1$ yields

$$\lambda_i|_{T_X} = \lambda_{2n-i}|_{T_X} = \omega_i, \quad \lambda_{n-1}|_{T_X} = \lambda_{n+1}|_{T_X} = \omega_{n-1} + \omega_n, \quad \lambda_n|_{T_X} = 2\omega_n, \quad (7.1)$$

this for every $1 \leq i \leq n-2$.

Let $V = L_Y(\lambda)$ be an irreducible KY -module having p -restricted highest weight λ . As stated in Chapter 1, a complete classification of the pairs (λ, p) such that X acts with exactly two composition factors on V was not obtained for a general n . However, by restricting the possibilities for λ , we were able to show the following result, where we consider the case $\lambda = a\lambda_i$ for some $a \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq 2n-1$. The methods used in the proof are similar to those introduced in Chapter 6.

Theorem 7.1

Let K, Y, X be as above and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_i \in X^+(T_Y)$, where $a \in \mathbb{Z}_{>0}$ and $1 \leq i \leq 2n - 1$. Then X has exactly two composition factors on V if and only if λ and p are as in Table 7.1, where we give λ up to graph automorphisms. Moreover, if (λ, p) is recorded in Table 7.1, then $V|_X$ is completely reducible if and only if $(\lambda, p) \neq (\lambda_i, 2)$.

λ	p	$V _X$	Dimensions
$2\lambda_1$	$p \nmid n$	$2\omega_1/0$	$(n+1)(2n-1), 1$
$3\lambda_1$	$p \nmid n+1$	$3\omega_1/\omega_1$	$\frac{2}{3}n(n+2)(2n-1), 2n$
λ_2 (n odd)	$p = 2$	$\omega_2/0$	$n(2n-1), 1$
λ_3 (n even)	$p = 2$	ω_3/ω_1	$\frac{2}{3}(n-2)n(2n+1), 2n$
λ_n	$p \neq 2$	$2\omega_{n-1}/2\omega_n$	$\frac{1}{2} \binom{2n}{n}, \frac{1}{2} \binom{2n}{n}$

Table 7.1: The case $\lambda = a\lambda_i$, where $a \in \mathbb{Z}_{>0}$, $1 \leq i \leq 2n - 1$.

Fix $1 \leq i < j \leq 2n - 1$ and consider the T_Y -weight $\lambda = \lambda_i + \lambda_j$. In order to prove a result similar to Theorem 7.1 in this particular situation, we start by studying the structure of the Weyl module $V_X(\lambda|_{T_X})$ for $i = 1$ and $1 < j < n$. The investigation of such KX -modules came to their full description in the case where $p \neq 2$, thus is recorded here for completeness.

Theorem 7.2

Assume $p \neq 2$ and let X be as above. Also fix $1 < j < n$ and consider the dominant T_X -weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. Then the following assertions hold.

1. If $1 < j < n - 2$, we have $V_X(\omega) = \omega/\omega_{j+1}^{\epsilon_p(j+1)}/\omega_{j-1}^{\epsilon_p(2n-j+1)}$. Furthermore, if p divides both $j+1$ and $2n-j+1$, then $V_X(\omega) \supset L_X(\omega_{j+1}) \oplus L_X(\omega_{j-1}) \supset L_X(\omega_{j+1}) \supset 0$ is a composition series of $V_X(\omega)$.
2. If $j = n - 2$, we have $V_X(\omega) = \omega/(\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)}/\omega_{n-3}^{\epsilon_p(n+3)}$. Moreover, if p divides $(n-1)(n+3)$, then $V_X(\omega) \supset L_X(\omega_{n-3})^{\epsilon_p(n+3)} \oplus L_X(\omega_{n-1} + \omega_n)^{\epsilon_p(n-1)} \supset 0$ is a composition series of $V_X(\omega)$.
3. If $\omega = \omega_1 + \omega_{n-1} + \omega_n$, we have $V_X(\omega) = \omega/2\omega_{n-1}^{\epsilon_p(n)}/2\omega_n^{\epsilon_p(n)}/\omega_{n-2}^{\epsilon_p(n+2)}$. Moreover, if p divides n , then $V_X(\omega) \supset L_X(2\omega_{n-1}) \oplus L_X(2\omega_n) \supset L_X(2\omega_{n-1}) \supset 0$ is a composition series of $V_X(\omega)$.

Next we focus our attention on $\omega = \omega_2 + \omega_j$, where $2 < j < n - 1$. In this case, our study does not lead to a full description of the structure of $V_X(\omega)$, as in Theorem 7.2, but still concludes with a complete knowledge of its composition factors.

Theorem 7.3

Assume $p \neq 2$ and let X be as above. Also fix $2 < j \leq n - 2$ and consider the dominant T_X -weight $\omega = \omega_2 + \omega_j$. If $j < n - 2$, then

$$V_X(\omega) = \omega / (\omega_1 + \omega_{j+1})^{\epsilon_p(j)} / (\omega_1 + \omega_{j-1})^{\epsilon_p(2n-j)} / (\omega_{j+2} + \delta_{j,n-3}\omega_n)^{\epsilon_p(j+1)} / \omega_j^{\epsilon_p(n)} / \omega_{j-2}^{\epsilon_p(2n-j+1)},$$

while if $j = n - 2$, then

$$V_X(\omega) = \omega / (\omega_1 + \omega_{n-1} + \omega_n)^{\epsilon_p(n-2)} / (\omega_1 + \omega_{n-3})^{\epsilon_p(n+2)} / 2\omega_{n-1}^{\epsilon_p(n-1)} / 2\omega_n^{\epsilon_p(n-1)} / \omega_{n-2}^{\epsilon_p(n)} / \omega_{n-4}^{\epsilon_p(n+3)}.$$

We next give a list of pairs (λ, p) such that X acts with exactly two composition factors on $L_Y(\lambda)$ in the case where $p \neq 2$ and $\lambda = \lambda_i + \lambda_j$ for some $1 \leq i < j \leq 2n - 1$.

Theorem 7.4

Assume $p \neq 2$ and let Y, X be as above. Also fix $1 \leq i < j < 2n$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_i + \lambda_j \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if λ and p are as in Table 7.2. Furthermore, if (λ, p) is recorded in Table 7.2, then $V|_X$ is completely reducible.

λ	p	$V _X$
$\lambda_1 + \lambda_j$ ($1 < j < n - 1$)	$p \nmid 2n - j + 1$	$\omega_1 + \omega_j / \omega_{j-1}$
$\lambda_1 + \lambda_{n-1}$	$p \nmid n + 2$	$\omega_1 + \omega_{n-1} + \omega_n / \omega_{n-2}$
$\lambda_1 + \lambda_{n+2}$	$p \nmid n - 1$	$\omega_1 + \omega_{n-2} / \omega_{n-1} + \omega_n$
$\lambda_1 + \lambda_j$ ($n + 2 < j < 2n$)	$p \nmid 2n - j + 1$	$\omega_1 + \omega_{2n-j} / \omega_{2n-j+1}$

Table 7.2: The case $\lambda = \lambda_i + \lambda_j$, where $1 \leq i < j \leq 2n - 1$.

Theorem 7.5

Let K, Y, X be as above, with $p \nmid n + 1$ and let (λ, p) be as in Table 7.3. Then X has exactly two composition factors on $V = L_Y(\lambda)$. Moreover, if (λ, p) is recorded in Table 7.3, then $V|_X$ is completely reducible.

λ	p	$V _X$
$2\lambda_1 + \lambda_j$ ($1 < j < n - 1$)	$p \mid j + 2, p \nmid n + 2$	$2\omega_1 + \omega_j / \omega_1 + \omega_{j-1}$
$2\lambda_1 + \lambda_{n-1}$	$p \mid n + 1$	$2\omega_1 + \omega_{n-1} + \omega_n / \omega_1 + \omega_{n-2}$
$2\lambda_1 + \lambda_{n+2}$	$p \mid n + 4$	$2\omega_1 + \omega_{n-2} / \omega_1 + \omega_{n-1} + \omega_n$
$2\lambda_1 + \lambda_j$ ($n + 1 < j < 2n$)	$p \mid j + 2, p \nmid n + 2$	$2\omega_1 + \omega_{2n-j} / \omega_1 + \omega_{2n-j+1}$

Table 7.3: The case $\lambda = 2\lambda_1 + \lambda_j$.

Finally, we record the following conjecture, based on computations and examples from [Lüb01], [Lüb15]. In particular, Theorems 5.1 and 6.1 show that the conjecture holds in the case where $n = 3$ or 4.

Conjecture 7.6

Let K, Y, X be as above, and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$. Then X has exactly two composition factors on V if and only if (λ, p) is recorded in Table 7.1, Table 7.2 or Table 7.3.

Remark

Let (λ, p) be as in Table 7.1, Table 7.2 or 7.3. Then $L_Y(\lambda)|_X$ is completely reducible if and only if $(\lambda, p) \notin \{(\lambda_2, 2), (\lambda_3, 2)\}$.

7.1 Preliminaries

Let K, Y, X be as above and write $\mathcal{L}(Y), \mathcal{L}(X)$ to denote the Lie algebras of Y and X respectively. As in Section 2.5.1, let $\mathcal{B}_Y = \{e_\alpha, f_\alpha, h_{\alpha_i} : \alpha \in \Phi(Y)^+, 1 \leq i \leq 2n - 1\}$ and $\mathcal{B}_X = \{e_\beta, f_\beta, h_{\beta_i} : \beta \in \Phi^+(X), 1 \leq i \leq n\}$ be standard Chevalley bases of $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ respectively. For $1 \leq i < j \leq 2n - 1$, write $e_{i,j} = e_{\alpha_i + \dots + \alpha_j}$. One can check (see [Sei87, Section 8]) that we may assume $e_{\beta_r} = e_{\alpha_r} - e_{\alpha_{2n-r}}$ for every $1 \leq r < n$, as well as $e_{\beta_n} = e_{n-1,n} - e_{n,n+1}$. Using the latter observation, the reader easily deduces that if V is a rational KY -module and $e_{\beta_r}v = 0$ for some $1 \leq r \leq n$, then $e_{\delta_{r,n}\alpha_{n-1} + \alpha_r}v = e_{\alpha_{2n-r} + \delta_{r,n}\alpha_{n+1}}v$ for every $v \in V$. In particular, the following result holds.

Lemma 7.1.1

Let V be a rational KY -module and suppose that v^+ is a maximal vector in V for B_X . Then $e_{\alpha_r}v^+ = e_{\alpha_{2n-r}}v^+$ for every $1 \leq r < n$ and $e_{n-1,n}v^+ = e_{n,n+1}v^+$.

The following consequence of Lemma 7.1.1 shall provide us with a way of proceeding by induction in the proof of Proposition 7.1.3

Lemma 7.1.2

Let V be a KY -module, fix $1 \leq i \leq j \leq 2n - 1$ such that $j - i \leq 2n - 3$ and suppose that $v^+ \in V$ is a maximal vector in V for B_X . If $0 \neq e_{i,j}v^+$ is not a maximal vector in V for B_X , then either $e_{r,j}v^+ \neq 0$ for some $1 \leq r < i$ or $e_{i,s}v^+ \neq 0$ for some $j < s \leq 2n - 1$.

Proof. Assume $i \notin \{n+1, n+2\}$ and $j \notin \{n-2, n-1\}$. (Observe that in this situation, we have neither $2n-i+1 = i-1$ nor $2n-j-1 = j+1$.) One first notices that $[e_{\beta_n}, e_{i,j}] = 0$ and hence $e_{\beta_n}e_{i,j}v^+ = 0$. On the other hand, writing $N_1 = N_{(\alpha_{i-1}, \alpha_i + \dots + \alpha_j)}$ and $N_2 = N_{(\alpha_{j+1}, \alpha_i + \dots + \alpha_j)}$,

one gets

$$\begin{aligned}
e_{\beta_r} e_{i,j} v^+ &= (e_{\alpha_r} - e_{\alpha_{2n-r}}) e_{i,j} v^+ \\
&= e_{\alpha_r} e_{i,j} v^+ - e_{\alpha_{2n-r}} e_{i,j} v^+ \\
&= [e_{\alpha_r}, e_{i,j}] v^+ - [e_{\alpha_{2n-r}}, e_{i,j}] v^+ + e_{i,j} (e_{\alpha_r} v^+ - e_{\alpha_{2n-r}} v^+) \\
&= N_1 (\delta_{r,i-1} - \delta_{r,2n-i+1}) e_{i-1,j} v^+ + N_2 (\delta_{r,j+1} - \delta_{r,2n-j-1}) e_{i,j+1} v^+
\end{aligned}$$

for every $1 \leq r < n$, where the last equality follows from Lemma 7.1.1, thus proving the assertion in this situation. We leave the remaining cases to the reader, as they can be dealt with in a similar fashion. \square

The following result (inspired by [Sei87, Proposition 8.5] and [BGT15, Lemma 4.3.6]) shows that in an irreducible KY -module V , there is always a maximal vector of weight ν for B_X “not too far” from a given maximal vector of weight μ for B_X , in the sense that $\text{ht}(\mu - \nu)$ is “small”. We recall that for any $\gamma = \sum_{r=1}^n c_r \alpha_r \in \mathbb{Z}\Pi$, the height $\text{ht}(\gamma)$ of γ is defined by

$$\text{ht}(\gamma) = \sum_{r=1}^n c_r.$$

Also, we denote by $\mathfrak{b}_X = \mathfrak{h}_X + \sum_{\alpha \in \Phi^+(Y)} K e_\alpha$ the Borel subalgebra of X , where \mathfrak{h}_X is the Lie algebra of T_X and write $V_{\max} = \{v^+ \in V : v^+ \text{ is a maximal vector for } \mathfrak{b}_X\}$.

Proposition 7.1.3

Consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight λ . Also write $\omega = \lambda|_{T_X}$ and let $\omega \neq \mu \in X^+(T_X)$ be such that $V_\mu \cap V_{\max} \neq \emptyset$. Then there exist $\nu \in X^+(T_X)$ and $0 \neq v^+ \in V_\nu$ such that $\mu \prec \nu \preceq \mu + 2\beta_1 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n$ and $e_{\beta_r} v^+ = 0$ for every $1 \leq r \leq n$ (that is, v^+ is a maximal vector in V for \mathfrak{b}_X).

Proof. Let $u^+ \in V_\mu \cap V_{\max}$ and assume for a contradiction that

$$V_\nu \cap V_{\max} = \emptyset \tag{7.2}$$

for every $\nu \in X^+(T_X)$ such that $\mu \prec \nu \preceq \mu + 2\beta_1 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n$. We first claim that $e_{i,j} u^+ = 0$ for every $1 \leq i \leq j \leq 2n-1$, proceeding by induction on $l = 2n+i-j-2$. If $l = 0$, then $i = 1$, $j = 2n-1$, and since $[e_\alpha, e_{1,2n-1}] = 0$ for every $\alpha \in \Phi^+(Y)$, one immediately gets $e_{\beta_r} e_{1,2n-1} u^+ = 0$ for every $1 \leq r \leq n$. Hence $e_{1,2n-1} u^+ = 0$ by (7.2). Assume then the claim true for $0 \leq l < l_0 < 2n-1$ (where $1 \leq l_0 < 2n-1$ is fixed) and let $1 \leq i_0 \leq j_0 \leq 2n-1$ be such that $2n-j_0+i_0-2 = l_0$ and $e_{i_0,j_0} u^+ \neq 0$. If there exists $1 \leq r \leq n$ such that $e_{\beta_r} e_{i_0,j_0} u^+ \neq 0$, then an application of Lemma 7.1.2 shows the existence of $1 \leq s < i_0$ or $j_0 < t \leq n$ such that $e_{s,j_0} u^+ \neq 0$ or $e_{i_0,t} u^+ \neq 0$, contradicting our induction hypothesis. Therefore $e_{\beta_r} e_{i,j} u^+ = 0$ for every $1 \leq r \leq n$ and $1 \leq i \leq j \leq 2n-1$, forcing $e_{i,j} u^+ = 0$ for every $1 \leq i \leq j \leq 2n-1$ by our initial assumption. In particular, we get that $e_{\alpha_r} u^+ = 0$ for every $1 \leq r \leq 2n-1$ and since $u^+ \notin V_\lambda$, we have $u^+ = 0$, giving our final contradiction and thus completing the proof. \square

Remark 7.1.4

A result similar to Proposition 7.1.3 can be proven in the case where X is of type B_n or (respectively, C_n) over K and embedded in $Y = \mathrm{SL}_{2n+1}(K)$ (respectively, $Y = \mathrm{SL}_{2n}(K)$) in the usual way.

We next give two consequences of [Sei87, Theorem 1, Table 1 (I_4, I_5)] on certain weight multiplicities.

Lemma 7.1.5

Assume $p \neq 2$ and fix $1 < j < n$. Also let $\omega = \omega_j + \delta_{j,n-1}\omega_n$ and adopt the notation $\omega_0 = 0$. Then $m_{L_X(\omega)}(\omega_{j-2}) = n - j + 2$. Similarly, if $3 < j < n$ then

$$m_{L_X(\omega)}(\omega_{j-4}) = \frac{1}{2}(n - j + 3)(n - j + 4).$$

Proof. We shall prove the first assertion and leave the second to the reader, as it can be dealt with in a similar fashion. By considering a suitable Levi subgroup of X , it is enough to prove the assertion in the case where $\omega = \omega_2$. Write $\lambda = \lambda_2$ and observe that the T_Y -weights in $L_Y(\lambda)$ restricting to $0 \in X^+(T_X)$ are $\lambda - (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r + \alpha_{r+1} + \cdots + \alpha_{2n-r-1})$ ($2 \leq r \leq n-1$), $\lambda - (\alpha_2 + \cdots + \alpha_{2n-2})$ and $\lambda - (\alpha_2 + \cdots + \alpha_{2n-1})$, all having multiplicity 1 in $L_Y(\lambda)$. Therefore $m_{L_Y(\lambda)|_X} = n$ and since $L_Y(\lambda)|_X \cong L_X(\omega)$ by [Sei87, Theorem 1, Table 1 (I_4, I_5)], the result follows. \square

Finally, consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda \in X^+(T_Y)$ and let v^+ denote a maximal vector in V for B_Y . Since $B_X \subset B_Y$, we get that v^+ is a maximal vector for B_X as well, so that $\omega = \lambda|_{T_X}$ affords the highest weight of a KX -composition factor of V . We conclude this section by recording a generalization of Lemma 5.2.1. Its proof being very similar to that of the latter, we omit the details here.

Lemma 7.1.6

Let ω be as above, and suppose that $\langle \lambda, \alpha_n \rangle = 0$. Then every T_Y -weight of $V = L_Y(\lambda)$ satisfies $\mu|_{T_X} \preceq \omega$.

7.2 Proof of Theorem 7.1

In this section, we give a complete proof of Theorem 7.1, starting by recording some general information on weight multiplicities and the structure of certain Weyl modules for a simple algebraic group of type D_n over K .

7.2.1 Preliminary considerations

Let G be a simple algebraic group of type D_n ($n \geq 5$) over K and as usual, fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U the unipotent radical of B . Also let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ denote a corresponding base of the root system Φ of G and $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental dominant weights for T corresponding to our choice of base Π . We start by investigating the multiplicity of $\mu = \sigma - (2\gamma_1 + \dots + 2\gamma_{n-2} + \gamma_{n-1} + \gamma_n)$ in a given irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = a\sigma_1$, where $a \in \mathbb{Z}_{>1}$, using information on the structure of $V_G(\sigma)$ as an \mathcal{L} -module, where \mathcal{L} denotes the Lie algebra of G . As usual, let $\mathcal{B} = \{e_\gamma, f_\gamma, h_{\gamma_i} : \gamma \in \Phi^+, 1 \leq i \leq n\}$ be a standard Chevalley basis of \mathcal{L} as in Section 2.5.1. By (2.14) and our choice of ordering on Φ^+ , one checks that the weight space $V_G(\sigma)_\mu$ is spanned by

$$\begin{aligned} & \{f_{1,i}F_{1,i+1}v^\sigma\}_{1 \leq i \leq n-3} \cup \{f_{1,n-2}f_{1,n}v^\sigma\} \\ & \cup \{f_{1,n-1}f_{\gamma_1+\dots+\gamma_{n-2}+\gamma_n}v^\sigma\}, \end{aligned} \quad (7.3)$$

where $v^\sigma \in V_G(\sigma)_\sigma$ denotes a maximal vector in $V_G(\sigma)$ for G (and thus for the corresponding Borel subalgebra \mathfrak{b} of \mathcal{L} as well). An application of Theorem 2.3.11 yields $m_{V_G(\sigma)}(\mu) = n-1$, forcing the generating elements of (7.3) to be linearly independent, so that the following holds.

Proposition 7.2.1

Let G be a simple algebraic group of type D_n over K and let $\sigma = a\sigma_1$, where $a \in \mathbb{Z}_{>1}$. Also consider $\mu = \sigma - (2\gamma_1 + \dots + 2\gamma_{n-2} + \gamma_{n-1} + \gamma_n)$. Then μ is dominant and the set (7.3) forms a basis of $V_G(\sigma)_\mu$.

Lemma 7.2.2

Let V be as above and write $\mu = \sigma - (2\gamma_1 + \dots + 2\gamma_{n-2} + \gamma_{n-1} + \gamma_n)$. Then μ is dominant, $\chi^\mu(\sigma) = \text{ch } L_G(\sigma) + \epsilon_p(a+n-2) \text{ch } L_G(\mu)$ and

$$m_V(\mu) = \begin{cases} n-2 & \text{if } p \mid a+n-2; \\ n-1 & \text{otherwise.} \end{cases}$$

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. One easily checks that $\nu_c^\mu(T_\sigma) = \nu_p(a+n-2)\chi^\mu(\mu)$ and since $\chi^\mu(\mu) = \text{ch } L_G(\mu)$, an application of Proposition 2.7.8 shows that $m_V(\mu) = m_{V_G(\sigma)}(\mu)$ if $p \nmid a+n-2$, so that the result follows from Proposition 7.2.1. We thus assume $p \mid a+n-2$ for the remainder of the proof, in which case μ affords the highest weight of a composition factor of $V_G(\sigma)$ by Proposition 2.7.8. Since $[V_G(\sigma), L_G(\nu)] = 0$ for every $\nu \in X^+(T)$ such that $\mu \prec \nu \prec \sigma$, we get that $[V_G(\sigma), L_G(\mu)]$ equals the dimension of the subspace of $V_G(\sigma)$ spanned by all maximal vectors in $V_G(\sigma)_\mu$ for B . Therefore, it only remains to show that the latter is 1-dimensional.

For $(A_r)_{1 \leq r \leq n-1} \in K^{n-1}$, set

$$u(A) = \sum_{r=1}^{n-3} A_r f_{1,r} F_{1,r+1} v^\sigma + A_{n-2} f_{1,n-2} f_{1,n} v^\sigma + A_{n-1} f_{1,n-1} f_{\gamma_1 + \dots + \gamma_{n-2} + \gamma_n} v^\sigma.$$

Clearly $u(A) \in V_G(\sigma)_\mu$ for every $A = (A_r)_{1 \leq r \leq n-1} \in K^{n-1}$ and by Lemma 2.5.5, we get

$$e_{\gamma_1} u(A) = \left(aA_1 + \sum_{i=2}^{n-1} A_i \right) F_{1,2} v^\sigma,$$

while

$$e_{\gamma_r} u(A) = (A_r - A_{r-1}) f_{1,r-1} F_{1,r+1} v^\sigma$$

for every $2 \leq r \leq n-3$, as well as

$$\begin{aligned} e_{\gamma_{n-2}} u(A) &= (A_{n-2} - A_{n-3}) f_{1,n-3} f_{1,n} v^\sigma, \\ e_{\gamma_{n-1}} u(A) &= (A_{n-1} - A_{n-2}) f_{1,n-2} f_{\gamma_1 + \dots + \gamma_{n-2} + \gamma_n} v^\sigma, \\ e_{\gamma_n} u(A) &= (A_{n-1} - A_{n-2}) f_{1,n-2} f_{1,n-1} v^\sigma. \end{aligned}$$

One checks that each of the vectors $F_{1,2} v^\sigma$, $f_{1,r-1} F_{1,r+1} v^\sigma$ ($2 \leq r \leq n-3$), $f_{1,n-3} f_{1,n} v^\sigma$, $f_{1,n-2} f_{\gamma_1 + \dots + \gamma_{n-2} + \gamma_n} v^\sigma$ and $f_{1,n-2} f_{1,n-1} v^\sigma$ is non-zero, so that $e_\gamma u(A) = 0$ for every $\gamma \in \Pi$ if and only if $A \in K^{n-1}$ is a solution to the system of equations

$$\begin{cases} aA_1 &= -\sum_{i=2}^{n-1} A_i \\ A_{r-1} &= A_r \text{ for every } 2 \leq r \leq n-1. \end{cases} \quad (7.4)$$

Now one easily sees that (7.4) admits a non-trivial solution $A \in K^{n-1}$ if and only if p divides $a + n - 2$, in which case $A \in \langle (1, \dots, 1) \rangle_K$. Therefore any two maximal vectors $u, u' \in V_G(\sigma)_\mu$ in $V_G(\sigma)$ for \mathfrak{b} satisfy $\langle u \rangle_K = \langle u' \rangle_K$, thus completing the proof. \square

We next consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = 2\sigma_1$. Here $\Lambda^+(\sigma) = \{\sigma, \sigma_2, 0\}$ and proceeding as in the proof of Lemma 2.4.8 (replacing Lemma 2.3.19 by Lemma 7.2.2), one easily shows the following result. The details are left to the reader.

Corollary 7.2.3

Let G be a simple algebraic group of type D_n over K and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = 2\sigma_1$. Then $V_G(\sigma) = \sigma / 0^{\epsilon_p(n)}$ and

$$\dim V = (n+1)(2n-1) - \epsilon_p(n).$$

A similar result holds in the case where $V = L_G(\sigma)$ is an irreducible KG -module having p -restricted highest weight $\sigma = 3\sigma_1$. Again, one proceeds as in the proof of Lemma 2.4.8, observing that

$$\Lambda^+(\sigma) = \{\sigma, \sigma_1 + \sigma_2, \sigma_3, \sigma_1\}$$

in this situation and $m_{V_G(\sigma)}(\sigma) = m_{V_G(\sigma)}(\sigma_1 + \sigma_2) = m_{V_G(\sigma)}(\sigma_3) = 1$. We leave the proof to the reader.

Corollary 7.2.4

Let G be a simple algebraic group of type D_n over K and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = 3\sigma_1$. Then $V_G(\sigma) = \sigma/\sigma_1^{\epsilon_p(n+1)}$ and

$$\dim V = \frac{2}{3}n((n+2)(2n-1) - 3\epsilon_p(n+1)).$$

In order to give a proof of Theorem 7.1, we need a better understanding of the structure of the Weyl module $V_G(\sigma_i)$ for $2 \leq i < n-1$, as well as $V_G(\sigma_{n-1} + \sigma_n)$. Now the composition factors of $V_G(\sigma_2)$ are well-known (see [Lüb01, Table 2], for example). Also observe that $\Lambda^+(\sigma_3) = \{\sigma_3, \sigma_1\}$ and thus applying Lemma 2.3.7 to the D_{n-1} -parabolic of G corresponding to the simple roots $\gamma_2, \dots, \gamma_n$ shows that the structure of $V_G(\sigma_3)$ is entirely determined by the structure of $V_G(\sigma_2)$. Those observations are recorded in the following Lemma.

Lemma 7.2.5

Assume $n \geq 5$ and consider the dominant T -weight $\sigma = \sigma_i$, where $i = 2$ or 3 . Then the KG -composition factors of $V_G(\sigma)$ are as in Tables 7.4 and 7.5, respectively.

p	Composition factors	Dimensions
$p \neq 2$	σ_2	$n(2n-1)$
$p = 2, n$ odd	$\sigma_2/0$	$n(2n-1) - 1, 1$
$p = 2, n$ even	$\sigma_2/0^2$	$n(2n-1) - 2, 1, 1$

Table 7.4: Composition factors of $V_G(\sigma_2)$ for G of type D_n ($n \geq 5$).

p	Composition factors	Dimensions
$p \neq 2$	σ_3	$\frac{2}{3}(n-1)n(2n-1)$
$p = 2, n$ even	σ_3/σ_1	$\frac{2}{3}(n-2)n(2n+1), 2n$
$p = 2, n$ odd	σ_3/σ_1^2	$\frac{2}{3}n(n+1)(2n-5), 2n, 2n$

Table 7.5: Composition factors of $V_G(\sigma_3)$ for G of type D_n ($n \geq 5$).

Now if $\sigma = \sigma_i + \delta_{i,n-1}\sigma_n$ for some $3 < i < n$, then not much is known about $V_G(\sigma)$ when $p = 2$. Fortunately, the following Lemma provides us with enough information to prove Theorem 7.1. Since it is referred to in Section 2.7 as an example of an application of the Jantzen p -sum formula, a detailed proof is recorded here.

Lemma 7.2.6

Assume $p = 2$ and consider the weight $\sigma = \sigma_4 + \delta_{n,5}\sigma_5 \in X^+(T)$. Then each of σ_2 and 0 affords the highest weight of a composition factor of $V_G(\sigma)$.

Proof. We assume $n > 6$ and refer to [Lüb01, Appendix A.42] and [Lüb15] for the cases where $n = 5, 6$. Let then $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4. We proceed as indicated in Section 2.7, starting by determining the coefficients in the truncated sum $\nu_c^0(T_\sigma)$ in (2.21). (Observe that since $\Lambda^+(\sigma) = \{\sigma, \sigma_2, 0\}$, we have $\nu_c^0(T_\sigma) = \nu_c(T_\sigma)$.) According to Lemma 2.7.11, we must find every $\gamma \in \Phi^+$ and $1 < r < \langle \sigma + \rho, \gamma \rangle$ for which $A_{\gamma,r} \sim_G B_{\sigma_2}$ or B_0 . (We refer the reader to Section 2.7.3 for a definition of $A_{\gamma,r}$ and B_ν .)

First consider the T -weight $\nu_1 = \sigma_2$. Here

$$B_{\nu_1} = (n, n - 1, n - 3, \dots, 1, 0)$$

and since $\sigma - \nu_1$ has support $\{\gamma_3, \dots, \gamma_n\}$, we can focus our attention on roots belonging to the subset $I = \{\varepsilon_3 + \varepsilon_l\}_{3 < l < n}$ of Φ^+ . For such $\gamma = \varepsilon_3 + \varepsilon_l \in I$ and $1 < r < \langle \sigma + \rho, \gamma \rangle$, we get

$$A_{\gamma,r} = (n, n - 1, n - 2 - r, n - 3, n - 5, \dots, 1, 0) - (r\delta_{jl})_{j=1}^n,$$

and one easily checks that $A_{\gamma,r} \sim_G B_{\nu_1}$ if and only if γ and r appear in Table 7.6. For such pairs (γ, r) , we also record the contribution to $\nu_c(T_\sigma)$ obtained by applying Corollary 2.7.7 for completeness.

γ	r	Contribution to $\nu_c(T_\sigma)$
$\varepsilon_3 + \varepsilon_4$	$2(n - 3)$	$\nu_p(2(n - 3))$
$\varepsilon_3 + \varepsilon_{n-1}$	2	$\nu_p(2)$
$\varepsilon_3 + \varepsilon_{n-1}$	$n - 3$	$-\nu_p(n - 3)$

Table 7.6: Contribution of ν_1 to $\nu_c(T_\sigma)$.

Next consider $\nu_2 = 0 \in \Lambda^+(\sigma)$, in which case $B_{\nu_2} = (n - 1, \dots, 1, 0)$. Here $\sigma - \nu_2$ has support Π , so that we only need to consider roots belonging to the subset $J = \{\varepsilon_1 + \varepsilon_l\}_{1 < l < n}$ of Φ^+ . For such $\gamma = \varepsilon_1 + \varepsilon_l \in J$ and $1 < r < \langle \sigma + \rho, \gamma \rangle$, we have

$$A_{\gamma,r} = (n - r, n - 1, n - 2, n - 3, n - 5, \dots, 1, 0) - (r\delta_{jl})_{j=1}^n$$

and again one checks that $A_{\gamma,r} \sim_G B_{\nu_2}$ if and only if γ and r appear in Table 7.7.

γ	r	Contribution to $\nu_c(T_\sigma)$
$\varepsilon_1 + \varepsilon_3$	2	$-\nu_p(2)$
$\varepsilon_1 + \varepsilon_3$	$2(n-2)$	$\nu_p(2(n-2))$
$\varepsilon_1 + \varepsilon_{n-2}$	4	$\nu_p(4)$
$\varepsilon_1 + \varepsilon_{n-2}$	$n-2$	$-\nu_p(n-2)$

Table 7.7: Contribution of ν_2 to $\nu_c(T_\sigma)$.

Therefore as we assumed $p = 2$, we get $\nu_c(T_\sigma) = 2\chi(\sigma_2) + 2\chi(0)$ and an application of Lemma 7.2.5 yields $\nu_c(T_\sigma) = 2 \operatorname{ch} L_G(\sigma_2) + 2(2 + \varepsilon_2(n)) \operatorname{ch} L_G(0)$. We then conclude thanks to Proposition 2.7.5. \square

7.2.2 Conclusion

Let K, Y, X be as in the statement of Theorem 7.1, fix $a \in \mathbb{Z}_{>0}$, $1 \leq i \leq 2n-1$, and consider the T_Y -weight $\lambda = a\lambda_i \in X^+(T_Y)$. Proceeding by induction on n (using Lemma 2.3.10 together with Theorems 5.1 and 6.1), we first give a small list of candidates (λ, p) with X acting with exactly two composition factors on $L_Y(\lambda)$.

Lemma 7.2.7

Consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_i$ for some $a \in \mathbb{Z}_{>0}$ and $1 \leq i \leq 2n-1$. Also suppose that X has exactly two composition factors on V . Then λ and p are as in Table 7.8, where we give λ up to graph automorphisms.

λ	p
$a\lambda_1$ ($a \in \mathbb{Z}_{\geq 2}$)	any
λ_2	$= 2$
$2\lambda_2$	$\nmid n-1$
$3\lambda_2$	$\nmid n$
λ_3 (n even)	$= 2$
λ_4 (n odd)	$= 2$
λ_n	$\neq 2$

Table 7.8: Candidates for (λ, p) to occur in Table 7.1.

Proof. Assume Theorem 7.1 is true for $Y = Y_k$ of type A_{2k-1} over K and every $3 \leq k < n$ (by Theorems 5.1 and 6.1, the result holds for $k = 3, 4$), and let $Y = Y_n$ be of type A_{2n-1} over K . Set $J = \{\beta_2, \dots, \beta_n\} \subset \Pi(X)$ and adopting the notation introduced in Section 2.3.2, consider the D_{n-1} -parabolic subgroup $P_J = Q_J L_J$ of X .

Denote by $P_Y = Q_Y L_Y$ the parabolic subgroup of Y given by Lemma 2.3.9 and notice that L_Y has type A_{2n-3} over K , with $\Pi(L_Y) = \{\alpha_2, \dots, \alpha_{2n-2}\}$. Write $\tilde{Y} = L'_Y$, $\tilde{X} = L'_J$ and finally $\tilde{\lambda} = \lambda|_{T_{\tilde{Y}}}$. An application of Lemma 2.3.10 then shows that \tilde{X} acts with at most two composition factors on $L_{\tilde{Y}}(\tilde{\lambda})$. If the latter is irreducible for \tilde{X} , then up to graph automorphisms, either $\lambda = a\lambda_1$ for some $a \in \mathbb{Z}_{\geq 0}$ or $\lambda = \lambda_r$ for some $1 < r < n$ thanks to [Sei87, Theorem 1, Table 1 (I_4, I_5)]. If on the other hand \tilde{X} acts with exactly two composition factors on $L_{\tilde{Y}}(\tilde{\lambda})$, then one easily concludes thanks to our induction hypothesis. \square

We now study all candidates (λ, p) given by Lemma 7.2.7, starting with the case where $\lambda = \lambda_i$ for some $1 \leq i < n$. As usual, we write ω to denote the restriction of $\lambda \in X^+(T_Y)$ to T_X . Observe that $L_Y(\lambda) \cong \bigwedge^i V_Y(\lambda_1)$ and hence $L_Y(\lambda)|_X \cong \bigwedge^i V_X(\omega_1)$. By Proposition 2.6.3, the latter admits a Weyl filtration, yielding

$$L_Y(\lambda)|_X \cong V_X(\omega). \quad (7.5)$$

Proposition 7.2.8

Consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = \lambda_i$, where $1 \leq i \leq n$. Then X has exactly two composition factors on V if and only if (λ, p) appears in Table 7.1.

Proof. First assume $p \neq 2$ and suppose that X has exactly two composition factors on V . Applying Lemma 7.2.7 then forces $\lambda = \lambda_n$, in which case $\omega = 2\omega_n$ by (7.1). Also the T_Y -weight $\lambda - \alpha_n$ restricts to $\omega' = 2\omega_{n-1}$, which is neither above nor under ω . One then checks that there is no weight $\nu \in \Lambda^+(V|_X)$ such that $\omega' \prec \nu$, showing that ω' is a highest weight of $V|_X$. Lemmas 2.4.5 and 2.4.7 then yield $\dim V = \dim L_X(\omega) + \dim L_X(\omega')$, showing that X has exactly two composition factors on V as desired.

Next assume $p = 2$ and suppose that X has exactly two composition factors on V . As above, an application of Lemma 7.2.7 yields $1 < i \leq 4$. Also observe that by Lemma 7.1.6, we have $\Lambda^+(V) = \Lambda^+(\omega)$, and by (7.5), the number of composition factors of $V|_X$ equals the number of composition factors of $V_X(\omega)$. Lemma 7.2.6 then rules out the possibility $\lambda = \lambda_4$, while the two remaining cases can be dealt with using Lemma 7.2.5. We leave the details to the reader. \square

Next we tackle the case $\lambda = a\lambda_2$, where $1 < a < 4$ and $p \nmid a+n-3$. Considering the D_{n-1} -parabolic subgroup of X corresponding to the roots β_2, \dots, β_n as in the proof of Lemma 7.2.7, one easily sees (using induction) that the T_X -weight $\omega' = \omega - (2\beta_2 + \dots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$ affords the highest weight of a second KX -composition factor of $L_Y(\lambda)$, namely

$$L_X(\omega') = L_X(2\omega_1 + (a-2)\omega_2).$$

Lemma 7.2.9

Consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_2$, where $1 < a < 4$. Then X has more than two composition factors on V .

Proof. Let ω, ω' be as above. Applying (2.9) (and Proposition 2.4.2 if desired), one first checks that $\dim V_Y(\lambda) > \dim V_X(\omega) + \dim V_X(\omega')$, which shows that in particular the result holds in characteristic zero. On the other hand, in the case where $a = 3$, one checks that

$$\Lambda^+(V) = \{\lambda, \lambda_1 + \lambda_2 + \lambda_3, 2\lambda_1 + \lambda_4, 2\lambda_3, \lambda_2 + \lambda_4, \lambda_1 + \lambda_5, \lambda_6\},$$

and using [Lüb01, Appendix A.10] together with Lemma 2.3.7 (applied to the A_5 -Levi subgroup of Y corresponding to the simple roots $\alpha_1, \dots, \alpha_5$) yields $V_Y(\lambda) \cong L_Y(\lambda)$. Therefore the assertion holds in this situation as well and we may assume $a = 2$ for the remainder of the proof. Here $\Lambda^+(\lambda) = \{\lambda, \lambda_1 + \lambda_3, \lambda_4\}$ and by Lemma 5.1.1 applied to the Levi subgroup of Y corresponding to the simple roots $\alpha_1, \alpha_2, \alpha_3$, we get $V_Y(\lambda) = \lambda/\lambda_4^{\epsilon_p(3)}$. If $p \neq 3$, then the result follows from the characteristic zero case, while if on the other hand $p = 3$, we have $[V_X(\omega), L_X(\omega_4)] = 1$ (by Lemma 5.1.1 again) and thus

$$\begin{aligned} \dim V &= \dim V_Y(\lambda) - \dim L_Y(\lambda_4) \\ &> \dim V_X(\omega) + \dim V_X(\omega') - \dim L_Y(\lambda_4) \\ &= \dim V_X(\omega) + \dim V_X(\omega') - \dim L_X(\omega_4) \\ &\geq \dim L_X(\omega) + \dim L_X(\omega'), \end{aligned}$$

where the last equality follows from [Sei87, Theorem 1, Table 1 (I₄, I₅)]. Consequently X has more than two composition factors on V in this situation as well, completing the proof. \square

We now deal with the case where $\lambda = a\lambda_1$ for some $a \in \mathbb{Z}_{\geq 0}$, by showing the following generalization of Proposition 6.2.3.

Proposition 7.2.10

Let $a, b \in \mathbb{Z}_{\geq 0}$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = a\lambda_1 + b\lambda_{2n-1}$. Then X has exactly two composition factors on V if and only if λ and p appear in Table 7.9, where we give λ up to graph automorphisms.

λ	p	$V _X$
$2\lambda_1$	$p \nmid n$	$2\omega_1/0$
$3\lambda_1$	$p \nmid n + 1$	$3\omega_1/\omega_1$
$\lambda_1 + \lambda_{2n-1}$	$p \neq 2$	$2\omega_1/\omega_2$
$2\lambda_1 + \lambda_{2n-1}$	$p \mid 2n + 1, p \neq 3$	$3\omega_1/\omega_1 + \omega_2$

Table 7.9: The case $\lambda = a\lambda_1 + b\lambda_{2n-1}$, where $a, b \in \mathbb{Z}_{\geq 0}$.

Proof. We proceed exactly as in the proof of Proposition 6.2.3, first considering the case where $b = 0$, so that $\lambda = a\lambda_1$, $\omega = a\omega_1$, and suppose that X has exactly two composition factors on V . We may assume $a > 1$, so $\omega' = \omega - (2\beta_1 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n) \in \Lambda^+(\omega)$. As in the proof of Proposition 6.2.3, one easily checks that exactly n weights for T_Y restrict to ω' , each having multiplicity 1 in V , hence $m_{V|_X}(\omega') = n$. Applying Lemmas 6.2.1 and 7.2.2 shows that ω' affords the highest weight of a second KX -composition factor of V and that $p \nmid a + n - 2$. Now if $a > 3$, consider $\omega'' = \omega - (4\beta_1 + \cdots + 4\beta_{n-2} + 2\beta_{n-1} + 2\beta_n)$. Using Theorem 2.3.11 (we refer to the proof of Proposition 6.2.3 once more), we get

$$m_{V|_X}(\omega'') \geq \frac{n(n+1)}{2}, \quad m_{V_X(\omega)}(\omega'') = \frac{n(n-1)}{2}, \quad \text{and} \quad m_{V_X(\omega')}(\omega'') = n-1,$$

giving the existence of a third composition factor of $V|_X$. Therefore $a = 2$ or 3 , in which case every T_X -weight but ω' has multiplicity 1 in $L_X(\omega)$. Applying Lemmas 2.4.4 together with Corollary 7.2.3 (respectively, 7.2.4) then yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ unless p divides $a + n - 2$. Under the latter condition on p , we have $\dim V = \dim L_X(\omega) + \dim L_X(\omega')$, so that X has exactly two composition factors on V as desired.

Now consider the case where $ab \neq 0$ and again first assume that X has exactly two composition factors on V . Here $\omega = (a+b)\omega_1$ and the T_Y -weights restricting to $\omega' = \omega - \beta_1$ are $\lambda - \alpha_1$ and $\lambda - \alpha_{2n-1}$. As $m_{L_X(\omega)}(\omega') = 1$, Lemma 7.1.6 shows that ω' affords the highest weight of a second composition factor of $V|_X$. Also, if $a, b > 1$, then the T_Y -weights $\lambda - 2\alpha_1$, $\lambda - \alpha_1 - \alpha_{2n-1}$, and $\lambda - 2\alpha_{2n-1}$ restrict to $\omega - 2\beta_1$, whose multiplicity in both $L_X(\omega)$ and $L_X(\omega')$ equals 1, giving the existence of a third composition factor of $V|_X$, a contradiction. Without loss of generality, we may then suppose that $\lambda = a\lambda_1 + \lambda_{2n-1}$, so that $\omega = (a+1)\omega_1$ and $\omega' = (a-1)\omega_1 + \omega_2$. Three situations may occur.

1. If $a = 1$ and $p = 2$, then $\omega' \notin L_X(\omega)$ by Theorem 2.3.2, so that $[V|_X, L_X(\omega')] = 2$ and X has more than two composition factors on V . If on the other hand $p \neq 2$, then one easily checks (using Lemma 2.4.8, Corollary 7.2.3 and Lemma 2.4.6) that $\dim L_Y(\lambda) = \dim L_X(\omega) + \dim L_X(\omega')$, hence the result.
2. If $a = 2$ and $p = 3$, then proceeding exactly as in the previous case shows that X has more than two composition factors on V . If on the other hand $p \neq 3$, then one checks (using Lemma 2.4.9, Corollary 7.2.4 and [McN98, Remark 4.9.3 (a)]) that $\dim L_Y(\lambda) = \dim L_X(\omega) + \dim L_X(\omega')$ if and only if $p \mid 2n+1$, thus yielding the desired assertion.
3. Finally if $a > 2$, let $\omega'' = \omega - (3\beta_1 + 2\beta_2 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$ and observe (as in the proof of Proposition 6.2.3) that at least $2n-1$ weights of V restrict to ω'' . As one of those ($\lambda - (2\alpha_1 + \alpha_2 + \cdots + \alpha_{2n-1})$) has multiplicity greater than or equal to $2(n-1)$ in $V_Y(\lambda)$, we get that $m_{V|_X}(\omega'') \geq 4(n-1)$, while by Theorem 2.3.11, we have $m_{V_X(\omega)}(\omega'') = n-1$, and $m_{V_X(\omega')}(\omega'') = 3n-4$, giving the existence of a third KX -composition factor of V .

□

Using Lemmas 7.2.7, 7.2.9, and Propositions 7.2.8 and 7.2.10, we are finally able to give a proof of the main result of this Section.

Proof of Theorem 7.1: Let K, Y, X be as in Theorem 7.1 and first suppose that X acts with exactly two composition factors on $V = L_Y(\lambda)$, where $\lambda = a\lambda_i$ for some $a \in \mathbb{Z}_{>0}$ and $1 \leq i \leq 2n - 1$. By Lemma 7.2.7, we get that λ and p are as in Table 7.8, and applying Lemma 7.2.9 shows that X has more than two composition factors on $L_Y(a\lambda_2)$ if $a > 1$. The result then follows from Propositions 7.2.8 and 7.2.10. Finally, we leave to the reader to check the assertion on the structure of $V|_X$, using Proposition 2.6.5 together with (7.5).

7.3 Proof of Theorem 7.2

Let X be as in the statement of Theorem 7.2 and assume throughout this section that $p \neq 2$. Here we determine the structure of $V_X(\omega)$, for $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_{n-1}$, where $1 < j < n - 1$. We start by considering the embedding of X in Y , where Y is as in the preamble of the chapter, and find a decomposition of $V_Y(\lambda_1 + \lambda_j)|_X$ ($1 < j < 2n$) in terms of irreducibles, assuming K has characteristic zero. As we shall see, doing so leads to a nice expression for the formal character of $V_X(\omega_1) \otimes V_X(\omega_j)$. We then study $V_X(\omega)$ via the embedding $\iota : V_X(\omega) \hookrightarrow V_X(\omega_1) \otimes V_X(\omega_j)$ given by Proposition 2.6.4.

7.3.1 Restriction of Weyl modules

Fix $1 < j < n$ and set $\lambda = \lambda_1 + \lambda_j$, which by (7.1) restricts to $\omega_1 + \omega_j + \delta_{j,n-1}\omega_n \in X^+(T_X)$. We first find a description of $\text{ch } V_Y(\lambda)|_X$ in terms of the \mathbb{Z} -basis $\{\chi(\mu)\}_{\mu \in X^+(T_X)}$ of $\mathbb{Z}[X(T_X)]^{\#X}$. In order to avoid the use of Theorem 2.3.11, we shall apply Proposition 7.1.3.

Proposition 7.3.1

Fix $1 < j < n$, write $\lambda = \lambda_1 + \lambda_j$ and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X , that is $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_{j-1}).$$

Proof. Write $V = V_Y(\lambda)$ and first observe that $\text{ch } V|_X$ is independent of p , so we may and will assume K has characteristic zero for the remainder of the proof. We start by treating the situation where $j = 2$, in which case $\omega = \omega_1 + \omega_2$ is the highest weight of $V|_X$ by Lemma 7.1.6 and

$$\Lambda^+(V|_X) = \{\omega, \omega_3, \omega_1\}.$$

One then shows that the only T_Y -weight restricting to ω_3 is λ_3 , and $\text{m}_{V|_X}(\omega_3) = \text{m}_{V_X(\omega)}(\omega_3)$. An application of Theorem 2.4.3 then yields $\dim V = \dim V_X(\omega) + \dim V_X(\omega_1)$, from which the assertion follows.

Next assume $2 < j < n - 2$, in which case $\Lambda^+(V|_X) = \Lambda^+(\omega)$ by Lemma 7.1.6, and write $\omega' = \omega_{j-1}$. An elementary computation (using Theorem 2.4.1) yields $\dim V_Y(\lambda) > \dim V_X(\omega)$, showing the existence of $\omega' \in X^+(T_X)$ such that $[V_Y(\lambda)|_X, L_X(\omega')] \neq 0$. Since we assumed K has characteristic zero, this translates to the existence of a maximal vector in $V_{\omega'}$ for B_X . By Proposition 7.1.3, we have $\omega - (2\beta_1 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n) \preceq \omega' \prec \omega$ and we leave to the reader to check that this forces

$$\omega' \in \{\omega_1 + \omega_{j-2}, \omega_{j+1}, \omega_{j-1}\}.$$

Now applying Lemma 2.3.7 to the D_{n-j+1} -parabolic subgroup of X corresponding to the simple roots $\beta_{j-1}, \dots, \beta_n$, followed by [Sei87, Theorem 1, Table 1 (I₄, I₅)] shows that $\omega_1 + \omega_{j-2}$ cannot afford the highest weight of a KX -composition factor of V . Also the only T_Y -weight in V restricting to ω_{j+1} is λ_{j+1} , from which one easily sees that $m_{V|_X}(\omega_{j+1}) = m_{V_X(\omega)}(\omega_{j+1})$ and hence $[V|_X, L_X(\omega_{j+1})] = 0$. Consequently $\omega' = \omega_{j-1}$ and an elementary computation (using Theorem 2.4.1 again, for example) yields $\dim V_Y(\lambda) = \dim V_X(\omega) + \dim V_X(\omega_{j-1})$, so the result holds in this case as well. The situations where $j = n - 2$ or $n - 1$ can be dealt with in a similar fashion and hence the details are left to the reader. \square

Corollary 7.3.2

Fix $1 < j < n$ and consider $\omega = a\omega_1 + \omega_j + \delta_{j,n-1}\omega_n$, where $a \in \mathbb{Z}_{>0}$. Then the T_X -weight $\mu = \omega - (\beta_1 + \cdots + \beta_{j-1} + 2\beta_j + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$ satisfies

$$m_{V_X(\omega)}(\mu) = j(n - j + 2) - 2.$$

Proof. First assume $a = 1$ as well as $1 < j < n - 1$ and consider the Weyl module $V = V_Y(\lambda)$ having highest weight $\lambda = \lambda_1 + \lambda_j$, so that $\lambda|_{T_X} = \omega$. The T_Y -weights in V restricting to μ are $\lambda - (\alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_r + \alpha_{r+1} + \cdots + \alpha_{2n-r-1})$ ($j \leq r \leq n - 1$), $\lambda - (\alpha_1 + \cdots + \alpha_{2n-j})$, $\lambda - (\alpha_1 + \cdots + \alpha_s + \alpha_j + \cdots + \alpha_{2n-s-1})$ ($1 \leq s \leq j - 2$), and $\lambda - (\alpha_j + \cdots + \alpha_{2n-1})$. Therefore Theorem 2.3.11 yields $m_{V|_X}(\mu) = j(n - j + 2) - 1$, while on the other hand, an application of Proposition 7.3.1 yields $m_{V_X(\omega)}(\mu) = m_{V_Y(\lambda)|_X}(\mu) - 1$, yielding the desired assertion in this situation. We leave to the reader to deal with the case where $\omega = a\omega_1 + \omega_{n-1} + \omega_n$ and then conclude using Proposition 2.3.12. \square

We now investigate the case where $\lambda = \lambda_1 + \lambda_n$, writing $\omega = \lambda|_{T_X}$. Here contrary to what we had in Proposition 7.3.1, the T_X -weight ω is not the unique highest weight of $V|_X$. (For example $\lambda - \alpha_n$ restricts to $\omega_1 + 2\omega_{n-1}$, which is neither under nor above ω .) The following result provides an alternative to Lemma 7.1.6 in this specific situation.

Lemma 7.3.3

Consider the T_Y -weight $\lambda = \lambda_1 + \lambda_n \in X^+(T_Y)$. Then each of $\omega = \lambda|_{T_X}$ and $\omega' = \omega - \alpha_n|_{T_X}$ affords the highest weight of a KX -composition factor of V . Furthermore, every T_X -weight $\nu \in \Lambda^+(V|_X)$ either satisfies $\nu \preceq \omega$ or $\nu \preceq \omega'$.

Proof. The fact that ω affords the highest weight of a KX -composition factor of V follows from the fact that any maximal vector in V for B_Y is a maximal vector for B_X as well. Now let $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r \in \Lambda(V)$ be such that $\omega' \preceq \mu|_{T_X}$. Recalling the restrictions to T_X of the simple roots for T_Y , we immediately get $c_r = 0$ for every $1 \leq r \leq 2n-1$ different from $n-1, n, n+1$. One then easily shows that $c_n = 1$ and hence $c_{n-1} = c_{n+1} = 0$, forcing $\mu|_{T_X} = \omega'$ as desired. In order to prove the second assertion, assume for a contradiction the existence of $\mu = \lambda - \sum_{r=1}^{2n-1} c_r \alpha_r \in \Lambda(V)$ such that neither $\mu|_{T_X} \preceq \omega$ nor $\mu|_{T_X} \preceq \omega'$. Recalling the restrictions to T_X of the simple roots for T_Y , one checks that

$$\mu|_{T_X} = \omega - \sum_{r=1}^{n-2} (c_r + c_{2n-r}) \beta_r - (c_{n-1} - c_n + c_{n+1}) \beta_{n-1} - c_n \beta_n.$$

Since we assumed $\mu|_{T_X} \not\preceq \omega$, we have $c_{n-1} - c_n + c_{n+1} < -1$. In particular $\langle \mu, \alpha_n \rangle < -c_n$, showing that $s_{\alpha_n}(\mu) \in \Lambda(V)$ is not under λ , a contradiction. \square

Thanks to Lemma 7.3.3, we are now able to prove a result similar to Proposition 7.3.1 in the case where $\lambda = \lambda_1 + \lambda_n$.

Proposition 7.3.4

Consider $\lambda = \lambda_1 + \lambda_n$ and denote by ω (respectively, ω') the restriction of λ (respectively, $\lambda - \alpha_n$) to T_X , that is $\omega = \omega_1 + 2\omega_n$ and $\omega' = \omega_1 + 2\omega_{n-1}$. Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega') + \chi(\omega_{n-1} + \omega_n).$$

Proof. Write $V = V_Y(\lambda)$ and notice that $\text{ch } V|_X$ is independent of p , so we may assume K has characteristic zero for the remainder of the proof. Also observe that the T_Y -weights restricting to $\omega'' = \omega_{n-1} + \omega_n$ are $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_n + \cdots + \alpha_{2n-r-1})$ (for $1 \leq r \leq n-1$) and $\lambda - (\alpha_n + \cdots + \alpha_{2n-1})$. An application of Lemma 2.3.19 then yields $m_{V|_X}(\omega'') = 2n-1$ as well as $m_{V_X(\omega)}(\omega'') = m_{V_X(\omega')}(\omega'') = n-1$, thus showing that ω'' occurs in a third KX -composition factor of V . Now one easily checks that every T_X -weight $\nu \in \Lambda^+(V|_X)$ such that $\omega'' \prec \nu \prec \omega$ or $\omega' \prec \nu \prec \omega'$ satisfies

$$m_{V|_X}(\nu) = m_{V_X(\omega)}(\nu) + m_{V_X(\omega')}(\nu),$$

showing that ω'' affords the highest weight of a third KX -composition factor of V by Lemma 7.3.3. As in the proof of Proposition 7.3.1, an application of Theorem 2.4.1 then yields the desired result. \square

Finally we give a result similar to Propositions 7.3.1 and 7.3.4 in the case where $\lambda = \lambda_1 + \lambda_j$ for some $n < j < 2n-1$, even though it shall not be used in the proof of Theorem 7.2. As a matter of fact, Proposition 7.3.5 provides us with an alternative proof of Corollary 7.3.2, which does not rely on Proposition 7.1.3.

Proposition 7.3.5

Fix $n < j < 2n - 1$, write $\lambda = \lambda_1 + \lambda_j$ and denote by $\omega \in X^+(T_X)$ the restriction of the T_Y -weight λ to T_X , so $\omega = \omega_1 + \omega_{2n-j} + \delta_{j,n+1}\omega_n$. Then

$$\text{ch } V_Y(\lambda)|_X = \begin{cases} \chi(\omega) + \chi(2\omega_{n-1}) + \chi(2\omega_n) & \text{if } j = n + 1; \\ \chi(\omega) + \chi(\omega_{2n-j+1} + \delta_{j,n+2}\omega_n) & \text{otherwise.} \end{cases}$$

Proof. As in the proofs of Propositions 7.3.1 and 7.3.4, we may assume K has characteristic zero throughout the proof. Start by supposing $j = n + 1$ and consider the dominant T_X -weight $\omega' = \omega - (\beta_1 + \cdots + \beta_{n-1}) = 2\omega_n \in X^+(T_X)$. Then the T_Y -weights restricting to ω' are $\lambda - (\alpha_1 + \cdots + \alpha_{n-1})$, $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_{n+1} + \cdots + \alpha_{2n-r-1})$ ($1 \leq r < n - 1$), and $\lambda - (\alpha_{n+1} + \cdots + \alpha_{2n-1})$. Therefore $m_{V|_X}(\omega') = n$, while $m_{V_X(\omega)}(\omega') = n - 1$ by Lemma 2.3.19, showing that ω' occurs in a second KX -composition factor of V . Since there is no dominant weight $\nu \in X^+(\omega)$ such that $\omega' \prec \nu \prec \omega$, an application of Lemma 7.1.6 yields $[V|_X, L_X(\omega')] = 1$ as desired. We leave to the reader to show that $[V|_X, L_X(2\omega_{n-1})] = 1$ as well, from which one easily concludes thanks to Theorem 2.4.1, for example.

Next assume $n + 1 < j < 2n - 1$ and let $\omega' = \omega - (\beta_1 + \cdots + \beta_{2n-j}) = \omega_{2n-j+1} \in X^+(T_X)$. Then one easily checks that the T_Y -weights restricting to ω' are $\lambda - (\alpha_j + \cdots + \alpha_{2n-1})$, $\lambda - (\alpha_1 + \cdots + \alpha_r + \alpha_j + \cdots + \alpha_{2n-r-1})$ ($1 \leq r \leq 2n - j - 1$), and $\lambda - (\alpha_1 + \cdots + \alpha_{2n-j})$. Therefore $m_{V|_X}(\omega') = 2n - j + 1$, while an application of Lemma 2.3.19 yields $m_{V_X(\omega)}(\omega') = 2n - j$, showing that ω' occurs in a second KX -composition factor of V . As above, there is no dominant weight $\nu \in X^+(\omega)$ such that $\omega' \prec \nu \prec \omega$ and thus $[V|_X, L_X(\omega')] = 1$ by Lemma 7.1.6. Again, applying Theorem 2.4.1 completes the proof. \square

7.3.2 Weyl filtrations and tensor products

Let G be a simple algebraic group of type A_n over K , fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U the unipotent radical of B . Also let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ be a corresponding base of the root system Φ of G and $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental weights for T corresponding to our choice of base Π . An expression for the formal character of the tensor product of exterior powers of the natural KG -module can be given by the following well-known result, whose proof is recorded here for completeness.

Lemma 7.3.6

Fix $1 < j < n$, write $\sigma = \sigma_1 + \sigma_j$ and consider $T(\sigma_1, \sigma_j) = V_G(\sigma_1) \otimes V_G(\sigma_j)$. Then $T(\sigma_1, \sigma_j)$ is tilting (see Definition 2.6.2) and

$$\text{ch } T(\sigma_1, \sigma_j) = \chi(\sigma) + \chi(\sigma_{j+1}).$$

Proof. The first assertion directly follows from Proposition 2.6.4. Also $\text{ch}T(\sigma_1, \sigma_j)$ is independent of p and thus we shall assume K has characteristic zero for the remainder of the proof. By Proposition 2.6.4 (part 1), σ is the highest weight of $T(\sigma_1, \sigma_j)$, so that $\Lambda^+(T(\sigma_1, \sigma_j)) = \{\sigma, \sigma_{j+1}\}$, and using Lemma 2.3.19, one gets $m_{T(\sigma_1, \sigma_j)}(\sigma_{j+1}) = j + 1$, while $m_{V_G(\sigma)}(\sigma_{j+1}) = j$. Therefore σ_{j+1} affords the highest weight of a second KG -composition factor of $T(\sigma_1, \sigma_j)$, allowing us to conclude. \square

In the remainder of this section, we assume $p \neq 2$ and let Y, X be as usual. Also for $1 < j < n - 1$, we set

$$T(\omega_1, \omega_j) = V_X(\omega_1) \otimes V_X(\omega_j)$$

and recall that the T_Y -weight $\lambda = \lambda_1 + \lambda_j$ restricts to $\omega_1 + \omega_j$. We now use Proposition 7.3.1 together with Lemma 7.3.6 to determine the formal character of $T(\omega_1, \omega_j)$.

Lemma 7.3.7

Assume $p \neq 2$ and for $1 < j < n - 1$, write $\omega = \omega_1 + \omega_j$. Then $T(\omega_1, \omega_j)$ is tilting and its formal character is given by

$$\text{ch}T(\omega_1, \omega_j) = \chi(\omega) + \chi(\omega_{j+1} + \delta_{j, n-2}\omega_n) + \chi(\omega_{j-1}).$$

Proof. The fact that $T(\omega_1, \omega_j)$ is tilting follows from Lemma 2.4.6 together with Proposition 2.6.4 (part 3). Also $\text{ch}T(\omega_1, \omega_j)$ is independent of p by Lemma 2.4.7 and hence it is enough to find a decomposition of $T(\omega_1, \omega_j)$ into a direct sum of irreducibles in characteristic zero. Now by [Sei87, Theorem 1, Table 1 (I_4, I_5)], the KX -module $V_Y(\lambda_i)|_X$ is isomorphic to $V_X(\omega_i + \delta_{i, n-1}\omega_n)$ for every $1 \leq i < n$ and thus Lemma 7.3.6 yields

$$T(\omega_1, \omega_j) \cong V_Y(\lambda)|_X \oplus V_X(\omega_{j+1} + \delta_{j, n-2}\omega_n),$$

where $\lambda = \lambda_1 + \lambda_j$ as above. Finally $V_Y(\lambda)|_X \cong V_X(\omega) \oplus V_X(\omega_{j-1})$ by Proposition 7.3.1, yielding the desired result. \square

Finally set $T(\omega_1, \omega_{n-1} + \omega_n) = V_X(\omega_1) \otimes V_X(\omega_{n-1} + \omega_n)$ and again, observe that the T_Y -weight $\lambda = \lambda_1 + \lambda_{n-1}$ restricts to $\omega_1 + \omega_{n-1} + \omega_n$. Arguing as in the proof of Lemma 7.3.7, observing that by Theorem 7.1, we have $\text{ch}V(\lambda_n)|_X = \chi(2\omega_{n-1}) + \chi(2\omega_n)$, one obtains the following result. The details are left to the reader.

Lemma 7.3.8

Assume $p \neq 2$ and write $\omega = \omega_1 + \omega_{n-1} + \omega_n$. Then $T(\omega_1, \omega_{n-1} + \omega_n)$ is tilting and its formal character is given by

$$\text{ch}T(\omega_1, \omega_{n-1} + \omega_n) = \chi(\omega) + \chi(\omega_{n-2}) + \chi(2\omega_{n-1}) + \chi(2\omega_n).$$

7.3.3 Conclusion

Fix $1 < j < n$ and consider the T_X -weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. In order to completely describe the structure of $V_X(\omega)$, we start by determining an upper bound for $[V_X(\omega) : L_X(\mu)]$ ($\mu \in X^+(T_X)$), following the ideas of [McN98]. The next assertion, being a consequence of Proposition 2.6.4 and Lemmas 7.3.7, 7.3.8, is referred to as a corollary.

Corollary 7.3.9

Assume $p \neq 2$, fix $1 < j < n$ and let $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$. Also suppose that $\omega \neq \mu$ affords the highest weight of a composition factor of $V_X(\omega)$. Then $[V_X(\omega), L_X(\mu)] = 1$ and one of the following holds.

1. If $j < n - 1$, then $\mu = \omega_{j+1} + \delta_{j,n-2}\omega_n$ or ω_{j-1} .
2. If $j = n - 1$, then $\mu = 2\omega_{n-1}$, $2\omega_n$ or ω_{n-2} .

Proof. First assume $1 < j < n - 2$, write $T(\omega_1, \omega_j) = V_X(\omega_1) \otimes V_X(\omega_j)$, and identify $\text{rad}(\omega)$ with $\iota(\text{rad}(\omega))$, where $\iota : V_X(\omega) \hookrightarrow T(\omega_1, \omega_j)$ is the injection given by Proposition 2.6.4 (part 2). By Lemmas 2.4.6 and 7.3.7, we have

$$\text{ch}(T(\omega_1, \omega_j)/\text{rad}(\omega)) = \text{ch} L_X(\omega) + \text{ch} L_X(\omega_{j+1}) + \text{ch} L_X(\omega_{j-1}),$$

and Proposition 2.6.4 (part 3) applies, yielding a surjective morphism of KX -modules $\phi : T(\omega_1, \omega_j) \rightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch} H^0(\omega) = \chi(\omega)$, the result follows. The cases where $j = n - 2$ or $n - 1$ can be dealt with in a similar fashion (replacing Lemma 7.3.7 by Lemma 7.3.8 in the latter situation), so the details are left to the reader. \square

For ω as in Theorem 7.2, we determine every $\mu \in \Lambda^+(\omega)$ such that $[V_X(\omega), L_X(\mu)] \neq 0$, using the method introduced in Section 2.7. We start by the case where $\omega = \omega_1 + \omega_j$ for some $1 < j < n - 1$.

Proposition 7.3.10

Assume $p \neq 2$, fix $1 < j < n - 1$ and consider the T_X -weight $\omega = \omega_1 + \omega_j$. Also set $\mu_1 = \omega_{j+1} + \delta_{j,n-2}\omega_n$, $\mu_2 = \omega_{j-1}$. Then μ_1 affords the highest weight of a composition factor of $V_X(\omega)$ if and only if $p \mid j + 1$. Similarly, μ_2 affords the highest weight of a composition factor of $V_X(\omega)$ if and only if $p \mid 2n - j + 1$.

Proof. Let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4. Proceeding as usual, we get $\nu_c^{\mu_1}(T_\omega) = \nu_p(j + 1)\chi(\mu_1) + \nu_p(2n - j + 1)\chi(\mu_2)$ and Lemma 2.4.6 then yields $\nu_c^{\mu_2}(T_\omega) = \nu_p(j + 1)\text{ch} L_X(\mu_1) + \nu_p(2n - j + 1)\text{ch} L_X(\mu_2)$. Finally, an application of Proposition 2.7.8 completes the proof. \square

Finally, we consider the dominant T_X -weight $\omega = \omega_1 + \omega_{n-1} + \omega_n$. Again the proof of the following result is omitted, being similar to the proof of Proposition 7.3.10. (Recall that $V_X(2\omega_{n-1}) \cong L_X(2\omega_{n-1})$ if $p \neq 2$ by Lemma 2.4.7.)

Proposition 7.3.11

Assume $p \neq 2$ and consider the T_X -weight $\omega = \omega_1 + \omega_{n-1} + \omega_n$. Also set $\mu_1 = 2\omega_n$, $\mu_2 = 2\omega_{n-1}$ and $\mu_3 = \omega_{n-2}$. Then each of μ_1 and μ_2 affords the highest weight of a composition factor of $V_X(\omega)$ if and only if $p \mid n$. Similarly, μ_3 affords the highest weight of a composition factor of $V_X(\omega)$ if and only if $p \mid n + 2$.

Proof of Theorem 7.2: First assume $1 < j < n - 1$ and let $\omega = \omega_1 + \omega_j$. Then applying Corollary 7.3.9 together with Proposition 7.3.10 yields the result on the composition factors of $V_X(\omega)$ in this case. If on the other hand $\omega = \omega_1 + \omega_{n-1} + \omega_n$, replacing Proposition 7.3.10 by Proposition 7.3.11 allows us to conclude as well. Therefore in order to complete the proof, it only remains to show the assertions on the composition series of $V_X(\omega)$, which directly follow from Lemma 2.6.5.

7.4 Proof of Theorem 7.3: the case $j = 3$

Let X be as in the statement of Theorem 7.3 and assume throughout this section that $p \neq 2$ and $n > 5$ (we refer the reader to [Lüb01, Appendix A.42] for the case where $n = 5$). Here we determine the composition factors of $V_X(\omega)$, for $\omega = \omega_2 + \omega_3$. Again, we let Y be as in the preamble of this chapter and setting $\lambda = \lambda_2 + \lambda_3$, we proceed as in Section 7.3, starting by finding a decomposition of $V_Y(\lambda)|_X$ in terms of irreducibles in characteristic zero.

7.4.1 Restriction of Weyl modules

Set $\lambda = \lambda_2 + \lambda_3$, which by (7.1) restricts to $\omega = \omega_2 + \omega_3 \in X^+(T_X)$. We first find a description of $\text{ch } V_Y(\lambda)|_X$ in terms of the \mathbb{Z} -basis $\{\chi(\mu)\}_{\mu \in X^+(T_X)}$ of $\mathbb{Z}[X(T_X)]^{\mathscr{W}}$. As in the proof of Proposition 7.3.1 (the case where $\omega = \omega_1 + \omega_2$), we take advantage of the fact that

$$\Lambda^+(\omega) = \{\omega, \omega_1 + \omega_4, \omega_1 + \omega_2, \omega_5 + \delta_{n,6}\omega_6, \omega_3, \omega_1\}. \tag{7.6}$$

Proposition 7.4.1

Consider $\lambda = \lambda_2 + \lambda_3 \in X^+(T_Y)$ and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + \omega_2) + \chi(\omega_1).$$

Proof. As in the proof of Proposition 7.3.1, first observe that $\text{ch } V_Y(\lambda)|_X$ is independent of p and thus we may assume K has characteristic zero for the remainder of the proof. Also if $\mu \in X^+(T_X)$ affords the highest weight of a KX -composition factor of $V_Y(\lambda)$, then $\mu \in \Lambda^+(\omega)$ by Lemma 7.1.6. One then easily sees (by applying Proposition 7.3.1 to the D_{n-1} -Levi subgroup of X corresponding to the simple roots $\beta_2, \dots, \beta_n \in \Pi(X)$) that $\omega_1 + \omega_2 \in X^+(T_X)$ affords the highest weight of a second KX -composition factor of $V_Y(\lambda)$. Now an elementary computation (using Theorem 2.4.1) yields $\dim V_Y(\lambda) = \dim V_X(\omega) + \dim V_X(\omega_1 + \omega_2) + 2n$, while $\dim V_X(\omega_5 + \delta_{n,6}\omega_6), \dim V_X(\omega_3) > 2n$. Hence neither $\omega_5 + \delta_{n,6}\omega_6$ nor ω_3 can afford the highest weight of a third composition factor of $V_Y(\lambda)|_X$ and the result follows from (7.6). \square

Corollary 7.4.2

Set $\omega = \omega_2 + \omega_3$ and let $\mu = \omega - (\beta_1 + 2\beta_2 + \dots + 2\beta_{n-2} + \beta_{n-1} + \beta_n) \in X(T_X)$. Then $\mu = \omega_3$ is dominant and satisfies $m_{V_X(\omega)}(\mu) = 5n - 11$.

Proof. Write $\lambda = \lambda_2 + \lambda_3$ and let $V = V_Y(\lambda)$. One checks that T_Y -weights restricting to μ are $\lambda - (\alpha_1 + 2\alpha_2 + \dots + 2\alpha_r + \alpha_{r+1} + \dots + \alpha_{2n-r-1})$ ($1 < r < n$), $\lambda - (\alpha_1 + \dots + \alpha_{2n-2})$ and $\lambda(\alpha_2 + \dots + \alpha_{2n-1})$. Therefore Theorem 2.3.11 yields $m_{V|_X}(\mu) = 5n - 9$, while on the other hand, an application of Proposition 7.4.1 yields $m_{V_X(\omega)}(\mu) = m_{V_Y(\lambda)|_X}(\mu) - 2$, thus completing the proof. \square

A result similar to Proposition 7.4.1 holds in the situation where $\lambda = \lambda_2 + \lambda_{2n-3}$, as the following Proposition shows. Its proof, being very similar to the proof of Proposition 7.3.5, is omitted here.

Proposition 7.4.3

Consider $\lambda = \lambda_2 + \lambda_{2n-3} \in X^+(T_Y)$ and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + \omega_4) + \chi(\omega_5 + \delta_{n,6}\omega_6).$$

7.4.2 Weyl filtrations and tensor products

Let G be a simple algebraic group of type A_n over K , fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U the unipotent radical of B . Also let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ be a corresponding base of the root system Φ of G and $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental weights for T corresponding to our choice of base Π . We first determine the formal character of the tensor product $V_G(\sigma_2) \otimes V_G(\sigma_3)$.

Lemma 7.4.4

Let $\sigma = \sigma_2 + \sigma_3$ and consider the tensor product $T(\sigma_2, \sigma_3) = V_G(\sigma_2) \otimes V_G(\sigma_3)$. Then $T(\sigma_2, \sigma_3)$ is tilting and its formal character is given by

$$\text{ch } T(\sigma_2, \sigma_3) = \chi(\sigma) + \chi(\sigma_1 + \sigma_4) + \chi(\sigma_5).$$

Proof. Proceed exactly as in the proof of Lemma 7.3.6, working in characteristic zero and observing that $\Lambda^+(T(\sigma_2, \sigma_3)) = \{\sigma, \sigma_1 + \sigma_4, \sigma_5\}$. We leave the details to the reader. \square

Adopting the notation $\sigma_{n+1} = 0$, we next focus our attention on the multiplicity of $\mu = \sigma_5$ in the irreducible KG -module having highest weight $\sigma = \sigma_2 + \sigma_3 \in X^+(T_X)$.

Proposition 7.4.5

Consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = \sigma_2 + \sigma_3$ and let $\mu = \sigma_5$. Then $V_G(\sigma) = \sigma / (\sigma_1 + \sigma_4)^{\delta_{p,3}} / \mu^{\delta_{p,2}}$ and

$$m_V(\mu) = \begin{cases} 1 & \text{if } p = 3; \\ 4 & \text{if } p = 2; \\ 5 & \text{otherwise.} \end{cases}$$

Proof. First observe that $\Lambda^+(\sigma) = \{\sigma, \sigma_1 + \sigma_4, \sigma_5\}$ and considering the A_4 -Levi subgroup of G corresponding to the simple roots $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ together with [Lüb15] completes the proof. \square

In the remainder of this section, we assume $p \neq 2$ and let Y, X be as usual. Also, we set $T(\omega_2, \omega_3) = V_X(\omega_2) \otimes V_X(\omega_3)$ and recall that the T_Y -weight $\lambda = \lambda_2 + \lambda_3$ restricts to $\omega_2 + \omega_3$. As in Section 7.3, we use Proposition 7.4.1 together with Lemma 7.4.4 to determine the formal character of $T(\omega_2, \omega_3)$.

Lemma 7.4.6

Assume $p \neq 2$ and set $\omega = \omega_2 + \omega_3$. Then $T(\omega_2, \omega_3)$ is tilting and its formal character is given by

$$\begin{aligned} \text{ch } T(\omega_2, \omega_3) &= \chi(\omega) + \chi(\omega_1 + \omega_4) + \chi(\omega_1 + \omega_2) \\ &\quad + \chi(\omega_5 + \delta_{n,6}\omega_6) + \chi(\omega_3) + \chi(\omega_1). \end{aligned}$$

Proof. The fact that $T(\omega_2, \omega_3)$ is tilting follows from Proposition 2.6.4 (part 3) and Lemma 2.4.6. Also, since $\text{ch } T(\omega_2, \omega_3)$ is independent of p , it is enough to find a decomposition of $T(\omega_2, \omega_3)$ into a direct sum of irreducibles in characteristic zero, so we assume $\text{char } K = 0$ for the remainder of the proof. Now $V_Y(\lambda_5)|_X$ is isomorphic to $V_X(\omega_5 + \delta_{n,6}\omega_6)$ by (7.5) and thus Lemma 7.4.4 yields

$$T(\omega_2, \omega_3) \cong V_Y(\lambda)|_X \oplus V_Y(\lambda_1 + \lambda_4)|_X \oplus V_X(\omega_5 + \delta_{n,6}\omega_6),$$

where $\lambda = \lambda_2 + \lambda_3$ as above. Finally, applying Propositions 7.3.1 and 7.4.1 completes the proof. \square

Using Lemma 7.4.6 together with Proposition 2.6.4 will enable us to determine the composition factors of $V_X(\omega_2 + \omega_3)$. The situation in which $\delta_{p,3}\epsilon_p(n) = 0$ can be dealt with in a pretty straightforward fashion, while in the case where $\delta_{p,3}\epsilon_p(n) = 1$, some more work is required in order to prove that $[V_X(\omega_2 + \omega_3), L_X(\omega_3)] \leq 1$. We thus treat the two cases separately.

7.4.3 Conclusion: the case $\delta_{p,3}\epsilon_p(n) = 0$

Assume $p \neq 2$, $\delta_{p,3}\epsilon_p(n) = 0$ and consider the T_X -weight $\omega = \omega_2 + \omega_3$. We are ready to determine the composition factors of $V_X(\omega)$, starting by finding an upper bound for $[V_X(\omega) : L_X(\mu)]$, for every $\mu \in X^+(T_X)$, and then conclude using Proposition 2.7.8. The first assertion is a consequence of Proposition 2.6.4 and Lemma 7.4.6, so is referred to as a corollary.

Corollary 7.4.7

Assume $p \neq 2$ as well as $\delta_{p,3}\epsilon_p(n) = 0$. Also let $\omega = \omega_2 + \omega_3$ and suppose that $\omega \neq \mu \in X^+(T_X)$ affords the highest weight of a composition factor of $V_X(\omega)$. Then $[V_X(\omega), L_X(\mu)] = 1$ and $\mu \in \{\omega_1 + \omega_4, \omega_1 + \omega_2, \omega_5 + \delta_{n,6}\omega_6, \omega_3, \omega_1\}$.

Proof. Write $T(\omega_2, \omega_3) = V_X(\omega_2) \otimes V_X(\omega_3)$ and identify $\text{rad}(\omega)$ with its image under ι , where $\iota : V_X(\omega) \hookrightarrow T(\omega_2, \omega_3)$ is the injection given by Proposition 2.6.4 (part 2). Now by Lemmas 7.4.6, Lemma 2.4.6 and Theorem 7.2, we have

$$\begin{aligned} \text{ch}(T(\omega_2, \omega_3)/\text{rad}(\omega)) &= \text{ch } L_X(\omega) + \text{ch } L_X(\omega_1 + \omega_4) \\ &\quad + \text{ch } L_X(\omega_1 + \omega_2) \\ &\quad + (1 + \delta_{p,5}) \text{ch } L_X(\omega_5 + \delta_{n,6}\omega_6) \\ &\quad + (1 + \delta_{p,3} + \epsilon_p(2n - 3)) \text{ch } L_X(\omega_3) \\ &\quad + (1 + \epsilon_p(2n - 1)) \text{ch } L_X(\omega_1) \end{aligned}$$

and Proposition 2.6.4 (part 3) applies, yielding the existence of a surjective morphism of KX -modules $\phi : T(\omega_2, \omega_3) \twoheadrightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch } H^0(\omega) = \chi(\omega)$, we get

$$\mu \in \{\omega_1 + \omega_4, \omega_1 + \omega_2, \omega_5 + \delta_{n,6}\omega_6, \omega_3, \omega_1\}$$

as desired as well as $[V_X(\omega), L_X(\omega_1 + \omega_4)][V_X(\omega), L_X(\omega_1 + \omega_2)] \leq 1$. Also, if $\mu = \omega_5 + \delta_{n,6}\omega_6$, then one can use Proposition 7.4.5 to see that $[V_X(\omega), L_X(\mu)] = 1$, while Corollary 2.7.3 forces $p \mid n$ if $\mu = \omega_3$, so that $p \neq 3$ by our initial assumption and hence $[V_X(\omega), L_X(\mu)] = 1$ as desired. Finally, if $\mu = \omega_1$, then an application of Corollary 2.7.3 forces $p \mid n - 1$, from which one deduces that $[V_X(\omega), L_X(\mu)] = 1$, completing the proof. \square

By determining every $\mu \in \Lambda^+(\omega)$ such that $[V_X(\omega), L_X(\mu)] \neq 0$ (using the method introduced in Section 2.7), we are now able to give the set of composition factors of $V_X(\omega_2 + \omega_3)$ in the case where $p \neq 2$ and $\delta_{p,3}\epsilon_p(n) = 0$.

Theorem 7.4.8

Assume $p \neq 2$ as well as $\delta_{p,3}\epsilon_p(n) = 0$. Also consider the T_X -weight $\omega = \omega_2 + \omega_3 \in X^+(T_X)$. Then

$$V_X(\omega) = \begin{cases} \omega/(\omega_1 + \omega_2)^{\epsilon_p(2n-3)}/\omega_1^{\epsilon_p(n-1)} & \text{if } p \nmid 3n; \\ \omega/\omega_1 + \omega_4/\omega_1^{\epsilon_p(n-1)} & \text{if } p = 3; \\ \omega/\omega_3. & \text{otherwise.} \end{cases}$$

Proof. Let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4 and write $\tau_1 = \omega_1 + \omega_4$, $\tau_2 = \omega_1 + \omega_2$, $\tau_3 = \omega_3$, $\mu = \omega_1$. We first assume $p \nmid 3n$ and leave to the reader to check that in this case $\nu_c^\mu(T_\omega) = \nu_p(2n-3)\chi^\mu(\tau_2) + \nu_p(2n-2)\chi^\mu(\mu)$, while an application of Theorem 7.2 yields $\chi^\mu(\tau_2) = \text{ch } L_X(\tau_2) + \epsilon_p(2n-1) \text{ch } L_X(\mu)$. Therefore since $p \neq 2$, we have

$$\nu_c^\mu(T_\omega) = \nu_p(2n-3) \text{ch } L_X(\tau_2) + \nu_p(2n-2) \text{ch } L_X(\mu),$$

which by Proposition 2.7.8 shows that τ_2 (respectively, μ) affords the highest weight of a composition factor of $V_X(\omega)$ if and only if p divides $2n-3$ (respectively, $2n-2$), while $[V_X(\omega), L_X(\nu)] = 0$ for every T_X -weight $\nu \in X^+(T_X)$ such that $\nu \neq \tau_2$ and $\mu \prec \nu \prec \omega$. One then concludes in this situation thanks to Corollary 7.4.7.

Next assume $p = 3$ (so that $p \nmid n$) and as above, we leave the reader to check that $\nu_c^\mu(T_\omega) = \nu_p(3)\chi^\mu(\tau_1) + \nu_p(2n-2)\chi^\mu(\mu)$, while applying Theorem 7.2 yields $\chi^\mu(\tau_1) = \text{ch } L_X(\tau_1)$ since $p \nmid n$. Therefore

$$\nu_c^\mu(T_\omega) = \nu_p(3) \text{ch } L_X(\tau_1) + \nu_p(2n-2) \text{ch } L_X(\mu)$$

and again one concludes using Proposition 2.7.8 together with Corollary 7.4.7.

Finally assume $p \mid n$ (so $p \neq 3$), in which case one can check (using Lemma 2.4.6) that we have $\nu_c^\mu(T_\omega) = \nu_p(2n) \text{ch } L_X(\tau_3) + \nu_p(2n-2) \text{ch } L_X(\mu)$. Applying Proposition 2.7.8 and Corollary 7.4.7 then completes the proof. \square

7.4.4 Conclusion: the case $\delta_{p,3}\epsilon_p(n) = 1$

We conclude this section by determining the composition factors of $V_X(\omega_2 + \omega_3)$, together with their multiplicity, under the assumption that $\delta_{p,3}\epsilon_p(n) = 1$. In order to do so, we first study the decomposition of $V_Y(2\lambda_2)|_X$ in terms of irreducibles in characteristic zero and then deduce the multiplicity of ω_3 in $V_X(2\omega_2)$.

Lemma 7.4.9

Consider the T_Y -weight $\lambda = 2\lambda_2 \in X^+(T_Y)$ and let $\omega = \lambda|_{T_X}$. Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(2\omega_1) + \chi(0)$$

Proof. Write $V = V_Y(\lambda)$ and first observe (see the discussion before Lemma 7.2.9) that $[V_Y(\lambda)|_X, L_X(2\omega_1)] = 1$. On the other hand one easily checks (using Theorem 2.4.1 and (2.9), for example) that $\dim V = \dim V_X(\omega) + \dim V_X(2\omega_1) + 1$. Therefore $[V|_X, L_X(0)] = 1$ and thus the assertion follows. \square

Corollary 7.4.10

Set $\omega = 2\omega_2 \in X(T_X)$ and let $\mu = \omega - (\beta_1 + 2\beta_2 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$. Then $\mu = \omega_2$ is dominant and $m_{V_X(\omega)}(\mu) = 2n - 3$.

Proof. Proceeding exactly as in the proofs of Corollaries 7.3.2 and 7.4.2 (replacing Propositions 7.3.1, 7.4.1 by Lemma 7.4.9) yields the desired result. The details are left to the reader. \square

Lemma 7.4.11

Assume $\delta_{p,3}\epsilon_p(n+1) = 1$ and consider an irreducible KX -module $V = L_X(\omega)$ having p -restricted highest weight $\omega = 2\omega_2$. Then $\mu = \omega_2$ satisfies $\chi^\mu(\omega) = \text{ch } V + \text{ch } L_X(\omega_4)$ and $m_V(\mu) = n - 1$.

Proof. Here the T_X -weights $\nu \in X^+(T_X)$ such that $\mu \preceq \nu \prec \omega$ and $m_{V_X(\omega)}(\nu) > 1$ are ω_4 and μ itself. An application of Lemma 5.1.1 then yields $[V_X(\omega), L_X(\omega_4)] = 1$, while $[V_X(\omega), L_X(\mu)] = 0$ by Corollary 2.7.3. Therefore the assertion on $\chi^\mu(\omega)$ holds and hence $m_V(\mu) = m_{V_X(\omega)}(\mu) - m_{L_X(\omega_4)}(\mu)$. One then checks (using Lemma 7.1.5, for example) that $m_{L_X(\omega_4)}(\mu) = n - 2$, while an application of Corollary 7.4.10 completes the proof. \square

Using Lemma 6.1.1 together with Lemma 7.4.11, we now give an upper bound for the multiplicity of ω_3 in $L_X(\omega_2 + \omega_3)$ in the case where $\delta_{p,3}\epsilon_p(n) = 1$.

Proposition 7.4.12

Assume $\delta_{p,3}\epsilon_p(n) = 1$ and consider the T_X -weight $\omega = \omega_2 + \omega_3 \in X^+(T_X)$. Also write $\mu = \omega - (\beta_1 + 2\beta_2 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$. Then $\mu = \omega_3$ is dominant and $m_V(\mu) \geq n - 2$.

Proof. Let $J = \{\gamma_1, \dots, \gamma_{n-1}\}$, where $\gamma_1 = \beta_1$, $\gamma_2 = \beta_2 + \beta_3$, $\gamma_r = \beta_{r+1}$ for every $3 \leq r < n-1$, so that $H = \langle U_{\pm\gamma_r} : 1 \leq r \leq n-1 \rangle$ is simple of type D_{n-1} over K , and denote by $\{\omega'_1, \dots, \omega'_{n-1}\}$ the set of fundamental weights corresponding to our choice of base. Adopting the latter notation, we get $\omega' = \omega|_{T_H} = 2\omega'_2$, $\mu' = \mu|_{T_H} = \omega' - (\gamma_1 + 2\gamma_2 + \cdots + 2\gamma_{n-2} + \gamma_{n-1} + \gamma_n)$, and as $\delta_{p,3}\epsilon_p(n) = 1$, Lemma 7.4.11 applies, yielding $m_{L_H(\omega')}(\mu') = n - 2$. The result then follows from Lemma 6.1.1. \square

By determining every $\mu \in \Lambda^+(\omega)$ such that $[V_X(\omega), L_X(\mu)] \neq 0$ (using the method introduced in Section 2.7), we are now able to give the structure of $V_X(\omega_2 + \omega_3)$ in the case where $\delta_{p,3}\epsilon_p(n) = 1$.

Theorem 7.4.13

Assume $\delta_{p,3}\epsilon_p(n) = 1$ and consider $\omega = \omega_2 + \omega_3 \in X^+(T_X)$. Then

$$V_X(\omega) = \omega/\omega_1 + \omega_4/\omega_1 + \omega_2/\omega_3.$$

Proof. First observe that an application of Corollary 2.7.3 yields $[V_X(\omega), L_X(\omega_1)] = 0$ and let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4. Also write $\tau_1 = \omega_1 + \omega_4$, $\tau_2 = \omega_1 + \omega_2$ and $\mu = \omega_3$. As usual, we leave to the reader to check that $\nu_c^\mu(T_\omega) = \chi^\mu(\tau_1) + \nu_3(2n-3)\chi^\mu(\tau_2) + \nu_3(2n)\chi^\mu(\mu)$. Also by Theorem 7.2, we get $\chi^\mu(\tau_1) = \text{ch } L_X(\tau_1) + \text{ch } L_X(\mu)$ as well as $\chi^\mu(\tau_2) = \text{ch } L_X(\tau_2) + \text{ch } L_X(\mu)$, from which one deduces (using Lemma 2.3.19, Proposition 2.7.8 and Theorem 7.2) that

$$m_{L_X(\omega)}(\mu) = m_{V_X(\omega)}(\mu) - m_{L_X(\tau_1)}(\mu) - m_{L_X(\tau_2)}(\mu) - [V_X(\omega), L_X(\mu)],$$

where $[V_X(\omega), L_X(\mu)] \neq 0$. An application of Corollary 7.4.2 then yields $m_{V_X(\omega)}(\mu) = 5n - 11$, while $m_{L_X(\tau_1)}(\mu) = 2(2n - 5) - 1$ by Theorem 7.2 and Corollary 7.3.2. The result thus follows from Proposition 7.4.12. \square

7.5 Proof of Theorem 7.3: the general case

Let X be as in the statement of Theorem 7.3 and assume throughout this section that $p \neq 2$. Here we determine the composition factors of $V_X(\omega)$, for $\omega \in \{\omega_2 + \omega_j\}_{3 < j < n-1}$. Let Y be as usual and setting $\lambda = \lambda_2 + \lambda_j$ for $3 < j < n - 1$, we proceed as in Section 7.4, starting by finding a decomposition of $V_Y(\lambda)|_X$ in terms of irreducibles, assuming K has characteristic zero.

7.5.1 Restriction of Weyl modules

Fix $3 < j < n$ and set $\lambda = \lambda_2 + \lambda_j$, which by (7.1) restricts to $\omega_2 + \omega_j + \delta_{j,n-1}\omega_n \in X^+(T_X)$. We first find a description of $\text{ch } V_Y(\lambda)|_X$ in terms of the \mathbb{Z} -basis $\{\chi(\mu)\}_{\mu \in X^+(T_X)}$ of $\mathbb{Z}[X(T_X)]^{\mathscr{W}}$.

Proposition 7.5.1

Fix $3 < j < n$ and write $\lambda = \lambda_2 + \lambda_j$. Also denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + \omega_{j-1}) + \chi(\omega_{j-2}).$$

Proof. Write $\omega' = \omega_1 + \omega_{j-1}$. Proceeding as in the proof of Proposition 7.3.1 shows that $[V_X(\omega), V_X(\omega')] = 1$ and that if ω'' affords the highest weight of a third KX -composition factor of $V_Y(\lambda)$, then $\omega_{j-2} \preccurlyeq \omega'' \prec \omega'$. One then easily checks (using Theorem 2.4.1, for example) that $\dim V_Y(\lambda) = \dim V_X(\omega) + \dim V_X(\omega') + \dim V_X(\omega_{j-2})$ and arguing as in the proof of Proposition 7.4.1 then completes the proof. We leave the details to the reader. \square

A similar result holds in the situation where $\lambda = \lambda_2 + \lambda_j$ for some $n + 1 < j < 2n - 2$, as the following Proposition shows. Its proof, being very similar to the proof of Proposition 7.3.5, is omitted here.

Proposition 7.5.2

Fix $n + 1 < j < 2n - 2$ and consider $\lambda = \lambda_2 + \lambda_j$. Also denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_{X^+} = \begin{cases} \chi(\omega) + \chi(\omega_1 + \omega_{n-1} + \omega_n) + \chi(2\omega_{n-1}) + \chi(2\omega_n) & \text{if } j = n + 2; \\ \chi(\omega) + \chi(\omega_1 + \omega_{2n-j+1}) + \chi(\omega_{2n-j+2} + \delta_{j,n+3}\omega_n) & \text{otherwise.} \end{cases}$$

7.5.2 Weyl filtrations and tensor products

Let G be a simple algebraic group of type A_n over K , fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U the unipotent radical of B . Also let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ be a corresponding base of the root system Φ of G and $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental weights for T corresponding to our choice of base Π . We first determine the formal character of the tensor product $V_G(\sigma_2) \otimes V_G(\sigma_j)$ for $2 < j < n$.

Lemma 7.5.3

Fix $2 < j < n$, write $\sigma = \sigma_2 + \sigma_j$ and consider the tensor product $T(\sigma_2, \sigma_j) = V_G(\sigma_2) \otimes V_G(\sigma_j)$. Then $T(\sigma_2, \sigma_j)$ is tilting and

$$\text{ch } T(\sigma_2, \sigma_j) = \chi(\sigma) + \chi(\sigma_1 + \sigma_{j+1}) + \chi(\sigma_{j+2}).$$

Proof. The first assertion directly follows from Proposition 2.6.4 (part 3). Also observe that $\text{ch } T(\sigma_2, \sigma_j)$ is independent of p and thus we may as well assume K has characteristic zero. By Proposition 2.6.4 (part 1), σ is the highest weight of $T(\sigma_2, \sigma_j)$, so that

$$\Lambda^+(T(\sigma_2, \sigma_j)) = \{\sigma, \sigma_1 + \sigma_{j+1}, \sigma_{j+2}\}$$

and as in the proof of Lemma 7.3.6, one easily sees that $m_{T(\sigma_1, \sigma_j)}(\sigma_1 + \sigma_{j+1}) = j + 1$, while $m_{V_G(\sigma)}(\sigma_1 + \sigma_{j+1}) = j$, showing that $[T(\sigma_2, \sigma_j), L_G(\sigma_1 + \sigma_{j+1})] = 1$. Finally, we leave to the reader to check that $\dim T(\sigma_2, \sigma_j) = \dim V_G(\sigma) + \dim V_X(\sigma_1 + \sigma_{j+1}) + \dim V_G(\sigma_{j+2})$, so that σ_{j+2} also affords the highest weight of a third composition factor of $T(\sigma_2, \sigma_j)$, completing the proof. \square

Proceeding exactly as in the proof of Corollary 7.3.2 (using Lemma 7.5.3 and replacing Proposition 7.3.1 by Proposition 7.5.1), one gets the following result. We leave the details to the reader.

Corollary 7.5.4

Fix $2 < j < n - 1$, write $\sigma = \sigma_2 + \sigma_j$ and let $\mu = \sigma_{j+2}$. Then $m_{V_G(\sigma)}(\mu) = \frac{1}{2}(j - 1)(j + 2)$.

Adopting the notation $\sigma_{n+1} = 0$, we next find a lower bound for the multiplicity of $\mu = \sigma_{j+2}$ in the irreducible KG -module having highest weight $\sigma = \sigma_2 + \sigma_j \in X^+(T_X)$, where $2 < j < n$.

Corollary 7.5.5

Let $2 < j < n-1$ and consider the T -weight $\sigma = \sigma_2 + \sigma_j \in X^+(T)$. Also suppose the existence of $\sigma \neq \mu \in X^+(T)$ such that μ affords the highest weight of a composition factor of $V_G(\sigma)$. Then $\mu \in \{\sigma_1 + \sigma_{j+1}, \sigma_{j+2}\}$ and $[V_G(\sigma), L_G(\mu)] = 1$.

Proof. Let $\sigma \neq \mu \in X^+(T)$ be such that $[V_G(\sigma), L_G(\mu)] \neq 0$ and observe that since $\Lambda^+(\sigma) = \{\sigma, \sigma_1 + \sigma_{j+1}, \sigma_{j+2}\}$, we immediately get $\mu \in \{\sigma_1 + \sigma_{j+1}, \sigma_{j+2}\}$ as desired. Also write $T(\sigma_2, \sigma_j) = V_G(\sigma_2) \otimes V_G(\sigma_j)$ and identify $\text{rad}(\sigma)$ with $\iota(\text{rad}(\sigma))$, where $\iota : V_G(\sigma) \hookrightarrow T(\sigma_2, \sigma_j)$ is the injection given by Proposition 2.6.4 (part 2). Applying Lemma 7.5.3 together with Lemma 2.3.19 then yields

$$\begin{aligned} \text{ch}(T(\sigma_2, \sigma_j)/\text{rad}(\sigma)) &= \text{ch } L_G(\sigma) + \text{ch } L_G(\sigma_1 + \sigma_{j+1}) \\ &\quad + (1 + \epsilon_p(j+2)) \text{ch } L_G(\sigma_{j+2}), \end{aligned}$$

and Proposition 2.6.4 (part 3) applies, yielding the existence of a surjective morphism of KG -modules $\phi : T \rightarrow H^0(\sigma)$ with $\text{rad}(\sigma) \subset \ker(\phi)$. As $\text{ch } H^0(\sigma) = \chi(\sigma)$, we then get $[V_G(\sigma), L_G(\sigma_1 + \sigma_{j+1})] \leq 1$ as desired and an application of Corollary 2.7.3 yields $[V_G(\sigma), L_G(\sigma_{j+2})] \leq 1$ as well, thus completing the proof. \square

Let $2 < j < n$ and as above, adopt the notation $\sigma_{n+1} = 0$. We next determine the composition factors of $V_G(\sigma_2 + \sigma_j)$ as well as the multiplicity of $\mu = \sigma_{j+2}$ in the irreducible KG -module $L_G(\sigma_2 + \sigma_j)$ in the case where $p \neq 2$.

Proposition 7.5.6

Assume $p \neq 2$ and consider an irreducible KG -module $V = L_G(\sigma)$ having p -restricted highest weight $\sigma = \sigma_2 + \sigma_j$, where $2 < j < n$. Then $V_G(\sigma) = \sigma/(\sigma_1 + \sigma_{j+1})^{\epsilon_p(j)}/\sigma_{j+2}^{\epsilon_p(j+1)}$ and

$$m_V(\sigma_{j+2}) = \frac{1}{2} \begin{cases} (j-2)(j+1) - 2 & \text{if } p \mid j; \\ (j-1)(j+2) - 2 & \text{if } p \mid j+1; \\ (j-1)(j+2) & \text{otherwise.} \end{cases}$$

Proof. Let $V_G(\sigma) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_G(\sigma)$ given by Proposition 2.7.4 and write $\tau = \sigma - (\gamma_2 + \dots + \gamma_j)$, $\mu = \tau - (\gamma_1 + \dots + \gamma_{j+1})$. One then checks that $\nu_c^\mu(T_\sigma) = \nu_p(j)\chi^\mu(\tau) + \nu_p(j+1)\chi^\mu(\mu)$ and hence an application of Lemma 2.3.19 yields

$$\nu_c^\mu(T_\sigma) = \nu_p(j) \text{ch } L_G(\tau) + (\nu_p(j+1) + \nu_p(j)\epsilon_p(j+2)) \text{ch } L_G(\mu).$$

Therefore $\chi^\mu(\sigma) = \text{ch } L_G(\sigma)$ if $p \nmid j(j+1)$ by Proposition 2.7.8 and Corollary 7.5.4 yields the assertion in this case. If on the other hand $p \mid j$ or $j+1$, then one concludes using Lemma 2.3.19, Proposition 2.7.8 and Corollary 7.5.5. We leave the details to the reader. \square

In the remainder of this section, we assume $p \neq 2$ and let Y, X be as in the statement of Theorem 7.3. In order to prove a result similar to Corollary 7.4.2 concerning the multiplicity of $\omega_j \in X^+(T_X)$ in $V_X(\omega_2 + \omega_j)$, one proceeds exactly as in the proof of Corollary 7.4.2 (using Lemma 7.5.3 and replacing Proposition 7.4.1 by Proposition 7.5.1). We leave the details to the reader.

Corollary 7.5.7

Fix $2 < j < n - 1$, let $\omega = \omega_2 + \omega_j$ and write $\mu = \omega - (\beta_1 + 2\beta_2 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$. Then the T_X -weight $\mu = \omega_j$ is dominant and

$$m_{V_X(\omega)}(\mu) = (j - 1) \left(\frac{1}{2}(j + 2)(n - j) + j - 1 \right).$$

For $2 < j < n - 1$, set $T(\omega_2, \omega_j) = V_X(\omega_2) \otimes V_X(\omega_j)$ and recall that the T_Y -weight $\lambda = \lambda_2 + \lambda_j$ restricts to $\omega_2 + \omega_j$. We now use Lemma 7.5.3 together with Proposition 7.5.1 to determine the formal character of $T(\omega_2, \omega_j)$ in terms of $\chi(\mu)$ ($\mu \in X^+(T)$).

Lemma 7.5.8

Assume $p \neq 2$ and for $2 < j < n - 2$, write $\omega = \omega_2 + \omega_j$. Then $T(\omega_2, \omega_j)$ is tilting and its formal character is given by

$$\text{ch } T(\omega_2, \omega_j) = \chi(\omega) + \chi(\omega_1 + \omega_{j+1}) + \chi(\omega_1 + \omega_{j-1}) + \chi(\omega_{j+2} + \delta_{j,n-3}\omega_n) + \chi(\omega_j) + \chi(\omega_{j-2}).$$

Similarly, if $\omega = \omega_2 + \omega_{n-2}$, then $T(\omega_2, \omega_{n-2})$ is tilting and its formal character is given by

$$\begin{aligned} \text{ch } T(\omega_2, \omega_{n-2}) &= \chi(\omega) + \chi(\omega_1 + \omega_{n-1} + \omega_n) + \chi(\omega_1 + \omega_{n-3}) + \chi(2\omega_{n-1}) + \chi(2\omega_n) \\ &\quad + \chi(\omega_{n-2}) + \chi(\omega_{n-4}). \end{aligned}$$

Proof. The fact that $T(\omega_2, \omega_j)$ is tilting follows from Lemma 2.4.6 and Proposition 2.6.4 (part 3). Also, since $\text{ch } T(\omega_2, \omega_j)$ is independent of p , it is enough to find a decomposition of $T(\omega_2, \omega_j)$ into a direct sum of irreducibles in characteristic zero, so we may and will assume $\text{char } K = 0$ for the remainder of the proof. First assume $2 < j < n - 2$. By (7.5), the KX -module $V_Y(\lambda_{j+2})|_X$ is isomorphic to $V_X(\omega_{j+2} + \delta_{j,n-3}\omega_n)$, thus Lemma 7.5.3 yields

$$T(\omega_2, \omega_j) \cong V_Y(\lambda)|_X \oplus V_Y(\lambda_1 + \lambda_{j+1})|_X \oplus V_X(\omega_{j+2} + \delta_{j,n-3}\omega_n),$$

where $\lambda = \lambda_2 + \lambda_j$. Propositions 7.3.1 and 7.5.1 then allow us to conclude in this situation. The case where $j = n - 2$ can be dealt with in a similar fashion (using the fact that $V_Y(\lambda_n)|_X \cong V_X(2\omega_{n-1}) \oplus V_X(2\omega_n)$ by Theorem 7.1), so the details are left to the reader. \square

7.5.3 Conclusion: the case $\epsilon_p(j)\epsilon_p(n) = 0$

Assume $\epsilon_p(j)\epsilon_p(n) = 0$ and consider the T_X -weight $\omega = \omega_2 + \omega_j$. We proceed as in Section 7.4, starting with the following consequence of Proposition 7.5.6 and Lemma 7.5.8.

Corollary 7.5.9

Assume $2 \neq p$ divides both j and n and for $2 < j < n - 1$, write $\omega = \omega_2 + \omega_j$. Also suppose that $\omega \neq \mu \in X^+(T_X)$ affords the highest weight of a composition factor of $V_X(\omega)$. Then $\mu \in \{\omega_1 + \omega_{j+1}, \omega_1 + \omega_{j-1}, \omega_{j+2} + \delta_{j,n-3}\omega_n, \omega_j, \omega_{j-2}\}$ and $[V_X(\omega), L_X(\mu)] = 1$.

Proof. First assume $2 < j < n - 2$, write $T(\omega_2, \omega_j) = V_X(\omega_2) \otimes V_X(\omega_j)$ and identify $\text{rad}(\omega)$ with $\iota(\text{rad}(\omega))$, where $\iota : V_X(\omega) \hookrightarrow T(\omega_2, \omega_j)$ is the injection given by Proposition 2.6.4 (part 2). Now by Lemmas 7.5.8, 2.4.6 and Theorem 7.2, we have

$$\begin{aligned} \text{ch}(T(\omega_2, \omega_j)/\text{rad}(\omega)) &= \text{ch } L_X(\omega) + \text{ch } L_X(\omega_1 + \omega_{j+1}) \\ &\quad + \text{ch } L_X(\omega_1 + \omega_{j-1}) \\ &\quad + (1 + \epsilon_p(j+2)) \text{ch } L_X(\omega_{j+2} + \delta_{j,n-3}\omega_n) \\ &\quad + (1 + \epsilon_p(j) + \epsilon_p(2n-j)) \text{ch } L_X(\omega_j) \\ &\quad + (1 + \epsilon_p(2n-j+2)) \text{ch } L_X(\omega_{j-2}) \end{aligned}$$

and Proposition 2.6.4 (part 3) applies, yielding the existence of a surjective morphism of KX -modules $\phi : T(\omega_2, \omega_j) \rightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch } H^0(\omega) = \chi(\omega)$, the first assertion holds and $[V_X(\omega), L_X(\omega_1 + \omega_{j+1})][V_X(\omega), L_X(\omega_1 + \omega_{j-1})] \leq 1$ as desired. Now if $\mu = \omega_{j+2} + \delta_{j,n-3}\omega_n$, then Corollary 2.7.3 yields $p \mid j+1$ and hence $[V_X(\omega), L_X(\mu)] = 1$ by Proposition 7.5.6 (applied to a suitable A_{j+2} -Levi subgroup of X). Also Corollary 2.7.3 forces $p \mid n$ if $\mu = \omega_j$, so that $p \nmid j$ and finally, if $\mu = \omega_{j-2}$, then an application of Corollary 2.7.3 forces $p \mid 2n-j+1$, from which one deduces that $[V_X(\omega), L_X(\mu)] = 1$. The case $j = n-2$ can be dealt with in a similar fashion. We leave the details to the reader. \square

Finally, we determine the composition factors of $V_X(\omega_2 + \omega_j)$ for $2 < j < n - 3$, using the method introduced in Section 2.7 together with Corollary 7.5.9.

Theorem 7.5.10

Assume $p \neq 2$ and let X be as in the statement of Theorem 7.3. Also let $\omega = \omega_2 + \omega_j$ for some $2 < j < n - 2$ and consider an irreducible KX -module $V = L_X(\omega)$ having p -restricted highest weight ω . Assume in addition $\epsilon_p(j)\epsilon_p(n) = 0$. Then

$$V_X(\omega) = \begin{cases} \omega/(\omega_1 + \omega_{j-1})^{\epsilon_p(2n-j)}/(\omega_{j+2} + \delta_{j,n-3}\omega_n)^{\epsilon_p(j+1)}/\omega_{j-2}^{\epsilon_p(2n-j+1)} & \text{if } p \nmid jn; \\ \omega/\omega_1 + \omega_{j+1}/\omega_{j-2}^{\epsilon_p(2n+1)} & \text{if } p \mid j; \\ \omega/(\omega_{j+2} + \delta_{j,n-3}\omega_n)^{\epsilon_p(j+1)}/\omega_j/\omega_{j-2}^{\epsilon_p(j-1)} & \text{if } p \mid n. \end{cases}$$

Proof. Let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supseteq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4, write $\tau_1 = \omega_1 + \omega_{j+1}$, $\tau_2 = \omega_1 + \omega_{j-1}$, $\tau_3 = \omega_{j+2} + \delta_{j,n-3}\omega_n$, $\tau_4 = \omega_j$, $\mu = \omega_{j-2}$ and first assume $p \nmid jn$. As usual, one checks that

$$\nu_c^\mu(T_\omega) = \nu_p(2n - j)\chi^\mu(\tau_2) + \nu_p(j + 1)\chi^\mu(\tau_3) + \nu_p(2n - j + 1)\chi^\mu(\mu),$$

while $\chi^\mu(\tau_2) = \text{ch } L_X(\tau_2) + \epsilon_p(2n - j + 2) \text{ch } L_X(\mu)$ by Theorem 7.2 and $\chi^\mu(\tau_3) = \text{ch } L_X(\tau_3)$ by Lemma 2.4.6. Hence we have

$$\nu_c^\mu(T_\omega) = \nu_p(2n - j) \text{ch } L_X(\tau_2) + \nu_p(j + 1) \text{ch } L_X(\tau_3) + \nu_p(2n - j + 1) \text{ch } L_X(\mu).$$

Therefore by Proposition 2.7.8, each of τ_2 , τ_3 and μ affords the highest weight of a composition factor of $V_X(\omega)$ and every other T_X -weight $\nu \in X^+(T_X)$ such that $\mu \preceq \nu \prec \omega$ satisfies $[V_X(\omega), L_X(\nu)] = 0$. One then concludes in this situation thanks to Corollary 7.5.9. Next assume $p \mid j$ and as above, we leave to the reader to check (using Lemma 2.4.6 and Theorem 7.2) that

$$\nu_c^\mu(T_\omega) = \nu_p(j) \text{ch } L_X(\tau_1) + \nu_p(2n - j + 1) \text{ch } L_X(\mu).$$

Applying Proposition 2.7.8 together with Corollary 7.5.9 yields the desired result in this situation as well. Finally assume $p \mid n$, in which case one can check (using Lemma 2.4.6 again) that we have

$$\nu_c^\mu(T_\omega) = \nu_p(j + 1) \text{ch } L_X(\tau_3) + \nu_p(2n) \text{ch } L_X(\tau_4) + \nu_p(2n - j + 1) \text{ch } L_X(\mu).$$

Proposition 2.7.8 and Corollary 7.5.9 then complete the proof. \square

Proceeding as in the proof of Theorem 7.5.10 yields the following result. We leave the details to the reader.

Theorem 7.5.11

Assume $p \neq 2$ and consider an irreducible KX -module $V = L_X(\omega)$ having p -restricted highest weight $\omega = \omega_2 + \omega_{n-2}$. Then

$$V_X(\omega) = \begin{cases} \omega/(\omega_1 + \omega_{n-3})^{\epsilon_p(n+2)}/2\omega_{n-1}^{\epsilon_p(n-1)}/2\omega_n^{\epsilon_p(n-1)}/\omega_{n-4}^{\epsilon_p(n+3)} & \text{if } p \nmid n(n-2); \\ \omega/\omega_1 + \omega_{n-1} + \omega_n/\omega_{n-4}^{\epsilon_p(5)} & \text{if } p \mid n-2; \\ \omega/\omega_{n-2}/\omega_{n-4}^{\epsilon_p(3)} & \text{if } p \mid n. \end{cases}$$

7.5.4 Conclusion: the case $\epsilon_p(j)\epsilon_p(n) = 1$

We now investigate the structure of $V_X(\omega)$ in the case where $2 \neq p$ divides both j and n (in which case $2 < j < n - 2$), starting by the following generalization of Lemma 7.4.11.

Lemma 7.5.12

Assume $p \neq 2$, let $2 < j < n - 2$ be such that $\epsilon_p(j + 1) = \epsilon_p(n + 1) = 1$ and consider an irreducible KX -module $V = L_X(\omega)$ having highest weight $\omega = \omega_2 + \omega_j$. Then

$$m_V(\omega_j) = \frac{1}{2}(j - 2)(j + 1)(n - j) + (j - 1)(n - 1).$$

Proof. First assume $2 < j < n - 3$ and observe that an application of Corollary 2.7.3 yields $[V_X(\omega), L_X(\nu)] = 0$ for $\nu = \omega_1 + \omega_{j+1}$, $\omega_1 + \omega_{j-1}$ and ω_j , while on the other hand $[V_X(\omega), L_X(\omega_{j+2})] = 1$ by Proposition 7.5.6. Therefore $\chi^{\omega_j}(\omega) = \text{ch } L_X(\omega) + \text{ch } L_X(\omega_{j+2})$ and hence $m_V(\omega_j) = m_{V_X(\omega)}(\omega_j) - m_{L_X(\omega_{j+2})}(\omega_j)$. Now an application of Corollary 7.5.7 yields $m_{V_X(\omega)}(\omega_j) = (j - 1)(\frac{1}{2}(j + 2)(n - j) + j - 1)$, while $m_{L_X(\omega_{j+2})}(\omega_j) = n - j$ by Lemma 7.1.5, from which the result follows. \square

Using Lemma 6.1.1 together with Lemma 7.5.12, we now give a lower bound for the multiplicity of ω_j in $L_X(\omega_2 + \omega_j)$ in the case where $p \neq 2$ and $\epsilon_p(j)\epsilon_p(n) = 1$.

Proposition 7.5.13

Assume $p \neq 2$, let $2 < j < n - 2$ be such that $\epsilon_p(j) = \epsilon_p(n) = 1$ and consider an irreducible KX -module $V = L_X(\omega)$ having highest weight $\omega = \omega_2 + \omega_j$. Then

$$m_V(\omega_j) \geq \frac{1}{2}j(j - 3)(n - j) + (j - 2)(n - 2).$$

Proof. Let $J = \{\gamma_1, \dots, \gamma_{n-1}\}$, where $\gamma_1 = \beta_1$, $\gamma_2 = \beta_2 + \beta_3$, $\gamma_r = \beta_{r+1}$ for every $3 \leq r \leq n - 1$, so that $H = \langle U_{\pm\gamma_r} : 1 \leq r < n - 1 \rangle$ is simple of type D_{n-1} over K . Also denote by $\{\omega'_1, \dots, \omega'_{n-1}\}$ the set of fundamental weights corresponding to our choice of base for the root system of H . Adopting the latter notation, we get $\omega' = \omega|_{T_H} = \omega_2 + \omega_{j-1}$, $\mu' = \mu|_{T_H} = \omega' - (\gamma_1 + 2\gamma_2 + \dots + 2\gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1})$, and as $\epsilon_p((j-1)+1) = \epsilon_p((n-1)+1) = 1$, Lemma 7.5.12 applies and the result then follows from Lemma 6.1.1. \square

Using the method introduced in Section 2.7, we determine the composition factors of $V_G(\omega)$ in the case where $\omega = \omega_2 + \omega_j$ for some $2 < j < n - 2$ such that $p \neq 2$ and $\epsilon_p(j)\epsilon_p(n) = 1$.

Theorem 7.5.14

Assume $p \neq 2$, let $2 < j < n - 2$ be such that $\epsilon_{p,j}\epsilon_p(n) = 1$ and consider the dominant T_X -weight $\omega = \omega_2 + \omega_j \in X^+(T_X)$. Then

$$V_X(\omega) = \omega/\omega_1 + \omega_{j+1}/\omega_1 + \omega_{j-1}/\omega_j.$$

Proof. First observe that $[V_X(\omega), L_X(\omega_{j-2})] = [V_X(\omega), L_X(\omega_{j+2})] = 0$ by Corollary 2.7.3 and let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4.

Also write $\tau_1 = \omega_1 + \omega_{j+1}$, $\tau_2 = \omega_1 + \omega_{j-1}$ and $\mu = \omega_j$. As usual, we leave to the reader to check that $\nu_c^\mu(T_\omega) = \nu_p(j)\chi^\mu(\tau_1) + \nu_p(2n-j)\chi^\mu(\tau_2) + \nu_p(2n)\chi^\mu(\mu)$. By Theorem 7.2, we get $\chi^\mu(\tau_1) = \text{ch } L_X(\tau_1) + \text{ch } L_X(\mu)$ as well as $\chi^\mu(\tau_2) = \text{ch } L_X(\tau_2) + \text{ch } L_X(\mu)$, from which one deduces (using Lemma 2.3.19, Proposition 2.7.8 and Theorem 7.2) that

$$m_{L_X(\omega)}(\mu) = m_{V_X(\omega)}(\mu) - m_{L_X(\tau_1)}(\mu) - m_{L_X(\tau_2)}(\mu) - [V_X(\omega), L_X(\mu)],$$

where $[V_X(\omega), L_X(\mu)] \neq 0$. One easily checks using Corollary 7.3.2, Theorem 7.2 and Lemma 2.3.19 that $m_{L_X(\omega)}(\mu) \leq \frac{1}{2}j(j-3)(n-j) + (j-2)(n-2)$. An application of Proposition 7.5.13 then yields $[V_X(\omega), L_X(\mu)] = 1$, thus completing the proof. \square

Proof of Theorem 7.3: In the case where $\omega = \omega_2 + \omega_3$, the result immediately follows from Theorems 7.4.8 and 7.4.13, while if $\omega = \omega_2 + \omega_j$ for some $3 < j < n-1$, applying Theorems 7.5.10, 7.5.11 and 7.5.14 yields the desired assertion.

7.6 Proof of Theorem 7.4

In this section, we assume $p \neq 2$, let Y, X be as in the statement of Theorem 7.4 and give a proof of the latter. We first consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_1 + \lambda_j$ for some $1 < j < 2n$.

7.6.1 The case $\lambda = \lambda_1 + \lambda_j$ ($1 < j < 2n$)

We start by giving a result similar to Lemma 7.3.7 in the case where $\omega = \omega_1 + 2\omega_n$. As usual, write $T(\omega_1, 2\omega_{n-1}) = V_X(\omega_1) \otimes V_X(2\omega_{n-1})$ and $T(\omega_1, 2\omega_n) = V_X(\omega_1) \otimes V_X(2\omega_n)$.

Lemma 7.6.1

Assume $p \neq 2$ and write $\omega = \omega_1 + 2\omega_n$. Then $T(\omega_1, 2\omega_n)$ is tilting and its formal character is given by

$$\text{ch } T(\omega_1, 2\omega_n) = \chi(\omega) + \chi(\omega_{n-1} + \omega_n).$$

Proof. The fact that $T(\omega_1, 2\omega_n)$ is tilting follows from Lemma 2.4.7 and Proposition 2.6.4 (part 3). Also, since $\text{ch } T(\omega_1, 2\omega_n)$ is independent of p , it is enough to find a decomposition of $T(\omega_1, 2\omega_n)$ in terms of irreducibles in characteristic zero. Now by Theorem 7.1, we have

$$T(\lambda_1, \lambda_n)|_X \cong T(\omega_1, 2\omega_{n-1}) \oplus T(\omega_1, 2\omega_n),$$

where $T(\lambda_1, \lambda_n) = V_Y(\lambda_1) \otimes V_Y(\lambda_n)$, while $T(\lambda_1, \lambda_n) \cong V_Y(\lambda_1 + \lambda_n) \oplus V_Y(\lambda_{n+1})$ by Proposition 7.3.6. Now by [Sei87, Theorem 1, Table 1 (I₄, I₅)] and Proposition 7.3.1, we have

$$T(\omega_1, 2\omega_{n-1}) \oplus T(\omega_1, 2\omega_n) \cong V_Y(\lambda_1 + \lambda_n)|_X \oplus V_X(\omega_{n-1} + \omega_n),$$

and an application of Proposition 7.3.4 then completes the proof. \square

Proposition 7.6.2

Assume $p \neq 2$ and write $\omega = \omega_1 + 2\omega_n$. Then $V_X(\omega) = \omega/(\omega_{n-1} + \omega_n)^{\epsilon_p(n+1)}$.

Proof. As above, write $T(\omega_1, 2\omega_n) = V_X(\omega_1) \otimes V_X(2\omega_n)$ and identify $\text{rad}(\omega)$ with $\iota(\text{rad}(\omega))$, where $\iota : V_X(\omega) \hookrightarrow T(\omega_1, 2\omega_n)$ is the injection given by Proposition 2.6.4 (part 2). Now by Lemmas 7.6.1 and 2.4.6, we have

$$\text{ch}(T(\omega_1, 2\omega_n)/\text{rad}(\omega)) = \text{ch} L_X(\omega) + \text{ch} L_X(\omega_{n-1} + \omega_n)$$

and Proposition 2.6.4 (part 3) applies, yielding the existence of a surjective morphism of KX -modules $\phi : T(\omega_1, 2\omega_n) \twoheadrightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch} H^0(\omega) = \chi(\omega)$, the only T_X -weight that could possibly afford the highest weight of a composition factor of $V_X(\omega)$ is $\omega_{n-1} + \omega_n$ and thus an application of Lemma 2.3.19 completes the proof. \square

Theorem 7.6.3

Assume $p \neq 2$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = \lambda_1 + \lambda_j$, where $1 < j < 2n$. Then

$$V|_X = \begin{cases} \omega/\omega_{j-1}^{1+\epsilon_p(2n-j+1)} & \text{if } 1 < j < n; \\ \omega/\omega_1 + 2\omega_{n-1}/\omega_{n-1} + \omega_n^{1+\epsilon_p(n+1)} & \text{if } j = n; \\ \omega/2\omega_{n-1}^{1+\epsilon_p(n)}/2\omega_n^{1+\epsilon_p(n)} & \text{if } j = n + 1; \\ \omega/(\omega_{2n-j+1} + \delta_{j,n+2}\omega_n)^{1+\epsilon_p(2n-j+1)} & \text{otherwise.} \end{cases}$$

In particular, X acts with exactly two composition factors on V if and only if $j \neq n, n + 1$, and $p \nmid 2n - j + 1$.

Proof. First assume $1 < j < n - 1$ and write $\omega' = \omega_{j-1}$. As in the proof of Corollary 7.3.2, one sees (using Lemma 2.3.19 instead of Theorem 2.3.11) that

$$m_{V|_X}(\omega') = \begin{cases} (j-1)(n-j+2) & \text{if } p \mid j+1; \\ j(n-j+2) - 1 & \text{otherwise.} \end{cases}$$

Notice that $\omega' = \omega - (\beta_1 + \cdots + \beta_{j-1} + 2\beta_j + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n)$. On the other hand, applying Theorem 7.2 yields

$$m_{L_X(\omega)}(\omega') = m_{V_X(\omega)}(\omega') - \epsilon_p(j+1) m_{L_X(\omega_{j+1})}(\omega') - \epsilon_p(2n-j+1).$$

Now $m_{L_X(\omega_{j+1})}(\omega') = m_{V_X(\omega_{j+1})}(\omega') = n - j + 1$ by Lemma 7.1.5, while an application of Corollary 7.3.2 gives $m_{V_X(\omega)}(\omega') = j(n-j+2) - 2$, thus showing that

$$m_{V|_X}(\omega') = m_{L_X(\omega)}(\omega') + (1 + \epsilon_p(2n-j+1)).$$

Consequently ω' occurs in a second KX -composition factor of V . One then checks that $m_{V|X}(\nu) = m_{L_X(\omega)}(\nu)$ for every $\nu \in \Lambda(V|_X)$ such that $\omega' \prec \nu \prec \omega$ and hence by Lemma 7.1.6, we get $[V|_X, L_X(\omega')] = 1 + \epsilon_p(2n - j + 1)$. Finally, an application of Proposition 7.3.1 yields $\dim V_Y(\lambda) = \dim V_X(\omega) + \dim V_X(\omega')$, while $\dim V = \dim V_Y(\lambda) - \epsilon_p(j + 1) \dim L_Y(\lambda_{j+1})$ by Lemma 2.4.8 and

$$\begin{aligned} \dim L_X(\omega) &= \dim V_X(\omega) - \epsilon_p(j + 1) \dim L_X(\omega_j + 1 + \delta_{j,n-2}\omega_n) \\ &\quad - \epsilon_p(2n - j + 1) \dim L_X(\omega_{j-1}) \end{aligned}$$

by Theorem 7.2, so that the result follows from (7.5). We leave the remaining cases to the reader, since they can be dealt with in a similar fashion. \square

In the remainder of this chapter, we shall write $\lambda - (c_r)_{r=1}^k$ to denote the T_Y -weight $\lambda - \sum_{r=1}^{2n-1} c_r \alpha_r$, where $c_r \in \mathbb{Z}_{\geq 0}$ for $1 \leq r \leq 2n - 1$ and $1 \leq k \leq 2n - 1$ is maximal such that $c_k \neq 0$. The following consequence of Theorem 7.6.3 shall be of use later in this chapter.

Corollary 7.6.4

Assume $p \neq 2$ and fix $3 < j < n$ such that $p \mid j + 1$ but $p \nmid n + 1$. Also consider an irreducible KX -module $L_X(\omega)$ having p -restricted highest weight $\omega = \omega_1 + \omega_j + \delta_{j,n-1}\omega_n$ and set $\mu = \omega_{j-3}$. Then

$$m_{L_X(\omega)}(\mu) = \frac{1}{2}(n - j + 3)(jn - j^2 + 5j - n - 10).$$

Proof. Consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_1 + \lambda_j$ and observe that the T_Y -weights recorded in Table 7.10 are the only T_Y -weights of V restricting to $\mu = \omega - (\beta_1 + \cdots + \beta_{j-3} + 2\beta_{j-2} + 3\beta_{j-1} + 4\beta_j + \cdots + 4\beta_{n-2} + 2\beta_{n-1} + 2\beta_n)$.

ν	Conditions
$\lambda - (1^{j-3}, 2, 3, 4^{r-j+1}, 3^{s-r}, 2^{2(n-s)-1}, 1^{s-r})$	$j \leq r \leq n - 2, r + 1 \leq s \leq n - 1$
$\lambda - (1^{j-3}, 2, 3^{r-j+2}, 2^{2(n-r)-1}, 1^{r-j+1})$	$j \leq r \leq n - 1$
$\lambda - (1^{j-3}, 2^2, 3^{r-j+1}, 2^{2(n-r)-1}, 1^{r-j+2})$	$j \leq r \leq n - 1, \lambda - (1^{j-3}, 2^{2(n-j)+3}, 1)$
$\lambda - (1^{j-2}, 2, 3^{r-j+1}, 2^{2(n-r)-1}, 1^{r-j+3})$	$j \leq r \leq n - 1$
$\lambda - (1^{j-2}, 2^{2(n-j+1)}, 1^2)$	
$\lambda - (1^{j-1}, 2^{2(n-j+1)}, 1)$	
$\lambda - (1^r, 0^{j-r-3}, 1, 2, 3^{s-j+1}, 2^{2(n-s)-1}, 1^{s-r})$	$0 \leq r \leq j - 4, j \leq s \leq n - 1$
$\lambda - (1^r, 0^{j-r-3}, 1, 2^{2(n-j+1)}, 1^{j-r-1})$	$0 \leq r \leq j - 4$
$\lambda - (1^r, 0^{j-r-3}, 1^2, 2^{2(n-j+1)}, 1^{j-r-2})$	$0 \leq r \leq j - 4$
$\lambda - (1^r, 0^{j-r-2}, 1, 2^{2(n-j)+3}, 1^{j-r-3})$	$0 \leq r \leq j - 4$

Table 7.10: T_Y -weights $\nu \in \Lambda(V)$ such that $\nu|_{T_X} = \mu$.

Furthermore, one sees that each T_Y -weight appearing in Table 7.10 is \mathscr{W}_Y -conjugate to either λ or λ_{j+1} and one calculates (using Lemma 2.3.19) that

$$m_{V|X}(\mu) = (n - j + 3) \left(\frac{1}{2}(j - 1)(n - j + 2) + j - 3 \right).$$

Finally, an application of Theorem 7.6.3 yields $m_{L_X(\omega)}(\mu) = m_{V|X}(\mu) - m_{L_X(\omega_{j-1})}(\mu)$ and one then concludes thanks to Lemma 7.1.5. \square

7.6.2 The case $\lambda = \lambda_2 + \lambda_j$ ($2 < j < 2n - 1$) and conclusion

Assume $p \neq 2$ and let Y, X be as in the statement of Theorem 7.4. We now aim at showing that X has more than two composition factors on $V = L_Y(\lambda)$ if $\lambda = \lambda_2 + \lambda_j$ for some $2 < j < 2n - 1$. We start by treating the case where $2 < j < n$, in which case each of $\omega = \lambda|_{T_X} = \omega_2 + \omega_j + \delta_{j,n-1}\omega_n$. Clearly ω affords the highest weight of a KX -composition factor of V , while arguing as in the proof of Theorem 7.6.3 shows that $\omega' = \omega_1 + \omega_{j-1}$ affords the highest weight of a second KX -composition factor of V .

Proposition 7.6.5

Let $2 < j < n - 2$ be such that $p \nmid 2n - j$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_j$. Then X has more than two composition factors on V .

Proof. If $p \nmid j(j+1)$, then $V = V_Y(\lambda)$ by Proposition 7.5.6 and the result directly follows from Proposition 7.5.1. If on the other hand $p \mid j$, then $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_{j+1})$ by Proposition 7.5.6 and an application of Theorem 7.6.3 yields ($p \nmid 2n - j$ and so $p \nmid n$)

$$\dim V = \dim V_Y(\lambda) - \dim L_X(\omega_1 + \omega_{j+1}) - \dim L_X(\omega_j).$$

Now $[V_X(\omega), L_X(\omega_1 + \omega_{j+1})] = [V_X(\omega'), L_X(\omega_j)] = 1$ by Lemma 2.3.19 and hence one gets $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ thanks to Proposition 7.5.1, thus showing that X has more than two composition factors on V in this situation. Finally, assume $p \mid j + 1$, in which case

$$\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_{j+2})$$

by Proposition 7.5.6, while the same result applied to the A_{j+2} -Levi subgroup of X corresponding to the simple roots $\beta_1, \dots, \beta_{j+2}$ yields

$$\dim L_X(\omega) \leq \dim V_X(\omega) - \dim L_X(\omega_{j+2} + \delta_{j,n-3}\omega_n).$$

As $L_Y(\lambda_{j+2})|_X \cong L_X(\omega_{j+2} + \delta_{j,n-3}\omega_n)$ by [Sei87, Theorem 1, Table 1 (I₄, I₅)], an application of Proposition 7.5.1 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, thus completing the proof. \square

Proposition 7.6.6

Assume $p \neq 2$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_{n-2}$. Then X has more than two composition factors on V .

Proof. If $p \nmid (n-1)$, then one can proceed exactly as in the proof of Proposition 7.6.5 (we leave the details to the reader) and hence we assume $p \mid n-1$ for the remainder of the proof. Here $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_n)$ by Proposition 7.5.6, while on the other hand, we also get (as in the proof of Proposition 7.6.5) that

$$\dim L_X(\omega) \leq \dim V_X(\omega) - \dim L_X(2\omega_{n-1}) - \dim L_X(2\omega_n).$$

Since $L_Y(\lambda_n)|_X \cong L_X(2\omega_{n-1}) \oplus L_X(2\omega_n)$ by Theorem 7.1, an application of Proposition 7.5.1 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, from which the result follows. \square

Lemma 7.6.7

Assume $2 \neq p \mid n$ and consider the T_X -weight $\omega = \omega_2 + \omega_{n-1} + \omega_n \in X^+(T_X)$. Then $\mu = \omega_{n-1} + \omega_n$ affords the highest weight of a composition factor of $V_X(\omega)$.

Proof. Let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4. As usual, we leave to the reader to check that $\nu_c^\mu(T_\omega) = \nu_p(n)\chi^\mu(\mu)$ and hence the assertion follows from Proposition 2.7.8. \square

Proposition 7.6.8

Assume $2 \neq p \nmid n+1$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_{n-1}$. Then X has more than two composition factors on V .

Proof. If $p \nmid (n-1)n$, then again the assertion directly follows from Propositions 7.5.1 and 7.5.6. If on the other hand $p \mid n-1$, then $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_n)$ by Proposition 7.5.6 and an application of Theorem 7.6.3 yields (recall that $p \nmid n+1$ by assumption)

$$\begin{aligned} \dim V &= \dim V_Y(\lambda) - \dim L_X(\omega_1 + 2\omega_{n-1}) \\ &\quad - \dim L_X(\omega_1 + 2\omega_n) \\ &\quad - \dim L_X(\omega_{n-1} + \omega_n). \end{aligned}$$

Now $[V_X(\omega), L_X(\omega_1 + 2\omega_i)] = [V_X(\omega'), L_X(\omega_{n-1} + \omega_n)] = 1$ for $i = n-1, n$ by Lemma 2.3.19 and hence $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ by Proposition 7.5.1, thus showing that X has more than two composition factors on V in this situation. Finally, assume $p \mid n$, in which case $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_{n+1})$ by Proposition 7.5.6, while on the other hand $\dim L_X(\omega) \leq \dim V_X(\omega) - \dim L_X(\omega_{n-1} + \omega_n)$ by Lemma 7.6.7. Since $L_Y(\lambda_{n+1})|_X$ is isomorphic to $L_X(\omega_{n-1} + \omega_n)$ by [Sei87, Theorem 1, Table 1 (I_4, I_5)], an application of Proposition 7.5.1 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, thus completing the proof. \square

We now assume $n+1 < j < 2n-1$, in which case the T_X -weight $\omega = \lambda|_{T_X} = \omega_2 + \omega_{2n-j}$ obviously affords the highest weight of a KX -composition factor of V . Again, arguing as in the proof of Theorem 7.6.3 shows that $\omega' = \omega_1 + \omega_{2n-j+1} + \delta_{j,n+2}\omega_{n-1}$ affords the highest weight of a second KX -composition factor of V .

Proposition 7.6.9

Assume $2 \neq p \nmid n - 2$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_{n+2}$. Then X has more than two composition factors on V .

Proof. If $p \nmid (n+2)(n+3)$, then $V = V_Y(\lambda)$ by Proposition 7.5.6 and thus the result directly follows from Proposition 7.5.2. Next if $p \mid n+2$, then $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_{n+3})$ by Proposition 7.5.6 and an application of Theorem 7.6.3 yields (recall that $p \nmid n - 2$ by assumption)

$$\dim V = \dim V_Y(\lambda) - \dim L_X(\omega_1 + \omega_{n-3}) - \dim L_X(\omega_{n-2}).$$

Now $[V_X(\omega), L_X(\omega_1 + \omega_{n-3})] = [V_X(\omega'), L_X(\omega_{n-2})] = 1$ by Theorem 7.2 and hence $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$ by Proposition 7.5.2, thus showing that X has more than two composition factors on V in this situation. Finally, assume $p \mid n+3$, in which case

$$\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_{n+4})$$

by Proposition 7.5.6, while on the other hand $\dim L_X(\omega) \leq \dim V_X(\omega) - \dim L_X(\omega_{n-4})$ by Theorem 7.3. Since $L_Y(\lambda_{n+4})|_X \cong L_X(\omega_{n-4})$ by [Sei87, Theorem 1, Table 1 (I₄, I₅)], an application of Proposition 7.5.2 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, thus completing the proof. \square

Proposition 7.6.10

Assume $p \neq 2$, fix $n+2 < j < 2n-2$ such that $p \nmid 2n-j$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_j$. Then X has more than two composition factors on V .

Proof. If $p \nmid j(j+1)$, then $V = V_Y(\lambda)$ by Proposition 7.5.6 and thus the result directly follows from Proposition 7.5.2. Next if $p \mid j$, then $\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_{j+1})$ by Proposition 7.5.6 and an application of Theorem 7.6.3 yields (recall that $p \nmid n$ by assumption)

$$\dim V = \dim V_Y(\lambda) - \dim L_X(\omega_1 + \omega_{2n-j-1}) - \dim L_X(\omega_{2n-j}).$$

Now $[V_X(\omega), L_X(\omega_1 + \omega_{2n-j-1})] = 1$ by Theorem 7.3, while similarly, applying Theorem 7.2 gives $[V_X(\omega'), L_X(\omega_{2n-j})] = 1$. Consequently an application of Proposition 7.5.2 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, thus showing that X has more than two composition factors on V in this situation. Finally assume $p \mid j+1$, in which case

$$\dim V = \dim V_Y(\lambda) - \dim L_Y(\lambda_{j+2})$$

by Proposition 7.5.6. Also $\dim L_X(\omega) \leq \dim V_X(\omega) - \dim L_X(\omega_{2n-j-2})$ thanks to Theorem 7.3. Since $L_Y(\lambda_{j+2})|_X \cong L_X(\omega_{2n-j-2})$ by [Sei87, Theorem 1, Table 1 (I₄, I₅)], an application of Proposition 7.5.2 yields $\dim V > \dim L_X(\omega) + \dim L_X(\omega')$, thus completing the proof. \square

Finally, we study the situation where $\lambda = \lambda_2 + \lambda_{2n-2}$, in which case each of $\omega = 2\omega_2$ and $\omega' = \omega_1 + \omega_3$ affords the highest weight of a KX -composition factor of V .

Proposition 7.6.11

Assume $p \neq 2$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = \lambda_2 + \lambda_{2n-2}$. Then X has more than two composition factors on V .

Proof. Let $\omega'' = \omega - \beta_1 - 2\beta_2 - \beta_3 \in X^+(T_X)$. Then one easily sees that the T_Y -weights $\lambda - \alpha_1 - 2\alpha_2 - \alpha_3$, $\lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_{2n-2}$, $\lambda - \alpha_1 - \alpha_2 - \alpha_{2n-3} - \alpha_{2n-2}$, $\lambda - \alpha_2 - \alpha_3 - \alpha_{2n-2} - \alpha_{2n-1}$, $\lambda - \alpha_2 - \alpha_{2n-3} - \alpha_{2n-2} - \alpha_{2n-1}$ and $\lambda - \alpha_{2n-3} - 2\alpha_{2n-2} - \alpha_{2n-1}$ all restrict to ω'' , so that $m_{V|X}(\omega'') \geq 6$. On the other hand, one checks (using Theorem 2.3.11, for example) that $m_{L_X(\omega)}(\omega'') \leq 2$ and $m_{L_X(\omega')}(\omega'') = 3$, showing the existence of a third KX -composition factor of V . \square

Proof of Theorem 7.4: Assume the result true for $Y = Y_k$ of type A_{2k-1} over K and every $3 \leq k < n$ and let $Y = Y_n$ be of type A_{2n-1} over K . By Theorems 5.1 and 6.1, the result holds for $k = 3, 4$. Set $J = \{\beta_2, \dots, \beta_n\} \subset \Pi(X)$ and adopting the notation introduced in Section 2.3.2, consider the D_{n-1} -parabolic subgroup $P_J = Q_J L_J$ of X . Also denote by $P_Y = Q_Y L_Y$ the parabolic subgroup of Y given by Lemma 2.3.9 and notice that L_Y has type A_{2n-3} over K , with $\Pi(L_Y) = \{\alpha'_1, \dots, \alpha'_{2n-3}\} = \{\alpha_2, \dots, \alpha_{2n-2}\}$. Writing $X' = L'_J$, $Y' = L'_Y$ and $\lambda' = \lambda_{T_Y \cap Y'}$, an application of Lemma 2.3.10 together with our induction assumption then shows that up to graph automorphisms, either $\lambda = \lambda_1 + \lambda_j$ for some $1 < j < 2n$ or $\lambda = \lambda_2 + \lambda_j$ for some $2 < j < 2n - 1$ such that $j \neq n, n + 1$ and $p \nmid 2n - j$ in the latter case. Applying Theorem 7.6.3 for the weights $\lambda_1 + \lambda_j$ and one of Propositions 7.6.5, 7.6.6, 7.6.8, 7.6.9, 7.6.10 or 7.6.11 for the weights $\lambda_2 + \lambda_j$ then completes the proof.

7.7 Proof of Theorem 7.5

In this section, we give a complete proof of Theorem 7.5. As in Section 7.3, we start by investigating the restriction of various Weyl modules in characteristic zero.

7.7.1 Restriction of Weyl modules

Fix $n + 1 < j < 2n - 1$ and set $\lambda = 2\lambda_1 + \lambda_j$, which by (7.1) restricts to the dominant T_X -weight $2\omega_1 + \omega_{2n-j} \in X^+(T_X)$. We first find a description of $\text{ch } V_Y(\lambda)|_X$ in terms of the \mathbb{Z} -basis $\{\chi(\mu)\}_{\mu \in X^+(T_X)}$ of $\mathbb{Z}[X(T_X)]^{\mathcal{M}}$.

Proposition 7.7.1

Fix $n + 1 < j < 2n - 1$, consider $\lambda = 2\lambda_1 + \lambda_j$ and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + \omega_{2n-j+1} + \delta_{j,n+2}\omega_n) + \chi(\omega_{2n-j}).$$

Proof. Write $V = V_Y(\lambda)$ and first observe that $\text{ch } V|_X$ is independent of p , so we may assume K has characteristic zero for the remainder of the proof. Here $\Lambda^+(V|_X) = \Lambda^+(\omega)$ by Lemma 7.1.6 and proceeding as in the proof of Proposition 7.3.5, one easily shows that $\omega' = \omega_1 + \omega_{j+1} \in X^+(T_X)$ affords the highest weight of a second KX -composition factor. An elementary computation (using Theorem 2.4.1, for example) yields

$$\dim V > \dim V_X(\omega) + \dim(\omega_1 + \omega_{j+1}),$$

showing the existence of $\omega'' \in X^+(T_X)$ such that $[V_X(\omega), L_X(\omega'')] \neq 0$. As K has characteristic zero, this translates to the existence of a maximal vector in $(V|_X)_{\omega''}$ for B_X . Now Proposition 7.1.3 yields $\omega' - (2\beta_1 + \cdots + 2\beta_{n-2} + \beta_{n-1} + \beta_n) \preceq \omega'' \prec \omega'$ and we leave to the reader to check that this forces $\omega'' \in \{\omega_1 + \omega_{j-1}, \omega_j\}$. Arguing as in the proof of Corollary 7.3.2, one sees that $\text{m}_{V|_X}(\omega_1 + \omega_{j-1}) = \text{m}_{V_X(\omega)}(\omega_1 + \omega_{j-1}) + \text{m}_{V_X(\omega')}(\omega_1 + \omega_{j-1})$, so that $[V|_X, L_X(\omega_1 + \omega_{j-1})] = 0$. Therefore $[V|_X, L_X(\omega_j)] \neq 0$ and an application of Theorem 2.4.1 completes the proof. \square

Proceeding as in the proof of Proposition 7.7.1, one obtains the following result. The details are left to the reader.

Proposition 7.7.2

Consider $\lambda = 2\lambda_1 + \lambda_{n+1}$ and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + 2\omega_{n-1}) + \chi(\omega_1 + 2\omega_n) + \chi(\omega_{n-1} + \omega_n).$$

Fix $n < j < 2n - 1$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having highest weight $\lambda = 2\lambda_1 + \lambda_j$. Also write $\omega = \lambda|_{T_X}$ and set $\mu = \omega_{2n-j}$. Then one easily checks that $\nu \in \Lambda(V)$ restricts to μ if and only if either $\nu = \lambda - (\alpha_1 + \cdots + \alpha_{2n-1})$ or ν is recorded in Table 7.11.

ν	Conditions
$\lambda - (2^r, 1^{2(n-r)-1})$	$1 \leq r \leq n - 1$
$\lambda - (2^r, 1^{s-r}, 0^{t-s-1}, 1^{2(n-t)+1}, 2^{t-s-1}, 1^{s-r})$	$1 \leq r \leq 2n - j - 2$ $r + 1 \leq s \leq 2n - j - 1$ $2n - j + 1 \leq t \leq n$
$\lambda - (1^s, 0^{t-s-1}, 1^{2(n-t)+1}, 2^{t-s-1}, 1^s)$	$1 \leq s \leq 2n - j - 1$ $2n - j + 1 \leq t \leq n$

Table 7.11: T_Y -weights in $L_Y(2\lambda_1 + \lambda_j)$ restricting to ω_{2n-j} .

One then sees that each T_Y -weight appearing in Table 7.11 is \mathscr{W}_Y -conjugate to either λ or $\lambda_1 + \lambda_{j+1}$ and applying Lemma 2.3.19 yields the following result on $\text{m}_{V|_X}(\mu)$ in this situation, to which we shall refer later in this chapter.

Lemma 7.7.3

Fix $n < j < 2n - 1$ and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = 2\lambda_1 + \lambda_j$. Also write $\omega = \lambda|_{T_X}$ and set $\mu = \omega_{2n-j}$. Then

$$m_{V|_X}(\mu) = (2n - j) \left(\frac{1}{2}(j - n)(2n - j - 1) + j - \epsilon_p(j + 2) \right) + j - n.$$

Corollary 7.7.4

Let $\omega = 2\omega_1 + \omega_j + \delta_{j,n-1}\omega_n \in X^+(T_X)$ for some $1 < j < n$ and consider the T_X -weight $\mu = \omega_j + \delta_{j,n-1}\omega_n$. Then

$$m_{V_X(\omega)}(\mu) = j \left(\frac{1}{2}(n - j)(j - 1) + n - 1 \right).$$

Proof. First assume $1 < j < n - 1$ and write $\lambda = 2\lambda_1 + \lambda_{2n-j} \in X^+(T_Y)$, so that $\lambda|_{T_X} = \omega$. An application of Proposition 7.7.1 then yields

$$m_{V_X(\omega)}(\mu) = m_{V_Y(\lambda)|_X}(\mu) - m_{V_X(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n)}(\mu) - 1$$

and one easily concludes using Corollary 7.3.2 and Lemma 7.7.3. In the case where $j = n - 1$, proceeding in the exact same fashion (replacing Proposition 7.7.1 by Proposition 7.7.2 and Corollary 7.3.2 by Lemma 2.3.19) yields the desired assertion. The details are left to the reader. \square

Proposition 7.7.5

Fix $1 < j < n$, consider $\lambda = 2\lambda_1 + \lambda_j$, and denote by $\omega \in X^+(T_X)$ the restriction of λ to T_X . Then

$$\text{ch } V_Y(\lambda)|_X = \chi(\omega) + \chi(\omega_1 + \omega_{j-1}) + \chi(\omega_j + \delta_{j,n-1}\omega_n).$$

Proof. Write $V = V_Y(\lambda)$ and first observe that $\text{ch } V|_X$ is independent of p , so we may assume K has characteristic zero for the remainder of the proof. By Lemma 7.1.6, $\Lambda^+(V|_X) = \Lambda^+(\omega)$ and we leave to the reader to check (using Corollaries 7.3.2 and 7.7.4 respectively) that each of $\omega_1 + \omega_{j-1}$ and $\omega_j + \delta_{j,n-1}\omega_n$ affords the highest weight of a KX -composition factor of V . As usual, applying Theorem 2.4.1 completes the proof. \square

Before going any further, we record the following consequence of Proposition 7.7.5, which explains the need for our assumption on p in Theorem 7.5.

Corollary 7.7.6

Fix $1 < j < 2n - 1$ (with $j \neq n, n + 1$) and consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = 2\lambda_1 + \lambda_j \in X^+(T_Y)$. Also suppose that X has exactly two composition factors on V . Then $p \mid j + 2$.

Proof. One first checks that $\Lambda^+(\lambda) = \{\lambda, \lambda_1 + \lambda_{j+1}, \lambda_{j+2}\}$ and hence an application of Lemma 6.1.3 yields $V = V_Y(\lambda)$ if $p \nmid j + 2$. The result then follows from Propositions 7.7.1 and 7.7.5. \square

Corollary 7.7.7

Fix $1 < j < n$ such that $p \mid j + 2$, consider an irreducible KY -module $V = L_Y(\lambda)$ having p -restricted highest weight $\lambda = 2\lambda_1 + \lambda_j$ and write $\mu = \omega_{j-2}$. Then

$$m_{V_Y(\lambda)|_X}(\mu) - m_{V|_X}(\mu) = \frac{1}{2}(n - j + 2)(jn - j^2 + 3j - 4).$$

Proof. First observe that our assumption on p forces $2 < j < n$. Also, one checks that the T_Y -weights $\nu \in \Lambda(\lambda)$ such that $\nu|_{T_X} = \mu$ and $m_{L_Y(\lambda)}(\nu) > 1$ are as in Table 7.12.

$\lambda - (2^{j-2}, 3, 4^{r-j+1}, 3^{s-r}, 2^{2(n-s)-1}, 1^{s-r})$	$j \leq r \leq n - 2$ $r + 1 \leq s \leq n - 1$
$\lambda - (2^{j-2}, 3^{r-j+2}, 2^{2(n-r)-1}, 1^{r-j+1})$	$j \leq r \leq n - 1$
$\lambda - (2^{j-1}, 3^{r-j+1}, 2^{2(n-r)-1}, 1^{r-j+2})$	$j \leq r \leq n - 1$
$\lambda - (2^r, 1^{j-r-2}, 2, 3^{s-j+1}, 2^{2(n-s)-1}, 1^{s-r})$	$1 \leq r \leq j - 3$ $j \leq s \leq n - 1$
$\lambda - (2^r, 1^{j-r-2}, 2^{2(n-j)+2}, 1^{j-r-1})$	$1 \leq r \leq j - 3$
$\lambda - (2^r, 1^{j-r-1}, 2^{2(n-j)+2}, 1^{j-r-2})$	$1 \leq r \leq j - 3$
$\lambda - (1^{j-2}, 2, 3^{r-j+1}, 2^{2(n-r)-1}, 1^r)$	$j \leq r \leq n - 1$

Table 7.12: T_Y -weights $\nu \in \Lambda(\lambda)$ such that $\nu|_{T_X} = \omega_{j-2}$ and $m_{V_Y(\lambda)}(\nu) > 1$.

Now by Theorem 2.3.4, $m_V(\nu) = m_{V_Y(\lambda)}(\nu)$ for every $\nu \in \Lambda(\lambda)$ such that $m_{V_Y(\lambda)}(\nu) = 1$. Applying Lemmas 2.3.19 and 6.1.3 then completes the proof, since any T_Y -weight appearing in Table 7.12 is \mathscr{W}_Y -conjugate to either $\lambda_1 + \lambda_{j+1}$ or λ_{j+2} . \square

7.7.2 Weyl filtrations and tensor products

Let G be a simple algebraic group of type A_n over K , fix a Borel subgroup $B = UT$ of G , where T is a maximal torus of G and U the unipotent radical of B . Also let $\Pi = \{\gamma_1, \dots, \gamma_n\}$ be a corresponding base of the root system Φ of G and $\{\sigma_1, \dots, \sigma_n\}$ be the set of fundamental weights for T corresponding to our choice of base Π .

Lemma 7.7.8

Assume $p \neq 2$ and fix $1 < j < n$. Also consider $\sigma = 2\sigma_1 + \sigma_j$ and denote by $T(2\sigma_1, \sigma_j)$ the tensor product $V_G(2\sigma_1) \otimes V_G(\sigma_j)$. Then $T(2\sigma_1, \sigma_j)$ is tilting and

$$\text{ch } T(2\sigma_1, \sigma_j) = \chi(\sigma) + \chi(\sigma_1 + \sigma_{j+1}).$$

Proof. The first assertion directly follows from Lemmas 2.4.4, 2.4.5 and Proposition 2.6.4 (part 3). Also observe that $\text{ch } T(2\sigma_1, \sigma_j)$ is independent of p and thus we may assume K has characteristic zero for the remainder of the proof. By Proposition 2.6.4 (part 1), σ is the highest weight of $T(2\sigma_1, \sigma_j)$, so that $\Lambda^+(T(2\sigma_1, \sigma_j)) = \{\sigma, \sigma_2 + \sigma_j, \sigma_1 + \sigma_{j+1}, \sigma_{j+2}\}$, and using Lemma 2.3.19, one easily sees that $m_{T(2\sigma_1, \sigma_j)}(\sigma_1 + \sigma_{j+1}) = j + 1$, while $m_{V_G(\sigma)}(\sigma_1 + \sigma_{j+1}) = j$. Therefore σ_{j+1} affords the highest weight of a second KG -composition factor of $T(2\sigma_1, \sigma_j)$ and applying Theorem 2.4.1 then yields the desired result. \square

Lemma 7.7.9

Assume $p \neq 2$ and fix $1 < j < n$. Also assume $p \nmid j + 1$, consider $\sigma = 2\sigma_1 + \sigma_j$ and write $T(\sigma_1 + \sigma_j, \sigma_1) = V_G(\sigma_1 + \sigma_j) \otimes V_G(\sigma_1)$. Then $T(\sigma_1 + \sigma_j, \sigma_1)$ is tilting and

$$\text{ch } T(\sigma_1 + \sigma_j, \sigma_1) = \chi(\sigma) + \chi(\sigma_2 + \sigma_j) + \chi(\sigma_1 + \sigma_{j+1}).$$

Proof. As usual, the first assertion directly follows from Lemma 2.3.19 together with Proposition 2.6.4 (part 3) and $\text{ch } T(\sigma_1 + \sigma_j, \sigma_1, \sigma_1)$ is independent of p . We thus proceed as in the proof of Lemma 7.7.8, first noticing that exactly two T_G -weights restrict to $\sigma_2 + \sigma_j \in X^+(T_G)$, whose multiplicity in $V_G(\sigma)$ equals 1. Therefore $[V|_G, L_G(\sigma_2 + \sigma_j)] = 1$ as desired and one easily shows that $\sigma_1 + \sigma_{j+1}$ affords the highest weight of a third KG -composition factor of $T(\sigma_1 + \sigma_j, \sigma_1, \sigma_1)$ as well. An application of Theorem 2.4.1 then completes the proof. \square

In the remainder of this section, we assume $p \neq 2$ and let Y, X be as in the statement of Theorem 7.5 and for $1 < j < n - 1$, we set $T(2\omega_1, \omega_j) = V_X(2\omega_1) \otimes V_X(\omega_j)$. (Recall that the T_Y -weight $\lambda = 2\lambda_1 + \lambda_j$ restricts to $2\omega_1 + \omega_j$.) We now use Proposition 7.3.1 together with Lemma 7.7.8 to determine the formal character of $T(2\omega_1, \omega_j)$.

Lemma 7.7.10

Fix $1 < j < n - 1$ and write $\omega = 2\omega_1 + \omega_j$. Then the formal character of $T(2\omega_1, \omega_j)$ is given by

$$\text{ch } T(2\omega_1, \omega_j) = \chi(\omega) + \chi(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n) + \chi(\omega_1 + \omega_{j-1}) + \chi(\omega_j).$$

Proof. Observe that $\text{ch } T(2\omega_1, \omega_j)$ is independent of p and hence it is enough to find a decomposition of $T(2\omega_1, \omega_j)$ into a direct sum of irreducibles in characteristic zero. Now $V_Y(2\lambda_1)|_X \cong V_X(2\omega_1) \oplus V_X(0)$ by Theorem 7.1, so that

$$T(2\lambda_1, \lambda_j)|_X \cong T(2\omega_1, \omega_j) \oplus V_Y(\lambda_j)|_X.$$

Also $V_Y(\lambda_i)|_X$ is isomorphic to $V_X(\omega_i + \delta_{i,n-1}\omega_n)$ for every $1 \leq i < n$ by (7.5) and thus Proposition 7.3.1 and Lemma 7.7.8 yield

$$\text{ch } T(2\omega_1, \omega_j) = \text{ch } V_Y(\lambda)|_X + \chi(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n),$$

where $\lambda = \lambda_1 + \lambda_j$ as above. Finally, an application of Proposition 7.7.5 yields the desired result. \square

Similarly, set $T(2\omega_1, \omega_{n-1} + \omega_n) = V_X(2\omega_1) \otimes V_X(\omega_{n-1} + \omega_n)$ and again observe that the T_Y -weight $\lambda = 2\lambda_1 + \lambda_{n-1}$ restricts to $2\omega_1 + \omega_{n-1} + \omega_n$. Arguing exactly as in the proof of Proposition 7.7.10 (replacing Proposition 7.3.1 by Proposition 7.3.4) then yields the following result. We leave the details to the reader.

Lemma 7.7.11

Write $\omega = 2\omega_1 + \omega_{n-1} + \omega_n$. Then the formal character of $T(2\omega_1, \omega_{n-1} + \omega_n)$ is given by

$$\begin{aligned} \text{ch } T(2\omega_1, \omega_{n-1} + \omega_n) &= \chi(\omega) + \chi(\omega_1 + 2\omega_{n-1}) + \chi(\omega_1 + 2\omega_n) \\ &\quad + \chi(\omega_1 + \omega_{n-2}) + \chi(\omega_{n-1} + \omega_n). \end{aligned}$$

Next let $1 < j < n - 2$ and set $T(\omega_1 + \omega_j, \omega_1) = V_X(\omega_1 + \omega_j) \otimes V_X(\omega_1)$. Using Proposition 7.3.1 and Lemma 7.7.9, we determine the formal character of $T(\omega_1 + \omega_j, \omega_1)$.

Lemma 7.7.12

Fix $1 < j \leq n - 2$ and let $\omega = 2\omega_1 + \omega_j$. Then the formal character of $T(\omega_1 + \omega_j, \omega_1)$ is given by

$$\begin{aligned} \text{ch } T(\omega_1 + \omega_j, \omega_1) &= \chi(\omega) + \chi(\omega_2 + \omega_j) + \chi(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n) \\ &\quad + \chi(\omega_1 + \omega_{j-1}) + \chi(\omega_j). \end{aligned}$$

Proof. Observe that $\text{ch } T(\omega_1 + \omega_j, \omega_1)$ is independent of p , hence it is enough to find a decomposition of $T(\omega_1 + \omega_j, \omega_1)$ into a direct sum of irreducibles in characteristic zero. Now $V_Y(\lambda_1 + \lambda_j)|_X \cong V_X(\omega_1 + \omega_j) \oplus V_X(\omega_{j-1})$ by Proposition 7.3.1, so that

$$T(\lambda_1 + \lambda_j, \lambda_1)|_X \cong T(\omega_1 + \omega_j, \omega_1) \oplus V_X(\omega_{j-1}) \otimes V_Y(\lambda_1)|_X.$$

Now $V_Y(\lambda_i)|_X$ is isomorphic to $V_X(\omega_i + \delta_{i,n-1}\omega_n)$ for every $1 \leq i < n$ by (7.5) and thus Propositions 7.3.1, 7.5.1 and Lemma 7.7.9 yield

$$\begin{aligned} \text{ch } T(\omega_1 + \omega_j, \omega_1) &= \text{ch } V_Y(\lambda)|_X + \text{ch } V_Y(\lambda_2 + \lambda_j)|_X \\ &\quad + \chi(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n), \end{aligned}$$

where $\lambda = 2\lambda_1 + \lambda_j$ as above. Again, applying Proposition 7.7.5 completes the proof. \square

Set $T(\omega_1 + \omega_{n-1} + \omega_n, \omega_1) = V_X(\omega_1 + \omega_{n-1} + \omega_n) \otimes V_X(\omega_1)$ and again observe that the T_Y -weight $\lambda = 2\lambda_1 + \lambda_{n-1}$ restricts to $2\omega_1 + \omega_{n-1} + \omega_n$. Arguing exactly as in the proof of Proposition 7.7.12 (using Propositions 7.3.1, 7.3.4, 7.5.1, 7.7.5 and Lemma 7.7.9) then yields the following result. We leave the details to the reader.

Lemma 7.7.13

Let $\omega = 2\omega_1 + \omega_{n-1} + \omega_n$. Then the formal character of $T(\omega_1 + \omega_{n-1} + \omega_n, \omega_1)$ is given by

$$\begin{aligned} \text{ch } T(\omega_1 + \omega_{n-1} + \omega_n, \omega_1) &= \chi(\omega) + \chi(\omega_2 + \omega_{n-1} + \omega_n) + \chi(\omega_1 + 2\omega_{n-1}) \\ &\quad + \chi(\omega_1 + 2\omega_n) + \chi(\omega_1 + \omega_{n-2}) \\ &\quad + \chi(\omega_{n-1} + \omega_n). \end{aligned}$$

Finally, we leave to the reader to show the following result, to which we shall refer throughout the remainder of this chapter.

Lemma 7.7.14

Assume $2 \neq p \nmid n + 1$, fix $1 < j \leq n - 2$ and let $\omega = 2\omega_1 + \omega_j$. Also consider the filtration $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ of $V_X(\omega)$ given by Proposition 2.7.4 and set $\mu = \omega_{j-2}$. If $1 < j < n - 2$, then

$$\begin{aligned} \nu_c^\mu(T_\omega) &= \nu_p(j + 2)\chi^\mu(\omega_1 + \omega_{j+1}) - \nu_p(j + 2)\chi^\mu(\omega_{j+2} + \delta_{j,n-3}\omega_n) \\ &\quad + \nu_p(2n - j + 2)\chi^\mu(\omega_1 + \omega_{j-1}) \\ &\quad - \nu_p(2n - j + 2)\chi^\mu(\omega_{j-2}), \end{aligned}$$

while if $j = n - 2$, then

$$\begin{aligned} \nu_c^\mu(T_\omega) &= \nu_p(j + 2)\chi^\mu(\omega_1 + \omega_{j+1}) - \nu_p(j + 2)\chi^\mu(\omega_{j+2} + \delta_{j,n-3}\omega_n) \\ &\quad + \nu_p(2n - j + 2)\chi^\mu(\omega_1 + \omega_{j-1}) \\ &\quad - \nu_p(2n - j + 2)\chi^\mu(\omega_{j-2}). \end{aligned}$$

7.7.3 Conclusion: the case $p \nmid n(n + 1)$

Let K, Y, X be as in the statement of Theorem 7.5, fix $1 < j < n$ and assume $p \nmid n(n + 1)$. Considering the T_X -weight $\omega = 2\omega_1 + \omega_j$, we proceed as in Section 7.4, starting with the following consequence of Lemma 7.7.10.

Corollary 7.7.15

Assume $p \nmid n(n + 1)$ and let $1 < j < n - 1$ be such that $p \mid (j + 2)(2n - j + 2)$. Also write $\omega = 2\omega_1 + \omega_j$ and suppose that $\omega \neq \mu \in X^+(T_X)$ affords the highest weight of a composition factor of $V_X(\omega)$. Then $\mu \in \{\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n, \omega_1 + \omega_{j-1}, \omega_{j+2} + \delta_{j,n-3}\omega_n, \omega_{j-2}\}$ and $[V_X(\omega), L_X(\mu)] = 1$.

Proof. Write $T(2\omega_1, \omega_j) = V_X(2\omega_1) \otimes V_X(\omega_j)$, which is tilting by Lemma 2.4.6, Proposition 2.6.4 (part 3) and Corollary 7.2.3, and identify $\text{rad}(\omega)$ with $\iota(\text{rad}(\omega))$, where

$$\iota : V_X(\omega) \hookrightarrow T(2\omega_1, \omega_j)$$

is the injection given by Proposition 2.6.4 (part 2). First assume $p \mid j+2$ (so $j < n-2$) and observe that by Lemmas 7.7.10, 2.4.6 and Theorem 7.2, we have

$$\begin{aligned} \text{ch}(T(2\omega_1, \omega_j)/\text{rad}(\omega)) &= \text{ch } L_X(\omega) + \text{ch } L_X(\omega_1 + \omega_{j+1}) \\ &\quad + \text{ch } L_X(\omega_1 + \omega_{j-1}) \\ &\quad + \text{ch } L_X(\omega_{j+2} + \delta_{j,n-3}\omega_n) \\ &\quad + \text{ch } L_X(\omega_j) \\ &\quad + \epsilon_p(n+2) \text{ch } L_X(\omega_{j-2}). \end{aligned}$$

Now clearly Proposition 2.6.4 (part 3) applies, yielding a surjective morphism of KX -modules $\phi : T(2\omega_1, \omega_j) \twoheadrightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch } H^0(\omega) = \chi(\omega)$ and since $[V_X(\omega), L_X(\omega_j)] = 0$ by Corollary 2.7.3, the result follows in this case. A similar argument in the situation where $p \mid 2n-j+2$ then completes the proof. The details are left to the reader. \square

Proposition 7.7.16

Assume $p \nmid n(n+1)$ and let $1 < j < n-1$ be such that $p \mid (j+2)(2n-j+2)$. Also consider an irreducible KX -module $V = L_X(\omega)$ having p -restricted highest weight $\omega = 2\omega_1 + \omega_j$. Then

$$V_X(\omega) = \begin{cases} \omega/\omega_1 + \omega_{j+1}/(\omega_1 + \omega_{j-1})^{\epsilon_p(n+2)} & \text{if } p \mid j+2; \\ \omega/(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n)^{\epsilon_p(n+2)}/\omega_1 + \omega_{j-1} & \text{if } p \mid 2n-j+2. \end{cases}$$

Proof. First assume $p \mid j+2$ and let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4. Observe that $j < n-2$ and write $\tau_1 = \omega_1 + \omega_{j+1}$, $\tau_2 = \omega_1 + \omega_{j-1}$, $\tau_3 = \omega_{j+2} + \delta_{j,n-3}\omega_n$ and $\mu = \omega_{j-2}$. Now $\chi^\mu(\tau_1) = \text{ch } L_X(\tau_1) + \text{ch } L_X(\tau_3)$ and $\chi^\mu(\tau_2) = \text{ch } L_X(\tau_2) + \epsilon_p(n+2) \text{ch } L_X(\mu)$ by Theorem 7.2, while $\chi^\mu(\tau_3) = \text{ch } L_X(\tau_3)$ by Lemma 2.4.6. Therefore an application of Lemma 7.7.14 yields

$$\nu_c^\mu(T_\omega) = \nu_p(j+2) \text{ch } L_G(\tau_1) + \nu_p(2n-j+2) \text{ch } L_X(\tau_2).$$

If $p \nmid n+2$ then by Proposition 2.7.8, τ_1 affords the highest weight of a composition factor of $V_X(\omega)$ and every other T_X -weight $\nu \in X^+(T_X)$ such that $\mu \preceq \nu \prec \omega$ satisfies $[V_X(\omega), L_X(\nu)] = 0$. One then concludes in this situation thanks to Corollary 7.7.15. We leave to the reader to conclude in the case where $p \mid 2n-j+2$, as in it can be dealt with in a similar fashion. \square

Proof of Theorem 7.5: the case $p \nmid n(n+1)$. Let K, X, Y be as in the statement of the Theorem, with $p \mid j+2$ and $p \nmid n(n+1)$. We start by considering the situation where $\lambda = 2\lambda_1 + \lambda_j$ for some $1 < j < n$ (in which case one notices that $j < n-2$). As in the proof of Theorem 7.6.3, one shows that $\omega' = \omega_1 + \omega_{j-1}$ affords the highest weight of a KX -composition factor of V and that $[V|_X, L_X(\nu)] = 0$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \prec \omega$. Lemma 6.1.3, Theorems 7.4 and Proposition 7.7.5 then yield

$$\begin{aligned} \dim V &= \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_{j+1}) \\ &= \dim V_Y(\lambda) - \dim L_X(\omega_1 + \omega_{j+1}) - \dim L_X(\omega_j) \\ &= \dim V_X(\omega) - \dim L_X(\omega_1 + \omega_{j+1}) + \dim V_X(\omega_1 + \omega_{j-1}). \end{aligned}$$

Therefore $\dim V = \dim L_X(\omega) + (1 + \epsilon_p(n+2)) \dim L_X(\omega')$ by Proposition 7.7.16 and Theorem 7.2, so that the assertion holds in this situation. (In particular X has more than two composition factors on V if $p \mid n+2$.) Assume $n+1 < j < 2n-1$ for the remainder of the proof and let $\omega = \lambda|_{T_X} = 2\omega_1 + \omega_{2n-j}$, $\omega' = \omega_1 + \omega_{j+1}$. Arguing as above (replacing Proposition 7.7.5 by Proposition 7.7.1), one checks that each of ω and ω' affords the highest weight of a composition factor of $V|_X$ as well as $\dim V = \dim V_X(\omega) - \dim L_X(\omega_1 + \omega_{2n-j-1}) + \dim V_X(\omega')$. Again, applying Theorem 7.2 and Proposition 7.7.16 then completes the proof.

7.7.4 Conclusion: the case $p \neq 3$, $p \mid n$

Let K, Y, X be as in the statement of Theorem 7.5, assume $p \neq 3$ and let $1 < j < n$ be such that $\epsilon_p((j+2)(2n-j+2))\epsilon_p(n) = 1$. Considering the T_X -weight $\omega = 2\omega_1 + \omega_j$, we start by investigating the structure of $V_X(\omega)$ for $\omega = 2\omega_1 + \omega_j$.

Proposition 7.7.17

Assume $p \neq 3$ and let $1 < j < n-1$ be such that p divides both $(j+2)(2n-j+2)$ and n . Also consider an irreducible KX -module $V = L_X(\omega)$ having p -restricted highest weight $\omega = 2\omega_1 + \omega_j$. Then

$$V_X(\omega) = \begin{cases} \omega/\omega_1 + \omega_{j+1} & \text{if } p \mid j+2; \\ \omega/\omega_1 + \omega_{j-1} & \text{if } p \mid 2n-j+2. \end{cases}$$

Proof. Write $T(\omega_1 + \omega_j, \omega_1) = V_X(\omega_1 + \omega_j) \otimes V_X(\omega_1)$, which is tilting by Lemma 2.4.6, Theorem 7.2 and Proposition 2.6.4 (part 3), and identify $\text{rad}(\omega)$ with $\iota(\text{rad}(\omega))$, where $\iota : V_X(\omega) \hookrightarrow T(\omega_1 + \omega_j, \omega_1)$ is the injection given by Proposition 2.6.4 (part 2). First assume $0 \leq j < n-1$ such that $p \mid j+2$ and observe that by Lemmas 7.7.12, 2.4.6, Theorems 7.2 and Theorem 7.3, we have

$$\begin{aligned} \text{ch}(T(\omega_1 + \omega_j, \omega_1)/\text{rad}(\omega)) &= \text{ch } L_X(\omega) + \text{ch } L_X(\omega_2 + \omega_j) \text{ch } L_X(\omega_1 + \omega_{j+1}) \\ &\quad + \text{ch } L_X(\omega_1 + \omega_{j-1}) + \text{ch } L_X(\omega_{j+2}) \\ &\quad + 2 \text{ch } L_X(\omega_j). \end{aligned}$$

Proposition then clearly 2.6.4 (part 3) applies, yielding a surjective morphism of KX -modules $\phi : T(\omega_1 + \omega_j, \omega_1) \twoheadrightarrow H^0(\omega)$ with $\text{rad}(\omega) \subset \ker(\phi)$. As $\text{ch } H^0(\omega) = \chi(\omega)$, we get that if μ affords the highest weight of a composition factor of V , then

$$\mu \in \{\omega_2 + \omega_j, \omega_1 + \omega_{j+1}, \omega_1 + \omega_{j-1}, \omega_{j+2}, \omega_j\}.$$

Now clearly $\omega_2 + \omega_j$ cannot afford the highest weight of a composition factor of $V_X(\omega)$ and an application of Corollary 2.7.3 shows that $[V_X(\omega), L_X(\omega_1 + \omega_{j-1})] = [V_X(\omega), L_X(\omega_j)] = 0$. Considering the A_{j+2} -Levi subgroup of X corresponding to the simple roots $\beta_1, \dots, \beta_{j+2}$, one gets $[V_X(\omega), L_X(\omega_{j+2})] = 0$ as well by Lemma 6.1.3. An application of Lemma 2.3.19 then yields the desired result in this case. A similar argument in the case where $p \mid 2n - j + 2$ completes the proof. The details are left to the reader. \square

Proof of Theorem 7.5: the case $3 \neq p \mid n$. Let K, Y, X be as in the statement of the Theorem, with p dividing both $j + 2$ and n . We start by considering the situation where $\lambda = 2\lambda_1 + \lambda_j$ for some $1 < j < n - 1$. As in the case where $p \nmid n(n - 1)$, one shows that each of ω and $\omega' = \omega_1 + \omega_{j-1}$ affords the highest weight of a KX -composition factor of V and that $[V|_X, L_X(\nu)] = 0$ for every $\nu \in X^+(T_X)$ such that $\omega' \prec \nu \prec \omega$. Lemma 6.1.3, Theorem 7.2 and Proposition 7.7.5 then yield

$$\begin{aligned} \dim V &= \dim V_Y(\lambda) - \dim L_Y(\lambda_1 + \lambda_{j+1}) \\ &= \dim V_Y(\lambda) - \dim L_X(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n) - \dim L_X(\omega_j) \\ &= \dim V_X(\omega) - \dim L_X(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n) + \dim V_X(\omega_1 + \omega_{j-1}). \end{aligned}$$

and the result follows from Proposition 7.7.17 in this situation. Finally, a similar argument allows us to conclude in the case where $n + 1 < j < 2n - 1$. We leave the details to the reader.

7.7.5 Conclusion: the case $\delta_{p,3}\epsilon_3(n) = 1$

In this section, we give a proof of Theorem 7.5 under the assumption that $p = 3$ divides n . First let $1 < j < n$ be such that $j \equiv 1 \pmod{3}$ and consider the T_Y -weight

$$\lambda = 2\lambda_1 + \lambda_j \in X^+(T_Y).$$

As usual, write $\omega = \lambda|_{T_X}$ as well as $\omega' = \omega_1 + \omega_{j-1}$ and let $v^+ \in V_\lambda$ be a maximal vector in V for B_Y (hence also for B_X). As in the proof of Theorem 7.6.3, one easily sees that ω' affords the highest weight of a second KX -composition factor of V .

Lemma 7.7.18

Adopt the notation introduced above and suppose that $\text{Ext}_X^1(L_X(\omega'), L_X(\mu)) = 0$ for every $\mu \in \Lambda^+(V|_X)$. In addition, suppose that X has more than two composition factors on V . Then replacing V by V^ if necessary, there exists a third maximal vector in V for B_X .*

Proof. First observe that the result obviously holds if either $\langle Xv^+ \rangle$ or $\langle Xw^+ \rangle$ is reducible. Therefore we shall assume $\langle Xv^+ \rangle \cong L_X(\omega)$ as well as $\langle Xw^+ \rangle \cong L_X(\omega')$ for the remainder of the proof. Now by assumption, we have $V|_X \cong L_X(\omega') \oplus M$, where $M \cong V|_X/L_X(\omega')$, and if M contains a maximal vector not in $\langle v^+ \rangle$, then the result follows. Otherwise, let U be any irreducible KX -submodule of M (hence $v^+ \in U$) and observe that $v^- = w_0v^+ \in U$ as well, where w_0 denotes the longest element in \mathscr{W}_X . Let $f^+ \in M^*$ be defined by $f^+(v^-) = 1$, $f^+(M_\mu) = 0$ for every $\mu \in \Lambda(M)$ such that $\mu \neq -\omega$. Then for every $t \in T$ and $v \in M_\mu$, we have $(tf^+)(v) = f^+(t^{-1}v)$, from which one easily deduces that $f^+ \in M_\omega^*$. Also for every $\alpha \in X^+(T_X)$, $c \in K$, and $v \in M_\mu$ ($\mu \preceq \omega$), we have

$$\begin{aligned} (u_\alpha(c)f^+)(v) &= f^+(u_\alpha(-c)v) \\ &= f^+\left(v + \sum_{\mu \prec \nu \preceq \omega} v_\nu\right), \end{aligned}$$

where $v_\nu \in M_\nu$ for every $\nu \in X^+(T_X)$ such that $\mu \prec \nu \preceq \omega$. Therefore f^+ is a maximal vector in M^* for B_X having weight ω and clearly $f^+ \notin \text{Ann}_{M^*}(U)$, as $f^+(v^-) = 1$, showing that $\text{Ann}_{M^*}(U)$ contains a maximal vector not in $\langle f^+ \rangle_K$, say g^+ . Since $L_X(\omega')$ is self-dual, we have $V^* \cong L_X(\omega') \oplus M^*$ and hence get the existence of 3 maximal vectors in V^* for B_X , thus completing the proof. \square

Next let $n+1 < j < 2n-1$ be such that $j \equiv 1 \pmod{3}$ and consider the T_Y -weight $\lambda = 2\lambda_1 + \lambda_j \in X^+(T_Y)$. Also write $\omega = \lambda|_{T_X} = 2\omega_1 + \omega_{2n-j}$ as well as $\omega' = \omega_1 + \omega_{2n-j+1}$ and let $v^+ \in V_\lambda$ be a maximal vector in V for B_Y (hence for B_X as well). As above, one easily checks that each of ω and ω' affords the highest weight of a KX -composition factor of V . Now $\text{Ext}_X^1(L_X(\omega), L_X(\omega')) = 0$ by Lemma 2.6.5, so there exists a maximal vector in $(V|_X)_{\omega'}$ for B_X , say w^+ . We leave to the reader to show the following result, whose proof is similar to that of Lemma 7.7.18.

Lemma 7.7.19

Adopt the notation introduced above and suppose that $\text{Ext}_X^1(L_X(\omega'), L_X(\mu)) = 0$ for every $\mu \in \Lambda^+(V|_X)$. In addition, suppose that X has more than two composition factors on V . Then replacing V by V^ if necessary, there exists a third maximal vector in V for B_X .*

Using Lemma 7.7.14, we also study the expression $\chi^{\omega_j-2}(2\omega_1 + \omega_j)$ in terms of characters of irreducibles in the case where $3 < j < n$.

Proposition 7.7.20

Assume $\delta_{p,3}\epsilon_p(n) = 1$ and let $3 < j < n-1$ be such that $\epsilon_p((j+2)(2n-j+2)) = 1$. Also consider the T_X -weight $\omega = 2\omega_1 + \omega_j$ and set $\mu = \omega_{j-2}$. Then

$$\chi^\mu(\omega) = \begin{cases} \text{ch } L_X(\omega) + \text{ch } L_X(\omega_1 + \omega_{j+1}) & \text{if } j \equiv 1 \pmod{3}; \\ \text{ch } L_X(\omega) + \text{ch } L_X(\omega_1 + \omega_{j-1}) & \text{if } j \equiv 2 \pmod{3}. \end{cases}$$

Proof. First assume $j \equiv 1 \pmod{3}$ and let $V_X(\omega) = V^0 \supset V^1 \supset \dots \supset V^k \supsetneq 0$ be the filtration of $V_X(\omega)$ given by Proposition 2.7.4. Using Lemma 2.4.6, Theorem 7.2 and Lemma 7.7.14, one checks that

$$\nu_c^\mu(T_\omega) = \nu_3(j+2) \operatorname{ch} L_X(\omega_1 + \omega_{j+1} + \delta_{j,n-2}\omega_n).$$

Therefore the result follows from Lemma 2.3.19 together with Proposition 2.7.8. Arguing in a similar fashion yields the result in the case where $j \equiv 2 \pmod{3}$, thus completing the proof. The details are left to the reader. \square

Lemma 7.7.21

Assume $\delta_{p,3}\epsilon_p(n) = 1$ and let $3 < j < n-1$ be such that $j \equiv 1 \pmod{3}$. Also consider an irreducible KY-module $V = L_Y(\lambda)$ having highest weight $\lambda = 2\lambda_1 + \lambda_j$ and write $\omega = \lambda|_{T_X}$, $\omega' = \omega_1 + \omega_{j-1}$, as well as $\mu = \omega_{j-2}$. Then

$$\mathfrak{m}_{V|_X}(\mu) = \mathfrak{m}_{L_X(\omega)}(\mu) + \mathfrak{m}_{L_X(\omega')}(\mu).$$

Proof. Write $l = \mathfrak{m}_{V_Y(\lambda)|_X}(\mu) - \mathfrak{m}_{V|_X}(\mu)$. Then applying Proposition 7.7.5, Lemma 2.4.6, Theorem 7.2 and Proposition 7.7.20 yields

$$\begin{aligned} \mathfrak{m}_{V|_X}(\mu) &= \mathfrak{m}_{V_Y(\lambda)|_X}(\mu) - l \\ &= \mathfrak{m}_{V_X(\omega)}(\mu) + \mathfrak{m}_{V_X(\omega')}(\mu) + \mathfrak{m}_{V_X(\omega_j)}(\mu) - l \\ &= \mathfrak{m}_{L_X(\omega)}(\mu) + \mathfrak{m}_{L_X(\omega')}(\mu) + \mathfrak{m}_{L_X(\omega_1 + \omega_{j+1})}(\mu) + \mathfrak{m}_{L_X(\omega_j)}(\mu) - l. \end{aligned}$$

Finally, we leave to the reader to check (using Lemma 7.1.5 together with Corollaries 7.6.4 and 7.7.7) that $\mathfrak{m}_{L_X(\omega_1 + \omega_{j+1})}(\mu) + \mathfrak{m}_{L_X(\omega_j)}(\mu) = l$, thus completing the proof. \square

Lemma 7.7.22

Assume $\delta_{p,3}\epsilon_p(n) = 1$ and let $n+2 < j < 2n-1$ be such that $j \equiv 1 \pmod{3}$. Also consider an irreducible KY-module $V = L_Y(\lambda)$ having highest weight $\lambda = 2\lambda_1 + \lambda_j$ and write $\omega = \lambda|_{T_X} = 2\omega_1 + \omega_{2n-j}$, $\omega' = \omega_1 + \omega_{2n-j+1}$, as well as $\mu = \omega_{2n-j}$. Then

$$\mathfrak{m}_{V|_X}(\mu) = \mathfrak{m}_{L_X(\omega)}(\mu) + \mathfrak{m}_{L_X(\omega')}(\mu).$$

Proof. We proceed as in the proof of Lemma 7.7.21, writing $l = \mathfrak{m}_{V_Y(\lambda)|_X}(\mu) - \mathfrak{m}_{V|_X}(\mu)$. Then applying Proposition 7.7.1, Lemma 2.4.6, Theorem 7.2 and Proposition 7.7.20 yields

$$\begin{aligned} \mathfrak{m}_{V|_X}(\mu) &= \mathfrak{m}_{V_Y(\lambda)|_X}(\mu) - l \\ &= \mathfrak{m}_{V_X(\omega)}(\mu) + \mathfrak{m}_{V_X(\omega')}(\mu) + \mathfrak{m}_{V_X(\mu)}(\mu) - l \\ &= \mathfrak{m}_{L_X(\omega)}(\mu) + \mathfrak{m}_{L_X(\omega')}(\mu) + \mathfrak{m}_{L_X(\omega_1 + \omega_{2n-j-1})}(\mu) + 1 - l. \end{aligned}$$

Finally, we leave to the reader to check (using Lemmas 2.3.19 and 7.7.3, for example) that $\mathfrak{m}_{L_X(\omega_1 + \omega_{2n-j-1})}(\mu) + 1 = l$, thus completing the proof. \square

Proof of Theorem 7.5: the case $\delta_{p,3\epsilon_p}(n) = 1$. First assume $1 < j < n$, in which case each of $\omega = \lambda|_{T_X}$ and $\omega' = \omega_1 + \omega_{j-1}$ affords the highest weight of a KX -composition factor of V as seen above. By Lemma 7.7.18, either $V|_X \cong L_X(\omega) \oplus L_X(\omega')$ or there exists a third maximal vector u^+ in V for B_X . Now in the latter case, by Proposition 7.1.3, there exists $\omega'' \in X^+(T_X)$ such that $\omega_{j-2} \preceq \omega'' \preceq \omega'$ and $u^+ \in V_{\omega''}$. Therefore $\omega'' \in \{\omega_1 + \omega_{j-3}, \omega_j, \omega_{j-2}\}$ and by Theorem 2.3.4, we get that $m_{L_X(\omega'')}(\omega_{j-2}) > 0$ in each case. An application of Lemma 7.7.21 then allows us to conclude in this situation.

Finally, assume $n < j < 2n - 1$ and first observe that if $j = n + 1$, then X has more than two composition factors on V . Also, arguing as above (replacing Lemma 7.7.18 by Lemma 7.7.19, ω_{j-2} by ω_{2n-j} and Lemma 7.7.21 by Lemma 7.7.22) completes the proof.

List of notations

$(-, -)$	19
$[G : H]$	33
$[i, j]$	46
$[V, L_G(\mu)]$	40
$[V, Q_j^i]$	25
$\text{ch } V$	44
$\chi(\lambda)$	44
$\chi^\mu(\lambda)$	44
$d(\lambda, \mu)$	27
$\epsilon_i(z)$	34
$\text{Ext}_G^1(V_2, V_1)$	42
$\Lambda(\lambda)$	24
$\Lambda(V)$	22
$\Lambda^+(\lambda)$	27
$\Lambda^+(V)$	22
$\langle -, - \rangle$	19
$\text{m}_V(\mu)$	22

$\mathfrak{b}_{\mathcal{L}}$	35
$\mathfrak{h}_{\mathcal{L}}$	35
\mathfrak{S}_l	20
\mathcal{B}	35
\mathcal{D}	45
$\mathcal{L}(G)$	38
$\mathcal{W} = \mathcal{W}_G$	19
\mathcal{W}_p	43
\mathcal{W}_μ	33
$\text{CL}_n(K)$	32
$\mu_1/\mu_2/\dots/\mu_s$	40
$\mu_{i,x}$	27
$\nu_c(T_\lambda)$	44
$\nu_c^\mu(T_\lambda)$	46
$\nu_p(n)$	44
$\Phi(G)$	19
$\Phi^+(G)$	19
Φ_i^+	32
$\Phi^-(G)$	19
Φ_j^+	24
$\Pi(G)$	19
\preceq	22
$\text{rad}(\lambda)$	24
ρ	20
$\text{soc}(\lambda)$	41

$\mathbb{Z}[X]$	28
$\mathbb{Z}[X]_d$	28
$A \sim_G B$	47
$A_{\alpha,r}$	47
B_G	19
B_μ	47
d_G	47
e_α	35
$f_{(m)}$	80
f_α	35
$H^0(\lambda)$	41
h_{α_i}	35
K_λ	40
$L_G(\lambda)$	23
L_J	24
L_r	49
$N_{(\alpha,\beta)}$	35
$P(i,j)$	80
P_J	24
P_r	49
Q_J	24
Q_r	51
$s_{\alpha,r}$	43
s_α	19
T_G	19

U_G	19
U_α	24
$u_\alpha(t)$	24
v^+	23
V^F	23
v^λ	38
$V_G(\lambda)$	24
V_{\max}	143
V_μ	22
$w \cdot \lambda$	43
w_0	41
$X(T_G)$	19
$X(T_G)_{\mathbb{R}}$	19
$X^+(T_G)$	19

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CITIZENSHIP Switzerland

LANGUAGES French (mother tongue), English, German.

RESEARCH INTERESTS Representation theory of algebraic groups.

EDUCATION **Ecole Polytechnique Federale de Lausanne (EPFL)**, Switzerland
The University Of York, UK

P.H.D., Mathematics, 2011-2015, EPFL

- Thesis Topic: *Restricting representations of classical algebraic groups to maximal subgroups*
- Adviser: Professor Donna Testerman
- Area of Study: Representation theory of algebraic groups

M.S., Mathematics, February 2011, EPFL and University of York

- Thesis Topic: *Representations of $SL_2(K)$*
- Advisers: Professor Stephen Donkin and Lecturer Donna Testerman
- Area of Study: Representation theory of algebraic groups

B.S., Mathematics, July 2009, EPFL