Abstract—Continuous-domain visual signals are usually captured as discrete (digital) images. This operation is not invertible in general, in the sense that the continuous-domain signal cannot be exactly reconstructed based on the discrete image, unless it satisfies certain constraints (e.g., bandlimitedness). In this paper, we study the problem of recovering shape images with smooth boundaries from a set of samples. Thus, the reconstructed image is constrained to regenerate the same samples (consistency), as well as forming a shape (bilevel) image. We initially formulate the reconstruction technique by minimizing the shape perimeter over the set of consistent binary shapes. Next, we relax the non-convex shape constraint to transform the problem into minimizing the total variation over consistent non-negative-valued images. We also introduce a requirement (called reducibility) that guarantees equivalence between the two problems. We illustrate that the reducibility property effectively sets a requirement on the minimum sampling density. One can draw analogy between the reducibility property and the so-called restricted isometry property (RIP) in compressed sensing which establishes the equivalence of the $\ell_0$ minimization with the relaxed $\ell_1$ minimization. We also evaluate the performance of the relaxed alternative in various numerical experiments.

Index Terms—Binary images, Cheeger sets, measurement-consistency, shapes, total variation.

I. INTRODUCTION

Sampling is at the heart of all digital signal acquisition devices. To store and process the data, we need to convert continuous-domain signals into a sequence of numbers. Since the continuous-domain signal will no longer be available after this stage, we expect the sequence to provide an exact or at least a fair representation of this signal. The Shannon sampling theory and its variations consider sampling strategies for the class of 1D signals studied in [7], [8], known as signals over a fixed background. A shape is—Continuous-domain visual signals can be modeled by filtering followed by sampling in space.

The shape images have only two different intensity values (0 and 1). However, the filtering effect caused by the PSF smooths out sharp intensity transitions and results in measurements with varied intensity levels. In short, the sampling process projects binary shape images onto gray-scale discrete images (Figure 3). In this paper, we are interested in a strategy to recover the original binary image or a binary approximation thereof from measurements. In any case, the recovered binary image should be able to regenerate the same measurements. This requirement assures us that we cannot discriminate between the original and the recovered images at least at the output of the sampling block. A reconstruction of the original image that satisfies this condition is called measurement-consistent or consistent for short [4], [5], [6]. The problem of consistent shape image reconstruction appears in applications where the aim is to exactly locate or describe the objects in a scene; astronomical imaging, quality monitoring in manufacturing, biomedical imaging and high-quality artwork rendering are a few examples.

A. Related Works

Due to the diverse shape geometries and sharp intensity transitions on the boundaries, this class of visual signals, like many other real-world signals, are neither bandlimited nor belong to a shift invariant subspace. Hence, the classical sampling results do not apply here. A similar scenario happens for the class of 1D signals studied in [7], [8], known as signals with finite rate of innovation (FRI). It is shown that the discrete samples can lead to perfect signal recovery, although the signals are not necessarily bandlimited. A generalization

---

Shapes From Pixels

Mitra Fatemi, Arash Amini, Loic Baboulaz, and Martin Vetterli
to 2D FRI signals is presented in [9], [10], [11], with the goal of recovering convex polygonal shapes from the gray-scale pixels. A different approach is devised in [12] by considering the boundary curves in a shape image as the zero-level-sets of specific 2D FRI signals. Due to the FRI requirements, exact recovery relies on the PSF satisfying the so-called Strang-Fix condition. Furthermore, the FRI model admits limited shape geometries.

The shape image recovery can also be viewed as fitting boundary curves to the interpolated gray-scale image (high-resolution version of the measurements). Such methods are widely known as segmentation techniques that fit deformable curves to gray-scale images, and include active contour algorithms also known as snakes. Based on the curve models, they are classified as point snakes [13], geodesic snakes [14], [15] and parametric snakes [16], [17]. In all cases, the segmentation algorithm is formulated by minimizing a snake energy functional that depends on the gray-scale image and the model of the boundary curves. However, it does not take the PSF into account [18]. As a consequence, the resulting binary image is likely to fail the consistency requirements.

**B. Contributions**

In this work, we propose a method to recover a measurement-consistent shape image that has continuously twice differentiable ($C^2$) boundary curves. Our approach is direct in the sense that it avoids intermediate curve fitting steps, and finds the shape with minimum perimeter. We formulate the method as an optimization problem constrained by the measurements (i.e. pixels), where the functional is the continuous-domain total variation (TV). Ideally, we should restrict the search domain to binary images. This leads to a non-convex problem which is computationally intractable. Hence, we consider the convex relaxation in which the search is over the set of all non-negative-valued images. Under a minimum resolution requirements (see Definition 4 for an explicit explanation), we prove that all the solutions to the non-convex problem are minimizers of the convex relaxation (see Theorem 2).

The number of constraints in the convex problem equals the number of pixels. However, we demonstrate that when the resolution requirement is satisfied, the multiple constraints can be replaced with a single one formed by a wisely chosen linear combination of them. This reduces the problem to an equivalent TV minimization problem with a single constraint, which is known in the literature as the Generalized Cheeger problem [19]. A generalized Cheeger set is a shape with minimum perimeter and a fixed weighted integral. This equivalence allows us to apply the existing results that show the Cheeger solutions are among the minimizers of the relaxed problem [19], [20].

The advantage of our method compared to the FRI works in [9]-[12] is that, we do not constrain the boundary curves by any specific model. Instead, we let the sampling kernel and the measurement values decide for them. As a result, there is less restriction on the achievable shape geometries. Besides, the choice of the PSF is arbitrary, and does not need to satisfy the Strang-Fix condition.

**C. Organization of the paper**

We explicitly define the problem and the used notations in Section II. We continue by reviewing the concept of Cheeger sets and the existing results in Section III. In Section IV, we present the theoretical results. We employ the primal-dual algorithm of [21] for the numerical approximation of the solutions to the convex minimization problem in Section V. This algorithm enables us to study the performance of the proposed shape recovery method in Section IV through numerical experiments. Finally, we conclude the paper in Section VI.

**II. PROBLEM DEFINITION**

We denote by $I(x, y)$ the continuous-domain image with domain $\Omega = [0, 1]^2$. Also, we represent the discrete (or measurement) image by $\mathbf{D}$. When $\mathbf{D}$ is the output of an $m \times m$-pixel digital camera with PSF $\phi(x, y) = \phi(-x, -y)$, we can relate the $m^2$ pixels $d_{ij}$, $1 \leq i, j \leq m$ of $\mathbf{D}$ to the image $I(x, y)$ as

$$d_{ij} = \frac{1}{T^2} \phi\left(\frac{x}{T}, \frac{y}{T}\right) * I(x, y) \big|_{(x, y) = (iT, jT)}$$

$$= \int_{\Omega} \frac{1}{T^2} \phi\left(\frac{x}{T} - j, \frac{y}{T} - i\right) I(x, y) \, dx \, dy,$$  \hspace{1cm} (1)

where $T$ is the sampling period (Figure 1).

In the consistent image recovery problem, we wish to find an approximation $\hat{I}$ of the original image that regenerates the
same measurement pixels. This means that the reconstruction error \( I - \hat{I} \) between the original image and its approximation is in the null space of the imaging process. Equivalently, the two images are perceived as identical by the imaging device.

Let \( k = (j - 1)m + i, \ 1 \leq k \leq m^2 \) represent the equivalent index of \( d_{ij} \) in the vertical raster scan of \( D \). Also, let

\[
f_k(x, y) = \frac{1}{T^2} \phi \left( \frac{x}{T} - \left\lfloor \frac{k}{m} \right\rfloor, \frac{y}{T} - ((k \mod m) + 1) \right)
\]

indicate the sampling kernel in (1) associated with \( d_k \). We denote by \( C_\Omega(D; f_1, ..., f_{m^2}) \) the set of all non-negative-valued images over the domain \( \Omega \) that are consistent with \( D = [d_k]_{1 \leq k \leq m^2} \).

\[
C_\Omega(D; f_1, ..., f_{m^2}) = \left\{ I \in BV(\Omega), I \geq 0 ; \int_\Omega I f_k \, dx \, dy = d_k, \ 1 \leq k \leq m^2 \right\}.
\]

Here, \( BV(\Omega) \) stands for the set of functions over \( \Omega \) with bounded variation; i.e., all elements of \( BV(\Omega) \) have well-defined and finite total variation values.

Consistent image recovery is equivalent to finding an element of \( C_\Omega(D; f_1, ..., f_{m^2}) \). In the consistent shape recovery problem, we limit the permissible solutions to the shape characteristic functions. Let \( S \) be a subset of \( \Omega \). We denote by \( \chi_S(x, y) \) the characteristic function of \( S \) over \( \Omega \)

\[
\chi_S(x, y) = \begin{cases} 
1, & \text{if } (x, y) \in S \\
0, & \text{if } (x, y) \in \Omega \setminus S.
\end{cases}
\]

We call \( S \) a shape if it is the union of a finite number of connected subsets of \( \Omega \). In this case, we call \( \chi_S \) a shape image.

We assume that the sampling kernel in (1) is always positive and has unit \( \ell_1 \) norm

\[
\iint \phi(x, y) \, dx \, dy = 1.
\]

Consequently, the pixels of the discrete image associated with a shape characteristic function shall take continuous values in the range \([0, 1]\). An example of a shape characteristic function and its associated \( 10 \times 10 \) discrete image is shown in Figure 3. We recall that Figure 3b provides a pictorial representation of the 100 pixels in \( D \) and it should not be mistaken with a piecewise constant approximation of the original image.

The consistent shape reconstruction problem is equivalent to finding a shape image \( I = \chi_S(x, y) \in C_\Omega(D; f_1, ..., f_{m^2}) \) for the set of \( m^2 \) pixels \( 0 \leq d_k \leq 1 \) in \( D \). Among all possible candidates, we are interested in shape images with minimum perimeter. This way we reject shapes with extra connected components and excessive boundary details (see Figure 4).

Minimum-perimeter consistent shapes are the global minimizers of the following problem

\[
\inf_{S \subseteq \Omega, \chi_S \in BV} \ Per(S), \quad (P_0)
\]

\[
s.t. \ I = \chi_S \in C_\Omega(D; f_1, ..., f_{m^2}),
\]

where \( Per(S) \) is the perimeter of \( S \). Problem \((P_0)\) is a variational non-convex problem and it is prone to having many local minima. This makes it very likely that common gradient descent methods get trapped in local minima. While in problems of this sort, global minimizers are usually all reasonable solutions, the local minima can be blatantly false. In the next sections, we show that if the discrete image \( D \) satisfies a resolution requirement defined in Definition 4, the minimum-perimeter consistent shapes are the minimizers of a convex relaxation of \((P_0)\). Furthermore, we conjecture that under this condition, there is a unique minimum-perimeter consistent shape which is also the unique solution of the convex problem. In the experimental section, we present an algorithm for the recovery of this solution.

III. CHEEGER SETS

An image is called consistent with the measurements if it complies with all the constraints in \((P_0)\). Essentially, each pixel of the discrete image enters \((P_0)\) as a constraint, resulting in an optimization with many constraints. In addition, we are also restricting the search domain to bilevel images, which further complicates the minimization task. The simplest scenario of having only one single pixel (measurement) is a well-studied topic known as the Cheeger problem. There is already a rich literature regarding the existence, uniqueness properties, regularity (smoothness) of the boundary and numerical evaluation of such sets for almost arbitrary kernels \( f \). In this section, we present a brief review of the Cheeger problem and related results upon which we build our general multi-constraint minimization problem. The details for the latter will be discussed in the next section.

The Cheeger problem can be directly extended to higher dimensions; however, for the purpose of image recovery, we focus on 2D signals in this paper. Let \( \Omega \) be a subset of \( \mathbb{R}^2 \). The Cheeger sets of \( \Omega \) are defined as those \( S \subseteq \Omega \) that minimize
is more aligned with our approach in the next section.

is a convex set. The following statement of this result by [20] implies that

where

\[ BV_\Omega(f) = \{ I \in BV(\Omega), I \geq 0 : \iint_\Omega f I \, dx \, dy = 1 \}. \]

Then, for every \( \mu \geq 0 \) such that the level-set \( E(I; \mu) \) is nonempty,

\[ \frac{1}{\iint_{E(I; \mu)} f \, dx \, dy} \chi_{E(I; \mu)} \]

is also a minimizer of (5).

In a nutshell, Theorem 1 states that the minimizer set of (5) is closed under level-set evaluation; i.e., normalized (scaled) non-empty level-sets of a minimizer also belong to the set of minimizers. This helps in finding a bilevel solution to (4), as finding any minimizer of the convex problem (5) necessarily leads to (at least) a bilevel image.

Another result proved in [20] indicates that the Cheeger sets are closed under set union. This immediately establishes the existence of a unique maximal Cheeger set that contains all the other ones [23]. Thus, we can remove the inherent ambiguity caused by non-uniqueness of the solutions to (5) by searching for the maximal set. However, finding the maximal set is not generally easy by considering the minimization of (5). A regularization technique is proposed in [23] that applies asymptotically vanishing penalty terms to the cost function (5) and achieves the maximal set at the limit of the minimizers. Based on this idea, a numerical method is introduced in [24] that approximates the maximal Cheeger set on a finite grid. The method is robust to discretization as the approximations converge point-wise to the continuous-domain Cheeger set when the grid resolution increases.

As a final note, we discuss the influence of the weight kernels \( f \) and \( g \). In fact, Cheeger sets consist of smooth \( C^2 \) boundaries, irrespective of the choice of \( f \) and \( g \) [19]. Nevertheless, it is known that the curvature of the boundaries is tightly controlled by these weight kernels. Formally, at each boundary point we have that [19]

\[ |\kappa| \leq \frac{\mathcal{J}(S) \sup f + \sup \| \nabla g \|}{\inf g} \]

where \( \kappa \) stands for the curvature and \( \mathcal{J}(S) \) is the cost value of the Cheeger set determined by the ratio in (4). As we set \( g \equiv 1 \) in the rest of the paper, the effective bound on the curvature simplifies to \( |\kappa| \leq \mathcal{J}(S) \sup f \). We will just briefly comment on employing a non-constant weight kernel \( g \) in Section VI.

IV. CONSISTENT SHAPE RECOVERY

Let us consider the problem \((P_0)\) for the case where the measurement image \( D = [d_k]_{1 \leq k \leq m^2} \) consists of more than one pixel. Similar to the single-measurement setting, non-convexity of the problem is a computational barrier. Therefore, we opt to use a convex relaxation in the form of

\[ \inf_{I \in C_0(D; f_1, \ldots, f_m)} \iint_\Omega |\nabla I| \, dx \, dy. \quad (P_1) \]

By extending the search domain from binary (bilevel) shapes to all non-negative-valued images, the problem becomes convex. Nevertheless, due to multiple measurement constraints,
this scenario obviously deviates from the conventional Cheeger problem.

In Theorem 2 we show that under certain conditions, the minimization in (P1) constrained by multiple measurements can be replaced with a similar minimization subject to a single constraint; i.e., we prove that (P1) could potentially have an equivalent Cheeger problem. In fact, we use a wisely chosen linear combination of the measurements as the single measurement. The interpretation of (P1) in form of a Cheeger problem automatically implies the existence of a bilevel minimizer (e.g., the maximal Cheeger set) for (P1). Thus, all shape minimizers of (P1) are also minimizers of the relaxed problem (P1). Further, it proves the existence and uniqueness of a maximal consistent shape. In Theorem 3, we provide a simple test to verify whether a minimizer of (P1) is the maximal shape. This helps us to validate a numerical solution obtained via minimizing (P1)—which might not have a unique minimizer—as a binary consistent shape.

The mathematical requirements for the equivalence of (P1) with a Cheeger problem (existence of a suitable linear combination of the measurements) is stated in Definition 4; essentially these requirements imply that the sampling density used for obtaining the measurement image needs to be fine enough.

A. Theoretical Results

We start by defining the maximal consistent shape.

Definition 1. A maximal consistent shape with minimum perimeter, or MCSMP in short, is a solution to (P0) whose support contains the support of all other minimizers of (P0).

Note that a MCSMP does not always exist. In general, the support union of two minimizers of (P0) does not necessarily generate a minimizer by scaling. It is evident by this fact that the claimed equivalent Cheeger problem plays a crucial role in our results. In Definition 4 below we will describe the sufficient conditions that enable us to associate (P0) or (P1) to a Cheeger problem.

Before stating Definition 4, we introduce a few notations used in the rest of this section. As we need to linearly combine the measurement constraints, we represent the n-dimensional coefficient set for the convex combinations by \( \Delta_n \):

\[
\Delta_n \triangleq \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid 0 \leq \lambda_i, \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

For non-negative-valued images, a zero measurement can only happen when the image vanishes over the support of the corresponding sampling kernel. Thus, we can exclude the support region from our search domain.

Definition 2. For the measurements \( \mathbf{D} = [d_k]_{1 \leq k \leq m^2} \) corresponding to the pixels \( 0 \leq d_k \leq 1 \) and sampling kernels \( f_1, \ldots, f_{m^2} \), let \( \rho \) denote the number of non-zero pixels and

\[
A = \{ i \mid d_i > 0 \} = \{ a_1, \ldots, a_\rho \}
\]

stand for the index set of active pixels. We define the reduced domain \( \Omega_r \) by

\[
\Omega_r = \Omega_r(\mathbf{D}; f_1, \ldots, f_{m^2}) = \Omega \setminus \bigcup_{i \in \{1, \ldots, m^2\} \setminus A} \text{supp}(f_i).
\]

Definition 3. For the measurements \( \mathbf{D} = [d_k]_{1 \leq k \leq m^2} \), sampling kernels \( f_1, \ldots, f_{m^2} \), and a vector \( \lambda \in \Delta_m \), we define the reduced kernel \( f^\lambda : \Omega_r \to \mathbb{R}^\geq 0 \) by

\[
f^\lambda = \left( \sum_{k=1}^{\rho} \lambda_k f_{a_k} \right) / \left( \sum_{k=1}^{\rho} \lambda_k d_{a_k} \right).
\]

Here, \( \rho, \Omega_r \), and \( a_k \) are as defined in Definition 2.

Now, we are prepared to state the Cheeger problem equivalence requirements.

Definition 4. As before, let \( \mathbf{D} = [d_k]_{1 \leq k \leq m^2} \) be the measurements captured by sampling kernels \( f_1, \ldots, f_{m^2} \) with \( 0 \leq d_i \), leading to \( \rho, A, \Omega_r \) as in Definition 2. For an arbitrary \( \lambda \in \Delta_m \), we define \( \lambda \in \Delta \subseteq \mathbb{R}^n \) to be the solution of (5) corresponding to the maximal Cheeger set \( S \) with \( f^\lambda \), when the domain is restricted to \( \Omega_r \). We call \( (\mathbf{D}; f_1, \ldots, f_{m^2}) \) reducible if \( A \) can be partitioned into \( K_1 \) and \( K_2 \) such that

(i) \( \forall k \in K_1, \lambda_k \in \Delta \), \( \lambda_k = 0 \) : \( \int_{\Omega_r} f^\lambda_k \mathbf{d}x \mathbf{d}y < d_k \),

(ii) \( \forall k \in K_2, \lambda \in \Delta \) : \( \int_{\Omega_r} f^\lambda_k \mathbf{d}x \mathbf{d}y \leq d_k \).

It was explained earlier that the measurements \( d_i \) obtained from a binary shape through normalized sampling kernels satisfy \( 0 \leq d_i \leq 1 \). The requirements in Definition 4 simply indicate that the maximal Cheeger solution corresponding to any convex combination of the kernels except a given one, should result in a strictly smaller measurement observed by the excluded kernel. Intuitively, we expect the Cheeger solution to have less contribution over the support of the excluded kernel. However, there are some exceptions; imagine the case where the support of a \( 3 \times 3 \) block of measurement kernels completely coincide with the interior of the binary shape. Thus, we shall have a block of all-one measurements. Now, it is likely that the maximal Cheeger set corresponding to a linear combination of the 8 surrounding kernels (but missing the middle one) using symmetric weights fully covers the support of the kernel in the middle. Hence, measuring this solution via the middle kernel results in \( d_i = 1 \), instead of being strictly less than 1. The partitions \( K_1 \) and \( K_2 \) in Definition 4 are introduced to distinguish between the ordinary (\( K_1 \)) and exceptional (\( K_2 \)) cases. We postpone further discussion and clarifications about this definition to Section IV-B.

Theorem 2. Let \( (\mathbf{D}; f_1, \ldots, f_{m^2}) \) be reducible according to Definition 4. Then, all solutions of the non-convex problem (P0) are included in the minimizers of its convex relaxation (P1). Moreover, the solution set of (P1) contains a unique MCSMP.

Our proof of Theorem 2 relies on the following lemma, the proof of which is provided in the appendix.

Lemma 1. For a given dimension \( n \) and a set \( \{ d_k \}_{k=1}^{n} \subset \mathbb{R} \), let \( K_1, K_2 \) be a partition of \( \{1, \ldots, n\} \), with the possibility of \( K_1 = \emptyset \) or \( K_2 = \emptyset \), and let \( v : \Delta_n \to \mathbb{R}^\geq \) be a continuous function that satisfies

(i) \( \forall \lambda \in \Delta_n : \lambda^T \cdot v(\lambda) = \sum_{k=1}^{n} \lambda_k d_k \),

(ii) \( \forall k \in K_1, \lambda \in \Delta_n : v_k(\lambda) < d_k \),

(iii) \( \forall k \in K_2, \lambda \in \Delta_n : v_k(\lambda) \leq d_k \).

Then, there exists \( \lambda^* \in \Delta_n \) such that \( v(\lambda^*) = [d_1, \ldots, d_n]^T \).
The first condition directly follows from

\[ \mathcal{C}_\Omega(D; f_1, \ldots, f_{m^2}) \subseteq \mathcal{C}_\Omega(1; f^A) = BV_{\Omega_1}(f^A). \]

Therefore, any minimizer of (5) that falls inside \( \mathcal{C}_\Omega(D; f_1, \ldots, f_{m^2}) \) is also a minimizer of (P1). Besides, if (P1) and (9) have a common minimizer, then, all the solutions of (P1) shall be among the solutions of (9). This is indeed, what we aim to prove.

Let \( f^A \) be the maximal Cheeger set solution of (9) on \( \Omega_r \), corresponding to the weight kernel \( f^A \). Consider the function

\[ v(\lambda) = \left( \int_{\Omega_r} f^A_{\lambda a_1}, \ldots, \int_{\Omega_r} f^A_{\lambda a_p} \right)^T. \]

We demonstrate that \( v(\cdot) \) satisfies the conditions of Lemma 1. The first condition directly follows from

\[ 1 = \int_{\Omega_r} f^A = \frac{1}{\gamma_k d_{ak}} \int_{\Omega_r} f^A \sum_{k=1}^{\rho} \lambda_k f_{ak}. \]

The reducibility property of \( (D; f_1, \ldots, f_{m^2}) \) also establishes Conditions (ii) and (iii) of Lemma 1. Consequently, there exists \( \lambda^* \in \Delta_\rho \), such that

\[ v_k(\lambda^*) = \int_{\Omega_r} f^A_{\lambda a_k} \, dx \, dy = d_{ak}, \quad 1 \leq k \leq \rho. \]

This means that the bilevel maximal Cheeger solution \( f^A \), which minimizes (9), is also consistent with the measurements \( D_A \). Hence, \( f^A \) is also a minimizer of (P1) as well as (P0); i.e., the three problems (5) with \( f^A \) over \( \Omega_r \), (P1) and (P0) share a minimizer. This proves the first part of the claim.

As for the second part, note that all minimizing shapes of (P0) are Cheeger solutions of (9). Thus, their shape should be included in the support of the maximal Cheeger solution \( f^A \). In words, \( f^A \) is a MCSMP.

Theorem 2 states that under reducibility, the solution set of (P1) is guaranteed to contain a MCSMP. Although we believe that the MCSMP is the unique solution of (P1) under reducibility, it is yet to be proven. However, we introduce a test in Theorem 3 to verify whether an obtained solution to (P1) is the MCSMP. This test helps us in simulation results, where we implement a minimization technique and eventually obtain a solution with a numerical precision. First, it is difficult to make sure whether the result is precisely bilevel, and second, even if it is bilevel, is it the MCSMP?

**Theorem 3.** Let \( (D; f_1, \ldots, f_{m^2}) \) with \( d_i = 1 \) for some \( i \) be reducible (at least one measurement equal to one). If the point values of a solution to (P1) never exceed 1, then, this solution is the MCSMP and it is binary (non-zero values are all 1).

**Proof.** Let \( I(x, y) \leq 1 \) be a solution to (P1), and let \( i \) be the index of a measurement equal to 1, i.e., \( d_i = 1 \). By comparing \( I(x, y) \leq 1 \) and \( d_i = 1 \), we conclude that for all \( (x, y) \in \text{supp}(f_i) \) we should have \( I(x, y) = 1 \) (the kernels are normalized). If \( I \) is the MCSMP, as it takes the value 1, it needs to be binary and the proof is complete. Therefore, let us assume the MCSMP to be \( \hat{I} \neq I \). As previously shown, the support of \( \hat{I} \) contains the support of \( I \), which obviously contains the support of \( f_i \). As \( \hat{I} \) is constant over its support and is also consistent with measurement \( d_i \), we should have that \( I(x, y) = 1 \) for all \( (x, y) \in \text{supp}(f_i) \). Thus, \( I \) is binary. However, this implies that \( I \) never exceeds \( \hat{I} \) at any point, while they generate the same set of measurements. In turn, this suggests that \( I \) cannot be less than \( \hat{I} \) on a set of non-zero measure. In other words, \( I \) and \( \hat{I} \) are essentially equal at all points.

For recovering a binary shape from discrete measurements, we infer the following: when the sampling density is high enough to provide the reducibility condition for the measurements, the studied convex problem is potentially able to return a consistent binary shape with minimum perimeter. Besides, the boundary of the output shall be a \( C^2 \) curve.

**B. The Sampling Density Requirement**

Earlier, we claimed that the reducibility condition in Definition 4 is effectively a requirement on the minimum sampling density. Here, we illustrate this intuition by some examples.

First, we consider the sampling of the shape in Figure 6a over a \( 3 \times 3 \)-pixel grid, employing the bilinear B-spline sampling kernel depicted in Figure 6b. This generates the
discrete measurements on a
we would like to recover the shape image in Figure 7a from its
f
a single reduced kernel
neighbors. (b) prevents the value of a pixel dropping substantially below its
parts with high curvature). Similarly, the reducibility condition
density for sampling the boundary curve of the shape (possibly,
between neighboring pixel values indicate lack of sufficient
leakage over its region from the neighboring pixels. Thus,
(d) corresponding maximal Cheeger solution. Although
the reduced sampling kernel
maximal Cheeger solution with levels 0 and 0.9891.

Fig. 7: The measurements of shapes with internal holes never
satisfy the reducibility requirement, no matter how high is the
measurement density, unless the original domain is replaced
with the reduced domain. (a) A binary shape image with an
internal hole, (b) the corresponding 10 × 10 measurements,
with \( d_{55} = d_{56} = d_{65} = d_{66} = 0 \), (c) reduced kernel \( f^A \),
with equal contributions from the 20 kernels associated with
pixels on the borders of the central 6 × 6 sub-grid, and (d) the
maximal Cheeger solution with levels 0 and 0.9891.

measurements
\[
D = \begin{bmatrix}
0.5634 & 0.0523 & 0.5750 \\
0.8996 & 0.9016 & 0.8882 \\
0.5247 & 0.8817 & 0.5097
\end{bmatrix}.
\]

Particularly, we focus on the \( d_4 \) measurement pixel (or \( d_{12} \)
in the usual matrix indexing format). It is evident that the value
of this measurement is considerably lower than its neighboring
measurement pixels. Intuitively, this sharp transition violates
the resolution requirement. Now, we check the reducibility
condition: let us exclude the \( d_4 \) pixel and apply equal weights
for a convex combination of the remaining measurements,
i.e., \( \lambda = [1, 1, 1, 0, 1, 1, 1, 1]/8 \in \Delta_0 \). Figure 6c depicts
the reduced sampling kernel \( f^A \), and Figure 6d shows the
corresponding maximal Cheeger solution. Although \( d_4 \) did not
contribute in this Cheeger solution, we observe substantial
leakage over its region from the neighboring pixels. Thus,
\((D, f_1, \ldots, f_9)\) is not reducible. Oftentimes, sharp transitions
between neighboring pixel values indicate lack of sufficient
density for sampling the boundary curve of the shape (possibly,
parts with high curvature). Similarly, the reducibility condition
prevents the value of a pixel dropping substantially below its
neighbors.

One of the shortcomings of reformulating \((P_1)\) as \((5)\) using
a single reduced kernel \( f^A \) is that the Cheeger solution never
admits a hole. Figure 7 provides a pictorial explanation. Here,
we would like to recover the shape image in Figure 7a from its
discrete measurements on a 10 × 10 grid. The hole causes four
vanishing middle pixels (Figure 7b), which make it obvious
that the shape content is 0 in the middle. We now consider
a reduced kernel by linearly combining (with equal weights)
only the 20 kernels associated with the pixels on the perimeter
of the central 6 × 6 sub-grid (Figure 7c). As claimed, the
Cheeger solution to \((5)\) depicted in Figure 7d has no holes and
completely covers the middle part. This seems to violate the
reducibility condition, no matter how high we set the sampling
density. However, note that we remove the 0 pixels from the
domain in Definition 4. Therefore, the Cheeger solution over
the reduced domain is forced to have a hole, although it is not
considered as hole with respect to the reduced domain.

The reducibility condition in Definition 4 is a useful guaran-
tee for recovering a shape image. However, verifying it
for a given set of measurements and sampling kernels is a
combinatorial problem in general. For the purpose of illustra-
tion, we investigate the simple case with 2 × 2 measurement
pixels. Let \( D = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \) with elements in [0, 1] represent
the measurement matrix. Without loss of generality, we assume
that \( d_4 = \rho \leq 1 \) is the largest element. To verify
the reducibility condition, we need to exclude each pixel, apply
an arbitrary convex combination on the rest and check an
inequality. As we can categorize \( d_4 \) to the \( K_2 \) set in Definition
4, the inequalities when \( d_4 \) is excluded are trivial. To verify
other inequalities, note that we can scale all measurements
by the factor \( \frac{1}{\rho} \) (or any other positive real). In fact, the
scaling does not affect the support set of the Cheeger solutions.
Consequently, the reducibility condition for \( D \), boils down to
a set of inequalities on each of \( \frac{d_1}{\rho}, \frac{d_2}{\rho}, \frac{d_3}{\rho} \) in terms of the other
two:
\[
\begin{align*}
\frac{d_1}{\rho} & > \rho Z(\frac{d_2}{\rho}, \frac{d_3}{\rho}), \\
\frac{d_2}{\rho} & > \rho Y(\frac{d_1}{\rho}, \frac{d_3}{\rho}), \\
\frac{d_3}{\rho} & > \rho Y(\frac{d_1}{\rho}, \frac{d_2}{\rho}).
\end{align*}
\]

The symmetries of the problem indicate that the lower-bounds
on \( d_2 \) and \( d_3 \) can be represented using the same function
\((Y(\cdot, \cdot))\), and the lower-bound \( Z(\cdot, \cdot) \) on \( d_1 \) is symmetric
with respect to the two inputs. In Figures 8 and 9 we depict the
functions \( Y, Z \) for two choices of the sampling kernel, namely,
the box-spline (Figure 8) with non-overlapping kernels and
bilinear B-spline kernels with 50% overlap (Figure 9). The
overlap introduces correlation among the neighboring pixels,
which naturally leads to tighter regions for validity of the
reducibility condition. This is indicated by larger \( Y \) and \( Z \) val-
ues. For instance, the measurement set \( D = \begin{bmatrix} 0.576 & 0.72 \\ 0.216 & 0.216 \end{bmatrix} \)
is reducible under the box-spline sampling kernels, but not
under the bilinear B-spline kernels.

V. NUMERICAL EXPERIMENTS

In this section, we aim at numerically calculating the optimal
solution(s) of the convex problem \((P_1)\). For this purpose,
we restrict the simulations to the discrete setting. Below, we
first explain the equivalent problem in the discrete domain and
then, present the simulation results.
results obtained with the discretized model converge to their respective.

It is shown that in the asymptotic regime of $N \to \infty$, the results obtained with the discretized model converge to their continuous-domain counterpart introduced in $(P_1)$ [25].

A. Discrete Formulation

For conducting computer simulations, we are limited to discrete scenarios. Therefore, we discretize the domain $\Omega$ (and subsequently all the functions defined on $\Omega$) with a finite step-size $h \sim \frac{1}{N}$ for some large integer $N$. This will approximate $\Omega$ and the continuous-domain objects $I, f_1, ..., f_m$ by their pseudo samples at the 2D grid

$$\{(ih, jh); \ i, j = 1, 2, ..., N\}$$

resulting in $\mathbb{R}^{N \times N}$ matrices. In the discretized version, we approximate the gradient operator by evaluating the forward differences; for instance we approximate $\nabla I$ with an $\mathbb{R}^{N \times N \times 2}$ tensor defined as

$$(\nabla I)_{i,j,k} = (\nabla I)_{i,j}^k$$

(8)

where

$$\begin{align*}
(\nabla I)_{i,j} &= \begin{cases} 
I_{i+1,j} - I_{i,j} & \text{if } i < N, \\
0 & \text{if } i = N,
\end{cases} \\
(\nabla I)^2_{i,j} &= \begin{cases} 
I_{i,j+1} - I_{i,j} & \text{if } j < N, \\
0 & \text{if } j = N.
\end{cases}
\end{align*}$$

(9)

(10)

It is shown that in the asymptotic regime of $N \to \infty$, the results obtained with the discretized model converge to their continuous-domain counterpart introduced in $(P_1)$ [25].

One of the standard approaches for solving the associated discrete minimization is the gradient descent algorithm, which is rather slow in high dimensions (small step-size $h$). In this paper, we use the recently proposed primal-dual algorithm in [21], [26], which is significantly faster and enjoys convergence guarantees. This scheme is based on the weak formulation of the total variation:

$$\min_{I \in C} \int g |\nabla I| = \min_{I \in C} \max_{|\zeta| \leq g} \left\{ \int -I \text{ div } \zeta \right\}$$

with the dual variable $\zeta : \Omega \to \mathbb{R}^2$. Here, $\text{div } \zeta$ stands for the divergence and is defined as the negative of the gradient adjoint. Each iteration of the optimization algorithm alternates between a gradient descent and a gradient ascent on the primal and dual variables, respectively. In short, the update equations are as follows:

$$\zeta^{(k+1)} = \text{Proj}_{B(\theta)}(\zeta^k + \sigma_k \nabla I^{(k)})$$

(12)

$$I^{(k+1)} = \text{Proj}_{C_\rho(D,f_1,...,f_m)}(I^{(k)} + \tau_k \text{ div } \zeta^{(k+1)})$$

(13)

$$\theta_k = \frac{1}{\sqrt{1 + 4\tau_k}}, \quad \tau_{k+1} = \theta_k \tau_k, \quad \sigma_{k+1} = \frac{\sigma_k}{\theta_k},$$

(14)

$$I^{(k+1)} = I^{(k+1)} + \theta_k (I^{(k+1)} - I^{(k)})$$

(15)
where $k$ represents the iteration index. Here, the notation $\text{Proj}_A(\cdot)$ stands for the orthogonal projection of the argument onto the set $A$ and $B(g)$ represents the ball with radius $g$ in the space of $N \times N \times 2$ tensors:

$$B(g) = \left\{ u \in \mathbb{R}^{N \times N \times 2} : \sqrt{u_{i,j,1}^2 + u_{i,j,2}^2} \leq g_{i,j} \right\}.$$ 

Hence, $\text{Proj}_{B(g)}(\cdot)$ scales only the points outside the ball $B(g)$. The somewhat more complicated projection of $\text{Proj}_{c_0(D_1,\ldots,D_{m}q)}(\cdot)$ is also implemented using the POCs algorithm [27]. The initial values $I^{(0)}$ and $\zeta^{(0)}$ are arbitrary, with $T = I^{(0)}$ and time steps $\tau = 1$ [21]. By analogy (continuous setting), the divergence in (13) shall be the negated adjoint of the discrete gradient used in (12). For the forward difference gradient in equations (8)-(10), this leads to

$$\text{(div}\zeta\text{)}_{ij} = \begin{cases} c_{1,j}^1 - \zeta_{i-1,j}^1 & \text{if } 1 < i < N, \\ c_{1,j}^1 & \text{if } i = 1, \\ \zeta_{i-1,j}^1 & \text{if } i = N, \\ -c_{i,j}^1 & \text{if } 1 < j < N, \\ c_{j}^2 - \zeta_{i,j-1}^2 & \text{if } j = 1, \\ \zeta_{i,j-1}^2 & \text{if } j = N. \\ \end{cases}$$

B. Simulation Results

In the first experiment, we study the effect of the number of measurements on the reconstructed images obtained with the proposed algorithm. Recalling the result of the previous section, we expect the solution of $(P_1)$ to be binary, given adequate number of measurement pixels. In this experiment, we employ a shape image with a parametric description, composed of a semicircle laid on one side of an equilateral triangle (Figure 10a). This enables us to precisely access and display the image at arbitrary fine resolutions as a reference. Figure 10a shows the image at the resolution $2000 \times 2000$. Figs. 10b, 10c and 10d show the solutions of algorithm $(P_1)$ with the same resolution applied to the measurement of sizes $40 \times 40$, $50 \times 50$ and $80 \times 80$, respectively. All measurements are generated with a box-spline kernel. The original shape has non-smooth details around the corners and thus, to facilitate comparison, we enlarged the reconstructed images around these areas. The results reveal that with lack of enough measurements, the reconstructed images have more than two levels. It seems that the $80 \times 80$-pixel image provides enough measurements to have a binary optimal solution for $(P_1)$.

In a similar experiment, we examine the performance of our algorithm in recovering circles from different number of measurements. For this purpose, we run a Monte Carlo experiment by generating 20 circles with fixed radius and random centers in the image plane. We then consider outputs of the algorithm at resolution $600 \times 600$ constrained with $m \times m$ analytic measurements of the circles with box-spline PSFs and different values of $m$. Figure 11 shows the average mean squared errors of the reconstructed images (after thresholding at level 0.5) versus $m$ for two different radii. The plots in this figure clearly indicate that the algorithm always perfectly recovers the circles from $m \times m$ measurements when $m$ is greater than 10. Next, we examine the solutions of $(P_1)$ to $200 \times 200$-pixel discrete images of the shapes depicted in Figures 12a and 12c at resolution $1000 \times 1000$ (Figure 12c is taken from...
Fig. 12: Consistent shape reconstruction with the proposed algorithm: Figs. 12a and 12c display two shapes at the resolution $1000 \times 1000$ that will be approximated from $200 \times 200$ discrete images, generated with biquadratic B-spline sampling kernels. Figs. 12b, 12e and 12f show the same enlarged sections of the original shape 12a, the corresponding discrete image and reconstructed shape, respectively. Figs. 12d, 12g and 12h display the same for the shape in Figure 12c.

Table I: Quantitative evaluation of the proposed algorithm (numbers in dB)

<table>
<thead>
<tr>
<th></th>
<th>shape in Figure 12a</th>
<th>shape in Figure 12c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>image PSNR</td>
<td>measurement PSNR</td>
</tr>
<tr>
<td>proposed solution</td>
<td>29.1507</td>
<td>58.3316</td>
</tr>
<tr>
<td>linear interpolation</td>
<td>26.4397</td>
<td>41.4224</td>
</tr>
</tbody>
</table>

For a given shape image, the resolution requirement in Definition 4 mainly depends on the sampling grid, rather than the PSF. To examine this fact, we repeat the experiment in Figure 12 by regenerating a $200 \times 200$ discrete image from Figure 12a using a stretched biquadratic B-spline sampling kernel with an effective support of $40 \times 40$ pixels. The result is the highly blurred image in Figure 13a. Also, Figure 13b shows the enlarged section equivalent to Figure 12b. The quality of the reconstructed image in Figure 13c (PSNR = 33.8dB) the middle part of Figure 2). The sampling kernel for this experiment is the biquadratic B-spline. Figure 12 presents the same enlarged sections of the original shapes, the discrete images (just for a visual comparison) and their reconstructions with the proposed algorithm. The figures demonstrate that both reconstructed images are almost binary. Also, Table I shows the quantitative evaluation of the reconstructed images. In this table, we also compare our results with the ones obtained by the interpolation of the measurement images with the bilinear B-spline kernel, followed by a thresholding at level 0.5. For a fair comparison, we also threshold our results to calculate the PSNRs. The numbers in this table clearly indicate the success of our proposed algorithm for consistent shape reconstruction.
confirms that the sampling grid outweighs the choice of the PSF in determining the performance.

Finally in the last experiment, we study the performance of the proposed method in a setting severely deficient in the number of measurements. For this purpose, we consider a recent image by the *New Horizons* spacecraft in July 2015 from a moon of Pluto named *Hydra*. Figure 14a depicts the received measurements. Although a high resolution imager is used, due to the long distance of the spacecraft to Hydra compared to the size of Hydra, we observe a highly pixelated image. According to the available data, the effective PSF width of the imager is around 1.5 pixels, which we model by a dilated biquadratic B-spline. Figure 14b shows the output of the convex program to the measurements by applying the approximate PSF. As the measurements are too few, the reconstructed image is not bilevel (indeed, it is not unlikely to assume the image of Hydra being binary from this distance). Nevertheless, it is interesting to note that the obtained multilevel image is not far from the processed image released by NASA in Figure 14c.

VI. Conclusion

In this paper, we studied the problem of reconstructing a continuous-domain shape image from the samples in a gray-scale discrete image. This is essentially equivalent to the interpolation of pixels in a way that generates a binary image. We formulated this problem as a minimization problem where the functional is the continuous-domain total variation and the constraints encode the sampling relation between the continuous-domain image and the pixels of the discrete image. When the search is over binary images, the minimizers will be shapes with minimum perimeter and smooth boundaries that satisfy the measurements. However, the search over shape images is computationally intractable. We introduced the reducibility condition on the samples of the discrete image and proved that when it is satisfied, extending the search domain to the non-negative-valued images would not omit any of the binary minimizers. The reducibility condition essentially calls for smooth changes in the values of the neighboring pixels. From this perspective, this is an intuitive requirement on the minimum sampling density that is needed for tracking local changes in the shape boundaries.

We conjecture that under the reducibility condition, the convex problem has a unique binary solution. Nevertheless, we introduced a test to verify whether an obtained solution to the convex minimization problem is binary. This test is mainly useful in the numerical calculation of the minimizers where the recovered solutions might not be precisely bilevel due to the numerical precision.

Our approach in this paper was based on minimization of the total variation, but all the results remain valid if we use a weighted total variation. A carefully designed weighting kernel might locally adjust the recovered shapes and lead to shapes with higher mean curvature.

APPENDIX

PROOF OF LEMMA 1

In the following, we reserve the notation $e^n_i$ for the canonical basis of $\mathbb{R}^n$:

$$e^n_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T \in \mathbb{R}^n.$$  

We prove the lemma by induction on $n$. We set the basis of the induction on $n = 1$. It is trivial to check that Condition (i) for $n = 1$ implies the claim in this case. Next, by assuming the validity of Lemma 1 for some $n \geq 1$, we demonstrate the validity for $n + 1$.

For the case $K_{1}^{(n+1)} = \emptyset$, it is not difficult to see that $\lambda = \left[\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right]^T$ satisfies the requirement. Here, Condition (i) implies that all the inequalities of Condition (iii) are in fact equalities. Hence, we focus on $K_{1}^{(n+1)} \neq \emptyset$. Without loss of generality, we assume that $n+1 \in K_{1}^{(n+1)}$. Next, we will try to reduce the $(n+1)$-dimensional problem into a similar $n$-dimensional one with $K_{1}^{(n)} = K_{1}^{(n+1)} \setminus \{n+1\}$ and $K_{2}^{(n)} = K_{2}^{(n+1)}$.

According to Condition (ii), at $\lambda = e^{n+1}_{n+1} \in \Delta_{n+1}$ we have that

$$\forall i \in K_{1}^{(n+1)} \setminus \{n+1\} : \quad v_i(e^{n+1}_{n+1}) < d_i.$$  

If $K_{1}^{(n+1)} \setminus \{n+1\} = \emptyset$, set $\epsilon = \frac{1}{2}$. Otherwise, set $0 < \epsilon \leq \frac{1}{2}$ such that for all $i \in K_{1}^{(n+1)} \setminus \{n+1\}$ and all $\lambda \in \Delta_{n+1}$ with $\|\lambda - e^{n+1}_{n+1}\| < \epsilon$ (i.e., $\epsilon$-neighborhood of $e^{n+1}_{n+1}$ inside $\Delta_{n+1}$), we have that $v_i(\lambda) < d_i$. The existence of such $\epsilon$ follows from the continuity of $v$ and consequently $v_i$s. Furthermore, Condition (iii) implies $v_i(\lambda) \leq d_i$ for all $i \in K_{2}^{(n+1)}$ and the same set of $\lambda$ vectors. In summary, we conclude the existence of $0 < \epsilon \leq \frac{1}{2}$ such that

$$\forall 1 \leq i \leq n, \lambda \in \Delta_{n+1} : \quad \|\lambda - e^{n+1}_{n+1}\| < \epsilon : \quad v_i(\lambda) \leq d_i.$$  

By taking Condition (i) into account, we observe that

$$\forall \lambda \in \Delta_{n+1} : \quad \|\lambda - e^{n+1}_{n+1}\| < \epsilon : \quad v_{n+1}(\lambda) \geq d_{n+1}. \quad (16)$$  

In words, the value of $v_{n+1}$ in a neighborhood of $e^{n+1}_{n+1}$ never drops below the desired value $d_{n+1}$. In contrast, the values of $v_{n+1}$ on the facet of the simplex $\Delta_{n+1}$ opposite to $e^{n+1}_{n+1}$ ($\lambda \in \Delta_{n+1}, \lambda_{n+1} = 0$) are strictly below $d_{n+1}$ according to Condition (ii). Since $v_{n+1}$ is continuous, by starting from any point on this facet and gradually moving towards $e^{n+1}_{n+1}$ on the line connecting the two points, $v_{n+1}$ will eventually attain the value $d_{n+1}$. By considering the points on all such lines that $v_{n+1}$ attains the value $d_{n+1}$ for the first time (when moving away from the facet towards the vertex $e^{n+1}_{n+1}$), we shall have a manifold intersecting with all the facets except possibly the studied one. To mathematically represent this manifold we employ the following definition:

$$\forall t \in \Delta_n : \quad \beta(t) \triangleq \inf \left\{ \beta \in [0, 1] \left| \forall \gamma, \beta \leq \gamma \leq 1 : \right. \right.$$

$$v_{n+1}(\gamma t_{1, \ldots, } , \gamma t_{n, 1 - \gamma} ) < d_{n+1} \right\}. \quad (17)$$
It is not difficult to apply the continuity of $v_{n+1}$ to conclude the continuity of $\beta(t)$ and the fact that
\[
\forall t \in \Delta_n : \quad v_{n+1}(\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t)) = d_{n+1}.
\]
(18)

Moreover, we invoke (16) to demonstrate that $\beta(t) \geq \frac{\lambda_1}{\sqrt{2}}$, i.e., $\beta(t)$ is strictly positive for all $t \in \Delta_n$.

Now we are ready to reduce the dimension to $n$. For this purpose, we define the function $u : \Delta_n \mapsto \mathbb{R}^n$ as
\[
u(t) = [t_1, \ldots, t_n]^T \in \Delta_n : 
\]
\[
u(t) \triangleq \left[ \begin{array}{c}
u_1(\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t)) \\
\vdots \\
\nu_n(\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t)) \end{array} \right] = \left[ \begin{array}{c}u_1(t) \\
\vdots \\
u_n(t) \end{array} \right]
\]
The continuity of $u(t)$ directly follows from the continuity of $v$ and $\beta$. To verify Condition (i) for $u$ note that
\[
\sum_{i=1}^{n} \beta(t)i v_i(\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t)) + (1 - \beta(t)) v_{n+1}(\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t))
\]
\[
= \sum_{i=1}^{n} \beta(t)i d_i + (1 - \beta(t))d_{n+1} = \sum_{i=1}^{\beta(t) \neq 0} t_i \sum_{i=1}^{\beta(t) \neq 0} \nu_i(t) = \sum_{i=1}^{\beta(t) \neq 0} t_i d_i.
\]

Also, let $t \in \Delta_n$ be such that $t_i = 0$ for some $1 \leq i \leq n$. Recalling the definition of $u$, we have that
\[
u_i(t) = v_i(\tilde{\lambda}),
\]
where
\[
\tilde{\lambda} = [\beta(t)t_1, \ldots, \beta(t)t_n, 1 - \beta(t)]^T,
\]
\[
\sum_{i=1}^{n} \tilde{\lambda}_i = \beta(t) \sum_{i=1}^{n} t_i + 1 - \beta(t) = 1 \Rightarrow \tilde{\lambda} \in \Delta_{n+1}.
\]

As $t_i = 0$ results in $\tilde{\lambda}_i = 0$, the Conditions (i) and (iii) directly carry over to the functions $u_i$ with $K_1^{(n)} = K_1^{(n+1)} \setminus \{n+1\}$ and $K_2^{(n)} = K_2^{(n+1)}$.

To sum up, $u$ is a continuous function that satisfies Conditions (i)-(iii). Therefore, we conclude by the assumption of the induction that there exists $t^* \in \Delta_n$ such that
\[
u(t^*) = [d_1, \ldots, d_n]^T.
\]

Finally, by plugging this result into (19) and using (18), we obtain that
\[
u(\beta(t^*)t_1^*, \ldots, \beta(t^*)t_n^*, 1 - \beta(t^*)) = [d_1, \ldots, d_n]^T.
\]

\section*{References}
\begin{thebibliography}{99}
\end{thebibliography}


