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GENERALIZED BOMBIERI–LAGARIAS' THEOREM AND GENERALIZED LI'S CRITERION WITH ITS ARITHMETIC INTERPRETATION

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We show that Li's criterion equivalent to the Riemann hypothesis, i.e., the statement that the sums

$$k_n = \Sigma_\rho \left(1 - \left(1 - \frac{1}{\rho} \right)^n \right)$$

over zeros of the Riemann xi-function and the derivatives

$$\lambda_n \equiv \frac{1}{(n-1)!} \frac{d^n}{dz^n} (z^{n-1} \ln(\xi(z)))\Big|_{z=1}, \quad \text{where} \quad n = 1, 2, 3, \ldots,$$

are nonnegative if and only if the Riemann hypothesis is true, can be generalized and the nonnegativity of certain derivatives of the Riemann xi-function estimated at an *arbitrary* real point a, except $a = 1/2$, can be used as a criterion equivalent to the Riemann hypothesis. Namely, we demonstrate that the sums

$$k_{n,a} = \Sigma_\rho \left(1 - \left(\frac{\rho - a}{\rho + a - 1} \right)^n \right)$$

for any real a such that $a < 1/2$ are nonnegative if and only if the Riemann hypothesis is true (correspondingly, the same derivatives with $a > 1/2$ should be nonpositive). The arithmetic interpretation of the generalized Li's criterion is given. Similarly to Li's criterion, the theorem of Bombieri and Lagarias applied to certain multisets of complex numbers is also generalized along the same lines.

1. Introduction

In 1997, Li established the following criterion equivalent to the Riemann hypothesis concerning nontrivial zeros of the Riemann ζ-function (see, e.g., [1] for the standard definitions and discussion of the general properties of this function; this criterion is now called Li's criterion) [2]:

Li's Criterion. *The Riemann hypothesis is equivalent to the nonnegativity of the following numbers:*

$$\lambda_n \equiv \frac{1}{(n-1)!} \frac{d^n}{dz^n} \left(z^{n-1} \ln(\xi(z))\Big|_{z=1} \right) \tag{1}$$

for any nonnegative integer n.

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Here, $\xi(z)$ is the Riemann xi-function connected with the Riemann ζ-function by the well-known relation [1]

$$\xi(z) = \frac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\varsigma(z) . \tag{2}$$

Two years later, Bombieri and Lagarias generalized Li's criterion [3]. If $\rho = 1/2 + iT$, T is real, and $i = \sqrt{-1}$, then $|(\rho-1)/\rho| = 1$ and, hence, can be rewritten in the form $\exp(i\vartheta_i)$, where

$$\vartheta_i = \arctan\frac{T}{T^2 - 1/4} .$$

We now introduce the sum

$$k_n = \Sigma_\rho\left(1-\left(1-\frac{1}{\rho}\right)^n\right) = \Sigma_\rho\left(1-\left(\frac{\rho-1}{\rho}\right)^n\right)$$

over the nontrivial zeros of the Riemann function (n is a nonnegative integer, zeros are counted with regard for their multiplicities; for $n = 1$, the contributions of complex conjugate zeros should be paired in finding the sum). For two complex conjugate "correct" zeros of the Riemann function $\rho = 1/2 \pm iT$, we easily see that their contribution to the sum k_n is equal to $2(1-\cos(n\vartheta_i))$ and, hence, is nonnegative; thus, the sum k_n is also nonnegative. On the contrary, if some nontrivial zero of the Riemann function with $\operatorname{Re}\rho \neq 1/2$ exists, then, for sufficiently large n, we get arbitrarily large (in modulus) negative contributions of these zeros, and one can directly show that, for infinitely many n, these contributions cannot be compensated by all other "correct" $1-\cos(n\vartheta_i)$ terms of the sum [3]. Hence, infinitely many sums k_n must be negative.

This consideration immediately shows that the nonnegativity of the sums k_n is equivalent to the Riemann hypothesis. Li also demonstrated that these sums are equal to the derivatives presented in Eq. (1) (certainly, this is the most technically difficult part of his work; another derivation of this relation is presented shortly in what follows).

2. Generalized Li's and Bombieri–Lagarias' Criteria

We now note that, for $\rho = 1/2 + iT$ and any real a,

$$\left|\frac{\rho-a}{\rho+a-1}\right| = \left|\frac{-a+1/2+iT}{a-1/2+iT}\right| = 1 .$$

Thus, we introduce the sum

$$k_{n,a} = \sum_\rho\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right) = \sum_\rho\left(1-\left(1-\frac{2a-1}{\rho+a-1}\right)^n\right).$$

To demonstrate that all these sums are nonnegative on RH, we just replace

$$\vartheta_i = \arctan \frac{T}{T^2 - 1/4}$$

considered above by

$$\vartheta_i = \arctan \frac{T(2a-1)}{T^2 - a^2 + a - 1/4}$$

and repeat the reasoning used above. To demonstrate the inverse implication, we briefly reproduce a slightly modified reasoning of Bombieri and Lagarias [3]; see their original paper for more detail.

Let $a < 1/2$. We observe that, for any Riemann zero $\rho = \sigma + iT$,

$$\left| \frac{\rho - a}{\rho + a - 1} \right|^2 = 1 + \frac{(1-2a)(2\sigma - 1)}{|\rho + a - 1|^2}$$

and, thus, for $\sigma > 1/2$, we can find at least one zero for which $\left| \frac{\rho - a}{\rho + a - 1} \right| > 1$. Since $\frac{(1-2a)(2\sigma - 1)}{|\rho + a - 1|^2}$ tends to zero as $|\rho_k|$ tends to infinity, the maximum of this expression over ρ is attained and there are only finitely many, say K, zeros ρ_k for which $\left| \frac{\rho - a}{\rho + a - 1} \right| = 1 + t = \max$. For all other zeros, we have

$$\left| \frac{\rho - a}{\rho + a - 1} \right| \le 1 + t - \delta$$

for some fixed positive δ. Clearly, if we take sufficiently large n, then the term

$$1 - \left(\frac{\rho_k - a}{\rho_k + a - 1} \right)^n = 1 - (1+t)^n \exp(in\vartheta_k)$$

$\left(\vartheta_k \text{ is the argument of } \left(\frac{\rho_k - a}{\rho_k + a - 1} \right) \right)$ can be made very large in modulus and negative. Thus, in view of the Dirichlet's theorem on the simultaneous Diophantine approximation, the sum of $1 - \left(\frac{\rho_k - a}{\rho_k + a - 1} \right)^n$ over all ρ_k can be made arbitrary close to $K(1 - (1+t)^n)$, while the sum over all other zeros has the order of $O(n^2(1+t-\delta)^n)$, just due to their known density. The case $a > 1/2$ is quite similar. Thus, we have proved the following theorem:

Theorem 1. *The Riemann hypothesis is equivalent to the nonnegativity of the sums*

$$k_{n,a} = \sum_\rho \left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right) = \sum_\rho \left(1-\left(1-\frac{2a-1}{\rho+a-1}\right)^n\right)$$

taken over the zeros of the Riemann xi-function for any real a, *except* $a = 1/2$. *Here,* n *is a nonnegative integer, the zeros are counted with regard for their multiplicities, and, for* $n = 1$, *the contributions of complex conjugate zeros should be paired when summing.*

Indeed, we have proved this statement not only for the zeros of the Riemann ξ-function but also for certain multisets of complex numbers; see [3]. For completeness, we now formulate this result in the form of a theorem:

Theorem 2 (Generalized Bombieri–Lagarias' Theorem). *Let* a *and* σ *are arbitrary real numbers,* $a < \sigma$, *and let* R *be a multiset of complex numbers* ρ *such that*

(i) $2\sigma - a \notin R$;

(ii) $\sum_\rho (1+|\mathrm{Re}\,\rho|)/(1+|\rho+a-2\sigma|^2) < +\infty$.

Then the following conditions are equivalent:

(a) $\mathrm{Re}\,\rho \le \sigma$ *for every* ρ;

(b) $\sum_\rho \mathrm{Re}\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right) \ge 0$ *for* $n = 1, 2, 3, \ldots$;

(c) for every fixed $\varepsilon > 0$, *there is a positive constant* $c(\varepsilon)$ *such that*

$$\sum_\rho \mathrm{Re}\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right) \ge -c(\varepsilon)e^{\varepsilon n}, \quad n = 1, 2, 3, \ldots.$$

If, under the same conditions, $a > \sigma$ *is taken, then the point (a) should be changed into*

(a') $\mathrm{Re}\,\rho \ge \sigma$ *for every* ρ;

the points (b) and (c) remain unchanged.

It is easy to see that the statement of Theorem 2 is formulated for any σ, not only for $\sigma = 1/2$, provided that $\sigma \ne a$. To demonstrate this, we just note that, for $\rho = \sigma + iT$,

$$\left|\frac{\rho-a}{\rho+a-2\sigma}\right| = \left|\frac{\sigma-a+iT}{a-\sigma+iT}\right| = 1$$

and, for $\rho = q + iT$,

$$\left|\frac{\rho - a}{\rho + a - 2\sigma}\right|^2 = 1 + \frac{4(\sigma - a)(q - \sigma)}{|\rho + a - 2\sigma|^2},$$

and then repeat the reasoning used above.

Assume that, in addition to the above-mentioned conditions of the generalized Bombieri–Lagarias' theorem, we also have the following condition:

(iii) If $\rho \in R$, then $\bar{\rho} \in R$ with the same multiplicity as ρ.

Then one can omit the operation of taking the real part in (b) and (c); the corresponding expressions are real. (Here, as usual, $\bar{\rho}$ denotes the complex conjugate of ρ.)

Following again the paper by Bombieri and Lagarias [3], we conclude this section with the following theorem:

Theorem 3 (Generalized Li's Criterion). *Let a be an arbitrary real number,* $a \neq \sigma$, *and let* R *be a multiset of complex numbers* ρ *such that*

(i) $2\sigma - a \notin R$, $a \notin R$;

(ii) $\sum_\rho (1 + |\operatorname{Re}\rho|)/\left(1 + |\rho + a - 2\sigma|^2\right) < +\infty$, $\sum_\rho (1 + |\operatorname{Re}\rho|)/\left(1 + |\rho - a|^2\right) < +\infty$;

(iii) if $\rho \in R$, *then* $2\sigma - \rho \in R$.

Then the following conditions are equivalent:

(a) $\operatorname{Re}\rho = \sigma$ *for every* ρ;

(b $\sum_\rho \operatorname{Re}\left(1 - \left(\frac{\rho - a}{\rho + a - 2\sigma}\right)^n\right) \geq 0$ *for any* a *and* $n = 1, 2, 3, \ldots$;

(c) for every fixed $\varepsilon > 0$ *and any* a, *there is a positive constant* $c(\varepsilon, a)$ *such that*

$$\sum_\rho \operatorname{Re}\left(1 - \left(\frac{\rho - a}{\rho + a - 2\sigma}\right)^n\right) \geq -c(\varepsilon, a)e^{\varepsilon n}$$

for $n = 1, 2, 3, \ldots$.

Clearly, in the conditions of the theorem, for all ρ, we have $\operatorname{Re}\rho \leq \sigma$ and $\operatorname{Re}(2\sigma - \rho) \leq \sigma$, whence $\operatorname{Re}\rho = \sigma$. If, in addition, to the above-mentioned conditions of the generalized Li's criterion, the following condition is also satisfied:

(iv) if $\rho \in R$, then the complex conjugate $\overline{\rho} \in R$ with the same multiplicity as ρ;

the one can omit the operation of taking the real part in (b) and (c); the indicated expressions are real.

Remark 1. By analogy with Li's criterion, the generalized Li's criterion can also be applied to numerous other zeta-functions, as shown, for the first time, by Li himself for the Dedekind zeta-function [2]. Later, this was the subject of numerous works written by different authors. We do not pursue this line of research in the present paper.

3. Connection between Generalized Li's Sums and Certain Derivatives of the Riemann Xi-Function

Our next aim is to establish a "Li's-type" relation similar to Eq. (1), i.e., the relation between the sums $k_{n,a}$ and certain derivatives of the Riemann xi-function. To do this, we use the generalized Littlewood theorem concerning contour integrals of the logarithm of an analytic function recently used in our paper to establish numerous equalities equivalent to the Riemann hypothesis [4] (for the sake of completeness, they are reproduced in what follows). The proof presented in [4] is a straightforward modification of the corresponding well-known proof of the Littlewood theorem (or lemma) (see, e.g., [5]). Actually, this theorem has been more or less explicitly used in the Riemann research already by Wang who established the first integral equality equivalent to the Riemann hypothesis in 1946 [6].a

Theorem 4 (Generalized Littlewood Theorem). *Let C be a rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, and $y = Y_2$, where $X_1 < X_2$ and $Y_1 < Y_2$, and let $f(z)$ be analytic and nonzero on C and meromorphic inside this rectangle. Also let $g(z)$ be analytic on C and meromorphic inside it. Let $F(z) = \ln(f(z))$, where the logarithm is defined as follows: We start from a particular determination on $x = X_2$ and obtain the values at the other points by continuous variations along $y = \mathrm{const}$ from $\ln(X_2 + iy)$. If, however, this path crosses a zero or a pole of $f(z)$, then we take $F(z)$ to be $F(z \pm i0)$ by analogy with approaching the path from above or from below. In addition, assume that the poles and zeros of the functions $f(z)$ and $g(z)$ do not coincide.*

Then

$$\int_C F(z)g(z)\,dz = 2\pi i\left(\sum_{\rho_g} \operatorname{res}\,(g(\rho_g)\cdot F(\rho_g)) - \sum_{\rho_f^0} \int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)\,dz + \sum_{\rho_f^{\mathrm{pol}}} \int_{X_1+iY_\rho^{\mathrm{pol}}}^{X_\rho^{\mathrm{pol}}+iY_\rho^{\mathrm{pol}}} g(z)\,dz\right),$$

where the sum is taken over all poles ρ_g of the function $g(z)$ lying inside C, all $\rho_f^0 = X_\rho^0 + iY_\rho^0$, which are zeros of the function $f(z)$ counted with regard for their multiplicities (i.e., the corresponding term is multiplied by m if this is a zero of the order m) and lying inside C, and all $\rho_f^{\mathrm{pol}} = X_\rho^{\mathrm{pol}} + iY_\rho^{\mathrm{pol}}$, which are poles of the function $f(z)$ counted with regard for their multiplicities and lying inside C. In order that this assertion be true, all relevant integrals on the right-hand side of the equality must exist.

Remark 2. Actually, the case of coincidence of the poles and zeros of the functions $f(z)$ and $g(z)$ often does not create any real problems and can easily be studied. Several cases of this kind were considered in [4].

A subtle moment connected with this generalized Littlewood theorem is a circumstance that the function $\arg(F(z))$ (the imaginary part of $\ln(f(z))$ is not continuous on the left boundary of the contour (segment X_1+iY_1, X_1+iY_2) if there are zeros or poles of the function $f(z)$ inside the contour. This is explicitly stated in the condition of the theorem:

If, however, this path crosses a zero or pole of $f(z)$, then we take $F(z)$ to be $F(z \pm i0)$ by analogy with approaching the path from above or from below.

In practice, this means that when finding the corresponding part of the contour integral, i.e., the integral $-\int_{X_1+iY_1}^{X_1+iY_2} \arg(F(z)) g(z)\,dz$ (the "minus" sign is explained by the necessity of traversing the contour counterclockwise), $\pm 2\pi i l$ jumps should be added to the argument of the function at a point $X_1 + iY_{z,p}$ whenever a zero or a pole of order l of the function $f(z)$ occurs somewhere at a point $X + iY_{z,p}$ inside the contour. The corresponding integral should be properly modified if the use of a continuous branch of the argument is desirable; see our paper [7] for details. It is also worth noting that the appropriateness of necessary modifications of the argument was numerically tested (and confirmed) by us for numerous integrals, e.g., for the integral

$$\int_0^\infty \frac{t \arg(\varsigma(1/4+it))}{(1/16+t^2)^2}\,dt = \pi \frac{\varsigma'(1/2)}{\varsigma(1/2)} - 9\pi - \pi \sum_{\rho,\sigma_k>1/4,t_k>0} \left(\frac{1}{t_k^2+1/4} \right)$$

(a similar equality in the form

$$\int_0^\infty \frac{t \arg(\varsigma(1/2+it))}{(1/16+t^2)^2}\,dt = \pi \frac{\varsigma'(3/4)}{\varsigma(3/4)} - \frac{32\pi}{3}$$

is equivalent to the Riemann hypothesis; see our Theorem 5 in [4]). However, for what follows, the asymptotics of the function $g(z)$ for large values of X_1 tending to minus infinity makes this modification irrelevant; the value of the integral $-\int_{X_1+iY_1}^{X_1+iY_2} \arg(F(z)) g(z)\,dz$ tends to zero anyway.

First, as an exercise, we use this theorem to establish the Li's relation (1). To this end, we consider a rectangular contour C with vertices at $\pm X \pm iX$ and real $X \to +\infty$. If a Riemann zero lies on the contour, we just shift it a bit to avoid this situation and consider a contour integral $\int_C g(z) \ln(\xi(z))\,dz$, where

$$g(z) = \frac{n}{(z-1)^2} \left(\frac{z}{z-1} \right)^{n-1} - \frac{n}{(z-1)^2}. \tag{3}$$

The known asymptotics of the logarithm of the xi-function for large $|z|$, $\cong O(z \ln z)$, guaranties that the value of the contour integral vanishes (tends to zero as $X \to \infty$ due to the asymptotics $g(z) \cong O(1/z^3)$). Thus, after the division by $2\pi i$, we conclude that

$$n\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1} - n\frac{\xi'}{\xi}(1) - \sum_{\rho}\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right) - n\sum_{\rho}\frac{1}{\rho} = 0. \tag{4}$$

The complex conjugate zeros should be paired when finding

$$\sum\nolimits_{\rho}\frac{1}{\rho} \quad \text{and} \quad \sum\nolimits_{\rho}\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right)$$

for $n=1$. Here, the first term is the contribution of the $(n+1)$th order pole of $g(z)$ at $z=1$; the second term is the contribution of the second-order pole arising from the term $-\frac{n}{(z-1)^2}$ in (3); the third and fourth terms are the integrals $-\int_{-\infty+iT_i}^{\rho_i} g(z)\,dz$. Clearly,

$$\frac{n}{(z-1)^2}\left(\frac{z}{z-1}\right)^{n-1} = \frac{d}{dz}\left(1-\left(1-\frac{1}{1-z}\right)^n\right).$$

This explains why we use the function $g(z)$ in the form (3). The term $-\frac{n}{(z-1)^2}$ is added just to ensure the asymptotics $g(z)\cong O(1/z^3)$ necessary to make the value of the contour integral equal to zero. It is also evident that

$$\sum_{\rho}\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right) = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right).$$

We know that $\frac{\xi'}{\xi}(1) = -\sum_{\rho}\frac{1}{\rho}$ [1]. Therefore,

$$n\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1} = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right),$$

which is just the required relation.

A quite similar consideration is applied to the analyzed case, where we now introduce the function

$$\tilde{g}(z) = -\frac{n(2a-1)(z-a)^{n-1}}{(z+a-1)^{n+1}} + \frac{n(2a-1)}{(z+a-1)^2} \tag{5}$$

and consider a contour integral $\int_C \tilde{g}(z)\ln(\xi(z))\,dz$ taken around the same contour as above. The application of Theorem 4 (generalized Littlewood theorem) gives

$$-\frac{n(2a-1)}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=1-a}+n(2a-1)\frac{\xi'}{\xi}(z)\Big|_{z=1-a}$$

$$-\sum_\rho\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right)+n(2a-1)\sum_\rho\frac{1}{\rho+a-1}=0. \tag{6}$$

Here, the complex conjugate zeros should also be paired whenever necessary. By using the well-known relation [1]:

$$\frac{\xi'}{\xi}(z)\Big|_{z=1-a}=-\sum\nolimits_\rho\frac{1}{\rho+a-1}$$

and Theorem 1, we arrive at the following theorem:

Theorem 5. *The Riemann hypothesis is equivalent to the nonnegativity of all derivatives*

$$\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=1-a}$$

for all nonnegative integers n and any real $a<1/2$; hence, it is also equivalent to the nonpositivity of all derivatives

$$\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=1-a}$$

for all nonnegative integers n and any real $a>1/2$.

Remark 3. Another possibility to get the same conclusions is to consider the formula

$$\left|\frac{\rho-a}{\rho+a-1}\right|=\left|\frac{-a+1/2+iT}{a-1/2+iT}\right|=1$$

as a precursor for the conformal mapping $s=\dfrac{z-a}{z+a-1}$. For $a<1/2$ and $\operatorname{Re} z\le 1/2$, the module of s is always less than or equal to 1. This equality is realized only in the line $z=1/2+it$. Hence, on RH, the function $\ln\xi\left(\dfrac{z-a}{z+a-1}\right)$ is analytic in the interior of the disk $|s|<1$. We do not pursue this line of research in the present work (see [2, 3] and our paper [8], where a similar idea was used to generalize the Balazard–Saias–Yor criterion equivalent to the Riemann hypothesis [9]). Similarly, in a more general case $s=\dfrac{z-a}{z+a-2\sigma}$, if $a<\sigma$ and $\operatorname{Re} z\le\sigma$, then the module of s is always less than or equal to 1. At the same time, if $a>\sigma$ and $\operatorname{Re} z\ge\sigma$, then this module is also always less than or equal to 1.

This also illustrates our Theorem 2 (the generalized Bombieri–Lagarias theorem).

Remark 4. In the same way, similar formulas connecting generalized Li's sums and certain derivatives of the logarithm can be established for numerous other zeta-functions. We do not pursue this line of research in the present work.

We now prove the following minor theorem:

Theorem 6. *The statement that there are no nontrivial zeros of the Riemann function with* $\operatorname{Re}\rho > \sigma > 1/2$ *is equivalent to the statement that, for any* $a < \sigma$, *all derivatives*

$$\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2\sigma-a}$$

are nonnegative and, for any $a > 1-\sigma$, *all derivatives*

$$\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2-2\sigma-a}$$

are nonpositive.

Proof. From Theorem 2 (the generalized Bombieri–Lagarias theorem), we know that the condition that there are no nontrivial zeros of the Riemann function with $\operatorname{Re}\rho = \sigma > 1/2$ is equivalent to the statement that, for any $a < \sigma$, all

$$\sum_\rho\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right) \geq 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

These sums are calculated by using Theorem 4 (the generalized Littlewood theorem) in exactly the same way as above with the only difference that, in this case, we use the function

$$\tilde{\tilde{g}}(z) = \frac{n(2\sigma-2a)(z-a)^{n-1}}{(z+a-2\sigma)^{n+1}} - \frac{n(2\sigma-2a)}{(z+a-2\sigma)^2}$$

instead of the function $\tilde{g}(z)$ given by (5). This change yields the equality between the sums

$$\sum_\rho\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right) \geq 0$$

and the derivatives

$$\frac{n(2\sigma-2a)}{(n-1)!}\,\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2\sigma-a}$$

which, consequently, should also be nonnegative.

If there are no zeros with $\operatorname{Re}\rho > \sigma > 1/2$, then there are no zeros with $\operatorname{Re}\rho < 1-\sigma$ and it is possible to apply Theorem 2 with $a > 1-\sigma$. Thus, all corresponding sums are nonnegative:

$$\sum_{\rho}\left(1-\left(\frac{\rho-a}{\rho-2+2\sigma+a}\right)^n\right) \geq 0$$

and given by the formula

$$\frac{n(2(1-\sigma)-2a)}{(n-1)!}\,\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2-2\sigma-a};$$

hence, the derivatives

$$\frac{1}{(n-1)!}\,\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2-2\sigma-a}$$

should be nonpositive.

Theorem 6 is proved.

4. Arithmetic Interpretation of the Generalized Li's Criterion

In [3], Bombieri and Lagarias revealed the relationship between Li's criterion, the so-called Weil explicit formula in the theory of prime numbers, and the Weil criterion for the validity of the Riemann hypothesis (see [10, 11]) and gave an arithmetic interpretation of Li's criterion. Later, interpretations of this kind were given for some other zeta-functions (see, e.g., [12]). For completeness, we would like to conclude our paper by establishing an arithmetic interpretation of the generalized Li's criterion. Here, we closely follow the paper [3].

For a suitable function f, the Mellin transform is defined as $\hat{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx$, while inverse Mellin transform formula gives $f(x) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=c}\hat{f}(s)x^{-s}\,ds$ with an appropriate value of c. The following assertion is actually a repetition of Lemma 2 from [3], which is, in fact, a special case corresponding to $a=1$:

Lemma 1. *For* $n = 1, 2, 3, \ldots$, *and any complex number* a, *the inverse Mellin transform of the function* $k_{n,a}(s) = 1-\left(1-\frac{2a-1}{s+a-1}\right)^n$ *is*

$$g_{n,a}(x) = P_{n,a}(x) \quad \textit{for} \quad 0 < x < 1,$$

$$g_{n,a}(x) = \frac{n}{2}(2a-1) \quad \text{if} \quad x = 1\,, \tag{7}$$

$$g_{n,a}(x) = 0 \qquad \text{if} \quad x > 1\,,$$

where

$$P_{n,a}(x) = x^{a-1} \sum\nolimits_{j=1}^{n} C_n^j \frac{(2a-1)^j \ln^{j-1} x}{(j-1)!} \qquad \text{and} \qquad C_n^j = \frac{n!}{j!(n-j)!}$$

is a binomial coefficient.

Proof. For $\operatorname{Re}(s+a) > 1$, we have

$$\sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!} \int_0^1 (\ln^{j-1} x) x^{s+a-2}\, dx = \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \int_0^1 x^{s+a-2}\, dx$$

$$= \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j (-1)^{j-1}}{(s+a-1)^j} = 1 - \left(1 - \frac{2a-1}{s+a-1}\right)^n .$$

If a is an arbitrary complex number with $\operatorname{Re} a > 1$, then, for the function $g_n(x)$, we can apply the so-called Weil explicit formula (see [3, 10, 11]) in the following form [3]:

$$\sum_{\rho} \hat{f}(\rho) = \int_0^\infty f(x)\,dx + \int_0^\infty \tilde{f}(x)\,dx - \sum_{n=1}^{\infty} \Lambda(n)(f(n) + \tilde{f}(n))$$

$$- (\ln \pi + \gamma) f(1) - \int_1^\infty \left\{ f(x) + \tilde{f}(x) - \frac{2}{x^2} f(1) \right\} \frac{x dx}{x^2 - 1}\,, \tag{8}$$

where $\Lambda(n)$ is a van-Mangoldt function (recall that, for $\operatorname{Re} s > 1$, we have [1]

$$\frac{\varsigma'(s)}{\varsigma(s)} = -\sum\nolimits_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}\Big),$$

$\gamma = 0.572\ldots$ is the Euler–Mascheroni constant, and $\tilde{f}(x) := \dfrac{1}{x} f\left(\dfrac{1}{x}\right)$. Thus, in our case, the function

$$\tilde{P}_{n,a}(x) = x^{-a} \sum\nolimits_{j=1}^{n} C_n^j \frac{(-1)^{j-1} (2a-1)^j \ln^{j-1} x}{(j-1)!}$$

should be used whenever appropriate. Clearly, $\tilde{P}_{n,a}(x)$ is the inverse Mellin transform of

$$k_{n,a}(1-s) = 1 - \left(1 - \frac{2a-1}{a-s}\right)^n .$$

This application is justified because it is easy to see that, for $\operatorname{Re} a > 1$ the functions $g_{n,a}(x)$ have the following necessary properties for Eq. (8) to be used for a function $f(x)$ [3, 10, 11]:

(A) $f(x)$ is continuous and continuously differentiable everywhere except finitely many points a_i at which both $f(x)$ and $f'(x)$ have at most a discontinuity of the first kind, where we set

$$f(a_i) = \frac{1}{2}\left[\, f(a_i+0) + f(a_i-0)\,\right];$$

(B) there is $\delta > 0$ such that $f(x) = O(x^{\delta})$ as $x \to 0+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$.

The use of Eq. (8) gives

$$\begin{aligned}
\sum_{\rho}\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right) &= \sum_{\rho}\left(1-\left(\frac{\rho+a-1}{\rho-a}\right)^n\right) \\
&= \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!}\left\{\int_0^1 x^{a-1}\ln^{j-1} x dx + (-1)^{j-1}\int_1^{\infty} x^{-a}\ln^{j-1} x\, dx \right. \\
&\quad \left. - (-1)^{j-1}\sum_{m=1}^{\infty}\frac{\Lambda(m)\ln^{j-1} m}{m^a}\right\} \\
&\quad - \frac{n}{2}(2a-1)(\ln\pi + \gamma) \\
&\quad - \int_1^{\infty}\left\{\sum_{j=1}^{n} C_n^j \frac{(-1)^{j-1}(2a-1)^{j-1}}{(j-1)!} x^{-a}\ln^{j-1} x - \frac{n}{x^2}(2a-1)\right\}\frac{xdx}{x^2-1} .
\end{aligned} \tag{9}$$

Further, in the second and third integrals on the right-hand side of (9), we perform the change of variables from x to $1/x$. As a result, these integrals take the forms $I_2 = \int_0^1 x^{a-2}\ln^{j-1}(x)\,dx$ and

$$I_3 = \int_0^1\left\{\sum_{j=2}^{n} C_n^j \frac{\ln^{j-1} x}{(j-1)!}(2a-1)^j x^{a-1} + n(2a-1)(x^{a-1}-x)\right\}\frac{dx}{1-x^2} .$$

(Note that, in the expression for I_3, we move the summation term corresponding to $j=1$ from the sum to the second term under the integral sign.) The first two integrals are taken by using Example 4.272.6 in [13]:

$$\int_0^1 \ln^{\mu-1}(1/x)x^{\nu-1}\,dx = \frac{1}{\nu^{\mu}}\Gamma(\mu)\,; \qquad \operatorname{Re}\mu>0 \quad \text{and} \quad \operatorname{Re}\nu>0\,.$$

In our case, we get

$$\int_0^1 \ln^{j-1}(x)x^{a-1}\,dx = \frac{(-1)^{j-1}}{a^j}(j-1)!\,, \qquad \int_0^1 \ln^{j-1}(x)x^{a-2}\,dx = \frac{(-1)^{j-1}}{(a-1)^j}(j-1)!\,.$$

By virtue of Example 3.244.3 in [13], the "second part" of the third integral I_3 is equal to

$$I_{32} = n(2a-1)\int_0^1 \frac{x^{a-1}-x}{1-x^2}\,dx = -\frac{n}{2}(2a-1)(\gamma+\psi(a/2))\,,$$

where ψ is a digamma function. In the first part of this integral, we perform the change of variables $x=\exp(-t)$:

$$I_{31} = \int_0^1 \sum_{j=2}^{n} C_n^j \frac{\ln^{j-1}x}{(j-1)!}(2a-1)^j x^{a-1}\frac{dx}{1-x^2} = \sum_{j=2}^{n} C_n^j(-1)^{j-1}\frac{(2a-1)^j}{(j-1)!}\int_0^\infty t^{j-1}\frac{e^{-at}}{1-e^{-2t}}\,dt\,.$$

Applying the Taylor expansion $(1-e^{-2t})^{-1} = 1+e^{-2t}+e^{-4t}+e^{-6t}+\dots$, we get

$$I_{31} = \sum_{j=2}^{n} C_n^j(-1)^{j-1}\frac{(2a-1)^j}{(j-1)!}\sum_{m=0}^{\infty}\frac{(j-1)!}{(2m+a)^j} = \sum_{j=2}^{n} C_n^j(-1)^{j-1}2^{-j}(2a-1)^j\varsigma(j,a/2)\,,$$

where

$$\varsigma(s,a) := \sum\nolimits_{m=0}^{\infty}\frac{1}{(m+a)^s}$$

is the Hurwitz zeta-function.

By using the relations

$$\sum_{j=1}^{n} C_n^j(-1)^{j-1}(2a-1)^j a^{-j} = 1-\sum_{j=0}^{n} C_n^j(-1)^j\left(\frac{2a-1}{a}\right)^j = 1-\left(-1+\frac{1}{a}\right)^n$$

and

$$\sum_{j=1}^{n} C_n^j (-1)^{j-1}(2a-1)^j (a-1)^{-j} = -1-\left(-1-\frac{1}{a-1}\right)^n$$

and collecting everything together, we arrive at the following theorem:

Theorem 7. *For* $n = 1,2,3,\ldots$ *and any complex a with* $\operatorname{Re} a > 1$, *the following relation is true:*

$$\sum_{\rho}\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right) = \sum_{\rho}\left(1-\left(\frac{\rho+a-1}{\rho-a}\right)^n\right) = 2-\left(-1+\frac{1}{a}\right)^n-\left(-1-\frac{1}{a-1}\right)^n$$

$$+\sum_{j=1}^{n} C_n^j (2a-1)^j \frac{(-1)^j}{(j-1)!}\sum_{m=1}^{\infty}\frac{\Lambda(m)\ln^{j-1} m}{m^a}$$

$$+\frac{n}{2}(2a-1)(\psi(a/2)-\ln\pi)+\sum_{j=2}^{n} C_n^j (-1)^j 2^{-j}(2a-1)^j \varsigma(j,a/2)\,. \qquad (10)$$

Remark 5. The case $n = 1$ of the Theorem 1 gives the following well-known equality:

$$\sum_{\rho}\frac{1}{a-\rho} = \frac{1}{a}+\frac{1}{a-1}-\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^a}+\frac{1}{2}(\psi(a/2)-\ln\pi)$$

(see, e.g., [1]).

The same relationship between Li's criterion and the Weil criterion for the validity of the Riemann hypothesis discussed in [3] takes place also for the generalized Li's criterion. This can be shown as follows:

The multiplicative convolution of the functions $f(x)$ and $g(x)$ satisfying the conditions (A) and (B) formulated above, is defined as

$$(f*g)(x) = \int_0^{\infty} f\left(\frac{x}{y}\right)g(y)\frac{dy}{y}$$

and the Mellin transform of this convolution is $\hat{f}(s)\cdot\hat{g}(s)$. For the multiplicative convolution $f*\bar{\tilde{f}}$ (we use the signs of complex conjugation and the definition $\tilde{f}(x) := \frac{1}{x}f\left(\frac{1}{x}\right)$), the Mellin transform is given by the formula $\hat{f}(s)\cdot\hat{\bar{f}}(1-s)$; this expression is clearly real and positive for $\operatorname{Re} s = 1/2$. Hence, for any function admitting the expression $f*\bar{\tilde{f}}$ in the RH, the sum over the nontrivial Riemann zeros should be positive. Weil showed that this is also a sufficient condition for the RH to be true.

We now recall that if $h(s) = \dfrac{s-a}{s+a-1}$, then $h(1-s) = 1/h(s)$ and, thus,

$$k_{n,a}(s) \cdot k_{n,a}(1-s) = k_{n,a}(s) + k_{n,a}(1-s)\,, \tag{11}$$

where

$$k_{n,a}(s) = 1 - \left(\frac{s-a}{s+a-1} \right)^n .$$

By construction, $k_{n,a}(s) = \hat{g}_{n,a}(s)$ and, in view of the general properties of the Mellin transform, we get $\hat{\tilde{g}}_{n,a}(s) = \hat{g}_{n,a}(1-s)$. Thus, (11) can be rewritten as $\hat{g}_{n,a}(s) \cdot \hat{\tilde{g}}_{n,a}(s) = \hat{g}_{n,a}(s) + \hat{\tilde{g}}_{n,a}(s)$. Hence, by applying the inverse Mellin transform, we find

$$g_{n,a}(x) + \tilde{g}_{n,a}(x) = (g_{n,a} * \tilde{g}_{n,a})(x)$$

This establishes the above-mentioned connection: the right-hand side of Eq. (8) is invariant under the change of $f(x)$ into $\tilde{f}(x)$.

5. Conclusions

Thus, we see that to judge the validity of the Riemann hypothesis, the evaluation of certain derivatives of the Riemann xi-function can be used at any point of the real axis apart from the point $z = 1/2$. In particular, this point can lie arbitrarily far to the right from the critical strip: For arbitrarily large numbers $b > -1/2$, all derivatives

$$\frac{1}{(n-1)!} \frac{d^n}{dz^n} \left((z+b)^{n-1} \ln(\xi(z)) \right) \Big|_{z=b+1}$$

should be nonnegative to guarantee that the RH is true, and vice versa.

The author sincerely hopes that this and other related interesting possibilities might be useful for the Riemann research. Finally, we are also sure that there is a room to use the approach presented in the paper for the investigation of analytic functions other than Riemann functions.

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