LOCALIZING POINT SOURCES IN DIFFUSION FIELDS FROM SPATIOTEMPORAL SAMPLES

Yue M. Lu[1], *Pier Luigi Dragotti*[2] *and Martin Vetterli*[3]

[1]Harvard School of Engineering and Applied Sciences, Cambridge, MA, USA
[2]Imperial College London, UK
[3]Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland
Emails: yuelu@seas.harvard.edu; p.dragotti@imperial.ac.uk; martin.vetterli@epfl.ch

ABSTRACT

Consider a diffusion field induced by a finite number of localized and instantaneous sources. In this paper, we study the problem of estimating these sources (including their intensities, spatial locations, and activation time) from the spatiotemporal samples taken by a network of spatially distributed sensors. We propose two estimation algorithms, depending on whether the activation time of the sources is known. For the case of known activation time, we present an annihilating filter based method to estimate the Euclidean distances between the sources and sensors, which can be subsequently used to localize the sources. For the case of a single source but with unknown activation time, we show that the diffusion field at any spatial location is a scaled and shifted version of a common prototype function, and that this function is the unique solution to a particular differential equation. This observation leads to an efficient algorithm that can estimate the unknown parameters of the source by solving a system of linear equations. For both algorithms proposed in this work, the minimum number of sensors required is $d + 1$, where d is the spatial dimension of the field. This requirement is independent of the number of active sources.

Keywords— diffusion field, source localization, finite rate of innovation, spatiotemporal sampling

1. INTRODUCTION

Diffusion models many important physical, biological and social phenomena, including temperature variations, air pollution dispersion, biochemical substance release, and epidemic dynamics. Sampling and reconstructing diffusion processes using a network of sensors have applications ranging from data center temperature monitoring [2] (detecting cold and hot spots responsible for energy inefficiencies) to environmental monitoring [1] and homeland security [5].

At the microscopic scale, diffusion describes the random motion of a large number of particles, migrating from regions of high concentration to those of low concentration. At the macroscopic scale, the "average" statistical distributions of these particles are governed by the diffusion equation, originally derived by Fick [6] in 1855. In its simplest form—when the underlying medium is isotropic—the diffusion equation can be written as

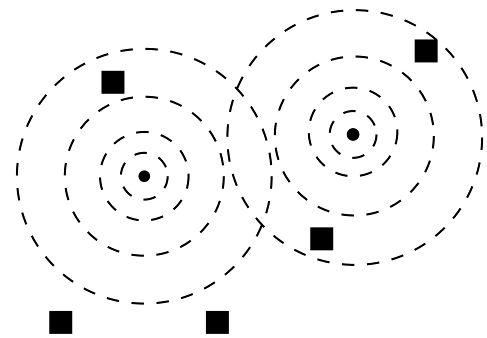

Fig. 1. Localization of point-like sources (shown as black dots) in a 2-D diffusive field based on the spatiotemporal samples taken by a network of sensors (shown as squares). Dashed lines represent the isolines of the diffusion fields induced by the two sources, respectively.

$$\frac{\partial}{\partial t} f(\boldsymbol{x}, t) \;=\; \mu \, \nabla^2 f(\boldsymbol{x}, t) + s(\boldsymbol{x}, t), \qquad (1)$$

where $f(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \mathbb{R}^d$ is the spatiotemporal distribution of the field, μ is the diffusion coefficient, $\nabla^2 f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$ is the (spatial) Laplacian of $f(\boldsymbol{x}, t)$, and $s(\boldsymbol{x}, t)$ represents the sources of the field.

In this paper, we consider the case where the unknown sources $s(x, t)$ are localized and instantaneous, *i.e.*,

$$s(\boldsymbol{x}, t) = \sum_{k=1}^{K} c_k \, \delta(\boldsymbol{x} - \boldsymbol{x}_k, t - t_k), \qquad (2)$$

for some unknown parameters $\{c_k, \boldsymbol{x}_k, t_k\}_{k=1}^{K}$. In practice, this source model can describe sudden events (*e.g.*, explosions or accidental release of pollutants) that are the key targets of various environmental monitoring or security applications.

As shown in Figure 1, we use a network of spatially distributed sensors to monitor the field, with each sensor taking samples over time. Our goal is to estimate the unknown sources (including their intensities $\{c_k\}_k$, locations $\{\boldsymbol{x}_k\}_k$ and activation time $\{t_k\}_k$) from the spatiotemporal sensor measurements.

P. L. Dragotti is supported in part by a Global Research Award from the Royal Academy of Engineering.

In this paper, we propose two algorithms for estimating the unknown sources, corresponding to two different scenarios. In Section 2, we consider the case when multiple sources are activated at a *common* and *known* time instant. We show that, under this assumption, the temporal samples at each sensor can be arranged in the form of a linear combination of exponentials. We propose to use the annihilating filter method [7, 8, 3] to estimate the exponents, which reveal the Euclidean distances between the sources and sensors. Incorporating the distance information from multiple sensors, we describe a simple linear algorithm in Section 2.3 to estimate the spatial locations of the sources.

Section 3 deals with the case when there is a single source in the field, but the activation time of the source is unknown. We show that the induced diffusion field at each sensor location can be written as a scaled and shifted version of a common prototype function. We further observe that this prototype function is the unique solution to a particular differential equation. This observation enables us to build a set of linear equations linking the unknown parameters (time instants and source-to-sensor distances) to local sensor measurements. Estimating the parameters of the source then boils down to solving a set of linear equations.

As a desirable property, the minimum number of sensors required by both algorithms is $d+1$, independent of the number of active sources in the field. This demonstrates the benefit of taking multiple temporal samples at each sensor.

In the rest of the paper, we assume that the field $f(\boldsymbol{x},t)$ is supported on two spatial dimensions (*i.e.*, $d=2$). However, our discussion and proposed algorithms can be easily extended to the general d-dimensional cases.

2. LOCATING POINT SOURCES WITH KNOWN ACTIVATION TIME

In this section, we consider the case when K sources are activated at a common and known time instant. Our goal is then to estimate the intensities and spatial locations of these sources.

2.1. A Parametric Form of the Diffusion Field

We start by presenting a parametric form of the diffusion field governed by (1). Due to the linearity of (1), its solutions have the form

$$f(\boldsymbol{x},t) \;=\; (s * g)(\boldsymbol{x},t), \tag{3}$$

where $g(\boldsymbol{x},t)$ is the Green's function of the equation and $*$ denotes convolutions along the spatial and temporal dimensions [4]. In this sense, the diffusion equation behaves exactly like a linear, shift-invariant system, linking the "input" $s(\boldsymbol{x},t)$ to the induced "output" $f(\boldsymbol{x},t)$ through a spatiotemporal impulse response $g(\boldsymbol{x},t)$.

A closed-form expression for the Green's function is

$$g(\boldsymbol{x},t) = \frac{1}{4\pi\mu t} e^{-\frac{\|\boldsymbol{x}\|^2}{4\mu t}} U(t), \tag{4}$$

where $\|\cdot\|$ denotes Euclidean norm and $U(t)$ is a unit-step function. On substituting (4) and (2) into (3), we get

$$f(\boldsymbol{x},t) = \sum_{k=1}^{K} \frac{c_k}{4\pi\mu(t-t_k)} e^{-\frac{\|\boldsymbol{x}-\boldsymbol{x}_k\|^2}{4\mu(t-t_k)}} U(t-t_k). \tag{5}$$

The diffusion field described above is not bandlimited in space or time; however, it is completely determined by a finite number of parameters, namely, the intensities, locations and activation time of the sources. For this reason, the induced field can be seen as a parametric signal with finite rate of innovation [8].

2.2. Multiple Sources with Known Activation Time

Assume now, that the field in (5) is observed with N spatially-distributed sensors, located at $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\} \subset \mathbb{R}^2$, respectively. For a fixed time t, the measurement $y_n(t)$ obtained with the nth sensor can be written as

$$y_n(t) = \sum_{k=1}^{K} \frac{c_k}{4\pi\mu(t-t_k)} e^{-\frac{D_{k,n}}{4\mu(t-t_k)}} U(t-t_k), \tag{6}$$

where

$$D_{k,n} \stackrel{\text{def}}{=} \|\boldsymbol{x}_k - \boldsymbol{p}_n\|^2 \tag{7}$$

is the squared Euclidean distance between the kth source and the nth sensor.

The estimation of both the positions and the activation time of the sources is in general difficult. In this section, we consider a simplified scenario where the K sources are all activated at the same time, *i.e.*, $t_0 = t_1 = \ldots t_{k-1} = \tau$, with τ either known or been correctly estimated. Without loss of generality, set $\tau = 0$ and (6) then becomes

$$y_n(t) = \sum_{k=1}^{K} \frac{c_k}{4\pi\mu t}\, e^{-\frac{D_{k,n}}{4\mu t}} \quad \text{for } t > 0. \tag{8}$$

Suppose that the sensors take uniform samples over time, with sampling interval T. Let J be the least common multiple of the integers $1, 2, \ldots, 2K$. In what follows, we show that taking J samples at each sensor is sufficient to recover $\{c_k\}$ and $\{D_{k,n}\}_{k,n}$, the latter of which can be subsequently used to infer the sensor locations $\{\boldsymbol{x}_k\}_k$.

To this end, consider the sampling time instants $\{m_\ell T : \ell = 1, 2, \ldots, 2K\}$, where $m_\ell \stackrel{\text{def}}{=} J/\ell$. For example, if $K=2$, we choose $J = \text{lcm}(1,2,3,4) = 12$ and pick the time instants $\{12T, 6T, 4T, 3T\}$. From (8), the corresponding sensor measurements taken at these time instants are

$$y_n(m_\ell T) = \sum_{k=1}^{K} \frac{c_k}{4\pi\mu JT/\ell}\, e^{-\frac{D_{k,n}}{4\mu JT}\ell}.$$

Defining a new sequence $w_\ell \stackrel{\text{def}}{=} y_n(m_\ell T)(4\pi\mu JT/\ell)$, we have

$$w_\ell = \sum_{k=1}^{K} c_k u_k^\ell, \quad \text{for } \ell = 1, 2, \ldots, 2K \tag{9}$$

where $u_k = e^{-\frac{D_{k,n}}{4\mu JT}}$.

Note that the sequence w_ℓ above is a linear combination of exponentials u_k. Signals of the form (9) are often encountered in array signal processing [7] and many algorithms exist that permit the retrieval of the parameters $\{c_k, u_k\}_{k=1}^{K}$ from $2K$ consecutive samples of w_ℓ. In our algorithm, we choose to use the annihilating filter method to estimate these parameters. Due to space limitations, we omit here further discussion on this method, and refer readers to [8, 3, 7] for details.

2.3. Source Localization from Distance Information

From the parameters $\{u_k\}_k$ in (9), we can obtain the squared distance $D_{k,n}$ between any pair of source and sensor. This information can be used to determine the spatial locations $\{\boldsymbol{x}_k\}$ of the sources. For simplicity, we assume that the source intensities $\{c_k\}_k$ are distinct, and thus there is no ambiguity in the correct labeling of the sources among different sensors.

Expanding (7), we have the following set of equations

$$D_{k,n} = \langle \boldsymbol{x}_k - \boldsymbol{p}_n, \boldsymbol{x}_k - \boldsymbol{p}_n \rangle = \|\boldsymbol{x}_k\|^2 - 2\boldsymbol{p}_n^T \boldsymbol{x}_k + \|\boldsymbol{p}_n\|^2, \quad (10)$$

where $\{D_{k,n}\}$ and $\{\boldsymbol{p}_n\}$ are known parameters, and $\{\boldsymbol{x}_k\}$ are the unknowns to be estimated. To be clear, (10) is a quadratic equation of $\boldsymbol{x}_k$. However, by treating $\left\{\|\boldsymbol{x}_k\|^2\right\}$ and $\{\boldsymbol{x}_k\}$ as if they were independent free variables, we can view (10) as a set of KN linear equations with $3K$ unknowns. When $N \geq d + 1 = 3$, we can obtain $\{\boldsymbol{x}_k\}$ by simply solving a system of linear equations.

3. ESTIMATING A SINGLE SOURCE WITH UNKNOWN ACTIVATION TIME

In this section, we consider the case when there is a single source in the field, but the activation time of the source is unknown. We leave the discussion on multiple independent sources to a future work.

3.1. The Temporal Prototype Function of Diffusion Fields

Our proposed algorithm is based on the following observation: at any spatial location, the diffusion field induced by sparse sources can be represented by a linear combination of a prototype function. To see this, we define

$$\varphi(t) \stackrel{\text{def}}{=} \frac{1}{4\pi t}\, e^{-1/(4t)}\, U(t). \quad (11)$$

We can easily verify that, at any given spatial location $\boldsymbol{x}$ ($\boldsymbol{x} \neq \boldsymbol{x}_k, 1 \leq k \leq K$), the field $f(\boldsymbol{x}, t)$ in (5) can be written as

$$f(\boldsymbol{x}, t) = \sum_{k=1}^{K} c'_k\, \varphi\left(\frac{t - t_k}{s_k}\right), \quad (12)$$

where $s_k \stackrel{\text{def}}{=} \|\boldsymbol{x} - \boldsymbol{x}_k\|^2/\mu$ and $c'_k \stackrel{\text{def}}{=} c_k/(s_k \mu)$. In words, the diffusion field at any spatial location can be written as a linear combination of K scaled and shifted versions of the prototype

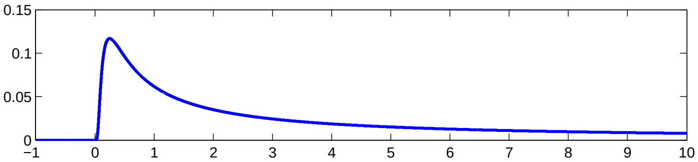

Fig. 2. The temporal prototype function $\varphi(t)$ of the diffusion process.

function $\varphi(t)$. Meanwhile, the scale and shift parameters correspond to the spatial locations and activation time of the sources, respectively.

In what follows, we present several important properties of the prototype function $\varphi(t)$. Figure 2 shows the values of $\varphi(t)$ over $t \in [-1, 10]$. We can easily verify that $\varphi(t)$ reaches its maximum (e^{-1}/π) at $t = 1/4$, and that it decays at the rate of $1/t$ for sufficiently large t.

A somewhat surprising fact is that, despite the existence of the unit step $U(t)$ in the definition (11), the function $\varphi(t)$ is *smooth* on the entire real line, *i.e.*, it has derivatives of all orders over $t \in \mathbb{R}$.

Lemma 1 *The function $\varphi(t)$ belongs to C^∞ on the real line. Meanwhile, at $t = 0$, the derivatives of $\varphi(t)$ of all orders are equal to zero.*

Proof We prove by induction. Assume that $\varphi^{(k)}(0) = 0$ for some $k \geq 0$. It is easy to verify that, for $t > 0$,

$$\varphi^{(k)}(t) = P(t^{-1})\, e^{-1/(4t)},$$

where $P(t^{-1})$ is some polynomial of t^{-1}. It follows that

$$\varphi^{(k+1)}(0) \stackrel{\text{def}}{=} \lim_{t \to 0^+} \left(P(t^{-1})\, e^{-1/(4t)} - 0\right)/t = 0.$$

□

Lemma 2 *The function $\varphi(t)$ satisfies the following differential equation:*

$$4t^2\, \varphi'(t) + (4t - 1)\, \varphi(t) = 0, \quad \textit{for } t \in \mathbb{R}. \quad (13)$$

Proof This can be straightforwardly verified from the definition of $\varphi(t)$ in (11). □

3.2. Sampling and Reconstructing Diffusion Fields with One Localized Source

Based on the properties of the prototype function $\varphi(t)$ presented above, we propose a sampling and reconstruction algorithm for diffusion fields induced by a *single* source $c\,\delta(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)$ with unknown location and activation time.

Consider $y_n(t)$, the diffusion field observed at the nth sensor at location $\boldsymbol{p}_n$. It follows from (12) that

$$y_n(t) = \frac{c}{D_n}\, \varphi\left(\frac{t - \tau}{D_n/\mu}\right),$$

where $D_n = \|\boldsymbol{\xi} - \boldsymbol{p}_n\|^2$ is the squared distance between the source location $\boldsymbol{\xi}$ and the sensor location $\boldsymbol{p}_n$.

Proposition 1 *The function $y_n(t)$ satisfies the following differential equation*

$$\frac{y_n(t)}{4\mu} D_n + \Big(y_n(t) + 2\,t\,y_n'(t)\Big)\tau - y_n'(t)\,\tau^2 = t^2 y_n'(t) + t\,y_n(t), \tag{14}$$

for $t \in \mathbb{R}$.

Proof This follows directly from Lemma 2. In particular, we can obtain (14) by replacing t with $(t-\tau)\mu/D_n$ in (13) and rearranging the terms. □

Consider the unknown parameters $[D_n, \tau, \tau^2]$ as an unknown vector in $\mathbb{R}^3$. Then, for each t, (14) provides a different *linear* equation, whose coefficients can be obtained from $y_n(t)$ and $y_n'(t)$. In practice, reliably estimating the derivative of a function is problematic, due to the noise amplification property of the differentiation operation. However, this concern for robustness can be alleviated by exploiting the linearity of (14), as shown in the following proposition.

Proposition 2 *Let $w(t)$ be an arbitrary window function. Then*

$$\left(\int \frac{y_n(t)w(t)}{4\mu} dt\right) D_n + \left(\int \Big(y_n(t) + 2\,t\,y_n'(t)\Big) w(t)\,dt\right)\tau - \left(\int y_n'(t)w(t)\,dt\right)\tau^2 = \int \Big(t^2 y_n'(t) + t\,y_n(t)\Big) w(t)\,dt. \tag{15}$$

Let $\{y_{n,m}\}_m$ denotes the set of measurements taken by the nth sensor. To apply the result of Proposition 2, we set the measurements to

$$\begin{aligned} y_{n,1+4k} &= \int \frac{y_n(t)w(t-kT)}{4\mu} dt, \\ y_{n,2+4k} &= \int \Big(y_n(t) + 2\,t\,y_n'(t)\Big) w(t-kT)\,dt, \\ y_{n,3+4k} &= -\int y_n'(t)w(t-kT)\,dt, \\ y_{n,4+4k} &= \int \Big(t^2 y_n'(t) + t\,y_n(t)\Big) w(t-kT)\,dt, \end{aligned} \tag{16}$$

for $k = 0, 1$, and some fixed sampling interval $T > 0$. In practice, the derivatives and integrations in (16) can be approximated by finite differences and finite sums on a dense sampling grid. By choosing the window function $w(t)$ to be polynomial B-splines, it is also possible to compute *exactly* the sample values in (16). Detailed discussion on this implementation is left to a future paper.

After obtaining the samples $y_{n,m}$, it follows from (15) that

$$\begin{pmatrix} y_{n,1} & y_{n,2} & y_{n,3} \\ y_{n,5} & y_{n,6} & y_{n,7} \end{pmatrix} \begin{pmatrix} D_n \\ \tau \\ \tau^2 \end{pmatrix} = \begin{pmatrix} y_{n,4} \\ y_{n,8} \end{pmatrix}. \tag{17}$$

Note that (17) gives two linear constraints for three variables. To uniquely solve for the unknown parameters, we can either take more temporal samples [by setting $k = 2, 3, \ldots$ in (16)] or incorporate measurements taken at different sensors. The latter approach is feasible because each additional sensor brings in two more linear equations in the form of (17) but only one new variable (*i.e.*, D_n), while τ and τ^2 remain the same for all sensors.

Finally, once we obtain $\{D_n\}$, *i.e.*, the squared distances from the source to the sensors, we can use the same technique presented in Section 2.3 to recover the spatial location of the source. For that purpose, we need at least three different sensors.

4. CONCLUSION

We studied the problem of estimating the unknown sources (including their intensities, spatial locations, and activation time) of a diffusion field from the spatiotemporal samples taken by a network of spatially distributed sensors. When the sources are activated at a common and known time, we first estimate the Euclidean distances between the sources and sensors using the annihilating filter method. Then the distance information from multiple sensors is used to localize the sources. When there is a single source but with unknown activation time, we showed that the sensor measurements at any sensor location is a scaled and shifted version of a common prototype function. The properties of the prototype function leads to a set of linear constraints, linking the unknown parameters of the source to local sensor measurements. Estimating the spatial location and activation time of the pointwise sources then boils down to solving a set of linear equations.

5. REFERENCES

[1] G. Barrenetxea, F. Ingelrest, G. Schaefer, M. Vetterli, O. Couach, and M. Parlange. SensorScope: Out-of-the-box environmental monitoring. In *Proc. 7th Int. Conf. on Information Processing in Sensor Networks (IPSN 2008)*, St. Louis, USA, Apr. 2008.

[2] R. Bianchini and R. Rajamony. Power and energy management for server systems. *Computer*, 37(11):68–76, November 2004.

[3] P.L. Dragotti, M. Vetterli, and T. Blu. Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strang-Fix. *IEEE Trans. Signal Process.*, 55(5):1741–1757, May 2007.

[4] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, RI, 1998.

[5] A. Nehorai, B. Porat, and E. Paldi. Detection and localization of vapor-emitting sources. *IEEE Trans. Signal Process.*, 43(1):243–253, Jan. 1995.

[6] J. Philibert. One and a half century of diffusion: Fick, Einstein, before and beyond. *Diffusion Fundamentals*, 2(1):1–10, 2005.

[7] P. Stoica and R. L. Moses. *Introduction to spectral analysis*. Prentice-Hall, Englewood Cliffs, NJ, 1997.

[8] M. Vetterli, P. Marziliano, and T. Blu. Sampling signals with finite rate of innovation. *IEEE Trans. Signal Process.*, 50(6):1417–1428, Jun. 2002.