Analytic derivation of the (*k*/*j*)-order modulation envelopes in the sum of two mistuned (co)sine waves

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1. Summary

The beating effects which may occur in the sum of two mistuned cosine (or sine) functions whose frequencies f_1, f_2 satisfy $f_2 \approx (k/j)f_1$ (where k/j is a reduced integer ratio) are known in as "beats of mistuned consonances". They have already been investigated in the 19th century as for their beat frequencies; but interestingly, nothing has been said on the mathematical shapes and properties of these beats. In the present contribution we derive analytically the equations of the modulating envelope curves which accurately outline these beats. Although this may seem to be a straightforward problem, the modulating envelope curves in cases other than $k = j = 1$ turn out to be surprisingly complex. Denoting by n the value $\max(k, j)$, we show that for any k and j we have $2n$ modulating envelope curves, and that the equations of these curves are the $2n$ solutions of a polynomial equation of order $2n$ in y, whose coefficients are periodic functions of x. As expected, the simple case of $k = j = 1$ reduces into the classical cosine sum-to-product identity, where the modulating envelopes are cosinusoidal and have the frequency $(f_1 - f_2)/2$. Due to the complexity of the curve equations in cases other than $k = j = 1$, a simplified approximation using $2n$ cosinusoidal curves is also provided, which can be used when the exact analytic solutions are not really required. Further properties of the beating effects in question are also discussed and illustrated. Similar results hold for the modulation envelopes of mistuned sine waves, too. Applications may concern various fields, including acoustics, optics, etc.

Keywords: sum of mistuned cosines (or sines), beats, modulation envelopes, higher-order modulation, beats of mistuned consonances

2. Introduction

Two instances of a periodic function are said to be *mistuned* if they differ in their frequency. For example, if $f_1 \neq f_2$, then $\cos(2\pi f_1 x)$ and $\cos(2\pi f_2 x)$ are mistuned cosine functions, having frequencies f_1 and f_2.

It is a widely known fact that the sum of two slightly mistuned cosines (or sines) gives a low-frequency beating effect due to modulation (Fig. 1). Similar beating effects may also occur when the two cosine (or sine) frequencies f_1, f_2 satisfy $f_2 \approx (k/j)f_1$ with any small integer ratio k/j.[1] These beating effects are known in acoustics as "beats of mistuned consonances" [1-3]. For k/j ratios other than 1/1 the shapes of the resulting beats may become more complex and intricate than in Fig. 1. But although such phenomena have been illustrated graphically in various sources ([1, pp. 484-487]; [2, p. 48]; [3, p. 168]; [4]; [5, p. 610]), surprisingly no detailed discussion can be found there on the mathematical shapes and properties of these beats, and only their frequency f_b is derived: $f_b = j\delta$, where δ is the difference between the frequencies f_2 and $(k/j)f_1$ (see [3,

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[1] Note that by "integer ratio" we always mean a *reduced* integer fraction. We will tacitly use this convention throughout this paper, even when it is not mentioned explicitly.

pp. 167-168]; [2, pp. 46-49]; [6, pp. 41-42]; and in particular the detailed review provided in [7]).

In the present contribution we proceed in two steps: Denoting $n = \max(k,j)$, we first show that the envelope curves outlining the beats in the sum of two mistuned cosines with frequencies f_1 and $f_2 \approx (k/j)f_1$ can be approximated by $2n$ interlaced cosinusoidal curves (that we will henceforth call by abuse of language *cosinusoidal envelopes*).

These cosinusoidal curves being only an approximation, we then present a mathematical approach based on differential geometry, that allows us to find analytically the precise curve equations of the modulating envelopes for any (k/j)-order modulation. It turns out that the equations $y = \mathrm{en}_1(x)$, … ,$y = \mathrm{en}_{2n}(x)$ of these curves are the $2n$ solutions of a polynomial equation of order $2n$ in y, whose coefficients are periodic functions of x. Because the precise equations are quite complex, the cosinusoidal approximation presented in the first part of the paper can offer a useful simplification when full precision is not needed.

Our work is organized as follows: In Sec. 3 we provide the initial background, starting from the simple, classical (1/1)-order modulation which occurs in the sum of two cosines with frequencies $f_2 \approx f_1$, and proceeding to cases with higher or fractional (k/j)-order modulations. In Sec. 4 we investigate the main properties of these higher-order modulations and show how their envelopes can be approximated by an appropriate set of interlaced cosinusoidal curves. In Sec. 5 we arrive to the main part of our work, the analytic derivation of the exact envelope curves for any given (k/j)-order modulations. As expected, the simple case of $k = j = 1$ reduces into the classical cosine sum-to-product identity, where the modulating envelopes are simply cosinusoidal and have the frequency $(f_1 - f_2)/2$. Similar results are obtained for mistuned sine waves, too. In Sec. 6 we discuss the consequences of our results and suggest possible fields of application, and finally, we present our conclusions in Sec. 7. Several appendices are provided in the supplementary material, which shed some more light on the subjects under discussion.

In addition to the figures which illustrate the discussions in the paper itself, two interactive Mathematica® applications are also provided in the supplementary material, along with their user's guide. Readers are encouraged to use these applications while reading the paper in order to better examine the cases under discussion, and for experimenting with other cases. By manipulating the different parameters one may obtain a vivid graphic demonstration of the beating modulation effects in question and of their dynamic behaviour. The source program of these applications is provided, too.

3. The beating effect in the sum of two mistuned continuous cosines

Consider the sum of two continuous cosines having frequencies f_1 and f_2:

$$s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x) \tag{3.1}$$

When $f_2 = f_1$, the sum $s(x)$ is simply a cosine having the same frequency and twice the amplitude. But how does the sum look like when $f_2 \neq f_1$?

Consider first the simple case where $f_2 \approx f_1$ (or in other words $f_2 = f_1 + \delta$ for some small value δ).

Using the well-known trigonometric sum-to-product identity [8, p. 284]:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{3.2}$$

we can reformulate the sum (3.1) as follows:

$$s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x) = 2\cos(2\pi\tfrac{f_1+f_2}{2}x)\cos(2\pi\tfrac{f_1-f_2}{2}x) \tag{3.3}$$

In our present case, where $f_2 \approx f_1$, the cosine product in the right-hand side of (3.3) corresponds to a modulation effect, where the cosine with the higher frequency $\frac{f_1+f_2}{2}$ represents the carrier and the cosine with the low frequency $\frac{f_1-f_2}{2}$ represents the modulating envelope (see Fig. 1). More precisely, this modulating envelope consists of *two* cosinusoidal curves with the same frequency $f_{\mathrm{en}} = (f_2 - f_1)/2 = \delta/2$ and the same period $p_{\mathrm{en}} = 1/f_{\mathrm{en}} = 2/\delta$, one of these two curves being shifted by half a period, i.e. by $a = 1/\delta$:[2]

$$\begin{aligned} \mathrm{en}_1(x) &= 2\cos(2\pi(\delta/2)x) \\ \mathrm{en}_2(x) &= 2\cos(2\pi(\delta/2)[x+a]) \end{aligned} \tag{3.4}$$

Thanks to this modulation, the sum $s(x)$ of two cosines with close frequencies f_1 and $f_2 = f_1 + \delta$ gives rise to a low-frequency beating effect.

However, the modulation effect is not only limited to the simple case where $f_2 \approx f_1$. It turns out that whenever $f_2 \approx kf_1$ with an integer k (namely $f_2 = kf_1 + \delta$), a new modulation phenomenon appears in the sum $s(x)$. This is a simple generalization of the classical case with $k = 1$ described above, and we call it a *k-th order modulation*. Fig. 2(b) shows a modulation effect that occurs in the cosine sum $s(x)$ when $k = 2$. As we can see by comparing Figs. 2(b) and 1, the second-order modulation with $k = 2$ looks more complex and intricate than the first-order modulation with $k = 1$, and has four modulating envelopes rather than two.[3]

Furthermore, it turns out that a similar phenomenon may also occur whenever $f_2 \approx (k/j)f_1$ (namely $f_2 = (k/j)f_1 + \delta$), where k/j is considered as a reduced integer ratio. We call such cases (k/j)*-order modulation* effects. For example, Fig. 2(c) shows a (3/2)-order modulation effect. These modulation effects can be seen even more clearly in Figs. 3 and 4, which are drawn with higher values of the frequency f_1 and have therefore much denser oscillations. Figs. 3(a)-(c) show the (k/j)-order modulation that occurs in the cosine sum $s(x)$ when $(k/j) = (2/1)$, (3/1) and (3/2), respectively. Similarly, Figs. 4(a)-(c)

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[2] If $f_2 > f_1$, we may prefer to consider $f_2 - f_1$ rather than $f_1 - f_2$. This makes no difference here, since the cosine function is insensitive to the sign of δ.

[3] Obviously, identity (3.3) remains true for all values of f_1 and f_2, but in cases where $f_2 \approx (k/j)f_1$ with $k > j$ or $j > k$ its right-hand side no longer corresponds to the modulation effect we actually see in the cosine sum, and consequently it cannot help us in finding the desired envelope curves. Note in particular that the number of modulating curves in such cases is higher than 2, while identity (3.3) only provides two curves, which are given by Eq. (3.4). See [9] for a more detailed discussion on this point.

illustrate cases with the reciprocal (k/j) values, namely (1/2), (1/3) and (2/3), where $k < j$.

This phenomenon is well known in the field of acoustics. Already in the 19th century it has been shown that when two pure tones of $f_1 = jf$ and $f_2 = kf + \delta$ cycles per second (i.e. f_1 and $f_2 = (k/j)f_1 + \delta$) are sounded together, they give rise to $f_b = j\delta$ beats per second [3, pp. 167-168], [2, pp. 46-49]. A historical account on this result and its various demonstrations and interpretations in the field of acoustics, as well as an extended bibliography, can be found in [7]. These beats are known in acoustics as "beats of mistuned consonances". Nice pictures of such beats are plotted in [1, pp. 484-487]; other pictures, that have been photographed on an oscilloscope, can be found in [4]. But as already mentioned above, nothing is said there concerning the precise shapes of the resulting modulation envelopes or their mathematical equations.

4. A cosinusoidal approximation to the modulating envelope curves

Before we proceed to the formal analytic derivation of the envelope curves in the following section, let us first observe in more detail some of the main properties of the (k/j)-order modulation effects. This will also lead us to a simple cosinusoidal approximation to the modulating envelope curves for any given (k/j)-order modulation.

First of all, we note that when $k+j$ is odd, as in the case of the (2/1)-order modulation (see Fig. 3(a)), the modulation effect we obtain is not vertically symmetric about the x axis as in Figs. 1 and 3(b): The maxima and minima of the beats, i.e. the top ripples and bottom ripples of the envelopes, are not synchronized but rather intermittent. We will henceforth call the synchronized type of beats *even beats*, and the non-synchronized type *odd beats*. We will also use the terms *even* or *odd modulation*.

A further observation concerns the shapes of the modulating envelopes that are obtained when $k/j \neq 1/1$. As shown in Figures 2 and 3, these envelopes may be approximated by interlaced cosinusoidal curves. However, a careful inspection of these figures shows that when $k/j \neq 1/1$, the curves which truly follow the high-frequency oscillations of a mistuned cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ are *not* exactly sinusoidal or cosinusoidal. In particular, their global extrema that are tangent to the x axis are slightly sharper than their global extrema that are tangent to the lines $y = 2$ or $y = -2$ (much like the prolate and oblate extrema of an egg). To clearly illustrate this, notice the red curve that we have overprinted on top of $s(x)$ in Fig. 3(a); this curve is the raised cosine $1 + \cos(2\pi(\delta/2)x)$. Note the significant discrepancy between this cosine and the high-frequency oscillations of $s(x)$. A similar discrepancy can be observed in Figs. 2-4, in all of which we have only traced the cosinusoidal approximations to the true envelope curves.

Note that only in the particular case of $k/j = 1/1$ (see Figs. 1 and 2(a)) our cosinusoidal envelopes indeed coincide with the true modulating curves, and accurately envelop the signal $s(x)$ without any discrepancies. This happens thanks to the cosinusoidal modulation provided by the right-hand side of identity (3.3).

As we can see in our figures (or using the interactive applications provided in the supplementary material), the curves in question have some further interesting properties:

(1) These curves are disposed like a double braid – one braid above the horizontal x axis, and one braid below it. Each of these two braids consists of k interlaced curves (if $k > j$) or j interlaced curves (if $k < j$). Furthermore, these two braids may be either symmetric with respect to the x axis (like in Fig. 3(b)) or not (like in Figs. 3(a),(c)): this simply depends on the even or odd nature of the beats in the signal $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$. Note, however, that in the particular case of $k = j = 1$ (Fig. 1(a)) there are no braids at all.

(2) The number of modulating envelopes is $2k$ if $k > j$ (k envelope curves above the x axis and k envelope curves below it; see Fig. 3), and $2j$ if $k < j$ (j envelope curves above the x axis and j envelope curves below it; see Fig. 4). In the simple case of $k = j = 1$ the number of modulating envelopes is 2, as given by Eq. (3.4); see Fig. 1. Therefore, if we denote $n = \max(k,j)$, we see that the number of envelope curves is precisely $2n$ in all cases.

(3) The frequency and period of these envelope curves are given as follows:

$$
\begin{array}{lll}
\text{if } k > j: & f_{\text{en}} = (1/k) f_{\text{b}} = \frac{j}{k}\delta, & p_{\text{en}} = k p_{\text{b}} = k/(j\delta) \\
\text{if } k < j: & f_{\text{en}} = (1/j) f_{\text{b}} = \delta, & p_{\text{en}} = j p_{\text{b}} = 1/\delta \\
\text{if } k = j: & f_{\text{en}} = (f_2 - f_1)/2 = \delta/2, & p_{\text{en}} = 1/f_{\text{en}} = 2/\delta
\end{array}
\tag{4.1}
$$

(see Figs. 2-4). On the other hand, as we remember from acoustics (see the end of the previous section), the period and frequency of the beats that are generated by a (k/j)-order modulation are always given by:

$$f_{\text{b}} = j\delta, \qquad p_{\text{b}} = 1/(j\delta) \tag{4.2}$$

The periods p_{en} and p_{b} are clearly shown in Fig. 2 for several (k/j)-order modulation cases.

(4) It follows therefore that for any given k/j the period of the (k/j)-order modulating envelopes depends on δ only, and it does not depend on the frequency f_1. Modifying the frequency f_1 will only affect the density of the high-frequency oscillations within the existing envelopes, but the envelope curves themselves will remain unchanged. This can be clearly visualized using the interactive application "analytic_env" in the supplementary material.

(5) Looking carefully at the curve $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ (see for example Fig. 2(b)), we see that it oscillates intermittently between the $2n$ individual braid curves, touching them always in the same order. Thus, we can say that the curve $s(x)$ is modulated by a set of interlaced braid curves, jumping intermittently from curve to curve. This gives an intricate modulation with $2n$ modulating envelopes (n envelope curves above the x axis and n envelope curves below it), where $n = \max(k,j)$. See also Sec. C.1 and Fig. C1 in Appendix C.

Using the notation $n = \max(k,j)$ and Eqs. (4.1) for the frequency f_{en} and period p_{en} of the braid curves, the equations of our $2n$ cosinusoidal approximate envelope curves can be expressed as follows (see, for example, Figs. 3 and 4):

In cases with $k > j$ (Fig. 3) the n interlaced cosinusoidal curves above the x axis are:

$$
\begin{aligned}
\mathrm{en}_1(x) &= \cos(2\pi f_{\mathrm{en}} x) + 1 \\
\mathrm{en}_2(x) &= \cos(2\pi f_{\mathrm{en}}[x + a]) + 1 \\
&\;\;\cdots \\
\mathrm{en}_n(x) &= \cos(2\pi f_{\mathrm{en}}[x + (n-1)a]) + 1
\end{aligned}
\tag{4.3}
$$

and the n interlaced cosinusoidal curves below the x axis are:

$$
\begin{aligned}
\mathrm{en}_{n+1}(x) &= \cos(2\pi f_{\mathrm{en}}[x + (a/2)]) - 1 \\
\mathrm{en}_{n+2}(x) &= \cos(2\pi f_{\mathrm{en}}[x + (a/2) + a]) - 1 \\
&\;\;\cdots \\
\mathrm{en}_{2n}(x) &= \cos(2\pi f_{\mathrm{en}}[x + (a/2) + (n-1)a]) - 1
\end{aligned}
\tag{4.4}
$$

where the envelope shift a is given by:

$$a = p_{\mathrm{en}}(j/k) \tag{4.5}$$

In cases where $k < j$ (Fig. 4) these equations remain unchanged except that the roles of k and j are reversed, i.e. k replaces j and j replaces k everywhere in Eqs. (4.3)-(4.5). And for the simple case with $k = j = 1$ we already have the exact cosinusoidal envelope curves given by Eqs. (3.4).

Note that the braid curves shown in Figures 2-5 are plotted using this cosinusoidal approximation; the true envelope curves, having an egg-shaped nature, will be analytically derived in the next section.

5. Analytic derivation of the exact envelope curve equations

In order to find the exact mathematical equations of our envelope curves, we adopt here the following approach: In differential geometry, the envelope of a family of plane curves $F(x,y,c) = 0$ with a single parameter c is defined as a curve (or a set of several curves) which is tangent to each member of the family at some point [10, p. 559]; [11, Sec. 3.5]. If such an envelope exists, its equation $g(x,y) = 0$ is obtained by eliminating the parameter c from the two equations:[4]

$$F(x,y,c) = 0, \qquad \frac{\partial F(x,y,c)}{\partial c} = 0 \tag{5.1}$$

A simple example illustrating this method is provided in Appendix A. In order to use this approach, we have to identify a family of curves $F(x,y,c) = 0$ that remain tangent to our desired envelope curves while the parameter c is being varied. One way to define

[4] The results thus obtained may also include some singularity points or curves, that may be eliminated by adding here some further conditions. These additional conditions may depend on the precise definition of such singularities, and therefore they may vary between different references [12, pp. 255-260], [13, pp. 53-59]. We prefer not to add here any further formal conditions, and to simply check the nature of the resulting curves and see whether or not they meet our expectations.

such a curve family consists of choosing the frequency f_1 as our varying parameter c. This choice stands to reason in view of point (4) in the previous section.[5] For the sake of convenience we will continue using the letter c to denote our parameter, even though it will actually refer to the frequency f_1.

Our family of curves $F(x,y,c)$ for the general (k/j)-order case is therefore given by:

$$y = \cos(2\pi cx) + \cos(2\pi[(k/j)c + \delta]x) \tag{5.2}$$

where the parameter c varies continuously. We thus obtain from Eqs. (5.1):

$$F(x,y,c) = \cos(2\pi cx) + \cos(2\pi[(k/j)c + \delta]x) - y = 0 \tag{5.3}$$

$$\frac{\partial}{\partial c}F(x,y,c) = -2\pi x \sin(2\pi cx) - 2\pi(k/j)x \sin(2\pi[(k/j)c + \delta]x) = 0 \tag{5.4}$$

Using the well-known trigonometric identities for the sine and cosine of a sum these equations can be reformulated as follows:

$$\begin{aligned}&\cos(2\pi cx) + \\ &\quad \cos(2\pi(k/j)cx)\cos(2\pi\delta x) - \sin(2\pi(k/j)cx)\sin(2\pi\delta x) = y\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\sin(2\pi cx) + \\ &\quad (k/j)\sin(2\pi(k/j)cx)\cos(2\pi\delta x) + (k/j)\cos(2\pi(k/j)cx)\sin(2\pi\delta x) = 0\end{aligned} \tag{5.6}$$

In order to obtain the equation of our desired envelope curves (either in the explicit form $y = f(x)$ or in the implicit form $g(x,y) = 0$) we need to eliminate the parameter c from Eqs. (5.5) and (5.6). This elimination would give us the desired relationship between x and y alone. However, this task seems to be rather hopeless due to the complexity of our two equations. And yet, there exists in the mathematical literature a clue that can significantly simplify things (see [14, p. 68]; also taken up in the Wikipedia [15]).

It turns out that the task of finding the envelope equation becomes much simpler when $F(x,y,c)$ is a polynomial in c. In this case Eqs. (5.1) simply mean that c is a multiple root of the polynomial $F(x,y,c)$, since at the point c both $F(x,y,c)$ and its derivative $\frac{\partial}{\partial c}F(x,y,c)$ have the value 0. As we know from algebra, c is a multiple root of a polynomial $P(c)$ *if and only if* the discriminant of the polynomial vanishes at the point c [16]. Therefore, when $F(x,y,c)$ is a polynomial in c the equation of the envelope can be simply found by setting the discriminant of the polynomial $F(x,y,c)$ to 0, which is often much simpler than eliminating the parameter c from the two equations (5.1). Note in particular that in such cases the second equation (that of the derivative) is no longer used in the derivation process.

Now, although in our present case (see Eq. (5.3)) $F(x,y,c)$ is not a polynomial in the parameter c, it can be reduced to such a polynomial by using a judicious substitution. Following [14, p. 69], let us introduce a new parameter t that is obtained from our original parameter c as follows:

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[5] Note that other choices of the parameter c are also possible. Such an alternative approach, based on a different family of curves with a different parameter, is provided in Appendix B. Both approaches finally give the very same envelope curves.

$$t = e^{2\pi i cx/j} \tag{5.7}$$

where j is the denominator of our integer ratio k/j. We therefore have:

$$\begin{aligned} t^k &= e^{2\pi i(k/j)cx} = \cos(2\pi(k/j)cx) + i\,\sin(2\pi(k/j)cx) \\ 1/t^k &= e^{-2\pi i(k/j)cx} = \cos(2\pi(k/j)cx) - i\,\sin(2\pi(k/j)cx) \end{aligned} \tag{5.8}$$

By adding or subtracting these equations we obtain:

$$\begin{aligned} t^k + 1/t^k &= 2\cos(2\pi(k/j)cx) \\ t^k - 1/t^k &= 2i\sin(2\pi(k/j)cx) \end{aligned} \tag{5.9}$$

or in other words:

$$\begin{aligned} \cos(2\pi(k/j)cx) &= t^k/2 + 1/(2t^k) \\ \sin(2\pi(k/j)cx) &= t^k/(2i) - 1/(2it^k) \end{aligned} \tag{5.10}$$

and in particular, when $k = j$:

$$\cos(2\pi cx) = t^j/2 + 1/(2t^j)$$

Substituting these expressions in $F(x,y,c) = 0$ (i.e. in Eq. (5.5)) we obtain:

$$\begin{aligned} &[t^j/2 + 1/(2t^j)] + \\ &\quad \cos(2\pi\delta x)\,[t^k/2 + 1/(2t^k)] - \sin(2\pi\delta x)\,[t^k/(2i) - 1/(2it^k)] = y \end{aligned} \tag{5.11}$$

Now, in order to obtain a polynomial in t, we must get rid of the negative powers of t (the denominators).

Thus, assuming that $k \geq j$ (so that $k - j \geq 0$), we multiply all terms by $2it^k$, which gives:

$$\begin{aligned} &[it^{k+j} + it^{k-j}] + \\ &\quad \cos(2\pi\delta x)\,[it^{2k} + i] - \sin(2\pi\delta x)\,[t^{2k} - 1] = 2iyt^k \end{aligned} \tag{5.12}$$

By multiplying both sides by $-i$ and rearranging the terms in decreasing order of powers we finally obtain the following polynomial in t:

$$(\cos(2\pi\delta x) + i\sin(2\pi\delta x))t^{2k} + t^{k+j} - 2yt^k + t^{k-j} + (\cos(2\pi\delta x) - i\sin(2\pi\delta x)) = 0$$

namely:

$$At^{2k} + Bt^{k+j} + Ct^k + Dt^{k-j} + E = 0 \tag{5.13}$$

with the coefficients:

$$\begin{aligned} A &= e^{2\pi i\delta x} \\ B &= 1 \\ C &= -2y \\ D &= 1 \\ E &= e^{-2\pi i\delta x} \end{aligned} \tag{5.14}$$

Having thus obtained a polynomial in t, the equation of our envelope curves can be found by simply setting its discriminant Δ to 0, which is indeed much simpler than eliminating c from Eqs. (5.5) and (5.6). Note that we do not need to solve Eq. (5.13) itself; all that we need here is to solve the discriminant equation $\Delta = 0$, which is a polynomial equation of order $2k$ in y, whose own coefficients are functions of x.[6] This polynomial equation has therefore $2k$ solutions of the form $y = f_1(x), \ldots, y = f_{2k}(x)$, which are precisely our desired envelope curves.[7] The parameter t itself does not appear in these solutions (just as the parameter c disappears when using the standard parameter-elimination method).

Let us illustrate this with two simple examples:

Example 5.1 (*the simple case of* $(k/j) = (1/1)$):

This is, in fact, the classical case of first-order modulation, where $f_2 = f_1 + \delta$ (see Fig. 1). In this simple case our polynomial (5.13) is a quadratic:

$$(A + B)t^2 + Ct + (D + E) = 0$$

Its discriminant Δ is given by:

$$\Delta = C^2 - 4(A + B)(D + E)$$

Therefore in this simple case $\Delta = 0$ gives, using (5.14), the following quadratic polynomial equation in y:

$$4y^2 - 4(e^{2\pi i\delta x} + 1)(1 + e^{-2\pi i\delta x}) = 0$$

i.e. $$y^2 - e^{2\pi i\delta x} - e^{-2\pi i\delta x} - 2 = 0$$

and using the identity $e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ we get:

$$y^2 = 2 + 2\cos(2\pi\delta x)$$

We therefore obtain the two envelope curves:

$$y = \pm\sqrt{2 + 2\cos(2\pi\delta x)}$$

which can be reduced using the trigonometric identity $\cos(\alpha/2) = \pm\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos\alpha}$ into:

$$y = \pm 2\left|\cos(2\pi[\delta/2]x)\right| \tag{5.15}$$

This gives indeed the same curves as our envelopes $\mathrm{en}_1(x)$ and $\mathrm{en}_2(x)$ in Eq. (3.4), although expressed here in a different way, i.e. as an upper envelope curve above the x axis and a lower envelope curve below the x axis. ■

Example 5.2 (*the case of* $(k/j) = (2/1)$):

This case corresponds to a second-order modulation, where $f_2 = 2f_1 + \delta$ (see Fig. 3(a)). In this case our polynomial (5.13) is a quartic whose coefficients are given by (5.14):

$$At^4 + Bt^3 + Ct^2 + Dt + E = 0$$

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[6] Note that y appears in (5.13) in the coefficient C, while x appears there in the coefficients A and E – but only in their exponential parts. Therefore Δ is a polynomial in y, but not in x.

[7] When the solutions of the polynomial equation (5.13) cannot be found explicitly (which may be expected when the polynomial is of order > 4), it is still possible to calculate them numerically.

The discriminant Δ of a quartic equation has 16 terms and is given by (see, for example, [17, p. 405] or [18, pp. 257-258]):[8]

$$\begin{aligned}\Delta = {} & 16AC^4E - 4AC^3D^2 - 4B^2C^3E \\ & + B^2C^2D^2 - 128A^2C^2E^2 - 80ABC^2DE \\ & + 18ABCD^3 + 18B^3CDE + 144A^2CD^2E + 144AB^2CE^2 \\ & - 27A^2D^4 - 27B^4E^2 + 256A^3E^3 - 192A^2BDE^2 - 6AB^2D^2E - 4B^3D^3\end{aligned}$$

In this case $\Delta = 0$ gives, using (5.14), the following 4-th order polynomial equation in y:

$$\begin{aligned}\Delta = {} & 256y^4 + 32(e^{2\pi i\delta x} + e^{-2\pi i\delta x})y^3 - 828y^2 \\ & - 324(e^{2\pi i\delta x} + e^{-2\pi i\delta x})y - 27(e^{4\pi i\delta x} + e^{-4\pi i\delta x}) + 54 = 0\end{aligned}$$

Using the identity $e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ we get:

$$128y^4 + 32\cos(2\pi\delta x)y^3 - 414y^2 - 324\cos(2\pi\delta x)y - 27\cos(4\pi\delta x) + 27 = 0$$

which finally gives, using the triginometric identity $\cos(2\alpha) = 1 - 2\sin^2\alpha$:

$$64y^4 + 16\cos(2\pi\delta x)y^3 - 207y^2 - 162\cos(2\pi\delta x)y + 27\sin^2(2\pi\delta x) = 0 \qquad (5.16)$$

The 4 solutions of this 4-th order polynomial equation in y, $y = \mathrm{en}_1(x)$, … ,$y = \mathrm{en}_4(x)$, give indeed the 4 envelope curves of the second-order modulation with $f_2 = 2f_1 + \delta$. These 4 curves are plotted in Fig. 5. Notice how these 4 curves perfectly follow the high-frequency oscillations of the cosine sum, as shown in row (a) of this figure. Interestingly, once again, the envelope curves thus obtained are not represented as 4 smooth interlaced “braid curves” similar to our cosinusoidal approximations, but rather as 4 horizontal “slices” through these curves, which are continuous but not smooth (due to their cusp points). Both presentations are of course completely equivalent. ■

So far our development starting from Eq. (5.12) was based on the assumption that $k \geq j$. If, however, $k < j$, the power of t^{k-j} in Eq. (5.13) becomes negative, meaning that we no longer have a polynomial in t. In order to get rid of this negative power we simply have to multiply all terms of Eq. (5.13) by t^{j-k}. This gives:

$$At^{k+j} + Bt^{2j} + Ct^{j} + \mathrm{D} + Et^{j-k} = 0 \qquad (5.17)$$

By rearranging the terms in decreasing order of powers we finally obtain the following polynomial in t:

$$A't^{2j} + B't^{j+k} + C't^{j} + D't^{j-k} + E' = 0 \qquad (5.18)$$

whose coefficients A', B', C', D' and E' are given by:

$$\begin{aligned} & A' = B = 1 \\ & B' = A = e^{2\pi i\delta x} \\ & C' = C = -2y \\ & D' = E = e^{-2\pi i\delta x} \\ & E' = D = 1 \end{aligned} \qquad (5.19)$$

[8] For convenience we have rearranged the terms of the discriminant in decreasing order of powers of C, which is the only coefficient in (5.13) that includes the variable y.

Since this is again a polynomial in t, the equation of our envelope curves can be found by setting its discriminant Δ to 0, in the same manner as we have done above in the case of $k \geq j$.

Example 5.3 (*the case of* $(k/j) = (1/2)$):

This case corresponds to the situation shown in Fig. 4(a), where $f_2 = (1/2)f_1 + \delta$. In this case we obtain from Eq. (5.18) the same quartic polynomial as in Example 5.2, but this time its coefficients are given by (5.19) rather than by (5.14):

$$A't^4 + B't^3 + C't^2 + D't + E' = 0$$

In this case $\Delta = 0$ gives, using (5.19):

$$\begin{aligned}\Delta = 256y^4 &+ 32(e^{2\pi i\delta(2x)} + e^{-2\pi i\delta(2x)})y^3 - 828y^2 \\ &- 324(e^{2\pi i\delta(2x)} + e^{-2\pi i\delta(2x)})y - 27(e^{4\pi i\delta(2x)} + e^{-4\pi i\delta(2x)}) + 54 = 0\end{aligned}$$

and by virtue of the identity $e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ we finally get:

$$64y^4 + 16\cos(2\pi\delta(2x))y^3 - 207y^2 - 162\cos(2\pi\delta(2x))y + 27\sin^2(2\pi\delta(2x)) = 0 \qquad (5.20)$$

This is the same 4-th order polynomial equation in y as in Example 5.2, with $2x$ replacing x everywhere. And indeed, as we can see in Fig. 6, the 4 envelope curves we get in this case are simply a horizontally 2-fold denser version of the 4 envelope curves of Example 5.2 (i.e. they are shrinked by factor 2 along the x axis). ■

Note that the number of modulating envelopes we thus obtain is $2k$ if $k > j$, $2j$ if $k < j$, and 2 in the simple case of $k = j = 1$. Thus, if we denote $n = \max(k,j)$, we see that the number of envelopes is $2n$ in all cases. This indeed agrees with our earlier observations in Sec. 4 (see point (2) there).

We have therefore proved the following general theorem:

Theorem 5.1 (*the sum of two mistuned cosine functions*):

Suppose we are given two continuous cosine functions with frequencies f_1 and f_2, respectively:

$$g_1(x) = \cos(2\pi f_1 x) \qquad \text{and} \qquad g_2(x) = \cos(2\pi f_2 x),$$

where:

$$f_2 = \tfrac{k}{j} f_1 + \delta$$

(δ being positive or negative, k/j being a reduced integer ratio, and $n = \max(k,j)$). Then the sum of the two given cosines, $s(x) = g_1(x) + g_2(x)$, is modulated by $2n$ periodic envelope curves $y = f_1(x), \ldots, y = f_{2n}(x)$, that are given by the $2n$ solutions of the discriminant equation $\Delta = 0$ of order $2n$ belonging to the polynomial equation (5.13) if $k \geq j$, or to the polynomial equation (5.18) if $k < j$. ■

Remark 5.1: (*the equivalence of cases with $k < j$ and cases with $k > j$*):

As illustrated by Examples 5.2 and 5.3 above, (k/j)-order cases with $k < j$ are in fact completely equivalent to their counterparts with $k > j$, so that we do not need to treat them separately. The reason is that if $k < j$ we can simply rewrite $f_2 = (k/j)f_1 + \delta$ as

$(j/k)f_2 = f_1 + (j/k)\delta$, and hence as $f_1 = (j/k)f_2 - (j/k)\delta$. By renaming $f_1' = f_2$, $f_2' = f_1$, $k' = j$, $j' = k$ and $\delta' = -(j/k)\delta$ we see that our original case of $f_2 = (k/j)f_1 + \delta$ with $k < j$ is fully equivalent to the case of $f_2' = (k'/j')f_1' + \delta'$ with $k' > j'$. It follows therefore that any (k/j)-order case with $k < j$ will have the same envelope curves as its reciprocal counterpart of order (j/k), except that δ will be replaced by $\delta' = -(j/k)\delta$. This is clearly illustrated by Example 5.3, which shows that the envelope curves of the (1/2)-order case are identical to those of the reciprocal (2/1)-case except that their frequency is $\delta' = -(2/1)\delta$ rather than δ (compare also Fig. 6 with Fig. 5). ■

Very similar results can be obtained for sine waves, too, since sine waves only differ from their cosine counterparts in their phase. It turns out that Eq. (5.13) remains unchanged, and only the coefficients (5.14) vary: Thus, for the sum of two mistuned sines we obtain: $A = e^{2\pi i\delta x}$, $B = 1$, $C = -2iy$, $D = -1$, $E = -e^{-2\pi i\delta x}$; and for the difference of two sines $\sin(2\pi f_2 x) - \sin(2\pi f_1 x)$ (which may sometimes be more pertinent; see, for example, [9, Sec. 4]) we obtain: $A = e^{2\pi i\delta x}$, $B = -1$, $C = -2iy$, $D = 1$, $E = -e^{-2\pi i\delta x}$. Similarly, for the difference $\sin(2\pi f_1 x) - \sin(2\pi f_2 x)$ we obtain: $A = -e^{2\pi i\delta x}$, $B = 1$, $C = -2iy$, $D = -1$, $E = e^{-2\pi i\delta x}$.

6. Discussion

Having obtained analytically the exact envelope curve equations for any (k/j)-order case, let us now discuss some consequences of our results.

A first interesting consequence of Theorem 5.1 is that the resulting envelope curves do not depend on the frequency f_1, but only on the difference δ. Note that δ appears explicitly in the coefficients (5.14), while f_1 does not appear at all in Eqs. (5.13) and (5.14). This result perfectly agrees with our expectations (see point (4) in Sec. 4). Indeed, as we can observe using the enclosed interactive application "analytic_env", modifying the frequency f_1 will only affect the density of the oscillations within the existing envelopes, but the envelope curves themselves will remain unchanged.

Another consequence of Theorem 5.1 is that the resulting curve equations $y = f_i(x)$ are trigonometric functions of x, or more precisely, algebraic combinations of $\cos(2\pi\delta x)$ in various powers (positive, negative, or fractional, i.e. roots). The reason is that the discriminant Δ of the polynomial equation (5.13) is symmetric in terms of the polynomial's coefficients A and E. This is guaranteed by the symmetry properties of the discriminant of a polynomial [19, p. 205], [20, p. 481]. This symmetry implies that in the polynomial equation in y, $\Delta = 0$, the coefficients A and E can always be cancelled out using the identity $e^{2\pi i\delta x} + e^{-2\pi i\delta x} = 2\cos(2\pi\delta x)$, giving instead a simple cosine (this is clearly illustrated in the developments provided in Examples 5.1-5.3 above).

Another interesting remark concerns the shapes of the plotted envelope curves. Based on our observations in Sec. 4, we expected the (k/j)-order modulation envelopes to have the form of smooth interlaced cosine-like curves that are disposed like a double braid, one braid above the x axis and one braid below it. However, it is interesting to note that the envelope equations we have obtained analytically are not plotted as $2n$ interlaced smooth cosine-like curves, but rather as $2n$ horizontal "slices" through these braid curves. Thus, rather than being smooth and rounded each of the plotted curves has sharp edges or cusp points (compare Figs. 5-7 with Figs. 2-4, which were plotted using the cosinusoidal approximations). Why does this happen?

Just as we have noted in the case of Example 5.1, this is simply a different but completely equivalent representation of the very same envelope curves. This happens due to the different possible ways to apply the ± signs of the root (see Eq. (5.15)). The straightforward way to apply the ± signs gives indeed the expression $\pm|\cos(\ldots)|$, which is interpreted as two cuspidal curves (half cosines), one above the x axis and one below it. But a different assignment of the ± curve segments could just as well give us the two expected smooth cosinusoidal curves rather than the equivalent cuspidal-curve representation of the envelopes. Both representations finally give the very same set of curves, and only the assignment of each of the $2n$ curve segments at any point along the x axis to one of the $2n$ envelopes (namely, which of the $2n$ curve segments at any point x belongs to which envelope) is different.[9] If required, it is always possible to reshuffle or "unwrap" the curve segments into the braid-curve configuration, as we have done for example in Fig. C1 of Appendix C. This representation of the exact envelope curves may sometimes be advantageous, as illustrated in Sec. C.1 of Appendix C.

Another interesting remark concerns the case of the (3/1)-order modulation, which is shown in Fig. 7. The envelope curves $\mathrm{en}_2(x)$ and $\mathrm{en}_4(x)$, which are plotted in our figure in green and magenta, surprisingly resemble triangular waves consisting of straight line segments. However, a close inspection reveals that each of these line segments is indeed slightly curved. Similar effects may occur in higher-order modulations, too (try, for example, the (4/1)-order case using the enclosed interactive applications).

One further remark is due at this point concerning the complexity of the exact curve equations. Just for the sake of illustration, the equation $y = \mathrm{en}_1(x)$ of the top (red) envelope curve plotted in Fig. 5 is given by the truly stunning expression shown in Fig. C2 of Appendix C. Other curve equations can be obtained and plotted using the Mathematica® application "analytic_eq" that is provided in the supplementary material. Due to the high complexity of the exact curve equations in all cases other than $k = j = 1$, the cosinusoidal approximation presented in Sec. 4 can offer a useful simplification when full precision is not needed. It also has the advantage of being directly expressed in terms of the "braid curve" representation; so no further efforts are needed for unwrapping the plots.

Finally, it should be mentioned that the cosinusoidal braid-curve approximation presented in Sec. 4 is not the only possible approximation to the exact envelope curves. A different cosinusoidal approximation has been presented in [9], which is somewhat less faithful to the exact envelopes, but on the other hand has the advantage of lending itself more easily to the Fourier-series formulation; this opens the way to the generalization of that approximation to (k/j)-order beats between any mistuned periodic waves (such as square waves, triangular waves, etc.).

[9] This is also equivalent to the assignment of colours to the different curve segments; compare Figs. 5-7 with Figs. 2-4.

Our findings in the present work may have both theoretical and practical significance. First of all, they have an interesting theoretical value – as an extension of the classical (1/1)-order modulating envelopes given by Eq. (3.3) to any (*k/j*)-order modulations. But our results may be also used in various applications, in the field of acoustics or in other scientific disciplines, for modelling the beating effects between two mistuned instances of a periodic signal. This could be useful, for example, in the research of the human auditory system. Beating effects between mistuned waves also have important applications in optics [22, pp. 295-296]. Other possible fields of application may include any situations in which two mistuned periodic signals may occur, for example in radioastronomy, geophysics, communications, radar applications, etc.

7. Conclusions

In the present contribution we study the beating modulation effects which may occur in the continuous world in the sum of two mistuned cosine (or sine) functions whose frequencies f_1, f_2 satisfy $f_2 \approx (k/j)f_1$, where k/j is a reduced integer ratio. This is in fact a generalization of the simple (1/1)-order modulation effect, the well-known beating phenomenon which occurs in the sum of two cosines (or sines) having close frequencies $f_2 \approx f_1$.

We first provide a simple approximation to the (*k/j*)-order modulation envelopes by means of $2n = 2\max(k,j)$ interlaced cosinusoidal curves, that are disposed like a double braid: one braid consisting of n interlaced cosines above the horizontal x axis, and another such braid below the x axis. But our main result consists of the analytic derivation of the general (*k/j*)-order modulation envelopes in the sum of two mistuned cosine (or sine) waves. Denoting by n the value $\max(k,j)$, we show that for any k and j we have $2n$ modulating envelope curves, and that the equations $y = \mathrm{en}_1(x)$, … ,$y = \mathrm{en}_{2n}(x)$ of these curves are the $2n$ solutions of a polynomial equation of order $2n$ in y, whose coefficients are periodic functions of x. As expected, the simple case of $k = j = 1$ reduces into the classical sum-to-product identity, where the modulating envelopes are simply cosinusoidal and have the frequency $(f_1 - f_2)/2$. In cases where the solutions of the polynomial equation cannot be expressed in the explicit form $y = \mathrm{en}(x)$, they still can be found numerically, and hence the exact envelope curves can still be plotted accurately.

Of course, the simple approximation by means of cosinusoidal curves is no longer needed once the exact analytic expressions have been derived. But due to the complexity of the analytic expressions, this qualitative approximation can still be advantageous when high precision is not required.

Finally, it should be noted that although only the one-dimensional case has been considered here, our results can be also extended to two or higher dimensional settings. A first step in the characterization of the (1/1)-order modulation effect in the two dimensional setting can be found in [21, pp. 302-306].

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Figure 1: The classical (1/1)-order modulation effect in the sum of two slightly mistuned cosines. (a) The cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies $f_1 = 3$ and $f_2 = f_1 + \delta$, where $\delta = 0.1$. (b) The two interlaced cosinusoidal envelope curves (plotted here in red and green) that outline the modulation effect shown in (a). Their equations are given by Eq. (3.4); their frequency is $f_{\mathrm{en}} = (f_1 - f_1)/2 = \delta/2 = 0.05$ and their period is $p_{\mathrm{en}} = 2/\delta = 20$. (c) The carrier of the modulation effect shown in (a) is given by the cosine $\cos(2\pi \frac{f_1 + f_2}{2} x)$. For the sake of completeness, we show in (d) the two original cosines themselves: The cosine $\cos(2\pi f_1 x)$ is plotted with a continuous line (left), while the cosine $\cos(2\pi f_2 x)$ is dotted (right); both curves are overprinted in the central part of row (d) to allow a better understanding of their sum in (a).

The (1/1)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:

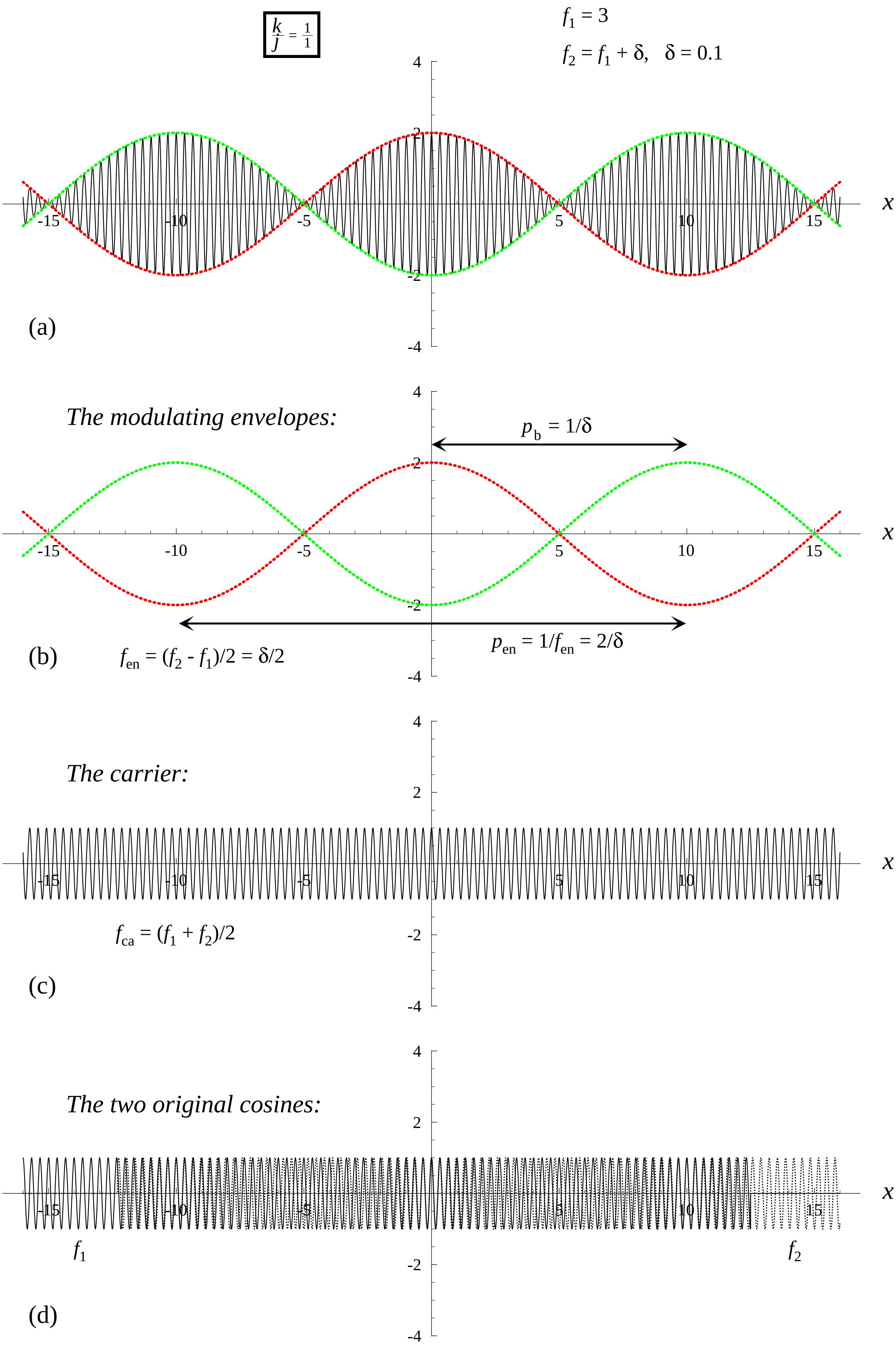


Figure 2: Illustration of various (k/j)-order modulation effects in the sum of two mistuned cosines. In all cases the modulating envelopes are represented by interlaced cosinusoidal curves (see the coloured curves). Each row shows the cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = \frac{k}{j} f_1 + \delta$, where: (a) $k/j = 1/1$; (b) $k/j = 2/1$; (c) $k/j = 3/2$. The beat period p_{b} and the cosinusoidal envelope period p_{en} are clearly indicated in each case. The cosinusoidal curves only give the exact modulating envelopes in the (1/1)-order case (row (a)), thanks to identity (3.3). In all other cases the cosinusoidal curves are only an approximation to the true envelope curves: Note the visible discrepancies between the coloured curves and the small peaks of the highly oscillating cosine sum $s(x)$. Although the cosine sum $s(x)$ itself is not necessarily periodic (note that the high-frequency oscillations inside consecutive beats are not necessarily identical), the (k/j)-order modulating envelopes *are* periodic.

Representation by cosinusoidal envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:

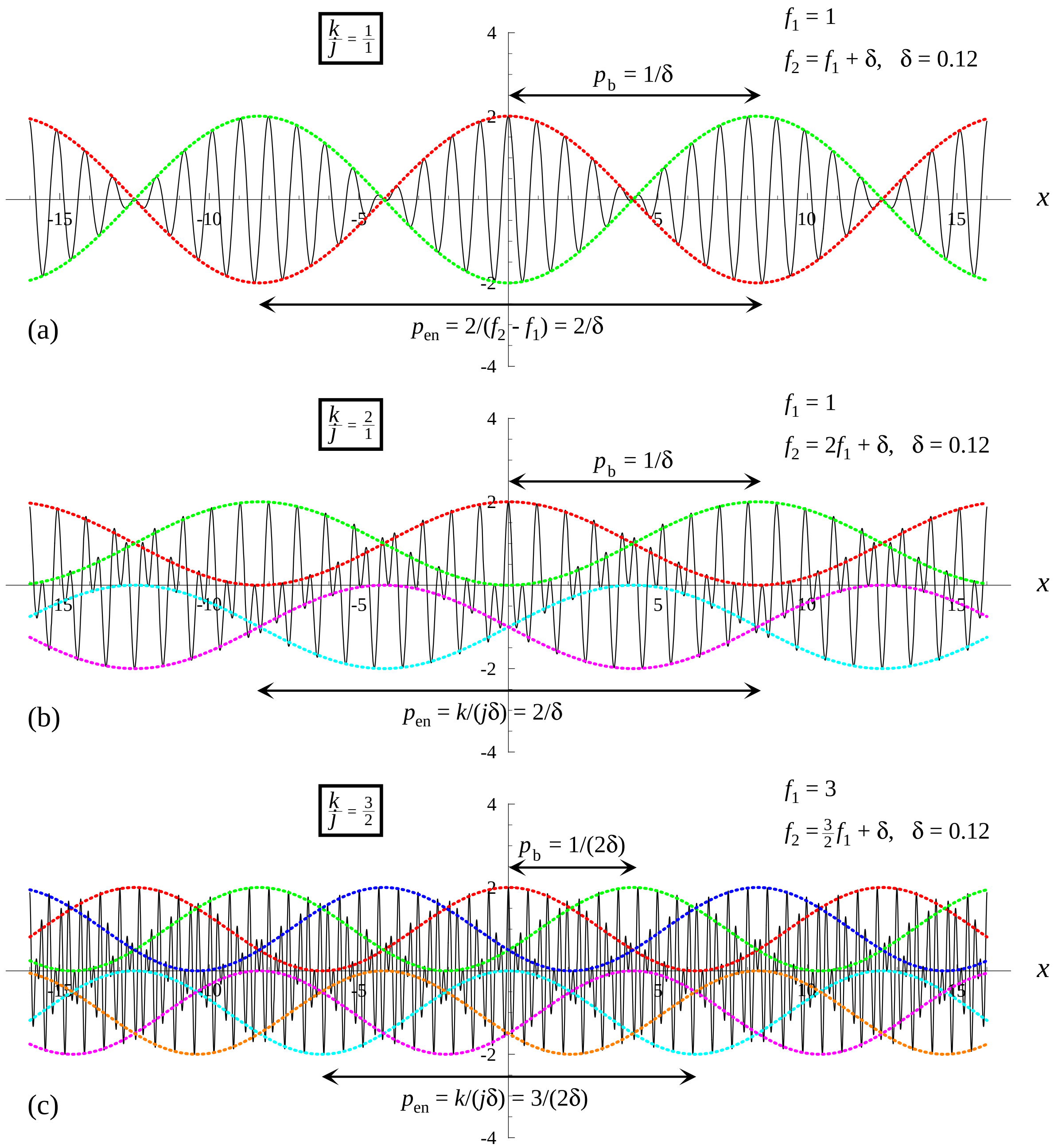


Figure 3: The double-braid disposition of the envelope curves which outline the high-frequency oscillations in various (k/j)-order modulation effects with $k > j$. Each row shows the cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = \frac{k}{j} f_1 + \delta$, where: (a) $k/j = 2/1$; (b) $k/j = 3/1$; (c) $k/j = 3/2$. All the braid curves are plotted using the cosinusoidal approximation (see the coloured curves); the inaccuracy of this approximation is particularly noticeable in rows (a) and (c). The highly oscillating function $s(x)$ is denser here than in Fig. 2 due to the higher value of the frequency f_1. Note that in row (c) we have doubled the frequency f_1 in order to keep the same oscillation density as in rows (a) and (b).

Representation by cosinusoidal envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:

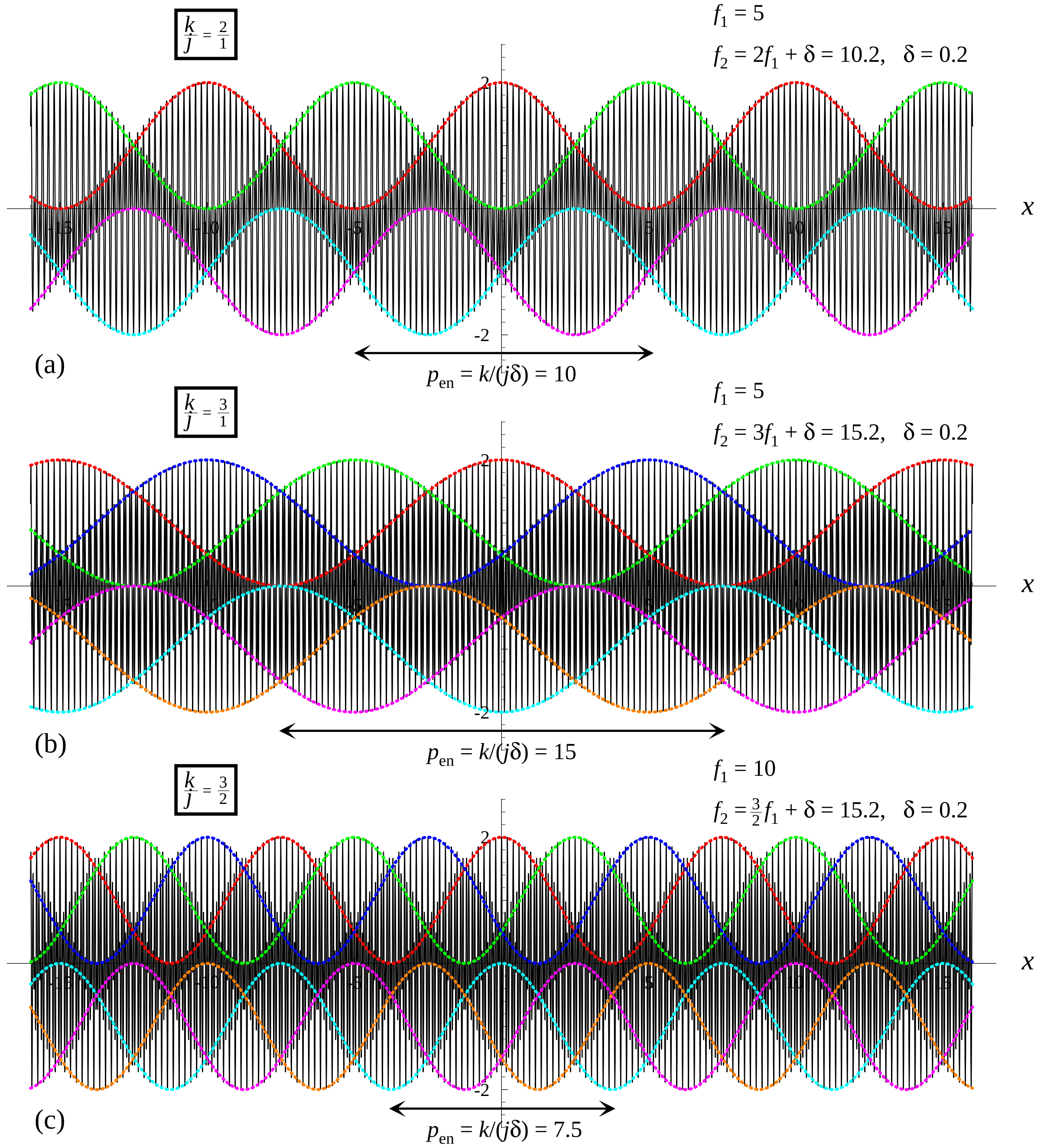


Figure 4: The double-braid disposition of the envelope curves which outline the high-frequency oscillations extrema in various (k/j)-order modulation effects with $k < j$. Each row shows the cosine sum $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = \frac{k}{j} f_1 + \delta$, where: (a) $k/j = 1/2$; (b) $k/j = 1/3$; (c) $k/j = 2/3$. Note that the (k/j) values in this figure are the reciprocals of those used in Fig. 3. Here, too, all the braid curves are plotted using the cosinusoidal approximation (see the coloured curves); the inaccuracy of this approximation is particularly visible in rows (a) and (c). Note that we have adapted the frequency f_1 in each of the rows so as to keep the same oscillation density as in Fig. 3.

Representation by cosinusoidal envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:

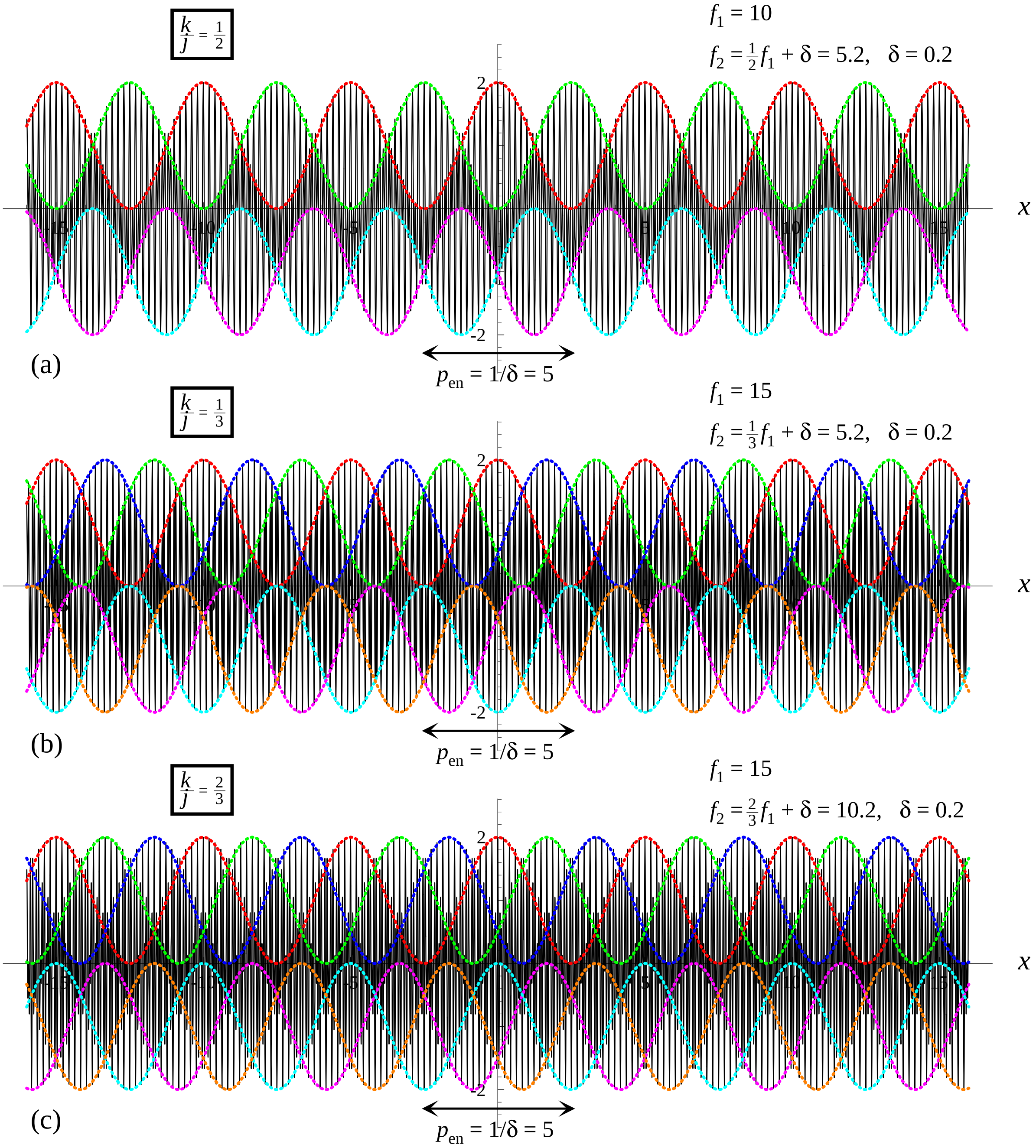
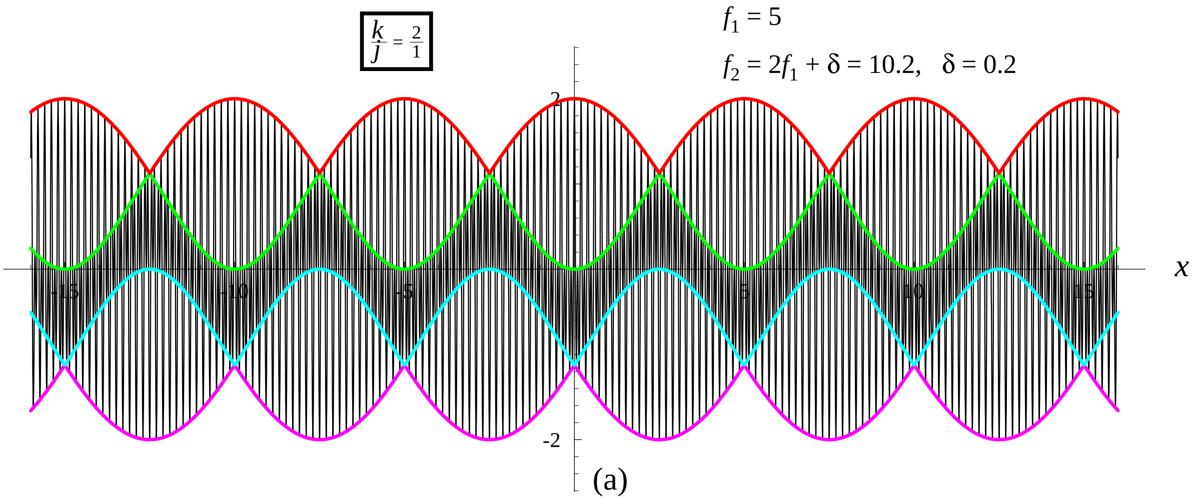


Figure 5: The (2/1)-order modulation effect in the sum of two mistuned cosines $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = (2/1)f_1 + \delta$, and its analytically obtained envelope curves $y = \mathrm{en}_1(x), \ldots, y = \mathrm{en}_4(x)$ (see Example 5.2). Each of the 4 analytically obtained curves is plotted in a different colour. Unlike the cosinusoidal envelope approximations (see Fig. 3(a)) the true envelope curves are clearly "egg-shaped" (i.e. sharper where they touch the x axis and flatter where they touch the lines $y = 2$ and $y = -2$), as we expected in Sec. 4. Note in row (a) the perfect agreement between these analytically obtained envelopes and the high-frequency oscillations of the sum $s(x)$. Interestingly, as shown in row (b), the analytically obtained envelopes do not follow the "double-braid" interlaced layout of our cosinusoidal approximate curves: Instead, they consist of 4 horizontal "slices" through these braid curves.

The true analytically obtained envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:



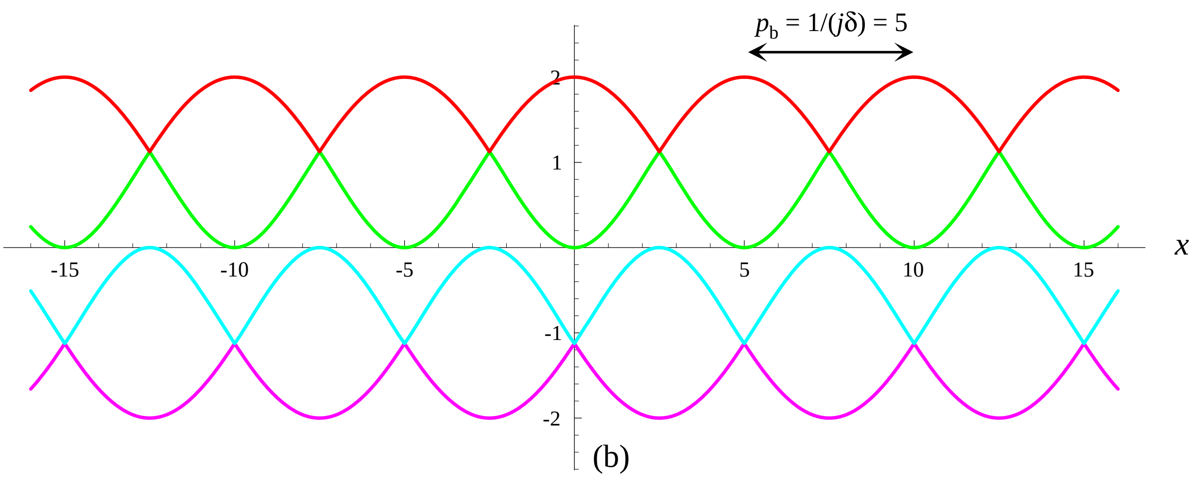


Figure 6: The (1/2)-order modulation effect in the sum of two mistuned cosines $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = (1/2)f_1 + \delta$, and its analytically obtained envelope curves $y = \mathrm{en}_1(x), \ldots, y = \mathrm{en}_4(x)$ (see Example 5.3). Each of the 4 analytically obtained curves is plotted in a different colour. Unlike the cosinusoidal envelope approximations (see Fig. 4(a)) the true envelope curves are clearly "egg-shaped", as expected. Note in row (a) the perfect agreement between these analytically obtained envelopes and the high-frequency oscillations of the sum $s(x)$. As shown in Example 5.3, these envelope curves are simply a 2-fold denser version of the envelope curves in Fig. 5.

The true analytically obtained envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with:

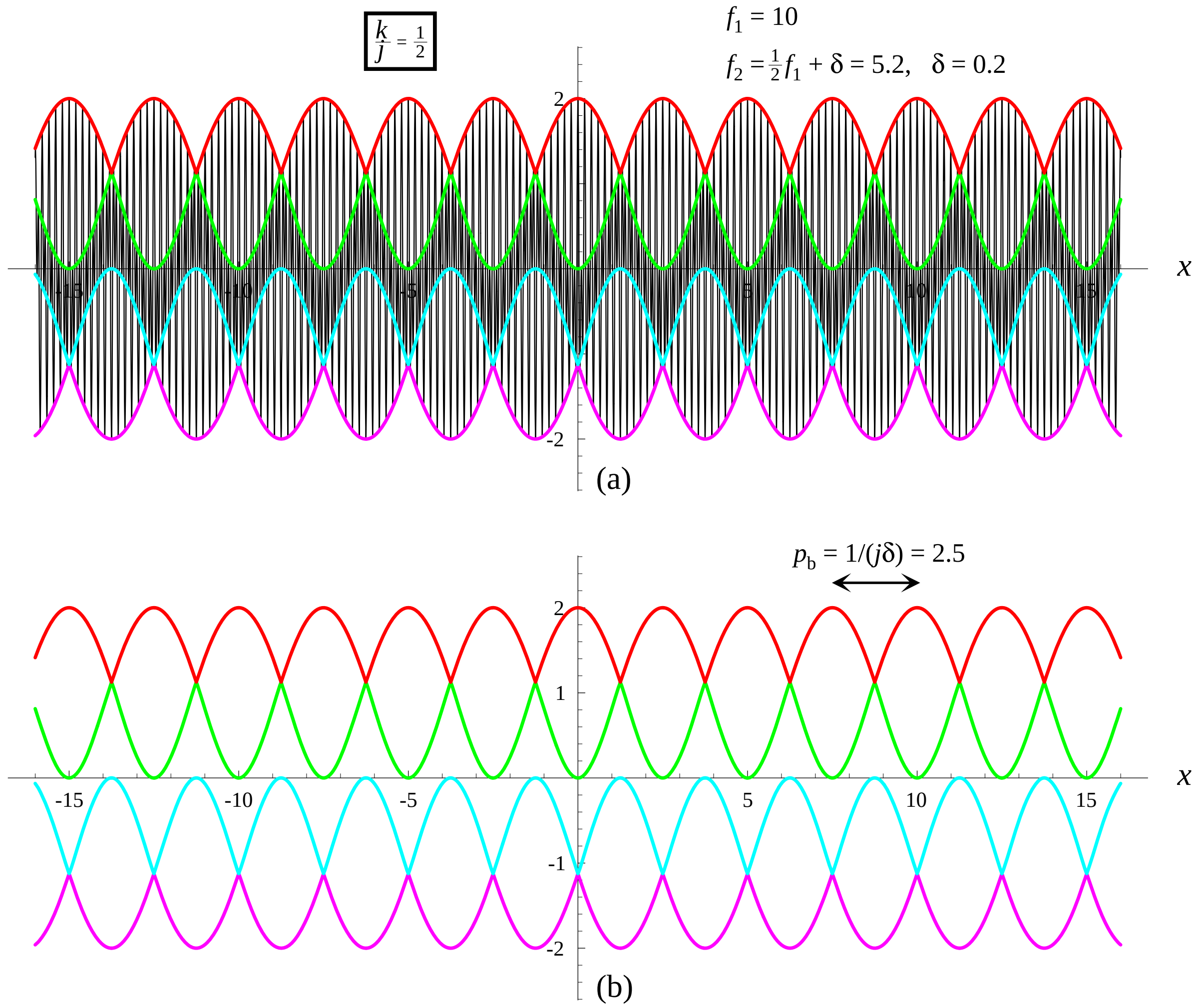


Figure 7: The (3/1)-order modulation effect in the sum of two mistuned cosines $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with frequencies f_1 and $f_2 = (3/1)f_1 + \delta$, and its 6 analytically obtained envelope curves. Each of these curves is plotted in a different colour. Note in row (a) the perfect agreement between these analytically obtained envelopes and the high-frequency oscillations of the sum $s(x)$ (compare with the cosinusoidal envelope approximations shown in Fig. 3(b)). Note that the green and magenta curves $\mathrm{en}_2(x)$ and $\mathrm{en}_4(x)$ which closely resemble triangular waves do not really consist of straight-line segments: In fact, each of their line segments is very slightly curved.

The true analytically obtained envelope curves of
the (k/j)-order modulation effect in $s(x) = \cos(2\pi f_1 x) + \cos(2\pi f_2 x)$ with: