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Fixed-order Control of LTI Systems Subject to Polytopic Uncertainty via the Concept of Strictly Positive Realness

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***Abstract*— This paper deals with the problem of fixed-order controller design of LTI continuous-time and discrete-time polytopic systems via homogeneous polynomially parameter-dependent Lyapunov matrices. The proposed method is based on the concept of Strictly Positive Realness (SPRness) of a transfer function depending on a parameter-dependent gain. To convert the problem to a set of LMI conditions, the parameter-dependent gain is determined *a priori* by means of a parameter-dependent state feedback controller. Simulation results and a comparison with recent existing methods demonstrate the effectiveness of the proposed approach.**

I. Introduction

The problem of fixed-order controller design is a theoretically challenging issue in control theory and it has attracted remarkable attention due to its great importance in practice. The non-convexity of the set of all fixed-order stabilizing controllers for a given plant is the main source of difficulty in solving such problem [1]. To address the problem, various approaches have been developed, e.g. non-smooth non-convex-based [2]–[4] and LMI-based methods [5]–[7].

The problem of fixed-order controller design becomes more challenging in case of uncertainties in the plant model due to parameter drifting, unmodeled dynamics, modeling errors, etc. In this case, the main objective is to design a fixed-order controller which guarantees the robust stability as well as the robust performance of the uncertain system. To solve the problem, several LMI-based methods have emerged in the literature, e.g. the methods of [8]–[11] in polynomial framework and the methods of [12]–[22] in state space framework. In the state space approaches, some slack variables are used as a tool to decouple the product of closed-loop matrices and Lyapunov matrices leading to a sequence of sufficient LMI conditions.

Recently, several slack variable-based approaches to fixed-order controller design of polytopic systems, which rely on the concept of Strictly Positive Realness (SPRness) of transfer functions, have been developed, e.g. [8]–[11], [20]–[22]. Moreover, it can be shown that most of the existing slack variable-based methods, e.g. [6], [12]–[19], implicitly rely on the concept of SPRness [23]. The main idea behind these approaches is to fix the slack variables *a priori* using some methods, e.g. initial output feedback controllers or a desired closed-loop characteristics polynomial. Therefore, the SPRness of the transfer functions is parameterized by some LMIs thanks to KYP Lemma [24]. The way the slack variables are fixed is very crucial and affects the performance as well as the conservatism of the fixed-order control approach.

In this paper, necessary and sufficient conditions for fixed-order controller design of both continuous-time and discrete-time systems by means of homogeneous polynomially parameter-dependent Lyapunov matrices are presented. The focus of this paper is on systems subject to polytopic uncertainty. The proposed approach relies on the concept of SPRness of a special transfer function in which the slack matrices are determined *a priori* using an initial parameter-dependent state feedback controller. Continuous-time and discrete-time fixed-order controller synthesis is treated in a unified manner. Furthermore, it is theoretically and numerically demonstrated that the proposed approach allows fixed-order stabilizing (H_∞) controller synthesis which potentially use less decision variables than some existing approaches, e.g. [18].

The organization of the paper is as follows. The problem formulation and preliminaries are given in Section II. Section III proposes the main results of the problem of fixed-order H_∞ controller design of LTI continuous-time and discrete-time systems with polytopic uncertainty. Simulation results are presented in Section IV. Finally, Section V concludes the paper.

Throughout the paper, matrices I and 0 are the identity matrix and the zero matrix of appropriate dimensions, respectively. The symbol T and $\star$ are used to show the matrix transpose and the symmetric blocks, respectively. For symmetric matrices, $P > 0$ ($P < 0$) indicates the positive-definiteness (the negative-definiteness).

II. Problem Formulation and Preliminaries

Consider a linear time-invariant (LTI) system described by the following dynamical equations:

$$\begin{aligned} \delta[x_g(t)] &= A_g x_g(t) + B_g u(t) + B_w w(t) \\ z(t) &= C_z x_g(t) + D_{zu} u(t) \\ y(t) &= C_g x_g(t) \end{aligned} \tag{1}$$

where $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^{n_o}$, and $z \in \mathbb{R}^s$ are the state, the control input, the exogenous input, the measured output, and the controlled output, respectively. The symbol $\delta[\cdot]$ presents the derivative term for continuous-time ($\delta[x(t)] = dx/dt$) and the forward operator for discrete-time systems ($\delta[x(t)] = x(t+1)$). Matrices A_g, B_g, B_w, C_z, C_g,

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and D_{zu} belong to the following uncertainty domain:

$$\begin{aligned}\Omega &= \{(A_g(\lambda), B_g(\lambda), B_w(\lambda), C_z(\lambda), C_g(\lambda), D_{zu}(\lambda)) \\ &= \sum_{i=1}^{q} \lambda_i (A_{g_i}, B_{g_i}, B_{w_i}, C_{z_i}, C_{g_i}, D_{zu_i})\}\end{aligned} \tag{2}$$

where $\lambda = [\lambda_1, \ldots, \lambda_q]$ belongs to the following unit simplex:

$$\Lambda_q = \left\{ \lambda_1, \ldots, \lambda_q \,\middle|\, \sum_{i=1}^{q} \lambda_i = 1, \quad \lambda_i \geq 0 \right\} \tag{3}$$

and matrices A_{g_i}, B_{g_i}, B_{w_i}, C_{z_i}, C_{g_i}, and D_{zu_i} are the i-th vertex of the polytope. The main objective of this paper is to design a robust fixed-order stabilizing (H_∞) controller for the polytopic system given by:

$$\begin{aligned}\delta[x_c(t)] &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t)\end{aligned} \tag{4}$$

where $A_c \in \mathbb{R}^{m \times m}$ and B_c, C_c, and D_c are of appropriate dimensions.

The problem of dynamic output-feedback controller synthesis can be equivalently transformed to a static output feedback one by introducing an augmented plant as follows [25]:

$$\begin{aligned}\delta[\bar{x}_g(t)] &= \bar{A}_g(\lambda)\bar{x}_g(t) + \bar{B}_g(\lambda)u(t) + \bar{B}_w(\lambda)w(t) \\ z(t) &= \bar{C}_z(\lambda)\bar{x}_g(t) + \bar{D}_{zu}(\lambda)u(t) \\ y(t) &= \bar{C}_g(\lambda)\bar{x}_g(t)\end{aligned} \tag{5}$$

where

$$\bar{A}_g(\lambda) = \begin{bmatrix} A_g(\lambda) & 0 \\ 0 & 0_m \end{bmatrix}, \quad \bar{B}_g(\lambda) = \begin{bmatrix} 0 & B_g(\lambda) \\ I_m & 0 \end{bmatrix}$$

$$\bar{B}_w(\lambda) = \begin{bmatrix} B_w(\lambda) \\ 0 \end{bmatrix}, \quad \bar{C}_g(\lambda) = \begin{bmatrix} 0 & I_m \\ C_g(\lambda) & 0 \end{bmatrix}$$

$$\bar{C}_z(\lambda) = \begin{bmatrix} C_z(\lambda) & 0 \end{bmatrix}, \quad \bar{D}_{zu}(\lambda) = \begin{bmatrix} 0 & D_{zu}(\lambda) \end{bmatrix}$$

Closed-loop system $H_{zw}(\lambda)$, transfer function from w to z, has the following state space realization:

$$\begin{aligned}\delta[x(t)] &= A(\lambda)x(t) + B(\lambda)w(t) \\ z(t) &= C(\lambda)x(t)\end{aligned} \tag{6}$$

where $x = \bar{x}_g$ and

$$K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{7}$$

$$\begin{aligned}A(\lambda) &= \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K\bar{C}_g(\lambda) \\ B(\lambda) &= \bar{B}_w(\lambda) \\ C(\lambda) &= \bar{C}_z(\lambda) + \bar{D}_{zu}(\lambda)K\bar{C}_g(\lambda)\end{aligned} \tag{8}$$

The remains of this section provide basic lemmas which will be used throughout this paper.

Lemma 1: (Kalman-Yakubovich-Popov (KYP) Lemma [24]) A square transfer function $H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SPR if and only if there exists a Lyapunov matrix $P > 0$ such that

For continuous-time systems:

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0 \tag{9}$$

For discrete-time systems:

$$\begin{bmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} < 0 \tag{10}$$

Lemma 2: The following statements are equivalent [20], [21]:

1) A square transfer function $H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SPR with Lyapunov matrix $P > 0$.
2) $H^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$ is SPR with Lyapunov matrix $P > 0$.

As a result, the following inequalities are equivalent: Continuous-time systems:

$$\begin{aligned}&\begin{bmatrix} A^T P + PA & \star \\ B^T P - C & -D - D^T \end{bmatrix} < 0 \\ &\begin{bmatrix} A^{*T} P + PA^* & \star \\ -D^{-T} B^T P - D^{-1} C & -D^{-1} - D^{-T} \end{bmatrix} < 0\end{aligned} \tag{11}$$

Discrete-time systems:

$$\begin{aligned}&\begin{bmatrix} A^T PA - P & \star \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} < 0 \\ &\begin{bmatrix} A^{*T} PA^* - P & \star \\ -(BD^{-1})^T PA^* - D^{-1}C & (BD^{-1})^T PBD^{-1} - (D^{-1} + D^{-T}) \end{bmatrix} < 0\end{aligned} \tag{12}$$

where $A^* = A - BD^{-1}C$.

III. Main Results

The following theorems present necessary and sufficient conditions for stabilizing (H_∞) static output feedback controller synthesis.

A. Fixed-order Stabilizing Controller Design

Theorem 1: The static output feedback controller $K = X^{-1}L$ stabilizes the augmented polytopic system in (5) if and only if there exist a parameter-dependent gain $K_{sf}(\lambda)$, a Lyapunov matrix $P(\lambda) > 0$, and two matrices X and L such that

For continuous-time systems:

$$\begin{bmatrix} M^T(\lambda)P(\lambda) + P(\lambda)M(\lambda) & \star \\ \bar{B}_g^T(\lambda)P(\lambda) - N(\lambda) & -X - X^T \end{bmatrix} < 0 \tag{13}$$

For discrete-time systems:

$$\begin{bmatrix} M^T(\lambda)P(\lambda)M(\lambda) - P(\lambda) & \star \\ \bar{B}_g^T(\lambda)P(\lambda)M(\lambda) - N(\lambda) & (\bar{B}_g^T(\lambda)P(\lambda)\bar{B}_g(\lambda) - X - X^T) \end{bmatrix} < 0 \tag{14}$$

where

$$\begin{aligned}M(\lambda) &= \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda) \\ N(\lambda) &= XK_{sf}(\lambda) - L\bar{C}_g(\lambda)\end{aligned} \tag{15}$$

Proof:

Sufficiency: According to KYP Lemma, the inequality (13)/(14) indicates that the following transfer function is SPR with Lyapunov matrix $P(\lambda)$:

$$H = \left[\begin{array}{c|c} M(\lambda) & \bar{B}_g(\lambda) \\ \hline X(K_{sf}(\lambda) - KC_g(\lambda)) & X \end{array}\right] \tag{16}$$

According to Lemma 2, the SPRness of H implies that H^{-1} with the following realization is also SPR with the same Lyapunov matrix $P(\lambda)$.

$$H^{-1} = \left[\begin{array}{c|c} A(\lambda) & -\bar{B}_g(\lambda)X^{-1} \\ \hline K_{sf}(\lambda) - K\bar{C}_g(\lambda) & X^{-1} \end{array}\right] \tag{17}$$

The SPRness of H^{-1} with $P(\lambda)$ implies that the closed-loop state matrix $A(\lambda)$ is stable with the parameter-dependent Lyapunov matrix $P(\lambda)$.

Necessity: Assume that K stabilizes the closed loop polytopic system in (6)-(8). Let's choose $X = I$ $(X + X^T > 0)$, $L = K$, and $K_{sf}(\lambda) = K\bar{C}_g(\lambda)$. Then, the following transfer function matrix H is SPR and (13)/(14) is satisfied.

$$\begin{aligned} H &= \left[\begin{array}{c|c} M(\lambda) & \bar{B}_g(\lambda) \\ \hline 0 & I \end{array}\right] \\ &= 0 \times (xI - M(\lambda))^{-1} \times \bar{B}_g(\lambda) + I \\ &= I > 0 \end{aligned} \tag{18}$$

where $x = s$ and $x = z$ for continuous-time and discrete-time case, respectively. Thus, this completes the proof. ■

B. Relation between the Proposed Stabilizing Static Output Feedback Controller Design Method in [18] and Theorem 1

In this subsection, we show that the proposed approach in [18] relies on the concept of SPRness of a transfer function where A-matrix is fixed by a parameter-dependent gain.

Lemma 3: Let $K_{sf}(\lambda)$ be a stabilizing parameter-dependent gain for the continuous-time augmented system subject to polytopic uncertainty described by (5). Then, the following statements are equivalent:

(a) If there exist two matrices X and L such that the following transfer function is SPR:

$$H(s) = \left[\begin{array}{c|c} \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda) & \bar{B}_g(\lambda) \\ \hline XK_{sf}(\lambda) - L\bar{C}_g(\lambda) & X \end{array}\right] \tag{19}$$

(b) If there exist a Lyapunov matrix $P(\lambda) > 0$, and two matrices X and L such that the following inequality holds:

$$\begin{bmatrix} M^T(\lambda)P(\lambda) + P(\lambda)M(\lambda) & \star \\ \bar{B}_g^T(\lambda)P(\lambda) - (XK_{sf}(\lambda) - L\bar{C}_g(\lambda)) & -X - X^T \end{bmatrix} < 0 \tag{20}$$

where $M(\lambda) = \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda)$.

(c) If there are a Lyapunov matrix $P(\lambda)$ and matrices $F(\lambda)$, $V(\lambda)$, X, and L such that (21) holds [18].

Then, $K = X^{-1}L$ is a robust static output feedback controller which stabilizes the augmented continuous-time system given in (5).

Proof: The statements (a) and (b) directly results from KYP Lemma. Therefore, it is enough to show that (21) is equivalent to (20). Post-multiplying (21) by $Q(\lambda)$

$$Q(\lambda) = \begin{bmatrix} I & 0 \\ \bar{A}_g + \bar{B}_g K_{sf} & \bar{B}_g \\ 0 & I \end{bmatrix} \tag{22}$$

and pre-multiplying by $Q^T(\lambda)$, the inequality given in (20) is obtained. ■

As a result, the parameter-dependent slack matrices $F(\lambda)$ and $V(\lambda)$ in (21) can be eliminated without conservatism. In fact, these matrices do not affect the final static output feedback controller; however, elimination of them leads to less computation time.

C. Fixed-order H_∞ Controller Design

Theorem 2: (Continuous-time case) The H_∞ static output feedback $K = X^{-1}L$ stabilizes the augmented polytopic system in (5) and ensures the robust performance $\|H_{zw}(\lambda)\|_\infty^2 < \mu$, for all $\lambda \in \Lambda_q$, if and only if there exist a stabilizing parameter-dependent gain $K_{sf}(\lambda)$, a Lyapunov matrix $P(\lambda) > 0$, and two auxiliary matrices X and L such that

$$\begin{bmatrix} M^T(\lambda)P(\lambda) + P(\lambda)M(\lambda) & \star & \star & \star \\ \bar{B}_g^T(\lambda)P(\lambda) - N(\lambda) & -X - X^T & \star & \star \\ B^T(\lambda)P(\lambda) & 0 & -\mu I & \star \\ \bar{C}_z(\lambda) + \bar{D}_{zu}(\lambda)K_{sf} & \bar{D}_{zu}(\lambda) & 0 & -I \end{bmatrix} < 0 \tag{23}$$

where $M(\lambda)$ and $N(\lambda)$ are defined in (15).

Theorem 3: (Discrete-time case) The H_∞ static output feedback $K = X^{-1}L$ stabilizes the augmented discrete-time polytopic system in (5) and guarantees $\|H_{zw}(\lambda)\|_\infty^2 < \mu$, for all $\lambda \in \Lambda_q$, if and only if there exist a stabilizing parameter-dependent gain $K_{sf}(\lambda)$, a Lyapunov matrix $P(\lambda) > 0$, and two auxiliary matrices X and L such that (24) with $M(\lambda)$ and $N(\lambda)$ defined in (15) holds.

Remark: If the parameter-dependent gain $K_{sf}(\lambda)$ is given *a priori* and $P(\lambda)$ is considered as a homogeneous polynomial w.r.t. λ, the parameter dependent conditions in Theorem 1-3 can be handled by a sequence of LMI relaxations. Based on the results mentioned in [26], parameter-dependent LMIs with parameters in the unit simplex always have homogenous polynomially parameter-dependent solutions of sufficiently high degree. Moreover, they can be solved with no conservatism by a set of LMI relaxations. Therefore, we assume that the gain $K_{sf}(\lambda)$ is initialized by means of a parameter-dependent stabilizing (H_∞) state feedback controller.

In following, two new theorems for parameter-dependent H_∞ state feedback controller design of both continuous-time and discrete-time augmented polytopic systems described by (5) are given.

D. Parameter-dependent State Feedback Control Design

A parameter-dependent H_∞ state feedback controller $K_{sf}(\lambda)$ for the continuous-time and discrete-time augmented systems in (5) can be determined by the following theorems.

Theorem 4: (Continuous-time case) The parameter-dependent state feedback controller $K_{sf}(\lambda) = Z(\lambda)F^{-1}(\lambda)$ stabilizes the augmented continuous-time polytopic systems in (5) and guarantees the desired performance $\|H_{zw}(\lambda)\|_\infty < \gamma$, for all $\lambda \in \Lambda_q$, if and only if there exist a parameter-dependent Lyapunov matrix $P(\lambda) > 0$, matrices $F(\lambda)$, and $Z(\lambda)$ and a positive scalar $\delta > 0$ such that (25) is satisfied.

Theorem 5: (Discrete-time case) The parameter-dependent state feedback $K_{sf}(\lambda) = Z(\lambda)F^{-1}(\lambda)$ guarantees the stability and closed-loop performance $\|H_{zw}(\lambda)\|_\infty^2 < \mu$

$$\begin{bmatrix} (\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))^T F^T(\lambda)+F(\lambda)(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)) & \star & \star \\ P(\lambda)-F^T(\lambda)+V(\lambda)(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)) & -V(\lambda)-V^T(\lambda) & \star \\ \bar{B}_g^T(\lambda)F^T(\lambda)+L\bar{C}_g(\lambda)-XK_{sf}(\lambda) & \bar{B}_g^T(\lambda)V^T(\lambda) & -X-X^T \end{bmatrix} < 0 \quad (21)$$

$$\begin{bmatrix} M^T(\lambda)P(\lambda)M(\lambda)-P(\lambda) & \star & \star & \star \\ \bar{B}_g^T(\lambda)P(\lambda)M(\lambda)-N(\lambda) & \bar{B}_g^T(\lambda)P(\lambda)\bar{B}_g(\lambda)-X-X^T & \star & \star \\ B^T(\lambda)P(\lambda)M(\lambda) & B^T(\lambda)P(\lambda)\bar{B}_g(\lambda) & -\mu I+B^T(\lambda)P(\lambda)B(\lambda) & \star \\ \bar{C}_z(\lambda)+\bar{D}_{zu}(\lambda)K_{sf}(\lambda) & \bar{D}_{zu}(\lambda) & 0 & -I \end{bmatrix} < 0 \quad (24)$$

$$\begin{bmatrix} \bar{A}_g(\lambda)F(\lambda)+F^T(\lambda)\bar{A}_g^T(\lambda)+\bar{B}_g(\lambda)Z(\lambda)+Z^T(\lambda)\bar{B}_g^T(\lambda) & \star & \star & \star \\ P(\lambda)-F(\lambda)+\delta(F^T(\lambda)\bar{A}_g^T(\lambda)+Z^T(\lambda)\bar{B}_g^T(\lambda)) & -\delta(F(\lambda)+F^T(\lambda)) & \star & \star \\ \bar{C}_w(\lambda)F(\lambda)+\bar{D}_{zu}(\lambda)Z(\lambda) & \delta(\bar{C}_w(\lambda)F(\lambda)+\bar{D}_{zu}(\lambda)Z(\lambda)) & -\gamma I & \star \\ B^T(\lambda) & 0 & 0 & -\gamma I \end{bmatrix} < 0 \quad (25)$$

of the augmented discrete-time polytopic systems in (5), for all $\lambda \in \Lambda_q$, if and only if there exist parameter-dependent matrices $P(\lambda) > 0$, $F(\lambda)$, and $Z(\lambda)$ such that (26) holds.

Remark: Theorem 4 and Theorem 5 can be used for the design of stabilizing parameter-dependent state feedback controllers for continuous-time and discrete-time systems by removing the third and forth rows and columns of matrices in (25) and (26), respectively.

E. Controller Design Procedure

The robust fixed-order H_∞ controller design procedure includes the following steps:

Step 1: Choose the order of controller (m) and construct the augmented system in (5).

Step 2: Set $j = 1$ and design the parameter-dependent gain $K_{sf}^{[1]}(\lambda)$ for the augmented system using Theorem 4/Theorem 5.

Step 3: Choose the degree of the homogenous Lyapunov matrix $P(\lambda)$ and solve the convex optimization problem proposed in Theorem 2/Theorem 3 by constructing the LMI constraints in (23)/(24) (using e.g. ROLMIP [27]) to obtain the static output feedback controller $K^{[j]}$.

Step 4: If $\mu^{[j-1]} - \mu^{[j]} > \epsilon$, update the parameter-dependent gain $K_{sf}(\lambda)$, i.e. $K_{sf}^{[j+1]}(\lambda) = K^{[j]}\bar{C}_g(\lambda)$, and go to Step 3 with $j \longleftarrow j+1$, else stop.

Theorem 6: The iterative algorithm leads to monotonic convergence of the upper bound on the H_∞ norm.

Remark: It should be mentioned that the set of LMI constraints from parameter-dependent LMIs in Theorem 1-5 with parameters in the unit simplex are constructed using ROLMIP (Robust LMI Parser) [27]. ROLMIP is a computational MATLAB package which provides an interface for the users to construct a finite set of LMIs from parameter-dependent LMIs with parameters in the unit simplex [27].

IV. Simulation Results

In this section, several examples from the literature are given to evaluate the effectiveness of the proposed fixed-order control approach. A comparison with the recent existing methods is made. It should be noted that in all tables, the set $\{d_Z, d_F, d_{P_{sf}}, d_P\}$, respectively, denotes the degrees of the homogeneous polynomials $Z(\lambda)$, $F(\lambda)$, and $P(\lambda)$ in Theorem 4 and 5 and the degree of homogeneous polynomially parameter-dependent Lyapunov matrix $P(\lambda)$ in Theorem 1-3[1] To solve the LMI problems in MATLAB, YALMIP [28] as the interface and SeDuMi [29]/MOSEK[2] as the solver are used.

TABLE I

Maximum Interval of Parameter a in Example 1

Method	$\{d_Z, d_F, d_{P_{sf}}, d_P\}$	a
Theorem 1	$\{1,0,1,1\}$	$[-17.8 \quad 122.2]$
[19]	$\{-,-,-,1\}$	$[3.6 \quad 82.2]$
[18]	$\{1,0,1,1\}$	$[-17.8 \quad 122.2]$
[30]	—	Non-applicable
[31]	—	Non-applicable

Example 1: Consider a third-order continuous-time polytopic system, borrowed from [19], with the following vertices:

$$\begin{aligned} &A_{g_1} = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 1 \\ a & 6 & -1 \end{bmatrix}; \quad A_{g_2} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -5 & 1 \\ 10 & 1 & -1 \end{bmatrix} \\ &B_{g_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad B_{g_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &C_{g_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad C_{g_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (27)$$

The main objective is to design a stabilizing static output feedback controller which leads to the largest interval of parameter a. Therefore, the system can be modeled with a polytope with three vertices, i.e. $A_{g_1}|_{a=a_{min}}$, $A_{g_1}|_{a=a_{max}}$, and A_{g_2}. The minimum and maximum values of a can be determined by a bisection algorithm. The results of Theorem 1 are compared with ones of [18], [19], [30], [31] in Table I. The proposed methods in Theorem 1 and [18] are both initialized with the same parameter-dependent state feedback $K_{sf}(\lambda)$.

As it is reported in [19], since matrix $C(\lambda)$ is not full row rank, the approaches of [30], [31] are not applicable. Results given in Table I indicate that the proposed method

[1]In this paper, the degree of the other slack variables of [18] is considered equal to one.

[2]Available online in http://www.mosek.com

$$\begin{bmatrix} P(\lambda) & \star & \star & \star \\ F^T(\lambda)\bar{A}_g^T(\lambda) + Z^T(\lambda)\bar{B}_g^T(\lambda) & F(\lambda) + F^T(\lambda) - P(\lambda) & \star & \star \\ 0 & \bar{C}_w(\lambda)F(\lambda) + \bar{D}_{zu}(\lambda)Z(\lambda) & I & \star \\ B^T(\lambda) & 0 & 0 & \mu I \end{bmatrix} > 0 \tag{26}$$

TABLE II

UPPER BOUND OF $\|H_{zw}(\lambda)\|_\infty$ IN EXAMPLE 2

Method	Iterations	$\gamma = \sqrt{\mu}$	K
[12]	1	9.73	[0.56 5.08]
[32]	1	6.80	[0.054 0.64]
[19]	1	2.33	[0.45 4.19]
[20]	5	1.79	[77.16 608.87]
[33]	30	1.66	[130.35 939.37]
Theorem 2	5	1.78	[9.36 69.57]

in this paper and [18] lead to the best results among the others.

Example 2: Consider the following continuous-time polytopic system with two vertices [19]:

$$A_{g_1} = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.39 \\ 0 & 0 & -30 \end{bmatrix}$$

$$A_{g_2} = \begin{bmatrix} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30 \end{bmatrix}$$

$$B_{g_1} = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}; B_{g_2} = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix}; B_w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; C_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; D_{zu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The objective here is to design a static output feedback H_∞ controller with linearly parameter-dependent Lyapunov matrices. To this end, an optimization problem, which is the minimization of the upper bound of H_∞ norm (γ) subject to a sequence of LMI constraints is solved. Resulting static output feedback initialized by parameter-dependent state feedback controller with $d_Z = 1$, $d_F = 0$, and $d_{P_{sf}} = 2$ is given in Table II. The results are compared with the LMI-based methods in [12], [19], [20], [32] and the BMI-based method in [33]. For all cases, the degree of Lyapunov matrix P is one.

Example 3: As the third example, consider the modified version of the pitch control of F4E, given in [34], described by the following state space matrices:

$$A_g = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & -30 & 30 \\ 0 & 0 & 0 & -10^4 \end{bmatrix}; \quad B_g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10^4 \end{bmatrix}$$

$$C_g = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix}; \quad B_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

$$C_w = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad D_{zu} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

TABLE III

PARAMETERS OF FOUR OPERATING POINTS IN EXAMPLE 3

Operating points	1	2	3	4
Mach number	0.5	0.9	0.85	1.5
Altitude(ft)	5000	35000	5000	35000
a_{11}	-0.9896	-0.6607	-1.0702	-0.5162
a_{12}	17.41	18.11	50.72	29.96
a_{13}	96.15	84.34	263.5	178.9
a_{21}	0.2648	0.08201	0.2201	-0.6896
a_{22}	-0.8512	-0.6587	-1.418	-1.225
a_{23}	-11.39	-10.81	-31.99	-30.38
b_1	-97.78	-272.2	-85.09	-175.6

TABLE IV

UPPER BOUND OF $\|H_{zw}(\lambda)\|_\infty$ IN EXAMPLE 3

Initialization	Theorem 4		HIFOO	
controller order	$m = 0$	$m = 1$	$m = 0$	$m = 1$
degree of P (d_P)	1	1	1	2
Iterations	4	2	30	30
γ	3.0780	3.1026	3.3378	2.2818

where $0.5 \leq c \leq 1$ and parameters a_{ij}, $i = 1, 2$; $j = 1, 2, 3$, and b_1 for four operating points are given in Table III.

The uncertainty of the system in (28) is in the form of a polytope with $q = 8$ vertices. The proposed approach in [34] as well as the full-order controller design method of [35] are employed for the comparison purposes. It should be noted since the main assumption of [19] is that $C_w^T D_{zu} = 0$, it cannot be applied to Example 3.

Theorem 2 is initialized by two different parameter-dependent gain $K_{sf}(\lambda)$. In the first case, an initial parameter-dependent state feedback controller is designed using Theorem 4 with $d_Z = 0$, $d_F = 0$, and $d_{P_{sf}} = 1$. In the second case, $K_{sf}(\lambda) = K_0^m C_g(\lambda)$ is considered, where K_0^m is a simultaneously stabilizing controller of order m designed by HIFOO [3]. The results of both cases are then summarized in Table IV.

Theorem 2 initialized by the parameter-dependent state feedback controller $K_{sf}(\lambda)$ of Theorem 4 results in the following reduced-order H_∞ controllers:

$$K^{m=0} = \begin{bmatrix} 1.9407 & 12.3911 \end{bmatrix}$$
$$K^{m=1} = \left[\begin{array}{c|cc} -4.3979 & 0 & 0 \\ \hline 0 & 2.2321 & 14.3543 \end{array}\right] \tag{29}$$

For the second case, the following HIFOO controllers are used to initialize $K_{sf}(\lambda)$:

$$K_0^{m=0} = \begin{bmatrix} 0.0958 & 0.7670 \end{bmatrix}$$
$$K_0^{m=1} = \left[\begin{array}{c|cc} -1.1754 & 0.2729 & -1.8780 \\ \hline 0.6998 & 0.3876 & 0.24434 \end{array}\right] \tag{30}$$

Theorem 2 initialized by $K_{sf}(\lambda)$ obtained by the above initial controllers leads to the following H_∞ controllers:

$$K^{m=0} = \left[\begin{array}{cc} 0.0965 & 0.8012 \end{array} \right]$$
$$K^{m=1} = \left[\begin{array}{c|cc} -1.8044 & 0.1048 & -3.1589 \\ \hline 0.9039 & 0.3921 & 3.1998 \end{array} \right] \tag{31}$$

As mentioned in [18], the proposed approach of [34] leads to the lowest H_∞ upper bound 37.20 for $m = 0, 1$ and the full-order control design method in [35] does not find any feasible solution.

V. Conclusions

In this paper, the problem of fixed-order H_∞ controller design of LTI systems subject to polytopic uncertainty is considered. To this end, necessary and sufficient conditions based on the concept of strictly positive realness (SPRness) of a special transfer function are developed. The proposed conditions depend on a parameter-dependent gain determined through a parameter-dependent state feedback controller. The robust stability and robust H_∞ performance of the closed-loop polytopic systems are guaranteed via homogeneous polynomially parameter-dependent Lyapunov matrices. Simulation examples show the efficiency of the proposed fixed-order H_∞ controller design approach.

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