Appendices

Appendix A: A simple example illustrating the derivation of envelopes

Let us illustrate here by a simple example the method being used in Sec. 5 for finding the envelope of a family of plane curves $F(x,y,c) = 0$ having a single parameter c.

Consider the family of all the straight lines in the x,y plane having distance 1 from the origin (see Fig. A1). These lines are given by

$$x\cos\alpha + y\sin\alpha = 1$$

with the parameter α. As the angle α varies continuously between 0 and 2π, these lines smoothly and tangently slide along the perimeter of a circle with radius 1 that is centered at the origin. The circle which is thus traced by our lines as the parameter α is being varied is the envelope curve of this family. In order to find its equation $g(x,y) = 0$, we have to eliminate the parameter α from the two equations given by (5.1):

$$F(x,y,\alpha) = x\cos\alpha + y\sin\alpha - 1 = 0 \qquad \text{(A.1)}$$

$$\frac{\partial}{\partial\alpha}F(x,y,\alpha) = y\cos\alpha - x\sin\alpha = 0 \qquad \text{(A.2)}$$

For this end, we first get from the second equation:

$$y = x\sin\alpha \,/\, \cos\alpha \qquad \text{(A.3)}$$

By substituting this in the first equation we obtain:

$$x\cos\alpha + x\sin^2\alpha \,/\, \cos\alpha = 1$$

so that $\quad x(\sin^2\alpha + \cos^2\alpha) = \cos\alpha$

whence $\quad x = \cos\alpha$

and by (A.3) $\quad y = \sin\alpha$

We can thus eliminate α from Eq. (A.1) by inserting there $\sin\alpha = y$ and $\cos\alpha = x$, which gives:

$$x^2 + y^2 = 1$$

This is, indeed, the equation of the expected envelope curve (a circle with radius 1 around the origin of the x,y plane): $g(x,y) = x^2 + y^2 - 1$.

As clearly illustrated by this simple example, in order to find the equation of our envelope curves, we need to identify a family of curves $F(x,y,c) = 0$ that smoothly and tangently slide along the desired envelopes while the parameter c varies. In other words, any member of the family should remain tangent to our desired envelope curves, and the tangent points should run along the desired envelopes continuously when the parameter c is being varied.

Let us now return to the case of our own envelope curves. According to point (4) of Sec. 4, for any given (k/j)-order modulation the envelopes depend on δ alone, and modifying the frequency f_1 while δ remains fixed only changes the density of the oscillations within the same envelopes.[1] Therefore we can choose the frequency f_1 to be our varying parameter c. An alternative approach, based on a different family of curves with a different parameter, is provided in Appendix B. Both approaches finally give the very same envelope curves.

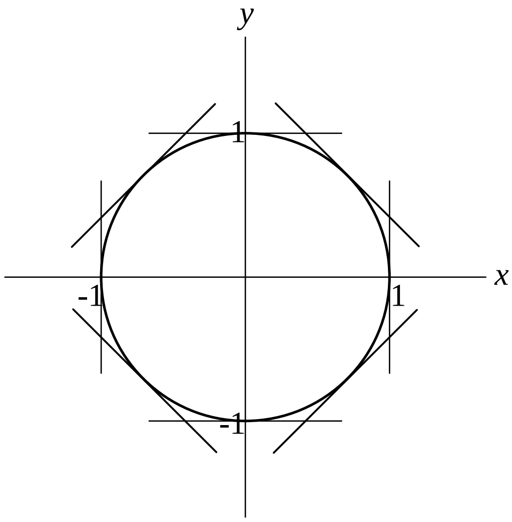

Figure A1: The family of straight lines $F(x,y,\alpha) = x\cos\alpha + y\sin\alpha - 1 = 0$ having a single parameter α (the angle), and their envelope curve $g(x,y) = x^2 + y^2 - 1$ (a circle of radius 1 around the origin).

[1] This can be clearly visualited using the enclosed interactive application “analytic_env”: Varying the frequency f_1 (using the “f_1” slider) only affects the density of the oscillations within the existing envelopes, but the envelope curves themselves remain unchanged.

Appendix B: An alternative derivation of the envelope curve equations

As already mentioned in Sec. 5 and in Appendix A, in order to find the equation of our envelope curves, we need to identify a family of curves $F(x,y,c) = 0$ that smoothly and tangently slide along the desired envelopes while the parameter c varies. If we can find such a family of curves, and if its envelope exists, then its equation $g(x,y) = 0$ can be obtained by eliminating the parameter c from the two equations:

$$F(x,y,c) = 0, \qquad \frac{\partial F(x,y,c)}{\partial c} = 0 \tag{B.1}$$

In general, there may exist many different families of curves $F(x,y,c) = 0$ that satisfy the above criteria for the very same envelopes. For instance, in the simple example of the unit circle given in Appendix A, we could also use any other family of curves that remain tangent to the desired circle while the parameter c is being varied. This is also true in our present case, where different families of curves may be used to find our envelope curves. For example, as we have seen in Appendix D of [9], this can be done by letting the phase of the high-frequency oscillations vary while keeping the envelopes unchanged; this is obtained in the general (k/j)-order case by the curve family

$$y = \cos(2\pi f_1 x + 2\pi j\phi) + \cos(-2\pi f_2 x - 2\pi k\phi) \tag{B.2}$$

where the parameter ϕ continuously varies (say, between 0 and 1). This can be clearly visualized using the enclosed application "analytic_env" (select the button phase = "carrier", and see what happens while varying ϕ). We therefore obtain from Eqs. (B.1):

$$F(x,y,\phi) = \cos(2\pi f_1 x + 2\pi j\phi) + \cos(-2\pi f_2 x - 2\pi k\phi) - y = 0 \tag{B.3}$$

$$\frac{\partial}{\partial\phi}F(x,y,\phi) = -2\pi j \sin(2\pi f_1 x + 2\pi j\phi) - 2\pi k \sin(2\pi f_2 x + 2\pi k\phi) = 0 \tag{B.4}$$

Using the well-known trigonometric identities for the sine and cosine of a sum these equations can be reformulated as follows:

$$\begin{aligned}&\cos(2\pi f_1 x)\cos(2\pi j\phi) - \sin(2\pi f_1 x)\sin(2\pi j\phi)\\ &\qquad + \cos(2\pi f_2 x)\cos(2\pi k\phi) - \sin(2\pi f_2 x)\sin(2\pi k\phi) = y\end{aligned} \tag{B.5}$$

$$\begin{aligned}&j\sin(2\pi f_1 x)\cos(2\pi j\phi) + j\cos(2\pi f_1 x)\sin(2\pi j\phi)\\ &\qquad + k\sin(2\pi f_2 x)\cos(2\pi k\phi) + k\cos(2\pi f_2 x)\sin(2\pi k\phi) = 0\end{aligned} \tag{B.6}$$

In order to obtain the equation of our desired envelope curves (either in the explicit form $y = f(x)$ or in the implicit form $g(x,y) = 0$) we need to eliminate the parameter ϕ from Eqs. (B.5) and (B.6). This elimination would give us the desired relationship between x and y alone. However, just like in Sec. 5, this task seems to be rather hopeless due to the complexity of our two equations. But here, too, we may use a similar substitution that can significantly simplify things [14, p. 68]. We provide below the detailed derivation, following the same lines as in Sec. 5 with the required adaptations.

As already mentioned in Sec. 5, the task of finding the envelope equation becomes much simpler when $F(x,y,c)$ is a polynomial in c. In this case Eqs. (B.1) simply mean that c is a multiple root of the polynomial $F(x,y,c)$, since at the point c both $F(x,y,c)$ and its derivative $\frac{\partial}{\partial c}F(x,y,c)$ have the value 0. As we know from algebra, c is a multiple root of a polynomial $P(c)$ *if and only if* the discriminant of the polynomial vanishes at the point c [16]. Therefore, when $F(x,y,c)$ is a polynomial in c the equation of the envelope can be simply found by setting the discriminant of the polynomial $F(x,y,c)$ to 0, which is often much simpler than eliminating the parameter c from the two equations (B.1). Note in particular that in such cases the second equation (that of the derivative) is no longer used in the derivation process.

Now, although in our present case (see Eq. (B.3)) $F(x,y,\phi)$ is not a polynomial in the parameter ϕ, it can be reduced to such a polynomial by using a judicious substitution. Following [14, p. 69], let us introduce a new parameter t that is obtained from our original parameter ϕ as follows:

$$t = e^{2\pi i\phi} \tag{B.7}$$

We therefore have:

$$\begin{aligned} t^k &= e^{2\pi ik\phi} = \cos(2\pi k\phi) + i\sin(2\pi k\phi) \\ 1/t^k &= e^{-2\pi ik\phi} = \cos(2\pi k\phi) - i\sin(2\pi k\phi) \end{aligned} \tag{B.8}$$

By adding or subtracting these equations we obtain:

$$\begin{aligned} t^k + 1/t^k &= 2\cos(2\pi k\phi) \\ t^k - 1/t^k &= 2i\sin(2\pi k\phi) \end{aligned} \tag{B.9}$$

or in other words:

$$\begin{aligned} \cos(2\pi k\phi) &= t^k/2 + 1/(2t^k) \\ \sin(2\pi k\phi) &= t^k/(2i) - 1/(2it^k) \end{aligned} \tag{B.10}$$

In a similar way, by using j rather than k, we obtain:

$$\begin{aligned} \cos(2\pi j\phi) &= t^j/2 + 1/(2t^j) \\ \sin(2\pi j\phi) &= t^j/(2i) - 1/(2it^j) \end{aligned}$$

Substituting these expressions in $F(x,y,\phi) = 0$ (i.e. in Eq. (B.5)) we obtain:

$$\begin{aligned} &\cos(2\pi f_1x)\,[t^j/2 + 1/(2t^j)] - \sin(2\pi f_1x)\,[t^j/(2i) - 1/(2it^j)] \\ &\qquad + \cos(2\pi f_2x)\,[t^k/2 + 1/(2t^k)] - \sin(2\pi f_2x)\,[t^k/(2i) - 1/(2it^k)] = y \end{aligned} \tag{B.11}$$

Now, in order to obtain a polynomial in t, we must get rid of the negative powers of t (the denominators).

Thus, assuming that $k \geq j$, we multiply all terms by $2it^k$, which gives:

$$\begin{aligned} &\cos(2\pi f_1x)\,[it^{k+j} + it^{k-j}] - \sin(2\pi f_1x)\,[t^{k+j} - t^{k-j}] \\ &\qquad + \cos(2\pi f_2x)\,[it^{2k} + i] - \sin(2\pi f_2x)\,[t^{2k} - 1] = 2iyt^k \end{aligned} \tag{B.12}$$

Multiplying both sides by $-i$ and rearranging the terms in decreasing order of powers we finally obtain the following polynomial in t:

$$(\cos(2\pi f_2 x) + i\sin(2\pi f_2 x))t^{2k} + (\cos(2\pi f_1 x) + i\sin(2\pi f_1 x))t^{k+j} - 2yt^k$$
$$+ (\cos(2\pi f_1 x) - i\sin(2\pi f_1 x))t^{k-j} + (\cos(2\pi f_2 x) - i\sin(2\pi f_2 x)) = 0$$

namely:

$$At^{2k} + Bt^{k+j} + Ct^k + Dt^{k-j} + E = 0 \tag{B.13}$$

with the coefficients:

$$\begin{aligned} A &= e^{2\pi i f_2 x} \\ B &= e^{2\pi i f_1 x} \\ C &= -2y \\ D &= e^{-2\pi i f_1 x} \\ E &= e^{-2\pi i f_2 x} \end{aligned} \tag{B.14}$$

Having thus obtained a polynomial in t, the equation of our envelope curves can be found by simply setting its discriminant Δ to 0, which is indeed much simpler than eliminating ϕ from Eqs. (B.5) and (B.6). Once again, note that we do not need to solve Eq. (B.13) itself; all that we need here is to solve the discriminant equation $\Delta = 0$, which is a polynomial equation of order $2k$ in y, whose own coefficients are functions of x. This polynomial equation has therefore $2k$ solutions of the form $y = f_1(x), \ldots, y = f_{2k}(x)$, which are precisely our desired envelope curves. The parameter t itself does not appear in these solutions (just as the parameter c disappears when using the standard parameter-elimination method).

Comparing our results here with the results obtained in Sec. 5 using a different family of curves, we see that we have obtained exactly the same polynomial in t, although its coefficients A, B, D and E are different (note that the coefficient C, the only one containing the variable y, remains unchanged). Nevertheless, both results are equivalent and finally give the very same envelope curves. Let us illustrate this with two simple examples:

Example B.1 (*the simple case of* $(k/j) = (1/1)$):

This is the classical case of first-order modulation, where $f_2 = f_1 + \delta$ (see Fig. 1). In this simple case our polynomial (B.13) is a quadratic:

$$(A + B)t^2 + Ct + (D + E) = 0$$

whose discriminant Δ is given by:

$$\Delta = C^2 - 4(A + B)(D + E)$$

Therefore in this simple case $\Delta = 0$ gives, using (B.14), the following quadratic polynomial equation in y:

$$4y^2 - 4(e^{2\pi i f_2 x} + e^{2\pi i f_1 x})(e^{-2\pi i f_1 x} + e^{-2\pi i f_2 x}) = 0$$

i.e.
$$y^2 - (e^{2\pi i(f_2 - f_1)x} + e^{2\pi i(f_1 - f_2)x}) + 2) = 0$$

Since in the case of $(k/j) = (1/1)$ we have $f_2 = f_1 + \delta$, this gives

$$y^2 - e^{2\pi i\delta x} - e^{-2\pi i\delta x} - 2 = 0$$

i.e. $$y^2 = e^{2\pi i\delta x} + e^{-2\pi i\delta x} + 2 = 2 + 2\cos(2\pi\delta x)$$

We therefore obtain the two envelope curves:

$$y = \pm\sqrt{2 + 2\cos(2\pi\delta x)}$$

which can be reduced using the trigonometric identity $\cos(\alpha/2) = \pm\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos\alpha}$ into:

$$y = \pm 2\left|\cos(2\pi[\delta/2]x)\right| \qquad \text{(B.15)}$$

This gives indeed the same curves as our envelopes $\mathrm{en}_1(x)$ and $\mathrm{en}_2(x)$ in Eq. (3.4), although expressed in a different way. ■

Example B.2 (*the case of* $(k/j) = (2/1)$):

This case corresponds to a second-order modulation, where $f_2 = 2f_1 + \delta$ (see Fig. 3). In this case our polynomial (B.13) is a quartic whose coefficients are given by (B.14):

$$At^4 + Bt^3 + Ct^2 + Dt + E = 0$$

The discriminant Δ of a quartic equation has 16 terms and is given by (see, for example, [17, p. 405] or [18, pp. 257-258]):[2]

$$\begin{aligned}\Delta = {} & 16AC^4E - 4AC^3D^2 - 4B^2C^3E \\ & + B^2C^2D^2 - 128A^2C^2E^2 - 80ABC^2DE \\ & + 18ABCD^3 + 18B^3CDE + 144A^2CD^2E + 144AB^2CE^2 \\ & - 27A^2D^4 - 27B^4E^2 + 256A^3E^3 - 192A^2BDE^2 - 6AB^2D^2E - 4B^3D^3\end{aligned}$$

In this case $\Delta = 0$ gives, using (B.14), the following 4-th order polynomial equation in y:

$$\begin{aligned}\Delta = {} & 256y^4 + 32(e^{2\pi i(f_2 - 2f_1)x} + e^{2\pi i(2f_1 - f_2)x})y^3 - 828y^2 \\ & - 324(e^{2\pi i(f_2 - 2f_1)x} + e^{2\pi i(2f_1 - f_2)x})y - 27(e^{4\pi i(f_2 - 2f_1)x} + e^{4\pi i(2f_1 - f_2)x}) + 54 = 0\end{aligned}$$

Using $f_2 = 2f_1 + \delta$ and the identity $e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ we get:

$$128y^4 + 32\cos(2\pi\delta x)y^3 - 414y^2 - 324\cos(2\pi\delta x)y - 27\cos(4\pi\delta x) + 27 = 0$$

which finally gives, using the triginometric identity $\cos(2\alpha) = 1 - 2\sin^2\alpha$:

$$64y^4 + 16\cos(2\pi\delta x)y^3 - 207y^2 - 162\cos(2\pi\delta x)y + 27\sin^2(2\pi\delta x) = 0 \qquad \text{(B.16)}$$

The 4 solutions of this 4-th order polynomial equation in y, $y = \mathrm{en}_1(x)$, … ,$y = \mathrm{en}_4(x)$, give indeed the 4 envelope curves ("braid curves") of the second-order modulation with $f_2 = 2f_1 + \delta$, which are shown in Fig. 3. These 4 curves are plotted in Fig. 5. Notice how these 4 curves perfectly follow the high-frequency oscillations of the cosine sum, as shown in row (a) of this figure. ■

As we can see, Examples 5.1 and B.1 as well as Examples 5.2 and B.2 finally give exactly the same results. The difference is that in the method provided in Sec. 5 the coefficients A, B, D and E are directly obtained in terms of δ, while in our alternative method here the coefficients A, B, D and E are given in terms of f_1 and f_2. But then

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[2] For convenience we have rearranged the terms of the discriminant in decreasing order of powers of C, which is the only coefficient in (5.13) that includes the variable y.

substituting $f_2 = (k/j)f_1 + \delta$ finally gives the very same results, formulated once again in terms of δ alone.

So far our development starting from Eq. (B.12) was based on the assumption that $k \geq j$. If, however, $k < j$, we simply have to multiply all terms of Eq. (B.11) by $2t^j$ rather than by $2t^k$. This gives:

$$\cos(2\pi f_1 x)\,[it^{2j} + i] - \sin(2\pi f_1 x)\,[t^{2j} - 1] + \cos(2\pi f_2 x)\,[it^{j+k} + it^{j-k}] - \sin(2\pi f_2 x)\,[t^{j+k} - t^{j-k}] = 2iyt^j \tag{B.17}$$

By multiplying both sides by $-i$ and rearranging the terms in decreasing order of powers we finally obtain the following polynomial in t:

$$(\cos(2\pi f_1 x) + i\sin(2\pi f_1 x))t^{2j} + (\cos(2\pi f_2 x) + i\sin(2\pi f_2 x))t^{j+k} - 2yt^j + (\cos(2\pi f_2 x) - i\sin(2\pi f_2 x))t^{j-k} + (\cos(2\pi f_1 x) - i\sin(2\pi f_1 x)) = 0$$

namely:

$$At^{2j} + Bt^{j+k} + Ct^j + Dt^{j-k} + E = 0 \tag{B.18}$$

where, this time:

$$\begin{aligned} A &= e^{2\pi i f_1 x} \\ B &= e^{2\pi i f_2 x} \\ C &= -2y \\ D &= e^{-2\pi i f_2 x} \\ E &= e^{-2\pi i f_1 x} \end{aligned} \tag{B.19}$$

Since this is again a polynomial in t, the equation of our envelope curves can be found by setting its discriminant Δ to 0, in the same manner as we did in the case of $k \geq j$.

Example B.3 (*the case of* $(k/j) = (1/2)$):

This case corresponds to the situation shown in Fig. 4(a), where $f_2 = (1/2)f_1 + \delta$. In this case we obtain from Eq. (B.18) the same quartic polynomial as in Example B.2, but this time its coefficients are given by (B.19) rather than by (B.14):

$$At^4 + Bt^3 + Ct^2 + Dt + E = 0$$

In this case $\Delta = 0$ gives, using this time (B.19):

$$\Delta = 256y^4 + 32(e^{2\pi i(f_1 - 2f_2)x} + e^{2\pi i(2f_2 - f_1)x})y^3 - 828y^2 - 324(e^{2\pi i(f_1 - 2f_2)x} + e^{2\pi i(2f_2 - f_1)x})y - 27(e^{4\pi i(f_1 - 2f_2)x} + e^{4\pi i(2f_2 - f_1)x}) + 54 = 0$$

Using now $f_2 = (1/2)f_1 + \delta$ and hence $2f_2 - f_1 = 2\delta$ we get this time:

$$64y^4 + 16\cos(2\pi[2\delta]x)y^3 - 207y^2 - 162\cos(2\pi[2\delta]x)y + 27\sin^2(2\pi[2\delta]x) = 0 \tag{B.20}$$

which is the same 4-th order polynomial equation in y as in Example B.2, with $2x$ replacing x everywhere. And indeed, as we can see in Fig. 6, the 4 envelope curves we get in this case are simply a horizontally 2-fold denser version of the 4 envelope curves of Example B.2 (i.e. they are shrinked by factor 2 along the x axis). ■

Appendix C: Miscellaneous issues

In this appendix we provide some additional remarks that will shed further light on our discussions so far.

C.1 *On the benefits of unwrapping the analytically obtained envelope plots*

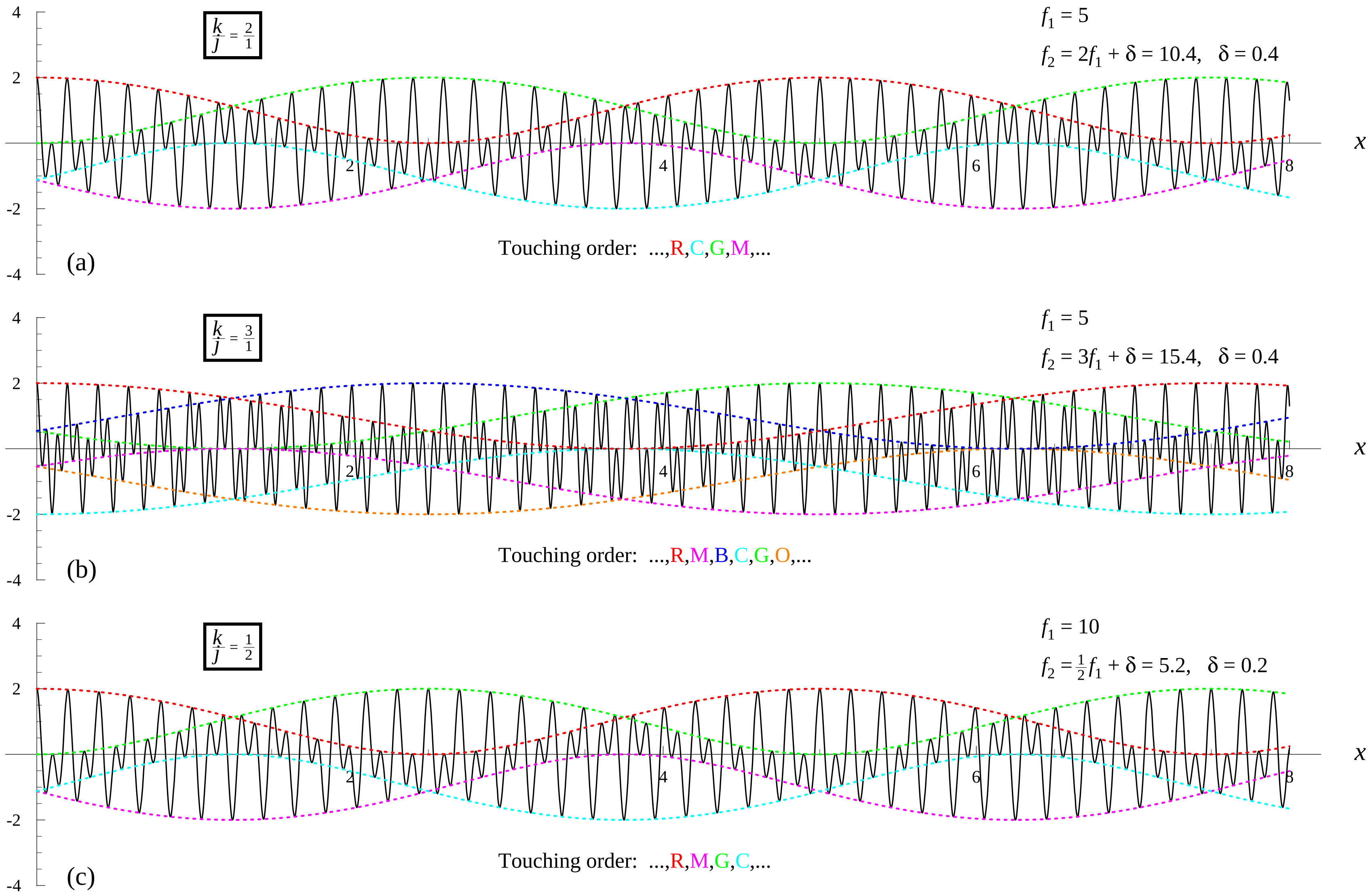
As already mentioned in Sec. 6, the envelope equations we obtain analytically are not plotted as $2n$ interlaced smooth cosine-like curves (i.e. as "braid curves"), but rather as $2n$ horizontal "slices" through these braid curves. Thus, rather than being smooth and rounded each of the plotted curves has sharp edges or cusp points (compare Figs. 5-7 with Figs. 2-4, which were plotted using the cosinusoidal approximations). However, if required, we may "unwrap" (i.e. unfold or re-deploy) the plotted envelope curves into their fully-equivalent but more natural braid-curve representation (much like in phase unwrapping [22, p. 167]). This unfolding may be advantageous in some situations, for example when studying in detail the behaviour of the underlying highly-oscillating curve $s(x)$.

To see this, consider first the signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ in the simplest (1/1)-order case, i.e. when $f_2 \approx (1/1)f_1$. As we can see in Fig. 2(a), this signal rapidly oscillates between its two envelope curves (the red and green lines in the figure). In each of its rapid oscillations it goes up until it touches the upper envelope cueve, and then it goes back down until it touches the bottom envelope curve, and so on. This behaviour is quite obvious and well known. But how does the signal $s(x)$ behave in higher (k/j)-order cases, when the number of envelope curves is larger than 2?

To get a first idea, we may have a look at figures which have been plotted with the analytically obtained envelope curves. The situation in such figures seems confusing, since at each point along the x axis the behaviour seems to be different. But things become much clearer if we take the pains to "unwrap" the analytically obtained envelopes, and to plot them in the more natural "braid-curve" representation, as shown in Fig. C1. Looking carefully at row (a) of this figure, which corresponds to the case of

Figure C1: Illustration of an advantage of using the "unwrapped" representation of the analytically obtained envelope curves. Each row shows the cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = \frac{k}{j}f_1 + \delta$, where: (a) $k/j = 2/1$; (b) $k/j = 3/1$; (c) $k/j = 1/2$. Note that in this figure we have "unwrapped" the "sliced" envelope curves so as to obtain the equivalent but more natural braid-curve representation of the $2n$ analytically derived envelopes. Having done so, we see that in each row the black curve of $s(x)$ oscillates intermittently between the $2n$ envelope curves, touching them always in the same order. For example, in row (a) ((2/1)-order modulation) the touching order is red, cyan, green, magenta, etc. while in row (c) (the reciprocal (1/2)-order modulation) the touching order is reversed: red, magenta, green, cyan, etc.

The "unwrapped" version of the analytically obtained envelope curves of the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:



$(k/j) = (2/1)$, we see that the behaviour of the signal $s(x)$ there is a straightforward generalization of its behaviour in the classical (1/1)-order case. Starting from the origin ($x = 0$), where $s(x)$ touches the red (R) envelope curve, $s(x)$ goes halfway down until it touches the cyan (C) envelope. Then it goes halfway up until it touches the green (G) envelope, and then it goes down again to touch the magenta (M) envelope. As we advance along the x axis, this behaviour of $s(x)$ repeats over and over again, always in the same order, even when the envelopes are interlaced and change their relative vertical positions. We see therefore that $s(x)$ oscillates intermittently between its $2n$ envelope curves, touching them always in the same predefined order. In other words, as we have seen in point (5) of Sec. 4, the curve $s(x)$ is modulated by a set of $2n$ interlaced envelopes (braid curves), and it oscillates intermittently from curve to curve.

Comparing the behaviour of $s(x)$ in the 3 rows of Fig. C1, we see that in each (k/j) case the "touching order" is different: In the (2/1) case (row (a)) $s(x)$ touches its envelope curves in the repeated order red, cyan, green, magenta, etc. (…,R,C,G,M,…); in the (3/1) case (row (b)) the touching order is …,R,M,B,C,G,O,…, and in the (1/2) case (row (c)) the touching order is …,R,M,G,C,… Note that in the reciprocal (2/1) and (1/2) cases (rows (a) and (c)) the touching order is simply reversed.

As we can see, the behaviour of $s(x)$ in the general (k/j)-order case is a straightforward generalization of its behaviour in the classical (1/1)-order case. But in order to be able to observe this behaviour in our figures, we must first "unwrap" the analytic envelope curves and plot them as a set of interlaced braid curves.

C.2 *On the remarkable complexity of the analytically obtained envelope equations*

As already mentioned above, the explicit mathematical expressions obtained analytically for each of the $2n$ envelope curves $y = \mathrm{en}_1(x)$, … ,$y = \mathrm{en}_{2n}(x)$ may become extremely complex. As an illustration, we show in Fig. C2 the explicit expression obtained by the Mathematica® software package (after simplification) for the envelope curve $y = \mathrm{en}_1(x)$ in the case of $(k/j) = (2/1)$, i.e. the red curve in Fig. 5. This curve is one of the 4 solutions of the 4-th order polynomial in y (5.16) (see Example 5.2).

The exact expressions of the other three envelope curves look similar, too. These expressions (as well as the envelope curve equations of other (k/j)-order cases) can be obtained using the interactive Mathematica® application "analytic_eq" that is provided in the supplementary material.

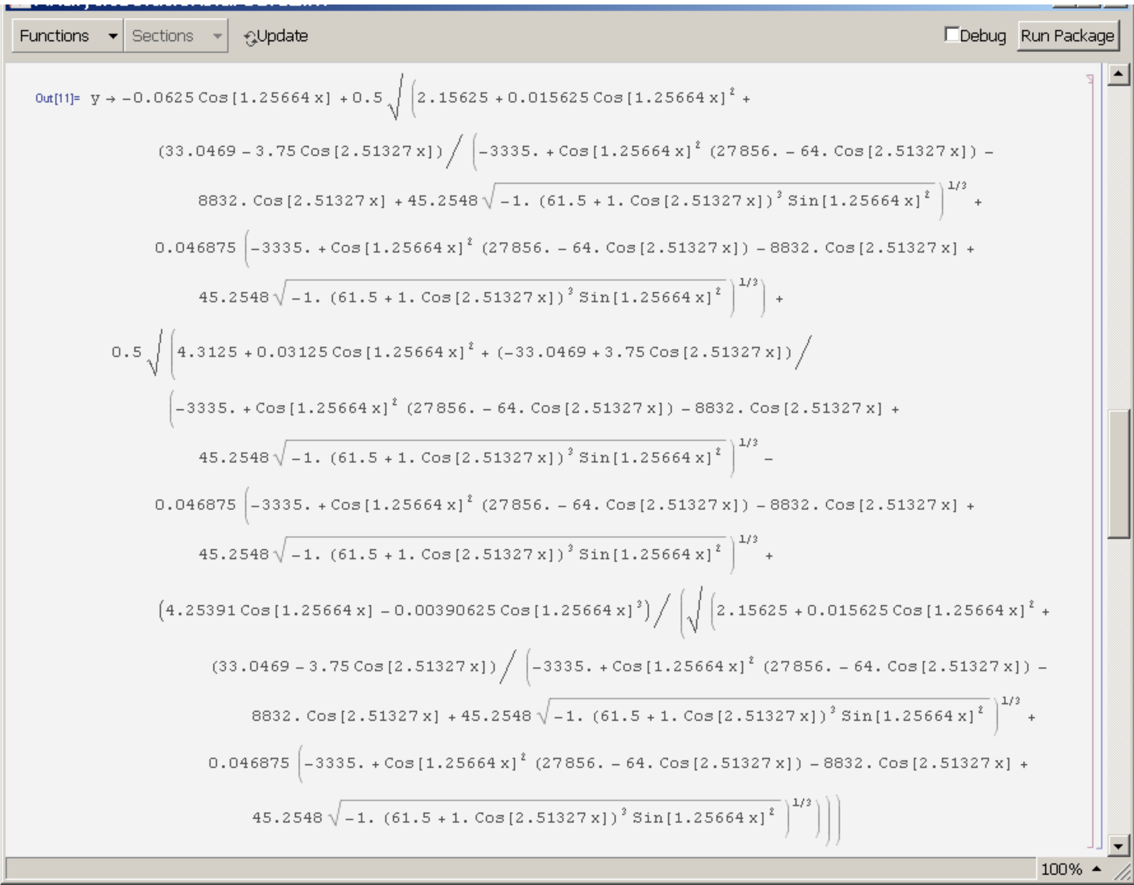


Figure C2: The explicit expression obtained by the Mathematica® software package (after simplification) for the envelope curve y = en$_1(x)$ in the case of $(k/j) = (2/1)$, i.e. for the red curve in Fig. 5.

C.3 *A closer look at the case of (k/j)* = (2/1)

As we have seen, the envelope curve equations obtained for all (k/j) cases except for the most trivial case of $k = j = 1$ are surprisingly complex. It would be therefore instructive to have a closer look at the simplest of these non-trivial cases, that of $(k/j) = (2/1)$ (see Fig. 5). Let us analyze here its various terms, in order to understand their individual contributions to the 4 envelope curves belonging to this case.

Fig. C3 shows the envelope equations obtained for the case $(k/j) = (2/1)$, using a generic δ value. As we can see, this expression consists of 3 terms (all of which are further scaled by 1/16):

(1) The term $-\cos(2\pi\delta x)$, that we denote here by a.

(2) A complex square root that we denote here by $\pm b$.

(3) An even more complex square root, that we denote by $\pm c$, which includes in the denominator of its own last term a nested copy of the square root $\pm b$ of point (2).

Figure C3: Top: the explicit expression of the 4 envelope curves in the case of $(k/j) = (2/1)$, with a generic δ value. Bottom: its terms a, b and c.

Note that both of the square roots b and c also contain several instances of a quite complex cubic root (the same cubic root in all cases).

We first remark that the 4 envelope curves obtained in the case of $(k/j) = (2/1)$ (see Fig. 5) simply correspond to the 4 possible combinations of the square-root signs $\pm b$ and $\pm c$, as shown in Table C1. A third square root, which is nested inside the above-mentioned cubic root, always takes the plus sign.

	$+b$	$-b$
$+c$	Red	Cyan
$-c$	Green	Magenta

Table C1: The 4 possible combinations of the square-root signs $\pm b$ and $\pm c$ and the corresponding 4 envelope curves (see Figs. 5 and C4). Note that the sign of b also applies to the nested copy of b inside the term c.

Let us now try to understand the individual role of each of the components a, b and c in the structure of the final envelope curves.

Term a being just a simple (scaled) cosine, we prefer to start our investigation with the other, more interesting terms; we will return to the role of term a later below.

Term b is already a much more complex expression; Taken alone, it gives a horizontal straight curve that is very close to $y = +17.015625$ or $y = -17.015625$, depending on the sign of the square root (or $y = \pm 1.063$, if we take into account the global scaling by 1/16). But although term b is very close to a straight line, in fact it corresponds to a slightly undulating cosine-like structure with frequency 2δ and a very small amplitude of less than 0.02 (see the two orange curves in Fig. C4(b)).

Term c, which is an even more complex expression (remember it includes a nested copy of the square root $\pm b$), provides the cuspidal shape of the envelope curves. Plotted alone, taking into account the global scaling by 1/16, it gives a periodic structure which is very similar to the rectified cosine $|\cos(2\pi[\delta/2]x)|$ (if we take inside c the '+' sign of the root b) or to the rectified sine $|\sin(2\pi[\delta/2]x)|$ (if we take inside c the '–' sign of b). But if the '–' sign of the square root c is taken, the periodic structures obtained are similar to $-|\cos(2\pi[\delta/2]x)|$ or to $-|\sin(2\pi[\delta/2]x)|$, respectively. These 4 curves are shown Fig. C4(a).

Combining together the pieces of the puzzle, we therefore get the following picture: Term c provides the basic cuspidal shape of each of the 4 envelope curves, depending on the 4 sign combinations of $\pm b$ and $\pm c$ as shown in Table C1. Then, each of these envelope curves is elevated to its own vertical height thanks to the addition of the term

b, which corresponds to about ±1.063 (depending on the sign of b). As shown in Fig. C4(b), the sum of c and b gives us indeed 4 cuspidal curves that are similar (but not identical) to those of Fig. 5. The addition of term a, $-\cos(2\pi\delta x)/16$, that we have so far ignored (see the black curve in Fig. C4(b)), provides some further fine tuning and gives us indeed the final envelope shapes of Fig. 5.

As we can see, the combination of terms (1)-(3) provides the precise envelope-curve shapes for our particular case of $(k/j) = (2/1)$. In more complex (k/j) cases more terms will be involved, and each of the terms may be even more complex than in the present case.

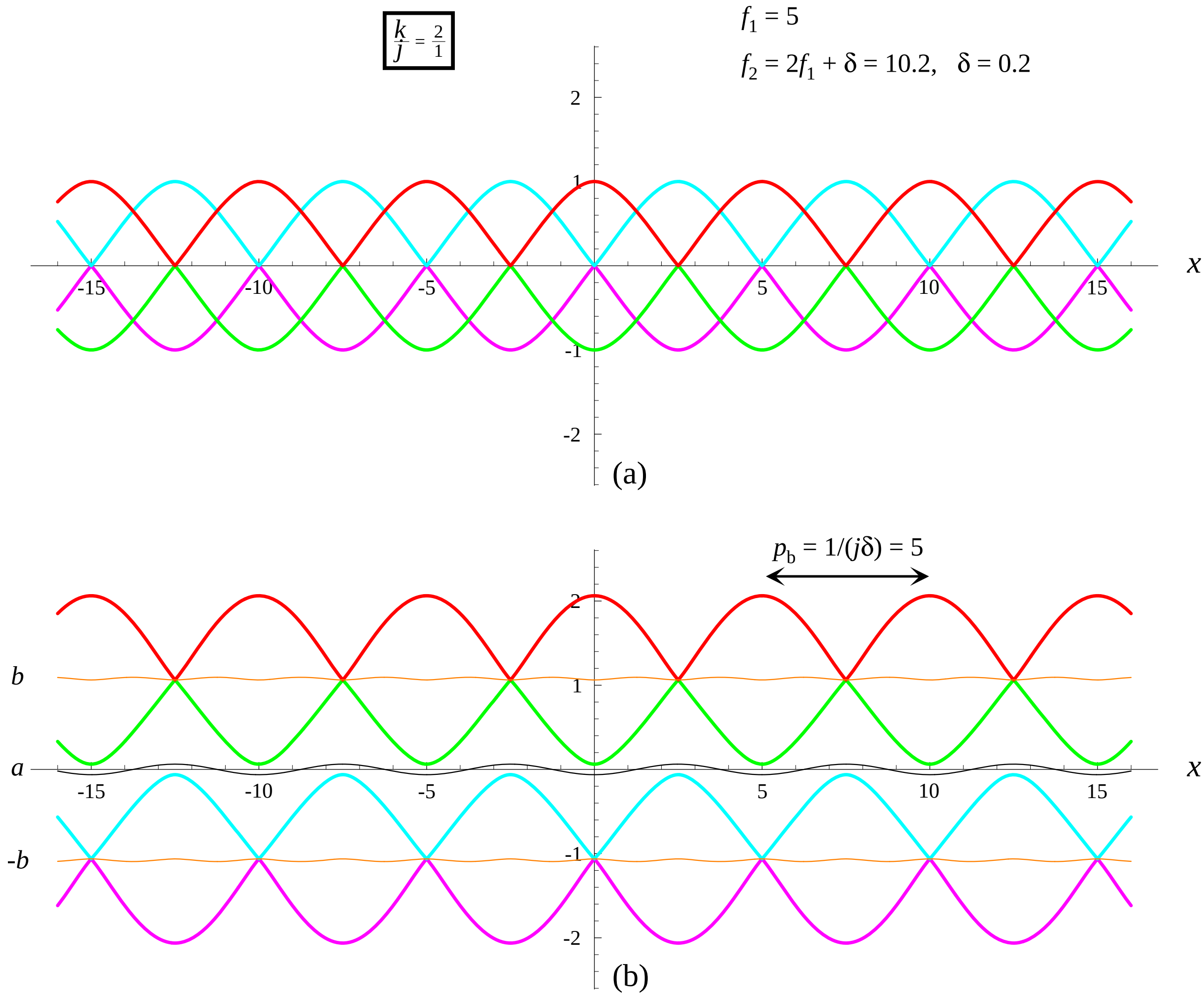


Figure C4: (a) Term (3) plotted alone; each of the 4 cuspidal curves is obtained by a different sign combination (see Table C1). (b) The sum of terms (3) and (2); note how the addition of $+b$ or $-b$ elevates each of the 4 curves to its own vertical position. The further addition of term (1), $-\cos(2\pi\delta x)/16$ (that is plotted here in black), gives the final envelope curves shown in Fig. 5.

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