Engineering Structures 89 (2015) 204–217

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Force-based higher-order beam element with flexural–shear–torsional interaction in 3D frames. Part I: Theory

António A. Correia [a,b,*], João P. Almeida [a,c,1], Rui Pinho [d,2]

[a] EUCENTRE, Pavia, Italy
[b] Current affiliation: Department of Structures, National Laboratory for Civil Engineering (LNEC), Lisbon, Portugal
[c] Current affiliation: École Polytechnique Fédérale de Lausanne, Switzerland
[d] Department of Civil Engineering and Architecture, University of Pavia, Italy

ARTICLE INFO

Article history:
Received 25 February 2014
Revised 30 August 2014
Accepted 6 October 2014
Available online 16 January 2015

Keywords:
Beam element
Force-based
Higher-order
Flexural–shear–torsional interaction
Warping
Timoshenko

ABSTRACT

An innovative higher-order beam theory, capable of accurately taking into account flexural–shear–torsional interaction, is originally combined with a force-based formulation to derive the corresponding finite element. The selected set of higher-order deformation modes leads to an explicit and direct interaction between three-dimensional shear and normal stresses. Namely, cross-sectional displacement and strain fields are composed of independent and orthogonal modes, which results in unambiguously defined generalised cross-sectional stress-resultants and in a minimisation of the coupling of equilibrium equations. On the basis of work-equivalency to three-dimensional continuum theory, dual one-dimensional higher-order equilibrium and compatibility equations are derived. The former, which govern an advanced form of beam equilibrium, are strictly satisfied via stress fields arising from the solution of the corresponding systems of coupled differential equations. The formulation, which is numerically validated in a companion paper for both linear and nonlinear material response, inherently avoids shear-locking and accurately accounts for span loads. Finally, the superiority of force-based approaches over displacement-based ones, well established for inelastic behaviour, is also demonstrated for the linear elastic case.

1. Introduction

1.1. Review of higher-order beam theories

The geometrical features of many structural engineering elements make it possible to construct a set of governing differential equations which are considerably easier to solve than their complex three-dimensional continuum counterparts. In particular, beam theory is the simplest and simultaneously one of the most widely employed structural mechanics theories. Classical beam theory was initially based on the plane sections assumption, which was stated by Hooke in the XVIIth century. Further developed in the XVIIIth century by Bernoulli and Euler, such classical beam theory is only applicable to slender beams since it neglects the effect of shear deformation. It would not reach generalised engineering application until the end of the XIXth century. Meanwhile, with a view to the application to thick or deep beams, Rankine [1] and Bresse [2] included the relaxation of the restriction on the angle of shearing deformation, allowing the cross-section not to remain perpendicular to the beam's centroidal line, despite remaining a plane section and rigid in its own plane. Following the work by Timoshenko [3,4], this theory eventually was named after him.

It is well known that such classical beam theories, namely the Euler–Bernoulli and Timoshenko ones, are often not sufficiently accurate to predict the global member response and its internal stress–strain state. For instance, in the Timoshenko beam theory (TBT), the shear strain distribution is incorrectly assumed to be constant throughout the beam height; considering a simple rectangular cross-section, it does not respect the zero shear strain and stress boundary conditions at its top and bottom. Therefore, a shear correction factor is required to accurately determine the strain energy of deformation. Mindlin and Deresiewicz [5] computed such correction factor for a variety of beam cross-sections. Cowper [6], based on a pioneering integration of the three-dimensional equilibrium equations to form beam governing relations, obtained a new definition for the shear coefficient and derived expressions for homogeneous, isotropic symmetric cross-sections—see also Cowper [7]. An account of the early history of the shear correction factor can be found in Kaneko [8]. Research

* Corresponding author at: LNEC-DE/NESDE, Av. do Brasil 101, 1700-066 Lisbon, Portugal.

E-mail addresses: aacorreia@lnec.pt (A.A. Correia), joao.almeida@epfl.ch (J.P. Almeida), rui.pinho@unipv.it (R. Pinho).

[1] Address: EPFL ENAC IIC EESD, GC B2 484 (Bâtiment GC), Station 18, 1015 Lausanne, Switzerland.

[2] Address: c/o EUCENTRE, Via Ferrata 1, 27100 Pavia, Italy.

http://dx.doi.org/10.1016/j.engstruct.2014.10.024
0141-0296/

on this field has continued throughout the following decades (e.g., Hutchinson and Zillmer [9]; Renton [10]; Hutchinson [11]) and up to the present day (Dong et al. [12]). Within the framework of this paper, classical beam theories are considered to be of the first-order, i.e., the cross-sectional displacement fields are linear functions on each of the cross-sectional coordinates.

However, shear deformation effects are best accounted for through higher-order beam theories (HOBTs), wherein the axial displacement field is represented by a power series expansion in the cross-sectional coordinates, thus relaxing the constraint in the cross-sectional warping. Therefore, out-of-plane displacements of the cross-sectional points are allowed by using shape functions for the cross-sectional axial displacements which are at least quadratic in one coordinate or bilinear in both. Planar beam theories can be found in the literature (Stephen and Levinson [13]) which are similar in form to the TBT but account also for 'shear curvature' and 'transverse direct stresses', using those authors' nomenclature. The early work by Soler [14], wherein Legendre polynomials were used for thick rectangular elastic isotropic beams, as well as for orthotropic beams (Tsai and Soler [15]), should be mentioned. This family of polynomials was employed because their completeness, convergence and orthogonality properties are well formulated. Furthermore, the usual stress-resultants of classical beam theory appear naturally. Even without previous knowledge of such approach, similar reasons also guided the use of Legendre polynomials in the theoretical developments of the present study.

Levinson [16] used a third-order beam theory satisfying zero shear strain conditions at both the upper and lower edges of the beam, obviating the need for the shear coefficient. The equations of motion therein derived are not variationally consistent, which was later corrected by other authors, either making use of Hamilton's principle (Bickford [17]) or the principle of virtual displacements (Reddy [18]). The variational consistency of Bickford's theory does not necessarily seem to imply, however, superior accuracy (Rychter [19,20]). The Euler–Lagrange equations of motion in Bickford's theory are typically displayed in terms of displacements, mechanical parameters (describing a linear elastic constitutive relation), and cross-sectional geometric properties (Petrolito [21]). Nevertheless, it is naturally possible to express them in a format wherein specific constitutive relations are not yet assumed (Reddy [22]). Such arrangement has the advantage of showing immediately the generalisation of stress-resultants that is required in higher-order theories. For example, in Bickford's theory the common definition of shear force gives place to a new definition of a higher-order shear force, involving the cross-sectional integral of the shear stresses; additionally, a higher-order moment of the normal stresses also shows up. It should be pointed out, however, that it is possible to construct HOBTs—such as the one herein proposed—wherein the classical definitions of the stress-resultants are preserved.

Based on Bickford's theory, a two-node beam finite element with three degrees of freedom per node was later developed and tested (Heyliger and Reddy [23]). Approximately at the same time, Kant and Manjunatha [24]—and later Manjunatha and Kant [25]—proposed beam theories with kinematic fields having different orders of variation for both the longitudinal and transverse displacements; the authors used Lagrangian four-noded cubic elements with different number of degrees of freedom per node (ranging from three to seven, according to the complexity of the underlying theory).

The Lo–Christensen–Wu theory (Lo et al. [26,27]) is an elegant theory that is widely used by researchers for the analyses of shear deformable beams and plates; it expands the axial displacement field as a cubic function in the thickness coordinate, while the polynomial expansion for the transverse displacement is truncated at one order lower. Vinayak et al. [28] and Prathap et al. [29] carry out a systematic evaluation of the Lo–Christensen–Wu theory, comparing the results of finite element analyses with available closed-form classical and elasticity solutions.

The refined model by Kim and White [30], developed for both thin- and thick-walled composite beams, is of interest since it accounts for transverse shear effects of the cross-section and of the beam walls, as well as primary and secondary warping. Rand [31] devised a model to handle arbitrary solid cross-sections or general thin-walled geometries; it considers five degrees of freedom, namely three cross-sectional displacements, a twist angle and a 3D warping function. The importance of the latter, which is made dependent on the boundary conditions (unlike traditional beam theories), is demonstrated in a subsequent study by the same author [32].

The number of proposals associated with composite beam modelling is countless and can be found in the literature reviews of Ghugal and Shimpi [33] and Volovoi et al. [34]. Nevertheless, more recent works deserve to be mentioned. In particular, the variational asymptotic beam sectional analysis (VABS) suggested by Yu et al. [35] is of relevance; therein, instead of assuming a 3D warping displacement, they compute it in terms of the 1D generalised strains. Also, in the context of the use of trigonometric functions, a new three-noded beam finite element was recently conceived by Vidal and Polit [36] for the analysis of laminated beams.

A significant improvement over previous HOBTs was accomplished on the so-called 'Carrera's Unified Formulation' (Carrera and Giunta [37]), also known as CUF, by allowing the order of the theory and, consequently, the number of cross-sectional displacement modes it takes into account, to be a free parameter. In view of the similarities to the current work and the additional fact that it will be considered for comparison purposes, a short introductory note on such formulation should be made. Originally applied to the modelling of anisotropic plate and shell structures (Carrera [38]), the method proposes a systematic manner of formulating axiomatically refined beam models by choosing the desired order of the theory. Using a concise notation for the kinematic field, the governing differential equations and the corresponding boundary conditions (BCs) are reduced to a 'fundamental nucleo' in terms of the displacement components, which does not depend upon the approximation order. The finite element formulation of the CUF for beam structures (Carrera et al. [39]) includes two, three and four-noded elements—using respectively linear, quadratic and cubic approximations along the beam axis—with different higher-order models for the cross-section displacement field. The displacement components are expanded in terms of the cross-section coordinates using Taylor-type expansions. The effectiveness of higher-order terms in the context of the CUF is analysed in a subsequent work (Carrera and Petrolo [40]), while its applicability to the free vibration of rotating beams is carried out in a new study (Carrera et al. [41]). A compilation of the beam formulations and results obtained with the CUF was recently published (Carrera et al. [42]).

1.2. Finite element formulations and objective of the study

In the context of finite element formulations for solid mechanics, it is well-known that low-order elements in classical displacement approximations lead to unsatisfactory performance, which can be due to either locking in the incompressible limit or to poor accuracy (namely in bending-dominated behaviour). The use of energy functionals representing multi-field variational principles provide a natural setting for the formulation of mixed finite element methods, and an approach to by-pass the aforementioned problems. In a mixed method it is possible to independently approximate all fields that exist in the functional, which opens the door to interesting methods of analysis. In particular, the three-field formulation proposed by de Veubeke [43,44] and commonly known as Hu–Washizu [45,46] allows to approximate the displacement, stress and strain as independent variables (see also

Chama and Reddy [47]). Another popular, two-field functional with displacement and stress as variables is the so-called Hellinger–Reissner formulation. Stolarski and Belytschko [48] have shown that the latter can be recovered as a special case of the classical Hu–Washizu formulation. The relationship between the several mixed formulations and other enhanced formulations can be found in the work by Djoko et al. [49].

The particular case of beam elements deserves special considerations. Stiffness or displacement-based methods (DB, as they will be henceforth called, see Bathe [50]), which make use of compatible displacement interpolation functions along the element length and the principle of virtual work (or virtual displacements), are still the most commonly used. They are also known as pure compatibility models in the literature (Pian and Tong [51]), and are popular since the inter-element continuity of the displacement field is trivially satisfied. On the other hand, the latter is difficult to enforce in flexibility or force-based formulations (FB, as they will be henceforth called), which are built on the derivation of self-equilibrated stress interpolation functions and the principle of complementary virtual work (or virtual forces). These approaches are also known as pure equilibrium models, and within its framework it is possible to find an exact solution of the beam equilibrium differential equations, except if geometric effects due to nonlinear terms in the strain–displacement compatibility relations are taken into account (Neuenhofer and Filippou [52]; Souza [53]). Hence, the main advantage of FB beam-column formulations, over the more common DB counterparts, is that equilibrium is always strictly verified. Such property holds independently of possible material nonlinear behaviour, explaining why flexibility methods have been progressively adopted by the structural engineering community for the inelastic analysis of members. An explicit consideration of the full three-dimensional response of beams and columns can frequently be of relevance for the referred inelastic behaviour, in order to account for the effects of bi-directional shear and torsion and to enable the use of arbitrary three-dimensional material models. Another advantage of FB formulations is that no shear-locking phenomena exist, contrary to what occurs in DB approaches. Although frame finite elements that are based on force interpolation functions alone have been in use for many decades (Menegotto and Pinto [54]; Ciampi and Carlesimo [55]), the development of efficient and stable state determination algorithms, wherein nodal compatibility is respected, is more recent (Spacone et al. [56]; Neuenhofer and Filippou [57]).

Comparative studies between FB and DB formulations can be found elsewhere (Neuenhofer and Filippou [57]; Alemdar and White [58]). Of interest are considerations on the bounding of the solution associated with these two different types of formulation, which are made and exemplified in the companion paper [59]. Equally fundamental is to understand the relation between DB and FB approaches, as they have been herein described, and energy principles. Such comparison should be carried out not just at the theoretical level, but also regarding numerical implementation and computational performance. While the application of the Hu–Washizu three-field functional seems to be a promising avenue (Frischkorn and Reese [60]), trade-offs between classical DB methods and mixed methods are far from being completely clarified (e.g., Hjelmstad and Taciroglu [61]). The merit of FB beam models is however clearly undisputed [62], and the developments of the present work rest on it.

To the authors' knowledge, pure equilibrium (FB) approaches have only been used, up to now, in association with classical elementary beam theory. In other words, the finite elements that were developed within the context of the HOBTs summarised in the previous section are, in their essence, displacement-based formulations. Bearing in mind the previous observations, the objective of the current paper is to present and explore, for the first time, a higher-order beam element developed within the framework of a pure force-based formulation.

It will be seen that the selected set of higher-order deformation modes leads to an explicit and direct interaction between three-dimensional shear and normal stresses. In particular, cross-sectional displacement and strain fields are composed of independent and orthogonal modes, herein described for solid rectangular sections, without loss of generality, by using normalised Legendre polynomials. Such orthogonality results in unambiguously defined generalised cross-sectional stress-resultants and in a minimisation of the coupling between equilibrium equations. On the basis of work-equivalency to three-dimensional continuum theory, dual one-dimensional higher-order equilibrium and compatibility equations are derived. The former, which govern an advanced form of beam equilibrium, are strictly satisfied via stress fields arising from the solution of the corresponding systems of coupled differential equations. Additionally, by using a force-based formulation, shear-locking problems are inherently avoided and span loads are accurately considered.

The proposed formulation is developed independently of the assumed constitutive behaviour. A companion paper [59] presents application examples to both linear elastic and nonlinear members with relevant shear and torsional deformations that validate the present theory. Moreover, an original but unexpected result obtained in this study is the verification of the superiority of FB models, for the linear elastic case, with respect to DB ones when HOBTs are adopted for both approaches. It is recalled that within classical elementary beam theory (Euler–Bernoulli beam models) the linear elastic results of FB and DB models were similar, unlike what happens in the analyses presented in the companion paper.

2. Development of equilibrium and compatibility relations

The distinctive trace in all beam theories, and following the basic idea from Vlassov [63], is the assumption of a displacement field composed by cross-sectional displacement modes defined a priori and multiplied by functions of the beam coordinate axis only. It allows the conversion of the 3D governing equations into the corresponding 1D beam theory, which is not always established in a clear way and thus not always guarantees power-conjugacy. In the current paper, this issue is addressed in a consistent manner, following the approach presented by Teixeira de Freitas and co-workers [64,65]. An illustrative derivation of an extended version of the classical TBT's compatibility and equilibrium equations, including distortion and warping, serves as an introductory example for the more complex higher-order formulation.

2.1. Classical continuum mechanics framework

The present paragraph introduces the adopted nomenclature and setting for the standard 3D continuum mechanics framework.

The components of the stress and strain tensors are assembled in the column vectors:

$$\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}]^T; \quad \boldsymbol{\varepsilon} = [\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}]^T$$

while the body forces, displacements and surface forces are represented respectively by:

$$\mathbf{b} = [b_x \quad b_y \quad b_z]^T; \quad \mathbf{u} = [u_x \quad u_y \quad u_z]^T; \quad \mathbf{t} = [t_x \quad t_y \quad t_z]^T$$

The classical local equilibrium equations can be expressed in the form:

$$\widehat{\mathbf{D}}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \text{ in } V \tag{1}$$

where V is the domain of the body and the linear differential equilibrium operator is:

$$\widehat{\mathbf{D}} = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & \partial/\partial z & 0 \\ 0 & \partial/\partial y & 0 & \partial/\partial x & 0 & \partial/\partial z \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial x & \partial/\partial y \end{bmatrix}$$

On the other hand, the classical local compatibility equations, or strain–displacement relations, take the form:

$$\boldsymbol{\varepsilon} = \widehat{\mathbf{D}}^{adj}\mathbf{u} \text{ in } V \tag{2}$$

where $\widehat{\mathbf{D}}^{adj}$ is the linear differential compatibility operator, adjoint to the differential equilibrium operator $\widehat{\mathbf{D}}$:

$$\widehat{D}_{ij}^{adj} = \frac{d^k}{ds^k} = (-1)^{k+1}\widehat{D}_{ji}, \quad s = x, y, z$$

That is, $\widehat{D}_{ij}^{adj} = \widehat{D}_{ji}$ if the derivative is of odd order or $\widehat{D}_{ij}^{adj} = -\widehat{D}_{ji}$ if it is of even order. In the present case $\widehat{\mathbf{D}}^{adj} = \widehat{\mathbf{D}}^T$, since only first derivatives are involved, which are the matrix equivalents of using the adjoint differential operators *div* and *grad* respectively applied to the stress and strain tensors. It should be noted that the expression just presented to compute the adjoint to a linear differential operator is, for convenience, symmetric to the usual definition (Wu [66]; Gao [67]; Estep [68]). The contragredient expressions (1) and (2) illustrate the duality between static and kinematic variables. Such adjointness or duality is complete when the appropriate BCs are considered.

It is noted that the differential operators $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{D}}^{adj}$ can be recast as follows:

$$\widehat{\mathbf{D}} = \widehat{\mathbf{N}}_x\frac{\partial}{\partial x} + \widehat{\mathbf{N}}_y\frac{\partial}{\partial y} + \widehat{\mathbf{N}}_z\frac{\partial}{\partial z}$$

$$\widehat{\mathbf{D}}^{adj} = \widehat{\mathbf{N}}_x^T\frac{\partial}{\partial x} + \widehat{\mathbf{N}}_y^T\frac{\partial}{\partial y} + \widehat{\mathbf{N}}_z^T\frac{\partial}{\partial z} = \widehat{\mathbf{D}}^T$$

where

$$\widehat{\mathbf{N}}_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad \widehat{\mathbf{N}}_y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\widehat{\mathbf{N}}_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The equilibrium BCs are:

$$\widehat{\mathbf{N}}\boldsymbol{\sigma} = \mathbf{t}^\dagger \text{ in } \Omega_t \tag{3}$$

where Ω_t is the part of the boundary where static or natural BCs are prescribed, the symbol † stands for prescribed quantities and

$$\widehat{\mathbf{N}} = \begin{bmatrix} n_x & 0 & 0 & n_y & n_z & 0 \\ 0 & n_y & 0 & n_x & 0 & n_z \\ 0 & 0 & n_z & 0 & n_x & n_y \end{bmatrix} = \widehat{\mathbf{N}}_x n_x + \widehat{\mathbf{N}}_y n_y + \widehat{\mathbf{N}}_z n_z$$

is the exterior unit normal matrix corresponding to the differential equilibrium operator $\widehat{\mathbf{D}}$.

Finally, the compatibility BCs can be expressed as:

$$\mathbf{u} = \mathbf{u}^\dagger \text{ in } \Omega_u \tag{4}$$

where Ω_u is the part of the boundary where kinematic or essential BCs are prescribed, with $\Omega = \Omega_t \cup \Omega_u$ and $\Omega_t \cap \Omega_u = \varnothing$.

The abovementioned dual properties of the static boundary value problem, composed by Eqs. (1) and (3), and its adjoint problem, i.e., the kinematic boundary value problem consisting of Eqs. (2) and (4), is expressed by the following identity:

$$\int_V \boldsymbol{\varepsilon}^T\boldsymbol{\sigma}\, dv = \int_V \mathbf{u}^T\mathbf{b}\, dv + \int_{\Omega_u} (\mathbf{u}^\dagger)^T(\widehat{\mathbf{N}}\boldsymbol{\sigma})\, da + \int_{\Omega_t} \mathbf{u}^T(\mathbf{t}^\dagger)\, da \tag{5}$$

which can, in fact, replace one of Eqs. (1)–(4). Moreover, since no specific requirements on the constitutive behaviour are imposed and any pair of statically admissible stress field and kinematically admissible displacement field can be used, the duality principle of expression (5) is equivalent to any form of the Potential Energy Theorems, of the Principle of Virtual Work, or of their complementary counterparts (Wu [66]; Gao [67]; Estep [68]; Washizu [69]).

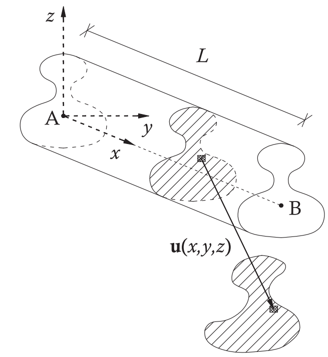

Fig. 1. Spatial beam element and deformed configuration of general cross-section.

2.2. Consistent derivation of beam theory equations

Consider the spatial beam element, of length L, represented in Fig. 1. The deformed configuration of a general cross-section is also schematically depicted.

As mentioned above, due to the inherent 1D character of any beam theory, its displacement field can be decomposed into a component function of the axis x of the beam and another varying in the cross-section (the coordinate axes of which are y and z):

$$\mathbf{u}(x, y, z) = \mathbf{U}(y, z)\mathbf{d}(x) \tag{6}$$

where $\mathbf{U}(y, z)$ is the matrix of the displacement interpolation functions over the cross-section (the cross-sectional displacement modes), and $\mathbf{d}(x)$ is the vector of weights associated with the interpolation functions, also called generalised displacements.

As an illustrative example of the derivation of the beam theory equations, without any loss of generality, an extended Timoshenko beam theory (ETBT) formulation is herein obtained. In addition to the classical TBT's cross-sectional displacement modes, this beam theory also considers the first-order in-plane distortion and out-of-plane warping, thus including all first-order terms for the cross-sectional displacements. The displacement field can be written in accordance with the format of Eq. (6):

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_{x0} + z\theta_y - y\theta_z + yz\eta \\ u_{y0} - z\theta_x + z\gamma \\ u_{z0} + y\theta_x + y\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y & yz & 0 \\ 0 & 1 & 0 & -z & 0 & 0 & 0 & z \\ 0 & 0 & 1 & y & 0 & 0 & 0 & y \end{bmatrix} \begin{bmatrix} u_{x0} \\ u_{y0} \\ u_{z0} \\ \theta_x \\ \theta_y \\ \theta_z \\ \eta \\ \gamma \end{bmatrix} = \mathbf{Ud}$$

where $\mathbf{d}$ includes the classical TBT's generalised displacements in terms of the axial and transverse displacements in the y and z directions of the reference axis (origin of the cross-section)—respectively u_{x0}, u_{y0} and u_{z0}, and the cross-sectional rotations along the x, y and z axes—respectively θ_x, θ_y and θ_z. Furthermore, $\mathbf{d}$ also includes the cross-sectional in-plane distortion—γ being the distortion intensity associated to a change of the cross-section's shape in its own plane, and the cross-sectional out-of-plane warping, the intensity of which is represented by η. The classical hypotheses of a rigid behaviour of the cross-section in its own plane and of the cross-sections remaining plane after deformation, are thus abandoned in this ETBT. It is noted that, in such first-order theory, the warping shape is taken as yz. Additionally, the classical TBT is readily obtained by neglecting these two additional degrees of freedom (γ and η). The cross-sectional displacement modes introduced in the previous expression are represented as modes 1–8 in Fig. 1 of the companion paper.

The previous expressions can be combined with Eq. (2) to derive the strain field:

$$\boldsymbol{\varepsilon}(x,y,z)=\widehat{\mathbf{D}}^{adj}\mathbf{u}=\left(\widehat{\mathbf{N}}_x\mathbf{U}\frac{\partial}{\partial x}+\widehat{\mathbf{N}}_y\frac{\partial \mathbf{U}}{\partial y}+\widehat{\mathbf{N}}_z\frac{\partial \mathbf{U}}{\partial z}\right)\mathbf{d}=\mathrm{B}\mathrm{D}^{adj}\mathbf{d}= \\ =\mathbf{B}(y,z)\mathbf{e}(x) \tag{7}$$

where $\mathbf{D}^{adj}$ is the beam differential compatibility operator and $\mathbf{B}(y,z)$ is the cross-section's strain interpolation matrix, i.e., a matrix that relates the generalised cross-sectional strains $\mathbf{e}(x)$ with the strains at each point of the cross-section. It thus contains the strain interpolation functions over the cross-section, which may be identified as its deformation modes.

The application of these expressions to the extended Timoshenko model results in the following strain vector:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}=\widehat{\mathbf{D}}^{adj}\mathbf{u}=\begin{bmatrix}\varepsilon_0+z\chi_y-y\chi_z+yz\chi_\eta\\ 0\\ 0\\ \gamma_{y0}-z\chi_x+z\chi_\gamma\\ \gamma_{z0}+y\chi_x+y\chi_\gamma\\ 2\gamma\end{bmatrix}= \\ =\begin{bmatrix}1&0&0&0&z&-y&yz&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&1&0&-z&0&0&0&z&0\\0&0&1&y&0&0&0&y&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\varepsilon_0\\ \gamma_{y0}\\ \gamma_{z0}\\ \chi_x\\ \chi_y\\ \chi_z\\ \chi_\eta\\ \chi_\gamma\\ 2\gamma\end{bmatrix}=\mathbf{Be}$$

where $\varepsilon_0=u'_{x0}$ is the average axial strain; $\gamma_{y0}=u'_{y0}-\theta_z$ and $\gamma_{z0}=u'_{z0}+\theta_y$ are the average shear strains; $\chi_x=\theta'_x$ is the torsional curvature; $\chi_y=\theta'_y$ and $\chi_z=\theta'_z$ are the flexural curvatures; and $\chi_\eta=\eta'$, $\chi_\gamma=\gamma'+\eta$ and 2γ are the generalised strains dual to the bimoment, bi-shear and distortional shear. The identification of the last group of kinematic variables will soon be clarified. It is pointed out that, similarly to the classical TBT, this ETBT presents no in-plane extensions ε_y and ε_z (although ε_{yz} is herein considered, as previously mentioned). Hence, it can be concluded that a simple behaviour like the Poisson's ratio effect is not correctly captured in first-order beam theories. Phenomena such as this, or other more complex ones, are naturally taken into account using higher-order approaches, as shown in the companion paper.

From expression (7) it becomes clear that the beam local compatibility equations in the domain, or generalised strain–displacement relations, are:

$$\mathbf{e}=\mathbf{D}^{adj}\mathbf{d}\ \text{in}\ L\iff\begin{bmatrix}\varepsilon_0\\ \gamma_{y0}\\ \gamma_{z0}\\ \chi_x\\ \chi_y\\ \chi_z\\ \chi_\eta\\ \chi_\gamma\\ 2\gamma\end{bmatrix}=\begin{bmatrix}\partial/\partial x&0&0&0&0&0&0&0\\0&\partial/\partial x&0&0&0&-1&0&0\\0&0&\partial/\partial x&0&1&0&0&0\\0&0&0&\partial/\partial x&0&0&0&0\\0&0&0&0&\partial/\partial x&0&0&0\\0&0&0&0&0&\partial/\partial x&0&0\\0&0&0&0&0&0&\partial/\partial x&0\\0&0&0&0&0&0&1&\partial/\partial x\\0&0&0&0&0&0&0&2\end{bmatrix}\begin{bmatrix}u_{x0}\\ u_{y0}\\ u_{z0}\\ \theta_x\\ \theta_y\\ \theta_z\\ \eta\\ \gamma\end{bmatrix}\ \text{in}\ L \tag{8}$$

Since the displacement modes of the cross-section were defined *a priori*, the corresponding beam local equilibrium equations may now be determined. In order to obtain power-conjugated generalised stress-resultants, such equations are derived through a projection of the classical local equilibrium Eq. (1) on the functional space of the cross-sectional displacement modes (Teixeira de Freitas et al. [65]). Such operation may also be regarded as weighting the residuals of the 3D local equilibrium Eq. (1), taking the cross-sectional displacement modes as weighting functions. Using the divergence theorem and integrating the terms $\partial\boldsymbol{\sigma}/\partial\mathbf{y}$ and $\partial\boldsymbol{\sigma}/\partial\mathbf{z}$ by parts, it leads to the following beam local equilibrium equations:

$$\int_A\mathbf{U}^T(\widehat{\mathbf{D}}\boldsymbol{\sigma}+\mathbf{b})\,da=\mathbf{0}\ \text{in}\ L\iff \\ \iff\int_A\mathbf{U}^T\left(\widehat{\mathbf{N}}_x\frac{\partial\boldsymbol{\sigma}}{\partial x}+\widehat{\mathbf{N}}_y\frac{\partial\boldsymbol{\sigma}}{\partial y}+\widehat{\mathbf{N}}_z\frac{\partial\boldsymbol{\sigma}}{\partial z}\right)da+\underbrace{\int_A\mathbf{U}^T\mathbf{b}\,da}_{\mathbf{p}_b}=\mathbf{0}\iff \\ \iff\int_A\left(\mathbf{U}^T\widehat{\mathbf{N}}_x\frac{\partial}{\partial x}-\frac{\partial\mathbf{U}^T}{\partial y}\widehat{\mathbf{N}}_y-\frac{\partial\mathbf{U}^T}{\partial z}\widehat{\mathbf{N}}_z\right)\boldsymbol{\sigma}\,da+\int_\Gamma\mathbf{U}^T(\widehat{\mathbf{N}}_y n_y+\widehat{\mathbf{N}}_z n_z)\boldsymbol{\sigma}\,ds+\mathbf{p}_b=\mathbf{0}\iff \\ \iff\mathbf{D}\int_A\mathbf{B}^T\boldsymbol{\sigma}\,da+\underbrace{\int_\Gamma\mathbf{U}^T\mathbf{t}^\dagger ds+\mathbf{p}_b}_{\mathbf{p}}=\mathbf{0}\iff \\ \iff\mathbf{Ds}+\mathbf{p}=\mathbf{0}\ \text{in}\ L \tag{9}$$

where A and Γ are, respectively, the area and boundary of the cross-section, $\mathbf{D}$ is the beam differential equilibrium operator (which is adjoint to $\mathbf{D}^{adj}$), $\mathbf{p}$ are the distributed loads and

$$\mathbf{s}=\int_A\mathbf{B}^T\boldsymbol{\sigma}\,da \tag{10}$$

are the generalised stress-resultants. Note that these are dual to the generalised strains since $\int_A\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}\,da=\mathbf{e}^T\mathbf{s}$.

For the extended Timoshenko beam model, the generalised stress-resultants and the distributed loads take the following form:

$$\mathbf{s}=\int_A\mathbf{B}^T\boldsymbol{\sigma}\,da=\int_A\begin{bmatrix}\sigma_x\\ \tau_{xy}\\ \tau_{xz}\\ -z\tau_{xy}+y\tau_{xz}\\ z\sigma_x\\ -y\sigma_x\\ yz\sigma_x\\ z\tau_{xy}+y\tau_{xz}\\ \tau_{yz}\end{bmatrix}da=\begin{bmatrix}N\\ V_y\\ V_z\\ T\\ M_y\\ M_z\\ B\\ Q\\ V_{yz}\end{bmatrix}$$

$$\mathbf{p} = \int_A \mathbf{U}^T \mathbf{b}\, da + \int_\Gamma \mathbf{U}^T \mathbf{t}^\dagger\, ds = \int_A \begin{bmatrix} b_x \\ b_y \\ b_z \\ -zb_y + yb_z \\ zb_x \\ -yb_x \\ yzb_x \\ zb_y + yb_z \end{bmatrix} da + \int_\Gamma \begin{bmatrix} t_x^\dagger \\ t_y^\dagger \\ t_z^\dagger \\ -zt_y^\dagger + yt_z^\dagger \\ zt_x^\dagger \\ -yt_x^\dagger \\ yzt_x^\dagger \\ zt_y^\dagger + yt_z^\dagger \end{bmatrix} ds = \begin{bmatrix} q_x \\ q_y \\ q_z \\ m_x \\ m_y \\ m_z \\ m_b \\ m_q \end{bmatrix}$$

where N is the axial force, V_y and V_z are the shear forces, T is the torsional moment, M_y and M_z are the flexural moments, B is the bimoment, Q is the bi-shear and V_{yz} is the distortional shear. It should be pointed out that the latter is related to the shear stresses τ_{yz}, which do not act in the cross-section of the beam. The elements of $\mathbf{p}$ are the distributed loads associated with the generalised displacements.

According to the definition of the generalised stress-resultants associated with the generalised cross-sectional strains, as visible in Eqs. (7) and (10), B corresponds to an axial stress distribution similar to the out-of-plane warping cross-sectional mode and defined by yz; Q is the resultant of the shear stresses associated with the bimoment B (similarly to the relation between V_z and M_y); finally, V_{yz} is the shear stress-resultant associated with the cross-sectional in-plane distortion.

The local equilibrium conditions (9) follow:

$$\mathbf{Ds} + \mathbf{p} = \mathbf{0} \text{ in } L \iff \begin{cases} N' + q_x = 0 \\ V_y' + q_y = 0 \\ V_z' + q_z = 0 \\ T' + m_x = 0 \\ M_y' - V_z + m_y = 0 \\ M_z' + V_y + m_z = 0 \\ B' - Q + m_b = 0 \\ Q' - 2V_{yz} + m_q = 0 \end{cases} \text{ in } L \tag{11}$$

Similarly to the local equilibrium equations, the three-dimensional boundary equilibrium Eq. (3) can also be weighted over the area of the cross-section, with $\mathbf{U}$ serving as the weighting function:

$$\int_A \mathbf{U}^T \underbrace{(\widehat{\mathbf{N}}\boldsymbol{\sigma})}_{(\widehat{\mathbf{N}}_x n_x \boldsymbol{\sigma})} da = \underbrace{\int_A \mathbf{U}^T \mathbf{t}^\dagger da}_{\mathbf{R}^\dagger} \text{ in } x = 0, L \iff \begin{bmatrix} n_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_x & 0 \end{bmatrix} \begin{bmatrix} N \\ V_y \\ V_z \\ T \\ M_y \\ M_z \\ B \\ Q \\ V_{yz} \end{bmatrix} = \begin{bmatrix} N^\dagger \\ V_y^\dagger \\ V_z^\dagger \\ T^\dagger \\ M_y^\dagger \\ M_z^\dagger \\ B^\dagger \\ Q^\dagger \end{bmatrix} \iff \mathbf{Ns} = \mathbf{R}^\dagger \text{ in } x = 0, L \tag{12}$$

where $\mathbf{N}$ is the exterior unit normal matrix corresponding to the beam differential equilibrium operator $\mathbf{D}$ (with $N_{ij} = n_x$ if $D_{ij} = \partial/\partial x$ and $N_{ij} = 0$ otherwise), while $\mathbf{R}^\dagger$ is the vector of generalised nodal forces applied at the beam ends. The previous expression depicts the beam boundary equilibrium equations.

Finally, the beam boundary compatibility conditions are simply given by:

$$\mathbf{d} = \mathbf{d}^\dagger \text{ in } x = 0, L \tag{13}$$

where $\mathbf{d}^\dagger$ is the vector of generalised nodal displacements applied at the beam ends. Its components are only defined at those degrees of freedom with imposed kinematic conditions, whereas the remaining degrees of freedom have static boundary conditions instead, with the corresponding elements of the vector $\mathbf{R}^\dagger$.

The adjointness of the beam theory equations presented above is expressed by the following identity:

$$\int_0^L \mathbf{e}^T \mathbf{s}\, dx = \int_0^L \mathbf{d}^T \mathbf{p}\, dx + [\mathbf{d}^T \mathbf{R}^\dagger]_0^L$$

for which similar comments to the ones relative to expression (5) can be produced. It is noted that the boundary terms considered refer to the case where only static BCs are imposed; otherwise the elements of the product $\mathbf{d}^T \mathbf{R}^\dagger$ corresponding to the kinematic BCs should be replaced by the ones of the product $(\mathbf{d}^\dagger)^T \mathbf{Ns}$.

Existing FB beam approaches are derived by solving the classical Timoshenko beam equilibrium equations, or equivalently the Euler–Bernoulli ones, which correspond to the first six equations of expression (11). Such system of six first-order linear differential equations with six unknowns, for a given set of known distributed loads, requires six static BCs. On the other hand, the corresponding TBT degrees of freedom sum up to 12 available BCs, as given by the first six equations of expression (12). Consequently, only six nodal forces at the beam ends are independent, while the others depend on the latter through beam equilibrium. Such dependency is equivalent to restraining the rigid-body motion of the beam. It is also noted that different sets of independent forces can be considered, corresponding to removing such rigid-body motion using different sets of statically determinate supports. Typically, a simply supported beam is considered, with the axial displacement and torsional rotation blocked at one end. Such conceptual statically determinate supports are also adopted in this work.

The remaining two beam equilibrium equations of expression (11), for given distributed loads, have three unknown generalised stress-resultants and four static BCs at the beam ends, where it is noted that V_{yz} plays no role in the equations of expression (12). Such differential equations cannot be solved unless, for instance, V_{yz} is defined *a priori*. Since these differential equations require only two static BCs, the remaining two can be used for defining an assumed variation for V_{yz} in a FB approach. Although the resulting FB formulation would strictly verify the equilibrium conditions, it is not unique in the sense that V_{yz} can be assumed to have different variations. In this work, the simplest possible polynomial function was assumed in such cases, as will be further discussed later, corresponding to a linear polynomial for V_{yz}.

3. Proposed beam element: governing equations

The current beam theory is derived, in this section, for elements with solid rectangular cross-section of dimensions h (height) $\times$ b

(width). Nevertheless, it can be extended to any other cross-sectional geometry, as discussed below.

3.1. Displacement field and Legendre polynomials

Similarly to expression (6), the displacement field is decomposed into a component varying along the axis of the beam and another varying in the cross-section:

$$\mathbf{u}(x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\mathbf{U}^{ij}(y,z)\mathbf{d}^{ij}(x) = \mathbf{U}(y,z)\mathbf{d}(x) \tag{14}$$

In the present formulation, the interpolation functions $\mathbf{U}(y,z)$ are based on Legendre polynomials. The latter, herein represented by $P_n(s)$ for integers $n = 0,1,2,\ldots$, form a sequence of solutions to Legendre's differential equation with the normalisation $P_n(1) = 1$, which results in $P_n(-1) = (-1)^n$. The Legendre polynomials are n^{th}-degree polynomials that can be expressed using Rodrigues' formula:

$$P_n(s) = \frac{1}{2^n n!}\frac{d^n}{ds^n}[(s^2-1)^n]$$

In particular, it is noted that $P_0(s) = 1$ and $P_1(s) = s$. The remaining polynomials can be obtained recursively according to:

$$P_n(s) = \frac{2n-1}{n}sP_{n-1}(s) - \frac{n-1}{n}P_{n-2}(s), \quad n \geqslant 2$$

In this paper, Legendre polynomials up to the fourth degree are used:

$$P_2(s) = \frac{3s^2-1}{2} \quad P_3(s) = \frac{5s^3-3s}{2} \quad P_4(s) = \frac{35s^4-30s^2+3}{8}$$

The present study also makes use of the derivatives of $P_n(s)$, for which the following recursive expression can be developed:

$$\frac{dP_n(s)}{ds} = (2n-1)P_{n-1}(s) + \frac{dP_{n-2}(s)}{ds}, \quad n \geqslant 2$$

The derivatives of the polynomials up to the fourth degree are thus given by:

$$P'_0(s) = 0 \quad P'_1(s) = 1 = P_0 \quad P'_2(s) = 3P_1 \quad P'_3(s) = 5P_2 + P_0$$
$$P'_4(s) = 7P_3 + 3P_1$$

One fundamental property of the Legendre polynomials is that they form a complete orthogonal set in the interval $-1 \leqslant s \leqslant 1$:

$$\int_{-1}^{1} P_m(s)P_n(s)ds = \delta_{mn}c_n \tag{15}$$

wherein δ_{mn} is the Kronecker delta and $c_n = 2/(2n+1)$.

The previous property is fundamental for the development of the current finite element, since it enables the definition of generalised stress-resultants which are not only independent between each other, but also completely orthogonal to one another. This leads to an unambiguous definition of the generalised stress-resultants and to a minimisation of the coupling of equilibrium equations, unlike all previous HOBTs, as will be discussed later. In view of the interval where the orthogonality property (15) holds, the following normalised cross-sectional coordinates are adopted:

$$y^* = y\frac{2}{b}, \quad z^* = z\frac{2}{h} \tag{16}$$

Consequently, Eq. (14) can be equivalently rewritten as:

$$\mathbf{u}(x,y^*,z^*) = \sum_{i=0}^{n}\sum_{j=0}^{n}\mathbf{U}^{*,ij}(y^*,z^*)\mathbf{d}^{*,ij}(x) = \mathbf{U}^*(y^*,z^*)\mathbf{d}^*(x) \tag{17}$$

The classical beam theory generalised displacements $\mathbf{d}^{ij}(x) = \begin{bmatrix} u_x^{ij} & u_y^{ij} & u_z^{ij}\end{bmatrix}^T$ are thus replaced by the normalised generalised displacements $\mathbf{d}^{*,ij}(x) = \begin{bmatrix} u_x^{*,ij} & u_y^{*,ij} & u_z^{*,ij}\end{bmatrix}^T$, which account for the normalisation procedure required to keep the orthogonality property of the cross-sectional interpolation functions. The latter are defined as follows:

$$\mathbf{U}^{*,ij}(y^*,z^*) = \begin{bmatrix} P_i(y^*)P_j(z^*) & 0 & 0 \\ 0 & P_{i-1}(y^*)P_j(z^*) & 0 \\ 0 & 0 & P_i(y^*)P_{j-1}(z^*)\end{bmatrix} \tag{18}$$

It can be observed that the transverse displacements have a one degree lower polynomial function than the axial displacements. This results in the same degree of approximation, considering the contributions of both the axial and transverse displacements in each set of modes *ij*, for the shear strain fields. A selected set of cross-sectional displacement shapes for a specific rectangular section is represented in the companion paper [59]. As already pointed out, this formulation may be extended to other cross-sectional geometries by using a power series expansion based on MacLaurin polynomials, for instance, and orthogonalising such functions through a Gram–Schmidt orthogonalisation procedure. In fact, the Legendre polynomials in y^* can be obtained as the result of such approach to the polynomial series $1, y^*, y^{*2}, y^{*3}, y^{*4}, \ldots$, considering a rectangular section.

The displacement field up to the first-order is thus composed of the following interpolation functions and associated weights, where $P_{-1}(s) = 0$:

$$\mathbf{U}^{*,00}(y^*,z^*) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad \mathbf{d}^{*,00}(x) = \begin{bmatrix} u_{x0}^* \\ 0 \\ 0\end{bmatrix}$$

$$\mathbf{U}^{*,10}(y^*,z^*) = \begin{bmatrix} y^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad \mathbf{d}^{*,10}(x) = \begin{bmatrix} -\theta_z^* \\ u_{y0}^* \\ 0\end{bmatrix}$$

$$\mathbf{U}^{*,01}(y^*,z^*) = \begin{bmatrix} z^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}, \quad \mathbf{d}^{*,01}(x) = \begin{bmatrix} \theta_y^* \\ 0 \\ u_{z0}^*\end{bmatrix}$$

$$\mathbf{U}^{*,11}(y^*,z^*) = \begin{bmatrix} y^*z^* & 0 & 0 \\ 0 & z^* & 0 \\ 0 & 0 & y^*\end{bmatrix}, \quad \mathbf{d}^{*,11}(x) = \begin{bmatrix} \eta^* \\ u_y^{*,11} \\ u_z^{*,11}\end{bmatrix}$$

which can be assembled into:

$$\mathbf{u}^{1st\,order} = \mathbf{U}^{*,1st\,order}\mathbf{d}^{*,1st\,order} \iff \begin{cases} u_x^{1st\,order} = u_{x0}^* + z^*\theta_y^* - y^*\theta_z^* + y^*z^*\eta^* \\ u_y^{1st\,order} = u_{y0}^* + z^*u_y^{*,11} \\ u_z^{1st\,order} = u_{z0}^* + y^*u_z^{*,11}\end{cases} \tag{19}$$

The generalised displacements $u_y^{*,11}$ and $u_z^{*,11}$ are associated with the torsional rotation θ_x^* and cross-sectional distortion γ^*, which are defined as:

$$\begin{cases} \theta_x^* = \frac{1}{2}\left(u_z^{*,11} - u_y^{*,11}\right) \\ \gamma^* = \frac{1}{2}\left(u_z^{*,11} + u_y^{*,11}\right)\end{cases} \Rightarrow \begin{cases} u_y^{*,11} = \gamma^* - \theta_x^* \\ u_z^{*,11} = \gamma^* + \theta_x^*\end{cases}$$

Replacing the previous relations in expression (19) yields, in matrix form:

$$\mathbf{u}^{1st\ order} = \mathbf{U}^{*,1st\ order}\mathbf{d}^{*,1st\ order} = \begin{bmatrix} 1 & 0 & 0 & 0 & z^* & -y^* & y^*z^* & 0 \\ 0 & 1 & 0 & -z^* & 0 & 0 & 0 & z^* \\ 0 & 0 & 1 & y^* & 0 & 0 & 0 & y^* \end{bmatrix} \begin{bmatrix} u_{x0}^* \\ u_{y0}^* \\ u_{z0}^* \\ \theta_x^* \\ \theta_y^* \\ \theta_z^* \\ \eta^* \\ \gamma^* \end{bmatrix} \quad (20)$$

The first-order compatibility matrix $\mathbf{A}^{*,1st\ order}$ transforming the vector of classical generalised displacements $\mathbf{d}^{1st\ order}$ (i.e., the one from the ETBT) into the normalised generalised displacements $\mathbf{d}^{*,1st\ order}$ can be readily derived, as well as its inverse relation:

$$\mathbf{d}^{*,1st\ order} = \mathbf{A}^{*,1st\ order}\mathbf{d}^{1st\ order} \Leftrightarrow \begin{cases} u_{x0}^* = u_{x0} \\ u_{y0}^* = u_{y0} \\ u_{z0}^* = u_{z0} \\ \theta_x^* = \dfrac{b+h}{4}\theta_x + \dfrac{b-h}{4}\gamma \\ \theta_y^* = \dfrac{h}{2}\theta_y \\ \theta_z^* = \dfrac{b}{2}\theta_z \\ \eta^* = \dfrac{bh}{4}\eta \\ \gamma^* = \dfrac{b-h}{4}\theta_x + \dfrac{b+h}{4}\gamma \end{cases} \qquad \mathbf{d}^{1st\ order} = \left(\mathbf{A}^{*,1st\ order}\right)^{-1}\mathbf{d}^{*,1st\ order} \Leftrightarrow \begin{cases} u_{x0} = u_{x0}^* \\ u_{y0} = u_{y0}^* \\ u_{z0} = u_{z0}^* \\ \theta_x = \left(\dfrac{1}{b}+\dfrac{1}{h}\right)\theta_x^* + \left(\dfrac{1}{b}-\dfrac{1}{h}\right)\gamma^* \\ \theta_y = \dfrac{2}{h}\theta_y^* \\ \theta_z = \dfrac{2}{b}\theta_z^* \\ \eta = \dfrac{4}{bh}\eta^* \\ \gamma = \left(\dfrac{1}{b}-\dfrac{1}{h}\right)\theta_x^* + \left(\dfrac{1}{b}+\dfrac{1}{h}\right)\gamma^* \end{cases} \quad (21)$$

The interpolation functions $\mathbf{U}^{*,ij}$ and associated weights $\mathbf{d}^{*,ij}$ corresponding to higher-order terms are not explicitly indicated herein due to space limitations (see Almeida [70]), but again they follow the form of expression (18). The number of terms and the combination of indices ij adopted in the present beam-column model was based on a balance between intended accuracy and computational cost: all terms $\mathbf{U}^{*,ij}$ up to the fourth-order in one coordinate and first-order in the other were considered. As demonstrated in the application examples of the companion paper [59], the terms adopted are sufficient to retrieve the intended flexural–shear–torsional interaction. Fig. 2 represents all 16 terms considered for the longitudinal displacement u_x in a Pascal triangle type of representation. The transverse displacements u_y and u_z contain 11 terms each, with one degree less in y^* and z^*, respectively, than the corresponding terms in u_x, resulting in a total of 38 displacement terms.

It is now possible to write down the complete expression of each component of the displacement field associated with the present formulation, including the higher-order terms:

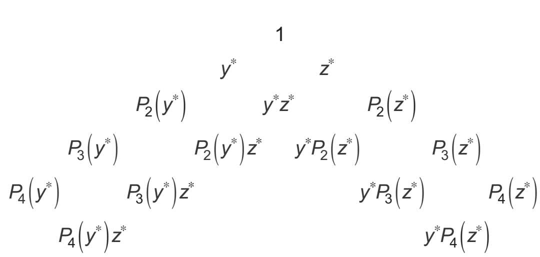

Fig. 2. Polynomial terms considered for the longitudinal displacement u_x.

$$\begin{aligned} u_x = {} & u_{x0}^* + z^*\theta_y^* - y^*\theta_z^* + y^*z^*\eta^* + P_2(y^*)u_x^{*,20} + P_2(z^*)u_x^{*,02} \\ & + P_2(y^*)z^*u_x^{*,21} + y^*P_2(z^*)u_x^{*,12} + P_3(y^*)u_x^{*,30} + P_3(z^*)u_x^{*,03} \\ & + P_3(y^*)z^*u_x^{*,31} + y^*P_3(z^*)u_x^{*,13} + P_4(y^*)u_x^{*,40} + P_4(z^*)u_x^{*,04} \\ & + P_4(y^*)z^*u_x^{*,41} + y^*P_4(z^*)u_x^{*,14} \end{aligned}$$

$$\begin{aligned} u_y = {} & u_{y0}^* - z^*\theta_x^* + z^*\gamma^* + y^*u_y^{*,20} + y^*z^*u_y^{*,21} + P_2(z^*)u_y^{*,12} + P_2(y^*)u_y^{*,30} \\ & + P_2(y^*)z^*u_y^{*,31} + P_3(z^*)u_y^{*,13} + P_3(y^*)u_y^{*,40} + P_3(y^*)z^*u_y^{*,41} \\ & + P_4(z^*)u_y^{*,14} \end{aligned}$$

$$\begin{aligned} u_z = {} & u_{z0}^* + y^*\theta_x^* + y^*\gamma^* + z^*u_z^{*,02} + P_2(y^*)u_z^{*,21} + y^*z^*u_z^{*,12} + P_2(z^*)u_z^{*,03} \\ & + P_3(y^*)u_z^{*,31} + y^*P_2(z^*)u_z^{*,13} + P_3(z^*)u_z^{*,04} + P_4(y^*)u_z^{*,41} \\ & + y^*P_3(z^*)u_z^{*,14} \end{aligned}$$

In expression (17), $\mathbf{U}^*(y^*, z^*)$ is a (3×38) matrix which, again, is not depicted due to space restrictions, while the normalised generalised displacements $\mathbf{d}^*(x)$ is the following (38×1) vector:

$$\begin{aligned} \mathbf{d}^* = [& u_{x0}^* \; u_{y0}^* \; u_{z0}^* \; \theta_x^* \; \theta_y^* \; \theta_z^* \; \eta^* \; \gamma^* \; u_x^{*,20} \; u_y^{*,20} \; u_x^{*,02} \\ & u_z^{*,02} \; u_x^{*,21} \; u_y^{*,21} \; u_z^{*,21} \; u_x^{*,12} \; u_y^{*,12} \; u_z^{*,12} \; u_x^{*,30} \; u_y^{*,30} \\ & u_x^{*,03} \; u_z^{*,03} \; u_x^{*,31} \; u_y^{*,31} \; u_z^{*,31} \; u_x^{*,13} \; u_y^{*,13} \; u_z^{*,13} \; u_x^{*,40} \\ & u_y^{*,40} \; u_x^{*,04} \; u_z^{*,04} \; u_x^{*,41} \; u_y^{*,41} \; u_z^{*,41} \; u_x^{*,14} \; u_y^{*,14} \; u_z^{*,14}]^T \end{aligned} \quad (22)$$

Similarly to the first-order components, it is possible to write down the complete compatibility matrix $\mathbf{A}^*$ (38×38) which converts the classical generalised displacements vector $\mathbf{d}$ (38×1) into the normalised generalised displacements vector $\mathbf{d}^*$ (38×1):

$$\mathbf{d}^* = \mathbf{A}^*\mathbf{d} \quad (23)$$

Considering common engineering reasoning, the meaningful classical generalised displacements are those of the first-order, which are dealt with in Eq. (21). Hence, the components of vector $\mathbf{d}$ related to higher-order generalised displacements are considered here to be identical to the corresponding normalised generalised displacements and the respective part of matrix $\mathbf{A}^*$ is an identity sub-matrix.

3.2. First and higher-order compatibility equations

In the compatibility equations herein developed, Legendre polynomials and their derivatives appear in the definition of generalised strains. However, those derivatives are not orthogonal either to the Legendre polynomials or between each other. Consequently, the generalised strains thus obtained would be independent from one another but they would not constitute an orthogonal set. Such lack of orthogonality would lead to a dubious definition for the generalised stress-resultants, according to Eq. (10), and to a large coupling of the equilibrium equations. This is undesirable in a FB approach since it would become extremely complex to obtain the self-equilibrated interpolation functions for the generalised stress-resultants. Moreover, the nodal BCs would also be coupled, which would unnecessarily increase the model's complexity. Hence, in order to have a unique definition for the generalised stress-resultants and to reduce the coupling between the equilibrium equations and BCs to a minimum, those derivatives may and should be decomposed on a basis of Legendre polynomials. Every single generalised strain will then be associated with a unique Legendre polynomial in each direction.

Analogously to the displacement field, the strain field can also be decomposed into first and higher-order components. The first-order strain field can be derived from Eqs. (20) and (2):

$$\boldsymbol{\varepsilon}^{1st\,order}=\begin{bmatrix}\varepsilon_x^{1st\,order}\\ \varepsilon_y^{1st\,order}\\ \varepsilon_z^{1st\,order}\\ \gamma_{xy}^{1st\,order}\\ \gamma_{xz}^{1st\,order}\\ \gamma_{yz}^{1st\,order}\end{bmatrix}=\widehat{\mathbf{D}}^{adj}\mathbf{u}^{1st\,order}=$$

$$=\begin{bmatrix}\varepsilon_0+z^*\,\chi_y^{*,1st\ order}-y^*\,\chi_z^{*,1st\ order}+y^*z^*\,\chi_\eta^{*,1st\ order}\\ 0\\ 0\\ \gamma_{y0}-z^*\,\chi_x^{*,1st\ order}+z^*\,\chi_\gamma^{*,1st\ order}\\ \gamma_{z0}+y^*\,\chi_x^{*,1st\ order}+y^*\,\chi_x^{*,1st\ order}\\ 2\gamma\end{bmatrix}\Longleftrightarrow$$

$$\Longleftrightarrow\boldsymbol{\varepsilon}^{1st\,order}=\begin{bmatrix}1&0&0&0&z^*&-y^*&y^*z^*&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&1&0&-z^*&0&0&0&z^*&0\\0&0&1&y^*&0&0&0&y^*&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\varepsilon_0\\ \gamma_{y0}\\ \gamma_{z0}\\ \chi_x^{*,1st\,order}\\ \chi_y^{*,1st\,order}\\ \chi_z^{*,1st\,order}\\ \chi_\eta^{*,1st\,order}\\ \chi_\gamma^{*,1st\,order}\\ 2\gamma\end{bmatrix}=\mathbf{B}^{*,1st\,order}\mathbf{e}^{*,1st\,order} \quad (24)$$

where $\varepsilon_0=u_{x0}^{*\prime}=u_{x0}^{\prime}$, $\gamma_{y0}=u_{y0}^{*\prime}-2\theta_z^*/b=u_{y0}^{\prime}-\theta_z$, $\gamma_{z0}=u_{z0}^{*\prime}+2\theta_y^*/h=u_{z0}^{\prime}+\theta_y$, $\chi_x^{*,1st\,order}=\theta_x^{*\prime}+\eta^*(1/h-1/b)$, $\chi_y^{*,1st\,order}=\theta_y^{*\prime}$, $\chi_z^{*,1st\,order}=\theta_z^{*\prime}$, $\chi_\eta^{*,1st\,order}=\eta^{*\prime}$, $\chi_\gamma^{*,1st\,order}=\gamma^{*\prime}+\eta^*(1/h+1/b)$, and 2γ are the generalised strains, and $\mathbf{e}^{*,1st\,order}$ is the vector of the first-order normalised generalised cross-sectional strains. The superscript '$*$, 1*st order*' has been appended only to the components that differ from the usual generalised cross-sectional strains associated with the ETBT, as indicated in Section 2.2.

The corresponding beam local compatibility equations are:

$$\mathbf{e}^{*,1st\,order}=\mathbf{D}^{adj*,1st\,order}\mathbf{d}^{*,1st\,order}\Longleftrightarrow\begin{bmatrix}\varepsilon_0\\ \gamma_{y0}\\ \gamma_{z0}\\ \chi_x^{*,1st\,order}\\ \chi_y^{*,1st\,order}\\ \chi_z^{*,1st\,order}\\ \chi_\eta^{*,1st\,order}\\ \chi_\gamma^{*,1st\,order}\\ 2\gamma\end{bmatrix}=$$

$$=\begin{bmatrix}\partial/\partial x&0&0&0&0&0&0&0\\0&\partial/\partial x&0&0&0&-\frac{2}{b}&0&0\\0&0&\partial/\partial x&0&\frac{2}{h}&0&0&0\\0&0&0&\partial/\partial x&0&0&\frac{1}{h}-\frac{1}{b}&0\\0&0&0&0&\partial/\partial x&0&0&0\\0&0&0&0&0&\partial/\partial x&0&0\\0&0&0&0&0&0&\partial/\partial x&0\\0&0&0&0&0&0&\frac{1}{h}+\frac{1}{b}&\partial/\partial x\\0&0&0&\frac{2}{b}-\frac{2}{h}&0&0&0&\frac{2}{b}+\frac{2}{h}\end{bmatrix}\begin{bmatrix}u_{x0}^*\\u_{y0}^*\\u_{z0}^*\\ \theta_x^*\\ \theta_y^*\\ \theta_z^*\\ \eta^*\\ \gamma^*\end{bmatrix}$$

Following an analogous rationale, the components of the complete strain field including higher-order terms can be obtained. Due to space limitations they are not depicted herein (see Almeida [70]), as neither is the subsequent form $\varepsilon=\mathbf{B}^*\mathbf{e}^*$. In the latter expression, $\mathbf{e}^*$ is the (57×1) vector of generalised cross-sectional strains and $\mathbf{B}^*$ is the (6×57) strain approximation matrix. Nevertheless, the following higher-order modifications to the first-order generalised cross-section strains of Eq. (24), due to the imposition of orthogonality between the deformation modes, should be noted:

$$\begin{cases}\gamma_{y0}^*=\gamma_{y0}+\frac{2}{b}u^{*,30}\\ \gamma_{z0}^*=\gamma_{z0}+\frac{2}{h}u^{*,03}\\ \chi_x^*=\theta_x^{*\prime}+\eta^*\left(\frac{1}{h}-\frac{1}{b}\right)+\frac{u^{*,13}}{h}-\frac{u^{*,31}}{b}\\ \chi_y^*=\theta_y^{*\prime}=\chi_y^{*,1st\ order}\\ \chi_z^*=\theta_z^{*\prime}=\chi_z^{*,1st\ order}\\ \chi_\eta^*=\eta^{*\prime}=\chi_\eta^{*,1st\ order}\\ \chi_\gamma^*=\gamma^{*\prime}+\eta^*\left(\frac{1}{h}+\frac{1}{b}\right)+\frac{u^{*,13}}{h}+\frac{u^{*,31}}{b}\end{cases}$$

The final local compatibility equations are, as expected, given by $\mathbf{e}^*=\mathbf{D}^{adj*}\mathbf{d}^*$, where $\mathbf{d}^*$ is the vector of normalised generalised displacements, as indicated in Eq. (22), and $\mathbf{D}^{adj*}$ is the (57×38) differential compatibility operator, not depicted herein for obvious reasons.

For what concerns the compatibility BCs, these are expressed by:

$$\mathbf{d}^*=\mathbf{d}^{*,\dagger}\ \text{in}\ x=0,L$$

3.3. First and higher-order equilibrium equations

The local equilibrium equations are obtained by using the same projection method applied in expression (9), resulting in power-conjugate or dual relationships to the local compatibility equations:

$$\int_A(\mathbf{U}^*)^T(\widehat{\mathbf{D}}\boldsymbol{\sigma}+\mathbf{b})da=\mathbf{0}\ \Longleftrightarrow$$

$$\Longleftrightarrow\mathbf{D}^*\underbrace{\int_A(\mathbf{B}^*)^T\boldsymbol{\sigma}\,da}_{\mathbf{s}^*}+\underbrace{\int_A(\mathbf{U}^*)^T\mathbf{b}\,da+\int_\Gamma(\mathbf{U}^*)^T\mathbf{t}^\dagger\,ds}_{\mathbf{p}^*}=\mathbf{0}\ \text{in}\ L$$

Once more, it is possible to isolate the first-order components. The corresponding local equilibrium equations are given by:

$$\mathbf{D}^{*,1st\,order}\mathbf{s}^{*,1st\,order}+\mathbf{p}^{*,1st\,order}=\mathbf{0}\Longleftrightarrow\begin{cases}N'+q_x=0\\ V_y'+q_y=0\\ V_z'+q_z=0\\ T^{*\prime}+2\left(\frac{1}{h}-\frac{1}{b}\right)V_{yz}+m_x^*=0\\ M_y^{*\prime}-\frac{2}{h}V_z+m_y^*=0\\ M_z^{*\prime}+\frac{2}{b}V_y+m_z^*=0\\ B^{*\prime}-\left(\frac{1}{b}+\frac{1}{h}\right)Q^*+\left(\frac{1}{b}-\frac{1}{h}\right)T^*+m_b^*=0\\ Q^{*\prime}-2\left(\frac{1}{h}+\frac{1}{b}\right)V_{yz}+m_q^*=0\end{cases}$$

The normalised generalised stress-resultants $\mathbf{s}^{*,1st\ order}=\begin{bmatrix}N&V_y&V_z&T^*&M_y^*&M_z^*&B^*&Q^*&V_{yz}\end{bmatrix}^T$ relate to the classical generalised stress-resultants, and conversely, through:

$$\begin{cases}M_y^*=\frac{2}{h}M_y\\ M_z^*=\frac{2}{b}M_z\\ B^*=\frac{4}{bh}B\\ Q^*=Q\left(\frac{1}{b}+\frac{1}{h}\right)+T\left(\frac{1}{b}-\frac{1}{h}\right)\\ T^*=Q\left(\frac{1}{b}-\frac{1}{h}\right)+T\left(\frac{1}{b}+\frac{1}{h}\right)\end{cases}\qquad\begin{cases}M_y=\frac{h}{2}M_y^*\\ M_z=\frac{b}{2}M_z^*\\ B=\frac{bh}{4}B^*\\ Q=Q^*\left(\frac{b}{4}+\frac{h}{4}\right)+T^*\left(\frac{b}{4}-\frac{h}{4}\right)\\ T=Q^*\left(\frac{b}{4}-\frac{h}{4}\right)+T^*\left(\frac{b}{4}+\frac{h}{4}\right)\end{cases}$$

For what concerns the distributed loads, it is assumed that m_b and m_q (classical beam theory quantities) are null. Bearing in mind these assumptions, the relevant first-order normalised distributed loads can be obtained from the classical ones, and conversely, by:

$$\begin{cases} m_x^* = m_x\left(\frac{1}{b}+\frac{1}{h}\right) \\ m_y^* = \frac{2}{h}m_y \\ m_z^* = \frac{2}{b}m_z \\ m_b^* = 0 \\ m_q^* = m_x\left(\frac{1}{b}-\frac{1}{h}\right) \end{cases} \quad \vdots \quad \begin{cases} m_x = m_q^*\left(\frac{b}{4}-\frac{h}{4}\right) + m_x^*\left(\frac{b}{4}+\frac{h}{4}\right) \\ m_y = \frac{h}{2}m_y^* \\ m_z = \frac{b}{2}m_z^* \\ m_b = 0 \\ m_q = 0 \end{cases}$$

The complete local equilibrium equations constitute a highly indeterminate system of differential equations and are composed of the (38×57) differential equilibrium operator $\mathbf{D}^*$ (adjoint to $\mathbf{D}^{adj*}$ and not explicitly represented due to space restrictions), the (38×1) vector of normalised distributed loads $\mathbf{p}^*$ and the (57×1) vector of normalised generalised stress-resultants $\mathbf{s}^*$:

$$\mathbf{s}^* = \left[N \quad M_y^* \quad M_z^* \quad B^* \quad \mathbf{N}^{*,ij}(12\times 1) \quad \mathbf{N}_y^{*,ij}(6\times 1) \quad \mathbf{N}_z^{*,ij}(6\times 1) \quad V_y \quad T^* \quad Q^* \quad \mathbf{V}_y^{*,ij}(9\times 1) \quad V_z \quad \mathbf{V}_z^{*,ij}(9\times 1) \quad V_{yz} \quad \mathbf{V}_{yz}^{*,ij}(6\times 1) \right]^T$$

where the higher-order components are given by:

$$\left.\begin{aligned} N^{*,ij} &= \textstyle\int_A P_i(y^*)P_j(z^*)\sigma_x\, da \ (12 \text{ terms}) \\ N_y^{*,ij} &= \textstyle\int_A P_{i-2}(y^*)P_j(z^*)\sigma_y\, da \ (6 \text{ terms}) \\ N_z^{*,ij} &= \textstyle\int_A P_i(y^*)P_{j-2}(z^*)\sigma_z\, da \ (6 \text{ terms}) \end{aligned}\right\} \text{for } i \geqslant 2 \text{ or } j \geqslant 2$$

$$\left.\begin{aligned} V_y^{*,ij} &= \textstyle\int_A P_{i-1}(y^*)P_j(z^*)\tau_{xy}\, da \ (9 \text{ terms}) \\ V_z^{*,ij} &= \textstyle\int_A P_i(y^*)P_{j-1}(z^*)\tau_{xz}\, da \ (9 \text{ terms}) \end{aligned}\right\} \text{for } i \geqslant 1 \text{ or } j \geqslant 1$$

$$V_{yz}^{*,ij} = \textstyle\int_A P_{i-1}(y^*)P_{j-1}(z^*)\tau_{yz}\, da \ (6 \text{ terms}) \text{ for } i \geqslant 1 \text{ and } j \geqslant 1$$

It is noted that $\mathbf{N}_y^{*,ij}, \mathbf{N}_z^{*,ij}$ and $\mathbf{V}_{yz}^{*,ij}$ are related to the stresses σ_y, σ_z and τ_{yz}, respectively, which do not act in the cross-section of the beam. Consequently, these 18 generalised stress-resultants, together with the first-order one V_{yz}, will not appear in the static BCs at the beam ends.

For what concerns the normalised distributed loads $\mathbf{p}^*$, the assumptions regarding the first-order loads $\mathbf{p}^{*,1st\ order}$ are valid. Additionally, it is also assumed that all higher-order components are zero.

After carrying out all the analytical developments [70,71], the final form of the local equilibrium equations can be expressed by eight uncoupled systems of dependent equations, in a total of 38 equations with 57 variables, which are depicted in Appendix A.

Again, the equilibrium BCs are obtained as in Eq. (12):

$$\int_A (\mathbf{U}^*)^T \begin{bmatrix} \sigma_x \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix} n_x da = \underbrace{\int_A (\mathbf{U}^*)^T \begin{bmatrix} t_x^\dagger \\ t_y^\dagger \\ t_z^\dagger \end{bmatrix} da}_{\mathbf{R}^{*,\dagger}} \iff \mathbf{N}^*\mathbf{s}^* = \mathbf{R}^{*,\dagger} \text{ in } x = 0, L \tag{25}$$

where $\mathbf{N}^*$ is the (38×57) exterior unit normal matrix corresponding to the differential equilibrium operator $\mathbf{D}^*$.

4. Force-based formulation

In the previous paragraphs the higher-order beam equilibrium and compatibility equations were derived. The choice of approximating either the field of generalised stress-resultants and strictly verifying the equilibrium equations, or the field of beam displacements together with the strict verification of the compatibility equations originates either a FB formulation (also known as equilibrium or flexibility-based) or a DB formulation (also known as compatibility or stiffness-based). Moreover, in a FB formulation, if the interpolated generalised stress-resultants' field is not self-equilibrated, then a simultaneous interpolation of the beam displacements' field is required and a mixed formulation is obtained. This is the case when geometrically nonlinear effects need to be considered. As previously stressed, a FB formulation is envisaged in this work. Hence, the field of generalised stress-resultants should respect the beam local equilibrium conditions previously derived.

As mentioned above, there are a total of 38 local equilibrium equations involving 57 unknown generalised stress-resultants (see Appendix A). Thus, in order to determine the complementary solution $\mathbf{s}_c^*$ of the homogeneous equilibrium equations $\mathbf{D}^*\mathbf{s}_c^* = \mathbf{0}$, a few assumptions have to be made concerning the functions describing the evolution of some generalised stress-resultants, as already discussed in Section 2.2 for the first-order ETBT. Given that the differential equations of equilibrium require only one BC each, since they involve only first derivatives, a total of 38 BCs are needed for this purpose. On the other hand, there are 76 available BCs at both extremities of the beam, corresponding to the nodal values of the 38 generalised stress-resultants related to the cross-sectional stresses σ_x, τ_{xy} and τ_{xz}. Moreover, as discussed in Section 2.2, six of these nodal values are dependent on the remaining ones since they are related to the six rigid-body motions of the beam. Hence, from the remaining 70 BCs, there are 32 which will not be used to solve the equations of equilibrium and that may be applied instead for defining *a priori* an assumed variation for the 19 generalised stress-resultants related to the stresses σ_y, σ_z and τ_{yz}. Such assumed variation is not unique, which means that different self-equilibrated approximations can be envisaged. In this work, the field of those 19 generalised stress-resultants is approximated by the simplest possible polynomial functions, namely linear and constant ones: the necessary assumptions on their variations are briefly indicated in Appendix B. The solution corresponding to such self-equilibrated higher-order stress-resultants is too lengthy to be included herein, but can be verified in Almeida [70]. Nevertheless, it is expressed in a compact form as:

$$\mathbf{s}_c^* = \mathbf{H_s}\mathbf{P}^*$$

where $\mathbf{H_s}$ is the (57×70) generalised stress-resultants' interpolation matrix, satisfying $\mathbf{D}^*\mathbf{H_s} = \mathbf{0}$, and $\mathbf{P}^*$ is the following (70×1) vector of independent or basic forces:

$$\mathbf{P}^* = \left[M_{yA}^* \quad M_{zA}^* \quad B_A^* \quad \mathbf{N}_A^{*,ij}(12\times 1) \quad T_A^* \quad Q_A^* \quad \mathbf{V}_{yA}^{*,ij}(9\times 1) \quad \mathbf{V}_{zA}^{*,ij}(9\times 1) \quad N_B \quad M_{yB}^* \quad M_{zB}^* \quad B_B^* \quad \mathbf{N}_B^{*,ij}(12\times 1) \quad Q_B^* \quad \mathbf{V}_{yB}^{*,ij}(9\times 1) \quad \mathbf{V}_{zB}^{*,ij}(9\times 1) \right]^T$$

In the previous definition, the subscripts 'A' and 'B' stand for the initial and final element nodes (refer to Fig. 1), respectively. Moreover, it is noted that the components of $\mathbf{P}^*$ were directly identified with the nodal forces at the beam ends, in the absence of span loads, by using the static BCs of Eq. (25). Additionally, the remaining nodal forces $(N_A, V_{yA}, V_{zA}, V_{yB}, V_{zB}$ and $T_B^*)$ are not included in $\mathbf{P}^*$ because they are dependent generalised forces.

A particular solution $\mathbf{s}_0^*$ equilibrating the possible span loads, i.e., verifying $\mathbf{D}^*\mathbf{s}_0^* + \mathbf{p}^* = \mathbf{0}$, can also be developed. Some of the application cases presented in the companion paper feature such distributed loading [59]. Nonetheless, the complete solution of the differential equilibrium equations is:

$$\mathbf{s}^* = \mathbf{s}_c^* + \mathbf{s}_0^* = \mathbf{H_s}\mathbf{P}^* + \mathbf{s}_0^* \tag{26}$$

With the above mentioned approximation, equilibrium in the domain and at the boundary is automatically satisfied. In order to correctly formulate a new beam element, the compatibility in the domain and the kinematic BCs have to be likewise satisfied. So as to maintain power-conjugacy, the domain compatibility equations are verified in a weighted form using the generalised stress-resultants' interpolation matrix as weighting functions, resulting in the following adjoint compatibility conditions:

$$\begin{aligned}
&\int_L \mathbf{H_s}^T(\mathbf{e}^* - \mathbf{D}^{adj*}\mathbf{d}^*)dx = \mathbf{0} \iff \\
&\iff \mathbf{v}^* \equiv \int_L \mathbf{H_s}^T\mathbf{e}^* dx = \int_L \mathbf{H_s}^T\mathbf{D}^{adj*}\mathbf{d}^* dx \iff \\
&\iff \mathbf{v}^* = -\int_L \Big(\underbrace{\mathbf{D}^*\mathbf{H_s}}_{=\mathbf{0}}\Big)^T \mathbf{d}^* dx + \left[(\mathbf{N}^*\mathbf{H_s})^T\mathbf{d}^*\right]_{x=0,L} \iff \\
&\iff \mathbf{v}^* = \underbrace{\left[(\mathbf{N}^*\mathbf{H_s})^T_{|x=0} | (\mathbf{N}^*\mathbf{H_s})^T_{|x=L}\right]}_{\mathbf{A}_b^*} \underbrace{\begin{bmatrix}\mathbf{d}^*_{|x=0}\\ \mathbf{d}^*_{|x=L}\end{bmatrix}}_{\mathbf{q}_b^*} \iff \\
&\iff \mathbf{v}^* = \mathbf{A}_b^*\mathbf{q}_b^*
\end{aligned}$$

where $\mathbf{A}_b^*$ is the (70×76) nodal compatibility matrix that transforms the (76×1) vector of nodal normalised displacements $\mathbf{q}_b^*$ into the (70×1) vector of independent deformations $\mathbf{v}^*$. The latter are power-conjugate to the independent forces $\mathbf{P}^*$, in the sense that

$$\int_L (\mathbf{e}^*)^T \mathbf{s}_c^* dx = (\mathbf{v}^*)^T\mathbf{P}^*,$$

and can be interpreted as the projection of the beam generalised strains in the space of interpolation functions for the generalised stress-resultants. In the above derivation, the compatibility BCs were introduced and the self-equilibrated properties of the generalised stress-resultants interpolation matrix were used. Recalling Eq. (23), the final compatibility equations can be written:

$$\mathbf{v}^* = \mathbf{A}_b^*\mathbf{q}_b^* = \mathbf{A}_b^* \underbrace{\begin{bmatrix}\mathbf{A}^* & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^*\end{bmatrix}}_{\mathbf{A}^*_{nodal}} \underbrace{\begin{bmatrix}\mathbf{d}_{|x=0}\\ \mathbf{d}_{|x=L}\end{bmatrix}}_{\mathbf{q}_b} = \mathbf{A}_b\mathbf{q}_b \tag{27}$$

The (70×76) final compatibility matrix $\mathbf{A}_b$ transforms the (76×1) vector of nodal classical displacements $\mathbf{q}_b$ into the (70×1) vector of independent deformations $\mathbf{v}^*$.

The final equilibrium equations, on the other hand, can be obtained as follows, making use of the equilibrium BCs (25) and the self-equilibrated stress-resultant interpolation field (26):

$$\begin{aligned}
&\mathbf{N}^*\mathbf{s}^* - \mathbf{R}^{*,\dagger} = \mathbf{0} \text{ in } x = 0, L \iff \\
&\iff \mathbf{N}^*\mathbf{H_s}\mathbf{P}^* + \mathbf{N}^*\mathbf{s}_0^* - \mathbf{R}^{*,\dagger} = \mathbf{0} \text{ in } x = 0, L \iff \\
&\iff \begin{bmatrix}(\mathbf{N}^*\mathbf{H_s})_{|x=0}\\ (\mathbf{N}^*\mathbf{H_s})_{|x=L}\end{bmatrix}\mathbf{P}^* + \begin{bmatrix}(\mathbf{N}^*\mathbf{s}_0^*)_{|x=0}\\ (\mathbf{N}^*\mathbf{s}_0^*)_{|x=L}\end{bmatrix} - \begin{bmatrix}\mathbf{R}^{*,\dagger}_{|x=0}\\ \mathbf{R}^{*,\dagger}_{|x=L}\end{bmatrix} = \mathbf{0} \iff \\
&\iff (\mathbf{A}_b^*)^T\mathbf{P}^* + \mathbf{Q}_{b0}^* - \mathbf{Q}_b^{*,\dagger} = \mathbf{0} \Rightarrow \\
&\Rightarrow (\mathbf{A}^*_{nodal})^T(\mathbf{A}_b^*)^T\mathbf{P}^* + (\mathbf{A}^*_{nodal})^T\mathbf{Q}_{b0}^* - (\mathbf{A}^*_{nodal})^T\mathbf{Q}_b^{*,\dagger} = \mathbf{0} \iff \\
&\iff \mathbf{A}_b^T\mathbf{P}^* + \mathbf{Q}_{b0} - \mathbf{Q}_b^\dagger = \mathbf{0}
\end{aligned} \tag{28}$$

where $\mathbf{A}_b^T$ is the (76×70) final equilibrium matrix, $\mathbf{Q}_b^\dagger$ is the (76×1) vector of external nodal classical forces and $\mathbf{Q}_{b0}$ are the nodal classical forces in equilibrium with the span loading considering the independent forces $\mathbf{P}^*$ null.

The adjointness or duality of the presented beam theory equations is thus expressed by the following identity:

$$\begin{aligned}
&(\mathbf{v}^*)^T\mathbf{P}^* = \mathbf{q}_b^T\left(\mathbf{Q}_b^\dagger - \mathbf{Q}_{b0}\right) \iff \\
&\iff (\mathbf{v}^*)^T\mathbf{P}^* + \int_L (\mathbf{e}^*)^T\mathbf{s}_0^* dx = \int_L (\mathbf{d}^*)^T\mathbf{p}^* dx + \mathbf{q}_b^T\mathbf{Q}_b^\dagger
\end{aligned}$$

Finally, it is possible to obtain the corresponding incremental equations required for nonlinear constitutive behaviour. The first result which is necessary for such derivation is:

$$\begin{aligned}
\Delta\mathbf{v}^* &\equiv \int_L \mathbf{H_s}^T\Delta\mathbf{e}^* dx = \int_L \mathbf{H_s}^T\mathbf{F}^*_{\text{section}}\Delta\mathbf{s}^* dx = \\
&= \int_L \mathbf{H_s}^T\mathbf{F}^*_{\text{section}}\mathbf{H_s} dx \Delta\mathbf{P}^* + \int_L \mathbf{H_s}^T\mathbf{F}^*_{\text{section}}\Delta\mathbf{s}_0^* dx = \\
&= \mathbf{F}^*_{elem}\Delta\mathbf{P}^* + \Delta\mathbf{v}_0^*
\end{aligned} \tag{29}$$

where $\Delta\mathbf{v}_0^*$ is the vector of increments of independent deformations related to an increment of span loads, and $\mathbf{F}^*_{\text{section}}$ and $\mathbf{F}^*_{elem}$ are the (57×57) and (70×70) tangent flexibility matrices, respectively, of the cross-section and of the beam element (with respect to the independent forces and deformations). The dual nature of $\mathbf{e}^*$ and $\mathbf{s}^*$ and the assumed symmetry of the constitutive material tangent stiffness operator imply that $\mathbf{F}^*_{\text{section}}$ is symmetric. Expression (29), on the other hand, also ensures the symmetry of $\mathbf{F}^*_{elem}$.

Making use of the previous relation, as well as of expressions (27) and (28), it can be written:

$$\begin{aligned}
&\begin{cases}\Delta\mathbf{v}^* = \mathbf{A}_b\Delta\mathbf{q}_b\\ \mathbf{A}_b^T\Delta\mathbf{P}^* + \Delta\mathbf{Q}_{b0} - \Delta\mathbf{Q}_b^\dagger = \mathbf{0}\end{cases} \iff \begin{cases}\Delta\mathbf{P}^* = (\mathbf{F}^*_{elem})^{-1}\mathbf{A}_b\Delta\mathbf{q}_b - (\mathbf{F}^*_{elem})^{-1}\Delta\mathbf{v}_0^* \Rightarrow\\ \mathbf{A}_b^T\Delta\mathbf{P}^* = \Delta\mathbf{Q}_b^\dagger - \Delta\mathbf{Q}_{b0}\end{cases} \\
&\Rightarrow \underbrace{\mathbf{A}_b^T(\mathbf{F}^*_{elem})^{-1}\mathbf{A}_b}_{\mathbf{K}_b}\Delta\mathbf{q}_b = \Delta\mathbf{Q}_b^\dagger \underbrace{- \Delta\mathbf{Q}_{b0} + \mathbf{A}_b^T(\mathbf{F}^*_{elem})^{-1}\Delta\mathbf{v}_0^*}_{-\Delta\tilde{\mathbf{Q}}_{b0}} \iff \\
&\iff \mathbf{K}_b\Delta\mathbf{q}_b = \Delta\mathbf{Q}_b^\dagger - \Delta\tilde{\mathbf{Q}}_{b0}
\end{aligned} \tag{30}$$

where $\mathbf{K}_b$ is the (76×76) symmetric stiffness matrix of the beam element, with respect to the classical nodal forces and displacements, and $\Delta\tilde{Q}_{b0}$ is the vector of increments of classical nodal forces in equilibrium with an increment of the span loads considering the incremental classical nodal displacements to be null.

It is noted that the governing system of 70 equations associated with the basic system (29) is six equations less than the global nodal governing equations (30). This difference corresponds to the rigid-body motions, as it would be expected.

5. Conclusions

A new beam element based on higher-order cross-sectional displacement modes was presented in this paper. It was derived using a normalised version of the so-called Legendre polynomials for the analysis of beams with solid rectangular cross-sections. These functions feature the important property of orthogonality, which is in turn reflected on the orthogonality of the cross-sectional displacement modes. A further careful mathematical manipulation also allows for an unambiguous definition of the corresponding generalised stress-resultants and for a minimisation of the coupling between higher-order equations of equilibrium, both in the domain and at the boundaries. The higher-order displacement field contains, of course, the six common displacement modes of classical beam theory—and consequently the usual cross-sectional stress-resultants as well, which appeals to concepts that are familiar to engineers. Note that any set of orthogonal displacement functions could be used instead of Legendre polynomials; such orthogonal set could be obtained from a set of independent base functions through a Gram–Schmidt orthogonalisation procedure. Moreover, the latter can be applied in order to extend this formulation to different cross-sectional geometries.

Besides the aforementioned innovative aspects, the main novelty is the development of a higher-order beam element based on a flexibility approach. Accordingly, self-equilibrated stress-resultants' interpolation functions were chosen in order to guarantee that the corresponding higher-order equilibrium conditions are always respected. The strict satisfaction of equilibrium is of the utmost importance to model the inelastic response of members, such as that caused by earthquake or blast loads. However, its advantages span to the elastic behaviour range as well, as it is made apparent in the companion paper. Those equilibrium conditions were obtained through a projection of the three-dimensional continuum equilibrium equations on the space of the higher-order displacement modes. This procedure guarantees the duality with the associated one-dimensional higher-order compatibility equations. Hence the beam formulation is completely consistent from the work-equivalency viewpoint.

The proposed model is inherently free from shear-locking issues, contrary to the more traditional displacement-based approaches, and is able to deal with arbitrary distributed loading conditions as consistently as with nodal loadings. It includes the interaction between the axial force, bi-directional shears, bending moments, and torsion with all higher-order generalised forces. Since explicit normal-shear stress interaction is directly accounted for, such element is also deemed suitable for shear critical member analysis.

The theory was developed independently of the constitutive behaviour, and application examples to both linear and nonlinear material response of members with relevant shear and torsional deformations are presented in the companion paper. These validate the present formulation, and demonstrate its accuracy and promising capabilities when compared to other existing approaches. Dynamic response and more general cross-sections will be considered in future work.

Acknowledgements

The financial support provided by the *Portuguese Science and Technology Foundation (FCT)* through the *2010 Science and Innovation Operational Programme* is gratefully acknowledged. The authors would also like to thank three anonymous reviewers, whose constructive comments helped improving significantly this paper.

Appendix A

This appendix depicts the eight uncoupled systems of dependent differential equations, involving only first derivatives, representing the final form of the complete higher-order beam equilibrium. The number of equations, unknowns and corresponding BCs at the beam ends are also shown together with each system of equations. Additionally, where pertinent, the number of rigid-body modes (or equivalently, redundant BCs) are also shown. Finally, the number of redundant BCs which are not necessary for solving each system of equations is computed:

$$(1)\quad N' + q_x = 0 \quad \left|\begin{array}{c} \text{1 equation}\\ \text{1 unknown}\\ \text{2 BCs}\\ \text{1 rigid-body mode}\\ (\textit{axial displacement}) \end{array}\right.$$

$$(2)\quad \left\{\begin{array}{c} V'_y + q_y = 0\\ M_z^{*\prime} + \frac{2V_y}{b} + m_z^* = 0\\ N^{*,30\prime} - \frac{2V_y}{b} - \frac{10}{b}V_y^{*,30} = 0\\ V_y^{*,30\prime} - \frac{6}{b}N_y^{*,30} = 0 \end{array}\right. \quad \left|\begin{array}{c} \text{4 equations}\\ \text{5 unknowns}\\ \text{8 BCs}\\ \text{2 rigid-body modes}\\ (\textit{displacement along } y,\\ \textit{rotation around } z)\\ \text{-------------------}\\ \text{2 unused } \textit{BCs} \end{array}\right.$$

$$(3)\quad \left\{\begin{array}{c} V'_z + q_z = 0\\ M_y^{*\prime} - \frac{2V_z}{h} + m_y^* = 0\\ N^{*,03\prime} - \frac{2V_z}{h} - \frac{10}{h}V_z^{*,03} = 0\\ V_z^{*,03\prime} - \frac{6}{h}N_z^{*,03} = 0 \end{array}\right. \quad \left|\begin{array}{c} \text{4 equations}\\ \text{5 unknowns}\\ \text{8 BCs}\\ \text{2 rigid-body modes}\\ (\textit{displacement along } z,\\ \textit{rotation around } y)\\ \text{-------------------}\\ \text{2 unused BCs} \end{array}\right.$$

$$(4)\quad \left\{\begin{array}{c} T^{*\prime} + 2\left(\frac{1}{h} - \frac{1}{b}\right)V_{yz} + m_x^* = 0\\ B^{*\prime} - Q^*\left(\frac{1}{b} + \frac{1}{h}\right) + T^*\left(\frac{1}{b} - \frac{1}{h}\right) = 0\\ Q^{*\prime} - 2\left(\frac{1}{b} + \frac{1}{h}\right)V_{yz} + m_q^* = 0\\ V_y^{*,31\prime} - \frac{6}{b}N_y^{*,31} - \frac{2}{h}V_{yz}^{*,31} = 0\\ V_z^{*,31\prime} - \frac{2}{b}V_{yz} - \frac{10}{b}V_{yz}^{*,31} = 0\\ N^{*,31\prime} - \frac{Q^*}{b} + \frac{T^*}{b} - \frac{10}{b}V_y^{*,31} - \frac{2}{h}V_z^{*,31} = 0\\ V_y^{*,13\prime} - \frac{2}{h}V_{yz} - \frac{10}{h}V_{yz}^{*,13} = 0\\ V_z^{*,13\prime} - \frac{6}{h}N_z^{*,13} - \frac{2}{b}V_{yz}^{*,13} = 0\\ N^{*,13\prime} - \frac{Q^*}{h} - \frac{T^*}{h} - \frac{2}{b}V_y^{*,13} - \frac{10}{h}V_z^{*,13} = 0 \end{array}\right. \quad \left|\begin{array}{c} \text{9 equations}\\ \text{14 unknowns}\\ \text{18 BCs}\\ \text{1 rigid-body mode}\\ (\textit{torsional rotation})\\ \text{-------------------}\\ \text{8 } \textit{unused BCs} \end{array}\right.$$

$$(5)\quad \left\{\begin{array}{c} N^{*,20\prime} - \frac{6}{b}V_y^{*,20} = 0\\ V_y^{*,20\prime} - \frac{2}{b}N_y^{*,20} = 0\\ N^{*,40\prime} - \frac{6}{b}V_y^{*,20} - \frac{14}{b}V_y^{*,40} = 0\\ V_y^{*,40\prime} - \frac{2}{b}N_y^{*,20} - \frac{10}{b}N_y^{*,40} = 0 \end{array}\right. \quad \left|\begin{array}{c} \text{4 equations}\\ \text{6 unknowns}\\ \text{8 BCs}\\ \text{------------}\\ \text{4 } \textit{unused BCs} \end{array}\right.$$

$$(6)\quad \left\{\begin{array}{c} N^{*,02\prime} - \frac{6}{h}V_z^{*,02} = 0\\ V_z^{*,02\prime} - \frac{2}{h}N_z^{*,02} = 0\\ N^{*,04\prime} - \frac{6}{h}V_z^{*,02} - \frac{14}{h}V_z^{*,04} = 0\\ V_z^{*,04\prime} - \frac{2}{h}N_z^{*,02} - \frac{10}{h}N_z^{*,04} = 0 \end{array}\right. \quad \left|\begin{array}{c} \text{4 equations}\\ \text{6 unknowns}\\ \text{8 BCs}\\ \text{-----------}\\ \text{4 } \textit{unused BCs} \end{array}\right.$$

$$(7)\quad \begin{cases} N^{*,21\prime} - \frac{6}{b}V_y^{*,21} - \frac{2}{h}V_z^{*,21} = 0 \\ V_y^{*,21\prime} - \frac{2}{b}N_y^{*,21} - \frac{2}{h}V_{yz}^{*,21} = 0 \\ V_z^{*,21\prime} - \frac{6}{b}V_{yz}^{*,21} = 0 \\ N^{*,41\prime} - \frac{6}{b}V_y^{*,21} - \frac{14}{b}V_y^{*,41} - \frac{2}{h}V_z^{*,41} = 0 \\ V_y^{*,41\prime} - \frac{2}{b}N_y^{*,21} - \frac{10}{b}N_y^{*,41} - \frac{2}{h}V_{yz}^{*,41} = 0 \\ V_z^{*,41\prime} - \frac{6}{b}V_{yz}^{*,21} - \frac{14}{b}V_{yz}^{*,41} = 0 \end{cases} \quad \begin{array}{|c} \text{6 equations} \\ \text{10 unknowns} \\ \text{12 BCs} \\ \text{-----------} \\ \text{6 } \textit{unused BCs} \end{array}$$

$$(8)\quad \begin{cases} N^{*,12\prime} - \frac{2}{b}V_y^{*,12} - \frac{6}{h}V_z^{*,12} = 0 \\ V_z^{*,12\prime} - \frac{2}{h}N_z^{*,12} - \frac{2}{b}V_{yz}^{*,12} = 0 \\ V_y^{*,12\prime} - \frac{6}{h}V_{yz}^{*,12} = 0 \\ N^{*,14\prime} - \frac{6}{h}V_z^{*,12} - \frac{14}{h}V_z^{*,14} - \frac{2}{b}V_y^{*,14} = 0 \\ V_z^{*,14\prime} - \frac{2}{h}N_z^{*,12} - \frac{10}{h}N_z^{*,14} - \frac{2}{b}V_{yz}^{*,14} = 0 \\ V_y^{*,14\prime} - \frac{6}{h}V_{yz}^{*,12} - \frac{14}{h}V_{yz}^{*,14} = 0 \end{cases} \quad \begin{array}{|c} \text{6 equations} \\ \text{10 unknowns} \\ \text{12 BCs} \\ \text{-----------} \\ \text{6 } \textit{unused BCs} \end{array}$$

Appendix B

The assumptions concerning the order of the approximating polynomials, for the solution of the sets of equilibrium equations depicted in Appendix A, are as follows:

(2) $N_y^{*,30}$: linear

(3) $N_z^{*,03}$: linear

(4) $V_{yz}^{*,13}, V_{yz}^{*,31}$: constant; $V_{yz}, N_y^{*,31}, N_z^{*,13}$: linear

(5) $N_y^{*,20}, N_y^{*,40}$: linear

(6) $N_z^{*,02}, N_z^{*,04}$: linear

(7) $V_{yz}^{*,21}, V_{yz}^{*,41}$: constant; $N_y^{*,21}, N_y^{*,41}$: linear

(8) $V_{yz}^{*,12}, V_{yz}^{*,14}$: constant; $N_z^{*,12}, N_z^{*,14}$: linear

References

[1] Rankine WJM. A manual of applied mechanics. Glasgow, UK: Richard Griffin and Company; 1858 [republished in 2008 by Hoar Press].
[2] Bresse JAC. Cours de mécanique appliquée. Paris, France: Mallet-Bachelier; 1859.
[3] Timoshenko SP. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philos Mag Ser. 6 1921;41:744–6. http://dx.doi.org/10.1080/14786442108636264.
[4] Timoshenko SP. On the transverse vibrations of bars of uniform cross-section. Philos Mag Ser. 6 1922;43:125–31. http://dx.doi.org/10.1080/14786442208633855.
[5] Mindlin RD, Deresiewicz H. Timoshenko's shear coefficient for flexural vibrations of beams. In: Proc. 2nd US National Congress Applied Mechanics; 1954. p. 175–8.
[6] Cowper GR. The shear coefficient in Timoshenko's beam theory. J Appl Mech 1966;33:335–40.
[7] Cowper GR. On the accuracy of Timoshenko's beam theory. J Eng Mech Div 1968;94:1447–53.
[8] Kaneko T. On Timoshenko's correction for shear in vibrating beams. J Phys D Appl Phys 1975;8:1927–36.
[9] Hutchinson JR, Zillmer SD. On the transverse vibration of beams of rectangular cross-section. J Appl Mech 1986;53:39–44. http://dx.doi.org/10.1115/1.3171735.
[10] Renton JD. Generalized beam theory applied to shear stiffness. Int J Solids Struct 1991;27:1955–67.
[11] Hutchinson JR. Shear coefficients for Timoshenko beam theory. J Appl Mech 2001;68:87–92. http://dx.doi.org/10.1115/1.1349417.
[12] Dong SB, Alpdogan C, Taciroglu E. Much ado about shear correction factors in Timoshenko beam theory. Int J Solids Struct 2010;47:1651–65. http://dx.doi.org/10.1016/j.ijsolstr.2010.02.018.
[13] Stephen NG, Levinson M. A second order beam theory. J Sound Vib 1979;67:293–305.
[14] Soler A. Higher order effects in thick rectangular elastic beams. Int J Solids Struct 1968;4:723–39.
[15] Tsai H, Soler A. Approximate theory for locally loaded plane orthotropic beams. Int J Solids Struct 1970;6:1055–68.
[16] Levinson M. A new rectangular beam theory. J Sound Vib 1981;74:81–7.
[17] Bickford WB. A consistent higher order beam theory. Dev Theor Appl Mech 1982;11:137–50.
[18] Reddy JN. A simple higher-order theory for laminated composite plates. J Appl Mech 1984;51:745–52.
[19] Rychter Z. On the accuracy of a beam theory. Mech Res Commun 1987;14:99–105.
[20] Rychter Z. A simple and accurate beam theory. Acta Mech 1988;75:57–62.
[21] Petrolito J. Stiffness analysis of beams using higher-order theory. Comput Struct 1995;55:33–9.
[22] Reddy JN. Nonlocal theories for bending, buckling and vibration of beams. Int J Eng Sci 2007;45:288–307. http://dx.doi.org/10.1016/j.ijengsci.2007.04.004.
[23] Heyliger PR, Reddy JN. A higher order beam finite element for bending and vibration problems. J Sound Vib 1988;126:309–26.
[24] Kant T, Manjunatha BS. Refined theories for composite and sandwich beams with C0 finite elements. Comput Struct 1989;33:755–64.
[25] Manjunatha BS, Kant T. New theories for symmetric/unsymmetric composite and sandwich beams with C0 finite elements. Compos Struct 1993;23:61–73.
[26] Lo KH, Christensen RM, Wu EM. A higher order theory for plate deformations, Part 1: Homogeneous plates. J Appl Mech 1977;44:663–8.
[27] Lo KH, Christensen RM, Wu EM. A higher order theory for plate deformations, Part 2: Laminated plates. J Appl Mech 1977;44:669–76.
[28] Vinayak RU, Prathap G, Naganarayana BP. Beam elements based on a higher order theory – I. Formulation and analysis of performance. Comput Struct 1996;58:775–89.
[29] Prathap G, Vinayak RU, Naganarayana BP. Beam elements based on a higher order theory – II. Boundary layer sensitivity and stress oscillations. Comput Struct 1996;58:791–6.
[30] Kim C, White SR. Thick-walled composite beam theory including 3-D elastic effects and torsional warping. Int J Solids Struct 1997;34:4237–59.
[31] Rand O. Fundamental closed-form solutions for solid and thin-walled composite beams including a complete out-of-plane warping model. Int J Solids Struct 1998;35:2775–93.
[32] Rand O. On the importance of cross-sectional warping in solid composite beams. Compos Struct 2000;49:393–7.
[33] Ghugal YM, Shimpi RP. A review of refined shear deformation theories for isotropic and anisotropic laminated beams. J Reinf Plast Compos 2001;20:255–72. http://dx.doi.org/10.1177/073168401772678283.
[34] Volovoi VV, Hodges DH, Cesnik CES, Popescu B. Assessment of beam modeling methods for rotor blade applications. Math Comput Model 2001;33:1099–112.
[35] Yu W, Hodges DH, Volovoi VV, Cesnik CES. On Timoshenko-like modeling of initially curved and twisted composite beams. Int J Solids Struct 2002;39:5101–21. http://dx.doi.org/10.1016/S0020-7683(02)00399-2.
[36] Vidal P, Polit O. A family of sinus finite elements for the analysis of rectangular laminated beams. Compos Struct 2008;84:56–72.
[37] Carrera E, Giunta G. Refined beam theories based on a unified formulation. Int J Appl Mech 2010;02:117–43. http://dx.doi.org/10.1142/S1758825110000500.
[38] Carrera E. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Arch Comput Methods Eng 2003;10:215–96.
[39] Carrera E, Giunta G, Nali P, Petrolo M. Refined beam elements with arbitrary cross-section geometries. Comput Struct 2010;88:283–93. http://dx.doi.org/10.1016/j.compstruc.2009.11.002.
[40] Carrera E, Petrolo M. On the effectiveness of higher-order terms in refined beam theories. J Appl Mech 2011;78. http://dx.doi.org/10.1115/1.4002207. 021013-1–17.
[41] Carrera E, Filippi M, Zappino E. Free vibration analysis of rotating composite blades via Carrera Unified Formulation. Compos Struct 2013;106:317–25. http://dx.doi.org/10.1016/j.compstruct.2013.05.055.
[42] Carrera E, Giunta G, Petrolo M. Beam structures: classical and advanced theories. Chichester, UK: John Wiley & Sons; 2011.
[43] de Veubeke BF. Diffusion des inconnues hyperstatiques dans les voilures à longerons couplés. In: Bull Serv Tech Aéronautique, 24. Brussels, Belgium: Imprimerie Marcel Hayez; 1951.
[44] de Veubeke BF. Displacement and equilibrium models in the finite element method. In: Zienkiewicz OC, Holister GS, editors. Stress Analysis. Chichester, UK: John Wiley & Sons; 1965. p. 145–97. http://dx.doi.org/10.1002/nme.339 [Chapter 9].
[45] Hu HC. On some variational principles in the theory of elasticity and the theory of plasticity. Scientia Sinica 1955;4:33–54.
[46] Washizu K. On the variational principles of elasticity and plasticity. Report 25-18, MIT Aeroelastic and Structures Research Laboratory; 1955.
[47] Chama A, Reddy BD. New stable mixed finite element approximations for problems in linear elasticity. Comput Methods Appl Mech Eng 2013;256:211–23. http://dx.doi.org/10.1016/j.cma.2012.12.006.
[48] Stolarski H, Belytschko T. Limitation principles for mixed finite elements based on the Hu–Washizu variational formulation. Comput Methods Appl Mech Eng 1987;60:195–216.
[49] Djoko JK, Lamichhane BP, Reddy BD, Wohlmuth BI. Conditions for equivalence between the Hu–Washizu and related formulations, and computational behavior in the incompressible limit. Comput Methods Appl Mech Eng 2006;195:4161–78. http://dx.doi.org/10.1016/j.cma.2005.07.018.
[50] Bathe K-J. Finite element procedures. Upper Saddle River, USA: Prentice Hall; 1996.
[51] Pian THH, Tong P. Rationalization in deriving element stiffness matrix by assumed stress approach. In: Proc 2nd Conf Matrix Methods Struct Analysis; 1968. p. 441–69.

[52] Neuenhofer A, Filippou FC. Geometrically nonlinear flexibility-based frame finite element. J Struct Eng 1998;124:704–11.
[53] Souza RM. Force-based finite element for large displacement inelastic analysis of frames. PhD thesis, University of California, Berkeley, USA; 2000.
[54] Menegotto M, Pinto PE. Method of analysis for cyclically loaded RC plane frames including changes in geometry and non-elastic behaviour of elements under combined normal force and bending. In: Proc IABSE Symp Resist Ultim Deform Struct acted by well defined Repeated Loads; 1973.
[55] Ciampi V, Carlesimo L. A nonlinear beam element for seismic analysis of structures. In: Proc 8th Eur Conf Earth Eng; 1986.
[56] Spacone E, Ciampi V, Filippou FC. Mixed formulation of nonlinear beam finite element. Comput Struct 1996;58:71–83. http://dx.doi.org/10.1016/0045-7949(95)00103-N.
[57] Neuenhofer A, Filippou FC. Evaluation of nonlinear frame finite-element models. J Struct Eng 1997;123:958–66.
[58] Alemdar BN, White DW. Displacement, flexibility, and mixed beam-column finite element formulations for distributed plasticity analysis. J Struct Eng 2005;131:1811–9.
[59] Almeida JP, Correia AA, Pinho R. Force-based higher-order beam element with flexural–shear–torsional interaction in 3D frames. Part II: Applications. Eng Struct 2015;89:218–35.
[60] Frischkorn J, Reese S. A solid-beam finite element and non-linear constitutive modelling. Comput Methods Appl Mech Eng 2013;265:195–212. http://dx.doi.org/10.1016/j.cma.2013.06.009.
[61] Hjelmstad KD. Mixed methods and flexibility approaches for nonlinear frame analysis. J Constr Steel Res 2002;58:967–93. http://dx.doi.org/10.1016/S0143-974X(01)00100-6.
[62] Hjelmstad KD, Taciroglu E. Variational basis of nonlinear flexibility methods for structural analysis of frames. J Eng Mech 2005;131:1157–69.
[63] Vlassov BZ. Pièces longues en voiles minces. Paris, France: Éditions Eyrolles; 1962.
[64] Teixeira de Freitas JA. Duality and symmetry in mixed integral methods of elastostatics. Int J Numer Methods Eng 1989;28:1161–79.
[65] Teixeira de Freitas JA, Moitinho de Almeida JP, Ribeiro Pereira EMB. Non-conventional formulations for the finite element method. Comput Mech 1999;23:488–501. http://dx.doi.org/10.1007/s004660050428.
[66] Wu JJ. On adjoint operators associated with boundary value problems. J Sound Vib 1975;39:195–206.
[67] Gao DY. Duality principles in nonconvex systems. Theory, methods and applications. Dordrecht, Netherlands: Kluwer Academic Publishers; 1999.
[68] Estep DJ. A short course on duality, adjoint operators, Green's functions, and a posteriori error analysis. Fort Collins, USA: Colorado State University; 2004.
[69] Washizu K. Variational methods in elasticity and plasticity. 3rd ed. Oxford, UK: Pergamon Press; 1982.
[70] Almeida JP. Flexure-shear element for 3D inelastic analysis of frames. PhD thesis, ROSE School, Università degli Studi di Pavia, Italy; 2009.
[71] Wolfram, Mathematica – Version 5.0; 2003.