Abstracts

Numerical methods for multiscale parabolic and hyperbolic problems

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In this report we summarize some recent developments of numerical homogenization methods for nonlinear parabolic equations and wave equations in heterogeneous medium.

Monotone parabolic problems. Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$ be a convex polygonal domain. Consider for $T > 0$, $f \in L^2(\Omega)$ the following nonlinear parabolic problem

$$\begin{aligned} \partial_t u^\varepsilon - \nabla \cdot (\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)) = f \qquad \text{in } \Omega \times (0,T), \\ u^\varepsilon|_{\partial\Omega \times (0,T)} = 0, \qquad u^\varepsilon|_{t=0} = g \qquad \text{in } \Omega. \end{aligned} \tag{1}$$

Assumptions:

- there exists $C_0 > 0$ such that $|\mathcal{A}^\varepsilon(x,0)| \leq C_0$ for a.e. $x \in \Omega$;
- Lipschitz continuity: $|\mathcal{A}^\varepsilon(x,\xi) - \mathcal{A}^\varepsilon(x,\eta)| \leq L|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^d$, a.e. $x \in \Omega$;
- Strong monotonicity: $[\mathcal{A}^\varepsilon(x,\xi) - \mathcal{A}^\varepsilon(x,\eta)] \cdot (\xi - \eta) \geq \lambda |\xi - \eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^d$, a.e. $x \in \Omega$.

Under theses assumptions (1) has a unique solution $u^\varepsilon \in E$, where

$$E = \{v \in L^2(0,T; H_0^1(\Omega)) \,|\, \partial_t v \in L^2(0,T; H^{-1}(\Omega))\},$$

and $\{u^\varepsilon\}$ is a bounded sequence in E which weakly converges (up to extracting a subsequence) to a function $u^0 \in E$ that is solution of a homogenized problem that takes a form similar to (1) with $\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)$ replaced by $\mathcal{A}^0(x, \nabla u^0(x,t))$ (see e.g. [10]). As $\mathcal{A}^0(x, \nabla u^0(x,t))$ is usually not explicitly known in closed form, it must be approximated numerically.

Multiscale methods. Let the time interval $(0,T)$ be uniformly divided into N subintervals of length $\Delta t = T/N$ and define $t_n = n\Delta t$ for $0 \leq n \leq N$ and $N \in \mathbb{N}_{>0}$. Let $u_0^H \in S_0^1(\Omega, \mathcal{T}_H) = \{v^H \in H_0^1(\Omega) \,|\, v^H|_K \in \mathcal{P}^1(K), \forall K \in \mathcal{T}_H\}$ be a given approximation of the initial condition $g(x)$.

Fully nonlinear method. Consider the multiscale method given by the recursion: for $0 \leq n \leq N-1$, find $u_{n+1}^H \in S_0^1(\Omega, \mathcal{T}_H)$ such that

$$\int_\Omega \frac{u_{n+1}^H - u_n^H}{\Delta t} w^H dx + B_H(u_{n+1}^H; w^H) = \int_\Omega f\, w^H dx, \qquad \forall\, w^H \in S_0^1(\Omega, \mathcal{T}_H), \tag{2}$$

with the nonlinear macro map B_H given by

(3)

$$B_H(v^H; w^H) = \sum_{K \in \mathcal{T}_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla v_K^h) dx \cdot \nabla w^H(x_K), \quad v^H, w^H \in S_0^1(\Omega, \mathcal{T}_H),$$

where $K_\delta \subset K$ are sampling domains of size δ (proportional to ε) and v_K^h solve the constrained micro problems: find $v_K^h - v^H \in S^1(K_\delta, \mathcal{T}_h)$ such that

$$\int_{K_\delta} \mathcal{A}^\varepsilon(x, \nabla v_K^h) \cdot \nabla z^h dx = 0, \qquad \forall\, z^h \in S^1(K_\delta, \mathcal{T}_h). \tag{4}$$

Here $S^1(K_\delta, \mathcal{T}_h)$ denotes a microscopic finite element subspace of $W^1_{per}(K_\delta) = \{v \in H^1_{per}(K_\delta) \,|\, \int_{K_\delta} v\, dx = 0\}$ or $H^1_0(K_\delta)$ and $\mathcal{T}_h$ is a partition of K_δ with micromesh size $h = \max_{T\in\mathcal{T}_h} \mathrm{diam} T$.

Linearized method. We next describe a method that relies only on *linear micro problems* and has the same computational cost than a numerical homogenization method for linear parabolic homogenization problems. Assume $\mathcal{A}^\varepsilon(x,\xi) = a^\varepsilon(x,\xi)\xi$. Define $\mathcal{S}^{H,h} = S^1_0(\Omega, \mathcal{T}_H) \times \prod_{K\in\mathcal{T}_H} S^1(K_\delta, \mathcal{T}_h)$, and the map $\hat{u}_n \mapsto \hat{u}_{n+1} = (u^H_{n+1}, \{u^h_{n+1,K}\}) \in \mathcal{S}^{H,h}$ given by the fully discrete space-time scheme

(1) evolution of the macroscopic state: find $u^H_{n+1} \in S^1_0(\Omega, \mathcal{T}_H)$ solution of the linear problem
$$\int_\Omega \frac{1}{\Delta t}(u^H_{n+1} - u^H_n)\, w^H dx + B_H(\hat{u}_n; u^H_{n+1}, w^H) = \int_\Omega f w^H dx;$$

(2) update the micro states: $\hat{u}_{n+1} = u^H_{n+1} + u^h_{n+1,K}$, where for $K \in \mathcal{T}_H$, $u^h_{n+1,K}$ satisfies $u^h_{n+1,K} - u^H_{n+1} \in S^1_h$ and
$$\int_{K_\delta} a^\varepsilon(\nabla \hat{u}^h_{n,K}) \nabla \hat{u}^h_{n+1,K} \cdot \nabla z^h dx = 0 \qquad \forall\, z^h \in S^1(K_\delta, \mathcal{T}_h).$$

A fully discrete space time analysis (including micro, macro and resonance errors) has been given in [4] and [5] for both methods under appropriate smoothness assumptions. We note that a new (linear) elliptic projection has been introduced in [4]. We note that for single scale problems, a nonlinear projection has been used with non-optimal L^2 convergence rates for low order finite elements [7]. Optimal convergence rates were obtained using weighted norm techniques for nonlinear problems [8]. Such weighted norm techniques are avoided in our analysis.

Multiscale wave problems. Let $\Omega \subset \mathbb{R}^d$, $d \le 3$ be a convex polygonal domain. Consider for $T > 0$, $f \in L^2(\Omega)$ the following wave equation with appropriate initial and boundary conditions

$$\partial_{tt} u^\varepsilon - \nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f \qquad \text{in } \Omega \times (0,T), \tag{5}$$

where $a^\varepsilon(x)$ is a rapidly varying tensor that is assumed to be uniformly elliptic and bounded. As for the parabolic problem, homogenisation theory ensures that $u_\varepsilon \rightharpoonup u_0$ weakly* in $L^\infty(0,T; H^1_0(\Omega))$, $\partial_t u_\varepsilon \rightharpoonup \partial_t u_0$ weakly* in $L^\infty(0,T; L^2(\Omega))$, where u_0 is the solution of a homogenized problem similar to (5) with $a^\varepsilon(x)$ replaced by and effective tensor $a^0(x)$ independent of the smallest scales [6].

Multiscale methods. Find $u^H \in [0,T] \to S^1_0(\Omega, \mathcal{T}_H)$ such that for $f \in L^2(\Omega)$

$$(\partial_{tt} u^H, v^H) + B_H(u^H, v^H) = (f, v^H)\ \forall v^H \in S^1_0(\Omega, \mathcal{T}_H) \tag{6}$$

with appropriate projection of the true initial conditions, where $B_H(u^H, v^H) = \sum_{K\in\mathcal{T}_H} \frac{|K|}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x)\nabla u_K^h \cdot \nabla v_K^h dx$ and u_K^h (respectively v_K^h) are solutions of the following micro problems: for $K \in \mathcal{T}_H$ find $(u_K^h - u^H) \in S^1(K_\delta, \mathcal{T}_h)$ such that

$$\int_{K_\delta} a^\varepsilon(x)\nabla u_K^h \cdot \nabla z^h dx = 0, \qquad \forall z^h \in S^1(K_\delta, \mathcal{T}_h), \tag{7}$$

where $S^1(K_{\delta_j}, \mathcal{T}_h)$ is similar as defined before. Optimal convergence rates towards the homogenized solution has been proved in [1] for a generalized version of the method presented above (continuous in time fully discrete in space) allowing for arbitrary macro and micro polynomial degrees and time dependent right-hand side.

For limited time the propagation of waves in a highly oscillatory medium is well-described by the classical homogenized wave equation. With increasing time, however, the true solution, u^ε, deviates from the classical homogenization limit, u^0, as dispersive effects develop [11, 9]. In [3],[2] we proposed a new multiscale method based on (6) but with a modified L^2 scalar product $(\partial_{tt}u^H, v^H)$. The method is proved to be consistent with the homogenized solution and is shown to capture (for one-dimensional problems) dispersive effects that appear in the true solution with increasing time but are not present in the homogenized model. Moreover, the computational cost of the new method is *identical* to the cost of the method (6).

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Aspects of discontinuous multiscale flow approximations on transport and a two-level mortar preconditioner

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(joint work with Hailong Xiao, Zhen Tao)

The equations governing fluid flow are generally of elliptic or parabolic type, such as the equation governing single-phase flow in a porous medium:

$$\nabla \cdot u = f \quad \text{and} \quad u = -a\nabla p.$$

Because porous media are notoriously heterogeneous, the permeability coefficient a varies greatly on very small scales, and so the velocity u and pressure p must be approximated by multiscale techniques. Often in practical applications, u transports some chemical species governed by another equation of nearly hyperbolic type. We study the use of multiscale numerical techniques for flow and their effect on the combined flow and transport system.

Because mass conservation is critical for most flow and transport problems, we concentrate on mixed methods for the elliptic flow problem. We consider three similar multiscale methods for flow problems, the mixed multiscale finite element method (MsFEM) [7, 10, 9], the variational multiscale method (VMM) [11, 3, 8], and the domain decomposition multiscale mortar mixed method [4]. MsFEM uses the original variational form but modifies the finite elements, whereas VMM uses standard finite elements but modifies the variational form. Modification of the variational form can be viewed as the original variational form with modified elements. In fact, the modified elements are standard multiscale elements, and so these two approaches are essentially the same [2].

The mortar domain decomposition method approximates the trace of the pressure λ on the interfaces Γ between the subdomains in a mortar space M. For a given $\lambda \in M$, one can easily solve the subdomain problems in parallel for $u(\lambda)$ and $p(\lambda)$, using λ as a boundary condition. The Schur complement system is

$$d(\lambda, \mu) = b(\mu) \quad \forall \mu \in M,$$

where $d(\lambda,\mu) - b(\mu) = \langle u(\lambda)\cdot\nu, \mu\rangle_\Gamma$ represents the weak form of the normal velocity flux jump on Γ. The method is efficient in parallel if M is not large.

Each mortar basis function $\lambda_i \in M$ gives rise to a multiscale finite element $\big(u(\lambda_i), p(\lambda_i)\big)$, and so the method can be viewed in terms of MsFEM or VMM. The resulting multiscale elements are very unusual, and have greater flexibility in approximating the pressure and velocity. However, it results in a velocity field that is merely weakly continuous in the direction normal to the subdomain interfaces.

Multiscale error analysis for a two-scale, locally periodic permeability coefficient suggests that the MsFEM has serious resonance error, which can be mitigated by using higher order elements [1]. The same is true of the mortar methods if