User's guide for the interactive Mathematica® applications `analytic_eq` and `analytic_env`

The interactive Mathematica® applications **`analytic_eq`** and **`analytic_env`** accompany the article "Analytic derivation of the (*k/j*)-order modulation envelopes in the sum of two mistuned (co)sine waves" by Isaac Amidror, 2015. These applications allow to dynamically observe the (*k/j*)-order modulation effects that may occur between two mistuned cosine waves having frequencies f_1 and $f_2 = (k/j)f_1 + \delta$. The application **`analytic_eq`** allows one to select any of the 2max(*k*,*j*) analytically-derived envelope curves, visualize it graphically, and display its equation $y = \mathrm{en}(x)$. The application **`analytic_env`** allows one to visualize either the analytically derived envelope curves or their cosinusoidal approximations ("braid curves"). In addition, it allows to vary the frequency f_1 or the phase ϕ and to observe the resulting effects on the (*k/j*)-order beats and their envelopes.

1. Introduction

The beating effects between two mistuned (co)sinusoidal waves can be best appreciated and understood graphically by means of hands-on experimentation. The figures in the main article only show a limited number of particular cases that were carefully chosen to best illustrate the text. But a truly interactive manipulation of the various parameters can provide a much deeper insight into the dynamics of the phenomena in question.

For this end, two dedicated interactive applications have been developed to accompany the main article. These applications allow one to observe the continuous signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ having frequencies f_1 and $f_2 = (k/j)f_1 + \delta$ (plotted in blue lines), as well as the (*k*/*j*)-order modulating envelopes that outline the corresponding beating effect. By manipulating the different parameters one may obtain a vivid graphic demonstration of the various effects that may occur in $s(x)$ and of their dynamic behaviour.

These applications are written in the Mathematica® programming language. The Mathematica® sourse code is also provided, to allow users make their own experiments by modifying other parameters inside the source program. Users who do not possess the Mathematica® software will be able to run the enclosed CDF versions of these applications (**`analytic_eq.cdf`** and **`analytic_env.cdf`**) using the free CDF player that can be downloaded from the Wolfram CDF web page; but in this case no modifications can be done in the source code itself. To run an "**`.m`**" file click on the filename and then on the "Run Package" button at the top of the frame; to run a "**`.cdf`**" file click on the filename and then, if required, on the "Enable Dynamics" button at the top. If the source code is displayed, scroll down to see the graphics.

2. How to use the applications

The interactive application **`analytic_env`** allows to dynamically visualize the analytically derived envelope curves, or their cosinusoidal approximations ("braid curves").

The buttons "***k/j***" and "$\boldsymbol{\delta}$" allow you to easily reach some of the most typical beating effects, such as those generated by (1/1)-order modulation, by (2/1)-order modulation, etc., with some preset values of δ.

The button "**`env`**" allows you to display the curves of the modulating envelopes of the preselected (k/j)-order modulation effect. Three options are available: "none" for no envelopes, "approx." for displaying the approximating cosinusoidal braid curves, and "analytic" for displaying the analytically derived envelope curves. Note that the displayed envelopes will be automatically updated whenever the "***k/j***" or "$\boldsymbol{\delta}$" button is set.

The slider "$\boldsymbol{f_1}$" allows you to dynamically observe the influence of the frequency f_1. You can vary f_1 between 0.5 and 5 by slowly moving the slider using the mouse. Note that the frequency f_2 is automatically updated to $f_2 = (k/j)f_1 + \delta$ using the current values of f_1, k/j and δ.

A second slider named "$\boldsymbol{\phi}$" allows to gradually modify the phase parameter ϕ of the two mistuned cosines between the values 0 and 1. As explained in Appendix D of [9], various types of phase effects may be obtained depending on the chosen definition of ϕ_1 and ϕ_2 (the relative phases of the two mistuned cosines) in terms of the phase parameter ϕ. One may select the desired type of phase effect using the button "**`phase`**"; this button allows to choose between rigid shift, shift of the envelopes only, shift of the carriers only, shift of both envelopes and carriers, and divergent shift (envelopes and carriers shifted in opposite directions). Obviously, only the rigid shift truly preserves the original function $s(x)$ itself; all the other phase variants will give somewhat different functions, but each of these variants is still interesting in its own right.

Finally, the button "**`plot density`**" at the top of the panel allows one to speed up the reactivity of the application **`analytic_env`**, by choosing a lower plot quality. Because this application has to calculate and plot quite complex functions, the density of the sampling points at which the function is evaluated for plotting may slow down the interactive reaction of the application. It is therefore recommended to use a rather speedy computer (for example, with an *i*7-2600 processor at 3.4 GHz). If the reactivity on your computer is too slow, you may set the button "**`plot density`**" to "low". This will reduce the plotting density and hence speed up the application. But on the other hand, the resulting plot may then "miss" some of the dense oscillations of the signal $s(x)$.

The interactive application **`analytic_eq`**, on its part, is designed to derive and display the explicit equation $y = \text{en}(x)$ of the analytic envelope curve you specify. This application allows you to select the desired (k/j)-order, and it then plots the corresponding signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ where $f_1 = 5$ and $f_2 = (2/1)f_1 + 0.1$. For any chosen k/j value, the "**`env_n`**" button allows you to select one of the corresponding 2max(k,j) analytically derived envelopes. The selected envelope curve is then plotted in red on the signal $s(x)$, and its analytically derived equation $y = \text{en}(x)$ is displayed below the plot. Note that the number of envelopes is automatically updated whenever you select a new k/j value; thus, for $k/j = 1/1$ the "**`env_n`**" button allows you to select between the envelope numbers 1 and 2, for $k/j = 2/1$ you may choose any number between 1 and 4, etc. The **`env_n`** value used by default is 1.

Due to the complexity of the derivations, this application, too, may be quite slow, depending on the chosen k/j value; it is therefore recommended to run it on a rather speedy computer.

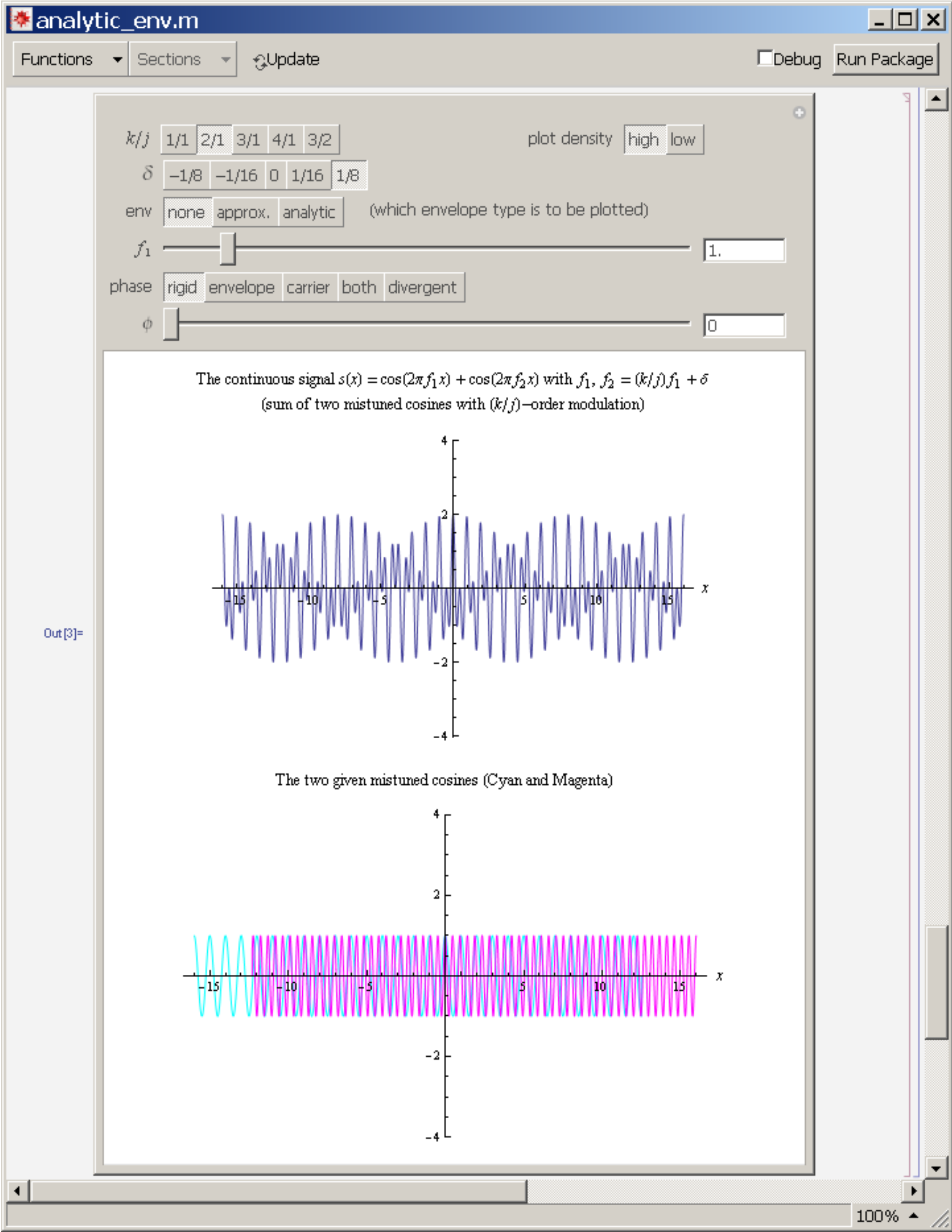
Remark: If you are running the application **`analytic_env`** on a fast computer and you wish to move a slider more slowly (which may be useful for fine tuning), simply hold down the Alt key (Windows) or the Option key (Macintosh) while dragging the mouse left or right. If you move the mouse outside the area of the slider, the value will move slowly in that direction as long as the mouse remains clicked. If you wish to move the slider even more slowly, you can hold down the Shift or Ctrl keys, or both, in addition to the Alt/Option key. It is also possible to enter numerically any desired value (within the permitted range): Simply enter the desired numerical value into the small box to the right of the slider; when hitting the Enter key the application will update the display accordingly. For example, when setting the relative phase to $\phi = 0.75$ the original cosine function will be shifted to the left by three quarters of its period; this simply gives a sine signal.

Remark: If you wish to increase or decrease the size of the displayed area, click with the mouse anywhere inside the figure; this will display a new thin gray frame around the entire displayed area, whose borders can be dragged with the mouse to obtain the desired dimensions.

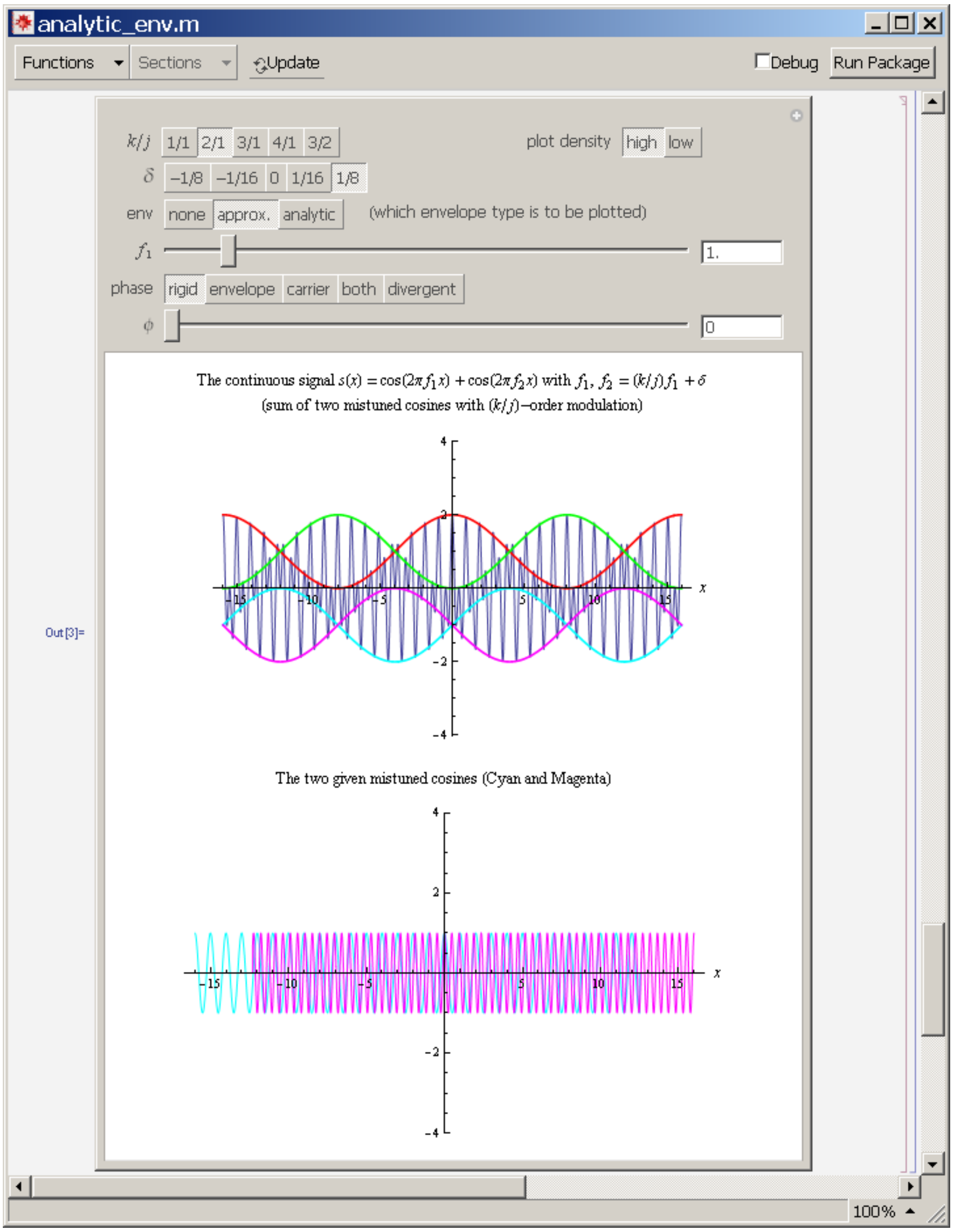
3. Examples

Let us now proceed to some examples illustrating the use of these two interactive applications.

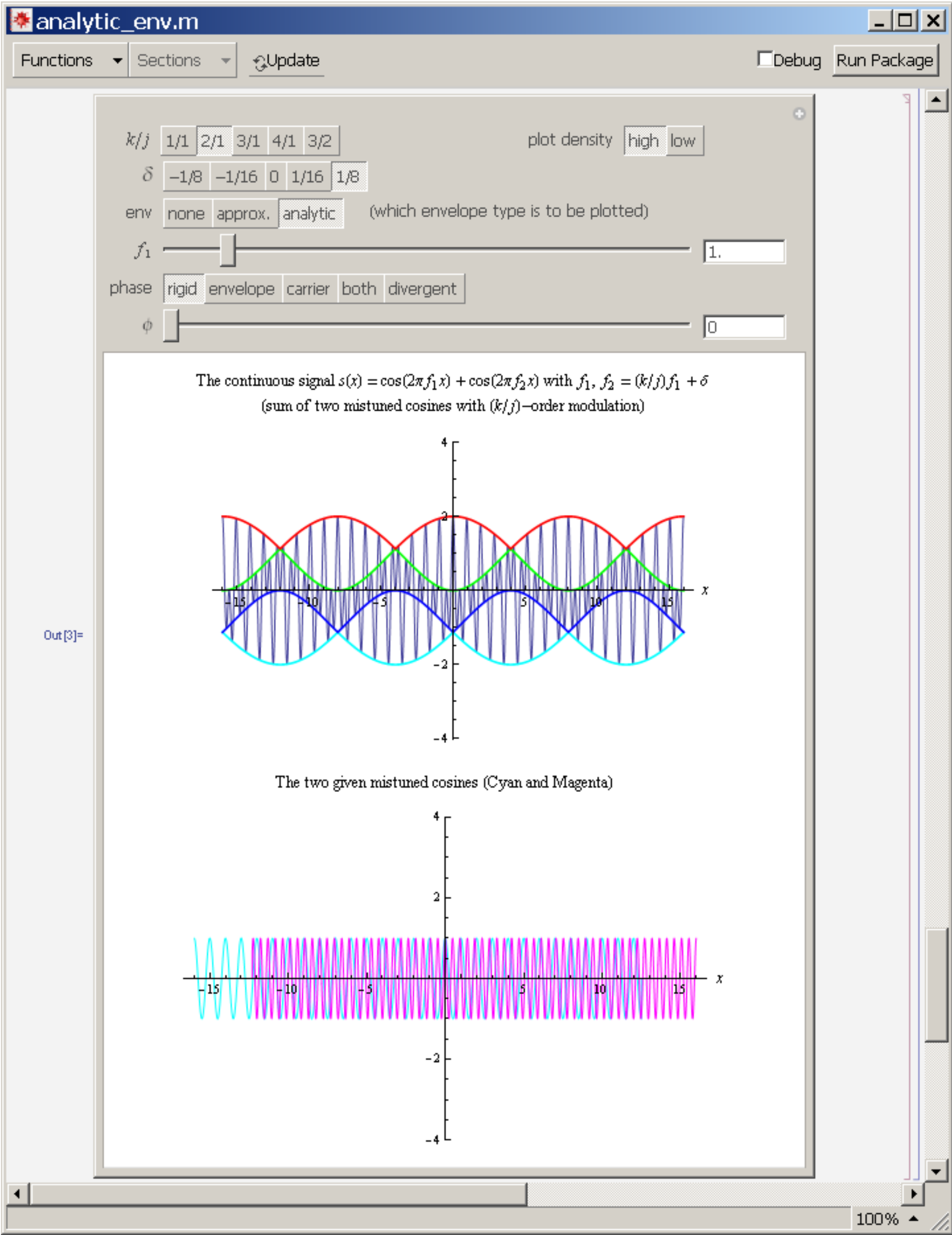
In both applications the parameter values are set by default to show a typical case, the (2/1)-order modulation effect. Thus, when you enter the application **`analytic_env`** you will get the following display, showing the signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ where $f_1 = 1$ and $f_2 = (2/1)f_1 + \delta$:



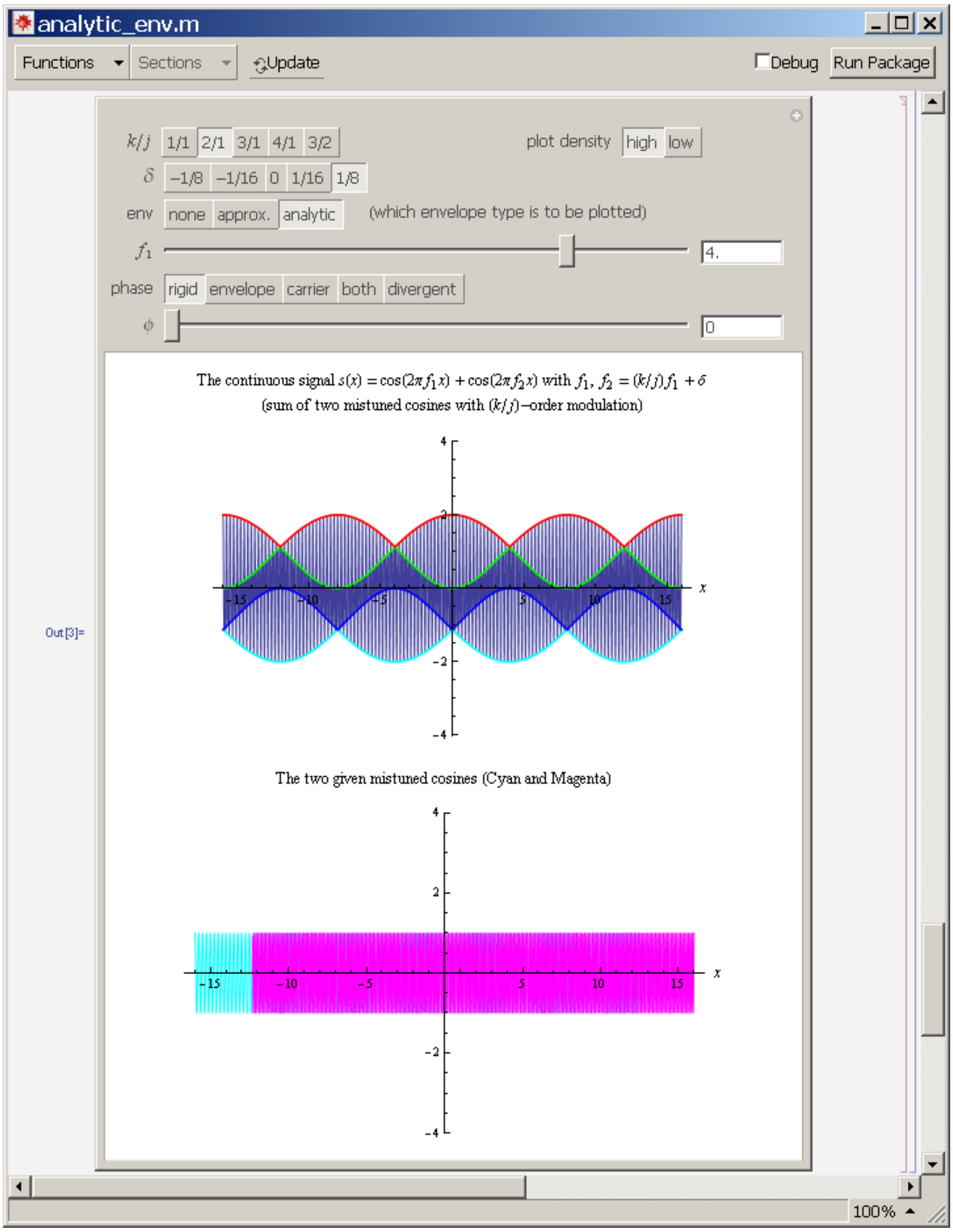
You may now try to view the same beating effect with another preset value of δ, or select another (k/j)-order modulation effect by clicking on a different k/j value. This is even more interesting when the "**`env`**" button is set to display the envelope curves. Two possible choices are available in addition to the default value "none" (no envelopes): "approx." for displaying the approximating cosinusoidal braid curves (see the figure on



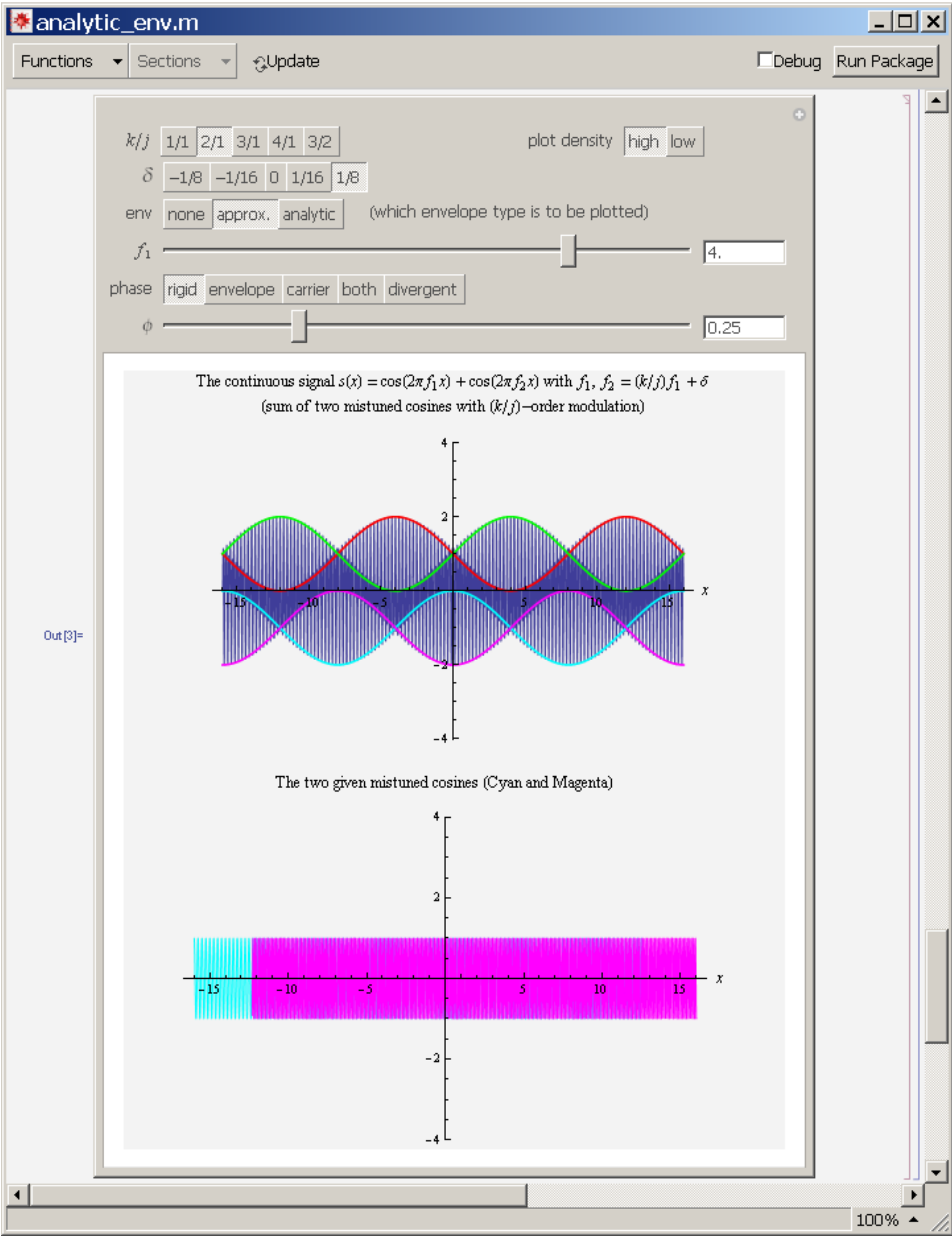
p. 5), or “analytic” for displaying the analytically derived envelope curves (see the figure on p. 6). Note that the analytically derived envelope curves are exact, but their derivation is longer and may be rather slow depending on the computer being used. If the reactivity on your computer is too slow, you may consider setting the “**`plot density`**” button to “low”, as explained in Sec. 2 above.



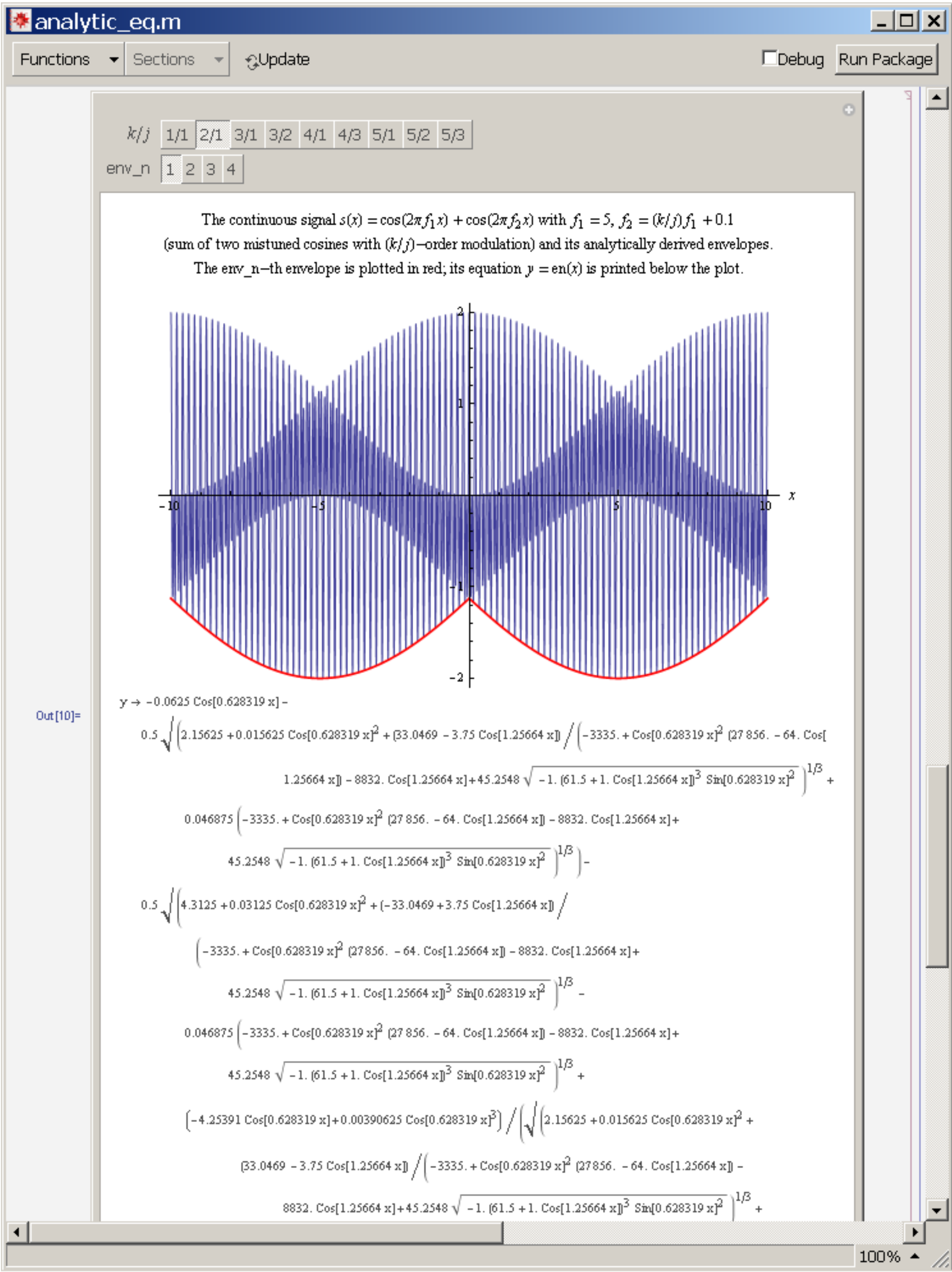
You can also use the slider "f_1" to gradually vary the frequency f_1 between 0.5 and 5, in order to see how f_1 influences the signal's beats and envelopes; the frequency f_2 will be automatically updated to $f_2 = (k/j)f_1 + \delta$ using the current values of f_1, k/j and δ. Note that varying the frequency f_1 only affects the density of the oscillations within the existing envelopes, but the envelope curves themselves remain unchanged:



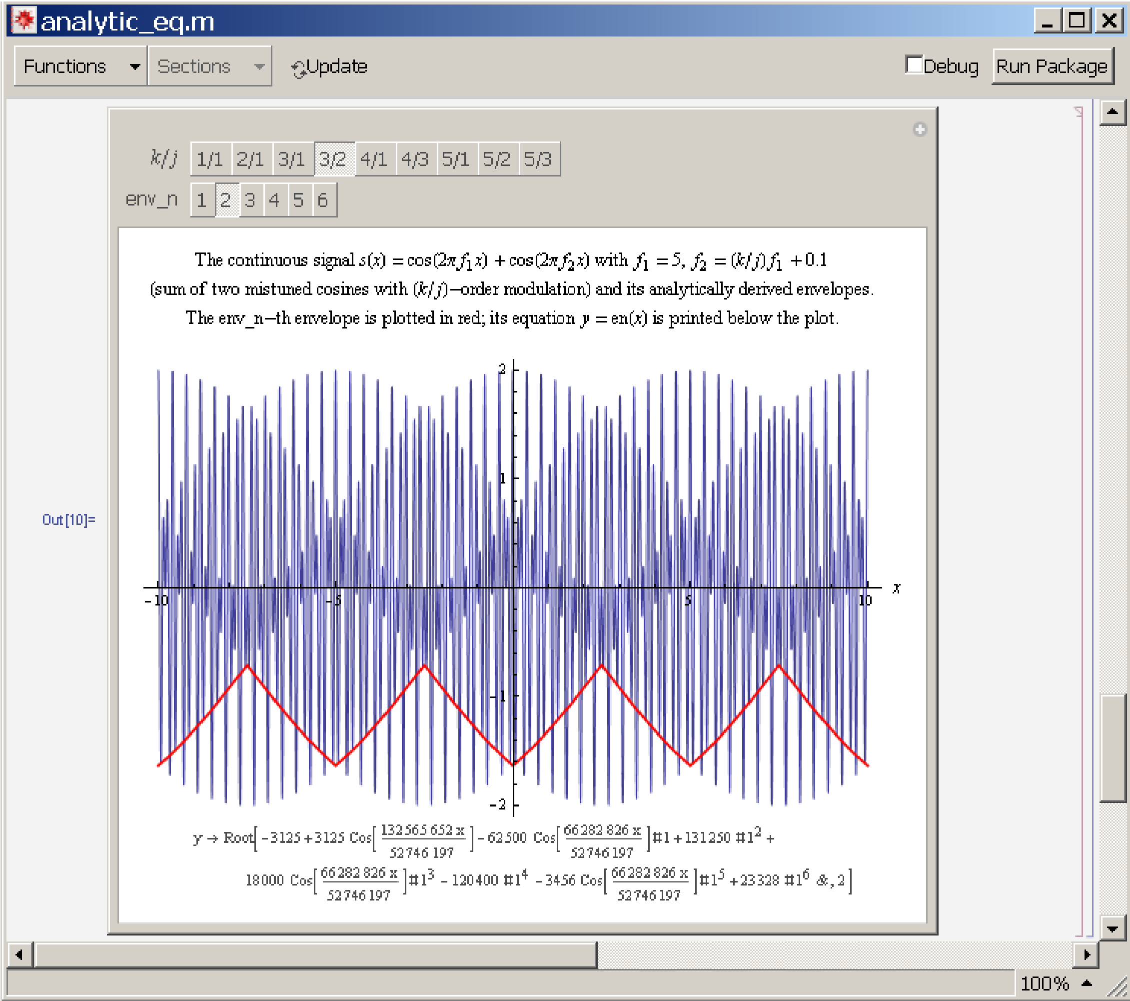
Finally, it is also possible to use the slider "$\boldsymbol{\phi}$" to gradually vary the phase parameter ϕ between 0 and 1. When the "**`phase`**" button is set to "rigid", varying ϕ will result in a rigid shift of the entire function $s(x)$, as shown in the figure on next page. When the "**`phase`**" button is set to "envelope", varying ϕ will only affect the phase of envelope curves, but the phase of the high-frequency oscillations will remain unchanged. And when the "carrier" option is selected, varying ϕ will only affect the high-frequency oscillations, but the envelope curves will remain unchanged.



We now proceed to the application **analytic_eq**. When you enter this application you will get the following display, showing the signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with $f_1 = 5$ and $f_2 = (2/1)f_1 + 0.1$ (see next page). The red curve shows one of the analytically derived envelope curves (which corresponds to the envelope number specified by the "**env_n**" button), and the equation $y = \mathrm{en}(x)$ of this envelope curve is printed below the plot. The envelope number **env_n** is by default 1, but you may select any other value between 1 and $2\max(k,j)$, as suggested by the values displayed next to the "**env_n**" button.



It is interesting to note the complexity of the analytic envelope-curve equations, even in relatively simple cases such as $k/j = 2/1$ (see the figure above). Due to the complexity of the analytic derivation, Mathematica® may sometimes fail to derive the explicit envelope curve equation $y = \mathrm{en}(x)$. In such cases the curve equation will be displayed in the form $y = \mathrm{Root}[\ldots]$, as shown in the figure on next page. That means Mathematica® cannot find analytically the roots of the polynomial equation (5.13) (which can be expected in particular when the polynomial is of order > 4). Nevertheless, in such cases the calculations are done numerically rather than symbolically, so that the envelope curve is still plotted correctly.



Acknowledgement

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