A PRIORI ERROR ANALYSIS OF THE FINITE ELEMENT HETEROGENENOUS MULTISCALE METHOD FOR THE WAVE EQUATION IN HETEROGENEOUS MEDIA OVER LONG TIME

ASSYR ABDULLE*‡ AND TIMOTHÉE POUCHON†‡

Abstract. A fully discrete a priori analysis of the finite element heterogenenous multiscale method (FE-HMM) introduced in [A. Abdulle, M. Grote, C. Stohrer, Multiscale Model. Simul. 2014] for the wave equation with highly oscillatory coefficients over long time is presented. A sharp a priori convergence rate for the numerical method is derived for long time intervals. The effective model over long time is a Boussinesq-type equation that has been shown to approximate the one-dimensional multiscale wave equation with ε-periodic coefficients up to time $\mathcal{O}(\varepsilon^{-2})$ in [Lamacz, Math. Models Methods Appl. Sci., 2011]. In this paper we also revisit this result by deriving and analysing a family of effective Boussinesq-type equations for the approximation of the multiscale wave equation that depends on the normalization chosen for certain micro functions used to define the macroscopic models.

Key words. a priori error analysis, multiscale method, heterogeneous media, effective equations, wave equation, long time behavior, dispersive waves

AMS subject classifications. 65M60, 65N30, (74Q10, 74Q15, 35L05)

1. Introduction.

The wave equation in heterogenous media is used in a number of scientific and engineering applications such as seismic inversion, medical imaging or the manufacture of composite materials. When the typical size of the heterogeneities (denoted here by ε) is much smaller than the scale of interest, standard numerical methods such as the finite difference method (FDM) or the finite element method (FEM) become prohibitively expensive as scale resolution is needed for the mesh sizes. In such situations, homogenization theory (see [12, 25, 17, 29]) provides a systematic procedure to derive an effective equation for the highly oscillatory wave equation, whose solution no longer oscillates on the ε-scale (see [13] for the specific case of the wave equation).

However in practice, no explicit solutions for the effective equation is available (usually obtained by the so-called G-limit of a sequence of differential operators [13]) hence multiscale numerical methods are required. The method considered in this paper is based on the heterogeneous multiscale methods (HMM) [21, 3, 4]. In this framework, a macroscopic effective equation is computed on a macroscopic grid that does not resolve the fine scale oscillation. The data of the effective equation are recovered "on the fly" by solving micro problems on sampling domains with size proportional to ε, hence at a cost independent of ε. A finite difference scheme based on the HMM (FD-HMM) was proposed by Engquist, Holst and Runborg [22] and a finite element heterogeneous multiscale method (FE-HMM) was later proposed in [5] together with a fully discrete analysis of the method. We mention also upscaling methods that do not rely on scale separation [33, 30, 14] but on coarse multiscale basis functions obtained by solving local problems on each macro element of the computational domain. In contrast to homogenized based methods the computational cost to obtain the macro model is no longer independent of ε.

Classical homogenization describes well the propagation of waves in a strongly heterogeneous medium for short time. The true oscillatory solution however deviates from the classical homogenization limit with increasing time as dispersive effects develop. To capture these longer time dispersive effects Santosa and Symes [32] proposed a higher order homogenized model that is a Boussinesq type equation which unfortunately is ill-posed. Based on this model, it was nevertheless shown in [23] that the FD-HMM can be modified to capture the long time dispersive effects of the fine scale problem using time dependent micro-solvers and space-time sampling domains with growing sizes as $\varepsilon \to 0$. The scheme based on an effective flux recovery needs well-prepared initial data and a high order micro-macro coupling. We also note that a regularization step is needed as

*assyr.abdulle@epfl.ch
†timothee.pouchon@epfl.ch
‡ANMC, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland

the limiting equation is an ill-posed problem. In [9] the numerical error arising from the approximation of the effective data in the FD-HMM has been analysed. We finally refer to [10] for recent development of this method.

Recently, Lamacz [27] derived a "good" Boussinesq equation for one-dimensional wave problem with periodic oscillatory coefficients and proved that the heterogeneous wave solution can be approximated with error $\mathcal{O}(\varepsilon)$ (in an $\mathrm{L}^\infty(\mathrm{L}^2)$ norm) up to time $\mathcal{O}(\varepsilon^{-2})$. The results in [27] and the construction of an appropriate correction of the L^2 scalar product of the FE-HMM, triggered the development of a new multiscale method called FE-HMM-L [6] that captures a well-posed effective equation with time-independent micro problems with a computational cost similar to that of the FE-HMM. In particular, only the micro functions needed for the classical elliptic homogenization problem are needed for the correction of the L^2 scalar product. This method was shown to be well-posed and consistent with the classical homogenization problem for the wave equation. Numerical evidence did indicate that the method is able to capture the long-term dispersive effects.

In this paper we aim at analysing the FE-HMM-L over long time. While the FE-HMM-L can be applied to multidimensional problems, we restrict our analysis to one-dimensional problems as this case already contains challenging issues. Let $T^\varepsilon = \varepsilon^{-2}T$ and consider $u^\varepsilon : [0,T^\varepsilon]\times\Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\partial_t^2 u^\varepsilon(t,x) = \partial_x\big(a^\varepsilon(x)\partial_x u^\varepsilon(t,x)\big) && \text{in } (0,T^\varepsilon]\times\Omega,\\
&x\mapsto u^\varepsilon(t,x)\ \ \Omega\text{-periodic} && \text{in } [0,T^\varepsilon],\\
&u^\varepsilon(0,x) = g^0(x),\quad \partial_t u^\varepsilon(0,x) = g^1(x) && \text{in } \Omega,
\end{aligned}\tag{1.1}$$

where for $Y\subset\mathbb{R}$, $a^\varepsilon(x) = a\big(\frac{x}{\varepsilon}\big) = a(y)$ is Y-periodic in y and g^0, g^1 are given initial conditions. We assume that the domain $\Omega\subset\mathbb{R}$ is a union of unit cells of size $\varepsilon|Y|$. The effective model derived in [27] is of the form

$$\partial_t^2\bar{u} = a^0\partial_x^2\bar{u} + \varepsilon^2 b^0\partial_x^2\partial_t^2\bar{u}.\tag{1.2}$$

It is well-known that the effective coefficient a^0 is obtained by $a^0 = \fint_Y a(y)(1+\partial\chi)\mathrm{d}y$, where χ is the periodic solution of an elliptic boundary value problem in Y. Next, while the effective correction b^0 in Lamacz's paper was obtained by a cascade of cell problems, it was simply obtained in the FE-HMM-L scheme by an L^2 average of χ^2 (see [6]). It is not difficult to see (as we show in this paper) that the two definitions are equivalent. But this rises then the following question : as the function χ is periodic we need to choose a normalization when solving the corresponding periodic boundary value problem. While this normalization does not matter for the definition of a^0 it determines the value of b^0. Precisely the value $\fint_Y \chi\mathrm{d}y = 0$ was chosen in [6] and for this normalization we obtain the effective equation (1.2) from [27]. In the first main result of this paper we show that *any* normalization yields an effective equation whose solution approximate the true solution with error $\mathcal{O}(\varepsilon)$ up to time $\mathcal{O}(\varepsilon^{-2})$. We obtain the following family of effective equations

$$\partial_t^2\tilde{u} = a^0\partial_x^2\tilde{u} - \varepsilon^2\big(\tilde{a}^2\partial_x^4\tilde{u} - \tilde{b}^0\partial_x^2\partial_t^2\tilde{u}\big).\tag{1.3}$$

Except for its theoretical interest, the effective equation with $\tilde{a}^2\neq 0$ seems of little practical use for numerical approximation. This is true for one-dimensional problems. It turns out that for multidimensional problem, fourth order differential operator cannot be avoided (already for the standard normalization) [19, 20] and there is some hope that the additional liberty from the above generalization might allow to construct efficient numerical schemes based on the FE-HMM-L. The generalization of the family of effective equations and its interplay with the construction of the FE-HMM-L for multidimensional wave problems over long time is currently under study and out of the scope of this paper.

For the second main result of the paper, we allow for locally periodic coefficients $a^\varepsilon(x) = a(x,x/\varepsilon) = a(x,y)$ Ω-periodic in x and Y-periodic in y and a source term f in equation (1.1). For the usual normalization (denoting by $\bar{u}$ the solution of the corresponding Boussinesq equation) we prove that

$$\begin{gathered}
\|\partial_t(\bar{u}-u_H)\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} + \|\bar{u}-u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^1(\Omega))} \le C\big(h/\varepsilon^2\big)^2 + \mathrm{e}^{\mathrm{FE}}_{\mathrm{H}^1},\\
\|\bar{u}-u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \le C\big(h/\varepsilon^2\big)^2 + \mathrm{e}^{\mathrm{FE}}_{\mathrm{L}^2},
\end{gathered}\tag{1.4}$$

where u_H is the solution obtained by the FE-HMM-L relying only on the equation (1.2) and h is the size of the mesh used in the sampling domains. The terms $\mathrm{e}_{\mathrm{H}^1}^{\mathrm{FEM}}$ and $\mathrm{e}_{\mathrm{L}^2}^{\mathrm{FEM}}$ are the standard error estimates between $\bar{u}$ and its FEM approximation on a mesh of size H independent of ε in the H^1 and L^2 norm, respectively. The constants C in (1.4) are independent of H, h, ε. Notice that the rate $(h/\varepsilon^2)^2$ is different than the corresponding rate for elliptic or parabolic problems for which $(h/\varepsilon)^2$ has been derived [1, 8]. The above rate for the long time integration of the wave equation is sharp as shown in our numerical experiments. Combining this result with the first main result we also obtain (for periodic coefficients)

$$\|u^\varepsilon - u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C\Big(\varepsilon + \big(h/\varepsilon^2\big)^2\Big) + \mathrm{e}_{\mathrm{L}^2}^{\mathrm{FE}}, \tag{1.5}$$

where C is again independent of H, h, ε. We note that this result is the first a priori error estimate for the numerical solution over long time $[0, \varepsilon^{-2}T]$ for an HMM or a numerical homogenization type method.

The rest of the paper is organized as follows. First, we describe the family of effective equations and prove the error estimate between its elements and u^ε in Section 2. In Section 3, we define the FE-HMM-L and perform the a priori analysis. Finally, numerical results illustrating our theoretical finding and the performance of the FE-HMM-L are presented in Section 4.

Notations and definitions. Let $\mathcal{O} \subset \mathbb{R}$ be a bounded open set and define the standard space of square integrable functions $\mathrm{L}^2(\mathcal{O})$ and the Sobolev space $\mathrm{H}^k(\mathcal{O}) = \{v \in \mathrm{L}^2(\mathcal{O}) : \partial_x^m v \in \mathrm{L}^2(\mathcal{O})\ 1 \leq m \leq k\}$. Equipped with their usual inner products, $\mathrm{L}^2(\mathcal{O})$ and $\mathrm{H}^k(\mathcal{O})$ are Hilbert spaces. We define the mean of an integrable function $v : \mathcal{O} \to \mathbb{R}$ as $\langle v \rangle_{\mathcal{O}} = |\mathcal{O}|^{-1} \int_{\mathcal{O}} v(x)\,\mathrm{d}x$. We define the quotient space $\mathcal{L}^2(\mathcal{O}) = \mathrm{L}^2(\mathcal{O})/\mathbb{R}$ and denote by a bold face letter $\boldsymbol{v}$ the equivalence class in $\mathrm{L}^2(\mathcal{O})/\mathbb{R}$. Equipped with the inner product

$$\big(\boldsymbol{v}, \boldsymbol{w}\big)_{\mathcal{L}^2(\mathcal{O})} = \big(v - \langle v \rangle_{\mathcal{O}}, w - \langle w \rangle_{\mathcal{O}}\big)_{\mathrm{L}^2(\mathcal{O})} = \big(v, w\big)_{\mathrm{L}^2(\mathcal{O})} - |\mathcal{O}|\langle v \rangle_{\mathcal{O}} \langle w \rangle_{\mathcal{O}} \quad \forall v \in \boldsymbol{v}, w \in \boldsymbol{w},$$

$\mathcal{L}^2(\mathcal{O})$ is a Hilbert space. The equivalence class of $v \in \mathrm{L}^2(\mathcal{O})$ is also noted $\big[v\big]$. Denote by $\mathcal{C}_{\mathrm{per}}^\infty(\mathcal{O})$ the $\mathcal{O}$-periodic functions of $\mathcal{C}^\infty(\mathcal{O})$ and define the space $\mathrm{H}_{\mathrm{per}}^1(\mathcal{O})$ as the closure of $\mathcal{C}_{\mathrm{per}}^\infty(\mathcal{O})$ for the H^1 norm. We define the quotient space $\mathcal{W}_{\mathrm{per}}(\mathcal{O}) = \mathrm{H}_{\mathrm{per}}^1(\mathcal{O})/\mathbb{R}$ and for $\boldsymbol{v} \in \mathcal{W}_{\mathrm{per}}(\mathcal{O})$ we define $\partial \boldsymbol{v} = \partial v \in \mathrm{L}^2(\mathcal{O})$ for all $v \in \boldsymbol{v}$. Equipped with the inner product $(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{W}_{\mathrm{per}}(\mathcal{O})} = (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{L}^2(\mathcal{O})} + (\partial \boldsymbol{v}, \partial \boldsymbol{w})_{\mathrm{L}^2(\mathcal{O})}$, $\mathcal{W}_{\mathrm{per}}(\mathcal{O})$ is a Hilbert space. Note that thanks to the Poincaré–Wirtinger inequality, $\boldsymbol{v} \mapsto \|\partial \boldsymbol{v}\|_{\mathrm{L}^2(\mathcal{O})}$ is a norm on $\mathcal{W}_{\mathrm{per}}(\mathcal{O})$, equivalent to $\|\cdot\|_{\mathcal{W}_{\mathrm{per}}(\mathcal{O})}$. The dual space $\mathcal{W}_{\mathrm{per}}^*(\mathcal{O})$ is characterized as follows : for $F \in \mathcal{W}_{\mathrm{per}}^*(\mathcal{O})$, there exists $\boldsymbol{f^0} \in \mathcal{L}^2(\mathcal{O})$, $f^1 \in \mathrm{L}^2(\mathcal{O})$ such that

$$\big\langle F, \boldsymbol{v} \big\rangle_{\mathcal{W}_{\mathrm{per}}^*(\mathcal{O}), \mathcal{W}_{\mathrm{per}}(\mathcal{O})} = \big(\boldsymbol{f^0}, \boldsymbol{v}\big)_{\mathcal{L}^2(\mathcal{O})} + \big(f^1, \partial \boldsymbol{v}\big)_{\mathrm{L}^2(\mathcal{O})}. \tag{1.6}$$

Furthermore, $\|F\|_{\mathcal{W}_{\mathrm{per}}^*(\mathcal{O})} = \inf\{\|\boldsymbol{f^0}\|_{\mathcal{L}^2(\mathcal{O})} + \|f^1\|_{\mathrm{L}^2(\mathcal{O})}\}$, where the infimum is taken over all $\boldsymbol{f^0} \in \mathcal{L}^2(\mathcal{O})$, $f^1 \in \mathrm{L}^2(\mathcal{O})$ satisfying (1.6). From characterization (1.6) we verify that a functional of $[\mathrm{H}_{\mathrm{per}}^1(\mathcal{O})]^*$ given by $w \mapsto (f^0, w)_{\mathrm{L}^2(\mathcal{O})} + (f^1, \partial w)_{\mathrm{L}^2(\mathcal{O})}$ for some $f^0, f^1 \in \mathrm{L}^2(\mathcal{O})$, belongs to $\mathcal{W}_{\mathrm{per}}^*(\mathcal{O})$ if and only if

$$(f^0, 1)_{\mathrm{L}^2(\mathcal{O})} = 0, \tag{1.7}$$

or equivalently f^0 has zero mean. Define $\mathrm{L}_0^2(\mathcal{O})$ (resp. $\mathrm{W}_{\mathrm{per}}(\mathcal{O})$) as the set constituted with the zero mean representative of $\mathcal{L}^2(\Omega)$ (resp. of $\mathcal{W}_{\mathrm{per}}(\mathcal{O})$). Equipped with the standard L^2 inner product (resp. H^1), $\mathrm{L}_0^2(\mathcal{O})$ is a Hilbert space (resp. $\mathrm{W}_{\mathrm{per}}(\mathcal{O})$). Note that the following embeddings are dense $\mathrm{W}_{\mathrm{per}}(\mathcal{O}) \subset \mathrm{L}_0^2(\mathcal{O}) \subset \mathrm{W}_{\mathrm{per}}^*(\mathcal{O})$.

For a Banach space X and $p \in [0, \infty)$, $\mathrm{L}^p(0,T;X)$ is the space of functions $v : [0,T] \to X$ such that $\|v\|_{\mathrm{L}^p(0,T;X)} = \big(\int_0^T \|v(t)\|_X^p\,\mathrm{d}t\big)^{1/p} < \infty$. The definition is similar for $p = \infty$, with the L^∞ norm in time. To simplify the notation we will often use the shorthand notation $\|\cdot\|_{\mathrm{L}^p}, \|\cdot\|_{\mathrm{L}^p(X)}$ instead of $\|\cdot\|_{\mathrm{L}^p(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{L}^p(0,T;X)}$ respectively.

2. A family of effective equations for the wave equation over long time. We present here our first main result. We consider problem (1.1), assume that a^ε belongs to $\mathrm{L}^\infty(\Omega)$ and that it is uniformly elliptic and bounded, i.e. there exists $\lambda, \Lambda > 0$ such that

$$\lambda \leq a^\varepsilon(x) \leq \Lambda \quad \text{ for a.e. } x \in \Omega \quad \forall \varepsilon > 0. \tag{2.1}$$

For the well-posedness of problem (1.1), we refer to Lions and Magenes in [28]. A detailed proof may be found in [24]. If $g^0 \in \mathrm{W}_{\mathrm{per}}(\Omega)$, $g^1 \in \mathrm{L}^2_0(\Omega)$ then there exists a unique weak solution $u^\varepsilon \in \mathrm{L}^2(0,T^\varepsilon;\mathrm{W}_{\mathrm{per}}(\Omega))$ with $\partial_t u^\varepsilon \in \mathrm{L}^2(0,T^\varepsilon;\mathrm{L}^2_0(\Omega))$ and $\partial_t^2 u^\varepsilon \in \mathrm{L}^2(0,T^\varepsilon;\mathrm{W}^*_{\mathrm{per}}(\Omega))$. We note that u^ε is proved to be even more regular, $u^\varepsilon \in \mathcal{C}([0,T^\varepsilon];\mathrm{W}_{\mathrm{per}}(\Omega))$ and $\partial_t u^\varepsilon \in \mathcal{C}([0,T^\varepsilon];\mathrm{L}^2_0(\Omega))$.

2.1. Convergence result for the family of effective equations. We describe next a family of equations whose solutions approximates well u^ε in the $\mathrm{L}^\infty(\mathrm{L}^2)$ norm over long times $\mathcal{O}(\varepsilon^{-2})$.

As mentioned in the introduction, an a priori error estimate between u^ε and the solution of a Boussinesq equation over times $\mathcal{O}(\varepsilon^{-2})$ was first proved by Lamacz in [27] for one particular effective equation among the family of effective equations presented below. The proof of our result is inspired by the techniques developed by Lamacz with several changes. First, the proof in [27] is done for an unbounded domain $\Omega = \mathbb{R}$, while we present here a proof in a bounded (periodic) domain. Second, in [27] the effective coefficient is defined through three auxiliary problems whereas the definition of our effective coefficients require one single cell-problem. Third, we define an adaptation of the effective solution with correction up to order ε^4, while in [27] it goes up to order ε^5. Fourth, we require the effective solution to satisfy $\tilde{u} \in \mathrm{L}^\infty(\mathrm{H}^5(\Omega)), \partial_t\tilde{u} \in \mathrm{L}^\infty(\mathrm{H}^4(\Omega)), \partial_t^2\tilde{u} \in \mathrm{L}^\infty(\mathrm{H}^3(\Omega))$ and the proof in [27] requires $\tilde{u} \in \mathrm{L}^\infty(\mathrm{H}^7(\Omega)), \partial_t^2\tilde{u} \in \mathrm{L}^\infty(\mathrm{H}^6(\Omega))$. Finally, in [27] well-prepared initial conditions are needed, but an intermediate error estimate in the $\mathrm{L}^\infty(\mathrm{H}^1)$ norm is obtained. In our proof, we avoid this assumption as we restrict ourself to an error estimate in the $\mathrm{L}^\infty(\mathrm{L}^2)$ norm.

We first define the usual cell problem in periodic elliptic homogenization. As the normalization of its solution will be important for the higher order effective equation, we define this cell problem in $\mathcal{W}_{\mathrm{per}}(Y)$. Let $\boldsymbol{\chi} \in \mathcal{W}_{\mathrm{per}}(Y)$ be the unique (equivalence class of) solution of the cell problem

$$\big(a(y)\partial\boldsymbol{\chi}, \partial\boldsymbol{w}\big)_{\mathrm{L}^2(Y)} = -\big(a(y), \partial\boldsymbol{w}\big)_{\mathrm{L}^2(Y)} \quad \forall \boldsymbol{w} \in \mathcal{W}_{\mathrm{per}}(Y), \tag{2.2}$$

and define the homogenized tensor by

$$a^0 = \big\langle a(y)(1+\partial\boldsymbol{\chi})\big\rangle_Y. \tag{2.3}$$

Note that a^0 is elliptic and bounded for the same λ, Λ as in (2.1). Let $\tilde{b}^0, \tilde{a}^2 \geq 0$ be non-negative coefficients and consider the following equation : find $\tilde{u} : [0,T^\varepsilon]\times\Omega \to \mathbb{R}$ such that

$$\begin{array}{ll} \partial_t^2\tilde{u} = a^0\partial_x^2\tilde{u} - \varepsilon^2\big(\tilde{a}^2\partial_x^4\tilde{u} - \tilde{b}^0\partial_x^2\partial_t^2\tilde{u}\big) & \text{in } (0,T^\varepsilon]\times\Omega, \\ x \mapsto \tilde{u}(t,x) \ \ \Omega\text{-periodic} & \text{in } [0,T^\varepsilon], \\ \tilde{u}(0,x) = g^0(x), \quad \partial_t\tilde{u}(0,x) = g^1(x) & \text{in } \Omega. \end{array} \tag{2.4}$$

The well-posedness of equation (2.4) can be proved using the Faedo–Galerkin method (see [24, 28]). In particular, if we assume that $g^0 \in \mathrm{W}_{\mathrm{per}}(\Omega)\cap\mathrm{H}^2(\Omega)$ and $g^1 \in \mathrm{L}^2_0(\Omega)\cap\mathrm{H}^1(\Omega)$ then, there exists a unique weak solution of (2.4), $\tilde{u} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{W}_{\mathrm{per}}(\Omega))$ with $\partial_t\tilde{u} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2_0(\Omega))$. The following theorem is our main result of the first part of this paper. It provides a sufficient condition on the coefficients $\tilde{b}^0, \tilde{a}^2$ so that $\tilde{u}$ describes the effective behaviour of u^ε.

Theorem 2.1. *Assume that the tensor $a^\varepsilon(x) = a\big(\frac{x}{\varepsilon}\big) = a(y)$ is uniformly Y-periodic and $a(y) \in \mathrm{W}^{1,\infty}(Y)$. Furthermore, assume that the solution $\tilde{u}$ of* (2.4) *and the initial conditions satisfy the regularity*

$$\tilde{u} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega)), \quad \partial_t\tilde{u} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^4(\Omega)), \quad \partial_t^2\tilde{u} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega)),$$
$$g^0 \in \mathrm{H}^4(\Omega), \quad g^1 \in \mathrm{H}^3(\Omega).$$

Let $\boldsymbol{\chi}$ be the solution of (2.2) *and assume that for a $\chi \in \boldsymbol{\chi}$, the coefficients $\tilde{b}^0, \tilde{a}^2$ satisfy the relation*

$$a^0\tilde{b}^0 - \tilde{a}^2 = a^0\langle\chi^2\rangle_Y - a^0\langle\chi\rangle_Y^2. \tag{2.5}$$

Then the following error estimate holds

$$\begin{aligned} \|u^\varepsilon - \tilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C\varepsilon\big(&\|g^1\|_{\mathrm{H}^3(\Omega)} + \|g^0\|_{\mathrm{H}^4(\Omega)} \\ &+ \|\tilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega))} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))}\big), \end{aligned} \tag{2.6}$$

where C depends only on Ω, T, Y, a, λ and Λ.

Theorem 2.1 shows that there exists a family of functions that describe well the behaviour of u^ε in the $\mathrm{L}^\infty(\mathrm{L}^2)$ norm. Let us express this family in an explicit way. First, note that the class $\boldsymbol{\chi}$ of solutions of (2.2) can be parametrized with the mean of its elements $\langle\chi\rangle_Y \in \mathbb{R}$. Second, we claim that (2.5) is equivalent to the definition

$$\tilde{b}^0 = b^0 + \langle\chi\rangle_Y^2, \qquad \tilde{a}^2 = a^0\langle\chi\rangle_Y^2, \tag{2.7}$$

where $b^0 = \big\langle(\chi - \langle\chi\rangle_Y)^2\big\rangle_Y$ is non-negative and independent of $\langle\chi\rangle_Y$. To see it, first verify by a direct computation that (2.7) implies (2.5). Next, let us show the converse implication. As $\tilde{a}^2$ is non-negative, we can write it as $\tilde{a}^2 = a^0\langle\chi\rangle_Y^2$ for some $\langle\chi\rangle_Y \in \mathbb{R}$. Hence, making use of (2.5) we have

$$a^0\tilde{b}^0 = a^0\langle\chi^2\rangle_Y - a^0\langle\chi\rangle_Y^2 + \tilde{a}^2 = a^0\big\langle(\chi - \langle\chi\rangle_Y)^2\big\rangle_Y + a^0\langle\chi\rangle_Y^2 = a^0(b^0 + \langle\chi\rangle_Y^2),$$

and as $a^0 > 0$, we recover (2.7). Now, the result of Theorem 2.1 ensures that the set

$$\mathcal{F} = \{\tilde{u}_{\langle\chi\rangle} \text{ solution of (2.4) with } \tilde{b}^0, \tilde{a}^2 \text{ defined in (2.7)}\}, \tag{2.8}$$

constitutes a family of effective solutions for u^ε. The elements of $\mathcal{F}$ are indexed by the parameter $\langle\chi\rangle = \langle\chi\rangle_Y \in \mathbb{R}$ because each distinct value $\langle\chi\rangle \in \mathbb{R}$ corresponds to a distinct effective solution $\tilde{u}_{\langle\chi\rangle} \in \mathcal{F}$.

2.2. Adaptation and correctors. To prepare the proof of Theorem 2.1, we start by defining correctors needed for the estimate (2.4). We show that relation (2.5) together with the definition of a^0 are necessary and sufficient conditions to prove the existence and unicity of the correctors.

We begin with the formal technique of asymptotic development, standard in homogenization theory. Let us summarize the process in our context. First, we make the ansatz that u^ε can be approximated by an adaptation of $\tilde{u}$ of the form

$$\mathcal{B}^\varepsilon\tilde{u}(t,x) = \tilde{u}(t,x) + \varepsilon\chi\big(\tfrac{x}{\varepsilon}\big)\partial_x\tilde{u}(t,x) + \varepsilon^2\theta\big(\tfrac{x}{\varepsilon}\big)\partial_x^2\tilde{u}(t,x) + \varepsilon^3\kappa\big(\tfrac{x}{\varepsilon}\big)\partial_x^3\tilde{u}(t,x) + \varepsilon^4\rho\big(\tfrac{x}{\varepsilon}\big)\partial_x^4\tilde{u}(t,x), \tag{2.9}$$

where χ, θ, κ and ρ are Y-periodic functions, the so-called correctors. Then, noting $\mathcal{A}^\varepsilon = -\partial_x\big(a\big(\tfrac{x}{\varepsilon}\big)\partial_x(\cdot)\big)$, we explicitly compute $(\partial_t^2 + \mathcal{A}^\varepsilon)\mathcal{B}^\varepsilon\tilde{u}$ using equation (2.4). For the term $\partial_t^2\mathcal{B}^\varepsilon\tilde{u}$, we obtain

$$\partial_t^2\big(\mathcal{B}^\varepsilon\tilde{u}\big) = \mathcal{B}^\varepsilon\big(\partial_t^2\tilde{u}\big) = a^0\partial_x^2\tilde{u} + \varepsilon a^0\chi\partial_x^3\tilde{u} + \varepsilon^2\big(a^0(\theta + \tilde{b}^0) - \tilde{a}^2\big)\partial_x^4\tilde{u} + \mathcal{O}(\varepsilon^3).$$

A direct computation for the term $\mathcal{A}^\varepsilon\big(\mathcal{B}^\varepsilon\tilde{u}\big)$ gives (recall that $y = \frac{x}{\varepsilon}$)

$$\begin{aligned}
-\partial_x\big(a\big(\tfrac{x}{\varepsilon}\big)\partial_x\big(\mathcal{B}^\varepsilon\tilde{u}\big)\big)(x) = \;& \varepsilon^{-1}\big(-\partial_y(a(y)(1+\partial_y\chi))\big)\partial_x\tilde{u} \\
&+ \quad \big(-\partial_y(a(y)(\chi+\partial_y\theta)) - a(y)(1+\partial_y\chi)\big)\partial_x^2\tilde{u} \\
&+ \varepsilon^1 \; \big(-\partial_y(a(y)(\theta+\partial_y\kappa)) - a(y)(\chi+\partial_y\theta)\big)\partial_x^3\tilde{u} \\
&+ \varepsilon^2 \; \big(-\partial_y(a(y)(\kappa+\partial_y\rho)) - a(y)(\theta+\partial_y\kappa)\big)\partial_x^4\tilde{u} + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Next, we impose the constraint $(\partial_t^2 + \mathcal{A}^\varepsilon)\mathcal{B}^\varepsilon\tilde{u} = \mathcal{O}(\varepsilon^3)$ and end up with elliptic PDEs for χ, θ, κ and ρ, the so-called cell problems. The cell problem for χ, given in (2.2), follows from the cancellation of the ε^{-1}-order term. The cell problems for θ, κ and ρ are given by

$$\varepsilon^0: \; \big(a(y)\partial\theta, \partial w\big)_Y = -\big(a(y)\chi, \partial w\big)_Y + \big(a(y)(1+\partial\chi) - a^0, w\big)_Y, \tag{2.10}$$

$$\varepsilon^1: \; \big(a(y)\partial\kappa, \partial w\big)_Y = -\big(a(y)\theta, \partial w\big)_Y + \big(a(y)(\chi+\partial\theta) - a^0\chi, w\big)_Y, \tag{2.11}$$

$$\varepsilon^2: \; \big(a(y)\partial\rho, \partial w\big)_Y = -\big(a(y)\kappa, \partial w\big)_Y + \big(a(y)(\theta+\partial\kappa) - a^0(\theta+\tilde{b}^0) + \tilde{a}^2, w\big)_Y, \tag{2.12}$$

for periodic test functions w, where we use the shorten notation $(\cdot,\cdot)_Y = (\cdot,\cdot)_{\mathrm{L}^2(Y)}$.

In order to show that these problems are well-posed in $\mathcal{W}_{\mathrm{per}}(Y)$, we verify the hypotheses of Lax–Milgram theorem. The assumptions on $a(y)$ imply that the bilinear form $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (a(y)\partial\boldsymbol{v}, \partial\boldsymbol{w})_Y$ is elliptic and bounded. We thus have to verify that the right hand sides are in the dual space $\mathcal{W}_{\mathrm{per}}^*(Y)$ and for that matter we are going to verify that they satisfy (1.7). We fix arbitrarily $\chi \in \boldsymbol{\chi}$. The definition of a^0 in (2.3) implies that $\big(a(y)(1+\partial\chi) - a^0, 1\big)_Y = 0$. Consequently, the right hand side of (2.10) is in $\mathcal{W}_{\mathrm{per}}^*(Y)$ and problem (2.10) has a unique solution $\boldsymbol{\theta} \in \mathcal{W}_{\mathrm{per}}(Y)$. We fix arbitrarily $\theta \in \boldsymbol{\theta}$ and consider now the solvability of problem (2.11). First, noting that $a(y)(1+\partial\chi) \in \mathrm{H}(\mathrm{div}, Y)$, we use integration by parts and equation (2.2) to obtain for any $y_1, y_2 \in Y$

$$a(y)(1+\partial\chi)\Big|_{y_1}^{y_2} = -\int_Y \big(H_{y_2} - H_{y_1}\big)\partial_y\big(a(y)(1+\partial\chi)\big)\mathrm{d}y = 0,$$

where H is the Heaviside step function. Hence, $a(y)(1+\partial\chi)$ is constant and thanks to the definition of a^0 we conclude that

$$a(y)\big(1+\partial\chi(y)\big) = a^0 \quad \forall y \in Y. \tag{2.13}$$

In a similar way, using (2.13) we show that $a(y)\big(\partial\theta + \chi\big) = C$ is constant on Y. Dividing this equality by $a(y)$, taking the mean value and using the fact that $a^0 = \langle 1/a(y)\rangle_Y^{-1}$ (thanks to (2.13)), we obtain the equality

$$a(y)\big(\chi(y) + \partial\theta(y)\big) = a^0\langle\chi\rangle_Y \quad \forall y \in Y. \tag{2.14}$$

The equality (2.14) implies that the right hand side in (2.11) satisfies (1.7) and hence (2.11) is well-posed in $\mathcal{W}_{\mathrm{per}}(Y)$. We fix arbitrarily $\kappa \in \boldsymbol{\kappa}$ and examine now the solvability of (2.12). Let $w = 1$ in the right hand side. Using successively equation (2.2) with κ as a test function and (2.11) with χ as a test function, we obtain

$$\big(a(y)(\theta + \partial\kappa), 1\big)_Y = \big(a(y)(1+\partial\chi), \theta\big)_Y - \big(a(y)(\chi + \partial\theta), \chi\big)_Y + a^0\big(\chi, \chi\big)_Y.$$

Now, using (2.13) and (2.14), we have

$$\big(a(y)(\theta + \partial\kappa) - a^0(\theta + \tilde{b}^0) + \tilde{a}^2, 1\big)_Y = a^0\big(\chi,\chi\big)_Y - a^0\langle\chi\rangle_Y\big(\chi, 1\big)_Y - |Y|(a^0\tilde{b}^0 - \tilde{a}^2).$$

Hence, problem (2.12) is well-posed if and only if $a^0\big(\chi,\chi\big)_Y - a^0\langle\chi\rangle_Y\big(\chi,1\big)_Y - |Y|(a^0\tilde{b}^0 - \tilde{a}^2) = 0$, or equivalently if and only if $\tilde{b}^0$ and $\tilde{a}^2$ satisfy the relation (2.5).

From now on, *correctors* will refer to functions $\chi, \theta, \kappa, \rho$ in the classes $\boldsymbol{\chi}, \boldsymbol{\theta}, \boldsymbol{\kappa}, \boldsymbol{\rho}$ of solutions of (2.2),(2.10),(2.11) and (2.12). Note that they satisfy the following hierarchical dependence : $\boldsymbol{\theta}$ depends on the choice of $\chi \in \boldsymbol{\chi}$, $\boldsymbol{\kappa}$ depends on the choices of $\chi \in \boldsymbol{\chi}$ and $\theta \in \boldsymbol{\theta}$, etc.

We close this section by noting that if we assume $a(y) \in \mathrm{W}^{1,\infty}(Y)$ then the correctors $\chi, \theta, \kappa, \rho$ are in $\mathrm{H}^2(Y)$ and since the embedding $\mathrm{H}^1(Y) \hookrightarrow \mathcal{C}^0(\bar{Y})$ is continuous for dimension $d = 1$ we have that $\chi, \theta, \kappa, \rho \in \mathcal{C}^1(\bar{Y})$.

2.3. A priori error estimate. In this section we prove Theorem 2.1. The steps are as follows : we first prove that $\mathcal{B}^\varepsilon\tilde{u} - u^\varepsilon$ solves (in an appropriate sense) an inhomogeneous wave equation in Lemma 2.2. We then apply an a priori error estimate for this equation in Lemma 2.3.

Consider the adaptation of $\tilde{u}$, $\mathcal{B}^\varepsilon\tilde{u}$, defined in (2.9) with the correctors $\chi, \theta, \kappa, \rho$. Since $|\Omega| = n\varepsilon|Y|$, observe that $\mathcal{B}^\varepsilon\tilde{u}$ is Ω-periodic and we have that $\mathcal{B}^\varepsilon\tilde{u} \in \mathrm{H}^1_{\mathrm{per}}(\Omega)$ but $\mathcal{B}^\varepsilon\tilde{u}$ is not in $\mathrm{W}_{\mathrm{per}}(\Omega)$ in general, as $\chi, \theta, \kappa, \rho \in \mathrm{H}^1_{\mathrm{per}}(\Omega)$ and hence $\langle\mathcal{B}^\varepsilon\tilde{u}\rangle_\Omega \neq 0$. We consider next the operator $\mathrm{H}^1_{\mathrm{per}}(\Omega) \cap \mathrm{H}^3(\Omega) \to \mathcal{W}^*_{\mathrm{per}}(\Omega)$, $v \mapsto \hat{\mathcal{B}}^\varepsilon v$ defined as

$$\big\langle\hat{\mathcal{B}}^\varepsilon v, \boldsymbol{w}\big\rangle = \big(\big[v + \varepsilon(\chi - \partial_y\theta)\partial_x v + \varepsilon^3(\kappa - \partial_y\rho)\partial_x^3 v\big], \boldsymbol{w}\big)_{\mathcal{L}^2} - \big(\varepsilon^2\theta\partial_x v + \varepsilon^4\rho\partial_x^3 v, \partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}, \tag{2.15}$$

where $\langle\cdot,\cdot\rangle$ denotes $\langle\cdot,\cdot\rangle_{\mathcal{W}^*_{\mathrm{per}},\mathcal{W}_{\mathrm{per}}}$, $[\cdot]$ is the equivalence class in $\mathcal{L}^2(\Omega)$ and the Y-periodic functions $\chi, \theta, \kappa, \rho$ are as usual defined on Ω by setting $y = x/\varepsilon$. We observe that for $v \in \mathrm{H}^1_{\mathrm{per}}(\Omega) \cap \mathrm{H}^4(\Omega)$ we have $\langle\hat{\mathcal{B}}^\varepsilon v, \boldsymbol{w}\rangle_{\mathcal{W}^*_{\mathrm{per}},\mathcal{W}_{\mathrm{per}}} = (\,[\mathcal{B}^\varepsilon v], \boldsymbol{w})_{\mathcal{L}^2}$. In the next lemma we show that $\hat{\mathcal{B}}^\varepsilon\tilde{u}$ satisfies a variational formulation of the wave equation with right hand side of order $\mathcal{O}(\varepsilon^2)$ in $\mathrm{L}^2(\mathcal{W}^*_{\mathrm{per}}(\Omega))$.

LEMMA 2.2. *Under the assumptions of Theorem 2.1, $\hat{\mathcal{B}}^\varepsilon\tilde{u}$ satisfies*

$$(\partial_t^2 + \mathcal{A}^\varepsilon)\hat{\mathcal{B}}^\varepsilon\tilde{u}(t) = \mathcal{R}^\varepsilon\tilde{u}(t) \quad \textit{in } \mathcal{W}^*_{\mathrm{per}}(\Omega) \quad \textit{for a.e. } t \in [0, T^\varepsilon], \tag{2.16}$$

where the right hand side $\mathcal{R}^\varepsilon\tilde{u} \in \mathrm{L}^2(0, T^\varepsilon; \mathcal{W}^*_{\mathrm{per}}(\Omega))$ satisfies the estimate

$$\|\mathcal{R}^\varepsilon\tilde{u}\|_{\mathrm{L}^2(0,T^\varepsilon;\mathcal{W}^*_{\mathrm{per}}(\Omega))} \le C\varepsilon^2\big(\|\tilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega))} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))}\big), \tag{2.17}$$

for a constant C that only depends on T, Y, a, λ and Λ.

Proof. To simplify the notation, $\langle\cdot,\cdot\rangle_{\mathcal{W}_{\mathrm{per}}^*,\mathcal{W}_{\mathrm{per}}}$ will be simply denoted $\langle\cdot,\cdot\rangle$. First, from the equation in (2.4) and the assumptions on the regularity of $\tilde{u}$, note that the following equalities hold in $\mathrm{L}^2(\Omega)$ for a.e. $t\in[0,T^\varepsilon]$,

$$\partial_t^2\tilde{u} = a^0\partial_x^2\tilde{u} - \varepsilon^2\tilde{a}^2\partial_x^4\tilde{u} + \varepsilon^2\tilde{b}^0\partial_x^2\partial_t^2\tilde{u}, \tag{2.18}$$

$$\partial_x\partial_t^2\tilde{u} = a^0\partial_x^3\tilde{u} - \varepsilon^2\tilde{a}^2\partial_x^5\tilde{u} + \varepsilon^2\tilde{b}^0\partial_x^3\partial_t^2\tilde{u}. \tag{2.19}$$

Next, for a $t\in[0,T^\varepsilon]$ let us develop the terms $\partial_t^2\big(\hat{\mathcal{B}}^\varepsilon\tilde{u}\big)(t)$ and $\mathcal{A}^\varepsilon\big(\hat{\mathcal{B}}^\varepsilon\tilde{u}\big)(t)$ separately. As $\hat{\mathcal{B}}^\varepsilon$ does not depend on time, note that $\big\langle\partial_t^2\big(\hat{\mathcal{B}}^\varepsilon\tilde{u}\big),\boldsymbol{w}\big\rangle = \big\langle\hat{\mathcal{B}}^\varepsilon\partial_t^2\tilde{u},\boldsymbol{w}\big\rangle$. From (2.18), using an integration by part we obtain

$$\big(\big[\partial_t^2\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2} = \big(\big[a^0\partial_x^2\tilde{u} - \varepsilon^2\tilde{a}^2\partial_x^4\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2} - \big(\varepsilon^2\tilde{b}^0\partial_x\partial_t^2\tilde{u},\partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}.$$

In $\big\langle\hat{\mathcal{B}}^\varepsilon\partial_t^2\tilde{u},\boldsymbol{w}\big\rangle$ (defined in (2.15)) we use this equality and get

$$\begin{aligned}\big\langle\hat{\mathcal{B}}^\varepsilon\partial_t^2\tilde{u},\boldsymbol{w}\big\rangle = {}&\big(\big[a^0\partial_x^2\tilde{u} + \varepsilon(\chi-\partial_y\theta)\partial_x\partial_t^2\tilde{u} - \varepsilon^2\tilde{a}^2\partial_x^4\tilde{u} + \varepsilon^3(\kappa-\partial_y\rho)\partial_x^3\partial_t^2\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &- \big(\varepsilon^2(\theta+b^0)\partial_x\partial_t^2\tilde{u} + \varepsilon^4\rho\partial_x^3\partial_t^2\tilde{u},\partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}.\end{aligned}$$

We use now (2.19) to substitute $\partial_x\partial_t^2\tilde{u}$ and obtain

$$\begin{aligned}\big\langle\hat{\mathcal{B}}^\varepsilon\partial_t^2\tilde{u},\boldsymbol{w}\big\rangle = {}&\big(\big[a^0\partial_x^2\tilde{u} + \varepsilon a^0(\chi-\partial_y\theta)\partial_x^3\tilde{u} - \varepsilon^2\tilde{a}^2\partial_x^4\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &- \big(\varepsilon^2 a^0(\theta+\tilde{b}^0)\partial_x^3\tilde{u},\partial_x\boldsymbol{w}\big)_{\mathrm{L}^2} + \big\langle\mathcal{R}_1^\varepsilon\tilde{u},\boldsymbol{w}\big\rangle,\end{aligned}$$

where

$$\begin{aligned}\big\langle\mathcal{R}_1^\varepsilon\tilde{u},\boldsymbol{w}\big\rangle = {}&\varepsilon^3\big(\big[(\kappa-\partial_y\rho+\tilde{b}^0(\chi-\partial_y\theta))\partial_x^3\partial_t^2\tilde{u} + \tilde{a}^2(\partial_y\theta-\chi)\partial_x^5\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &+ \varepsilon^4\big(\tilde{a}^2(\theta+\tilde{b}^0)\partial_x^5\tilde{u} - (\tilde{b}^0(\theta+\tilde{b}^0)+\rho)\partial_x^3\partial_t^2\tilde{u},\partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}.\end{aligned}$$

We integrate by parts to get

$$\big\langle\hat{\mathcal{B}}^\varepsilon\partial_t^2\tilde{u},\boldsymbol{w}\big\rangle = \big(\big[a^0\partial_x^2\tilde{u} + \varepsilon a^0\chi\partial_x^3\tilde{u} + \varepsilon^2(a^0(\theta+\tilde{b}^0)-\tilde{a}^2)\partial_x^4\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2} + \big\langle\mathcal{R}_1^\varepsilon\tilde{u},\boldsymbol{w}\big\rangle. \tag{2.20}$$

We develop now the second term :

$$\begin{aligned}\big\langle\mathcal{A}^\varepsilon\big(\hat{\mathcal{B}}^\varepsilon\tilde{u}\big),\boldsymbol{w}\big\rangle = \Big(\Big[\;&\varepsilon^{-1}\big(-\partial_y(a(y)(1+\partial_y\chi))\big)\partial_x\tilde{u}\\ &+\big(-\partial_y(a(y)(\chi+\partial_y\theta)) - a(y)(1+\partial_y\chi)\big)\partial_x^2\tilde{u}\\ &+\varepsilon^1\big(-\partial_y(a(y)(\theta+\partial_y\kappa)) - a(y)(\chi+\partial_y\theta)\big)\partial_x^3\tilde{u}\\ &+\varepsilon^2\big(-\partial_y(a(y)(\kappa+\partial_y\rho)) - a(y)(\theta+\partial_y\kappa)\big)\partial_x^4\tilde{u}\Big],\boldsymbol{w}\Big)_{\mathcal{L}^2}\\ &+\big\langle\mathcal{R}_2^\varepsilon\tilde{u},\boldsymbol{w}\big\rangle,\end{aligned} \tag{2.21}$$

where $\big\langle\mathcal{R}_2^\varepsilon\tilde{u},\boldsymbol{w}\big\rangle = -\varepsilon^3\big(\big[a(y)(\kappa+\partial_y\rho)\partial_x^5\tilde{u}\big],\boldsymbol{w}\big)_{\mathcal{L}^2} - \varepsilon^4\big(a(y)\rho\partial_x^5\tilde{u},\partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}$. Now, combining (2.20) and (2.21) and using the cell problems (2.2), (2.10), (2.11), (2.12), we verify that $(\partial_t^2+\mathcal{A}^\varepsilon)\hat{\mathcal{B}}^\varepsilon\tilde{u}(t) = \mathcal{R}^\varepsilon\tilde{u}(t)$ where $\mathcal{R}^\varepsilon\tilde{u} = \mathcal{R}_1^\varepsilon\tilde{u} + \mathcal{R}_2^\varepsilon\tilde{u}$. Consequently, using Hölder inequality and the regularity of the correctors, we obtain estimate (2.17)

$$\|\mathcal{R}^\varepsilon\tilde{u}\|_{\mathrm{L}^2(\mathcal{W}_{\mathrm{per}}^*)} \le T^{1/2}\varepsilon^{-1}\|\mathcal{R}^\varepsilon\tilde{u}\|_{\mathrm{L}^\infty(\mathcal{W}_{\mathrm{per}}^*)} \le C\varepsilon^2\big(\|\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^5)} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^3)}\big). \qquad\square$$

LEMMA 2.3. *Under the assumptions of Theorem 2.1, $\eta^\varepsilon = \hat{\mathcal{B}}^\varepsilon\tilde{u} - \big[u^\varepsilon\big]$ satisfies*

$$\|\partial_t\eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{W}_{\mathrm{per}}^*)} + \|\eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{L}^2)} \le C\big(\|\partial_t\eta^\varepsilon(0)\|_{\mathcal{W}_{\mathrm{per}}^*} + \|\eta^\varepsilon(0)\|_{\mathcal{L}^2} + \varepsilon^{-1}\|\mathcal{R}^\varepsilon\tilde{u}\|_{\mathrm{L}^2(\mathcal{W}_{\mathrm{per}}^*)}\big),$$

where C depends only on λ, Λ and T and $\mathcal{R}^\varepsilon\tilde{u}$ is given in Lemma 2.2.

Proof. We use again $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{\mathcal{W}_{\mathrm{per}}^*,\mathcal{W}_{\mathrm{per}}}$. Lemma 2.2 implies that

$$\big\langle(\partial_t^2+\mathcal{A}^\varepsilon)\eta^\varepsilon(t),\boldsymbol{w}\big\rangle = \big\langle\mathcal{R}^\varepsilon\tilde{u}(t),\boldsymbol{w}\big\rangle \quad \forall\boldsymbol{w}\in\mathcal{W}_{\mathrm{per}}(\Omega) \quad \text{for a.e. } t\in[0,T^\varepsilon]. \tag{2.22}$$

To simplify the notation let us drop the notation of the superscript ε. Thanks to Lax–Milgram theorem, define the inverse of $\mathcal{A}$, noted $\mathcal{A}^{-1}$. Using the properties of $\mathcal{A}$, we can show that $\mathcal{A}^{-1}$ is self-adjoint, elliptic ($\langle F, \mathcal{A}^{-1}F\rangle \geq \lambda/\Lambda^2\|F\|^2_{\mathcal{W}^*_{\rm per}}$) and bounded ($\|\mathcal{A}^{-1}\| \leq \lambda^{-1}$). Using (2.22) with the test function $\boldsymbol{w} = \mathcal{A}^{-1}\partial_t\eta(t)$, we obtain for a.e. $t \in [0, T^\varepsilon]$

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\big\langle \partial_t\eta(t), \mathcal{A}^{-1}\partial_t\eta(t)\big\rangle + \|\eta(t)\|^2_{\mathcal{L}^2}\Big) = \big\langle \mathcal{R}\tilde{u}(t), \mathcal{A}^{-1}\partial_t\eta(t)\big\rangle. \tag{2.23}$$

Noting $E\eta(t) = \big\langle \partial_t\eta(t), \mathcal{A}^{-1}\partial_t\eta(t)\big\rangle + \|\eta(t)\|^2_{\mathcal{L}^2}$, we integrate (2.23) over $[0,\xi]$ to get

$$E\eta(\xi) = E\eta(0) + 2\int_0^\xi \big\langle \mathcal{R}\tilde{u}(t), \mathcal{A}^{-1}\partial_t\eta(t)\big\rangle\,\mathrm{d}t \qquad \forall \xi \in [0, T^\varepsilon].$$

Using Hölder and Young inequalities and the bounededness of $\mathcal{A}^{-1}$, we obtain

$$E\eta(\xi) \leq E\eta(0) + 2\Lambda^2/\lambda^2\|\mathcal{R}\tilde{u}\|^2_{\mathrm{L}^1(\mathcal{W}^*_{\rm per})} + \lambda/(2\Lambda^2)\|\partial_t\eta\|^2_{\mathrm{L}^\infty(\mathcal{W}^*_{\rm per})}. \tag{2.24}$$

Using the ellipticity of $\mathcal{A}^{-1}$ we have $\lambda/\Lambda^2\|\partial_t\eta(\xi)\|^2_{\mathcal{W}^*_{\rm per}} \leq E\eta(\xi)$, hence, taking the L^∞ norm with respect to ξ, we obtain $\|\partial_t\eta\|^2_{\mathrm{L}^\infty(\mathcal{W}^*_{\rm per})} \leq C\big(E\eta(0) + \|\mathcal{R}\tilde{u}\|^2_{\mathrm{L}^1(\mathcal{W}^*_{\rm per})}\big)$. Estimate (2.24) and the boundedness of $\mathcal{A}^{-1}$ then gives

$$\|\partial_t\eta\|_{\mathrm{L}^\infty(\mathcal{W}^*_{\rm per})} + \|\eta\|_{\mathrm{L}^\infty(\mathcal{L}^2)} \leq C\big(\|\partial_t\eta(0)\|_{\mathcal{W}^*_{\rm per}} + \|\eta(0)\|_{\mathcal{L}^2} + \|\mathcal{R}\tilde{u}\|_{\mathrm{L}^1(\mathcal{W}^*_{\rm per})}\big).$$

Thanks to Hölder inequality we have $\|\mathcal{R}\tilde{u}\|_{\mathrm{L}^1(\mathcal{W}^*_{\rm per})} \leq T^{1/2}\varepsilon^{-1}\|\mathcal{R}\tilde{u}\|_{\mathrm{L}^2(\mathcal{W}^*_{\rm per})}$ and the proof of the lemma is complete. ∎

Proof of Theorem 2.1. First, note that $\eta^\varepsilon = \hat{\mathcal{B}}^\varepsilon\tilde{u} - [u^\varepsilon]$ satisfies at $t = 0$

$$\eta^\varepsilon(0) = \big[\varepsilon\chi\partial_x g^0 + \varepsilon^2\theta\partial_x^2 g^0 + \varepsilon^3\kappa\partial_x^3 g^0 + \varepsilon^4\rho\partial_x^4 g^0\big],$$

and for any $\boldsymbol{w} \in \mathcal{W}_{\rm per}(\Omega)$,

$$\big\langle \partial_t\eta^\varepsilon(0), \boldsymbol{w}\big\rangle = \big(\big[\varepsilon(\chi - \partial_y\theta)\partial_x g^1 + \varepsilon^3(\kappa - \partial_y\rho)\partial_x^3 g^1\big], \boldsymbol{w}\big)_{\mathcal{L}^2} - \big(\varepsilon^2\theta\partial_x g^1 + \varepsilon^4\rho\partial_x^3 g^1, \partial_x\boldsymbol{w}\big)_{\mathrm{L}^2}.$$

Hence, Lemma 2.3 together with Lemma 2.2 and the regularity of the correctors imply that η^ε satisfies $\|\eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{L}^2)} \leq C\varepsilon\big(\|g^1\|_{\mathrm{H}^3} + \|g^0\|_{\mathrm{H}^4} + \|\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^5)} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^3)}\big)$. Now note that $\|(\mathcal{B}^\varepsilon\tilde{u} - u^\varepsilon)(t)\|^2_{\mathrm{L}^2} = \|\eta^\varepsilon(t)\|^2_{\mathcal{L}^2} + |\Omega|\big\langle(\mathcal{B}^\varepsilon\tilde{u} - u^\varepsilon)(t)\big\rangle^2_\Omega$ and hence, using the normalization $\langle\tilde{u}(t)\rangle_\Omega = \langle u^\varepsilon(t)\rangle_\Omega = 0$, we obtain

$$\|\mathcal{B}^\varepsilon\tilde{u} - u^\varepsilon\|_{\mathrm{L}^\infty(\mathrm{L}^2)} \leq \|\eta^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{L}^2)} + C\varepsilon\|\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^4)} \leq C\varepsilon\big(\|g^1\|_{\mathrm{H}^3} + \|g^0\|_{\mathrm{H}^4} + \|\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^5)} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^3)}\big).$$

Finally, thanks to the triangle inequality and the obvious estimate $\|\tilde{u} - \mathcal{B}^\varepsilon\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{L}^2)} \leq C\varepsilon\|\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^4)}$, we obtain (2.6) and the proof of Theorem 2.1 is complete. ∎

2.4. The homogeneous equation on time intervals $\mathcal{O}(\varepsilon^{-1})$. We briefly discuss here the effective equation valid one time intervals $\mathcal{O}(\varepsilon^{-1})$. We show that in this situation, the "classical homogenized equation" is still valid, following the arguments of Theorem 2.1. Let u^0 be the solution of the homogenized equation

$$\begin{array}{ll} \partial_t^2 u^0(t,x) = a^0\partial_x^2 u^0(t,x) & \text{in } (0, \varepsilon^{-1}T] \times \Omega, \\ x \mapsto u^0(t,x) \ \ \Omega\text{-periodic} & \text{in } [0, \varepsilon^{-1}T], \\ u^0(0) = g^0, \quad \partial_t u^0(0) = g^1 & \text{in } \Omega. \end{array} \tag{2.25}$$

We define the adaptation operator for $v \in \mathrm{H}^1_{\rm per}(\Omega) \cap \mathrm{H}^3(\Omega)$ as

$$\big\langle \bar{\mathcal{B}}^\varepsilon v, \boldsymbol{w}\big\rangle_{\mathcal{W}^*_{\rm per}, \mathcal{W}_{\rm per}} = \big(\big[v + \varepsilon(\chi - \partial_y\theta)\partial_x v + \varepsilon^3\kappa\partial_x^3 v\big], \boldsymbol{w}\big)_{\mathcal{L}^2} - \big(\varepsilon^2\theta\partial_x v, \partial_x\boldsymbol{w}\big)_{\mathrm{L}^2},$$

where $\chi, \theta, \kappa \in \mathrm{H}^1_{\rm per}(Y)$ are the correctors defined in Section 2.2. Then, similarly to Lemma 2.2, we show that $\bar{\mathcal{B}}^\varepsilon u^0$ satisfies the variational wave equation (2.16) with a right hand side $\bar{\mathcal{R}}^\varepsilon u^0$ of order $\mathcal{O}(\varepsilon)$ (in the $\mathrm{L}^2(\mathcal{W}^*_{\rm per})$ norm). We set $\bar{\eta}^\varepsilon = \bar{\mathcal{B}}^\varepsilon u^0 - [u^\varepsilon]$ and repeat the proof of Lemma 2.3 in the case of a time interval $[0, \varepsilon^{-1}T]$ to obtain

$$\|\bar{\eta}^\varepsilon\|_{\mathrm{L}^\infty(\mathcal{L}^2(\Omega))} \leq C\big(\|\partial_t\bar{\eta}^\varepsilon(0)\|_{\mathcal{W}^*_{\rm per}} + \|\bar{\eta}^\varepsilon(0)\|_{\mathcal{L}^2} + \varepsilon^{-1/2}\|\bar{\mathcal{R}}^\varepsilon u^0\|_{\mathrm{L}^2(\mathcal{W}^*_{\rm per}(\Omega))}\big) \leq C\varepsilon^{1/2}.$$

As in the proof of Theorem 2.1, we conclude that $\|u^\varepsilon - u^0\|_{\mathrm{L}^\infty(0,\varepsilon^{-1}T;\mathrm{L}^2(\Omega))} \leq C\varepsilon^{1/2}$.

3. Analysis of the finite element heterogeneous multiscale method over long times. In this section we recall the FE-HMM-L, a numerical method that captures the long time behaviour of the multiscale wave equation at essentially the same cost (in the 1d case) as a classical numerical homogenization method. We then give the first a priori error analysis valid for time up to $\mathcal{O}(\varepsilon^{-2})$. We consider the oscillatory wave equation (1.1) with a source $f \in \mathrm{L}^2(0,T^\varepsilon;\mathrm{L}^2_0(\Omega))$: find $u^\varepsilon : [0,T^\varepsilon]\times\Omega \to \mathbb{R}$ such that

$$\begin{array}{ll} \partial_t^2 u^\varepsilon(t,x) - \partial_x\big(a^\varepsilon(x)\partial_x u^\varepsilon(t,x)\big) = f(t,x) & \text{in } (0,T^\varepsilon]\times\Omega, \\ x\mapsto u^\varepsilon(t,x) \ \ \Omega\text{-periodic} & \text{in } [0,T^\varepsilon], \\ u^\varepsilon(0,x) = g^0(x), \quad \partial_t u^\varepsilon(0,x) = g^1(x) & \text{in } \Omega. \end{array} \tag{3.1}$$

Most of the results in this section are valid for non-uniformly periodic coefficients of the form $a^\varepsilon(x) = a\big(x,\frac{x}{\varepsilon}\big)$, where $a(x,y)$ is Ω-periodic in x and Y-periodic in y.

3.1. An appropriate effective model for numerical approximation. We now want to approximate numerically an effective solution for u^ε. Among the family of effective equations $\mathcal{F}$ (defined in (2.8), see Section 2), we pick $\langle\chi\rangle_Y = 0$ which yields $\tilde{a}^2 = 0$ and removes the 4th order differential operator $\tilde{a}^2\partial_x^4$. As mentioned earlier this is the natural choice for one-dimensional problem. Note however that for the multidimensional multiscale wave equation, the current effective model [20] contains a fourth order term, also for the "natural normalization".

Let us denote $\bar{u}$ the effective solution of $\mathcal{F}$ corresponding to the parameter $\langle\chi\rangle_Y = 0$. As the numerical analysis is performed in the case of a locally periodic tensor, we define now explicitly $\bar{u}$ in this context. For each $x\in\Omega$, define $\chi(x,\cdot)\in \mathrm{W}_{\mathrm{per}}(Y)$ as the unique solution of the cell problem

$$\big(a(x,\cdot)\partial_y\chi(x,\cdot),\partial_y w\big)_{\mathrm{L}^2(Y)} = -\big(a(x,\cdot),\partial_y w\big)_{\mathrm{L}^2(Y)} \qquad \forall w\in \mathrm{W}_{\mathrm{per}}(Y). \tag{3.2}$$

Define for $x\in\Omega$ the tensors

$$a^0(x) = \big\langle a(x,\cdot)(1+\partial_y\chi(x,\cdot))\big\rangle_Y, \qquad b^0(x) = \big\langle\big(\chi(x,\cdot)\big)^2\big\rangle_Y, \tag{3.3}$$

and verify that $a^0(x)$ and $b^0(x)$ satisfy for λ,Λ as in (2.1) and some $\beta>0$,

$$\lambda \le a^0(x) \le \Lambda, \qquad 0\le b^0(x)\le \beta \qquad \text{for a.e. } x\in\Omega. \tag{3.4}$$

The equation for $\bar{u}$ is then : find $\bar{u} : [0,T^\varepsilon]\times\Omega\to\mathbb{R}$ such that

$$\begin{array}{ll} \partial_t^2\bar{u} - \partial_x\big(a^0(x)\partial_x\bar{u}\big) - \varepsilon^2\partial_x\big(b^0(x)\partial_x\partial_t^2\bar{u}\big) = f & \text{in } (0,T^\varepsilon]\times\Omega, \\ x\mapsto \bar{u}(t,x) \ \ \Omega\text{-periodic} & \text{in } [0,T^\varepsilon], \\ \bar{u}(0,x) = g^0(x), \quad \partial_t\bar{u}(0,x) = g^1(x) & \text{in } \Omega. \end{array} \tag{3.5}$$

Define the bilinear forms

$$A^0(v,w) = \big(a^0(x)\partial_x v,\partial_x w\big)_{\mathrm{L}^2(\Omega)}, \qquad B^0(v,w) = \big(b^0(x)\partial_x v,\partial_x w\big)_{\mathrm{L}^2(\Omega)}, \tag{3.6}$$

and the Hilbert space $\mathcal{S}(\Omega) = \{v\in \mathrm{L}^2(\Omega) : \partial_x v\in \mathrm{L}^2(\Omega)\}$ equipped with the inner product and corresponding norm

$$(v,w)_{\mathcal{S}(\Omega)} = (v,w)_{\mathrm{L}^2(\Omega)} + \varepsilon^2 B^0(v,w), \qquad \|v\|_{\mathcal{S}(\Omega)} = (v,v)^{1/2}_{\mathcal{S}(\Omega)}. \tag{3.7}$$

The well-posedness of (3.5) can be proved using the Faedo–Galerkin method (see [28, 24]). If we assume that $g^0\in \mathrm{W}_{\mathrm{per}}(\Omega)$, $g^1\in \mathrm{L}^2_0(\Omega)\cap\mathcal{S}(\Omega)$ and $f\in \mathrm{L}^2(0,T^\varepsilon;\mathrm{L}^2_0(\Omega))$, then, there exists a unique weak solution of (3.5) $\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{W}_{\mathrm{per}}(\Omega))$ with $\partial_t\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2_0(\Omega)\cap\mathcal{S}(\Omega))$. From now on, we assume that $\partial_t^2\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathcal{S}(\Omega))$. Hence, $\bar{u}$ satisfies

$$\begin{array}{l} \big(\partial_t^2\bar{u}(t),v\big)_{\mathcal{S}(\Omega)} + A^0\big(\bar{u}(t),v\big) = \big(f(t),v\big)_{\mathrm{L}^2(\Omega)} \quad \forall v\in \mathrm{W}_{\mathrm{per}}(\Omega) \quad \text{for a.e. } t\in[0,T^\varepsilon], \\ \bar{u}(0) = g^0, \quad \partial_t\bar{u}(0) = g^1. \end{array} \tag{3.8}$$

3.2. Definition of the FE-HMM-L. Following Abdulle, Grote and Stohrer in [7, 6], we introduce the finite element multiscale method for the numerical approximation of the wave equation (3.1) over long time.

Let $\mathcal{T}_H$ be a partition of Ω. Denote by H_K the diameter of the element $K \in \mathcal{T}_H$ and define $H = \max_{K\in\mathcal{T}_H} H_K$. The macro finite element space is defined, for a given $\ell \in \mathbb{N}_{>0}$, as

$$V_H(\Omega) = \{v_H \in \mathrm{W}_{\mathrm{per}}(\Omega) \ : \ v_H|_K \in \mathcal{P}^\ell(K) \ \ \forall K \in \mathcal{T}_H\}, \tag{3.9}$$

where $\mathcal{P}^\ell(K)$ is the space of polynomials on K of degree at most ℓ. Let $\hat{K}$ be the reference element and for every $K \in \mathcal{T}_H$ note F_K the unique continuous mapping such that $F_K(\hat{K}) = K$ with $\partial F_K > 0$. We are given a quadrature formula on $\hat{K}$ by a set of weights and quadrature points $\{\hat{\omega}_j, \hat{x}_j\}_{j=1}^J$. Note that it naturally induces a quadrature formula on K whose weights and quadrature points are given by $\{\omega_{K_j} = \partial F_K\hat{\omega}_j, \ x_{K_j} = F_K(\hat{x}_j)\}_{j=1}^J$. The following assumptions are required for the construction of the stiffness matrix to ensure the optimal convergence rate of FEM with numerical quadrature [16, 15] :

$$\begin{array}{lll} \textit{(i)} & \hat{\omega}_j > 0, \quad j = 1,\dots,J, & \\ \textit{(ii)} & \int_{\hat{K}} \hat{p}(\hat{x})\,\mathrm{d}\hat{x} = \sum_{j=1}^J \hat{\omega}_j \hat{p}(\hat{x}_j) \quad \forall \hat{p} \in \mathcal{P}^\sigma(\hat{K}), \ \sigma = \max\{2\ell-2,\ell\}. \end{array} \tag{3.10}$$

Furthermore, we assume that the quadrature formula $\{\hat{\omega}'_j, \hat{x}'_j\}_{j=1}^{J'}$, required for the computation of the mass matrix, fulfills the following hypothesis

$$\textit{(iii)} \quad \sum_{j=1}^{J'} \hat{\omega}'_j |\hat{p}(\hat{x}'_j)|^2 \geq \hat{\lambda}' \|\hat{p}\|_{\mathrm{L}^2(\hat{K})} \quad \forall \hat{p} \in \mathcal{P}^\ell(\hat{K}), \ \text{for a } \hat{\lambda}' > 0. \tag{3.11}$$

The quadrature formula $\{\hat{\omega}'_j, \hat{x}'_j\}_{j=1}^{J'}$ defines a scalar product (and associated norm) on $V_H(\Omega) \times V_H(\Omega)$ equivalent to the standard L^2 scalar product. For every macro element $K \in \mathcal{T}_H$ and every $j \in \{1,\dots,J\}$, we define around the quadrature point x_{K_j} a sampling domain $K_{\delta j} = x_{K_j} + \delta Y$, where δ is a positive real number such that $\delta \geq \varepsilon$. Each sampling domain $K_{\delta j}$ is discretized in a partition $\mathcal{T}_h$, where $h = \max_{Q\in\mathcal{T}_h} h_Q$ is the maximal diameter of the elements $Q \in \mathcal{T}_h$. The micro finite element space is defined, for a $q \in \mathbb{N}_{>0}$, as

$$V_h(K_{\delta j}) = \{z_h \in \mathrm{W}_{\mathrm{per}}(K_{\delta j}) \ : \ z_h|_Q \in \mathcal{P}^q(Q) \ \ \forall Q \in \mathcal{T}_h\}. \tag{3.12}$$

Remark 3.1. Other finite element spaces for the micro scale are possible. For example, we can use $\mathring{V}_h(K_{\delta j}) = \{z_h \in \mathrm{H}^1_0(K_{\delta j}) \ : \ z_h|_Q \in \mathcal{P}^q(Q) \ \ \forall Q \in \mathcal{T}_h\}$. The formulation of the FE-HMM-L has then to be adapted accordingly, e.g., replacing the function v_h by $(v_h - \langle v_h\rangle_{K_{\delta j}})$ in the FE-HMM-L formula below.

The FE-HMM-L. Let g^0_H, g^1_H be suitable approximations in $V_H(\Omega)$ of the initial conditions g^0, g^1. The FE-HMM-L is defined as follows : find $u_H : [0,T^\varepsilon] \to V_H(\Omega)$ such that

$$\begin{aligned} &\big(\partial_t^2 u_H(t), v_H\big)_Q + A_H\big(u_H(t), v_H\big) = \big(f(t), v_H\big)_{\mathrm{L}^2} \quad \forall v_H \in V_H(\Omega) \text{ for a.e. } t \in [0,T^\varepsilon], \\ &u_H(0) = g^0_H, \quad \partial_t u_H(0) = g^1_H. \end{aligned} \tag{3.13}$$

The bilinear forms are defined for $v_H, w_H \in V_H(\Omega)$ as

$$A_H(v_H, w_H) = \sum_{K\in\mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta j}|} \int_{K_{\delta j}} a^\varepsilon(x) \partial_x v_{h,K_j}(x) \partial_x w_{h,K_j}(x)\,\mathrm{d}x, \tag{3.14}$$

$$\big(v_H, w_H\big)_Q = \big(v_H, w_H\big)_H + \big(v_H, w_H\big)_M, \tag{3.15}$$

$$\big(v_H, w_H\big)_H = \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J'} \omega'_{K_j} v_H(x'_{K_j}) w_H(x'_{K_j}), \tag{3.16}$$

$$\big(v_H, w_H\big)_M = \sum_{K\in\mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta j}|} \int_{K_{\delta j}} \big(v_{h,K_j} - v^{\mathrm{lin}}_{H,K_j}\big)\big(w_{h,K_j} - w^{\mathrm{lin}}_{H,K_j}\big)(x)\,\mathrm{d}x, \tag{3.17}$$

where the piecewise linear approximation of v_H (resp. w_H) around x_{K_j} is given by

$$v^{\mathrm{lin}}_{H,K_j}(x) = v_H(x_{K_j}) + (x - x_{K_j})\partial_x v_H(x_{K_j}),$$

and the micro functions v_{h,K_j} for v_H (resp. w_H) are the solutions of the following micro problems in $K_{\delta j}$: find v_{h,K_j} such that $(v_{h,K_j} - v^{\mathrm{lin}}_{H,K_j}) \in V_h(K_{\delta j})$ and

$$\big(a^\varepsilon(x)\partial_x v_{h,K_j}, \partial_x z_h\big)_{\mathrm{L}^2(K_{\delta j})} = 0 \qquad \forall z_h \in V_h(K_{\delta j}). \tag{3.18}$$

3.3. Usefull reformulation of the FE-HMM-L. In this section, we give a reformulation of problem (3.13) needed in the a priori analysis. We refer to [6, 7] for the details and to [1, 2] for various estimates of the FE-HMM.

For every $(K,j) \in \mathcal{T}_H \times \{1,\dots,J\}$, define $\psi_{h,K_j} \in V_h(K_{\delta j})$ as the solution of the cell problem in the sampling domain $K_{\delta j}$:

$$\big(a^\varepsilon(x)\partial_x \psi_{h,K_j}, \partial_x z_h\big)_{\mathrm{L}^2(K_{\delta j})} = -\big(a^\varepsilon(x), \partial_x z_h\big)_{\mathrm{L}^2(K_{\delta j})} \qquad \forall z_h \in V_h(K_{\delta j}), \tag{3.19}$$

and define the approximated tensors a^0_K and b^0_K at the quadrature point x_{K_j} as

$$a^0_K(x_{K_j}) = \big\langle a^\varepsilon(x)\big(1+\partial_x\psi_{h,K_j}\big)\big\rangle_{K_{\delta j}}, \qquad b^0_K(x_{K_j}) = \varepsilon^{-2}\big\langle\big(\psi_{h,K_j}\big)^2\big\rangle_{K_{\delta j}}. \tag{3.20}$$

We then have the two following lemmas (see [1, 2] and [6, 7] for the proofs) :

Lemma 3.2. *The bilinear form A_H can be rewritten for $v_H, w_H \in V_H(\Omega)$ as*

$$A_H(v_H,w_H) = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J} \omega_{K_j} a^0_K(x_{K_j})\partial_x v_H(x_{K_j})\partial_x w_H(x_{K_j}). \tag{3.21}$$

Furthermore, A_H is elliptic and bounded, i.e., for any $v_H, w_H \in V_H(\Omega)$,

$$A_H(v_H,v_H) \geq \lambda\|\partial_x v_H\|^2_{\mathrm{L}^2(\Omega)}, \qquad A_H(v_H,w_H) \leq \Lambda^2/\lambda\|\partial_x v_H\|_{\mathrm{L}^2(\Omega)}\|\partial_x w_H\|_{\mathrm{L}^2(\Omega)}.$$

Lemma 3.3. *The product $(\cdot,\cdot)_M$ can be rewritten as $(v_H,w_H)_M = \varepsilon^2 B_H(v_H,w_H)$, where the bilinear form B_H is defined as*

$$B_H(v_H,w_H) = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J} \omega_{K_j} b^0_K(x_{K_j})\partial_x v_H(x_{K_j})\partial_x w_H(x_{K_j}),$$

and is positive semidefinite and bounded, i.e., for any $v_H, w_H \in V_H(\Omega)$,

$$B_H(v_H,v_H) \geq 0, \qquad B_H(v_H,w_H) \leq C\|\partial_x v_H\|_{\mathrm{L}^2(\Omega)}\|\partial_x w_H\|_{\mathrm{L}^2(\Omega)}, \tag{3.22}$$

where C is a constant independent of H.

Remark 3.4. We emphasize that although $b^0_K(x_{K_j})$ depends on ε the product $(\cdot,\cdot)_M$ does not. In fact, ψ_{h,K_j} is an approximation of $\varepsilon\chi(x_{K_j},\frac{\cdot}{\varepsilon})$, where χ is defined in (3.2) (see the proof of Lemma 3.12 for details). Hence, assuming $\delta = \varepsilon$, we have via the change of variable $x = \varepsilon y$

$$b^0_K(x_{K_j}) = \varepsilon^{-2}|K_{\delta j}|^{-1}\int_{K_{\delta j}}\big(\psi_{h,K_j}(x)\big)^2\,\mathrm{d}x \approx |Y|^{-1}\int_Y\big(\chi(x_{K_j},y)\big)^2\,\mathrm{d}y = b^0(x_{K_j}),$$

where $b^0(x)$ is defined in (3.3). Consequently, B_H is obtained from B^0 by approximating the integral with numerical quadrature and approximating $b^0(x_{K_j})$ with $b^0_K(x_{K_j})$.

Remark 3.5. As a consequence of Lemmas 3.2 and 3.3, problem (3.13) is equivalent to a regular second order ordinary differential equation. Therefore, existence and uniqueness of a solution of (3.13) is given by classical theory for ordinary differential equations [18] and the FE-HMM-L is well-posed. Furthermore, the solution u_H satisfies the regularity $u_H \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{W}_{\mathrm{per}}(\Omega)), \partial_t u_H \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2_0(\Omega))$.

3.4. A priori analysis of the FE-HMM-L. Let us first comment our analysis. Let $\bar{u}_H$ be the FEM approximation in $V_H(\Omega)$ of $\bar{u}$. Proceeding similarly as in [11], we can obtain optimal a priori estimates for $\mathrm{e}^{\mathrm{FE}} = \|\bar{u}-\bar{u}_H\|$ in both the H^1 and the L^2 norms. Our purpose is not to analyse e^{FE}, but to estimate $\mathrm{e}^{\mathrm{HMM}} = \|\bar{u}_H - u_H\|$, the error generated by the upscaling procedure.

However, in order to have regularity assumptions on $\bar{u}$ (and not on $\bar{u}_H$), we have to proceed to the "full analysis" and estimate $\|\bar{u}-u_H\|$.

Recall that ℓ is the degree of the macro finite element space $V_H(\Omega)$. Let I_H be an interpolation operator such that for $v \in \mathrm{W}_{\mathrm{per}}(\Omega)\cap \mathrm{H}^{s+1}(\Omega)$ where $1\le s\le \ell$,

$$\Big(\sum_{K\in\mathcal{T}_H}\|v-I_Hv\|^2_{\mathrm{H}^m(K)}\Big)^{1/2}\le CH^{s+1-m}\|v\|_{\mathrm{H}^{s+1}(\Omega)}\quad 0\le m\le s+1, \tag{3.23}$$

where C is a constant independent of H and v. For example, I_H can be the nodal interpolation operator introduced in [15]. The following a priori error estimates for the FE-HMM-L are our second main result. We start with an $\mathrm{L}^\infty(\mathrm{H}^1)$ estimate.

Theorem 3.6. *Assume that δ satisfies $\delta/\varepsilon \in \mathbb{N}_{>0}$, that the micro mesh size is $h\le\varepsilon$ and that the degree of the micro finite element space is $q=1$. Furthermore, assume that the tensor $a^\varepsilon\in \mathrm{W}^{1,\infty}(\Omega)$ is collocated in the slow variable, i.e. for all $(K,j)\in\mathcal{T}_H\times\{1,\dots,J\}$, $a^\varepsilon(x)=a\big(x_{K_j},\frac{x}{\varepsilon}\big)$ for a.e. $x\in K_{\delta j}$. Finally, assume that $a^0,b^0\in \mathrm{W}^{\ell,\infty}(\Omega)$ and $\partial_t^k\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^{\ell+1}(\Omega))$ for $0\le k\le 4$. Then, the error $e=\bar{u}-u_H$ satisfies the estimate*

$$\|\partial_t e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))}+\|e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^1(\Omega))}\le C_1\big(h/\varepsilon^2\big)^2+\mathrm{e}^{\mathrm{FE}}_{\mathrm{H}^1}, \tag{3.24}$$

where $\mathrm{e}^{\mathrm{FE}}_{\mathrm{H}^1}$ is the standard FEM error estimate,

$$\mathrm{e}^{\mathrm{FE}}_{\mathrm{H}^1}\le C_2\big(\|g^1-g^1_H\|_{\mathrm{L}^2(\Omega)}+\varepsilon\|g^1-g^1_H\|_{\mathrm{H}^1(\Omega)}+\|g^0-g^0_H\|_{\mathrm{H}^1(\Omega)}+H^\ell\big),$$

$C_1=\tilde{C}_1\sum_{k=0}^4\|\partial_t^k\bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})}$ and $C_2=\tilde{C}_2\sum_{k=0}^4\|\partial_t^k\bar{u}\|_{\mathrm{L}^1(\mathrm{H}^{\ell+1})}$ with $\tilde{C}_1,\tilde{C}_2$ independent of H, h, ε and δ.

The next result is an $\mathrm{L}^\infty(\mathrm{L}^2)$ estimate.

Theorem 3.7. *As in Theorem 3.6, assume that $h\le\varepsilon$, $q=1$ and $a^\varepsilon(x)=a\big(x_{K_j},\frac{x}{\varepsilon}\big)$ for a.e. $x\in K_{\delta j}$. Furthermore, assume that $\partial_t^k\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^{\ell+1}(\Omega))$ for $0\le k\le 3$, $g^1_H=I_Hg^1$ and $a^0\in \mathrm{W}^{\ell+1,\infty}(\Omega)$. Then, $e=\bar{u}-u_H$ satisfies the estimate*

$$\|e\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))}\le C_1\big(h/\varepsilon^2\big)^2+\mathrm{e}^{\mathrm{FE}}_{\mathrm{L}^2}, \tag{3.25}$$

where $\mathrm{e}^{\mathrm{FE}}_{\mathrm{L}^2}$ is the standard FEM error estimate,

$$\mathrm{e}^{\mathrm{FE}}_{\mathrm{L}^2}\le C_2\big(\|g^0-g^0_H\|_{\mathrm{L}^2(\Omega)}+\varepsilon\|g^0-g^0_H\|_{\mathrm{H}^1(\Omega)}+H^{\ell+1}+\varepsilon H^\ell\big),$$

$C_1=\tilde{C}_1\sum_{k=0}^3\|\partial_t^k\bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})}$ and $C_2=\tilde{C}_2\sum_{k=0}^3\|\partial_t^k\bar{u}\|_{\mathrm{L}^1(\mathrm{H}^{\ell+1})}$ with $\tilde{C}_1,\tilde{C}_2$ independent of H, h, ε and δ.

Combining now Theorems 2.1 and 3.7 we obtain the following estimate for the error between u^ε the solution of (1.1) and u_H the solution of the FE-HMM-L.

Corollary 3.8. *Assume that the tensor $a^\varepsilon(x)=a\big(\frac{x}{\varepsilon}\big)$ is periodic with $a\in \mathrm{W}^{1,\infty}(\Omega)$ and $f=0$. Also, assume that $g^0_H=I_Hg^0$, $g^1_H=I_Hg^1$ and let the settings of the FE-HMM-L be such that $\delta/\varepsilon\in\mathbb{N}_{>0}$, $h\le\varepsilon$, $q=1$ and $\ell=1$. Finally assume that the following regularity holds*

$$g^0\in \mathrm{H}^4(\Omega),\ g^1\in \mathrm{H}^3(\Omega),\quad \partial_t^k\bar{u}\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^{5-k}(\Omega))\ \ 0\le k\le 3.$$

Then, we have the following estimate

$$\|u^\varepsilon-u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))}\le C_1\Big(\varepsilon+\big(h/\varepsilon^2\big)^2\Big)+C_2\big(H^2+\varepsilon H\big),$$

where $C_1=\tilde{C}_1\sum_{k=0}^3\|\partial_t^k\bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{5-k})}$ and $C_2=\tilde{C}_2\sum_{k=0}^3\|\partial_t^k\bar{u}\|_{\mathrm{L}^1(\mathrm{H}^2)}$ with $\tilde{C}_1,\tilde{C}_2$ independent of H, h, ε and δ.

3.5. Proof of the a priori error estimates. The proofs of Theorems 3.6 and 3.7 are divided into four Lemmas. We split the error $\bar{u}_H-u_H$ as

$$\bar{u}-u_H=(\bar{u}-\pi_H\bar{u})-(u_H-\pi_H\bar{u})=\eta-\zeta_H, \tag{3.26}$$

where $\pi_H\bar{u}$ is the elliptic projection defined below. We first provide a priori estimates for η and ζ_H in Lemmas 3.9, 3.10 and 3.11. We then quantify the error made at the micro level by the FEM and the error coming from the upscaling procedure of the FE-HMM-L in Lemma 3.12.

In the whole proof, c and C represent generic constants independent of H, h, ε, δ, $\bar{u}$, e_{a^0}, e_{b^0} (defined below). Hypothesis (3.11) ensures that $\|v_H\|_H = (v_H, v_H)_H^{1/2}$ is a norm on $V_H(\Omega)$, equivalent to the L^2 norm independently of H. Hence, using the result of Lemma 3.3, the norm $\|v_H\|_Q = (v_H, v_H)_Q^{1/2}$ (where $(\cdot,\cdot)_Q$ is defined in (3.15)) satisfies

$$c\|v_H\|_{\mathrm{L}^2} \le \|v_H\|_Q \le C\big(\|v_H\|_{\mathrm{L}^2} + \varepsilon\|v_H\|_{\mathrm{H}^1}\big). \tag{3.27}$$

Let us introduce the following bilinear forms, for $v_H, w_H \in V_H(\Omega)$,

$$\begin{aligned} A_H^0(v_H, w_H) &= \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J} \omega_{K_j} a^0(x_{K_j})\partial_x v_H(x_{K_j})\partial_x w_H(x_{K_j}),\\ B_H^0(v_H, w_H) &= \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J} \omega_{K_j} b^0(x_{K_j})\partial_x v_H(x_{K_j})\partial_x w_H(x_{K_j}). \end{aligned}$$

where $a^0(x)$, $b^0(x)$ are the exact tensors defined in (3.3). Furthermore, we define the following inner product in $V_H(\Omega)$, $(v_H, w_H)_{Q^0} = (v_H, w_H)_H + \varepsilon^2 B_H^0(v_H, w_H)$. The HMM errors are defined as

$$\mathrm{e}_{a^0} = \sup_{K\in\mathcal{T}_H, 1\le j\le J} |a^0(x_{K_j}) - a_K^0(x_{K_j})|, \qquad \mathrm{e}_{b^0} = \sup_{K\in\mathcal{T}_H, 1\le j\le J} \varepsilon^2 |b^0(x_{K_j}) - b_K^0(x_{K_j})|,$$

where $a_K^0(x_{K_j})$, $b_K^0(x_{K_j})$ are defined in (3.20). Using Lemmas 3.2 and 3.3, we verify that for any $v_H, w_H \in V_H(\Omega)$,

$$\begin{aligned} |A_H(v_H, w_H) - A_H^0(v_H, w_H)| &\le \mathrm{e}_{a^0}\|\partial_x v_H\|_{\mathrm{L}^2}\|\partial_x w_H\|_{\mathrm{L}^2},\\ \varepsilon^2|B_H(v_H, w_H) - B_H^0(v_H, w_H)| &\le \mathrm{e}_{b^0}\|\partial_x v_H\|_{\mathrm{L}^2}\|\partial_x w_H\|_{\mathrm{L}^2}. \end{aligned} \tag{3.28}$$

Finally, the broken norm on $V_H(\Omega)$ is defined as $\|v_H\|_{\bar{\mathrm{H}}^k(\Omega)} = \big(\sum_{K\in\mathcal{T}_H}\|v_H\|^2_{\mathrm{H}^k(K)}\big)^{1/2}$. Thanks to assumptions (3.10) and (3.11) and provided sufficient regularity of a^0, b^0, we have the following estimates for the numerical integration errors (see [15, 31]) :

$$\begin{aligned} |A^0(v_H, w_H) - A_H^0(v_H, w_H)| &\le CH^{\ell+\mu}\|a^0\|_{\mathrm{W}^{\ell+\mu,\infty}}\|v_H\|_{\bar{\mathrm{H}}^{\ell+1}}\|w_H\|_{\bar{\mathrm{H}}^{1+\mu}},\\ |A^0(v_H, w_H) - A_H^0(v_H, w_H)| &\le CH\|a^0\|_{\mathrm{W}^{1,\infty}}\|v_H\|_{\mathrm{H}^1}\|w_H\|_{\mathrm{H}^1},\\ |B^0(v_H, w_H) - B_H^0(v_H, w_H)| &\le CH^{\ell}\|b^0\|_{\mathrm{W}^{\ell,\infty}}\|v_H\|_{\bar{\mathrm{H}}^{\ell+1}}\|w_H\|_{\bar{\mathrm{H}}^1},\\ |(v_H, w_H)_{\mathrm{L}^2} - (v_H, w_H)_H| &\le CH^{\ell+\mu}\|v_H\|_{\bar{\mathrm{H}}^{\ell+1}}\|w_H\|_{\bar{\mathrm{H}}^{1+\mu}}, \end{aligned} \tag{3.29}$$

for any $v_H, w_H \in V_H(\Omega)$ and $\mu = 0, 1$ (A^0, B^0 are defined in (3.6)). In what follows, we will use the following estimates for $v \in \mathrm{H}^{\ell+1}(\Omega)\cap \mathrm{W}_{\mathrm{per}}(\Omega)$ and $w_H \in V_H(\Omega)$, $\mu = 0, 1$,

$$\begin{aligned} |A^0(v, w_H) - A_H(I_H v, w_H)| &\le C\big(\mathrm{e}_{a^0}\|v\|_{\mathrm{H}^1} + H^{\ell+\mu}\|v\|_{\mathrm{H}^{\ell+1}}\big)\|w_H\|_{\bar{\mathrm{H}}^{1+\mu}},\\ |(v, w_H)_{\mathcal{S}} - (I_H v, w_H)_Q| &\le C\big(\mathrm{e}_{b^0}\|v\|_{\mathrm{H}^1} + (H^{\ell+\mu} + \varepsilon^2 H^{\ell})\|v\|_{\mathrm{H}^{\ell+1}}\big)\|w_H\|_{\bar{\mathrm{H}}^{1+\mu}}, \end{aligned} \tag{3.30}$$

where $(\cdot,\cdot)_{\mathcal{S}}$ is defined in (3.7). They are obtained by combining the triangle inequality, (3.23), (3.28) and (3.29).

Define now the elliptic projection $\pi_H\bar{u} : [0, T^\varepsilon] \to V_H(\Omega)$ of the function $\bar{u}$, solution of

$$A_H\big(\pi_H\bar{u}(t), v_H\big) = \big(f(t), v_H\big)_{\mathrm{L}^2} - \big(I_H\partial_t^2\bar{u}(t), v_H\big)_Q \;\forall v_H \in V_H(\Omega) \;\text{ for a.e. } t\in[0, T^\varepsilon]. \tag{3.31}$$

As A_H is elliptic and bounded $\pi_H\bar{u}(t)$ exists and is unique for a.e. $t \in [0, T^\varepsilon]$. Furthermore, using (3.8) we have $\big(f(t), v_H\big)_{\mathrm{L}^2} = A^0\big(\bar{u}(t), v_H\big) + \big(\partial_t^2\bar{u}(t), v_H\big)_{\mathcal{S}}$ and we obtain the estimate

$$\|\pi_H\bar{u}(t)\|_{\mathrm{H}^1} \le C\big(\|\bar{u}(t)\|_{\mathrm{H}^1} + \|\partial_t^2\bar{u}(t)\|_{\mathrm{H}^1}\big) \qquad \text{for a.e. } t\in[0, T^\varepsilon]. \tag{3.32}$$

Hence, provided $\partial_t^2\bar{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^1(\Omega))$, we get $\pi_H\bar{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^1(\Omega))$.

LEMMA 3.9. *Assume that for* $1 \le p \le \infty$, $\partial_t^k\bar{u}, \partial_t^{k+2}\bar{u} \in \mathrm{L}^p(0, T^\varepsilon; \mathrm{H}^{\ell+1}(\Omega))$ *for* $k \ge 0$. *Then,* $\partial_t^k\pi_H\bar{u} \in \mathrm{L}^p(0, T^\varepsilon; \mathrm{H}^1(\Omega))$ *and provided* $a^0, b^0 \in \mathrm{W}^{\ell,\infty}(\Omega)$, *the following estimate holds for* $\eta = \bar{u} - \pi_H\bar{u}$,

$$\begin{aligned} \|I_H\partial_t^k\eta\|_{\mathrm{L}^p(\mathrm{H}^1)} + \|\partial_t^k\eta\|_{\mathrm{L}^p(\mathrm{H}^1)} \le{}& C\Big(\big(\mathrm{e}_{a^0} + \mathrm{e}_{b^0}\big)\big(\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^1)} + \|\partial_t^{k+2}\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^1)}\big)\\ &+ H^{\ell}\big(\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})} + \|\partial_t^{k+2}\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})}\big)\Big). \end{aligned} \tag{3.33}$$

If in addition we assume $a^0 \in \mathrm{W}^{\ell+1,\infty}(\Omega)$, then

$$\|I_H\partial_t^k\eta\|_{\mathrm{L}^p(\mathrm{L}^2)} + \|\partial_t^k\eta\|_{\mathrm{L}^p(\mathrm{L}^2)} \leq C\big((1+\mathrm{e}_{a^0})(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}) + H^{\ell+1} + \varepsilon^2 H^\ell\big) \times \big(\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})} + \|\partial_t^{k+2}\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})}\big). \tag{3.34}$$

Proof. First, as the forms A^0, $(.,.)_{\mathcal{S}}$, A_H and $(.,.)_Q$ are time independent, the time differentiation of equations (3.31) and (3.8) yields, similarly to (3.32), the estimate

$$\|\partial_t^k\pi_H\bar{u}(t)\|_{\mathrm{H}^1} \leq C\big(\|\partial_t^k\bar{u}(t)\|_{\mathrm{H}^1} + \|\partial_t^{k+2}\bar{u}(t)\|_{\mathrm{H}^1}\big) \qquad \text{for a.e. } t\in[0,T^\varepsilon].$$

Hence in view of the assumption on $\partial_t^k\bar{u}, \partial_t^{k+2}\bar{u}$ we obtain $\partial_t^k\pi\bar{u}_H \in \mathrm{L}^p(0,T^\varepsilon;\mathrm{H}^1(\Omega))$. Second, we prove estimates (3.33) and (3.34) for $k=0$. The proof for $k>0$ is obtained in the same way by differentiating equations (3.31) and (3.8). Using (3.31) and (3.8) we have almost everywhere in $[0,T^\varepsilon]$,

$$A_H\big(I_H\eta, v_H\big) = A_H\big(I_H\bar{u}, v_H\big) - A^0\big(\bar{u}, v_H\big) + \big(\partial_t^2\bar{u}, v_H\big)_{\mathcal{S}} - \big(I_H\partial_t^2\bar{u}, v_H\big)_Q.$$

We make use of (3.30) to obtain for a.e $t\in[0,T^\varepsilon]$,

$$A_H\big(I_H\eta(t), v_H\big) \leq C\Big(\big(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}\big)\sum_{k=0,2}\|\partial_t^k\bar{u}(t)\|_{\mathrm{H}^1} + H^\ell\sum_{k=0,2}\|\partial_t^k\bar{u}(t)\|_{\mathrm{H}^{\ell+1}}\Big)\|v_H\|_{\mathrm{H}^1}.$$

Letting now $v_H = I_H\eta(t)$, using the ellipticity of A_H and taking the L^p norm with respect to t, we obtain

$$\|I_H\eta\|_{\mathrm{L}^p(\mathrm{H}^1)} \leq C\Big(\big(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}\big)\sum_{k=0,2}\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^1)}\big) + H^\ell\sum_{k=0,2}\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})}\Big).$$

Note that $\eta = \bar{u} - I_H\bar{u} + I_H\eta$ and $\|\bar{u}-I_H\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^1)} \leq CH^\ell\|\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})}$ and we have proved estimate (3.33) for $k=0$. To prove (3.34), we use a standard Aubin–Nitsche argument. For a.e. $t\in[0,T^\varepsilon]$, note that $\|\eta(t)\|_{\mathrm{L}^2} = \sup_{g\in\mathrm{L}^2(\Omega)}\|g\|_{\mathrm{L}^2}^{-1}\big|\big(\eta(t),g\big)_{\mathrm{L}^2}\big|$. Let now $g\in\mathrm{L}^2(\Omega)$ and define ψ_g as the solution of the elliptic problem $A^0(v,\varphi_g) = (g,v)_{\mathrm{L}^2}\ \ \forall v\in\mathrm{W}_{\mathrm{per}}(\Omega)$. The regularity of a^0 and the polygonal domain ensure that $\|\varphi_g\|_{\mathrm{H}^2} \leq C\|g\|_{\mathrm{L}^2}$ (see [26]). Using (3.31) and (3.8), we verify that

$$\begin{aligned} A^0\big(\eta(t),\varphi_g\big) &= A^0\big(\eta(t),\varphi_g - v_H\big) + \big(I_H\partial_t^2\bar{u}(t), v_H\big)_Q - \big(\partial_t^2\bar{u}(t), v_H\big)_{\mathcal{S}} \\ &\quad + A_H\big(\pi_H\bar{u}(t), v_H\big) - A^0\big(\pi_H\bar{u}(t), v_H\big), \end{aligned} \tag{3.35}$$

for any $v_H\in V_H(\Omega)$ and a.e. t. Note that we can rewrite the last two terms as

$$\begin{aligned} A_H\big(\pi_H\bar{u}(t), v_H\big) - A^0(\pi_H\bar{u}(t), v_H) &= A^0\big(I_H\eta(t), v_H\big) - A_H\big(I_H\eta(t), v_H\big) \\ &\quad + A_H\big(I_H\bar{u}(t), v_H\big) - A^0\big(I_H\bar{u}(t), v_H\big). \end{aligned}$$

Hence, using the triangle inequality and (3.23), (3.28) and (3.29), we have

$$\big|A_H\big(\pi_H\bar{u}(t), v_H\big) - A^0(\pi_H\bar{u}(t), v_H)\big| \leq C\big((\mathrm{e}_{a^0}+H)\|I_H\eta(t)\|_{\mathrm{H}^1} + (\mathrm{e}_{a^0}+H^{\ell+1})\|\bar{u}(t)\|_{\mathrm{H}^{\ell+1}}\big)\|v_H\|_{\bar{\mathrm{H}}^2}.$$

Now, as $(\eta(t),g)_{\mathrm{L}^2} = A^0(\eta(t),\varphi_g)$, from (3.35) with $v_H = I_H\varphi_g$, we use estimates (3.23) and (3.30) to obtain for a.e. t

$$\begin{aligned} \big|\big(\eta(t),g\big)_{\mathrm{L}^2}\big| &\leq C\Big(H\|\eta(t)\|_{\mathrm{H}^1} + (\mathrm{e}_{a^0}+H)\|I_H\eta(t)\|_{\mathrm{H}^1} + (\mathrm{e}_{a^0}+H^{\ell+1})\|\bar{u}(t)\|_{\mathrm{H}^{\ell+1}} \\ &\quad + \mathrm{e}_{b^0}\|\partial_t^2\bar{u}(t)\|_{\mathrm{H}^1} + (H^{\ell+1}+\varepsilon^2H^\ell)\|\partial_t^2\bar{u}(t)\|_{\mathrm{H}^{\ell+1}}\Big)\|\varphi_g\|_{\mathrm{H}^2}. \end{aligned}$$

Hence, as $\|\eta(t)\|_{\mathrm{L}^2} \leq \|g\|_{\mathrm{L}^2}^{-1}\big|\big(\eta(t),g\big)_{\mathrm{L}^2}\big|$ and recalling that $\|\varphi_g\|_{\mathrm{H}^2} \leq C\|g\|_{\mathrm{L}^2}$, we obtain a bound for $\|\eta(t)\|_{\mathrm{L}^2}$. Taking the L^p norm with respect to t and using estimate (3.33), we obtain

$$\|\eta\|_{\mathrm{L}^p(\mathrm{L}^2)} \leq C\Big((1+\mathrm{e}_{a^0}+H)\big(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}\big) + \mathrm{e}_{a^0}H^\ell + H^{\ell+1} + \varepsilon^2H^\ell\Big)\sum_{k=0,2}\|\partial_t^k\bar{u}\|_{\mathrm{L}^p(\mathrm{H}^{\ell+1})},$$

which yields estimate (3.34) for $\|\eta\|_{\mathrm{L}^p(\mathrm{L}^2)}$. Finally, note that $\|I_H\eta\|_{\mathrm{L}^p(\mathrm{L}^2)} \leq \|\bar{u}-I_H\bar{u}\|_{\mathrm{L}^p(\mathrm{L}^2)} + \|\eta\|_{\mathrm{L}^p(\mathrm{L}^2)}$ and use (3.23) to obtain (3.34) for $k=0$. That ends the proof of Lemma 3.9. ∎

LEMMA 3.10. *The following estimate holds for $\zeta_H = u_H - \pi_H \bar{u}$,*

$$\|\partial_t\zeta_H\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \|\zeta_H\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \leq C\big(\mathrm{e}^{\mathrm{data}}_{\mathrm{H}^1} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|\partial_t\eta\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \varepsilon\|\partial_t\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|I_H\partial_t^2\eta\|_{\mathrm{L}^1(\mathrm{L}^2)} + \varepsilon\|I_H\partial_t^2\eta\|_{\mathrm{L}^1(\mathrm{H}^1)}\big), \tag{3.36}$$

where $\mathrm{e}^{\mathrm{data}}_{\mathrm{H}^1} = \|g^0 - g^0_H\|_{\mathrm{H}^1} + \|g^1 - g^1_H\|_{\mathrm{H}^1} + \varepsilon\|g^1 - g^1_H\|_{\mathrm{L}^2}$.

Proof. Using equations (3.13) and (3.31), we verify that for any $v_H \in V_H(\Omega)$ and a.e. $t \in [0, T^\varepsilon]$ it holds

$$\big(\partial_t^2\zeta_H(t), v_H\big)_Q + A_H\big(\zeta_H(t), v_H\big) = \big(I_H\partial_t^2\eta(t), v_H\big)_Q. \tag{3.37}$$

Set $v_H = \partial_t\zeta_H(t)$ and use the symmetry of the forms $(\cdot,\cdot)_Q$ and A_H to get for a.e. t

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t\zeta_H(t)\|_Q^2 + A_H\big(\zeta_H(t), \zeta_H(t)\big)\Big) = \big(I_H\partial_t^2\eta(t), \partial_t\zeta_H(t)\big)_Q.$$

Setting $E_H\zeta_H(t) = \|\partial_t\zeta_H(t)\|_Q^2 + A_H\big(\zeta_H(t), \zeta_H(t)\big)$, we integrate this equality and get

$$E_H\zeta_H(\xi) = E_H\zeta_H(0) + 2\int_0^\xi \big(I_H\partial_t^2\eta(t), \partial_t\zeta_H(t)\big)_Q\,\mathrm{d}t \qquad \forall\xi \in [0, T^\varepsilon]. \tag{3.38}$$

We apply now Cauchy–Schwartz, Hölder and Young inequalities to bound the second term of the right hand side of (3.38) as

$$2\int_0^\xi \big(I_H\partial_t^2\eta(t), \partial_t\zeta_H(t)\big)_Q\,\mathrm{d}t \leq 2\|I_H\partial_t^2\eta\|^2_{\mathrm{L}^1(Q)} + \tfrac{1}{2}\|\partial_t\zeta_H\|^2_{\mathrm{L}^\infty(Q)}. \tag{3.39}$$

As $A_H\big(\zeta_H(\xi), \zeta_H(\xi)\big) \geq 0$, combining (3.38) and (3.39) and taking the L^∞ norm with respect to ξ, we obtain the estimate $\frac{1}{2}\|\partial_t\zeta_H\|^2_{\mathrm{L}^\infty(Q)} \leq E_H\zeta_H(0) + 2\|I_H\partial_t^2\eta\|^2_{\mathrm{L}^1(Q)}$. A similar bound can then be deduced for $\|\zeta_H\|^2_{\mathrm{L}^\infty(\mathrm{H}^1)}$ from (3.38), (3.39) and the ellipticity of A_H. Then, using the boundedness of A_H, we obtain

$$\tfrac{1}{2}\|\partial_t\zeta_H\|^2_{\mathrm{L}^\infty(Q)} + \lambda\|\zeta_H\|^2_{\mathrm{L}^\infty(\mathrm{H}^1)} \leq \|\partial_t\zeta_H(0)\|_Q^2 + \Lambda^2/\lambda\|\zeta_H(0)\|^2_{\mathrm{H}^1} + 2\|I_H\partial_t^2\eta\|^2_{\mathrm{L}^1(Q)}.$$

The first two terms satisfy (recall the splitting of the error (3.26))

$$\begin{aligned}
\|\partial_t\zeta_H(0)\|_Q &\leq \|g^1_H - g^1\|_Q + \|\partial_t\eta(0)\|_Q \leq \|g^1_H - g^1\|_Q + \|\partial_t\eta\|_{\mathrm{L}^\infty(Q)},\\
\|\zeta_H(0)\|_{\mathrm{H}^1} &\leq \|g^0_H - g^0\|_{\mathrm{H}^1} + \|\eta(0)\|_{\mathrm{H}^1} \leq \|g^0_H - g^0\|_{\mathrm{H}^1} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)}.
\end{aligned}$$

Finally, we make use of (3.27) to obtain estimate (3.36) and that concludes the proof of Lemma 3.10. ◻

LEMMA 3.11. *If we assume that $g^1_H = I_H g^1$, then $\zeta_H = u_H - \pi_H\bar{u}$ satisfies*

$$\|\zeta_H\|_{\mathrm{L}^\infty(\mathrm{L}^2)} \leq C\big(\mathrm{e}^{\mathrm{data}}_{\mathrm{L}^2} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \varepsilon\|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|I_H\partial_t\eta\|_{\mathrm{L}^1(\mathrm{L}^2)} + \varepsilon\|I_H\partial_t\eta\|_{\mathrm{L}^1(\mathrm{H}^1)}\big), \tag{3.40}$$

where $\mathrm{e}^{\mathrm{data}}_{\mathrm{L}^2} = \|g^0 - g^0_H\|_{\mathrm{L}^2} + \varepsilon\|g^0 - g^0_H\|_{\mathrm{H}^1}$.

Proof. Rewriting (3.37) with $v_H = w_H(t)$, where $w_H \in \mathrm{H}^1(0, T^\varepsilon; V_H(\Omega))$, we have almost everywhere in $[0, T^\varepsilon]$

$$-\big(\partial_t\zeta_H, \partial_t w_H\big)_Q + A_H\big(\zeta_H, w_H\big) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\partial_t(I_H\eta - \zeta_H), w_H\big)_Q - \big(\partial_t I_H\eta, \partial_t w_H\big)_Q.$$

For $\xi \in [0, T^\varepsilon]$, we define $\hat{w}_H(t) = \int_t^\xi \zeta_H(\tau)\mathrm{d}\tau$, which satisfies $\hat{w}_H \in \mathrm{H}^1(0, T^\varepsilon; V_H(\Omega))$, $\hat{w}_H(\xi) = 0$ and $\partial_t\hat{w}_H = -\zeta_H$. We set $w_H = \hat{w}_H$ in the previous equality and thanks to the symmetry of the forms A_H, $(\cdot,\cdot)_Q$ we get almost everywhere in $[0, T^\varepsilon]$

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\zeta_H\|_Q^2 + A_H\big(\hat{w}_H, \hat{w}_H\big)\Big) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\partial_t(I_H\eta - \zeta_H), \hat{w}_H\big)_Q + \big(I_H\partial_t\eta, \zeta_H\big)_Q.$$

Recalling that by assumption $\partial_t(I_H\eta - \zeta_H)(0) = I_H g^1 - g^1_H = 0$, we integrate over $[0, \xi]$ and obtain for all $\xi \in [0, T^\varepsilon]$,

$$\|\zeta_H(\xi)\|_Q^2 + A_H\big(\hat{w}_H(0), \hat{w}_H(0)\big) = \|\zeta_H(0)\|_Q^2 + 2\int_0^\xi \big(I_H\partial_t\eta(t), \zeta_H(t)\big)_Q\,\mathrm{d}t. \tag{3.41}$$

As A_H is elliptic, note that $A_H(\hat{w}_H(0), \hat{w}_H(0)) \geq 0$. The first term of the right hand side is bounded using the triangle inequality as $\|\zeta_H(0)\|_Q \leq \|g^0 - g^0_H\|_Q + \|\eta\|_{\mathrm{L}^\infty(Q)}$ and the second is bounded using Cauchy–Schwarz, Hölder and Young inequality as

$$2\int_0^\xi \big(I_H\partial_t\eta(t), \zeta_H(t)\big)_Q \,\mathrm{d}t \leq 2\|I_H\partial_t\eta\|^2_{\mathrm{L}^1(Q)} + \tfrac{1}{2}\|\zeta_H\|^2_{\mathrm{L}^\infty(Q)}.$$

Now, taking the L^∞ norm with respect to ξ in (3.41) and recalling (3.27), we obtain estimate (3.40) and the proof of Lemma 3.11 is complete. ∎

Lemma 3.12. *Under the hypotheses of Theorem 3.6,* e_{a^0} *and* e_{b^0} *satisfy*

$$\mathrm{e}_{a^0} \leq C(h/\varepsilon)^2, \qquad \mathrm{e}_{b^0} \leq C\varepsilon(h/\varepsilon)^2. \tag{3.42}$$

Proof. The proof of the estimate for e_{a^0} can be found in [1]. We prove here the estimate for e_{b^0} in a similar way. For $(K, j) \in \mathcal{T}_H \times \{1, \dots, J\}$, we introduce the exact solution of the cell problem in $K_{\delta j}$: $\psi_{K_j} \in \mathrm{W}_{\mathrm{per}}(K_{\delta j})$ is the solution of

$$\big(a^\varepsilon(x)\partial_x\psi_{K_j}, \partial_x z\big)_{\mathrm{L}^2(K_{\delta j})} = -\big(a^\varepsilon(x), \partial_x z\big)_{\mathrm{L}^2(K_{\delta j})} \qquad \forall z \in \mathrm{W}_{\mathrm{per}}(K_{\delta j}). \tag{3.43}$$

We define $\bar{b}^0_K(x_{K_j}) = \varepsilon^{-2}\langle\psi^2_{K_j}\rangle_{K_{\delta j}}$ and split e_{b^0} as $\mathrm{e}_{b^0} \leq \mathrm{e}^{\mathrm{mod}}_{b^0} + \mathrm{e}^{\mathrm{mic}}_{b^0}$ where

$$\mathrm{e}^{\mathrm{mod}}_{b^0} = \sup_{K,j}\varepsilon^2\big|b^0(x_{K_j}) - \bar{b}^0_K(x_{K_j})\big|, \quad \mathrm{e}^{\mathrm{mic}}_{b^0} = \sup_{K,j}\varepsilon^2\big|\bar{b}^0_K(x_{K_j}) - b^0_K(x_{K_j})\big|.$$

We show that : (i) $\mathrm{e}^{\mathrm{mod}}_{b^0} = 0$ and (ii) $\mathrm{e}^{\mathrm{mic}}_{b^0} \leq C\varepsilon\big(h/\varepsilon\big)^2$. Fix $(K, j) \in \mathcal{T}_H \times \{1, \dots, J\}$ and write $n = \frac{\delta}{\varepsilon} \in \mathbb{N}_{>0}$, $K_{n\varepsilon} = K_{\delta j}$, $x_K = x_{K_j}$, $\psi = \psi_{K_j}$, $\psi_h = \psi_{h,K_j}$, $b^0 = b^0(x_{K_j})$, and similarly for $\bar{b}^0_K$ and b^0_K. We verify that for any $z \in \mathrm{W}_{\mathrm{per}}(K_{n\varepsilon})$,

$$\Big(a\big(x_K, \tfrac{x}{\varepsilon}\big)\big(\partial_x\big(\varepsilon\chi(x_K, \tfrac{x}{\varepsilon})\big) + 1\big), \partial_x z\Big)_{\mathrm{L}^2(K_{n\varepsilon})} = 0. \tag{3.44}$$

In order to do this, we split the integral over $K_{n\varepsilon}$ into n integral over sub-cells of size $\varepsilon|Y|$, make the change of variable $x = \varepsilon y$ and use the equation for χ (3.2). We conclude from (3.44) that $\psi(x) = \varepsilon\chi\big(x_K, \frac{x}{\varepsilon}\big)$ a.e. on $K_{n\varepsilon}$. Similarly we show that

$$\bar{b}^0_K = (n\varepsilon)^{-1}|Y|^{-1}\int_{K_{n\varepsilon}}\big(\chi\big(x_K, \tfrac{x}{\varepsilon}\big)\big)^2\,\mathrm{d}x = (n\varepsilon)^{-1}|Y|^{-1}\sum_{k=1}^n\int_Y\big(\chi(x_K, y)\big)^2\,\varepsilon\mathrm{d}y = b^0,$$

and that proves (i). We now show (ii). First, as $a^\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ and $|a^\varepsilon|_{\mathrm{W}^{1,\infty}(\Omega)} \leq C\varepsilon^{-1}$, elliptic H^2-regularity ensures that $|\psi|_{\mathrm{H}^2(K_\delta)} \leq C\varepsilon^{-1}|K_\delta|^{1/2}$. Hence,

$$\|\psi - \psi_h\|_{\mathrm{L}^2(K_\delta)} \leq Ch^2|\psi|_{\mathrm{H}^2(K_\delta)} \leq Ch^2\varepsilon^{-1}|K_\delta|^{1/2}. \tag{3.45}$$

We then evaluate $|K_\delta|\varepsilon^2\big|\bar{b}^0_K - b^0_K\big| = \big|\|\psi\|^2_{\mathrm{L}^2(K_\delta)} - \|\psi_h\|^2_{\mathrm{L}^2(K_\delta)}\big|$ as

$$|K_\delta|\varepsilon^2\big|\bar{b}^0_K - b^0_K\big| \leq \|\psi - \psi_h\|_{\mathrm{L}^2(K_\delta)}\big(2\|\psi\|_{\mathrm{L}^2(K_\delta)} + \|\psi - \psi_h\|_{\mathrm{L}^2(K_\delta)}\big),$$

and using (3.45), we obtain

$$|K_\delta|\varepsilon^2|\bar{b}^0_K - b^0_K| \leq C\varepsilon\big(h/\varepsilon\big)^2|K_\delta|^{1/2}\Big(\|\psi\|_{\mathrm{L}^2(K_\delta)} + \varepsilon\big(h/\varepsilon\big)^2|K_\delta|^{1/2}\Big).$$

As we are in dimension 1, $\psi \in \mathrm{L}^\infty(K_\delta)$ and $\|\psi\|_{\mathrm{L}^2(K_\delta)} \leq |K_\delta|^{1/2}\|\psi\|_{\mathrm{L}^\infty(K_\delta)}$, hence,

$$|K_\delta|\varepsilon^2|\bar{b}^0_K - b^0_K| \leq C|K_\delta|\Big(\varepsilon\big(h/\varepsilon\big)^2 + \varepsilon^2\big(h/\varepsilon\big)^4\Big).$$

As we assume $h \leq \varepsilon$, that proves (ii) and the proof of Lemma 3.12 is complete. ∎

Proof of Theorem 3.6. Let $e = \bar{u} - u_H$ and denote the norm $\|v\| = \|\partial_t v\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \|v\|_{\mathrm{L}^\infty(\mathrm{H}^1)}$. Recalling the splitting (3.26), we apply the triangle inequality and Lemma 3.10 and obtain

$$\|e\| \leq \|\eta\| + \|\zeta_H\| \leq C\big(\mathrm{e}^{\mathrm{data}}_{\mathrm{H}^1} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|\partial_t\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|I_H\partial_t^2\eta\|_{\mathrm{L}^1(\mathrm{H}^1)}\big). \tag{3.46}$$

Let us focus on the last term of the right hand side. Using Lemma 3.9 and Hölder inequality, we have the bound

$$\|I_H\partial_t^2\eta\|_{\mathrm{L}^1(\mathrm{H}^1)} \le C\big(\varepsilon^{-2}(\mathrm{e}_{a^0}+\mathrm{e}_{b^0})\textstyle\sum_{k=2,4}\|\partial_t^k\bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + CH^\ell\sum_{k=2,4}\|\partial_t^k\bar{u}\|_{\mathrm{L}^1(\mathrm{H}^{\ell+1})}\big). \tag{3.47}$$

The second term of the right hand side of (3.47) is part of the FEM error $\mathrm{e}^{\mathrm{FE}}_{\mathrm{H}^1}$. For the first term, Lemma 3.12 ensures that $\varepsilon^{-2}(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}) \le C(h/\varepsilon^2)^2$. Finally, applying Lemma 3.9 on the other terms of (3.46), we obtain estimate (3.24) and the proof of Theorem 3.6 is complete. ∎

Proof of Theorem 3.7. First, note that as we assume $h\le\varepsilon$ Lemma 3.12 ensures that $(1+\mathrm{e}_{a^0})(\mathrm{e}_{a^0}+\mathrm{e}_{b^0}) \le C(h/\varepsilon)^2$. The rest of the proof follows the same line as for Theorem 3.6 : Using the triangle and Hölder inequalities and Lemma 3.11, we obtain

$$\begin{aligned}\|e\|_{\mathrm{L}^\infty(\mathrm{L}^2)} &\le C\big(\mathrm{e}^{\mathrm{data}}_{\mathrm{L}^2} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \varepsilon\|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|I_H\partial_t\eta\|_{\mathrm{L}^1(\mathrm{L}^2)} + \varepsilon\|I_H\partial_t\eta\|_{\mathrm{L}^1(\mathrm{H}^1)}\big)\\ &\le C\varepsilon^{-2}(h/\varepsilon)^2\textstyle\sum_{k=0}^3\|\partial_t^k\bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})} + \mathrm{e}^{\mathrm{FE}}_{\mathrm{L}^2}.\end{aligned}$$

That proves estimate (3.25) and the proof of Theorem 3.7 is complete. ∎

4. Numerical experiments.

This section is divided in two parts. First, we verify through several examples that the a priori estimate from Theorem 3.7 is sharp. Second, we illustrate the result of Theorem 2.1 for various examples. Furthermore, we show that in practice the result still holds in the case of Dirichlet boundary conditions and for a locally periodic tensor. Let us define the data for the model problem as :

$$\begin{gathered}\Omega = (\text{-}1,1), \qquad Y = (\text{-}1/2,1/2),\\ f=0, \qquad g^0(x) = e^{-x^2/0.05} - \big\langle e^{-x^2/0.05}\big\rangle_\Omega, \qquad g^1 = 0.\end{gathered} \tag{4.1}$$

4.1. Convergence of the FE-HMM-L to the effective solution.

We verify here that the term $(h/\varepsilon^2)^2$ in the estimate (3.25) of Theorem 3.7 is sharp. Let the data be as in the model problem (4.1) and define the oscillatory periodic tensor as

$$a^\varepsilon(x) = a\big(\tfrac{x}{\varepsilon}\big) = \sqrt{2} + \sin\big(\pi\big(2\tfrac{x}{\varepsilon} - 1/2\big)\big), \tag{4.2}$$

where we choose $\varepsilon = 1/10$. The homogenized tensor is given by $a^0 = \big\langle 1/a(y)\big\rangle_Y^{-1} = 1$ and the solution of the single cell problem can be computed as $\chi(y) = \frac{1}{\pi}\operatorname{atan}\big((1+\sqrt{2})\tan(\pi y)\big) - y + C_0$, where C_0 ensures $\langle\chi\rangle_Y = 0$. Consequently, b^0 can be approximated accurately as $b^0 = \langle\chi^2\rangle_Y = 0.00909633$. We partition Ω in a macro mesh of size $H = 2^{-6}$ and choose a $\mathcal{P}^4$ macro finite element method (i.e. $\ell = 4$) so that the micro error term $(h/\varepsilon^2)^2$ dominates in estimate (3.25). The setting at the micro scale of the FE-HMM-L are set as $\delta=\varepsilon$, $q=1$ and we define the size of the mesh at iteration n as $h_n = \varepsilon/2^{n+1}$. The reference solution $\bar{u}$ is computed with $\mathcal{P}^4$-FEM on a mesh of size $H_{\mathrm{ref}} = 2^{-7}$. We use a leap-frog scheme for the time discretization with $\Delta t = H_{\mathrm{ref}}/50$, which ensures stability and a negligible time discretization error. For $n = 1,\dots,12$, we compute $\bar{u}$ and u_H on the time interval $[0,\varepsilon^{-2}] = [0,100]$. The obtained error $\|\bar{u}-u_H\|_{\mathrm{L}^\infty(\mathrm{L}^2)}$ is reported in the left plot of Figure 4.1, where we observe that it converges with rate 2, as predicted by Theorem 3.7.

We perform the same experiment with homogeneous Dirichlet boundary conditions instead of periodic (g^0 must be replaced by $g^0 + c \in \mathrm{H}^1_0(\Omega)$, $c\in\mathbb{R}$). We observe in the center plot of Figure 4.1 that the result of Theorem 3.7 also holds in this case.

We consider now the locally periodic tensor

$$a^\varepsilon(x) = a\big(x,\tfrac{x}{\varepsilon}\big) = \tfrac{1}{2} + \tfrac{1}{6}\big(\sin(2\pi x) + \sin\big(2\pi\tfrac{x}{\varepsilon}\big)\big), \tag{4.3}$$

where we set $\varepsilon = 1/10$. Note that $a^0(x)$ and $\chi(x,y)$ can be computed analytically and hence $b^0(x)$ can be approximated very accurately. The right plot of Figure 4.1 shows that in this case again the error converges with rate 2, as predicted by Theorem 3.7.

4.2. Dispersion and Boussinesq corrections.

In this section, we first illustrate the fact that the family of effective equations analysed in Theorem 2.1 captures the long time dispersive effects of u^ε, while the homogenized solution u^0 does not. Second, we compare u^ε and $\bar{u}$ in the cases of Dirichlet boundary conditions and a locally periodic tensor. Finally, we show that for even longer

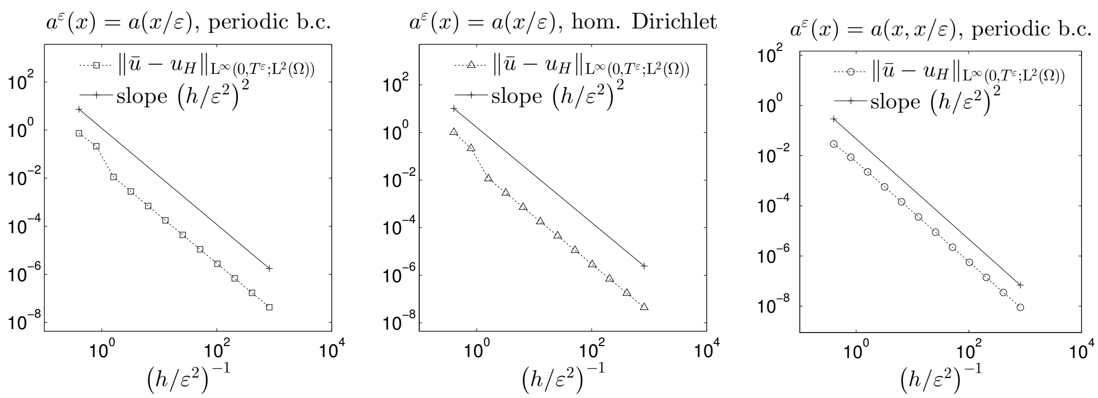

FIG. 4.1. *Log-log plot of* $\|\bar{u} - u_H\|_{\mathrm{L}^\infty(\mathrm{L}^2)}$ *for the following settings (from left to right) : tensor* (4.2)*, periodic b.c. ; tensor* (4.2)*, homogeneous Dirichlet b.c. ; tensor* (4.3)*, periodic b.c.*

times $T^\varepsilon = \mathcal{O}(\varepsilon^3)$ additional dispersion effects appear in u^ε which are not captured by $\bar{u}$. Note that $\bar{u}$ always represents the solution of (3.5), i.e., the effective equation with the normalization $\langle\chi\rangle_Y = 0$.

We set the data as in the model problem (4.1) with periodic boundary conditions and pick the oscillatory periodic tensor given by (4.2), where we choose $\varepsilon = 1/20$ (a^0 and b^0 are given in Section 4.1). We compute reference approximations of u^ε, u^0 and $\bar{u}$ with $\mathcal{P}^1$-FEM, on a mesh of size $H = \varepsilon/25$. The leap frog scheme is used for the time integration with a time step $\Delta t = H/50$.

We first compare u^ε and u^0. Observe in Figure 4.2 that as the time increases, dispersion effects appear in the macro behaviour of u^ε, while it does not appear in u^0. As expected, u^0 describes well the macro behaviour of u^ε at short times ($t = 20 = \varepsilon^{-1}$), while at long time it does not capture all its features. Next, we compare u^ε with $\bar{u}$. In Figure 4.3, we observe that the dispersion effects of u^ε are accurately captured by $\bar{u}$ even for a long time $T^\varepsilon = \varepsilon^{-2} = 400$, as predicted by Theorem 2.1.

For the same data, we compute the solutions $\tilde{u}_{\langle\chi\rangle}$ of (2.4) for several values of $\langle\chi\rangle$ in $[0, 0.6]$.

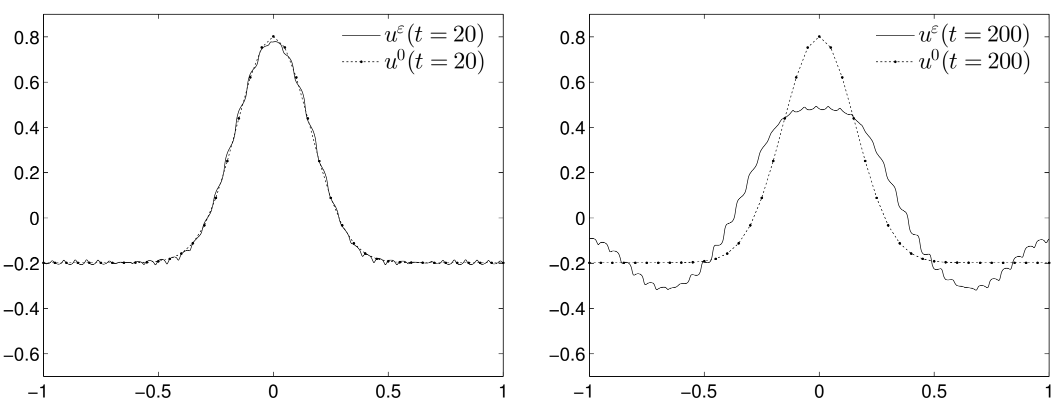

FIG. 4.2. *Comparison between the fine scale solution* u^ε *and the homogenized solution* u^0 *(tensor* (4.2)*, periodic b.c.).*

As (2.4) involves a 4th order differential operator in space, $\tilde{u}_{\langle\chi\rangle}$ is approximated using $\mathcal{C}^1$-FEM on a mesh of size $H = \varepsilon/16$. On Figure 4.4, we display u^ε, u^0, $\bar{u}$ and $\tilde{u}_{\langle\chi\rangle}$ at $t = 400 = \varepsilon^{-2}$ (each color corresponds to a particular $\langle\chi\rangle$). The result of Theorem 2.1 is verified : all the $\tilde{u}_{\langle\chi\rangle}$ and $\bar{u}$ captures the long time dispersive effects of u^ε.

Consider now the same problem but with homogeneous Dirichlet boundary conditions (instead of periodic). Even though this setting does not verify the hypotheses of Theorem 2.1, we verify for

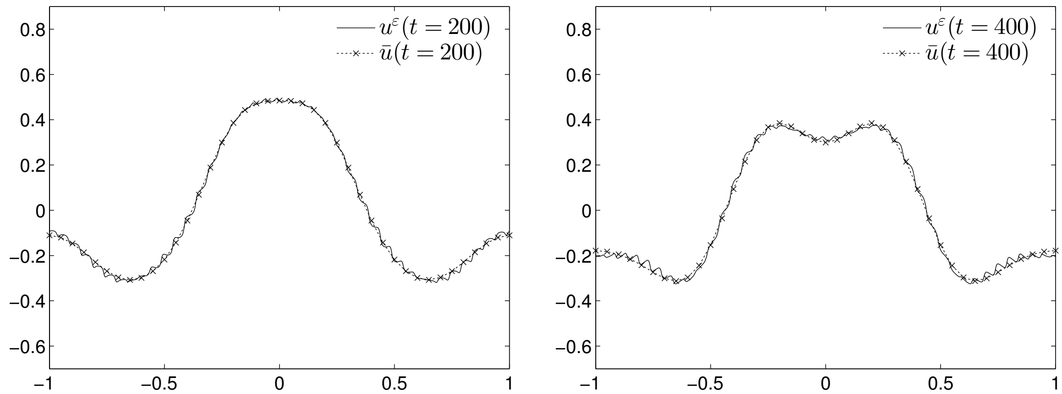

FIG. 4.3. *Comparison between the fine scale solution u^ε and the Boussinesq solution $\bar{u}$ (tensor (4.2), periodic b.c.).*

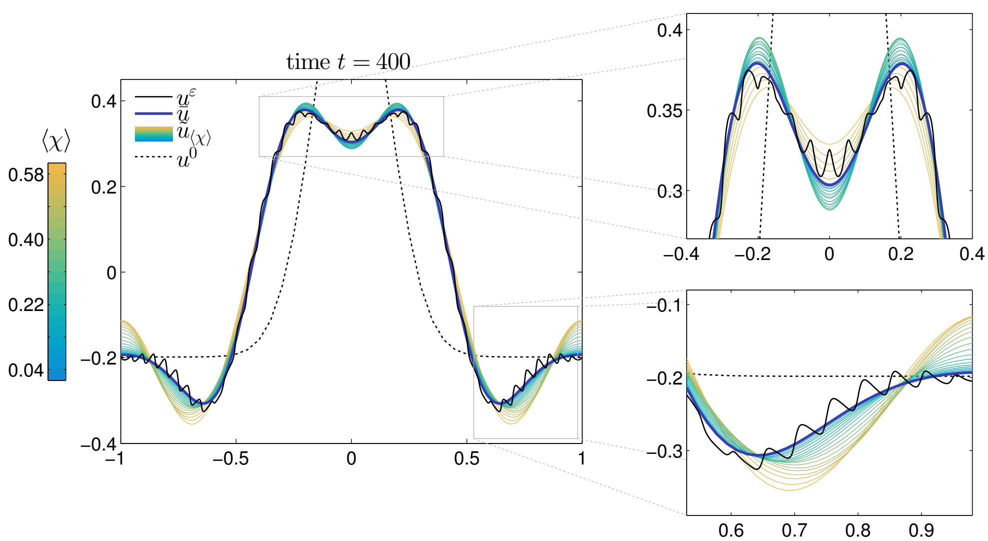

FIG. 4.4. *Comparison between the fine scale solution u^ε, the homogenized solution u^0 and the Boussinesq solutions $\tilde{u}_{\langle\chi\rangle}$ (equation (2.4)) for $\langle\chi\rangle \in [0, 0.6]$ (tensor (4.2), periodic b.c.).*

this example in Figure 4.5 that at times $\mathcal{O}(\varepsilon^{-2})$, $\bar{u}$ captures the dispersive effects of u^ε, while u^0 does not.

We consider now the locally periodic tensor (4.3) where we set $\varepsilon = 1/20$ and the model problem (4.1) with periodic boundary conditions. In Figure 4.6, we observe that the result of Theorem 2.1 is still verified numerically for this example.

In the last experiment, we show that the ε^2-order operator of the effective equation (3.5) is not sufficient for times $\mathcal{O}(\varepsilon^{-3})$ or greater. Set the data as (4.1) with periodic boundary conditions and pick the tensor (4.2) with $\varepsilon = 1/20$. In Figure 4.7, we can see that at $t = 2000 > \varepsilon^{2.5}$, dispersion effects that are not captured by $\bar{u}$ are visible. At $t = 7900 \sim \varepsilon^{-3}$, these effects are even more pronounced. It seems likely that at times $\mathcal{O}(\varepsilon^{-3})$, an additional correction operator is required in the effective equation in order to fully capture the dispersive effects.

REFERENCES

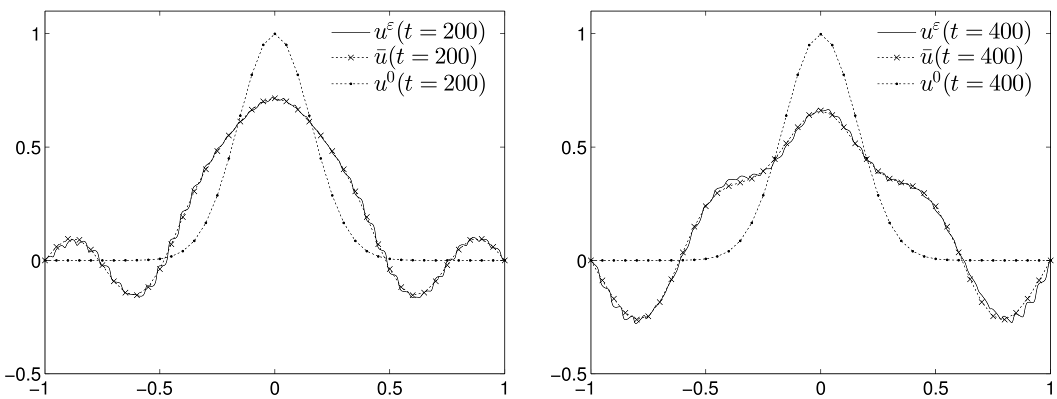

FIG. 4.5. *Comparison between the fine scale solution u^ε, the Boussinesq solution $\bar{u}$ and the homogeneous solution u^0 (tensor* (4.2)*, homogeneous Dirichlet b.c.).*

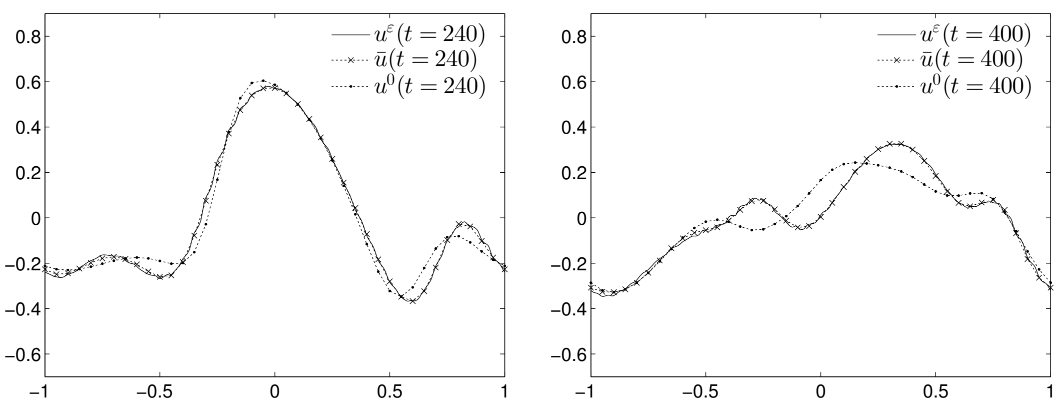

FIG. 4.6. *Comparison between the fine scale solution u^ε, the Boussinesq solution $\bar{u}$ and the homogeneous solution u^0 (tensor* (4.3)*, periodic b.c.).*

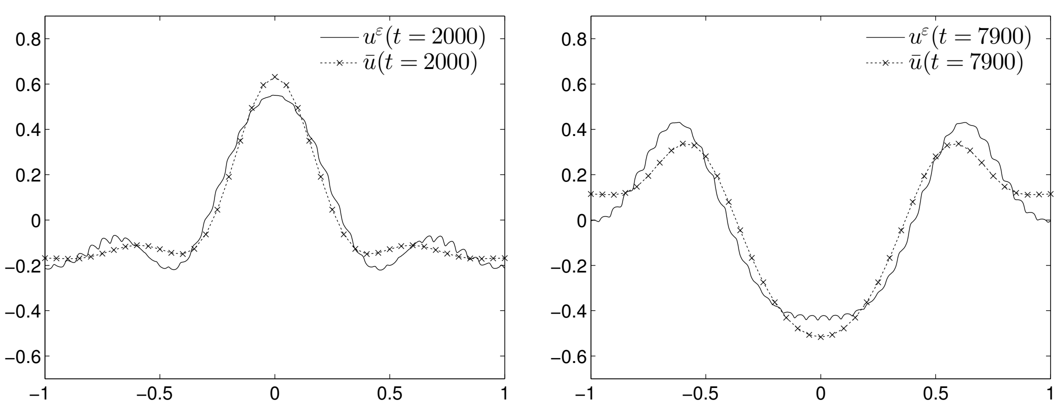

FIG. 4.7. *Comparison between the fine scale solution u^ε and the Boussinesq solution $\bar{u}$ (tensor* (4.2)*, periodic b.c.).*

[1] ASSYR ABDULLE, *On a priori error analysis of fully discrete heterogeneous multiscale FEM*, Multiscale Model. Simul., 4 (2005), pp. 447–459.

[2] ———, *Discontinuous Galerkin finite element heterogeneous multiscale method for elliptic problems with multiple scales*, Math. Comp., 81 (2012), pp. 687–713.

[3] Assyr Abdulle and Weinan E, *Finite difference heterogeneous multi-scale method for homogenization problems*, J. Comput. Phys., 191 (2003), pp. 18–39.

[4] Assyr Abdulle, Weinan E, Björn Engquist, and Eric Vanden-Eijnden, *The heterogeneous multiscale method*, Acta Numer., 21 (2012), pp. 1–87.

[5] Assyr Abdulle and Marcus J. Grote, *Finite element heterogeneous multiscale method for the wave equation*, Multiscale Model. Simul., 9 (2011), pp. 766–792.

[6] Assyr Abdulle, Marcus J. Grote, and Christian Stohrer, *FE heterogeneous multiscale method for long-time wave propagation*, C. R. Math. Acad. Sci. Paris, 351 (2013), pp. 495–499.

[7] ———, *Finite element heterogeneous multiscale method for the wave equation: long-time effects*. Accepted in Multiscale Model. Simul., 2013.

[8] Assyr Abdulle and Gilles Vilmart, *Coupling heterogeneous multiscale FEM with Runge-Kutta methods for parabolic homogenization problems: a fully discrete space-time analysis*, Math. Models Methods Appl. Sci., 22 (2012), pp. 1250002/1–40.

[9] Doghonay Arjmand and Olof Runborg, *Analysis of heterogeneous multiscale methods for long time wave propagation problems*, Multiscale Model. Simul., 12 (2014), pp. 1135–1166.

[10] ———, *Analysis of hmm for long time multiscale wave propagation problems in locally-periodic media*, (2015).

[11] Garth A. Baker and Vassilios Dougalis, *The effect of quadrature errors on finite element approximations for second order hyperbolic equations*, SIAM J. Numer. Anal., 13 (1976).

[12] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland Publishing Co., Amsterdam, 1978.

[13] Safia Brahim-Otsmane, Gilles A. Francfort, and François Murat, *Correctors for the homogenization of the wave and heat equations*, J. Math. Pures Appl., 71 (1992), pp. 197–231.

[14] Yann Capdeville, Laurent Guillot, and Jean-Jacques Marigo, *1-d non-periodic homogenization for the seismic wave equation*, Geophys. J. Int., 181 (2010), pp. 897–910.

[15] Philippe G. Ciarlet, *The finite element method for elliptic problems*, vol. 4 of Studies in Mathematics and its Applications, North-Holland, 1978.

[16] Philippe G. Ciarlet and Pierre Arnaud Raviart, *The combined effect of curved boundaries and numerical integration in isoparametric finite element methods*, in The mathematical foundations of the finite element method with applications to partial differential equations, 1972, pp. 409–474.

[17] Doina Cioranescu and Patrizia Donato, *An introduction to homogenization*, vol. 17 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, New York, 1999.

[18] A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, International series in pure and applied mathematics, McGraw-Hill, 1955.

[19] Tomas Dohnal, Agnes Lamacz, and Ben Schweizer, *Bloch-wave homogenization on large time scales and dispersive effective wave equations*, Multiscale Model. Simul., 12 (2014), pp. 488–513.

[20] ———, *Dispersive homogenized models and coefficient formulas for waves in general periodic media*, ArXiv e-prints, (2014).

[21] Weinan E and Björn Engquist, *The heterogeneous multiscale methods*, Commun. Math. Sci., 1 (2003), pp. 87–132.

[22] Björn Engquist, Henrik Holst, and Olof Runborg, *Multi-scale methods for wave propagation in heterogeneous media*, Commun. Math. Sci., 9 (2011).

[23] ———, *Multiscale methods for wave propagation in heterogeneous media over long time*, in Numerical analysis of multiscale computations, Springer, 2012, pp. 167–186.

[24] Lawrence C. Evans, *Partial Differential Equations*, Graduate studies in mathematics, American Mathematical Society, 1998.

[25] Vasilii V. Jikov, Sergei M. Kozlov, and Olga A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, Heidelberg, 1994.

[26] Olga Aleksandrovna Ladyzhenskaya, *The boundary value problems of mathematical physics*, vol. 49 of Applied Mathematical Sciences, Springer-Verlag, New York, 1985.

[27] Agnes Lamacz, *Dispersive effective models for waves in heterogeneous media*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1871–1899.

[28] Jacques-Louis Lions and Enrico Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1 of Travaux et recherches mathématiques, Dunod, Paris, 1968.

[29] François Murat and Luc Tartar, *H-convergence*, in Topics in the mathematical modelling of composite materials, vol. 31 of Progr. Nonlinear Differential Equations Appl., Birkhäuser Boston, Boston, MA, 1997, pp. 21–43.

[30] Houman Owhadi and Lei Zhang, *Numerical homogenization of the acoustic wave equations with a continuum of scales*, Comput. Methods Appl. Mech. Engrg., 198 (2008), pp. 397–406.

[31] Pierre Arnaud Raviart, *The use of numerical integration in finite element methods for solving parabolic equations*, in Topics in numerical analysis. Proceedings of the Royal Irish Academy, Conference on Numerical Analysis, 1972, John J. H. Miller, ed., Academic Press, 1973, pp. 233–264.

[32] Fadil Santosa and William Symes, *A dispersive effective medium for wave propagation in periodic composites*, SIAM J. Appl. Math., 51 (1991), pp. 984–1005.

[33] Tetyana Vdovina, Susan E. Minkoff, and Oksana Korostyshevskaya, *Operator upscaling for the acoustic wave equation*, Multiscale Model. Simul., 4 (2005), pp. 1305–1338 (electronic).