Forum Math. **27** (2015), 119–130
DOI 10.1515/forum-2012-0038 **Forum Mathematicum**

Goodwillie calculus and Whitehead products

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Communicated by Frederick R. Cohen

Abstract. We prove that iterated Whitehead products of length $(n+1)$ vanish in any value of an n-excisive functor in the sense of Goodwillie. We compare then different notions of homotopy nilpotency, from the Berstein–Ganea definition to the Biedermann–Dwyer one. The latter is strongly related to Goodwillie calculus and we analyze the vanishing of iterated Whitehead products in such objects.

Keywords. Nilpotent, homotopy nilpotent, algebraic theory, Goodwillie calculus, excisive functor.

2010 Mathematics Subject Classification. Primary 55Q15; secondary 18C10, 55P47, 55P65, 55P3.

Introduction

Goodwillie calculus, [8, 9], gives a systematic way to approximate a functor (say from spaces to spaces) by a tower of functors satisfying higher excision properties. When applied to the identity functor, this tower reflects remarkable periodicity properties, as investigated by Arone and Mahowald, [2]. More recently Biedermann and Dwyer, [5], used the stages of the very same tower to construct (simplicial) algebraic theories in the sense of Lawvere, [15]. The homotopy algebras over these theories are called homotopy nilpotent groups, and the class of nilpotency corresponds exactly to the chosen stage of the Goodwillie tower.

Our objectives in this article are twofold. First we investigate why n-excisive functors should be related to homotopy nilpotency in the classical sense. In the early sixties, Berstein and Ganea introduced a concept of nilpotent loop spaces, [4]. They require that an iterated commutator map be trivial up to homotopy, which implies in particular that iterated Samelson products vanish in the loop space ΩX, or equivalently, that iterated Whitehead products vanish in X. Already G. Whitehead [20] had the insight that the (J. H. C.) Whitehead products satisfy identities which reflect commutator identities for groups. Work of Hopkins, [12], drew renewed attention to such questions by relating this classical nilpotency notion with

The second author is supported by FEDER/MEC grant MTM2010-20692.

the Nilpotence Theorem of Devinatz, Hopkins, and Smith, [7]. We prove the following.

Theorem 2.1. *Let F be any n-excisive functor from the category of pointed spaces to pointed spaces. Then all $(n+1)$-fold iterated Whitehead products vanish in $F(X)$ for every finite space X.*

Our result shows in fact that $\Omega F(X)$ is a homotopy nilpotent loop space in the sense of Ganea and Bernstein for every n-excisive functor F and every finite space X.

The difficulty of the proof resides in finding a way to take into account the global property of the functor (to be n-excisive) and not to focus on a particular value $F(X)$. Except for this, the proof uses the general theory of Goodwillie calculus.

In the second part of the article we look more closely at the relationship between the different types of homotopy nilpotency available on the market. We start with the classical algebraic theory Nil_n describing nilpotent groups of class $\leq n$, and observe that Berstein–Ganea nilpotent loop spaces are Nil_n-algebras in the homotopy category of spaces. We show that homotopy Nil_n-algebras in the sense of Badzioch, [3], are always homotopy nilpotent in the sense of Biedermann and Dwyer. Finally, both are Nil_n-algebras in the homotopy category of spaces, so that the vanishing of Whitehead products applies to all kinds of homotopy nilpotent groups that appeared so far in the literature, and in particular to the Biedermann–Dwyer ones.

Theorem 2.1 can be rephrased thus as follows: All $(n+1)$-fold iterated Whitehead products vanish in X if ΩX is a homotopy nilpotent group of class $\leq n$. This provides a positive answer to a question asked by the authors of [5] and the proof depends on a non-trivial computation of sets of components they perform.

1 Samelson and Whitehead products

We recall briefly the definition of Samelson and Whitehead products and construct a "universal space" built from wedges of spheres in which higher Whitehead products vanish.

Let X be a pointed space. Given $\alpha \in \pi_{a+1}X$ and $\beta \in \pi_{b+1}X$, take the adjoint classes $\alpha' \in \pi_a(\Omega X)$ and $\beta' \in \pi_b(\Omega X)$. The composite of the product map $\alpha' \times \beta' : S^a \times S^b \to \Omega X \times \Omega X$ with the commutator map $\Omega X \times \Omega X \to \Omega X$ is null-homotopic when restricted to the wedge $S^a \vee S^b$ and thus factors through S^{a+b}, uniquely up to homotopy. This factorization represents the Samelson product $\langle \alpha', \beta' \rangle \in \pi_{a+b}\Omega X$ and the adjoint class is the so-called Whitehead product $[\alpha, \beta] \in \pi_{a+b+1}X$.

Remark 1.1. Iterated Whitehead products can be computed as adjoint to iterated Samelson products. For example a triple Whitehead product of the form $[[\alpha, \beta], \gamma]$ coincides with the adjoint of the Samelson product $\langle\langle\alpha', \beta'\rangle, \gamma'\rangle$. Let us also mention that the order of the classes in a Whitehead product does not matter (up to a sign). We will therefore concentrate on one standard choice of bracketing.

By definition, the Whitehead product $[\iota_1, \iota_2]$ of the two canonical inclusions $\iota_1 : S^a \hookrightarrow S^a \vee S^b$ and $\iota_2 : S^b \hookrightarrow S^a \vee S^b$ is the attaching map of the top cell in $S^a \times S^b$. Moreover, any Whitehead product $[\alpha, \beta] : S^{a+b+1} \to X$ factors through $[\iota_1, \iota_2]$. This motivates the construction of a space built from wedges of spheres which will be crucial for understanding when certain iterated Whitehead products vanish. We consider $n+1$ positive integers $k_1, \dots, k_{n+1}$ and the wedge of $n+1$ spheres $W = \vee S^{k_i}$. Denote by $\iota_i : S^{k_i} \to W$ the wedge summand inclusion. If $\mathcal{P}(\underline{n+1})$ denotes the poset of subsets of $\underline{n+1} = \{1, \dots, n+1\}$, define the $(n+1)$-cube of pointed spaces $V : \mathcal{P}(\underline{n+1}) \backslash \{\emptyset\} \to \text{Spaces}_*$ by sending a subset $S \subset \underline{n+1}$ to $\bigvee_{i \notin S} S^{k_i}$. Adding W as initial value $V(\emptyset)$ makes this diagram a strongly homotopy co-Cartesian cube as defined by Goodwillie in [9, p. 647], i.e. all 2-dimensional faces are homotopy pushouts. We let Q be the homotopy inverse limit of V, and to fix a representative we take Q to be the inverse limit of the fibrant replacement $V \overset{\sim}{\hookrightarrow} \hat{V}$ of this diagram in the injective model structure, [10, 13]. We will also write abusively $V(i)$ instead of $V(\{i\})$ to ease the notation.

Example 1.2. When $n = 1$, we have two spheres S^{k_1} and S^{k_2}. The diagram V is the pull-back diagram $S^{k_1} \to * \leftarrow S^{k_2}$ and $Q = S^{k_1} \times S^{k_2}$. The Whitehead product of the summand inclusions is trivial in Q.

The looped diagram ΩV is easier to analyze since the loop space on a wedge of spheres splits by the Hilton–Milnor Theorem, see the original article [11] or Milnor's unpublished article in [1]. So-called "basic words" w in $x_1, \dots, x_{n+1}$ form a basis of the free Lie algebra generated by $x_1, \dots, x_{n+1}$ and each of these determines a Whitehead product in $\pi_{N(w)}(S^{k_1} \vee \dots \vee S^{k_{n+1}})$ where $N(w)$ plus the number of letters in w is the sum of as many k_i's as there are x_i's in w plus 1. For example, when $n = 2$, the basic word $x_1x_2x_3$ corresponds to the Whitehead product $[[\iota_1, \iota_2], \iota_3]$ represented by a map $S^{k_1+k_2+k_3-2} \to S^{k_1} \vee S^{k_2} \vee S^{k_3}$. The Hilton–Milnor Theorem then states that

$$\Omega(S^{k_1} \vee \dots \vee S^{k_{n+1}}) \simeq \prod_w \Omega S^{N(w)}.$$

Lemma 1.3. *The loop space ΩQ is homotopy equivalent to a product of loop spaces on spheres, namely $\prod \Omega S^{N(w)}$ where the product is taken over all basic words in at most n of the letters $x_1, \dots, x_{n+1}$.*

Proof. We identify ΩQ with the homotopy inverse limit of the diagram ΩV, each value of which splits as a product of loop spaces on spheres:

$$\Omega V(S) \simeq \prod_{i \notin S} \Omega S^{k_i} \times \cdots \times \prod_{w \in W_S} \Omega S^{N(w)}$$

where W_S is the subset of those basic words written with all x_i's with $i \notin S$. We observe that each map $\Omega V(S) \to \Omega V(T)$, with $S \subset T$, is the projection on the summands $\Omega S^{N(w)}$ corresponding to the basic words not written with the letters in T. Therefore the diagram ΩV is a hypercube of which the homotopy inverse limit is the product of all $\prod_{w \in W_S} \Omega S^{N(w)}$ with $S \neq \emptyset$. □

For any choice of bracketing $n+1$ elements there is an $(n+1)$-fold Whitehead product $w : S^{k_1+\cdots+k_{n+1}-n} \to W$. We denote by C_w the homotopy cofiber of w.

Lemma 1.4. *The $(n+1)$-fold Whitehead product $[[\ldots[[\iota_1, \iota_2], \iota_3], \ldots], \iota_{n+1}]$ vanishes in Q.*

Proof. This Whitehead products vanishes in Q if and only if the adjoint Samelson product vanishes in ΩQ. Since ΩQ splits as a product of loop spaces on spheres, it is sufficient to prove that the projection on each factor is null-homotopic. By Lemma 1.3 each factor already appears in $\Omega V(S)$ for some non-empty subset S, so that, by adjunction again, it is enough to show that the image in $V(i)$ of our $(n+1)$-fold Whitehead product vanishes for any $1 \leq i \leq n+1$. This is so because the image of ι_i in $V(i)$ is the trivial map and any Whitehead product involving the trivial map is null-homotopic. □

2 The values of *n*-excisive functors

We perform our main computation in this section. Let F be an n-excisive functor from pointed spaces to pointed spaces (so F sends strongly homotopy co-Cartesian $(n+1)$-cubes to homotopy Cartesian ones). We prove that all $(n+1)$-fold Whitehead products vanish in $F(X)$ for any space X. Because it is very difficult to use the global property of excision by focusing on one single value of the functor F, we will use pointed representable functors R^X, defined by

$$R^X(Y) = \mathrm{map}_*(X, Y).$$

For any pointed space A a natural transformation $R^X \wedge A \to F$ corresponds by adjunction to a map $A \to \mathrm{hom}(R^X, F)$, i.e. to a map $A \to F(X)$ by the enriched Yoneda Lemma, see [14, 2.31]. Any functor G has a universal n-excisive approximation $G \to P_n G$, see [9].

Theorem 2.1. *Let F be any n-excisive functor from the category of pointed spaces to pointed spaces. Then all $(n+1)$-fold iterated Whitehead products vanish in $F(X)$ for every finite space X.*

Proof. Let us fix homotopy classes of maps $\alpha_i : S^{k_i} \to F(X)$ for $1 \leq i \leq n+1$. We need to prove that the iterated Whitehead product

$$[[\dots[[\alpha_1, \alpha_2], \alpha_3], \dots], \alpha_{n+1}]$$

is zero. This product is represented by a map

$$S^{k_1+\cdots+k_{n+1}+1} \xrightarrow{w} \bigvee_{i=1}^{n+1} S^{k_i} = W \to F(X)$$

which is null-homotopic if it factors through the homotopy cofiber C_w of the "universal" $(n+1)$-fold Whitehead product w. The use of representable functors translates then as follows: We need to show that any natural transformation $\eta : R^X \wedge W \to F$ factors through $R^X \wedge C_w$. As F is n-excisive, there exists a natural transformation $P_n(R^X \wedge W) \to F$ such that the composite

$$R^X \wedge W \to P_n(R^X \wedge W) \to F$$

coincides with η up to homotopy. It is thus enough to construct a natural transformation $R^X \wedge C_w \to P_n(R^X \wedge W)$.

Smashing the diagram V with a representable functor, we obtain a hypercube $R^X \wedge V$ of functors, which is strongly homotopy co-Cartesian since V is so. We focus on the natural transformations $R^X \wedge W \to R^X \wedge V(i)$. If $c = \dim X$, and Y is a k-connected space with $k \geq c$, then $R^X(Y)$ is $(k-c)$-connected and $(R^X \wedge W)(Y) \to (R^X \wedge V(i))(Y)$ is $(k - c + k_i)$-connected. Let G denote the homotopy inverse limit of the diagram of functors $R^X \wedge V$.

The generalized Blackers–Massey Theorem [8, Theorem 2.3] implies that the natural transformation $\theta : R^X \wedge W \to G$ is $[(n+1)k-(n+1)c+\sum k_i - n]$-connected when evaluated at a k-connected space with $k \geq c$. This implies that $R^X \wedge W$ and G agree to order n in the terminology of [9, Definition 1.2, Proposition 1.6], so that $P_n(R^X \wedge W) \simeq P_n(G)$.

Lemma 1.4 yields a map $C_w \to Q$ such that $W \to C_w \to Q$ is the natural map from W to the homotopy inverse limit of the diagram V (we fix the model $C_w = W \cup_w D^{k_1+\cdots+k_{n+1}+2}$ for the homotopy cofiber so that the factorization is strict). We interpret this map as a map from the constant diagram C_w to a fibrant replacement $\hat{V}$ of V in the injective model category of hypercubical diagrams. Smashing with a representable functor, we get a natural transformation

$$R^X \wedge C_w \to R^X \wedge V.$$

Taking homotopy inverse limits, we obtain finally a natural transformation

$$R^X \wedge C_w \to G$$

such that the composite $R^X \wedge W \to R^X \wedge C_w \to G$ coincides with θ. The natural transformation

$$R^X \wedge C_w \to G \to P_n G \simeq P_n(R^X \wedge W)$$

is the one we needed to conclude. □

Remark 2.2. The proof of Theorem 2.1 easily generalizes to show that iterated *generalized* Whitehead products vanish. It suffices to replace the Hilton Splitting Theorem for loop spaces on a wedge of spheres by Milnor's generalized version for wedges of suspensions.

3 Nilpotent groups and algebraic theories

Let us first recall the classical concept of algebraic theory due to Lawvere [15] and some of its modern variations.

Definition 3.1. A small category T is an algebraic theory if the objects of T are indexed by natural numbers $\{T_0, T_1, \ldots, T_n, \ldots\}$ and for all $n \in \mathbb{N}$ the n-fold categorical coproduct of T_1 is naturally isomorphic to T_n. The algebraic theory T is *simplicial* if it is a (pointed) simplicial category, i.e. T is enriched over sSets_*

Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*algebra* over a theory T is a functor $A\colon T^{\mathrm{op}} \to \mathcal{C}$ taking coproducts in T into products in $\mathcal{C}$.

If T is a simplicial algebraic theory and $\mathcal{C} = \mathrm{sSets}_*$, then we distinguish between *strict* and *homotopy* simplicial algebras, which are simplicial functors

$$A\colon T^{\mathrm{op}} \to \mathcal{C}$$

taking coproducts in T to products in $\mathcal{C}$ strictly or up to homotopy, respectively.

The categories of simplicial algebras and homotopy simplicial algebras were compared by Badzioch in [3]. He proved that any homotopy algebra can be rigidified to a strict algebra.

Of central interest for us will be algebras over algebraic theories defined in the homotopy category of simplicial sets $\mathcal{C} = \mathrm{Ho}(\mathrm{sSets}_*)$. We call them *algebras up to homotopy*, in order to distinguish them from the homotopy algebras defined above. There is a natural way to associate to every homotopy algebra A, an algebra up to homotopy: just compose the functor A with the product preserving functor

$$\Gamma\colon \mathrm{sSets}_* \to \mathrm{Ho}(\mathrm{sSets}_*).$$

Formally, we need to choose homotopy inverse maps

$$f_k : A(k) \to A(1)^k \quad \text{and} \quad g_k : A(1)^k \to A(k)$$

and replace each morphism $A(h) : A(m) \to A(n)$ by the composite $f_n \circ A(h) \circ g_m$. The converse is not true of course, and we will encounter examples of algebras up to homotopy which cannot be upgraded to homotopy algebras.

Lawvere in his seminal article [15] has discovered the fundamental fact that an algebraic theory defining a variety as the category of algebras is the dual of the subcategory of finitely generated free algebras in this variety. In this work we will look closer into the algebraic theories defining the concepts of groups and nilpotent groups of class $\leq n$ in various settings.

Thus, we will consider the category Nil_n whose objects are the natural numbers $0, 1, 2, \ldots$ and morphisms $k \to l$ are group homomorphisms

$$F_k / \Gamma_{n+1} F_k \to F_l / \Gamma_{n+1} F_l.$$

By identifying the object k with the free nilpotent groups $F_k / \Gamma_{n+1} F_k$ of class n, one embeds Nil_n as a full subcategory of the category of groups. In fact, as nilpotent groups of class $\leq n$, the group $F_k / \Gamma_{n+1} F_k$ is free in the sense that it can be identified with the coproduct of k copies of $\mathbf{Z} = F_1 / \Gamma_{n+1} F_1$. The set of morphisms from 1 to k is precisely the group $F_k / \Gamma_{n+1} F_k$. When $n = \infty$, we think of the objects of Nil_∞ to be the free groups F_k. A Nil_n*-algebra* in Sets is thus a product preserving contravariant functor $N : \mathrm{Nil}_n^{\mathrm{op}} \to \mathrm{Sets}$.

Proposition 3.2. *A* Nil_n*-algebra is a nilpotent group of class* $\leq n$.

Because it will play an important role in the sequel, let us be precise and say explicitly how the group structure arises and why it is nilpotent. By abuse of notation we write also N for the value $N(1)$. The multiplication $m : N \times N \to N$ is the morphism corresponding to the product of the two generators of F_2 in the quotient $F_2 / \Gamma_{n+1} F_2$ and the inverse is the morphism $N \to N$ corresponding to the inverse of the generator of F_1. It is easy to check that this equips N with a group structure. This is in fact equivalent to the structure of a Nil_∞-algebra: Given k elements $n_1, \ldots, n_k \in N$ and a word w in k letters, the product $w(n_1, \ldots, n_k)$ can be read off from the morphism $N^k \to N$ corresponding to w. The claim about the nilpotency class follows then from the fact that all words of the form

$$[[\ldots[[x_1, x_2], x_3], \ldots], x_{n+1}]$$

are identified to 1 in $F_{n+1} / \Gamma_{n+1} F_{n+1}$. Hence any iterated commutator of length $\geq n + 1$ must be trivial in a Nil_n-algebra.

Remark 3.3. A Nil_n-algebra in the category of simplicial sets, i.e. a product preserving contravariant functor $N : \mathrm{Nil}_n^{\mathrm{op}} \to \mathrm{sSets}$, is a simplicial nilpotent group of class $\leq n$. In particular when $n = 1$, we are considering simplicial abelian groups, i.e. generalized Eilenberg–Mac Lane spaces, so-called "GEMs", see for example [6]. Schwede also considers such objects and compares them stably, [19, Example 7.4], with a category of modules over a Gamma-ring.

Badzioch's rigidification result states in this context that any homotopy Nil_n-algebra is homotopy equivalent to a strict Nil_n-algebra. Again for $n = 1$, this means that all homotopy Nil_1-algebras are homotopy equivalent to GEMs. This is not quite what we would like to study when we are speaking about a homotopy version of abelian topological groups (what we understand under this name is rather an infinite loop space). The notion of Nil_n-algebras in simplicial sets is thus too rigid and we will need to relax it a little.

4 Nilpotent groups in the homotopy category

In the next section we will turn to the solution Biedermann and Dwyer found to describe homotopy nilpotency. But before we do that, we first describe the most naive way to define nilpotency in homotopy theory.

Definition 4.1. A *nilpotent group up to homotopy of class* $\leq n$ is a product preserving contravariant functor $N : \mathrm{Nil}_n^{\mathrm{op}} \to \mathrm{Ho}(\mathrm{Spaces}_*)$.

How do these nilpotent groups up to homotopy look like? They are pointed spaces G together with a homotopy associative multiplication and a homotopy inverse (i.e. group-like H-spaces) coming from the morphisms in $\mathrm{Nil}_n^{\mathrm{op}}(2,1)$ and $\mathrm{Nil}_n^{\mathrm{op}}(1,1)$ described in the previous section, such that all higher commutator maps of order $n+1$ are null-homotopic. Berstein and Ganea, [4, Definition 1.7], give a definition of nilpotency for group like spaces by requiring that the $(n+1)$-st commutator map be null-homotopic. Their work predates by two years the introduction by Lawvere of algebraic theories, and is therefore not stated in the language we have used, but it is equivalent.

Proposition 4.2. *A nilpotent group up to homotopy is a homotopy nilpotent group in the sense of Berstein and Ganea.* □

Example 4.3. When $n = 1$, a loop space is abelian (nilpotent of class ≤ 1) up to homotopy if the commutator map $\Omega X \times \Omega X \to \Omega X$ is null-homotopic, i.e. if the product is homotopy commutative. Thus any double loop space is abelian up homotopy. When $n = \infty$, groups up to homotopy are simply group objects in the homotopy category, i.e. homotopy associative H-spaces with inverse.

These examples show that the Berstein–Ganea definition is too flexible. When looking at loop spaces, the filtration given by nilpotency up to homotopy interpolates roughly between loop spaces and double loop spaces. However it allows us to read off the vanishing of iterated Samelson products. The following result is basically [4, Theorem 4.6].

Proposition 4.4. *Let X be a pointed space and assume that the loop space ΩX is nilpotent up to homotopy of class $\leq n$. Then all $(n+1)$-fold iterated Whitehead products vanish in X.*

Proof. The vanishing of iterated Whitehead products is equivalent to the vanishing of iterated Samelson products in the loop space. This follows now directly from the fact that in a Nil_n-algebra in the homotopy category the $(n+1)$-fold commutator map $(\Omega X)^{n+1} \to \Omega X$ is null-homotopic by definition. □

Example 4.5. Porter proved that S^3 is nilpotent up to homotopy of class 3, [16]. There is a non-trivial 3-fold Whitehead product in BS^3, but all 4-fold products vanish. However, the compact Lie group S^3 is not nilpotent as a group. More generally, Rao proved that compact Lie groups are nilpotent up homotopy if and only if their integral homology is torsion free, [17]. The if part is due to Hopkins, [12].

5 Enriched homotopy nilpotent groups

This section finally introduces the "correct" homotopy nilpotent groups. We recall their definition, show that iterated Samelson products vanish in such spaces, and compare them to homotopy Nil_n-algebras and spaces which are nilpotent up to homotopy in the sense of Berstein and Ganea.

In their recent work [5], Biedermann and Dwyer define homotopy nilpotent groups as homotopy $\mathcal{G}_n$-algebras in the category of pointed spaces, where $\mathcal{G}_n$ is a *simplicial* algebraic theory constructed from the Goodwillie tower of the identity. Concretely $\mathcal{G}_n^{\mathrm{op}}$ is the simplicial category whose objects are $0, 1, 2, \ldots$ and such that the simplicial set of morphisms $\mathcal{G}_n^{\mathrm{op}}(k, l)$ is the space of natural transformations

$$\prod_k \Omega(P_n(\mathrm{id}))^{\mathrm{inj}} \to \prod_l \Omega(P_n(\mathrm{id}))^{\mathrm{inj}}$$

The functor $P_n(\mathrm{id})$ lives in the category of functors from finite pointed spaces to pointed spaces, and $(P_n(\mathrm{id}))^{\mathrm{inj}}$ denotes the fibrant replacement in the injective model structure. Hence a homotopy nilpotent group of class $\leq n$ is the value at 1 of a simplicial functor $\tilde{X}$ from $\mathcal{G}_n^{\mathrm{op}}$ to pointed spaces which commutes up to homotopy with products. Homotopy algebras in an enriched context have been studied by Rosický in [18].

Proposition 5.1. *A homotopy* Nil_n*-algebra is always a homotopy* $\mathcal{G}_n$*-algebra. Both of them are* Nil_n*-algebras up to homotopy.*

Proof. The space of maps from k to 1 in the algebraic theory $\mathcal{G}_n$, which is by definition the space of all natural transformations from $(\Omega P_n(\mathrm{id}))^k$ to $\Omega P_n(\mathrm{id})$, is identified as the space $\Omega P_n(\mathrm{id})(\bigvee_k S^1)$, see [5, Corollary 4.7]. Biedermann and Dwyer's main computation shows that the group of connected components coincides with the free nilpotent group of class n on k generators:

$$\pi_0 \mathcal{G}_n(k,1) \cong \pi_0\big(\Omega P_n(\mathrm{id})(\textstyle\bigvee_k S^1)\big) \cong F_k/\Gamma_{n+1} F_k.$$

There is hence a functor of simplicial algebraic theories $\pi_0 \colon \mathcal{G}_n \to \mathrm{Nil}_n$. Thus any homotopy Nil_n-algebra can be seen as a homotopy $\mathcal{G}_n$-algebra by pulling back along π_0.

Consider now a homotopy nilpotent group ΩX of class $\leq n$ given as the value at 1 of a homotopy $\mathcal{G}_n$-algebra $\tilde{X} : \mathcal{G}_n^{\mathrm{op}} \to \mathrm{Spaces}_*$. The composite diagram

$$F : \mathcal{G}_n^{\mathrm{op}} \to \mathrm{Spaces}_* \to \mathrm{Ho}(\mathrm{Spaces}_*)$$

is now simply a diagram

$$F : \mathrm{Nil}_n^{\mathrm{op}} \to \mathrm{Ho}(\mathrm{Spaces}_*)$$

as we keep from the simplicial data only one homotopy class of maps

$$\tilde{X}(k) \to \tilde{X}(l)$$

for each connected component of $\mathcal{G}_n(k,l) \simeq \Omega P_n(\mathrm{id})(\bigvee_k S^1)^l$. The second claim then follows from the general procedure we described in Section 3 to get an algebra up to homotopy from a homotopy algebra. □

Theorem 5.2. *Let* ΩX *denote a homotopy nilpotent group of class* $\leq n$*. Then all* $(n+1)$*-fold iterated Whitehead products vanish in* X*.*

Proof. The Berstein–Ganea Proposition 4.4 implies the vanishing of all iterated Whitehead products of length $n+1$ in X. □

Remark 5.3. Observe here that a homotopy nilpotent group of class n is also a homotopy nilpotent group of class ∞ since we have a map of algebraic theories $\mathcal{G}_\infty \to \mathcal{G}_n$. This means that a homotopy nilpotent group of class n has the homotopy type of a loop space and the multiplication derived from the algebraic theory is this precise loop multiplication. This is what allows us to use the Berstein–Ganea result in the last line of the previous proof.

Example 5.4. Homotopy abelian groups, that is homotopy $\mathcal{G}_1$-algebras, are infinite loop spaces and homotopy groups, i.e. homotopy $\mathcal{G}_\infty$-algebras, are loop spaces. This is why the notion of homotopy nilpotency of Biedermann and Dwyer is better than the others. It interpolates between the “right” notions of groups and abelian groups in homotopy theory. In particular, BU is homotopy abelian, but not a homotopy Nil_1-algebra, and $\Omega^2 S^4$ is abelian up to homotopy, but not a homotopy abelian group. This illustrates how the different notions of nilpotency differ.

Remark 5.5. Biedermann and Dwyer prove that any n-excisive functor F from the category of pointed spaces to pointed spaces produces examples of homotopy nilpotent groups: $\Omega F(X)$ is homotopy nilpotent of class $\leq n$ for any finite space X, see [5, Corollary 9.3]. Hence Theorem 5.2 gives an alternative proof of Theorem 2.1. Biedermann and Dwyer also claim that all homotopy nilpotent groups are given as values of loops on n-excisive functors. This implies that both theorems are in fact equivalent.

Acknowledgments. We would like to thank Amnon Neeman and the Mathematical Sciences Institute at ANU where this project started. The hospitality of EPFL is greatly acknowledged by the first author, who visited the second author in order to complete this work. We learned about homotopy nilpotent groups from Georg Biedermann and Bill Dwyer and thank them for their interest in this project.

Bibliography

[1] J. F. Adams, *Algebraic Topology—A Student's Guide*, London Mathematical Society Lecture Notes 4, Cambridge University Press, London, 1972.

[2] G. Arone and M. Mahowald, The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres, *Invent. Math.* **135** (1999), no. 3, 743–788.

[3] B. Badzioch, Algebraic theories in homotopy theory, *Ann. of Math. (2)* **155** (2002), no. 3, 895–913.

[4] I. Berstein and T. Ganea, Homotopical nilpotency, *Illinois J. Math.* **5** (1961), 99–130.

[5] G. Biedermann and W. G. Dwyer, Homotopy nilpotent groups, *Algebr. Geom. Topol.* **10** (2010), no. 1, 33–61.

[6] E. B. Curtis, Simplicial homotopy theory, *Adv. Math.* **6** (1971), 107–209.

[7] E. S. Devinatz, M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. I, *Ann. of Math. (2)* **128** (1988), no. 2, 207–241.

[8] T. G. Goodwillie, Calculus. II. Analytic functors, *K-Theory* **5** (1991/92), no. 4, 295–332.

[9] T. G. Goodwillie, Calculus. III. Taylor series, *Geom. Topol.* **7** (2003), 645–711 (electronic).

[10] A. Heller, Homotopy theories, *Mem. Amer. Math. Soc.* **71** (1988), no. 383.

[11] P. J. Hilton, On the homotopy groups of the union of spheres, *J. Lond. Math. Soc.* **30** (1955), 154–172.

[12] M. J. Hopkins, Nilpotence and finite H-spaces, *Israel J. Math.* **66** (1989), no. 1–3, 238–246.

[13] J. F. Jardine, Simplicial presheaves, *J. Pure Appl. Algebra* **47** (1987), no. 1, 35–87.

[14] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Notes 64, Cambridge University Press, Cambridge, 1982.

[15] F. W. Lawvere, Functorial semantics of algebraic theories, *Proc. Natl. Acad. Sci. USA* **50** (1963), 869–872.

[16] G. J. Porter, Homotopical nilpotence of S^3, *Proc. Amer. Math. Soc.* **15** (1964), 681–682.

[17] V. K. Rao, Homotopy nilpotent Lie groups have no torsion in homology, *Manuscripta Math.* **92** (1997), no. 4, 455–462.

[18] J. Rosický, On homotopy varieties, *Adv. Math.* **214** (2007), no. 2, 525–550.

[19] S. Schwede, Stable homotopy of algebraic theories, *Topology* **40** (2001), no. 1, 1–41.

[20] G. W. Whitehead, On products in homotopy groups, *Ann. of Math. (2)* **47** (1946), 460–475.

Received March 30, 2012; revised June 21, 2012.

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