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ABSTRACT

When a finite element method is used to numerically solve partial differential equations, spectral pollution phenomena may appear. To clarify them, we study a simple physical situation : the wave propagation in a waveguide. We first discretize the Maxwell equations with bilinear and bicubic finite elements and find some spurious modes which do not satisfy the analytical dispersion relation and fail to converge to the true values, known analytically. To eliminate these numerical instabilities, one can choose to discretize the equations with non-polluting elements or to transform them so that the discretization scheme is pollution-free. To solve wave propagation problems, it may be convenient to consider a gauge transformation.

INTRODUCTION

The recent success of ion cyclotron resonance heating (ICRH) experiments has stimulated interest in the elaboration of theoretical and numerical models [1-6] calculating the wave fields and power deposition profiles in tokamaks. From the technical point of view there are two major difficulties in the numerical modeling of ICRH-fields, the first being simply a problem of resolution. The second one is a numerical instability known as "spectral pollution" and is common to all wave propagation problems in which the dominant spatial operator is of the ($\nabla x \nabla x$)-type. In general physical terms, spectral pollution may be described as being due to a discretization induced, very bad deformation of certain wave branches. In the problem of electromagnetic wave propagation in vacuo in particular, electrostatic solutions can appear at finite frequencies and "pollute" the spectrum of wave-guide modes, for instance.

Large two-dimensional codes should be based on a pollution-free method even if the primary target of the code is a problem where pollution does not appear. There are three reasons for that. First, potentially polluting methods seem to need for equal precision higher resolution than non-polluting ones [7, 15]. Second, a large code will very likely be used for parameters which have not been considered from the outset; the new physical parameter range might then be infested by pollution. Third, the same pollution-free method can be applied to both the plasma and the vacuum, a fact which might considerably ease the code construction. These are the motivations for the present detailed investigation of the spectral pollution phenomena in the discretization of Maxwell's equations.

The paper is essentially technical, concentrating on the finite element discretizations of known physics. We clarify the spectral pollution phenomena by means of the discrete dispersion relation and give a formulation that leads to reliable physical results when solving linear wave propagation problems. In sections 1 and 3, we are interested in the wave propagation problem in rectangular and cylindrical waveguides. One way of eliminating spectral pollution is to express the Maxwell equations in terms of the 4- vector potential $(\mathbf{A},\phi)$. These investigations are reported in sections 2 and 4 . Finally, the main conclusions are drawn.

1. Wave propagation in a rectangular waveguide

1.1 Theoretical considerations

We propose to solve a basic physical problem : the wave propagation in a rectangular waveguide free of charges and currents. We assume the guide walls to be ideal conductors.

We choose such a simple problem to be able to test the code by comparing the theoretical and numerical results. As we shall see later , this equation also permits to clarify " the spectral pollution phenomena ".

The system to be solved is written as :

$$\nabla\Lambda\nabla\Lambda\mathbf{E} - k_0^2\mathbf{E} = 0 \qquad (1)$$

where $k_0 = \frac{\omega}{c}$ (2)

with ω : the forcing RF (Radio Frequency) angular frequency

$\mathbf{E}$: the RF electric field vector

c : the speed of light

With the assumption that the waveguide is of infinite extent in the Oz direction, the components of the electric field $\mathbf{E}$ can be developed in Fourier harmonics in this direction :

$$E_j(x, y, z) = U_j(x, y) \exp(i(k_z z - \omega t)) \qquad (j = x, y, z) \qquad (3)$$

- where k_z is the wave number in the Oz direction;

In order to solve eq. (1) on the rectangular domain Ω, defined by

$$\Omega = \{(x, y) : x_{min} \leq x \leq x_{max} \text{ and } y_{min} \leq y \leq y_{max}\}$$

one imposes as boundary conditions that :

- the fields are periodic in the Oy direction

$$E_j(x, y_{min}) = E_j(x, y_{max}) \qquad (j = x, y, z) \qquad (4)$$

- the tangential components of **E** are continuous across the conductor walls so that:

$$E_j(x_{min}, y) = E_j(x_{max}, y) = 0 \qquad (j = y, z) \qquad (5)$$

Therefore the theoretical dispersion relation can be written:

$$\omega_{mn}^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) + c^2k_z^2 \qquad (6)$$

- where m and n are the integer number of half-wavelengths in the Ox and Oy directions. Because of the boundary conditions, n is even;
- where a and b are the inner dimensions of the waveguide in the Ox and Oy directions.

It is known that one can describe some arbitrary electromagnetic disturbance in a rectangular waveguide from the Transverse Electric TE_{mn} (E_z=0; everywhere) and Transverse Magnetic TM_{mn} modes (B_z=0;everywhere). Let us note that these modes satisfy the same dispersion relation and are degenerate for the same indices (m , n).

1.2 Numerical considerations

To solve numerically eq.(1), we employ a finite element method which has been widely recognized as a powerful tool to solve partial differential equations.The main merit of this method is the adaptability to various kinds of coordinate systems and arbitrary shapes of boundaries.Therefore it becomes possible to analyze the waves in different types of configurations without considerable modification of the numerical code.

As it was mentioned previously, we assume that the electric field **E** is developed in Fourier harmonics in the Oz direction :

$$E_j(x, y, z) = U_j(x, y) \exp(i(k_z z - \omega t)) \qquad (j = x, y, z) \qquad (7)$$

Two-dimensional finite elements are used to determine $U_j(x, y)$.

The principle of the finite element method can be summarized as follows :

1) We discretize the domain Ω into N_x and N_y rectangular mesh cells :

$$x_{min} = x_0 < x_1 < x_2 < \ldots < x_j < \ldots < x_{Nx} = x_{max}$$
$$y_{min} = y_0 < y_1 < y_2 < \ldots < y_j < \ldots < y_{Ny} = y_{max} \qquad (8)$$

2) We design each mesh cell by $[x_1, x_2] \times [y_1, y_2]$ and we write the components $U_j(x, y)$ in the form :
there m and n have nothing to do with those in eq. (6)!

$$U(x,y) = \sum_{k\,l=1}^{2} \sum_{m=0}^{m_{max}} \sum_{n=0}^{n_{max}} u_{kl}^{mn}\ \psi_{kl}^{mn}(x,y) \qquad (9)$$

where we have suppressed the subscript j for notational simplicity.

where u_{kl}^{mn} are the nodal values at the points (x_k,y_l) and hence the unknowns to be determined ;

where $(m_{max}+1)\,(n_{max}+1)$ is the number of unknowns by node.

The functions $\psi_{kl}^{mn}(x,y)$ are tensor-products of one-dimensional basis:

$$\psi_{kl}^{mn}(x,y) = \psi_k^m(x) \otimes \psi_l^n(y) \qquad (10)$$

For convenience we prefer to consider the rectangle Ω in the form of the standard unit square $\hat{\Omega}$ and use the variables p and q :

$$0 \le p\,,\, q \le 1$$

Therefore the interpolant is written:

$$\hat{U}(p,q) = \sum_{k\,l=1}^{2} \sum_{m=0}^{m_{max}} \sum_{n=0}^{n_{max}} \hat{u}_{kl}^{mn}\ \hat{\psi}_{kl}^{mn}(p,q) \qquad (11)$$

where $\hat{u}_{kl}^{mn}$ are the new unknowns of the problem.

The basis functions $\hat{\psi}_{kl}^{mn}(p,q)$ are also tensor-products of one-dimensional basis.

The map from a rectangular element Ω to the unit square $\hat{\Omega}$ is simply the composition of two linear one-dimensional transformations $x \rightarrow p$ and $y \rightarrow q$. Hence we can develop the interpolant on $\hat{\Omega}$ and then map back to Ω without any loss of generality.

Hybrid elements

Here the interpolant is :

$$\hat{U}(p,q) = \sum_{k\ l=1}^{2} \hat{u}_{kl}^{00}\ \hat{\psi}_{kl}^{00}(p,q) \tag{12.a}$$

and the derivatives of U(p,q) may be chosen to be :

$$\frac{\partial \hat{U}}{\partial p}(p,q) = \sum_{k\ l=1}^{2} \hat{u}_{kl}^{00}\ \hat{\psi}_{kl}^{10}(p,q) \tag{12.b}$$

$$\frac{\partial \hat{U}}{\partial q}(p,q) = \sum_{k\ l=1}^{2} \hat{u}_{kl}^{00}\ \hat{\psi}_{kl}^{01}(p,q) \tag{12.c}$$

with:

$$\hat{\psi}_{11}^{00} = \hat{\psi}_{12}^{00} = \hat{\psi}_{21}^{00} = \hat{\psi}_{22}^{00} = \frac{1}{4} \tag{13.a}$$

$$\tilde{\psi}_{11}^{10} = \hat{\psi}_{12}^{10} = -\hat{\psi}_{21}^{10} = -\hat{\psi}_{22}^{10} = \frac{1}{2} \tag{13.b}$$

$$\tilde{\psi}_{11}^{01} = -\hat{\psi}_{12}^{01} = \tilde{\psi}_{21}^{01} = -\hat{\psi}_{22}^{01} = -\frac{1}{2} \tag{13.c}$$

where the four corner points numbered (1,1),(2,1),(1,2),(2,2) are respectively located at the positions (0,0),(1,0),(0,1),(1,1) in (p,q) space.

Let us note that the unknowns $\hat{u}_{kl}^{00}$ are approximated by piecewise constants in each mesh cell and determined at the center of these cells.

It has been shown by R. Gruber and J. Rappaz that the eigenvalue spectrum is well represented when a finite hybrid discretization is used to solve the differential eigenvalue equations of the ideal linear magnetohydrodynamics [8].

Bilinear elements

Here the interpolant is

$$\hat{U}(p,q) = \sum_{k}^{2} \sum_{l=1} \hat{u}_{kl}^{00} \hat{\psi}_{kl}^{00}(p,q) \tag{14}$$

with:

$$\hat{\psi}_1^0(p) = 1 - p \tag{15}$$
$$\hat{\psi}_2^0(p) = p$$

so that:

$$\hat{\psi}_{11}^{00}(p,q) = (1-p).(1-q)$$
$$\hat{\psi}_{21}^{00} = p.(1-q)$$
$$\hat{\psi}_{12}^{00}(p,q) = (1-p).q \tag{16}$$
$$\hat{\psi}_{22}^{00} = p.q$$

where the four corner points numbered (1,1),(2,1),(1,2),(2,2) are respectively located at the positions (0,0),(1,0),(0,1),(1,1) in (p,q) space. The bilinear basis functions $\hat{\psi}_{kl}^{00}$ are piecewise bilinear and take on value unity at the (k,l) node

point and zero at all the other node points. The hybrid and bilinear elements have a convergence order in $\theta(h^2)$ so that it is not always possible to achieve good convergence properties . For this reason we also discuss the discretization with bicubic Hermite elements.

Bicubic Hermite elements.

First of all, let us define the cubic Hermite elements.

The interpolant is written as:

$$\hat{U}(p) = \sum_{j=1}^{2} (\hat{u}_j^0 \hat{\psi}_j^0(p) + \hat{u}_j^1 \hat{\psi}_j^1(p)) \qquad (17)$$

where the Hermite basis functions $\hat{\psi}_j^0(p)$ and $\hat{\psi}_j^1(p)$ satisfy the interpolation properties at end nodes p=0 and p=1:

$$\hat{\psi}_j^0(p_k) = \delta_{jk} \qquad \hat{\psi}_j^1(p_k) = 0$$

$$\frac{\partial \hat{\psi}_j^0}{\partial p}(p_k) = 0 \qquad \frac{\partial \hat{\psi}_j^1}{\partial p}(p_k) = \delta_{jk} \qquad (18)$$

where δ_{jk} is the Kronecker delta.

With these properties $\hat{U}(p_j)=\hat{u}_j^0$ and $\hat{U}'(p_j)=\hat{u}_j^1$. (19)

We can construct the Hermite cubics directly as :

$$\hat{\psi}_1^0(p) = (1-p)^2 (1+2p)$$

$$\hat{\psi}_1^1(p) = p (p-1)^2$$

$$\hat{\psi}_2^0(p) = p^2(3-2p) \qquad (20)$$

$$\hat{\psi}_2^1(p) = p^2(p-1)$$

By means of the linear map $p \rightarrow x$ we can transform the shape functions $\hat{\psi}_j^0$ and $\hat{\psi}_j^1$ on to the corresponding shape functions ψ_j^0 and ψ_j^1 on Ω.

The form of the approximation (17) on Ω becomes :

$$U(x) = \sum_{j=1}^{2} u_j \psi_j^0(x) + \Delta x \sum_{j=1}^{2} \left(\frac{du}{dx}\right)_j \psi_j^1(x) \qquad (21)$$

where $x_j \leq x \leq x_{j+1}$

and where we have used the chain rule to express $\frac{d\hat{u}}{dp}$ as $\Delta x \frac{du}{dx}$ with $\Delta x = x_j - x_{j-1}$ as the element length.

In the two-dimensional case the interpolant is:

$$U(p,q) = \sum_{k}^{2} \sum_{l=1}^{} \sum_{m}^{1} \sum_{n=0}^{} u_{kl}^{mn} \psi_{kl}^{mn}(p,q) \qquad (22)$$

where

$$\begin{aligned} \hat{u}_{kl}^{00} &= \hat{u}(p_k,q_l) \\ \hat{u}_{kl}^{10} &= \frac{\partial \hat{u}}{\partial p}(p_k,q_l) \\ \hat{u}_{kl}^{01} &= \frac{\partial \hat{u}}{\partial q}(p_k,q_l) \\ \hat{u}_{kl}^{11} &= \frac{\partial^2 \hat{u}}{\partial p \partial q}(p_k,q_l) \end{aligned} \qquad (23)$$

and

$$\hat{\psi}_{11}^{00}(p,q) = (p-1)^2(2p+1)(q-1)^2(2q+1)$$

$$\hat{\psi}^{00}_{12}(p,q) = -(p-1)^2(2p+1)q^2(2q-3)$$

$$\hat{\psi}^{00}_{21}(p,q) = -p^2(2p-3)(q-1)^2(2q+1)$$

$$\hat{\psi}^{00}_{22}(p,q) = p^2(2p-3)q^2(2q-3)$$

$$\hat{\psi}^{10}_{11}(p,q) = p(p-1)^2(q-1)^2(2q+1)$$

$$\hat{\psi}^{10}_{12}(p,q) = -p(p-1)^2q^2(2q-3) \qquad (24)$$

$$\hat{\psi}^{10}_{21}(p,q) = p^2(p-1)(q-1)^2(2q+1)$$

$$\hat{\psi}^{10}_{22}(p,q) = -p^2(p-1)q^2(2q-3)$$

$$\hat{\psi}^{01}_{11}(p,q) = (p-1)^2(2p+1)q(q-1)^2$$

$$\hat{\psi}^{01}_{12}(p,q) = (p-1)^2(2p+1)q^2(q-1)$$

$$\hat{\psi}^{01}_{21}(p,q) = -p^2(2p-3)q(q-1)^2$$

$$\hat{\psi}^{01}_{22}(p,q) = -p^2(2p-3)q^2(q-1)$$

$$\hat{\psi}^{11}_{11}(p,q) = p(p-1)^2q(q-1)^2$$

$$\hat{\psi}^{11}_{12}(p,q) = p(p-1)^2q^2(q-1)$$

$$\hat{\psi}^{11}_{21}(p,q) = p^2(p-1)q(q-1)^2$$

$$\hat{\psi}^{11}_{22}(p,q) = p^2(p-1)q^2(q-1)$$

where the four corner points numbered (1,1),(2,1),(1,2),(2,2) are respectively located at the positions (0,0),(1,0),(0,1),(1,1) in (p,q) space .

Using the maps $p \rightarrow x$ and $q \rightarrow y$, we obtain the corresponding tensor-product shape functions on the domain Ω. The derivatives in the p and q

directions in (22) transform by the chain rule to partial derivatives in x and y. Therefore there are four element shape functions associated with each mode :

$$(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y})$$

The interpolant transforms to :

$$U(x,y) = \sum_{k}^{2} \sum_{l=1}^{} \sum_{m}^{1} \sum_{n=0}^{} u_{kl}^{mn} \psi_{kl}^{mn}(x,y) \tag{25}$$

with :

$$\begin{aligned} u_{kl}^{00} &= u(x_k,y_l) \\ u_{kl}^{10} &= \frac{\partial u}{\partial x}(x_k,y_l) \\ u_{kl}^{01} &= \frac{\partial u}{\partial y}(x_k, y_l) \\ u_{kl}^{11} &= \frac{\partial^2 u}{\partial x \partial y}(x_k,y_l) \end{aligned} \tag{26}$$

One can show that:

$$\begin{aligned} \frac{\partial \hat{u}}{\partial p}(p,q) &= \Delta x \frac{\partial u}{\partial x}(x,y) \\ \frac{\partial \hat{u}}{\partial q}(p,q) &= \Delta y \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial^2 \hat{u}}{\partial p \partial q}(p,q) &= \Delta x \Delta y \frac{\partial^2 u}{\partial x \partial y}(x,y) \end{aligned} \tag{27}$$

with:

$$\begin{aligned} \Delta x &= x_{i+1} - x_i \\ \Delta y &= y_{i+1} - y_i \end{aligned} \tag{28}$$

The advantage of the cubic (or bicubic) Hermite elements is that they have a convergence order in $\theta(h^4)$.

3) We determine the weak variational form of the wave equation. This is usually done by multiplying the differential equations by sufficiently regular test functions **G**(x,y,z) and then integrating by parts.
One of the possibilities is to take **G**(x,y,z) as follows:

$$\mathbf{G}(x,y,z)= (V_x(x,y),V_y(x,y),V_z(x,y)) \exp(i(k_z z-\omega t)) \tag{29}$$

so that eq. (1) becomes:

$$\int_{\Omega} \text{rot}\, \mathbf{G}^*.\text{rot}\, \mathbf{E}\, d\tau - k_0^2 \int_{\Omega} \mathbf{G}^*.\mathbf{E}\, d\tau = \int_{S} (\mathbf{G}^* \wedge \text{rot}\, \mathbf{E}) . \mathbf{n}\, dl \tag{30}$$

where

n is the normal of the surface S,

Ω is the integration domain,

S is the surface which bounds Ω.

The boundary condition on the perfect conducting wall requires that the tangential component of **E** vanishes. Besides, we assume that **G** satisfies the same boundary conditions as **E** so that the surface integral equals zero. Let us note that the integrand of the surface integral corresponds to the Poynting vector representing the electromagnetic part of the energy flux.

The integration over Ω can be replaced by a sum over all mesh cells. For each cell, one calculates its contribution to the variational form (eq.30) using the formula (eq.9). This yields local matrices, namely element matrices in the literature, which have to be added in the proper way to construct the total

matrices of the discretized form. In our code, this is done so that the vector solution is automatically periodic along the OY axis. Then this code is particularly adapted to solve periodic problems .

The integrals are computed over each mesh cell with the Gaussian integration formulae, which express an integral over the interval [-1,1] × [-1,1] as :

$$\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\gamma)\, d\xi\, d\gamma \cong \sum_{k\,l}^{m} c_{kl}\; g(\xi_k, \psi_l) \qquad (31)$$

where c_{kl} are the integration weights and ξ_k and ψ_l integration abcissae.
In our case, one considers a transformation from the product intervals [-1,1]x[-1,1] to the unit square [0,1]x[0,1].

The m-points formula integrates exactly polynomials of order 2m-1. Hence the order of the error is $\theta(h^{2m})$. For the bilinear and bicubic elements,we choose respectively m=2 and m=4 while in the hybrid case,the integration reduces to a multiplication of the value of the integrand at the centre of a cell by the surface of this cell.

4) We solve the eigenvalue problem :

$$\mathbf{CX} = k_0^2\, \mathbf{DX} \qquad (32)$$

where C and D are the discretized versions of the operators defined by eq.(30) .
where k_0^2 and **X** are respectively the eigenvalues and the eigenvectors to be determined .

Let us note that the matrices C and D have a band structure when an adequate numbering is used. They have a block-tridiagonal structure whose bandwith increases linearly with N_y^3. Therefore, memory requirements increases as N_y^2 and computing time as N_y^3.

All these eigenvalues and eigenfunctions are determined using the QZ algorithm:

- C is reduced to upper Hessenberg form and at the same time D is reduced to upper triangular form.
- C is further reduced to triangular form while the triangular form of D is maintained and the diagonal elements of D are made real and non-negative.
- The eigenvectors are obtained from the triangular matrices .

5) The next step is to reconstruct the electric field **E** from the solution vector according to (9). Then, it can be evaluated at any arbitrary position by interpolation.

The finite element software is developed for the most general variational form by supposing that the unknowns lie on the nodes of the mesh. It is adapted to solve a system constituted by N partial differential equations and it is sufficiently modular to include other elements than hybrid,bilinear and bicubic Hermite ones .

1.3 Results

With the following values of the different parameters :

$$k_z = 0 \; ; \; x_{min} = y_{min} = 0 \; ; \; x_{max} = 2.33 \text{ cm} \; ; \; y_{max} = 2\pi \text{ cm} \qquad (33)$$

one obtains the spectra given in Figs. [1 - 2] when bilinear and bicubic Hermite elements are used to discretize the Maxwell equations.

All the theoretical investigations, with the assumption that k_z is equal to zero, are reported in the next sections and the Tables [1 - 2]. Another choice of the value of k_z does not conceptually modify our conclusions.

As it can be seen, spectra exhibit spurious eigenvalues which spread among physical ones . They fail to converge to the true values , known analytically, for any mesh density. This phenomenon, namely spectral pollution, was already found by Appert et al. [7] when they computed the eigenvalues of the ideal linear magnetohydrodynamics equations for a homogeneous cylindrical plasma.

One possibility to investigate the spectral pollution phenomena is to determine the discrete dispersion relation of the Maxwell equations.

1.4 Diagnosis of pollution

Let us define the refractive index $\mathbf{n}= \frac{\mathbf{k}}{k_0}$. There is no loss of generality in assuming $n_z=0$.

Eq.(1) decouples into two systems for components (E_x, E_y) and E_z, respectively. The analytical dispersion relation becomes:

$$n_x^2 + n_y^2 = 1 \qquad (34)$$

It can be obtained by assuming the solutions of the form $\exp(i\mathbf{k}.\mathbf{x})$ and replacing the operators ∂_j by ik_j (j=x,y) .The dispersion relation of the

discretized system is similarly obtained by substituting solutions proportional to exp(i**k.x**) and finding the zeros of the resulting determinant.

For the E_z component , which satisfies the Helmholtz equation, the modes belong to the branch (a) that is strongly distorted at high k_x and k_y by the discretization used (see Fig. 3) .

The system satisfied by (E_x,E_y) exhibit spectral pollution. Indeed there appears a second branch (b) whose the modes have a very short wavelength in the wave propagation region. So the discretization scheme of this system is said to be polluting.

Let us note that a polluting scheme may be very good for boundary value problems with k_0^2 fixed by the source. Indeed it may happen that the spurious modes are not excited by the source or that they are strongly damped in the region of interest. It is also possible that they are suppressed by the boundary conditions but in this case, the discretization scheme may yield good results for only a class of boundary conditions.

To make such a scheme non-polluting, the first possibility is to discretize the equations with non-polluting elements. It has been successfully suggested by R. Gruber and J. Rappaz [8] , H. Evequoz [9] and J. Jaccard [10]. The second possibility is to formulate the equations in another way so that the discretization scheme is pollution-free. To solve wave propagation problems, it may be convenient to consider a gauge transformation.

1.5 Eigenvalues

Before proposing a non-polluting scheme, let us analyze the degeneracy of the eigenvalues in the cases where $(m=0,n\neq0)$, $(m\neq0,n=0)$ and $(m\neq0,n\neq0)$.

In the general case where ($m \neq 0, n \neq 0$) this system admits four linearly independent solutions :

a)
$$E_x = E_{x01} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$
$$E_y = E_{y01} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$E_z = 0$$

b)
$$E_x = E_{x01} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$E_y = E_{y01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t) \qquad (35)$$
$$E_z = 0$$

c)
$$E_z = E_{z01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$
$$E_x = E_y = 0$$

d)
$$E_z = E_{z02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$E_x = E_y = 0$$

These solutions reduce to 2 in the cases where ($m \neq 0, n = 0$) and ($m = 0, n \neq 0$). Let us note that the degeneracy of the eigenvalues, obtained after a numerical computation with a polluting method, is not always in agreement with the theoretical predictions.

2 GAUGE TRANSFORMATION

2.1 Theoretical considerations

Batchelor et al. [12] have solved the Maxwell equations directly in terms of the electric field **E** with a finite difference method. They have found that the solutions do not always have a good physical structure. They have then formulated these equations in terms of the 4- vector potential ($\mathbf{A},\phi$) using the Coulomb gauge. They have developed an alternative differencing scheme for the Poisson equation such that $\nabla^2(\nabla.\mathbf{A})=0$ in the finite difference sense. They have noticed a good agreement in the field structures and in energy conservation when the Coulomb condition, $\nabla.\mathbf{A}=0$, is accurately satisfied.

From the first Maxwell equation,

$$\text{div}\,\mathbf{B} = 0 \tag{36}$$

so that **B** may be written as:

$$\mathbf{B} = \text{rot}\mathbf{A} \tag{37}$$

However, the vector potential **A** is not completely specified by this relation. It may be modified by the addition of any gradient without affecting **B** . Thus,

$$\mathbf{A(r)} = \mathbf{A}_0\,\mathbf{(r)} + \text{grad}\,\chi(\mathbf{r}) \tag{38}$$

The addition of a gradient is called a gauge transformation. Since the field **B** is unchanged, there are no physical consequences, so the theory is said

to be gauge invariant. This property allows complete freedom for the divergence of **A(r)**.

Since

$$\mathrm{div}\mathbf{A} = \mathrm{div}\mathbf{A}_0 + \nabla^2\chi(\mathbf{r}) \tag{39}$$

if we want div**A** to be some specified function f(**r**) , we have only to choose $\chi(\mathbf{r})$ as a solution of a " Poisson equation" :

$$\nabla^2\chi(\mathbf{r}) = f(\mathbf{r}) - \mathrm{div}\mathbf{A}_0(\mathbf{r}) \tag{40}$$

From the second Maxwell equation, we obtain:

$$\mathrm{rot}\,(\mathbf{E} - i\omega\mathbf{A}) = 0 \tag{41}$$

Hence (**E** - iω**A**) is a gradient so that :

$$\mathbf{E} = -\nabla\phi + i\omega\mathbf{A} \tag{42}$$

ϕ(r) is called the scalar potential.

In terms of the 4- vector potential, the Maxwell equations are readily found :

$$\nabla^2\mathbf{A} - \nabla(\nabla.\mathbf{A}) + \frac{\omega^2}{c^2}\mathbf{A} + i\frac{\omega}{c^2}\nabla\phi = 0 \tag{43.a}$$

$$\nabla^2\phi - i\omega\,\nabla.\mathbf{A} = 0 \tag{43.b}$$

This is quite a messy pair of equations. However, as we shall see later, we can achieve considerable simplification by taking advantage of the flexibility allowed by gauge transformation.

We assume that the components of the 4- vector potential ($\mathbf{A}$,ϕ) are developed in Fourier harmonics in the Oz direction :

$$\mathbf{A}_j(x, y, z) = \mathbf{A}_j(x, y)\ \exp(i(k_z z - \omega t)) \qquad (j = x, y, z) \qquad (44.a)$$

$$\phi(x, y, z) = \phi(x, y) \exp(i(k_z z - \omega t)) \qquad (44.b)$$

We solve eq. (36) with the boundary conditions

$$\mathbf{A}_j(x_{min}, y) = \mathbf{A}_j(x_{max}, y) = 0 \qquad (j = y, z) \qquad (45.a)$$

$$\phi(x_{min}, y) = \phi(x_{max}, y) = 0 \qquad (45.b)$$

and with the 4- vector potential supposed periodic in the Oy direction :

$$\mathbf{A}_j(x, y_{min}) = \mathbf{A}_j(x, y_{max}) \qquad (j = x, y, z) \qquad (46.a)$$

$$\phi(x, y_{min}) = \phi(x, y_{max}) \qquad (46.b)$$

The system (43) has the same dispersion relation as eq. (6) . Let us note that the divergence of eq.(43.a) leads to the Poisson equation, given by (43.b).

2.1.1 Lorentz gauge

Using the Lorentz condition :

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \qquad (47)$$

and replacing ϕ by $\phi_0 = \frac{\phi}{\omega}$, eq. (43) can be written :

$$\nabla^2 \mathbf{A} = -k_0^2 \mathbf{A} \tag{48.a}$$

$$\nabla^2 \phi_0 - i\, \nabla . \mathbf{A} = 0 \tag{48.b}$$

This system can be put in the form:

$$C\mathbf{x} = k_0^2\, D\, \mathbf{x} \tag{49}$$

where k_0^2 and $\mathbf{x}$ are respectively the eigenvalues and the eigenfunctions to be determined;

where C and D are matricial operators such that:

$$C = \begin{pmatrix} \nabla^2 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \\ 0 & 0 & \nabla^2 & 0 \\ -i\frac{\partial}{\partial x} & -i\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} & \nabla^2 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{50}$$

and $\mathbf{x}^T = (\mathbf{A}, \phi_0)$

2.1.2 Coulomb gauge

We propose to solve eq.(43.a) subject to the conditions (45) and (46) and with the Coulomb condition respected , div**A** equals zero everywhere. This system is thus written as:

$$\nabla^2\mathbf{A} = -k_0^2\mathbf{A} - ik_0^2\nabla\phi_0$$
$$\nabla.\mathbf{A} = 0$$
$$\phi_0 = 0 \qquad \text{on the boundaries}$$
$$\mathbf{A}_t = 0 \qquad \text{on the boundaries} \qquad (51)$$

with consequences:

$$\nabla^2(\nabla.\mathbf{A}) = 0$$
$$\nabla^2\phi_0 - i\,\nabla.\mathbf{A} = 0$$

It is equivalent to the following system :

$$\nabla^2\mathbf{A} = -k_0^2\mathbf{A} - ik_0^2\nabla\phi_0$$
$$\nabla^2\phi_0 - i\,\nabla.\mathbf{A} = 0$$
$$\nabla.\mathbf{A} = 0 \qquad \text{on the boundaries}$$
$$\phi_0 = 0 \qquad \text{on the boundaries}$$
$$\mathbf{A}_t = 0 \qquad \text{on the boundaries} \qquad (52)$$

with consequences:

$$\nabla.\mathbf{A} = 0$$
$$\nabla^2(\nabla.\mathbf{A}) = 0$$

In this last case, the matricial operators C , D and the eigenfunctions **x** are then :

$$C=\begin{pmatrix} -\nabla^2 & 0 & 0 & 0 \\ 0 & -\nabla^2 & 0 & 0 \\ 0 & 0 & -\nabla^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (53)$$

$$D=\begin{pmatrix} 1 & 0 & 0 & i\nabla \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{\partial}{\partial x} & -i\frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} & \nabla^2 \end{pmatrix}$$

$$\mathbf{x}^T = (\mathbf{A}, \phi_0)$$

2.2 Numerical considerations

The weak variational form of the eqs. (43) , expressed in the Coulomb gauge is given by :

$$\int_\Omega (-\text{ rot } \mathbf{G}^*.\text{rot } \mathbf{A} - \text{div } \mathbf{G}^*.\text{div } \mathbf{A} + k_0^2 (\text{grad } \phi_0^* + \mathbf{A}^*) (\text{grad } \phi_0 + \mathbf{A})) \, d\tau$$

$$= \int_{\delta\Omega} (\mathbf{G}^* \wedge \text{rot } \mathbf{A} + \mathbf{G}^*.\text{div}\mathbf{A} + k_0^2 \phi_0^* \text{grad } \phi_0) \, . \, \mathbf{n} \, dS \qquad (54)$$

where we can choose as test functions $\mathbf{G}(x,y,z)$:

$$\mathbf{G}(x,y,z)= (V_x (x,y), V_y (x,y), V_z(x,y), V_\phi (x,y)) \ \exp(i(k_z z - \omega t)) \qquad (55)$$

However, in the Lorentz gauge, the term $\mathbf{A}^*.\text{grad } \phi_0$ doesn't appear on the left-hand side while $i \phi_0^*.\mathbf{A}$ has to be added on the right- hand side.

2.3 Eigenvalues and eigenfunctions in the Lorentz and Coulomb gauges

We propose to determine the eigenvalues and analyze their degeneracy and the associated eigenfunctions.

2.3.1 Lorentz gauge

In the Lorentz gauge, the system (48) is not coupled when one inserts the Lorentz condition.

$$\nabla^2 \mathbf{A} = -k_0^2 \mathbf{A} \qquad (56.a)$$

$$\nabla^2 \phi_0 = -k_0^2 \phi_0 \qquad (56.b)$$

In this case one solves eq. (56.a) by taking into account the boundary conditions, eqs. (45) and (46). By using the Lorentz condition,one determines the electric potential ϕ .

In the general case where ($m \neq 0, n \neq 0$) this system admits six linearly independent solutions :

a)
$$A_x = A_{x01} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_y = A_z = 0 \qquad (57.a)$$
$$\phi_0 = i\frac{m\pi}{a}\frac{1}{k_0^2} A_{x01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$

b)
$$A_x = A_{x02} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_y = A_z = 0 \qquad (57.b)$$
$$\phi_0 = i\frac{m\pi}{a}\frac{1}{k_0^2} A_{x02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$

c) $$A_y = A_{y01} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$

$$A_x = A_z = 0 \qquad (57.c)$$

$$\phi_0 = -i\frac{n\pi}{b}\frac{1}{k_0^2} A_{y01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$

d) $$A_y = A_{y02} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t) \qquad (57.d)$$

$$A_x = A_z = 0$$

$$\phi_0 = i\frac{n\pi}{b}\frac{1}{k_0^2} A_{y02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$

e) $$A_z = A_{z01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t) \qquad (57.e)$$

$$A_x = A_y = \phi = 0$$

f) $$A_z = A_{z02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t) \qquad (57.f)$$

$$A_x = A_y = \phi = 0$$

These solutions reduce to 3 and 2 in the cases where ($m\neq0$,$n=0$) and ($m=0$,$n\neq0$). They are indicated in the third and eighth columns of Table 1.

One can show that the theoretical solutions TE_{mn} in **A** and ϕ_0 (eqs. 57a-57d) can be combinated to give rise to zero electric and magnetic fields , as follows.

a) For this, you multiply eq. 57a by $\frac{1}{A_{x01}}\frac{m\pi}{a}$ and eq.57c by $\frac{1}{A_{y01}}\frac{n\pi}{b}$ so that:

$$A_x = \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$

$$A_y = \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t) \qquad (58.a)$$

$$A_z = 0$$

$$\phi_0 = i \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$

b) Multiply eq. 57b by $\frac{1}{A_{x02}}\frac{m\pi}{a}$ and eq.57d by $\frac{1}{A_{y02}}\frac{n\pi}{b}$ so that :

$$A_x = \frac{m\pi}{a} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_y = \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t) \qquad (58.b)$$
$$A_z = 0$$
$$\phi_0 = -i \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$

From eqs (37) and (42), it is clear that these two 4-vector potential lead to zero electric and magnetic fields.

Let us note that these two modes, namely "ghosts modes", have no physical meaning . They pose no problem ,however, because all the physical fields, computed from them, are zero. Unlike the polluting modes, they satisfy the analytic dispersion relation everywhere.
Let us note that the Maxwell equations (eq.1) reduce to

$$\nabla^2 \mathbf{E} = -k_0^2 \mathbf{E} \qquad (59)$$

when inserting the Poisson equation, div$\mathbf{E}$ equals zero everywhere. So, this system subject to the boundary conditions (eqs.4 and 5) admits also "ghosts modes" as solutions. Unfortunately, it is not possible to remove them.

With the same choice of parameters as previously we determine the eigenvalue spectra when the Maxwell equations are discretized with hybrid, bilinear and bicubic elements. The figures [4 - 6] show clearly that these eigenvalues converge to the theoretical ones and that the spurious modes are completely eliminated. One can show by computing the discrete dispersion relation that the polluting branch (b) of the Fig. (3) does not appear. Let us also

mention that the degeneracy of the computational eigenvalues correspond to the theoretical predictions.

2.4.2 Coulomb gauge

In the Coulomb gauge, eq.(52) admits four independent solutions:

a) $$A_x = A_{x01} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_y = A_{y01} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_z = \phi = 0$$

b) $$A_x = A_{x02} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_y = A_{y02} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_z = \phi = 0$$

c) $$A_z = A_{z01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t) \qquad (60)$$
$$A_x = A_y = \phi = 0$$

d) $$A_z = A_{z02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$$
$$A_x = A_y = \phi = 0$$

This is seen by solving eq.(52.a) for the components (A_x, A_y). The third A_z is given by (52.b). Let us note that the scalar potential ϕ equals always zero when there is no source in the waveguide.

These four solutions reduce to 2 in the cases where $(m\neq 0, n=0)$ and $(m=0, n\neq 0)$.

Therefore the degeneracy of the eigenvalues is the same when the Maxwell equations are solved by using the field **E** or the 4- vector potential $(\mathbf{A}, \phi)$ as

variables. All these results are reported in the third and seventh columns of Table 2.

With the same choice of parameters as previously, the spectra are not polluted and the eigenvalues are approximately identical to the ones obtained in the Lorentz gauge (more than six digits for a mesh 8*8). Furthermore, agreement is noted between the computational and theoretical degeneracy of the eigenvalues.

3. Wave propagation in a cylindrical waveguide

3.1 Theoretical considerations

We propose to study wave propagation in vacuum and in a cylindrical waveguide with a circular cross section of radius a .
We assume that the components of the field **E** vary in one-dimensional geometry as:

$$E_j(r, \theta, z) = E_j(r) \exp(i(m\theta + k_z z - \omega t)) \qquad (j = r, \theta, z) \quad (61)$$

The field inside the waveguide must be periodic in θ with a period 2π. It is therefore necessary to choose m equal to an integer .

3.1.1 Transverse Magnetic modes

For the TM (transverse magnetic) modes a solution of

$$\nabla_t^2 E_z + k_c^2 E_z = 0 \qquad (62)$$

$$\text{where } k_c^2 = \frac{\omega^2}{c^2} - k_z^2 \qquad (63)$$

is required such that :

$$\mathbf{E}_z\,(r=a)=0 \tag{64}$$

When we express the transverse Laplacian ∇_t^2 in cylindrical geometry the equation above becomes :

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + k_c^2\, E_z = 0 \tag{65}$$

Equation (64) is Bessel's differential equation and has two solutions $J_m(k_c r)$ and $Y_m\,(k_c r)$, called Bessel functions of the first and second kind,respectively and of order m.

For the problem under investigation here, only $J_m(k_c r)$ is a physically acceptable solution since $Y_m(k_c r)$ becomes infinite at r=0.The final solution for E_z may thus be expressed as :

$$E_z(r,\theta,z) = A\, J_m(k_c r)\, \exp\,(i(m\theta + k_z z - \omega t)) \tag{66}$$

where A is an arbitrary constant.

Since E_z must vanish when r=a, it is necessary to choose $k_c a$ in such a manner that $J_m(k_c a)=0$.If the nth root of the equation $J_m(x)$ is designated p_{mn}, the allowed values of k_c are :

$$k_{c,m,n} = \frac{p_{mn}}{a} \tag{67}$$

and the dispersion relation can be written as :

$$\omega_{mn}^2 = c^2 k_z^2 + c^2 \frac{p_{mn}^2}{a^2} \tag{68}$$

Each choice of m and n specifies a particular TM mode. Expressions for the remaining field components ($E_r(r)$, $E_\theta(r)$) can be derived by using the general equations (37) and (42).

3.1.2 Transverse Electric modes

The solution for TE (transverse electric) modes parallels that for the TM modes with the exception that the boundary conditions require that:

$$\frac{\partial B_z}{\partial r}(r=a) = 0 \tag{69}$$

An appropriate solution for B_z is :

$$B_z(r,\theta,z) = B\, J_m'(k_c r)\ \exp(i(m\theta + k_z z - \omega t)) \tag{70}$$

where B is an arbitrary constant.

The roots are designated by p_{mn}' so the eigenvalues $k_{c,m,n}$ are given by :

$$k_{c,m,n} = \frac{p_{mn}'}{a} \tag{71}$$

The dispersion relation is written as :

$$\omega_{mn}^2 = c^2 k_z^2 + c^2 \frac{p_{mn}'^2}{a^2} \tag{72}$$

Expressions for the remaining field components ($E_r(r)$, $E_\theta(r)$) can be derived by using the general equations (37) and (42).

3.2 Numerical considerations

The weak variational form is the same as the one given by eq. (29) and the test functions $\mathbf{G}(r,\theta,z)$:

$$\mathbf{G}(r,\theta,z) = (G_r(r), G_\theta(r), G_z(r)) \exp(i(m\theta + k_z z - \omega t)) \tag{73}$$

are arbitrary functions in some functional space of sufficient regularity. Cubic Hermite elements are used to discretize the differential equations.

With the following values of the different parameters

$$k_z = 0,\ m = 0\ ,\ a = 1.\ cm \tag{74}$$

the spectrum is polluted as can be seen in Fig.(7).

4. Gauge transformation

4.1 Theoretical considerations

As previously , we propose to use the 4- vector potential as variables. The Maxwell equations are then given by eq. (43) ,which reduce to eq. (48) in the Lorentz gauge and eq. (52) in the Coulomb gauge. As boundary conditions one imposes : firstly, that the tangential components of the 4- vector potential vanish on the perfect conducting wall and secondly, that the normal components to the wave propagation direction vanish at the center of the circular waveguide.

4.2 Numerical considerations

The weak variational form is given by eq. (54) where the test functions $\mathbf{G}(r,\theta,z)$ are chosen such as :

$$\mathbf{G}(r,\theta,z) = (G_r(r),\, G_\theta(r),\, G_z(r),\, \phi(r)) \exp\left(i(m\theta + k_z z - \omega t)\right) \qquad (75)$$

As in the 2D case,the spectrum is not polluted (see Fig.8) and as previously, the eigenvalues are approximately the same in the two gauges (more than six digits for N_r=50; N_r being the number of points in the direction 0r).

5. CONCLUSION

When one discretizes the Maxwell equations with standard finite elements (such as bilinear or bicubic ones) , the eigenvalue spectrum appears clearly to be ill-represented : there appear spurious modes which do not satisfy the dispersion relation and do not converge. To remove these numerical instabilities, one can choose to discretize the equations with non-polluting elements or to modify them so that the discretization scheme is pollution-free. This last approach has led us to express the Maxwell equations in terms of the 4- vector potential and to solve them in the Coulomb or Lorentz gauge. When using this latter gauge, there appear analytical modes, namely ghost modes, which have no physical meaning. However, they pose no problem because all the physical fields computed from them are zero. Let us note that without considering a gauge transformation, the Maxwell equations also admit ghost modes for solutions but that it is not possible to remove them.

6. REFERENCES

[1] Villard L., Appert K., Gruber R. and Vaclavik J., Comput. Phys. Rep. **4** (1986) 95

[2] Jaeger E.F., Batchelor D.B., Weitzner H. and Whealton J.H., Comput. Phys. Commun. **40** (1986) 33

[3] Phillips M.W. and Todd A.M.M., Comput. Phys. Comm. **40** (1986) 65

[4] Edlery D. and Picq H., Comput. Phys. Comm. **40** (1986) 95

[5] Brambilla M. and Krücken T., Nucl. Fusion **28** (1988) 1813

[6] Fukuyama A., Itoh K. and Itoh S.I., Comput. Phys. Rep. **137** (1986).

[7] Appert K.,Berger D.,Gruber R.,Troyon F. and Rappaz J.,"Studium der Eigenschwingungen eines zylindrischen Plasmas mit der Methode der Finiten Elemente", J.appl. Math. Phys. (ZAMP) **25** (1974).

[8] Gruber R. and Rappaz J., "Finite Element Methods in Linear Ideal Magnetohydrodynamics" (Springer-Verlag,Berlin,1985).

[9] Lamalle P.U., "Finite Element Discretization of the ICRF wave equation", Proceedings of the joint Varenna-Lausanne international workshop (Chexbres, Switzerland, 1988), 699-705 (Editrice Compositori, Bologna, 1989).

[10] Evequoz H., "Approximation spectrale liée à l'étude de la stabilité MHD d'un plasma par une méthode d'éléments finis non conformes", Thèse No. **375**, EPF-Lausanne 1980.

[11] Jaccard J., "Approximation spectrale par la méthode des éléments finis conformes d'une classe d'opérations non compacts et partiellement réguliers", Thèse No. **374**, EPF-Lausanne 1980.

[12] Batchelor D.B., Jaeger E.F. and Weitzner H., International School of Plasma Physics, Chexbres 1988.

[13] Descloux J., Nassif N. and Rappaz J., On spectral approximation. Part 2, Error estimates for the Galerkin method, RAIRO Analyse Numerique **12** (1978) 113

[14] Strang G. and Fix G.J., "An analysis to the Finite Element Method", Prentice-Hall 1973

[15] Appert K., Balet B., Gruber R., Troyon F., Vaclavik J., Comput. Phys. Comm. 24 (1981) 329.

TABLE I

m	n	Eigenfunctions	$\nabla.A$	Electric field	Magnetic field	Mode	$m_{k_0}^2$ (A,ϕ)	$m_{k_0}^2$ (E)
0	0	$A_x = A_{x02} \exp(-i\omega t)$ $A_y = A_z = \phi_0 = 0$	$= 0$	$E_x \neq 0$ $E_y = E_z = 0$	$B_x = B_y =$ $B_z = 0$	electrostatic	1	1
0	$\neq 0$	$A_x = A_{x01} \sin \frac{n\pi y}{b} \exp(-i\omega t)$ $A_y = A_z = \phi_0 = 0$ --- $A_x = A_{x02} \cos \frac{n\pi y}{b} \exp(-i\omega t)$ $A_y = A_z = \phi_0 = 0$	$= 0$	$E_x \neq 0,$ $E_y = E_z = 0$	$B_x = B_y = 0$ $B_z \neq 0$	TE	2	2
$\neq 0$	$= 0$	$A_x = A_{x02} \cos \frac{m\pi x}{a} \exp(-i\omega t)$ $A_y = A_z = 0$ $\phi = i \frac{m\pi}{a} \frac{1}{k_0^2} A_{x02} \sin \frac{m\pi x}{a} \exp(-i\omega t)$	$\neq 0$	$E_x = E_y =$ $E_z = 0$	$B_x = B_y$ $= B_z = 0$		1	0
$\neq 0$	$= 0$	$A_y = A_{y01} \sin \frac{m\pi x}{a} \exp(-i\omega t)$ $A_x = A_z = \phi_0 = 0$	$= 0$	$E_x = 0$ $E_y \neq 0$ $E_z = 0$	$B_x = 0$ $B_y = 0$ $B_z \neq 0$	TE	1	1

		$A_z=A_{z02}\sin\frac{m\pi x}{a}\exp(-i\omega t)$ $A_x=A_y=\phi_0=0$	=0	$E_x=0$ $E_y=0$ $E_z\neq0$	$B_x=0$ $B_y=0$ $B_z\neq0$	TM	1	1
		$A_x=A_{x01}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $\phi_0=\frac{-im\pi}{a}\frac{1}{k^2_0}A_{x01}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_y=A_z=0$	≠0	$E_x\neq0$ $E_y\neq0$ $E_z=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	see paper		
		$A_x=A_{x02}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $\phi_0=\frac{-im\pi}{a}\frac{1}{k^2_0}A_{x02}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $A_x=A_z=0$	≠0	$E_x\neq0$ $E_y\neq0$ $E_z=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	see paper		
		$A_y=A_{y01}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $\phi_0=\frac{-in\pi}{b}\frac{1}{k^2_0}A_{y01}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_x=A_z=0$	≠0	$E_x\neq0$ $E_y\neq0$ $E_z=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	see paper		

≠0	≠0	$A_y = A_{y02} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$ $\phi = i\frac{n\pi}{b}\frac{1}{k^2_0} A_{y02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$ $A_x = A_z = 0$	≠0	$E_x \neq 0$ $E_y \neq 0$ $E_z = 0$	$B_x = 0$ $B_y = 0$ $B_z \neq 0$	see paper		
		$A_z = A_{z01} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \exp(-i\omega t)$ $A_x = A_y = \phi_0 = 0$	=0	$E_x = 0$ $E_y = 0$ $E_z \neq 0$	$B_x \neq 0$ $B_y \neq 0$ $B_z = 0$	TM	1	1
		$A_z = A_{z02} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \exp(-i\omega t)$ $A_x = A_y = \phi_0 = 0$	=0	$E_x = 0$ $E_y = 0$ $E_z \neq 0$	$B_x \neq 0$ $B_y \neq 0$ $B_z = 0$	TM	1	1

where $m_{k_0}{}^2(A,\phi)$ and $m_{k_0}{}^2(E)$ are respectively the multiplicity of the eigenvalue k^2_0 when the Maxwell equations are expressed in terms of the electromagnetic potentials or of the electric field

TABLE II

m	n	Eigenfunctions	Electric field	Magnetic field	Mode	$m_k 2_0$ (A,ϕ)	$m_k 2_0$ (E)
0	0	$A_x=A_{x01}\exp(-i\omega t)$ $A_y=A_z=\phi=0$	$E_x\neq0$ $E_y=E_z=0$	$B_x=B_y=$ $B_z=0$	electrostatic	1	1
0	$\neq0$	$A_x=A_{x01}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_y=A_z=\phi=0$	$E_x\neq0$ $E_y=E_z=0$	$B_x=B_y=0$ $B_z\neq0$	TE	1	1
		$A_x=A_{x02}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $A_y=A_z=\phi=0$	$E_x\neq0$ $E_y=E_z=0$	$B_x=B_y=0$ $B_z\neq0$	TE	1	1
$\neq0$	=0	$A_y=A_{y01}\sin\frac{m\pi x}{a}\exp(-i\omega t)$ $A_x=A_z=\phi=0$	$E_y\neq0$ $E_z=0$ $E=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	TE	1	1
		$A_z=A_{z02}\sin\frac{m\pi x}{a}\exp(-i\omega t)$ $A_x=A_y=\phi=0$	$E_x=0$ $E_y=0$ $E_z\neq0$	$B_x=0$ $B_y\neq0$ $B_z=0$	TM	1	1

≠0	≠0	$A_x=A_{x01}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_y=A_{y01}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $A_z=\phi=0$	$E_x\neq0$ $E_y\neq0$ $E_z=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	TE	1	1
		$A_x=A_{x02}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $A_y=A_{y02}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_z=\phi=0$	$E_x\neq0$ $E_y\neq0$ $E_z=0$	$B_x=0$ $B_y=0$ $B_z\neq0$	TE	1	1
		$A_z=A_{z01}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp(-i\omega t)$ $A_x=A_y=\phi=0$	$E_x=0$ $E_y=0$ $E_z\neq0$	$B_x\neq0$ $B_y\neq0$ $B_z=0$	TM	1	1
		$A_z=A_{z02}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp(-i\omega t)$ $A_x=A_y=\phi=0$	$E_x=0$ $E_y=0$ $E_z\neq0$	$B_x\neq0$ $B_y\neq0$ $B_z=0$	TM	1	1

where $m_{k^2_0}(A,\phi)$ and $m_{k^2_0}(E)$ are the multiplicity of the eigenvalue k^2_0 when the Maxwell equations are expressed in terms of the electromagnetic potentials or of the electric field .

Figure captions.

Figure 1. : The Maxwell equations are expressed in terms of the electric field in rectangular geometry and discretized with bilinear elements. The eigenvalue spectrum is polluted.
Convergence study of the eigenvalues ω^2 ($\times 10^{17}$) versus N^2 ($=N_x \times N_y$; respectively equals to 6×6, 8×8, 10×10)

Figure 2. : The Maxwell equations are expressed in terms of the electric field in rectangular geometry and discretized with bicubic elements. The eigenvalue spectrum is polluted.
Convergence study of the eigenvalues ω^2 ($\times 10^{17}$) versus N^2 ($=N_x \times N_y$; respectively equals to 4×4, 5×5, 6×6)

Figure 3. : Discrete dispersion relation. The horizontal axis is $\text{argsh}(n_x^2)$ and the vertical one is $\text{argsh}(n_y^2)$. Branch (a) is strongly distorted at high n_x and n_y . Branch (b) is polluted.

Figure 4. : The Maxwell equations are expressed in terms of the 4-vector potential in rectangular geometry and discretized with hybrid elements. The eigenvalue spectrum is pollution-free.
Convergence study of the eigenvalues ω^2 ($\times 10^{17}$) versus N^2 ($=N_x \times N_y$; respectively equals to 6×6, 8×8, 10×10)

Figure 5. : The Maxwell equations are expressed in terms of the 4-vector potential in rectangular geometry and discretized with bilinear elements. The eigenvalue spectrum is pollution-free.
Convergence study of the eigenvalues ω^2 ($\times 10^{17}$) versus N^2 ($=N_x \times N_y$; respectively equals to 6×6, 8×8, 10×10)

Figure 6. : The Maxwell equations are expressed in terms of the 4-vector potential in rectangular geometry and discretized with bicubic elements. The eigenvalue spectrum is pollution-free.
Convergence study of the eigenvalues ω^2 ($\times 10^{17}$) versus N^2 ($=N_x \times N_y$ = 5×5)

Figure 7. : The Maxwell equations are expressed in terms of the electric field in one-dimensional cylindrical geometry and discretized with cubic elements. The eigenvalue spectrum is polluted.
Convergence study of the eigenvalues ω^2 ($\times 10^{22}$) versus N (equals to 30)

Figure 8. : The Maxwell equations are expressed in terms of the 4-vector potential in one-dimensional cylindrical geometry and discretized with cubic elements. The eigenvalue spectrum is pollution-free.
Convergence study of the eigenvalues ω^2 ($\times 10^{22}$) versus N (equals to 30)

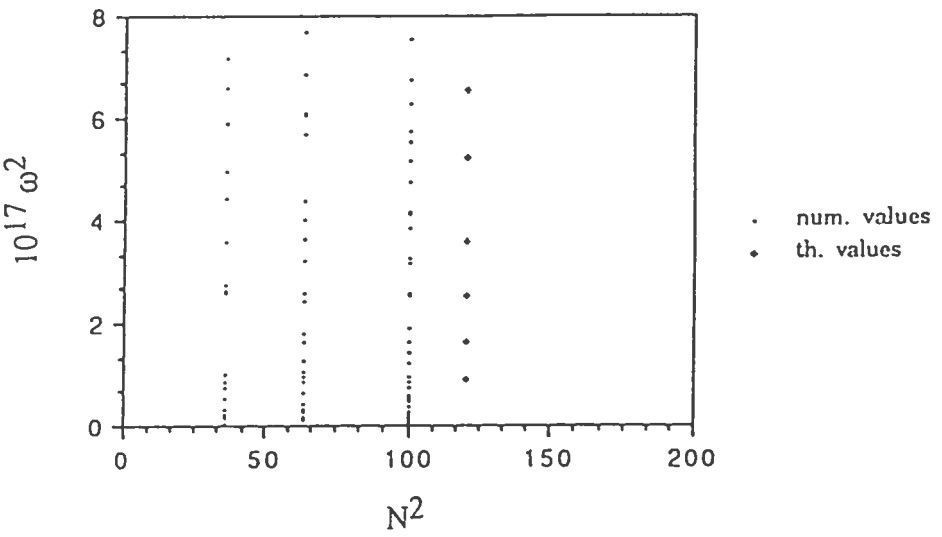

Figure 1.

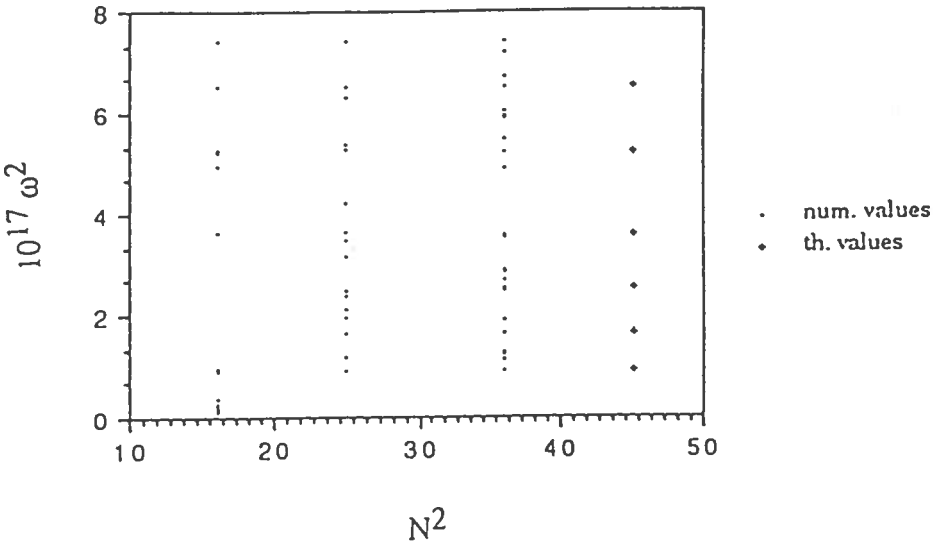

Figure 2.

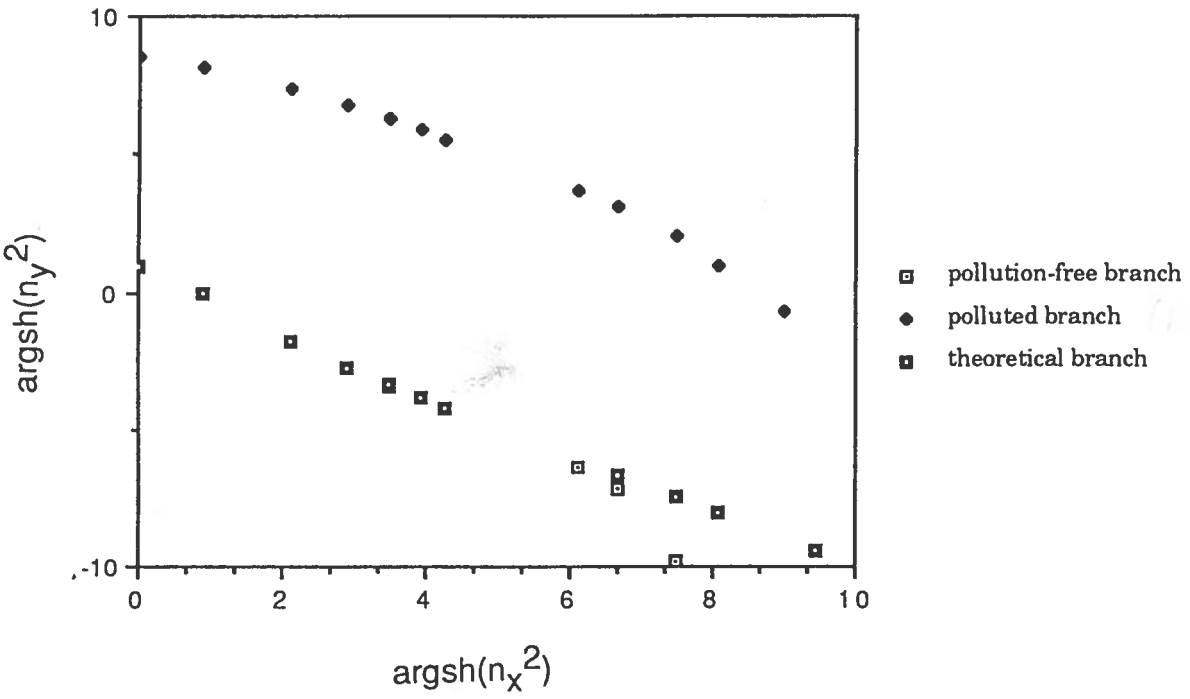
10
0
-10
0
2
4
6
8
10
argsh(n_y^2)
argsh(n_x^2)
pollution-free branch
polluted branch
theoretical branch

Figure 3.

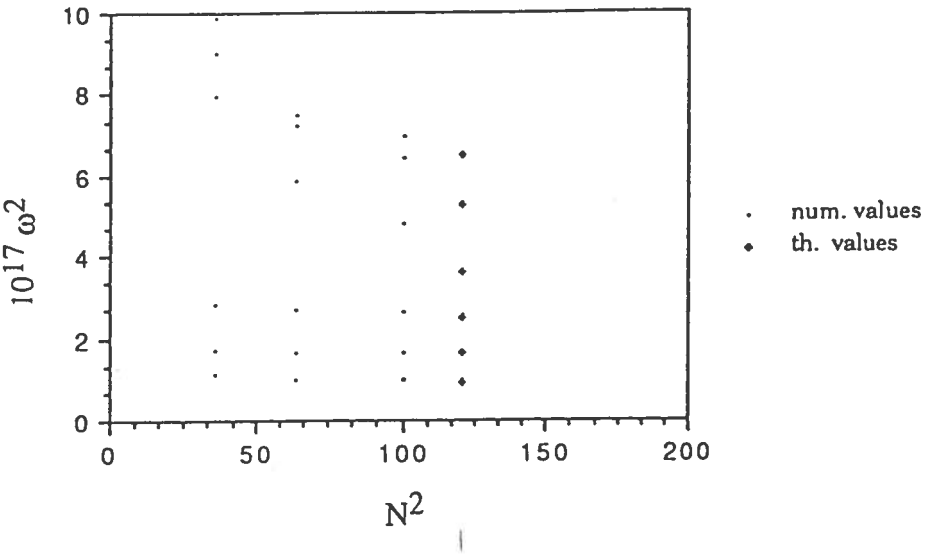

Figure 4.

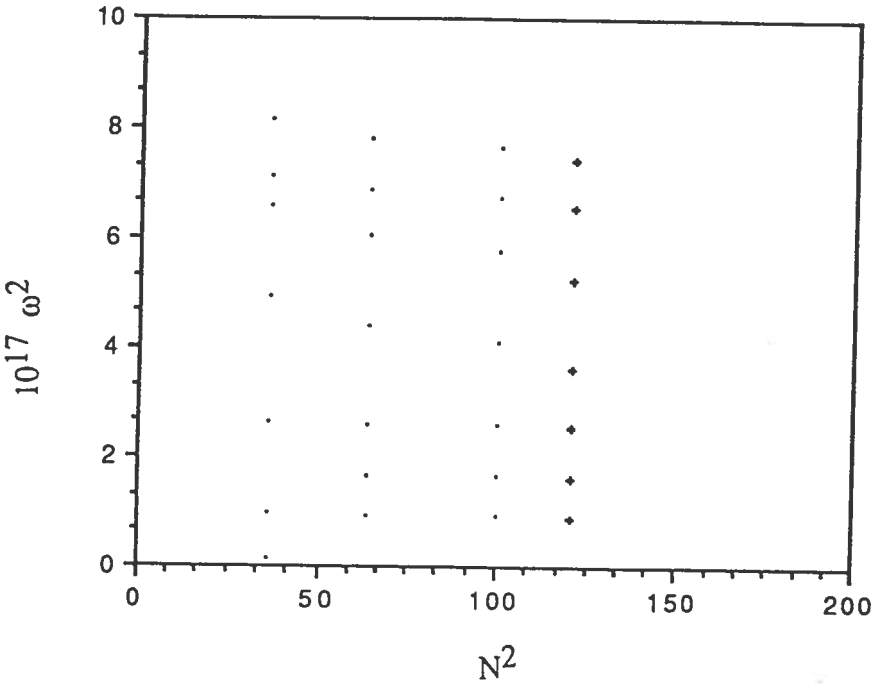

Figure 5.

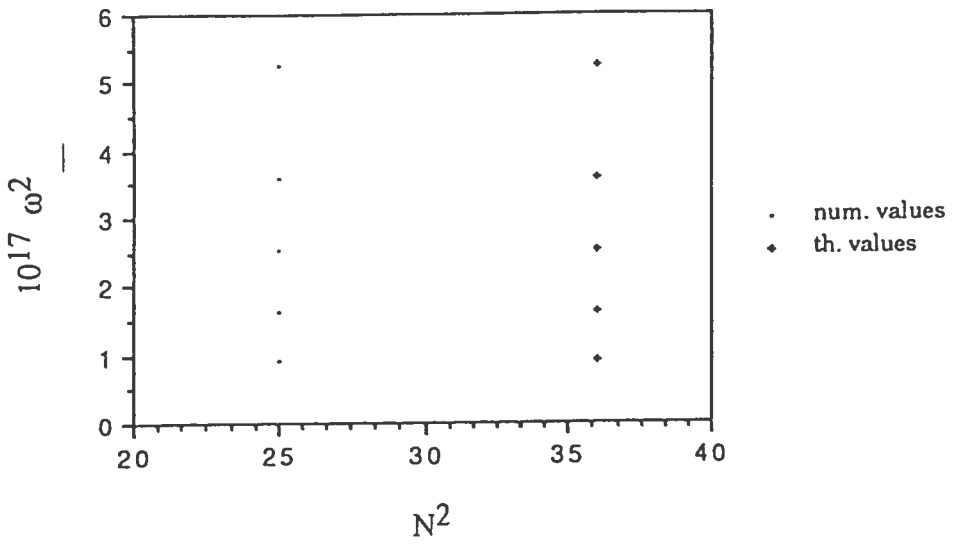

Figure 6.

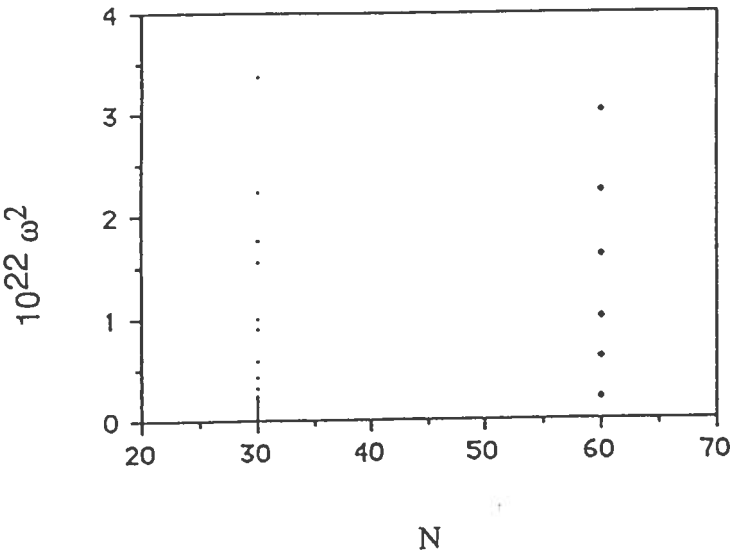

Figure 7.

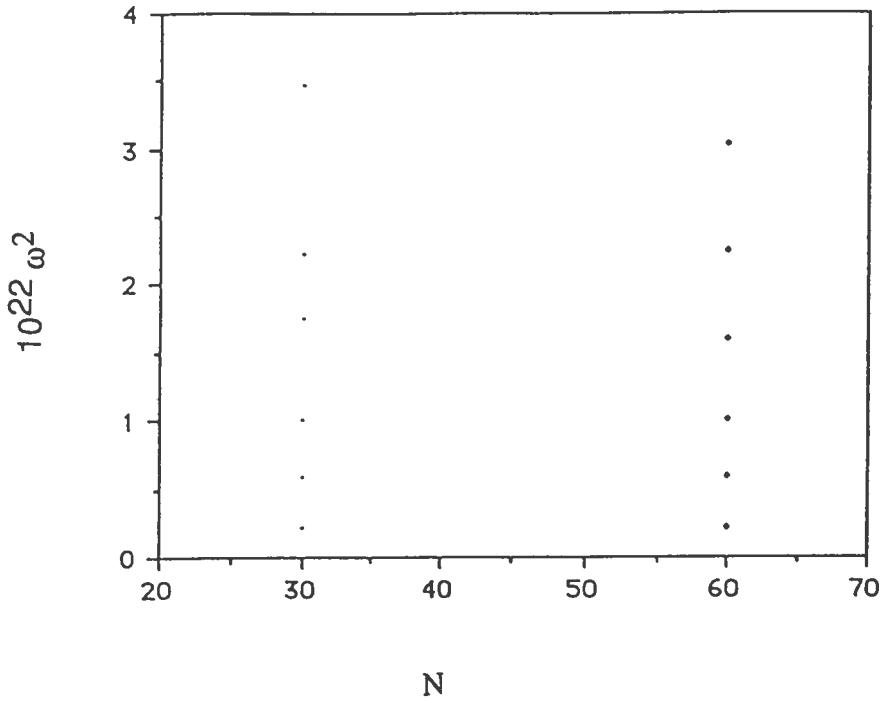

Figure 8.