Optimal polynomial blow up range for critical wave maps

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Whatever you do in life will be insignificant,
but it is very important that you do it...
— Mahatma Gandhi

To Li Wan, Wenjie Gao and Joachim Krieger…

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Abstract

We prove that the critical Wave Maps equation with target S^2 and origin $\mathbb{R}^{2+1}$ admits energy class blow up solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \varepsilon(t,r)$$

where $Q : \mathbb{R}^2 \to S^2$ is the ground state harmonic map and $\lambda(t) = t^{-1-\nu}$ for any $\nu > 0$. This extends the work [17], where such solutions were constructed under the assumption $\nu > \frac{1}{2}$. Also in the later chapter, we give the necessary remarks and key changes one needs to notice while the same problem is considered in a more general case while $\mathcal{N}$ is a surface of revolution. We are also able to extends the blow-up range in Carstea's work [3] to $\nu > 0$. In light of a result of Struwe [29], our results are optimal for polynomial blow up rates.

Key words: critical wave equation, hyperbolic dynamics, blow-up, scattering, stability, invariant manifold.

Résumé

Nous montrons que l'équation critique de la carte d'onde avec la cible S^2 et l'origine $\mathbb{R}^{2+1}$ admet des solutions pour l'explosion de la classe énergétique de la forme

$$u(t,r) = Q(\lambda(t)r) + \varepsilon(t,r)$$

où $Q : \mathbb{R}^2 \to S^2$ est le plan harmonique de l'état fondamental et $\lambda(t) = t^{-1-\nu}$ pour tout $\nu > 0$. Cela étend le travail [17], où de telles solutions ont été construites sous l'hypothèse $\nu > \frac{1}{2}$. Dans le chapitre suivant, nous offrons les remarques indispensables et les changements nécessaires lorsque le même problème est considéré dans un cas plus général, tandis que $\mathcal{N}$ est une surface de révolution. Nous sommes également en mesure d'étendre l'intervalle d'explosion dans [3] à $\nu > 0$. Compte tenu d'un résultat de Struwe [29], nos résultats sont optimaux pour des taux d'explosion polynomiaux.
Mots clés : équation d'onde critique, dynamique hyperboliques, blow-up, diffusion, stabilité, variété invariante.

Contents

Introduction

More than a hundred years ago, Poincaré wrote in his address to the first ICM (International Congress of Mathematicians) the following

"... The combinations that can be formed with numbers and symbols are an infinite multitude. In this thicket how shall we choose those that are worthy of our attention? Shall we be guided only by whimsy?...This would undoubtedly carry us far from each other, and we would rapidly cease to understand each other. But that is only the minor side of the problem. Not only will physics perhaps prevent us from getting lost, but it will also protect us from a more fearsome danger...turning around forever in circles. History shows that physics has not only forced us to choose from the multitude of problems which arise, but it has also imposed on us directions that would never have been dreamed of otherwise... What would be more useful!"

In that article, he gave a inspiring analysis of the interactions between Mathematics and Physics. Einstein also believe that any important advance in Physics will wake up a new major developments in Mathematics, which he proved that point himself via his marvelous *General Relativity Theory*. From the point of view of mathematicians in this context, in the end of his *Lecture notes on Differential Geometry*, Chern also argued that

"... without the theory of relativity, Riemannian geometry would hardly have enjoyed the status it does among mathematicians."

Looking into the connecting area between Mathematics and Physics, or we call *Mathematical Physics*, PDE is an unavoidable and important subject. However in his article *PDE as a unified subject*, Klainerman argued

"... It is the passage from local to global properties which forces us to abandon any generality and take full advantage of the special features of the important equations...The field of PDE, as a whole, has all but ceased to exist, except in some old fashioned textbooks. What we have instead is a large collection of loosely connected subjects... We can redraw the boundaries between the two subjects (Mathematics and Physics) in a way which allows us to view PDE as a core subject of Mathematics, with an important applied component."

Most of the basic PDEs are derived for the sake of combining the simple first principles with some of the underlying geometric principles of modern Physics. For example the heat, Schrodinger and wave operators $\partial_t - \Delta$, $\frac{1}{i}\partial_t - \Delta$ and $\partial^2 - \Delta$ are very simple evolution operators one can form from a more basic operator, what we call Laplacian Δ. Δ is a fundamental differential operator which is invariant under the group of isometries or rigid transformations of $\mathbb{R}^n$, the Euclidean space. The wave operator, which gives *wave equations*, my area of doctoral study, has a similar way of association to the Minkowski space $\mathbb{R}^{n+1}$ comparing to how Δ associates to $\mathbb{R}^n$. Moreover, the solutions to the equation $\Delta\phi = 0$ can be viewed as a time independent solution to $\Box\phi = 0$, where $\Box = \partial^2 - \Delta$.

Making an agreement that the speed of light is 1, we can write the free wave equation in $\mathbb{R}^n$ as

$$(\partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2) u(t,x) = 0$$

This equation describes the free motion of an n-dimensional surface in an ambient Euclidean space (a string, the surface of a drum, or the atmosphere are examples while $n = 1,2,3$). If we generalize this equation a bit, instead of considering the motion in Euclidean space, we consider the free motion of a point in a Riemannian manifold, then the relevant equation in this context is the geodesic flow equation

$$\nabla_t \frac{du}{dt}(t) = 0$$

where ∇_t is the *covariant derivatives*. The wave map equation

$$\nabla_t \frac{\partial u}{\partial t}(t,x) = \sum_{j=1}^{n} \nabla_{x_j} \frac{\partial u}{\partial x_j}(t,x)$$

is the natural combination of the these equations. The motion of a string that is constrained to lie on a sphere would be given as a *wave map*, as such, wave maps are one of the fundamental equations of geometric motion.

A wave map is formally defined as a map u from $n+1$ dimensional Minkowski space-time with signature $(-1,1,...,1)$ to a Riemannian Manifold $\mathcal{N}$. It is defined as a critical point of the action functional, which is the following Lagrangian

$$\mathcal{L}(u) := \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha u, \partial^\alpha u \rangle_{\mathcal{N}} \, d\sigma, \partial^\alpha = m^{\alpha\beta}\partial_\beta$$

where $\alpha = 0,1,...,n$, and $m^{\alpha\beta}$ is the Minkowski metric.

The wave map $u : \mathbb{R}^{3+1} \to S^3$ has application to *nonlinear sigma model*[14] from quantum field theory in modern physics, so it is very interesting to study the cases when target manifolds are spheres. The case $u : \mathbb{R}^{2+1} \to H^2$ is a model problem arising from the study of *Einstein's equation*[4]. The curvature of the target manifold plays an important role in the global well-posedness properties of the corresponding equation. One interesting fact is that because In the energy critical case (we will explain below what is energy critical) global well-posedness fails for the S^2 target, while it holds for H^2 (see below theorem 0.0.1 and see[20, 21] and references therein). Another important observation is wave maps are the natural hyperbolic analogues of the much studied *harmonic map* heat flow, which in local coordinates is described by

$$\partial_t u^i = \Delta u^i + \sum_{\alpha=1}^{n} \Gamma^i_{jk} \partial_\alpha u^j \partial^\alpha u^k$$

Considering the following model equation

$$\Box u = N(u, \nabla u), \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \tag{0.0.1}$$

for some smooth N(., .). Wave maps in local coordinates fall into this category. Major studies of this problem fall into the following directions: i) local existence theory(strong local well-posedness); ii) small data global existence theory(weak global well posed-ness); iii) approaching the large data problem in the critical dimension n=2 and hyperbolic target; iv) imposing symmetry: radial and equivariant wave maps in the case n=2; v) singularity formation in the critical dimension. For details of the main results upon to those domain, we refer the reader to a very well-written survey paper on wave maps by Krieger [19] and the references therein.

In this paper, we study the blow-up solutions of energy critical co-rotational wave map equation on $\mathbb{R}^{2+1} \to \mathcal{N}$ with polynomial blow-up rate in the case when $\mathcal{N}$ is a surface of revolution. Before we move further, we shall explain first about energy critical and definition of co-rotational.

Scaling constraints. Assume that the set of solutions $u(t,x)$ of (0.0.1) is invariant under the scaling transformation $u(t,x) \to \lambda^\alpha u(\lambda t, \lambda x)$.Then one introduces the *critical Sobolev index* $s_c = \frac{n}{2} - \alpha$. Observe that the norm

$$\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}$$

is left invariant under the re-scaling. Note that

$$s_c = \frac{n}{2}$$

for wave maps in the local coordinate formulation.

Energy constraints. A quantity

$$E[u] \gtrsim \|u\|_{H^{s_0}} + \|u_t\|_{H^{s_0-1}}$$

which is preserved under the flow. Then one distinguishes between: i) energy subcritical $s_c < s_0$: one expects global well-posedness, provided strong local well-posedness in the full subcritical range, or also just for some $s_c < s < s_0$; ii) energy critical $s_c = s_0$: global well-posedness hinges on fine structure of equation; iii) energy supercritical $s_c > s_0$: no global well-posedness for generic large data expected.

Note that when the background is $2+1$-dimensional, wave maps are *energy critical.* This means explicitly the following quantity

$$\mathscr{E}(u) := \int_{\mathbb{R}^2} \left[|u_t|^2 + |\nabla_x u|^2| \right] dx \tag{0.0.2}$$

is invariant under the intrinsic scaling (recall that $s_c = n/2$ in the local coordinate formulation)

$$u(t,x) \to u(\lambda t, \lambda x)$$

Co-rotational wave maps. A wave map $u : \mathbb{R}^{2+1} \to M$ is called *equivariant* provided we have

$$u(t, \omega x) = \rho(\omega) u(t,x), \forall \omega \in S^1$$

Here $\rho(\omega)$ acts as an isometry on M and $\omega \in S^1$ acts on $\mathbb{R}^2$ in the canonical fashion as rotations. For global well-posedness of equivariant wave maps we have the following important results by Shatah, Tahvildar-Zadeh [33]

Theorem 0.0.1 (Shatah, Tahvildar-Zadeh). *Let the target (M, g) be a warped product manifold satisfying a suitable geodesic convexity condition. Then equivariant wave maps $u : \mathbb{R}^{2+1} \to M$ with smooth data stay globally regular.*

However, the case $u : \mathbb{R}^{2+1} \to S^2$ does not satisfy the hypotheses of the preceding theorem. Thus the discovery of the singularity for this case is very crucial. We let S^1 act on S^2 by means of rotations around the z-axis via $\rho(\omega) = k\omega, k \in Z/\{0\}, \omega \in S^1$. Fixing a k, the wave map is then determined in terms of the polar angle, and becomes a scalar equation on $\mathbb{R}^{1+1}$ as follows:

$$-u_{tt} + u_{rr} + \frac{1}{r} u_r = k^2 \frac{\sin(2u)}{2r^2} \tag{0.0.3}$$

The case k = 1 in particular is called *co-rotational.*

The wave maps equation has a remarkable so-called null-structure, as evidenced by its explicit form

$$\Box u = -u_{tt} + \triangle u = -u(-|u_t|^2 + |\nabla_x u|^2),\ u(t,x) \in S^2 \subset \mathbb{R}^3 \tag{0.0.4}$$

This null-structure is responsible for the fact that (0.0.4) enjoys an almost optimal local well-posedness property: from [15], it is known that (0.0.4) is strongly locally well-posed (in the sense of real analytic dependence of the solution on the data) in any space H^s, $s > 1$. On the other hand, from [1], it is known that (0.0.4) is *ill-posed* (however, only in the sense of non-uniform continuous dependence of a local solution on the data) in any $H^s, s < 1$. In the delicate borderline case of data in H^1 (corresponding to the energy (0.0.2)), it is known[1], see

[1]For an earlier result in the equivariant context, see [25].

[30], and more recently [26], that for $s>1$, *H^s-smooth data of small enough energy result in a global H^s-smooth solution.* Furthermore, the solutions scatter at infinity like free waves, provided the initial data are C^∞-smooth and constant outside of a compact set, say. In fact, the recent result [26] furnishes the optimal energy threshold, namely that of the minimum energy non-trivial harmonic map Q from $\mathbb{R}^2 \to S^2$, without any symmetry assumptions on the map. An earlier result [5] derived such a result in the co-rotational context. See also [6], [7] for developments in the context of energy much above the ground state.

M. Struwe's fundamental work [29] on the structure of singularities of co-rotational Wave maps shows that

Theorem 0.0.2 (Struwe)**.** *If u is a smooth co-rotational wave map which cannot be smoothly extended past time T, $\exists t_i \to T$, $\lambda_i \to +\infty$ s.t. on each fixed time slice $t=t_i$, we can write*

$$u(t_i,x) = Q(\lambda(t_i)x) + \varepsilon(t_i,x)$$

where Q is ground state (harmonic map) $Q:\mathbb{R}^2 \to S^2$, while the local energy[2] *of ε converges to 0.*

Furthermore, Struwe established an upper bound on the blow up rate

$$\lim_{i\to\infty} \lambda(t_i)(T-t_i) = +\infty \tag{0.0.5}$$

The other side of the coin is to consider the issue of building a polynomial blow up dynamics for critical co-rotational wave maps from $\mathbb{R}^{2+1}$ into S^2, the standard two-dimensional sphere. Since the work [17], and later [23], it has been known that for any $\varepsilon>0$, there exist initial data[3] of energy $\mathscr{E}(Q)+\varepsilon$ and which lead to finite time singularity formation. See also [24] for blow up solutions with energy $>4\mathscr{E}(Q)$. In fact, the works [17], [23], produced different blow up rates, the former exhibiting a continuum of blow up rates, the latter a more rigid rate but in turn demonstrably stable (within the co-rotational class). The blow up rates exhibited in [17], [23] obey the asymptotic behavior described in [28], and in fact we have

$$\lambda(t) = (T-t)^{-\nu-1}$$

with $\nu > \frac{1}{2}$ for the solutions constructed in [17].

It then remains a very natural question to decide whether in fact all $\nu>0$ are admissible. In this thesis, we provide a positive answer to this. To formulate the main theorem, we recall that co-rotational wave maps may be parametrized in terms of a function $u(t,r)\to\mathbb{R}$ which solves the *scalar wave equation*

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u = \frac{\sin(2u)}{2r^2} \tag{0.0.6}$$

[2] 'local' refers to the area inside the light cone around the singular point.
[3] They may be chosen of any regularity $H^s, s>1$.

In terms of this representation, the ground state harmonic map (which corresponds to a static wave map) is given by

$$Q(r) = 2\arctan r$$

The function $u(t,r)$ is to be thought of as a function on $\mathbb{R}^2$, thus the conserved energy is given by

$$\int_0^\infty [u_t^2 + |u_r|^2 + \frac{\sin^2(u)}{r^2}]\, r\,dr$$

Theorem 0.0.3. *For any $\nu > 0$, there exist $T > 0$[4] and co-rotational initial data (f,g) with*

$$(f-\pi, g) \in H_{\mathbb{R}^2}^{1+\frac{\nu}{2}-} \times H_{\mathbb{R}^2}^{\frac{\nu}{2}-}$$

which result in a[5] solution $u(t,r)$, $t \in (0,T]$ which blows up at time $t=0$ and has the following representation:

$$u(t,r) = Q(\lambda(t)r) + \varepsilon(t,r)$$

where $\lambda(t) = t^{-1-\nu}$, and such that the function

$$(\theta, r) \longrightarrow \big(e^{i\theta}\varepsilon(t,r), e^{i\theta}\varepsilon_t(t,r)\big) \in H^{1+\nu-}(\mathbb{R}^2) \times H^{\nu-}(\mathbb{R}^2)$$

uniformly in t. Also, we have the asymptotic as $t \to 0$

$$\mathcal{E}_{loc}\big(\varepsilon(t,\cdot)\big) \lesssim (t\lambda(t))^{-1}\log^2 t$$

This thesis will be organized as following: in Chapter 1, we give our approach to the main theorem which is following closely the one in [17], with a key modification in the second part which essentially follows [16]. Specifically, we recall that the construction in [17] has two essentially distinct stages:

- In a first stage, we construct an approximate solution, denoted by

  $$u_{approx}(t,r) = Q(\lambda(t)r) + u^e(t,r)$$

  where the correction term $u^e(t,r)$ is obtained by iteratively solving certain 'elliptic approximations' to the wave equation (0.0.6). While $u_{approx}(t,r)$ is not an exact solution of (0.0.6), it is a very accurate solution, in that we can ensure that given $N \geq 0$, we can

[4] Note that we switch between forward and backward light cones from time to time, however, essentially the arguments are in the same context.

[5] Here we use the identification of the wave map with a function $u(t,r)$ as before.

ensure that the error

$$-\partial_{tt}u_{approx}+\partial_{rr}u_{approx}+\frac{1}{r}\partial_r u_{approx}-\frac{\sin(2u_{approx})}{2r^2}=O(t^N).$$

Of course the larger N, the more 'elliptic correction terms' need to be added to $Q(\lambda(t)r)$. It is important to observe here that the restriction $\nu>\frac{1}{2}$ imposed in [17] does not come in at this stage; in fact, any $\nu>0$ will suffice.

- In a second stage, we complete the approximate solution u_{approx} to an exact one by adding a correction term $\varepsilon(t,r)$. This latter correction term is now determined by solving an actual wave equation, albeit one with a time dependent potential term. Dealing with the latter forces one to develop some rather sophisticated spectral theory. To find ε, one implements a fixed point argument in a suitable Banach space, and it is here, in the treatment of the nonlinear terms with singular weights, that the restriction on ν comes in. Indeed, in Lemma 9.5 in [17], the bound (notation to be explained further below)

$$\|R^{-\frac{3}{2}}fg\|_{H_\rho^{\alpha+\frac{1}{4}}}\lesssim\|f\|_{H_\rho^{\alpha+\frac{1}{2}}}\|g\|_{H_\rho^{\alpha+\frac{1}{2}}}$$

  is derived which holds provided $\alpha>\frac{1}{4}$. Since the iterates for ε live naturally in the space $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$, the condition $\nu>\frac{1}{2}$ used in [17] follows.

In the present work, we overcome this restriction as follows:

- First, we analyze the 'zeroth iterate' (to be explained below) for (a suitable variant of)ε, and show that we can split this into the sum of two terms, one of which has a regularity gain which lands us in the regime where the Lemma 9.5 in [17] is applicable, the other of which does not gain regularity but satisfies an a priori L^∞-bound near the symmetry axis $R=0$. Note that the regularity requirement in Lemma 9.5 in [17] comes primarily from the singular weight $R^{-\frac{3}{2}}$ at $R=0$, and so an a priori bound on the (weighted) L^∞ norm will be seen to suffice to estimate an expression such as $R^{-\frac{3}{2}}\varepsilon^2$. Intuitively, the reason why we can control the part of the zeroth iterate near $R=0$ comes from the fact that the singular behavior of the approximate solution from the first part of the construction and the error it generates is localized to the boundary of the light cone.
- Second, by writing the equation for the distorted Fourier transform of (a variant of) ε in a way that subtly differs from the one in [17], we manage to show that the higher iterates all differ from the zeroth iterate by terms with a smoothness gain. This will then suffice to show the desired convergence.

In chapter 2 we will study the same problem in a different or more general case while $\mathcal{N}$ is a surface of revolution with polynomial blow-up rate. We are able to extend the blow-up range in [3] to $\nu>0$. This work relies on and generalizes the result in Chapter 2, where the target manifold is chosen as the standard sphere. The blow-up range we prove there is also optimal for polynomial blow up rates according to the result of Struwe [29].

1 Optimal polynomial blow up range for critical wave maps

1.1 Construction of an approximate solution

Here we shall follow closely the procedure in [17], but also correct for certain (inessential) algebraic errors in the latter reference. In particular, we shall slightly modify the function spaces used (again without any major consequence). Denote

$$R = \lambda(t)r, \lambda(t) = t^{-1-\nu}, \nu > 0$$

Also, write $u_0(R) := Q(R) = 2\arctan R$. The reader should be aware that we are building the approximate solution within the light cone. While picking a small enough neighborhood of 0, $R \sim 0$ refers to the area around t axis while $R \sim \infty$ refers to the inside part light cone.

We state the following, quite analogous to the result in [17]:

Theorem 1.1.1. *Assume $k \in \mathbf{N}$. There exists an approximate solution $u_{2k-1}(R)$ for* (0.0.6) *which can be written as*

$$u_{2k-1}(t,r) = Q(R) + \frac{c_k}{(t\lambda)^2} R\log(1+R^2) + \frac{\widetilde{c}_k}{(t\lambda)^2} R + O\Big(\frac{(\log(1+R^2))^2}{(t\lambda)^2}\Big)$$

with a corresponding error of size[1]

$$e_{2k-1} = (1 - \frac{R}{\lambda t})^{-\frac{1}{2}+\nu} O\Big(\frac{R(\log(1+R^2))^2}{(t\lambda)^{2k}}\Big)$$

Here the implied constant in the $O(\dots)$ symbols are uniform in $t \in (0,\delta]$ for some $\delta = \delta(k) > 0$ sufficiently small.

The construction of this solution follows very closely the treatment in [17]. Specifically, we

[1] The extra factor $(1 - \frac{R}{\lambda t})^{-\frac{1}{2}}$ here arises for $\nu < \frac{1}{2}$, and is not present in [17].

shall arrive at the k-th approximation by adding k correction terms to u_0:

$$u_k = u_0 + \sum_{j=1}^{k} v_j$$

Write

$$e_k = \partial_t^2 u_k - \partial_r^2 u_k - \frac{1}{r}\partial_r u_k + \frac{\sin(2u_k)}{2r^2}$$

From [17] we recall how the correction terms v_k are computed inductively: for each k, we employ a splitting

$$e_k = e_k^0 + e_k^1$$

where e_k^1 denotes certain higher order error terms relegated to a later stage of the inductive process. Then depending on whether k is even or not, we define

$$\big(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2u_0)}{r^2}\big) v_{2k+1} = e_{2k}^0 \tag{1.1.1}$$

$$\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big) v_{2k} = e_{2k-1}^0 \tag{1.1.2}$$

where we impose trivial Cauchy data at $r = 0$[2], resulting in the new error terms

$$e_{2k+1} = e_{2k}^1 - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1}),\ e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k})$$

Here we have introduced the expressions

$$N_{2k}(v) = \frac{1-\cos(2u_{2k-1})}{r^2} v + \frac{\sin(2u_{2k-1})}{2r^2}(1-\cos(2v)) + \frac{\cos(2u_{2k-1})}{2r^2}(2v - \sin(2v)) \tag{1.1.3}$$

$$N_{2k+1}(v) = \frac{\cos(2u_0) - \cos(u_{2k})}{r^2} v + \frac{\sin(2u_{2k})}{2r^2}(1-\cos(2v)) + \frac{\cos(2u_{2k})}{2r^2}(2v - \sin(2v)) \tag{1.1.4}$$

The key fact for this construction is that while (1.1.2) is a wave equation, the ansatz that we will use to construct v_{2k} will allow us to reformulate this problem as a singular elliptic Sturm-Liouville problem, which can be solved by standard ODE methods. It will then be seen that the errors are in fact decreasing near $t = 0$. The main challenge is to control the (increasingly complicated) corrections v_k by placing them in suitable function spaces.

We now define these spaces, implementing very subtle changes compared to [17], in the definition of the ingredients of $S^m(R^k(\log R)^l, \mathcal{Q}_n)$ below:

[2] To be more precise in Step 2 below.

Definition 1.1.2. *For $i \in \mathbf{N}$, let $j(i) = i$ if ν is irrational, respectively $j(i) = 2i^2$ if ν is rational. Then*

- *$\mathcal{Q}$ is the algebra of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:*
  (i) q is analytic in $[0,1)$ with even expansion around $a = 0$.
  (ii) near $a = 1$ we have an absolutely convergent expansion of the form

$$\begin{aligned} q(a) = & q_0(a) + \sum_{i=1}^{\infty}(1-a)^{\beta(i)+\frac{1}{2}} \sum_{j=0}^{j(i)} q_{i,j}(a)\big(\log(1-a)\big)^j \\ & + \sum_{i=1}^{\infty}(1-a)^{\widetilde{\beta}(i)+\frac{1}{2}} \sum_{j=0}^{j(i)} \widetilde{q}_{i,j}(a)\big(\log(1-a)\big)^j \end{aligned}$$

  with analytic coefficients $q_0, q_{i,j}$, and $\beta(i) = i\nu$, $\widetilde{\beta}(i) = \nu i + \frac{1}{2}$.

- *$\mathcal{Q}_n$ is the algebra which is defined similarly, but also requiring $q_{i,j}(1) = 0$ if $i \geq 2n+1$.*

We also define the space of functions obtained by differentiating $\mathcal{Q}_n$:

Definition 1.1.3. *Define $\mathcal{Q}'$ as in the preceding definition but replacing $\beta(i)$ by $\beta'(i) := \beta(i) - 1$, and similarly for $\mathcal{Q}'_n$.*

The next definition also differs slightly from the one in [17], see also [16]:

Definition 1.1.4. *$S^n(R^k(\log R)^l)$ is the class of analytic functions $v : [0,\infty) \to \mathbb{R}$ with the following properties:*
(i) v vanishes of order n at $R = 0$.
(ii) v has a convergent expansion near $R = \infty$

$$v = \sum_{\substack{0 \leq j \leq l+i \\ i \geq 0}} c_{ij} R^{k-i} (\log R)^j$$

Next, introduce the symbols

$$b_1 = \frac{\big(\log(1+R^2)\big)^2}{(t\lambda)^2}, \; b_2 = \frac{1}{(t\lambda)^2}$$

The final function space is also slightly different than the one in [17]:

Definition 1.1.5. *Pick t sufficiently small such that both b_1, b_2, when restricted to the light cone $r \leq t$ are of size at most b_0.*

- *$S^m(R^k(\log R)^l, \mathcal{Q}_n)$ is the class of analytic functions $v : [0,\infty) \times [0,1) \times [0,b_0]^2 \to \mathbb{R}$ so that*
  (i) v is analytic as a function of R, b_1, b_2,

$$v : [0,\infty) \times [0,b_0]^2 \to \mathcal{Q}_n$$

(ii) v vanishes to order m at $R = 0$.
(iii) v admits a convergent expansion at $R = \infty$,

$$v(R, \cdot, b_1, b_2) = \sum_{\substack{0 \le j \le l+i \\ i \ge 0}} c_{ij}(\cdot, b_1, b_2) R^{k-i} (\log R)^j$$

where the coefficients $c_{ij} : [0, b_0]^2 \to \mathcal{Q}_n$ are analytic with respect to $b_{1,2}$.

- $IS^m(R^k (\log R)^l, \mathcal{Q}_n)$ *is the class of analytic functions w inside the cone $r < t$ which can be represented as*

  $$w(t, r) = v(R, a, b_1, b_2),\ v \in S^m(R^k (\log R)^l, \mathcal{Q}_n)$$

  and $t > 0$ sufficiently small.

In the sequel, we shall show inductively that one can choose the corrections v_k to satisfy the following:

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^3\big(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}\big) \tag{1.1.5}$$

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1}\big) \tag{1.1.6}$$

$$v_{2k} \in \frac{1}{(t\lambda)^{2k+2}} IS^3\big(R^3(\log R)^{2k-1}, \mathcal{Q}_k\big) \tag{1.1.7}$$

$$t^2 e_{2k} \in \frac{1}{(t\lambda)^{2k}} \big[IS^1\big(R^{-1}(\log R)^{2k}, \mathcal{Q}_k\big) + \langle b_1, b_2\rangle IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_k\big)\big] \tag{1.1.8}$$

and the starting error e_0 satisfying

$$e_0 \in IS^1(R^{-1})$$

Here we denote by $\langle b_1, b_2\rangle$ the ideal generated by b_1, b_2 inside the algebra generated by b_1, b_2. We first explicitly compute the first and second corrections $v_{1,2}$, and then automate the process for the higher iterates. To begin with, from the calculation in [17], we find

$$e_0 = \frac{1}{t^2}\Big((\nu+1)^2 \frac{4R}{(1+R^2)^2} - \nu(\nu+1)\frac{2R}{1+R^2}\Big)$$

1.1.1 The first correction

If we try to make $u_1 = u_0 + \varepsilon$ an exact solution, then ε needs to solve

$$\big(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r\big)\varepsilon - \frac{\cos(2u_0)}{2r^2}\sin 2\varepsilon + \frac{\sin(2u_0)}{2r^2}(1-\cos(2\varepsilon)) = e_0 \tag{1.1.9}$$

Introduce the operator

$$\widetilde{\mathscr{L}} := \partial_R^2 + \frac{1}{R}\partial_R - \frac{\cos(2u_0)}{R^2} = \partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\frac{1-6R^2+R^4}{(1+R^2)^2}$$

Now if we neglect the time derivatives $-\partial_{tt}\varepsilon$ as well as the nonlinear term $\frac{\sin(2u_0)}{2r^2}(1-\cos(2\varepsilon))$ in (1.1.9) and replace the exact correction ε by an approximate one v_1, we obtain the following relation

$$(t\lambda)^2\widetilde{\mathscr{L}}v_1 = t^2 e_0$$

which is a non-degenerate second order ODE and hence solvable by standard methods. Introduce the conjugated operator $\widetilde{\mathscr{L}}$ by means of

$$-\mathscr{L}(\sqrt{R}v) = \sqrt{R}\widetilde{\mathscr{L}}v$$

Then one has

$$-\mathscr{L} = \partial_R^2 - \frac{3}{4R^2} + \frac{8}{(1+R^2)^2},$$

and a fundamental system for the operator $\mathscr{L}$ is given by (see [17])

$$\phi(R) = \frac{R^{3/2}}{1+R^2} \quad \theta(R) = \frac{-1+4R^2\log R + R^4}{\sqrt{R}(1+R^2)}.$$

With this choice, we have $W(\phi,\theta) = 2$. We have the variation of constants formula

$$(t\lambda)^2 v_1 = \frac{1}{2}R^{-\frac{1}{2}}\theta(R)\int_0^R \phi(R')\sqrt{R'}f(R')\,dR' - \frac{1}{2}R^{-\frac{1}{2}}\phi(R)\int_0^R \theta(R')\sqrt{R'}f(R')\,dR'$$

where we have put $f = t^2 e_0$. Then compute for large R and suitable constants $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$

$$\begin{aligned}
&R^{-1/2}\theta(R)\int_0^R \phi(R')\sqrt{R'}t^2 e_0(R')dR' \qquad (1.1.10)\\
&= \frac{-1+4R^2\log R+R^4}{R(1+R^2)}\int_0^R (c_1 + \frac{c_2}{1+R'^2})\Big(\frac{c_3}{1+R'^2} + \frac{c_4}{(1+R'^2)^2}\Big)d(1+R'^2)\\
&= \frac{-1+4R^2\log R+R^4}{R(1+R^2)}\Big(\frac{c_1}{1+R^2} + \frac{c_2}{(1+R^2)^2} + c_3\log(1+R^2) + c_4\Big)\\
&= d_1 R\log R + d_2 R + d_3 R^{-1}\log^2 R + d_4 R^{-1} + O(R^{-2}\log^2 R).
\end{aligned}$$

and similarly (with re-labelled coefficients)

$$
\begin{aligned}
&R^{-1/2}\phi(R)\int_0^R \theta(R')\sqrt{R'}t^2 e_0(R')dR' \qquad (1.1.11)\\
&= \frac{R}{1+R^2}\int_0^R \frac{R'^4+4R'^2\log R'-1}{1+R'^2}\Big((\nu+1)^2\frac{4R'^3}{(1+R'^2)^2}-\nu(\nu+1)\frac{2R'}{1+R'^2}\Big)dR'\\
&= \frac{R}{(1+R^2)}\int_0^R \Big(c_1+c_2(1+R'^2)+\frac{c_3R'^2\log R'}{1+R'^2}\Big)\Big(\frac{c_5}{1+R'^2}+\frac{c_6}{(1+R'^2)^2}\Big)d(1+R'^2)\\
&= R\Big(\sum_{i=-3}^{0} d_i(1+R^2)^i + d_3\log(1+R^2)+\frac{d_4\log(1+R^2)}{1+R^2}+\frac{d_5\log(1+R^2)}{(1+R^2)^2}+\frac{d_6(\log(1+R^2))^2}{1+R^2}\Big)\\
&+O(R^{-3}\log^2 R)\\
&= e_1 R\log R + e_2 R + e_3\log R + e_4 + O(R^{-1}\log^2 R)
\end{aligned}
$$

Furthermore, since e_0 vanishes to first order at $R=0$, it follows that v_1 vanishes to third order at zero, Combining these observations, we find that indeed

$$v_1 \in \frac{1}{(t\lambda)^2} IS^3(R\log R, \mathcal{Q}_0)$$

as required from (1.1.5).

1.1.2 The error generated after the first correction

Here we calculate $t^2 e_1$. This is given by

$$
\begin{aligned}
t^2 e_1 &= -t^2(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r)(u_0+v_1)+t^2\frac{\sin(2u_0+2v_1)}{2r^2} \qquad (1.1.12)\\
&= t^2\Big[\partial_{tt}v_1-\frac{\sin 2u_0}{2r^2}(1-\cos(2v_1))-\frac{\cos(2u_0)}{2r^2}(2v_1-\sin(2v_1))\Big]\\
&= t^2\partial_{tt}v_1-\frac{\sin 2u_0}{2R^2}(t\lambda)^2(1-\cos(2v_1))-\frac{\cos(2u_0)}{2R^2}(t\lambda)^2(2v_1-\sin(2v_1))
\end{aligned}
$$

Then we use that for $l\geq 1$

$$R^{-2}(t\lambda)^2 v_1^{2l+1} \in \frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_0),\ R^{-2}(t\lambda)^2 v_1^{2l}\in\frac{1}{(t\lambda)^2}IS^3(\log^2 R,\mathcal{Q}_0),$$

which in addition to the fact that u_0 admits an expansion in terms of inverse powers of R near $R=+\infty$ leads to

$$t^2 e_1 \in \frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_0)\subset\frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_0'),$$

as required.

1.1.3 The second correction

Now we intend to add a second correction v_2 in order to reduce the error e_1 from the first stage. More precisely, this time we reduce this error near the light cone. Write $t^2 e_1$ in terms of its expansion at $R = \infty$:

$$t^2 e_1 = \frac{1}{(t\lambda)^2}\big[c_1 R\log R + c_2 R + c_3 \log R + c_4 + O(R^{-1}\log^2 R)\big]$$

for suitable coefficients $c_1, \dots, c_4$. Neglecting the higher order error terms $O(R^{-1}\log^2 R)$, we have to solve the equation

$$t^2\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)v_2 = t^2 e_1^0,$$

where we write

$$t^2 e_1^0 := \frac{1}{(t\lambda)^2}\big[c_1 R\log R + c_2 R + c_3 \log R + c_4\big]$$

Homogeneity considerations suggest making the following ansatz: $v_2 = w_2 + \widetilde{w}_2$, where

$$w_2 = \frac{1}{t\lambda}(W_2^1(a)\log R + W_2^0(a)), \quad \widetilde{w}_2 = \frac{1}{(t\lambda)^2}(\widetilde{W}_2^1(a)\log R + \widetilde{W}_2^0(a)).$$

To obtain the equations for the functions $W_2^1(a)$, we match powers of R and $\log R$. We arrive at the following equations:

$$t^2\Box(\frac{1}{t\lambda}W_2^i(a)) = \frac{1}{t\lambda}(ac_{i+1} - F_i(a)), \quad i = 1, 0 \tag{1.1.13}$$

$$t^2\Box(\frac{1}{(t\lambda)^2}\widetilde{W}_2^i(a)) = \frac{1}{(t\lambda)^2}(c_{i+2} - \widetilde{F}_i(a)), \quad i = 1, 0 \tag{1.1.14}$$

where

$$\Box = -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}$$

as well as

$$F_1(a) = 0, \quad F_0(a) = ((\nu+1)\nu + a^{-2})W_2^1(a) + (a^{-1} - (1+\nu)a)\partial_a W_2^1(a)$$
$$\widetilde{F}_1(a) = 0, \quad \widetilde{F}_0 = (2(\nu+1)\nu + a^{-2})\widetilde{W}_2^1(a) + (a^{-1} - (1+\nu)a)\partial_a \widetilde{W}_2^1(a)$$

We conjugate out the power of t and rewrite the equations in the a variable

$$\mathscr{L}_\nu W_2^i(a) = ac_{i+1} - F_i(a)$$
$$\mathscr{L}_{2\nu}\widetilde{W}_2^i(a) = c_{i+2} - \widetilde{F}_i(a)$$

where the one parameter family of operators $\mathscr{L}_\beta$ is defined by

$$\mathscr{L}_\beta := (1-a^2)\partial_\alpha^2 + (a^{-1} + 2a\beta - 2a)\partial_a + (-\beta^2 + \beta - a^{-2}) \tag{1.1.15}$$

From [17], we know that there exist analytic solutions $W_2^i(a), \widetilde{W}_2^i(a)$ for (1.1.13) on $[0,1)$, such that

$$W_2^i(a),\ i = 0,1,$$

admits an odd power expansion around $a = 0$ starting with a^3, while $\widetilde{W}_2^i(a)$ admits an even expansion around $a = 0$, starting with a^2. Moreover, for a near 1, as shown in [17], we have expansions

$$W_2^1(a) = g_0(a) + g_1(a)(1-a)^{\nu+\frac{1}{2}} + g_2(a)(1-a)^{\nu+\frac{1}{2}}\log(1-a)$$

$$W_2^0(a) = h_0(a) + (1-a)^{\nu+\frac{1}{2}}\sum_{l=0}^{2} h_{l+1}(a)[\log(1-a)]^l + (1-a)^{2\nu+1}h_{l+4}(a)[\log(1-a)]^l,$$

where we have taken into account the most general case (when ν is irrational, there are fewer terms in the expansion). The result for $\widetilde{W}_2^{1,0}(a)$ is of course analogous. The expressions for $w_2, \widetilde{w}_2$ are not quite what we want, since we need ultimately functions which vanish to odd order at $R = 0$, in order to ensure the desired smoothness. Furthermore, we also have the logarithmic factors $\log R$, which of course become singular at $R = 0$. In order to deal with these issues, we now *re-define* the correction terms $w_2, \widetilde{w}_2$ in the following manner:

$$\begin{aligned} w_2 &= \frac{1}{t\lambda}\left(W_2^1(a)\frac{1}{2}\log(1+R^2) + W_2^0(a)\right),\\ \widetilde{w}_2 &= \frac{1}{(t\lambda)^2}\frac{R}{(1+R^2)^{\frac{1}{2}}}\left(\widetilde{W}_2^1(a)\frac{1}{2}\log(1+R^2) + \widetilde{W}_2^0(a)\right). \end{aligned}$$

Writing

$$\frac{1}{t\lambda}W_2^{1,0}(a) = \frac{1}{(t\lambda)^2}RZ_2^{1,0}(a)$$

where now $Z_2^{1,0}(a) \in \mathscr{Q}_1$, while from construction we have $\widetilde{W}_2^0(a) \in \mathscr{Q}_1$, and observing that $Z_2^{1,0}(a), \widetilde{W}_2^0(a)$ vanish quadratically at $a = 0$ we see that

$$v_2 = w_2 + \widetilde{w}_2 \in \frac{1}{(t\lambda)^4} IS^3(R^3\log R, \mathscr{Q}_1),$$

as required.

1.1.4 The error generated after the second correction v_2

We write $u_2 = u_1 + v_2 = u_0 + v_1 + w_2 + \widetilde{w}_2$, and need to estimate

$$t^2 e_2 = t^2(e_1 - e_1^0) + t^2\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)v_2 - t^2 e_1^0 + t^2 N_2(v_2)$$

We check that each of the terms on the right satisfies the property (1.1.8) with $k = 1$.

(1): The contribution of $t^2(e_1 - e_1^0)$. From our choice of e_1^0, we immediately get

$$t^2(e_1 - e_1^0) \in \frac{1}{(t\lambda)^2} IS^1(R^{-1}(\log R)^2, \mathcal{Q}_1)$$

(2): The contribution of $t^2\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)v_2 - t^2 e_1^0$. This error is produced by replacing $\log R$ by $\frac{1}{2}\log(1+R^2)$, as well as by including the factor $\frac{R}{(1+R^2)^{\frac{1}{2}}}$. Thus we write this contribution as[3]

$$\begin{aligned}
&t^2\Box\big[\frac{1}{t\lambda}W_2^1(a)(\frac{1}{2}\log(1+R^2) - \log R)\big]\\
&+ t^2\Box\big[\frac{1}{(t\lambda)^2}\frac{R}{(1+R^2)^{\frac{1}{2}}}(\widetilde{W}_2^1(a)(\frac{1}{2}\log(1+R^2) - \log R)\big]\\
&+ t^2\Box'\big[\frac{1}{(t\lambda)^2}\frac{R}{(1+R^2)^{\frac{1}{2}}}\widetilde{W}_2^1(a)\frac{1}{2}\log(1+R^2)\big]\\
&+ \big(\frac{R}{(1+R^2)^{\frac{1}{2}}} - 1\big)t^2 e_1^0
\end{aligned}$$

where the notation $\Box'$ means that at least one derivative falls on the factor $\frac{R}{(1+R^2)^{\frac{1}{2}}}$. Since $\frac{1}{2}\log(1+R^2) - \log R = O(R^{-2})$ as $R \to \infty$, and since $W_2^1(a)$ vanishes to third order at $a = 0$, we obtain easily that the first three expressions are in the space

$$\frac{1}{(t\lambda)^2} IS^1(R^{-1}, \mathcal{Q}_1')$$

and since $\frac{R}{(1+R^2)^{\frac{1}{2}}} - 1 = O(R^{-2})$, the same applies to the last term above. This is not quite of the form required for (1.1.8). However, we can rectify this by writing as in [17] for any $t^2 e \in \frac{1}{(t\lambda)^2} IS^1(R^{-1}, \mathcal{Q}_1')$

$$t^2 e = (1 - a^2)t^2 e + \frac{R^2}{(t\lambda)^2} t^2 e$$

This implies

$$\frac{1}{(t\lambda)^2} IS^1(R^{-1}, \mathcal{Q}_1') \subset \frac{1}{(t\lambda)^2} IS^1(R^{-1}, \mathcal{Q}_1) + b_2 \frac{1}{(t\lambda)^2} IS^1(R, \mathcal{Q}_1')$$

[3] Recall that $\Box = -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}$.

(3): The contribution of $t^2 N_2(v_2)$. Recall from (2.2.7) that we need to control three terms. First, we have

$$\begin{aligned}
& t^2 \frac{1-\cos(2u_1)}{r^2} v_2 \\
&= \frac{(t\lambda)^2}{R^2}\Big(S^1(R^{-1},\mathcal{Q}_1) + \frac{1}{(t\lambda)^2} S^3(R\log R,\mathcal{Q}_1)\Big)^2 \times \frac{1}{(t\lambda)^4} S^3(R^3(\log R),\mathcal{Q}_1) \\
&\in \frac{1}{(t\lambda)^2}\Big(S^3(R^{-1}(\log R),\mathcal{Q}_1) + \frac{1}{(t\lambda)^2} S^5(R(\log R)^2,\mathcal{Q}_1) + \frac{1}{(t\lambda)^4} S^7(R^3(\log R)^3,\mathcal{Q}_1)\Big) \\
&\subset \frac{1}{(t\lambda)^2}\Big(S^3(R^{-1}(\log R),\mathcal{Q}_1) + \frac{\langle b_1,b_2\rangle}{(t\lambda)^2} S^5(R(\log R),\mathcal{Q}_1)\Big),
\end{aligned}$$

as required. Further, just as in [17], one checks that

$$t^2 \frac{\sin(2u_1)}{2r^2}(1-\cos(2v_2)) \in \frac{1}{(t\lambda)^2}(S^1(R^{-1}(\log R)^2,\mathcal{Q}_1) + \langle b_1,b_2\rangle S^3(R(\log R),\mathcal{Q}_1))$$

and finally for the the cubic term

$$t^2 \frac{\cos(2u_1)}{2r^2}(2v_2 - \sin(2v_2)) \in \frac{\langle b_1,b_2\rangle}{(t\lambda)^2} S^1(R(\log R),\mathcal{Q}_1).$$

Combining all we have now, we conclude

$$t^2 e_2 \in \frac{1}{(t\lambda)^2}\big[S^1(R^{-1}(\log R)^2,\mathcal{Q}_1) + \langle b_1,b_2\rangle S^1(R(\log R),\mathcal{Q}_1)\big],$$

thus verifying (1.1.8) for $k=1$.

1.1.5 The higher order corrections v_k, $k \geq 3$.

Here we repeat the preceding steps, making sure that the corrections and errors remain in the appropriate function spaces. We essentially use the same inductive procedure as in [17], with the same subtle changes as before. We do this in a number of steps:

Step 1: *Given* u_{2k-2} *with generating error* $e_{2k-2}, k \geq 2$*, as in* (1.1.8)*, choose* v_{2k-1} *as in* (1.1.5) *with error* e_{2k-1} *satisfying* (1.1.6).

This is accomplished exactly as in **Steps 1,2** in [17]. We repeat them here to make this work self-consistent. The following arguments mimic those in [17].

If $k \geq 3$, we define the principal part e_{2k-2} of e^0_{2k-2} by letting $b=0$, i.e.,

$$e^0_{2k-2}(R,a) := e_{2k-2}(R,a,0)$$

we use to write

$$e_{2k_2} = e^0_{2k-2} + e^1_{2k-2}$$

where

$$\begin{aligned}
&t^2 e^0_{2k-2} \in \frac{1}{(t\lambda)^{2k-2}} IS^1(R^{-1}(\log R)^{2k-2}, \mathcal{Q}_{k-1}) \\
&t^2 e^1_{2k-2} \in \frac{b}{(t\lambda)^{2k-2}} \left[IS^1(R^{-1}(\log R)^{2k-2}, \mathcal{Q}_{k-1}) + IS^1(R(\log R)^{2k-3}, \mathcal{Q}'_{k-1})\right] \\
&\subset \frac{1}{(t\lambda)^{2k}} (R(\log R)^{2k-1}, \mathcal{Q}'_{k-1})
\end{aligned}$$

we can see that e^1_{2k-2} can be included in e_{2k-1}. We define v_{2k-1} by neglecting the a dependence of e^0_{2k-2}. In other words, a is treated as a parameter. Changing variables to R we need to solve the same equation

$$(t\lambda)^2 \widetilde{\mathcal{L}}(v_{2k-1}) = t^2 e^0_{2k-2} \in \frac{1}{(t\lambda)^{2k-2}} IS^1(R^{-1}(\log R)^{2k-2}, \mathcal{Q}_{k-1})$$

where the operator was already defined above. Then (1.1.5) is the consequence of the following ODE lemma which has been proved and identical to lemma 3.7 in [17]

Lemma 1.1.6. *Let* $k \geq 1$, *then the solution* v *to the equation*

$$\widetilde{\mathcal{L}} v = f \in S^1(R^{-1}(\log R)^{2k-2}), \quad v(0) = v'(0) = 0$$

has the regularity

$$v \in S^3(R(\log R)^{2k-1}).$$

Next we show that if v_{2k-1} is chosen as above then (1.1.6) holds. Thinking of v_{2k-1} as a function of t, R and a, we can write e_{2k-1} in the form

$$e_{2k-1} = N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

Here $N_{2k-1}(v_{2k-1})$ accounts for the contribution from the nonlinearity and is given by 2.2.7. $E^t v_{2k-1}$ contains the terms in $\partial_t^2 v_{2k-1}(t, R, a)$ where no derivative of variable a is applied, while $E^a v_{2k-1}$ contains the terms in $(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r) v_{2k-1}(t, R, a)$ where at least one derivative applies to the variable a. The analysis will be the same as in [17], we briefly recall the main results here. For the terms in N_{2k-1}, summing over v_j over $1 \leq j \leq 2k-2$

$$u_{2k-2} - u_0 \in \frac{1}{(t\lambda)^2} IS^3(R \log R, \mathcal{Q}_{k-1}).$$

And we need the following lemma to switch to trigonometric functions, which is identical to lemma 3.8 in [17]:

Lemma 1.1.7. *Let*

$$v \in \frac{1}{(t\lambda)^2} IS^3(R \log R, \mathcal{Q}_{k-1}).$$

Then

$$\sin v \in \frac{1}{(t\lambda)^2} IS^3(R\log R, \mathcal{Q}_{k-1}), \quad \cos v \in IS^0(1, \mathcal{Q}_{k-1}).$$

We repeat the results of computation from [17] here

$$\begin{aligned}
&t^2 \frac{\cos(2u_0) - \cos(2u_{2k-2})}{r^2} v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^5(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}) \\
&t^2 \frac{\sin(2u_{2k-2})}{2r^2}(1 - \cos(2v_{2k-1})) \in \frac{1}{(t\lambda)^{2k}} (R(\log R)^{2k-1}, \mathcal{Q}_{k-1}) \\
&t^2 \frac{\cos(2u_{2k-2})}{r^2}(2v_{2k-1} - \sin(2v_{2k-1})) \in \frac{1}{(t\lambda)^{2k}} IS^7(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})
\end{aligned}$$

this concludes the analysis of $N_{2k-1}(v_{2k-1})$. To continue with the terms in $E^t v_{2k-1}$, we can neglect the a dependence. Therefore, it suffices to compute

$$t^2 \partial_t^2 \Big(\frac{1}{(t\lambda)^{2k}} IS^3(R(\log R)^{2k-1}) \Big) \subset \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}).$$

Finally, one needs to consider the terms in $E^a v_{2k-1}$. For

$$v_{2k-1}(t,r) = \frac{1}{(t\lambda)^{2k}} w(R,a), \quad w \in S^3(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})$$

we get

$$\begin{aligned}
&t^2 E^a v_{2k-1} = \frac{1}{(t\lambda)^{2k}} \big[2k\nu a w_a(R,a) - (\nu+1) R a w_{Ra}(R,a) + 2Ra^{-1} w_{Ra}(R,a) \\
&+ a^{-1} w_a(R,a) + (1-a^2) w_{aa}(R,a) - a w_a(R,a) \big].
\end{aligned}$$

Because $\mathcal{Q}_{k-1}$ are even in a, we conclude

$$a\partial_a, a^{-1}\partial_a, (1-a^2)\partial_a^2 : \mathcal{Q}_{k-1} \to \mathcal{Q}'_{k-1}.$$

Meanwhile the R^{-1} factor simply removes one order of vanishing at $R = 0$. In the end, we obtain

$$t^2 E^a v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1}).$$

which concludes the proof of (1.1.6).

Step 2: *Given* e_{2k-1} *as in* (1.1.6), *construct* v_{2k} *as in* (1.1.7)

Here we have to diverge slightly from [17], since our definition of the algebra $S^m(R^l \log R^l)$ is different. Thus assume

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1})$$

is given. We begin by isolating the leading component e^0_{2k-1} which includes the terms of top degree in R as well as those of one degree less (which is where we differ from [17]). Thus we write

$$t^2 e^0_{2k-1} = \frac{R}{(t\lambda)^{2k}} \sum_{j=0}^{2k-1} q_j(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{q}_j(a)(\log R)^j, \quad q_j, \widetilde{q}_j \in \mathcal{Q}'_{k-1}$$

which we can rewrite as

$$t^2 e^0_{2k-1} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} a q_j(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{q}_j(a)(\log R)^j$$

Consider the following equation

$$t^2 \Box(v_{2k}) = t^2 e^0_{2k-1}.$$

Homogeneity considerations suggest that we should look for a solution v_{2k} which has the form

$$v_{2k} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} W^j_{2k}(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{W}^j_{2k}(a)(\log R)^j$$

The one-dimensional equations for W^j_{2k}, $\widetilde{W}^j_{2k}$ are obtained by matching the powers of $\log R$. We get the following systems

$$\begin{aligned}
&t^2 \Box\Big(\frac{1}{(t\lambda)^{2k-1}} W^j_{2k}(a)\Big) = \frac{1}{(t\lambda)^{2k-1}} (a q_j(a) - F_j(a)) \\
&t^2 \Box\Big(\frac{1}{(t\lambda)^{2k}} \widetilde{W}^i_{2k}(a)\Big) = \frac{1}{(t\lambda)^{2k}} (\widetilde{q}_i(a) - \widetilde{F}_i(a))
\end{aligned}$$

where $F_j(a)$, $\widetilde{F}_i(a)$ are

$$\begin{aligned}
&F_j(a) = (j+1)[((\nu+1)\nu(2k-1) + a^{-2}) W^{j+1}_{2k} + (a^{-1} - (1+\nu) a \partial_a W^{j+1}_{2k})] \\
&+ (j+2)(j+1)((\nu+1)^2 + a^{-2}) W^{j+1}_{2k} \\
&\widetilde{F}_i(a) = (i+1)[(2(\nu+1)\nu k + a^{-2}) W^{i+1}_{2k} + (a^{-1} - (1+\nu) a \partial_a W^{i+1}_{2k})] \\
&+ (i+2)(i+1)((\nu+1)^2 + a^{-2}) W^{i+1}_{2k}
\end{aligned}$$

Here we make the convention that $W^j_{2k}(a), \widetilde{W}^i_{2k} = 0$ for $j \geq 2k$ and $i \geq 2k+1$. Then we solve the systems successively for decreasing values of j, i. Conjugating out the power of t we get

$$\begin{aligned}
&t^2 \Big(- \Big(\partial_t + \frac{(2k-1)\nu}{t}\Big)^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big) W^j_{2k}(a) = a q_j - F_j(a) \\
&t^2 \Big(- \Big(\partial_t + \frac{2k\nu}{t}\Big)^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big) \widetilde{W}^i_{2k}(a) = \widetilde{q}_i - \widetilde{F}_i(a)
\end{aligned}$$

with the definition of $\mathscr{L}_\beta$ we give in (1.1.15), we rewrite them in the a variable

$$\mathscr{L}_{(2k-1)\nu} W_{2k}^j = a q_j(a) - F_j(a)$$
$$\mathscr{L}_{2k\nu} \widetilde{W}_{2k}^i = \widetilde{q}_i(a) - \widetilde{F}_i(a)$$

we claim that solving this system with Cauchy data at $a=0$ yields solutions which satisfy

$$W_{2k}^j(a) \in a^3 \mathscr{Q}_k, \quad j = \overline{0, 2k-1}$$
$$\widetilde{W}_{2k}^i \in a^2 \mathscr{Q}_k, \quad i = \overline{0, 2k}$$

and this claim is established as in the computation of v_2 above, we repeat lemma 3.9 from [17].

Lemma 1.1.8. *Let $0 \le m(j) \lesssim j^2$. Let f be an analytic function in $[0,1)$ with an odd expansion at 0 and on absolutely convergent expansion near $a = 1$ of the form*

$$f(a) = f_0(a) + \sum_{j=1}^{\infty} \Big[(1-a)^{(2j-1)\nu - \frac{1}{2}} \sum_{m=0}^{m(2j-1)} f_{2j-1,m}(a) [\log(1-a)]^m + (1-a)^{2j\nu} \sum_{m=0}^{m(2j)} f_{2j,m}(a) [\log(1-a)]^m \Big] \tag{1.1.16}$$

with f_{ij} analytic near $a = 1$. Then there is a unique solution w to the equation

$$\mathscr{L}_{(2k-1)\nu} w = f, \quad w(0) = 0, \partial_a w(0) = 0 \tag{1.1.17}$$

with the following properties:
(i) w is an analytic function in $[0,1)$
(ii) w is cubic at 0 and has an odd expansion at 0
(iii) w has an absolutely convergent expansion near $a = 1$ of the form

$$w(a) = w_0(a) + \sum_{j=1}^{\infty} \Big[(1-a)^{(2j-1)\nu + \frac{1}{2}} \sum_{l=0}^{l(2j-1)} w_{2j-1,l}(a) [\log(1-a)]^l + (1-a)^{2j\nu+1} \sum_{l=0}^{l(2j)} w_{2j,l}(a) [\log(1-a)]^l \Big] \tag{1.1.18}$$

with $w_{i,j}$ analytic near $a = 1$ and $l(i) = m(i)$ with one exception, namely $l(2k-1) = m(2k-1)+1$. If however $f_{2k-1,m(2k-1)}(1) = 0$, then $l(2k-1) = m(2k-1)$. In that case also $w_{2k-1,l} = 0$ if $l > 0$, but not necessarily when $l = 0$. Finally, if $f_{2i-1,j}(1) = 0$ for all $i > k$ and all j, then also $w_{2i-1,l}(1) = 0$ for all $i > k$ and all l.

We need to make a adjustment for v_{2k} because of the singularity of $\log R$ at $R = 0$. Also, we need to make sure that v_{2k} has order 3 vanishing at $R=0$. Thus we define v_{2k} as

$$v_{2k}:=$$
$$\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}W_{2k}^{j}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^{j}+\frac{1}{(t\lambda)^{2k}}\frac{R}{(1+R^2)^{\frac{1}{2}}}\sum_{j=0}^{2k}\widetilde{W}_{2k}^{j}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^{j}$$

By doing this we get a large error near $R=0$, but it is not very significant since the purpose of the correction is to improve the error for large R. Since $a=R/t\lambda$, it's easy to pull out a a^3 factor from W's and a^2 from $\widetilde{W}$'s to see that we have (1.1.7).

Step 3: *Show that the error e_{2k} generated by $u_{2k}=u_{2k-1}+v_{2k}$ satisfies* (1.1.8). Write

$$t^2e_{2k}=t^2(e_{2k-1}-e_{2k-1}^0)+t^2\big(e_{2k-1}^0-\Box(v_{2k})\big)+t^2N_{2k}(v_{2k})$$

where we recall (2.2.7). We begin with the first term on the right, which has the form

$$t^2(e_{2k-1}-e_{2k-1}^0)\in\frac{1}{(t\lambda)^{2k}}[IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1}')+\langle b_1,b_2\rangle IS^1(R(\log R)^{2k-1},\mathcal{Q}_{k-1}')]$$

and we conclude by observing that

$$IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1}')\subset IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1})+\langle b_1,b_2\rangle IS^1(R(\log R)^{2k-1},\mathcal{Q}_{k-1}')$$

. The reason for this is that for $w\in IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1}')$ we can write

$$w=(1-a^2)w+\frac{1}{(t\lambda)^2}R^2w.$$

For the second term in the definition of t^2e_{2k}, we have that by the same computation as in *(2)* of the preceding subsection

$$t^2\big(e_{2k-1}^0-\Box(v_{2k})\big)\in\frac{1}{(t\lambda)^{2k}}\big[IS^1\big(R^{-1}(\log R)^{2k},\mathcal{Q}_k\big)+\langle b_1,b_2\rangle[IS^1\big(R(\log R)^{2k-1},\mathcal{Q}_k'\big)$$

Finally, for the contribution of $t^2N_{2k}(v_{2k})$, we use as in [17] that

$$u_{2k-1}-u_0\in\frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_k)$$

and, reasoning as in [17], we find

$$
\begin{aligned}
& t^2 \frac{1-\cos(2u_{2k-1})}{r^2} v_{2k} \\
& \in \frac{1}{(t\lambda)^{2k}} \Big(IS^3(R^{-1}(\log R)^{2k-1}, \mathcal{Q}_k) + \frac{\langle b_1, b_2\rangle}{(t\lambda)^2} IS^5(R(\log R)^{2k-1}, \mathcal{Q}_k) \Big) \\
& t^2 \frac{\sin(2u_{2k-1})}{2r^2} (1-\cos(2v_{2k})) \\
& \in \frac{1}{(t\lambda)^{2k}} \Big(IS^1(R^{-1}(\log R)^{2k}, \mathcal{Q}_k) + \langle b_1, b_2\rangle IS^3(R(\log R)^{2k-1}, \mathcal{Q}_k) \Big) \\
& t^2 \frac{\cos(2u_{2k-1})}{2r^2} (2v_{2k} - \sin(2v_{2k})) \in \frac{\langle b_1, b_2\rangle}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_k)
\end{aligned}
$$

This shows that e_{2k} has the desired form.

Iteration of **Step 1** - **Step 3** immediately furnishes the proof of Theorem 2.1.1 .

1.2 The spectral analysis of the underlying strongly singular Sturm-Liouville operator

Here we gather the spectral theory needed for the construction of the precise solution. This is a summary of results contained in [17]. It is almost identical of the relevant section in [17]. In the sequel, we shall often invoke the operator

$$\mathcal{L} := -\frac{d^2}{dr^2} + \frac{3}{4r^2} - \frac{8}{(1+r^2)^2}$$

acting on (a subspace of) $L^2(0,\infty)$. The actual domain is given by

$$\mathrm{Dom}(\mathcal{L}) = \{f \in L^2(0,\infty) : f, f' \in AC_{loc}\big((0,\infty)\big), \mathcal{L}f \in L^2\big((0,\infty)\big)$$

The operator $\mathcal{L}$ is then self-adjoint with this domain. The spectrum $\mathrm{spec}(\mathcal{L}) = [0,\infty)$ is purely absolutely continuous. We have the following key results, identically stated and proved in [17].

Remark 1.2.1. Here we will repeat all the proofs, which are identical to those in [17] to make our work self-consistent, for those readers who are already familiar with these results or who want to move on faster to see the construction of the precise solutions, this section can be treated as a black box.

Theorem 1.2.2. *(a) For each $z \in \mathbf{C}$ there exists a fundamental system $\phi(r,z), \theta(r,z)$ for $\mathcal{L} - z$ which is analytic in z for each $r > 0$ and has the asymptotic behavior*

$$\phi(r,z) \sim r^{\frac{3}{2}}, \theta(r,z) \sim \frac{1}{2} r^{-\frac{1}{2}}, as\, r \to 0.$$

In particular, their Wronskian $W(\phi(\cdot,z), \theta(\cdot,z)) = 1$ for all $z \in \mathbf{C}$. Here $\phi(\cdot,z)$ is the Weyl-

Titchmarsh solution of $\mathscr{L} - z$ *at* $r = 0$. *The functions* $\phi(\cdot, z)$, $\theta(\cdot, z)$ *are real valued for* $z \in \mathbb{R}$.
(b) For each $z \in \mathbf{C}$, $\operatorname{Im} z > 0$, *let* $\psi^+(r, z)$ *denote the Weyl-Titchmarsh solution of* $\mathscr{L} - z$ *at* $r = +\infty$, *normalized such that*

$$\psi^+(r, z) \sim z^{-\frac{1}{4}} e^{iz^{\frac{1}{2}} r} \text{ as } r \to +\infty, \ \operatorname{Im} z^{\frac{1}{2}} > 0.$$

If $\xi > 0$, *then the limit* $\psi^+(r, \xi + i0)$ *exists point-wise for all* $r > 0$ *and we denote it by* $\psi^+(r, \xi)$. *Moreover, define* $\psi^-(\cdot, \xi) = \overline{\psi^+(\cdot, \xi)}$. *Then* $\psi^+(r, \xi), \psi^-(r, \xi)$ *form a fundamental system of* $\mathscr{L} - \xi$ *with asymptotic behavior* $\psi^{\pm}(r, \xi) \sim \xi^{-\frac{1}{4}} e^{\pm i \xi^{\frac{1}{2}} r}$ *as* $r \to \infty$.
(c) The spectral measure of $\mathscr{L}$ *is absolutely continuous and its density is given by*

$$\rho(\xi) = \frac{1}{\pi} \operatorname{Im} m(\xi + i0) \chi_{\xi > 0}$$

with the generalized Weyl-Titchmarsh function

$$m(\xi) = \frac{W(\theta(\cdot, \xi), \psi^+(\cdot, \xi))}{W(\psi^+(\cdot, \xi), \phi(\cdot, \xi))}.$$

(d) The distorted Fourier transform defined as

$$\mathscr{F} : f \to \widehat{f}(\xi) := \lim_{b \to \infty} \int_0^b \phi(r, \xi) f(r) \, dr$$

is a unitary operator from $L^2(\mathbb{R}_+)$ *to* $L^2(\mathbb{R}_+, \rho)$, *and its inverse is given by*

$$\mathscr{F}^{-1} : \widehat{f} \to f(r) = \lim_{\mu \to \infty} \int_0^\mu \phi(r, \xi) \widehat{f}(\xi) \rho(\xi) \, d\xi$$

Here $\lim$ *refers to the* $L^2(\mathbb{R}_+, \rho)$, *respectively the* $L^2(\mathbb{R}_+)$ *limit.*

The next two propositions detail the precise analytic structure of the functions $\phi(r, z)$, $\psi^{\pm}(r, z)$. This will be pivotal for our arguments below.

Theorem 1.2.3. *([17]) For any* $z \in \mathbf{C}$, *the fundamental system* $\phi(r, z)$, $\theta(r, z)$ *from the preceding theorem admits absolutely convergent asymptotic expansions*

$$\phi(r, z) = \phi_0(r) + r^{-\frac{1}{2}} \sum_{j=1}^{\infty} (r^2 z)^j \phi_j(r^2)$$

$$\theta(r, z) = r^{-\frac{1}{2}} \frac{1}{2} \big(1 - r^2 - \sum_{j=1}^{\infty} (r^2 z)^j \theta_j(r^2)\big) - (2 + \frac{4}{z}) \phi(r, z) \log r$$

where the functions ϕ_j, θ_j *are holomorphic in* $U = \{\operatorname{Re} u > -\frac{1}{2}\}$ *and satisfy the bounds*

$$|\phi_j(u)| \leq \frac{3C^j}{(j-1)!} \log(1 + |u|), \ |\phi_1(u)| > \frac{1}{2} \log u \ \text{ if } u \gg 1$$

$$|\theta_1(u)| \leq C|u|, |\theta_j(u)| \leq \frac{C^j}{(j-1)!}(1+|u|), \, u \in U.$$

Furthermore,

$$\phi_1(u) = -\frac{1}{4}\log u + \frac{1}{2} + O(u^{-1}\log^2 u) \text{ as } u \to \infty,$$

as well as

$$\phi_1(u) = -\frac{u}{8} + \frac{u^2}{12} + O(u^3) \text{ as } u \to 0.$$

As for the functions $\psi^{\pm}(r,z)$, they admit Hankel expansions at infinity, as follows:

Theorem 1.2.4. *([17]) For any $\xi > 0$, the solution $\psi^+(\cdot,\xi)$ from Theorem 1.2.2 is of the form*

$$\psi^+(r,\xi) = \xi^{-\frac{1}{4}} e^{ir\xi^{\frac{1}{2}}} \sigma(r\xi^{\frac{1}{2}}, r), \, r\xi^{\frac{1}{2}} \gtrsim 1,$$

where σ admits the asymptotic series approximation

$$\sigma(q,r) \sim \sum_{j=0}^{\infty} q^{-j}\psi_j^+(r), \, \psi_0^+ = 1, \, \psi_1^+ = \frac{3i}{8} + O(\frac{1}{1+r^2})$$

with zero order symbols $\psi_j^+(r)$ that are analytic at infinity,

$$\sup_{r>0} |(r\partial_r)^k \psi_j^+(r)| < \infty$$

in the sense that for all large integers j_0, and all indices α, β, we have

$$\sup_{r>0} |(r\partial_r)^{\alpha}(q\partial_q)^{\beta}\big[\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\big]| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}$$

for all $q > 1$.

Finally, the structure of the spectral measure is given by the following

Theorem 1.2.5. *([17]) (a) We have*

$$\phi(r,\xi) = a(\xi)\psi^+(r,\xi) + \overline{a(\xi)\psi^+(r,\xi)},$$

where a is smooth, always nonzero, and has size

$$|a(\xi)| \sim -\xi^{\frac{1}{2}}\log\xi, \, \xi \ll 1, \, |a(\xi)| \sim \xi^{-\frac{1}{2}}, \, \xi \gtrsim 1$$

Moreover, it satisfies the symbol bounds

$$|(\xi\partial\xi)^k a(\xi)| \leq c_k |a(\xi)|, \, \forall \xi > 0.$$

(b) The spectral measure $\rho(\xi)d\xi$ has density

$$\rho(\xi) = \frac{1}{\pi}|a(\xi)|^{-2}$$

and therefore satisfies

$$\rho(\xi) \sim \frac{1}{\xi \log^2 \xi}, \xi \ll 1, \rho(\xi) \sim \xi, \xi \gtrsim 1.$$

1.3 Construction of the precise solution

Our point of departure here is the assumption that an approximate solution u_{2k-1}, $k \gg 1$, has been constructed, with a corresponding error term e_{2k-1} which decays rapidly in the renormalized time $\tau := \nu^{-1} t^{-\nu}$. Note that with respect to this time, we get

$$\lambda(\tau) := \lambda(t(\tau)) = (\nu\tau)^{\frac{1+\nu}{\nu}}$$

We also have the re-scaled variable $R = \lambda(\tau)r$. We shall assume that

$$|e_{2k-1}(t,r)| \lesssim \tau^{-N}, r \leq t$$

for some sufficiently large N, which is possible if we choose k large enough. We shall also assume the fine structure of e_{2k-1} as in section 1.1, and more specifically as in (1.1.6). We try to complete the approximate solution u_{2k-1} to an exact solution $u = u_{2k-1} + \varepsilon$, where ε solves the following equation:

$$\big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)\widetilde{\varepsilon} - \mathscr{L}\widetilde{\varepsilon} = \lambda^{-2} R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big) \tag{1.3.1}$$

Our strategy is to formulate this equation in terms of the Fourier coefficients of $\widetilde{\varepsilon}$ with respect to the Fourier basis associated with $\mathscr{L}$, the latter as in the preceding section, given by

$$\mathscr{L} = -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$$

To introduce the operator

$$\mathscr{K} = -\big(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho(\eta)}\big)\delta_0(\xi - \eta) + \mathscr{K}_0,$$

see [17]. This operator is needed to describe the transition from (1.3.1) to the equivalent formulation in terms of the Fourier coefficients:

$$\begin{aligned}\mathscr{F}\big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\big)^2&=\big(\partial_\tau+\frac{\lambda_\tau}{\lambda}(-2\xi\partial_\xi+\mathscr{K})\big)^2\mathscr{F}\\&=\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)^2\mathscr{F}+\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\frac{\lambda_\tau}{\lambda}\mathscr{K}\mathscr{F}\\&+\frac{\lambda_\tau}{\lambda}\mathscr{K}\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\mathscr{F}+(\frac{\lambda_\tau}{\lambda})^2\mathscr{K}^2\mathscr{F}\\&=\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)^2\mathscr{F}+2\frac{\lambda_\tau}{\lambda}\mathscr{K}\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\mathscr{F}\\&+\partial_\tau(\frac{\lambda_\tau}{\lambda})\mathscr{K}\mathscr{F}-2(\frac{\lambda_\tau}{\lambda})^2[\xi\partial_\xi,\mathscr{K}]\mathscr{F}+(\frac{\lambda_\tau}{\lambda})^2\mathscr{K}^2\mathscr{F}\end{aligned}\tag{1.3.2}$$

To proceed further, we have to precisely understand the structure of the 'transference operator' $\mathscr{K}$. Make the

Definition 1.3.1. *We call an operator $\widetilde{\mathscr{K}}$ to be 'smoothing', provided it enjoys the mapping property*

$$\widetilde{\mathscr{K}}:L^{2,\alpha}_\rho\longrightarrow L^{2,\alpha+\frac12}_\rho\ \forall\alpha$$

For the definition of the weighted L^2-space $L^{2,\alpha}_\rho$, specifically we have

$$\|u\|_{L^{2,\alpha}_\rho}:=\big(\int_0^\infty|u(\xi)|^2\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi\big)^{\frac12}$$

The preceding definition means that applying $\widetilde{\mathscr{K}}$ we gain $1/2$ power of ξ of decay as $\xi\to\infty$. For future reference, we will also use the following notation: if

$$f(R)=\int_0^\infty\phi(R,\xi)x(\xi)\rho(\xi)\,d\xi,$$

then we write

$$\|f\|_{H^\alpha_\rho}:=\big(\int_0^\infty x^2(\xi)\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi\big)^{\frac12}$$

Now according to Proposition 5.2 in [17], both operators $\mathscr{K}_0$, $[\xi\partial_\xi,\mathscr{K}_0]$, are smoothing. [4] Our strategy shall be to move terms involving a smoothing operator to the right hand side, and keep those terms without smoothing property on the left, building them implicitly into the parametrix. This procedure is different than that employed in [17], and mimics the strategy in [16].

Write (see Theorem 1.2.2)

$$\widetilde{\varepsilon}(\tau,R)=\int_0^\infty\phi(R,\xi)x(\tau,\xi)\rho(\xi)\,d\xi$$

[4]This is not stated this way in the cited Proposition for the commutator term, but the same proof as for $\mathscr{K}_0$ yields the smoothing property for $[\xi\partial_\xi,\mathscr{K}_0]$.

whence $x(\tau,\xi) = (\mathscr{F}\widetilde{\varepsilon})(\tau,\xi)$. Then using $\mathscr{F}\big(\mathscr{L}\widetilde{\varepsilon}\big)(\tau,\xi) = \xi x(\tau,\xi)$, we get from (1.3.1) and (1.3.2)(all functions are to be evaluated at (τ,ξ))

$$\begin{aligned} -\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)^2 x - \xi x = 2\frac{\lambda_\tau}{\lambda}\mathscr{K}\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x + (\frac{\lambda_\tau}{\lambda})^2\big[\mathscr{K}^2 - 2[\xi\partial_\xi,\mathscr{K}]\big]x \\ -\big(\frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)x + \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathscr{K}x \\ + \lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big)\big] \end{aligned} \tag{1.3.3}$$

Here we want to remove all linear terms that do not have the smoothing property from the right hand side, which forces us to modify the left hand side. Observe the identity

$$\begin{aligned} \big(\partial_\tau - \frac{\lambda_\tau}{\lambda}[2\xi\partial_\xi + \frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]\big)^2 = \big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\big)^2 \\ -\big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\big)\frac{\lambda_\tau}{\lambda}[\frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}] \\ -\frac{\lambda_\tau}{\lambda}[\frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]\big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\big) \\ +[\frac{\lambda_\tau}{\lambda}]^2[\frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]^2 \end{aligned}$$

It follows that we have the relation

$$\begin{aligned} -\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}[2\xi\partial_\xi + \frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]\big)^2 x - \xi x = 2\frac{\lambda_\tau}{\lambda}\mathscr{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x \\ + (\frac{\lambda_\tau}{\lambda})^2\big[\mathscr{K}^2 - (\mathscr{K} - \mathscr{K}_0)^2 - 2[\xi\partial_\xi,\mathscr{K}_0]\big]x \\ -\big(\frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)x + \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathscr{K}_0 x \\ + \lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big)\big] \end{aligned} \tag{1.3.4}$$

Here the linear expression

$$\big(\frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)x = \tau^{-2}\big(\frac{1}{4}(\frac{\nu+1}{\nu})^2 - \frac{1}{2}\frac{\nu+1}{\nu}\big)x =: c\tau^{-2}x$$

still doesn't have the smoothing property. However, x has better decay than the source terms e_{2k-1}, and so we will gain smoothness once we apply the parametrix to this term. We shall therefore leave it on the right hand side. To simplify notation, introduce the operator

$$\mathscr{D}_\tau := \partial_\tau - \frac{\lambda_\tau}{\lambda}[2\xi\partial_\xi + \frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]$$

Then we can finally formulate (1.3.4) in the form

$$\mathscr{D}_\tau^2 x + \xi x = f, \tag{1.3.5}$$

where we have

$$
\begin{aligned}
-f =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x + (\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi,\mathcal{K}_0]\big]x \\
&+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x + \lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big)\big] - c\tau^{-2}x
\end{aligned}
\tag{1.3.6}
$$

In order to solve (**??**), we first formally replace $\mathcal{D}_\tau$ by ∂_τ and reduce to the simpler model problem

$$
L_{\xi,\tau}x := \partial_\tau^2 x + \lambda^{-2}(\tau)\xi x = f, \tag{1.3.7}
$$

The extra factor $\lambda^{-2}(\tau)$ comes from a change of scale. Introduce the symbol $S(\tau,\sigma,\xi)$ as in [17], via the requirements

$$
\partial_\tau^2 S + \lambda^{-2}(\tau)\xi S = 0,\ S(\tau,\tau,\xi) = 0,\ \partial_\tau S(\tau,\sigma,\xi)|_{\tau=\sigma} = -1.
$$

We commence by noting that it suffices to consider the case $\xi = 1$. In fact (see [17]),

Lemma 1.3.2. *We have the scaling relation*

$$
S(\tau,\sigma,\xi) = \xi^{\frac{\nu}{2}} S(\tau\xi^{-\frac{\nu}{2}},\sigma\xi^{-\frac{\nu}{2}},1)
$$

Proof. We verify that the expression on the right has the desired properties. This follows from

$$
\begin{aligned}
&\partial_\tau^2 \xi^{\frac{\nu}{2}} S(\tau\xi^{-\frac{\nu}{2}},\sigma\xi^{-\frac{\nu}{2}},1) = \xi^{-\frac{\nu}{2}}(\partial_\tau^2 S)(\tau\xi^{-\frac{\nu}{2}},\sigma\xi^{-\frac{\nu}{2}},1) \\
&\tau^{-2-\frac{2}{\nu}}\xi\big(\xi^{\frac{\nu}{2}} S(\tau\xi^{-\frac{\nu}{2}},\sigma\xi^{-\frac{\nu}{2}},1)\big) = \xi^{-\frac{\nu}{2}}(\tau\xi^{-\frac{\nu}{2}})^{-2-\frac{2}{\nu}} S(\tau\xi^{-\frac{\nu}{2}},\sigma\xi^{-\frac{\nu}{2}},1),
\end{aligned}
$$

where we recall that $\lambda(\tau) \sim \tau^{\frac{1+\nu}{\nu}}$. □

We now construct $S(\tau,\sigma,1)$ via the following

Lemma 1.3.3. *(a) Let $\nu \le \frac{1}{2}$. Then there exist two analytic solutions ϕ_0, ϕ_1 of $L_{1,\tau}\phi_j = 0$, $j = 0,1$, which admit a Puiseux series type representation*

$$
\phi_j(\tau) = \sum_{k=0}^{\infty} c_{jk}\tau^{j-\frac{2k}{\nu}},\ c_{j0} = 1,\ j \in \{0,1\}.
$$

The series converges absolutely for any $\tau > 0$. (b) There is a solution $\phi_2(\tau)$ for $L_{1,\tau}$ of the form

$$
\phi_2(\tau) = \tau^{\frac{1}{2}+\frac{1}{2\nu}} e^{i\nu\tau^{-\frac{1}{\nu}}}[1 + a(\tau^{\frac{1}{\nu}})]
$$

with $a(0) = 0$.

It implies the following key

Proposition 1.3.4. *([17]) There is a constant $C > 0$ such that we have the bounds*

$$|S(\tau,\sigma,\xi)| \lesssim \sigma(\frac{\sigma}{\tau})^C(1+\tau^{-\frac{2}{\nu}}\xi)^{-\frac{1}{2}},\ |\partial_\tau S(\tau,\sigma,\xi)| \lesssim (\frac{\sigma}{\tau})^C$$

We can now write down the explicit solution of (1.3.5):

Lemma 1.3.5. *The equation* (1.3.5) *is formally solved by the following parametrix*

$$x(\tau,\xi) = \int_\tau^\infty \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} S(\tau,\sigma,\lambda^2(\tau)\xi) f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d\sigma =: (Uf)(\tau,\xi) \tag{1.3.8}$$

This is a simple direct verification, as in [16]. For us, we will need the mapping properties of this parametrix with respect to suitable Banach spaces. We have

Lemma 1.3.6. *Introducing the norm*

$$\|f\|_{L^{2,\alpha;N}_\rho} := \sup_{\tau>\tau_0} \tau^N \|f(\tau,\cdot)\|_{L^{2,\alpha}_\rho},$$

we have

$$\|Uf\|_{L^{2,\alpha+\frac{1}{2};N-2}_\rho} \lesssim \|f\|_{L^{2,\alpha;N}_\rho}$$

provided N is sufficiently large.

Proof. This is a consequence of the bounds in the preceding proposition. Observe that

$$\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \lesssim \frac{\lambda(\sigma)}{\lambda(\tau)}$$

It follows that

$$\begin{aligned}
&|\langle\xi\rangle^{\alpha+\frac{1}{2}} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} S(\tau,\sigma,\lambda^2(\tau)\xi) f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)| \\
&\lesssim \langle\xi\rangle^{\alpha+\frac{1}{2}}\big(\frac{\lambda(\tau)}{\lambda(\sigma)}\big)^{\frac{1}{2}} \sigma(\frac{\sigma}{\tau})^C(1+\tau^2\xi)^{-\frac{1}{2}}|f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)| \\
&\lesssim \sigma(\frac{\sigma}{\tau})^C\big(\frac{\lambda(\sigma)}{\lambda(\tau)}\big)^{2\alpha-\frac{1}{2}}|\langle\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\rangle^\alpha f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)|
\end{aligned}$$

It follows that

$$\begin{aligned}
\|Uf\|_{L^{2,\alpha+\frac{1}{2};N-2}_\rho} &\lesssim \sup_{\sigma>\tau_0}\|f\|_{L^{2,\alpha;N}_\rho} \sup_{\tau>\tau_0}\tau^{N-2}\int_\tau^\infty \sigma(\frac{\sigma}{\tau})^C\big(\frac{\lambda(\sigma)}{\lambda(\tau)}\big)^{2\alpha+\frac{1}{2}}\sigma^{-N}\,d\sigma \\
&\lesssim \sup_{\sigma>\tau_0}\|f\|_{L^{2,\alpha;N}_\rho}
\end{aligned}$$

provided $N > (2\alpha + \frac{1}{2})\frac{\nu+1}{\nu} + C + 2$.

□

The goal is now to formulate (1.3.5), (1.3.6) as an integral equation and find a suitable fixed point, which will be the desired $x(\tau,\xi)$. The issue is that x will only have very weak regularity a priori, in fact $x(\tau,\cdot) \in L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}$ is optimal, see [17], and this does not suffice for good algebra estimates provided $\nu \le \frac{1}{2}$. We thus have to find some better space to place x into. The key for this will be the *zeroth iterate* for solving (1.3.5), (1.3.6). Thus solve these via

$$x(\tau,\xi) = (Uf)(\tau,\xi) \tag{1.3.9}$$

with f as in (1.3.6). To find such a fixed point, we use the iterative scheme

$$x_j(\tau,\xi) = (Uf_{j-1})(\tau,\xi),\ j \ge 1$$

where we put

$$\begin{aligned} -f_j =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_j + (\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi,\mathcal{K}_0]\big]x_j \\ &+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x_j + \lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_j) + e_{2k-1}\big)\big] - c\tau^{-2}x_j \end{aligned}$$

and of course we put

$$\widetilde{\varepsilon}_j(\tau,R) = \int_0^\infty \phi(R,\xi)x_j(\tau,\xi)\rho(\xi)\,d\xi$$

The zeroth iterate in turn is defined via

$$x_0(\tau,\xi) = (U\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(e_{2k-1}\big)\big])(\tau,\xi);$$

here we may also replace e_{2k-1} by a function which co-incides with it in the backward light cone $r \le t$, in light of Huyghen's principle. This shall be handy below.

1.3.1 The zeroth iterate

We claim in effect the following:

Proposition 1.3.7. *There exists $\widetilde{e}_{2k-1} \in H^{\frac{\nu}{2}-}_{RdR}$ such that $\widetilde{e}_{2k-1}|_{r\le t} = e_{2k-1}$, and such that if we put*

$$x_0(\tau,\xi) = (U\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(\widetilde{e}_{2k-1}\big)\big])(\tau,\xi),$$

we can write

$$x_0 = x_0^{(1)} + x_0^{(2)},$$

where we have

$$x_0^{(1)} \in \tau^{-N} L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}$$

as well as

$$\chi_{R<1}\widetilde{\varepsilon}_0^{(1)}(\tau,R) = \chi_{R<1}\int_0^\infty \phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi \in \tau^{-N}R^{\frac{3}{2}}L^\infty$$

while also

$$x_0^{(2)} \in \tau^{-N} L_\rho^{2,1+\frac{\nu}{2}-}$$

Remark 1.3.8. Note that for $R \geq 1$, we actually have the bound

$$|\widetilde{\varepsilon}_0^{(1)}(\tau,R)| \lesssim \tau^{-N}$$

since $\widetilde{\varepsilon}_0^{(1)}(\tau,\cdot) \in H_{dR}^{1+\nu}$.

Proof. Recall from structure of the error e_{2k-1} of the approximate solution u_{2k-1} that e_{2k-1} can be written as a sum of terms involving the singular expressions

$$\tau^{-N-2}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j,\ j \leq j(i),\ i \geq 1,$$

multiplied by smooth (in t,r) functions. In fact, up to an error of class $H_{\mathbb{R}^2}^{2+\nu-}$ (namely when $(2i-1)\nu > 2+\nu$), there are only finitely many such expressions. We now define $\widetilde{e}_{2k-1}$ by replacing each of the above expressions by their truncation

$$\tau^{-N-2}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j|_{r\leq t};$$

and the rest of e_{2k-1} is smoothly truncated to a dilate of the light cone $r \leq t$. Thus, specifically, we shall write

$$\widetilde{e}_{2k-1} = \widetilde{e}_{2k-1}^{(1)} + \widetilde{e}_{2k-1}^{(2)},$$

where we may assume

$$\widetilde{e}_{2k-1}^{(2)} \in \tau^{-N-2}H_{\mathbb{R}^2}^{2+\nu-}$$

while $\widetilde{e}_{2k-1}^{(1)}$ is a sum of singular terms of the above form with smooth bounded coefficients. Since $\mathscr{F}\circ T^{-1}(H_{\mathbb{R}^2}^{\alpha}) = L_\rho^{2,\frac{\alpha}{2}}$, where T is the map

$$u(R) \to e^{i\theta}R^{-\frac{1}{2}}u(R),$$

we see from lemma 1.3.6 that we have the bound

$$\|U\lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(\widetilde{e}^{(2)}_{2k-1}\big)\big]\|_{L^{2,1+\frac{\nu}{2}-;N}_{\rho}} \lesssim 1$$

and so we can include $U\lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(\widetilde{e}^{(2)}_{2k-1}\big)\big]$ into $x^{(2)}_0$.
Next, consider the more difficult contribution of $\widetilde{e}^{(1)}_{2k-1}$, where a more detailed analysis of the parametrix becomes necessary. For general f, consider the decomposition

$$\begin{aligned}
&\int_{\tau}^{\infty}\frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}S(\tau,\sigma,\lambda^2(\tau)\xi)f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma\\
&=\int_{\max\{\tau,C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}\}}^{\infty}\frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}S(\tau,\sigma,\lambda^2(\tau)\xi)f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma\\
&+\int_{\tau}^{\max\{\tau,C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}\}}\frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}S(\tau,\sigma,\lambda^2(\tau)\xi)f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma\\
&=:(Uf)^{(1)}+(Uf)^{(2)}
\end{aligned}$$

for some large constant C. In the first integral, we have

$$\sigma\geq C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}},$$

whence we obtain

$$\xi\leq(\frac{\sigma}{C})^{\frac{2}{\nu}}\lambda^{-2}(\tau)$$

It follows that

$$\|(Uf)^{(1)}\|_{L^{2,1+\frac{\nu}{2}-;N}_{\rho}}\lesssim\|f\|_{L^{2,\frac{\nu}{2}-;N+\frac{2}{\nu}+C_1}_{\rho}}$$

and so we have gained smoothness for this terms at the expense of temporal decay. It thus remains to consider the term $(Uf)^{(2)}$, which in fact requires most of the work. On account of lemma 1.3.2, we have

$$S(\tau,\sigma,\lambda^2(\tau)\xi)=(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}S(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},1)$$

Then from the proof of lemma 8.1, [17], we can write

$$S(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},1)=\operatorname{Im}\big(\phi_2(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\overline{\phi_2}(\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)$$

and so using the factorization[5] $\phi_2(\tau)=\tau^{\frac{1}{2}+\frac{1}{2\nu}}e^{i\nu^{-\frac{1}{\nu}}\tau^{-\frac{1}{\nu}}}[1+a(\tau^{\frac{1}{\nu}})]$ as in lemma 8.1[17], we obtain

[5]Here we correct a typo in [17]: we replace a factor ν by the correct $\nu^{-\frac{1}{\nu}}$

$$
\begin{aligned}
&(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}S(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}},1)\\
&=\tau^{\frac{1}{2}+\frac{1}{2\nu}}\sigma^{\frac{1}{2}+\frac{1}{2\nu}}(\lambda^2(\tau)\xi)^{-\frac{1}{2}}\sin\big(\nu\xi^{\frac{1}{2}}\tau\big(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\big)\big)\big(1+a(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)\big(1+a(\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)\\
&=\big(\frac{\sigma}{\tau}\big)^{\frac{1}{2}+\frac{1}{2\nu}}\xi^{-\frac{1}{2}}\sin\big(\nu\xi^{\frac{1}{2}}\tau\big(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\big)\big)\big(1+a(\tau(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)\big(1+a(\sigma(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)
\end{aligned}
\tag{1.3.10}
$$

Here the function $a(\tau)$ is smooth with bounded derivatives.
Our task now consists in checking how the oscillations of this expression potentially cancel against the oscillations of $f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)$ in $(Uf)^{(2)}$. Recall that

$$
f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)=\lambda^{-2}(\sigma)\mathscr{F}\big[R^{\frac{1}{2}}\big(\tilde{e}^{(1)}_{2k-1}(\sigma,\cdot)\big)\big](\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi))
$$

We need to analyze the large frequency asymptotics of this expression. We recall from theorem 1.2.5 that

$$
\phi(R,\xi)=a(\xi)\psi^+(R,\xi)+\overline{a(\xi)\psi^+(R,\xi)}
$$

where we have the large frequency asymptotics $|a(\xi)|\sim\xi^{-\frac{1}{2}}$, $\xi\gg 1$. The function $a(\xi)$ is smooth and in fact obeys symbol behavior, see theorem 1.2.5. Furthermore, the oscillatory function ψ^+ can be written in the form

$$
\psi^+(R,\xi)=\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\sigma(R\xi^{\frac{1}{2}},R),\ R\xi^{\frac{1}{2}}\gtrsim 1,
$$

where we have a symbolic expansion, see theorem 1.2.4,

$$
\sigma(q,r)=\sum_{j=0}^{\infty}q^{-j}\psi_j^+(r)
$$

and the functions ψ_j^+ are uniformly bounded and smooth with symbol behavior. We insert these asymptotics into the formula for the Fourier transform, using the singular source term

$$
\lambda^{-2}R^{\frac{1}{2}}\tilde{e}^{(1)}_{2k-1}=\tau^{-N-2}\chi_{r\leq t}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j,\ i\geq 1.
$$

In fact, we may replace all additional factors $R^{-k}(\log R)^l$ by $(\lambda(\sigma)\cdot\sigma)^{-k}(\log(\lambda(\sigma)\sigma))^l$, since the errors committed thereby gain one degree of smoothness, and are thus in $H^{1+\nu-}_{\mathbb{R}^2}$. By the same token, we can also include a smooth cutoff $\chi_{a\geq\frac{1}{2}}$.

We now find that (with $f(\sigma,\xi) = \mathscr{F}\big(\lambda^{-2}R^{\frac{1}{2}}\tilde{e}^{(1)}_{2k-1}(\sigma,\cdot)\big)(\xi)$ as well as $a = \frac{R}{\lambda(\sigma)\sigma}$)

$$\begin{aligned}f(\sigma,\xi) =&\sigma^{-N-2}\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\sigma(R\xi^{\frac{1}{2}},R)\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &+\sigma^{-N-2}\int_0^{\nu\sigma}\overline{a(\xi)}\xi^{-\frac{1}{4}}e^{-iR\xi^{\frac{1}{2}}}\overline{\sigma(R\xi^{\frac{1}{2}},R)}\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\end{aligned}$$

Using the asymptotic expansion

$$\sigma(R\xi^{\frac{1}{2}},R) = c_0 + O(\frac{1}{R\xi^{\frac{1}{2}}}),$$

where the O-term enjoys symbol behavior, we get

$$\begin{aligned}&\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\sigma(R\xi^{\frac{1}{2}},R)\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &= c_0\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &+\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}O(R^{-1}\xi^{-\frac{1}{2}})\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\end{aligned}$$

To bound the second term, observe that

$$\begin{aligned}&\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}O(R^{-1}\xi^{-\frac{1}{2}})\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &=\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}O(R^{-\frac{1}{2}}\xi^{-\frac{1}{2}})(\nu\sigma)^{-\frac{1}{2}}\chi_{a\geq\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &= O(\xi^{-\frac{7}{4}})\end{aligned}$$

after integration by parts with respect to R. In short, we have now shown that

$$\begin{aligned}f(\sigma,\xi) =&c_0\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &\overline{+c_0\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR}\\ &+O(\xi^{-\frac{7}{4}})\end{aligned}$$

We now analyze the integrals more closely. We introduce the variable $x = \nu\sigma - R$. Then we can write

$$\begin{aligned}&\int_0^{\nu\sigma} a(\xi)\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\chi_{a\geq\frac{1}{2}}a^{\frac{1}{2}}(1-a)^{i\nu-\frac{1}{2}}(\log(1-a))^j\,dR\\ &= e^{i\nu\sigma\xi^{\frac{1}{2}}}a(\xi)\xi^{-\frac{1}{4}}(\nu\sigma)^{\frac{1}{2}-i\nu}\int_0^{\infty}e^{ix\xi^{\frac{1}{2}}}\chi_{x\leq\frac{\nu\sigma}{2}}\big(1-\frac{x}{\nu\sigma}\big)^{\frac{1}{2}}x^{-\frac{1-2i\nu}{2}}\big(\log(\frac{x}{\nu\sigma})\big)^j\,dx\end{aligned}$$

Changing variables to $y = x\xi^{\frac{1}{2}}$ allows us to express this expression in the form

$$e^{i\nu\sigma\xi^{\frac{1}{2}}} a(\xi)\xi^{-\frac{1}{2}-i\nu} F(\sigma,\xi),$$

where we have

$$F(\sigma,\xi) := (\nu\sigma)^{\frac{1}{2}-i\nu} \int_0^\infty e^{iy} \chi_{y\leq\frac{\nu\sigma\xi^{\frac{1}{2}}}{2}} \big(1 - \frac{y}{\nu\sigma\xi^{\frac{1}{2}}}\big)^{\frac{1}{2}} y^{-\frac{1-2i\nu}{2}} \big(\log(\frac{y}{\nu\sigma\xi^{\frac{1}{2}}})\big)^j \, dy$$

Observe that $F(\sigma,\xi) \in C^\infty$, and we have

$$|\partial^l_{\xi^{\frac{1}{2}}} F(\sigma,\xi)| \lesssim (\nu\sigma)^{\frac{1}{2}-i\nu} \xi^{-\frac{l}{2}}, \, |\partial_\sigma F(\sigma,\xi)| \lesssim (\nu\sigma)^{\frac{1}{2}-i\nu} \sigma^{-1}.$$

Here it is of course important that we have the restriction $y \leq \frac{\nu\sigma\xi^{\frac{1}{2}}}{2}$. We thus now have the representation

$$\begin{aligned} f(\sigma,\xi) =& c_0 \sigma^{-N} e^{i\nu\sigma\xi^{\frac{1}{2}}} a(\xi)\xi^{-\frac{1}{2}-i\nu} F(\sigma,\xi) \\ &+ \overline{c_0 \sigma^{-N} e^{i\nu\sigma\xi^{\frac{1}{2}}} a(\xi)\xi^{-\frac{1}{2}-i\nu} F(\sigma,\xi)} \\ &+ \sigma^{-N} O(\xi^{-\frac{7}{4}}) \end{aligned} \tag{1.3.11}$$

where we keep in mind the restriction that $\xi > 1$, as we only care about the large frequency case. We shall now use this in the context of $(Uf)^{(2)}$, see above, with

$$f = \lambda^{-2}(\sigma) \mathscr{F}\big[R^{\frac{1}{2}} \big(\tilde{e}^{(1)}_{2k-1}(\sigma,\cdot)\big](\xi)$$

Begin by writing

$$\begin{aligned} (Uf)^{(2)}(\tau,\xi) =& \int_\tau^{\min\{C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}, \xi^{\frac{\nu}{2(1+\nu)}}\tau\}} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} S(\tau,\sigma,\lambda^2(\tau)\xi) f(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi) \, d\sigma \\ &+ \int_{\min\{C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}, \xi^{\frac{\nu}{2(1+\nu)}}\tau\}}^{\max\{\tau, C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}\}} \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} S(\tau,\sigma,\lambda^2(\tau)\xi) f(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi) \, d\sigma \\ &=: (Uf)^{(21)}(\tau,\xi) + (Uf)^{(22)}(\tau,\xi) \end{aligned}$$

In the second integral, we have

$$\xi < \big(\frac{\sigma}{\tau}\big)^{\frac{2(1+\nu)}{\nu}}$$

and so we get

$$\|(Uf)^{(22)}\|_{L^{2,1;N}_\rho} \lesssim \|f\|_{L^{2,\frac{\nu}{2}-;N-2}_\rho},$$

provided N is sufficiently large in relation to ν.

We have now reduced things to $(Uf)^{(21)}(\tau,\xi)$, where we have $\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi > 1$, and so we can use (1.3.11). We shall combine this with the asymptotic relation (1.3.10). Just recording the integrand of the resulting expression and omitting constants, we find the schematic expression

$$\frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\big(\frac{\sigma}{\tau}\big)^{\frac{1}{2}+\frac{1}{2\nu}}\xi^{-\frac{1}{2}}\sin\big(\nu\xi^{\frac{1}{2}}\tau\big(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\big)\big)\prod_{\kappa=\tau,\sigma}\big(1+a(\kappa(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\big)$$
$$\cdot\sigma^{-N}e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}a(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)^{-\frac{1}{2}-i\nu}F(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)$$

and so $(Uf)^{(21)}(\tau,\xi)$ is obtained by applying $\int_\tau^{\min\{C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}},\xi^{\frac{\nu}{2(1+\nu}}\tau\}}d\sigma$ to this integrand. Observe that the decay of this expression with respect to large ξ is

$$O(\xi^{-\frac{3}{2}-i\nu}),$$

but in order to obtain the desired $L_\rho^{2,1+\frac{\nu}{2}-;N}$-decay, we would need at least $\xi^{-2-\frac{\nu}{2}}$. The only way to eke out this extra decay in ξ is to exploit the integration in σ, for which the product of the oscillatory factors

$$\sin\big(\nu\xi^{\frac{1}{2}}\tau\big(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\big)\big)\cdot e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}=\frac{e^{i\left(\nu\xi^{\frac{1}{2}}\tau\left(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)}-e^{-i\left(\nu\xi^{\frac{1}{2}}\tau\left(1-(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)}}{2}\cdot e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}$$

is key. The resulting phase functions (upon developing this product) are either of the form

$$e^{\pm i\left(\nu\xi^{\frac{1}{2}}\tau\left(1-2(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)},$$

in which case we gain a factor $\xi^{-\frac{1}{2}}$ via integration by parts with respect to σ, or else of the form

$$e^{\pm i\nu\xi^{\frac{1}{2}}\tau},$$

in which case the σ-oscillation has been destroyed.

It is this last case we now investigate more closely. We shall essentially put

$$x_0^{(1)}=(Uf)^{(21)}(\tau,\xi)$$

Then the required inclusion $x_0^{(1)}\in L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-;N}$ is immediate, and so we now need to verify the sufficient vanishing of $\widetilde{\varepsilon}_0^{(1)}(\tau,R)$ at $R=0$. Thus consider

$$
\begin{aligned}
\widetilde{\varepsilon}_0^{(1)}(\tau,R) &= \int_0^\infty \phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi \\
&= \int_0^\infty \chi_{\xi<1}\phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi && (1.3.12)\\
&+ \int_0^\infty \chi_{1\le\xi<R^{-2}}\phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi && (1.3.13)\\
&+ \int_0^\infty \chi_{\xi\ge R^{-2}}\phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi && (1.3.14)
\end{aligned}
$$

We have included smooth cutoffs to dilates of the indicated regions. Here the first term (1.3.12) clearly is in $L_\rho^{2,1;N}$ and hence negligible. It remains to control the other two terms, for which we use the asymptotic expansions of $\phi(R,\xi)$. For the last term, use

$$
\phi(R,\xi) = a(\xi)\psi^+(R,\xi) + \overline{a(\xi)\psi^+(R,\xi)}
$$

with

$$
\psi^+(R,\xi) = \xi^{-\frac14}e^{iR\xi^{\frac12}}\sigma(R\xi^{\frac12},R),\ R\xi^{\frac12}\gtrsim 1,
$$

as well as $|a(\xi)|\lesssim \xi^{-\frac12}$. Then keeping in mind the structure of $x_0^{(1)} = (Uf)^{(21)}(\tau,\xi)$, we can write (schematically)

$$
\begin{aligned}
&\int_0^\infty \chi_{\xi\ge R^{-2}}\phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi \\
&= \int_0^\infty a(\xi)\xi^{-\frac14}\chi_{\xi\ge R^{-2}}e^{i[R\xi^{\frac12}\pm\nu\xi^{\frac12}\tau]}\sigma(R\xi^{\frac12},R)\Big(\int_\tau^{\kappa(\tau,\xi)} G_1(\sigma,\tau,\xi)\,d\sigma\Big)\rho(\xi)\,d\xi && (1.3.15)\\
&+ \int_0^\infty a(\xi)\xi^{-\frac14}\chi_{\xi\ge R^{-2}}e^{iR\xi^{\frac12}}\sigma(R\xi^{\frac12},R)\Big(\int_\tau^{\kappa(\tau,\xi)} G_2(\sigma,\tau,\xi)\,d\sigma\Big)\rho(\xi)\,d\xi && (1.3.16)
\end{aligned}
$$

where we have used the notation

$$
\kappa(\tau,\xi) = \min\{C(\lambda^2(\tau)\xi)^{\frac{\nu}{2}}, \xi^{\frac{\nu}{2(1+\nu}}\tau\}
$$

as well as

$$
\begin{aligned}
G_1(\sigma,\tau,\xi) = &\frac{\lambda^{\frac32}(\tau)}{\lambda^{\frac32}(\sigma)}\frac{\rho^{\frac12}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac12}(\xi)}\Big(\frac{\sigma}{\tau}\Big)^{\frac12+\frac{1}{2\nu}}\xi^{-\frac12}\prod_{\kappa=\tau,\sigma}\Big(1+a(\kappa(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\Big)\\
&\cdot\sigma^{-N}a\Big(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)\Big(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)^{-\frac12-i\nu}F\Big(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)
\end{aligned}
$$

Further, for the oscillatory second integral, we have

$$G_2(\sigma,\tau,\xi) = \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)} \frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \big(\frac{\sigma}{\tau}\big)^{\frac{1}{2}+\frac{1}{2\nu}} \xi^{-\frac{1}{2}} e^{i\left(\pm\nu\xi^{\frac{1}{2}}\tau\left(1-2(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)} \prod_{\kappa=\tau,\sigma} \left(1 + a(\kappa(\lambda^2(\tau)\xi)^{-\frac{\nu}{2}})\right)$$
$$\cdot \sigma^{-N} \xi^{\frac{1}{2}} a(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)^{-\frac{1}{2}-i\nu} F(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)$$

The idea now is that in the first integral (1.3.15), we can perform an integration by parts with respect to $\xi^{\frac{1}{2}}$, provided the phase $R \pm \nu\tau$ is large, which is certainly the case if we restrict to $R < \frac{\nu\tau}{2}$. More precisely, this becomes possible once we split the ξ-integral into two, where the limit $\kappa(\tau,\xi)$ is a smooth function of ξ. Observe that

$$|G_1(\sigma,\tau,\xi)| \lesssim \Lambda(\sigma,\tau)\xi^{-\frac{3}{2}}$$

for a suitable $\Lambda(\sigma,\tau)$. Performing an integration by parts with respect to $\xi^{\frac{1}{2}}$ in (1.3.15) and assuming N to be large enough (in relation to ν), as well as using the bound $\chi_{\xi \ge R^{-2}}\xi^{-\frac{3}{4}} \lesssim R^{\frac{3}{2}}$, we then find

$$|\chi_{R<\frac{\nu\tau}{2}} \int_0^\infty a(\xi)\xi^{-\frac{1}{4}} \chi_{\xi\ge R^{-2}} e^{i[R\xi^{\frac{1}{2}} \pm \nu\xi^{\frac{1}{2}}\tau]} \sigma(R\xi^{\frac{1}{2}}, R)\Big(\int_\tau^{\kappa(\tau,\xi)} G_1(\sigma,\tau,\xi)\, d\sigma\Big)\rho(\xi)\, d\xi|$$
$$\lesssim \tau^{-(N-1)} R^{\frac{3}{2}}$$

Next, consider the integral (1.3.16). Here we perform the integration by parts inside the σ-integral, due to the oscillatory phase

$$e^{i\left(\pm\nu\xi^{\frac{1}{2}}\tau\left(1-2(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)}$$

Indeed, we have

$$e^{i\left(\pm\nu\xi^{\frac{1}{2}}\tau\left(1-2(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)} = \mp\big(\frac{\sigma}{\tau}\big)^{1+\nu^{-1}} (2i\xi^{\frac{1}{2}})^{-1} \partial_\sigma\Big(e^{i\left(\pm\nu\xi^{\frac{1}{2}}\tau\left(1-2(\frac{\tau}{\sigma})^{\frac{1}{\nu}}\right)\right)}\Big)$$

and so we gain one inverse power $\xi^{-\frac{1}{2}}$ at the expense of a weight $\big(\frac{\sigma}{\tau}\big)^{1+\nu^{-1}}$, and this is enough to force absolute integrability with respect to ξ since $\rho(\xi) \sim \xi$ for large ξ. It follows that

$$|\int_0^\infty a(\xi)\xi^{-\frac{1}{4}} \chi_{\xi\ge R^{-2}} e^{iR\xi^{\frac{1}{2}}} \sigma(R\xi^{\frac{1}{2}}, R)\Big(\int_\tau^{\kappa(\tau,\xi)} G_2(\sigma,\tau,\xi)\, d\sigma\Big)\rho(\xi)\, d\xi|$$
$$\lesssim \tau^{-(N-1)} R^{\frac{3}{2}},$$

even irrespective of the size of R. This concludes the estimate for the term (1.3.14).

It remains to deal with (1.3.13), where we use the expansion

$$\phi(R,\xi) = \phi_0(R) + R^{-\frac{1}{2}} \sum_{j=1}^\infty (R^2\xi)^j \phi_j(R^2),$$

where the functions ϕ_j are smooth with very good bounds:

$$|\phi_j(u)| \leq \frac{3C^j}{(j-1)!}\log(1+|u|),$$

see Theorem **??**. Then as in (1.3.15), (1.3.16), we decompose

$$\begin{aligned}
&\int_0^\infty \chi_{\xi<R^{-2}}\phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)\,d\xi \\
&= \int_0^\infty \chi_{\xi<R^{-2}}\phi(R,\xi)e^{i\nu\xi^{\frac{1}{2}}\tau}\Big(\int_\tau^{\kappa(\tau,\xi)} G_1(\sigma,\tau,\xi)\,d\sigma\Big)\rho(\xi)\,d\xi \qquad (1.3.17)\\
&+ \int_0^\infty \chi_{\xi<R^{-2}}\phi(R,\xi)\Big(\int_\tau^{\kappa(\tau,\xi)} G_2(\sigma,\tau,\xi)\,d\sigma\Big)\rho(\xi)\,d\xi \qquad (1.3.18)
\end{aligned}$$

In the first integral on the right, we perform integration by parts with respect to $\xi^{\frac{1}{2}}$, gaining a factor τ^{-1}. If the derivative falls on the function $\phi(R,\xi)$, we obtain the differentiated series

$$\sum_{j=1}^\infty j(R^2\xi)^{j-1}R^{\frac{3}{2}}\xi^{\frac{1}{2}}\phi_j(R^2)$$

which is bounded in absolute value by

$$\Big|\sum_{j=1}^\infty j(R^2\xi)^{j-1}R^{\frac{3}{2}}\xi^{\frac{1}{2}}\phi_j(R^2)\Big| \lesssim R^{\frac{3}{2}}\log(2+R)$$

When the derivative falls on the inner integral, the bound is the same as before, and the last integral (1.3.18) is also bounded just like (1.3.16). This concludes the proof of the proposition. □

For later reference, we need somewhat more refined information, which however easily follows from the preceding proof. We mention

Corollary 1.3.9. *Denote by P_λ the frequency localizers*

$$\mathscr{F}\big(P_{<\lambda}f\big)(\xi) = \chi_{<\lambda}(\xi)\big(\mathscr{F}f\big)(\xi)$$

where $\chi_{<\lambda}(\xi)$ is a smooth cutoff function localizing to $\xi \lesssim \lambda$, as in [17]; here λ is a dyadic number. Then we have

$$\chi_{R<1}P_{<\lambda}\tilde{\varepsilon}_0^{(1)} \in \tau^{-N}R^{\frac{3}{2}}L^\infty$$

uniformly in $\lambda > 1$. Furthermore, for any integer $l \geq 0$, we have

$$\nabla_R^l R^{-\frac{3}{2}}P_{<\lambda}\tilde{\varepsilon}_0^{(1)} = O(\tau^{-N})$$

uniformly in $\lambda > 1$.

1.3.2 Analysis of the nonlinear source terms

From (1.1.3), we recall the following formula for the main source term:

$$\lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) =$$
$$\frac{4\sin(u_0-u_{2k})\sin(u_0+u_{2k})}{R^2}\widetilde{\varepsilon} \tag{1.3.19}$$
$$+\frac{\sin(2u_{2k})}{2R^{\frac{3}{2}}}\big(1-\cos(2R^{-\frac{1}{2}}\widetilde{\varepsilon})\big) \tag{1.3.20}$$
$$+\frac{\cos(2u_{2k})}{2R^{\frac{3}{2}}}\big(2R^{-\frac{1}{2}}\widetilde{\varepsilon}-\sin(2R^{-\frac{1}{2}}\widetilde{\varepsilon})\big) \tag{1.3.21}$$

According to the preceding proposition, we have

$$x_0 \in \tau^{-N}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}$$

whence

$$\widetilde{\varepsilon}_0(\tau,\cdot) \in \tau^{-N}H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$$

This means that for the source terms, we need at least $H_\rho^{\frac{\nu}{2}-}$-regularity. In fact, we can do much better for the term (1.3.19). Recall that

$$u_{2k} = u_0 + \sum_{j=1}^{2k} v_j$$

where we have

$$v_{2j-1} \in \frac{1}{(t\lambda)^{2j}} IS^3\big(R(\log R)^{2j-1}, \mathcal{Q}_{j-1}\big),\ v_{2j} \in \frac{1}{(t\lambda)^{2j+2}} IS^3\big(R(\log R)^{2j-1}, \mathcal{Q}_j\big)$$

This implies in particular that

$$\frac{\sin(u_0-u_{2k})}{R} \in (\lambda t)^{-2} IS(\log R, \mathcal{Q}),\ \frac{\sin(u_0+u_{2k})}{R} \in IS(R^{-1}, \mathcal{Q})$$

Then we recall lemma 8.1 from [17]:

Lemma 1.3.10. *[[17]] Assume* $|\alpha| < \frac{\nu}{2}+\frac{3}{4}$, $f \in IS(1,\mathcal{Q})$. *Then we have*

$$\|gf\|_{H_\rho^\alpha} \lesssim \|f\|_{H_\rho^\alpha}$$

Application of this lemma yields the bound

$$\|\frac{4\sin(u_0-u_{2k})\sin(u_0+u_{2k})}{R^2}\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim (\lambda t)^{-2}\|\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \tag{1.3.22}$$

To deal with the truly nonlinear source terms (1.3.20) and (1.3.21), we need the following multilinear estimates:

Lemma 1.3.11. *Assume* $f, g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}} L^\infty$, $P_{<\lambda} f, P_{<\lambda} g \in \log\lambda R^{\frac{3}{2}} L^\infty$ *uniformly in* $\lambda > 1$. *If also* $\chi_{R<1} \nabla^l \big(R^{-\frac{3}{2}} P_{<\lambda} f\big) \in L^\infty$ *uniformly in* $\lambda > 1$, $l \geq 0$, *then we have*

$$R^{-\frac{3}{2}} f g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-\delta-} \cap R^{\frac{3}{2}} L^\infty.$$

for arbitrarily small $\delta \in (0, \frac{\nu}{100}]$ *(with implicit constant depending on* δ*), and we also have*

$$R^{-\frac{3}{2}} P_{<\lambda}\big(R^{-\frac{3}{2}} f g\big) \in \log\lambda L^\infty,\ R^{-1} P_{<\lambda}\big(R^{-\frac{3}{2}} f g\big) \in L^\infty$$

uniformly in $\lambda > 1$. *If* $f \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-\delta-} \cap R^{\frac{3}{2}} L^\infty$, $P_{<\lambda} f \in \log\lambda R^{\frac{3}{2}} L^\infty$ *uniformly in* λ, *but* $g \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$, $\delta \in (0, \frac{\nu}{100}]$, *then*

$$R^{-\frac{3}{2}} f g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}} L^\infty,\ R^{-\frac{3}{2}} P_{<\lambda}\big(R^{-\frac{3}{2}} f g\big) \in \log\lambda R^{\frac{3}{2}} L^\infty,\ R^{-1} P_{<\lambda}\big(R^{-\frac{3}{2}} f g\big) \in L^\infty$$

The same conclusion obtains if both $f, g \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$. *Further, if* $f, g \in (H_\rho^{\frac{1}{2}+\frac{\nu}{2}-\delta-} \cap R^{\frac{3}{2}} L^\infty)$, *as well as*

$$P_{<\lambda} f \in R L^\infty,\ P_{<\lambda} g \in R L^\infty,\ \chi_{R<1} \nabla_R^l \big(R^{-1} P_{<\lambda} f\big) \in L^\infty,\ l \geq 0,$$

uniformly in $\lambda > 1$, *or else one of* $f, g \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$, *we get for* $j = 0, 1$

$$R^{-j} f g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-\delta-} \cap R L^\infty,\ P_{<\lambda}\big(R^{-j} f g\big) \in R L^\infty,$$

the latter inclusion uniformly in $\lambda > 1$.

Proof. Throughout $\lambda_{1,2}, \sigma$ are dyadic numbers. We mimic the proof of lemma 8.5 in [17]. Write

$$R^{-\frac{3}{2}} f g = \sum_{\lambda_{1,2}} \sum_{\sigma < \max\{\lambda_{1,2}\}} P_\sigma\big(R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\big) + \sum_{\lambda_{1,2}} \sum_{\sigma \geq \max\{\lambda_{1,2}\}} P_\sigma\big(R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\big)$$

To bound the first term, write

$$\begin{aligned}
&\sum_{\lambda_{1,2}} \sum_{\sigma < \max\{\lambda_{1,2}\}} P_\sigma\big(R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\big)\\
&= \sum_{\lambda_1 < \lambda_2} \sum_{\sigma < \max\{\lambda_{1,2}\}} P_\sigma\big(R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\big)\\
&+ \sum_{\lambda_1 \geq \lambda_2} \sum_{\sigma < \max\{\lambda_{1,2}\}} P_\sigma\big(R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\big)
\end{aligned} \tag{1.3.23}$$

Then we get for the first term (after summing over λ_1 only)

$$\sigma^{\frac{1}{2}+\frac{\nu}{2}-\delta-}\|R^{-\frac{3}{2}}P_{<\lambda_2}fP_{\lambda_2}g\|_{L^2} \le \sigma^{\frac{1}{2}+\frac{\nu}{2}-\delta-}\|R^{-\frac{3}{2}}P_{<\lambda_2}f\|_{L^\infty}\|P_{\lambda_2}g\|_{L^2}$$
$$\lesssim \Big(\frac{\sigma}{\lambda_2}\Big)^{\frac{1}{2}+\frac{\nu}{2}-\delta-}\lambda_2^{-\delta}\|R^{-\frac{3}{2}}P_{<\lambda_2}f\|_{L^\infty}\|P_{\lambda_2}g\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}$$

which is more than acceptable in the case $\sigma < \lambda_2$ (allowing for square summation over σ, λ_2), even taking into account the logarithmic loss from the factor $\|R^{-\frac{3}{2}}P_{<\lambda_2}f\|_{L^\infty}$ on the right, thus controlling the first term on the right of (1.3.23) in case $g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$. If on the other hand $g \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$, we get

$$\sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|R^{-\frac{3}{2}}P_{<\lambda_2}fP_{\lambda_2}g\|_{L^2} \le \sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|R^{-\frac{3}{2}}P_{<\lambda_2}f\|_{L^\infty}\|P_{\lambda_2}g\|_{L^2}$$
$$\le \Big(\frac{\sigma}{\lambda_2}\Big)^{\frac{1}{2}+\frac{\nu}{2}-}\lambda_2^{-\frac{1}{2}+2\delta}\|R^{-\frac{3}{2}}P_{<\lambda_2}f\|_{L^\infty}\|P_{\lambda_2}g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}$$

Here, we can again square-sum over σ, λ_2. Next, for the case $\lambda_1 \ge \lambda_2$ in (1.3.23), the argument is identical to the one above provided $g \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}}L^\infty$, $P_{<\lambda}f, g \in \log\lambda R^{\frac{3}{2}}L^\infty$ uniformly in $\lambda > 1$. On the other hand, if $g \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$, we have

$$\sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\|_{L^2} \le \sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|P_{\lambda_1}f\|_{L^2}\|R^{-\frac{3}{2}}P_{\lambda_2}g\|_{L^\infty}$$
$$\lesssim \sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|P_{\lambda_1}f\|_{L^2}\lambda_2\|P_{\lambda_2}g\|_{L^2}$$
$$\lesssim \Big(\frac{\sigma}{\lambda_1}\Big)^{\frac{1}{2}+\frac{\nu}{2}-}\lambda_2^{-\frac{\nu}{2}+2\delta}\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|P_{\lambda_2}g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}$$

by lemma 8.3 in [17]. Again this is more than enough to square-sum over σ, λ_1 and sum over λ_2. These observations handle the case of small σ. We note that the L^2-type estimates for

$$R^{-j}fg,\ j \in \{0,1\}$$

are just the same and in fact easier under the corresponding assumptions in the lemma. Next, consider the case $\sigma \ge \max\{\lambda_{1,2}\}$. If $\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}f\big) \in L^\infty$ uniformly in $\lambda > 1$, then we get

$$\|\mathscr{L}^k\big(\chi_{R<1}R^{-\frac{3}{2}}P_{<\sigma}fP_{\lambda_2}g\big)\|_{L^2} \lesssim \lambda_2^k\|P_{\lambda_2}g\|_{L^2}$$

Here we have used lemma 8.4 in [17]. It follows that

$$\|P_\sigma\big(\chi_{R<1}R^{-\frac{3}{2}}P_{<\sigma}fP_{\lambda_2}g\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \Big(\frac{\lambda_2}{\sigma}\Big)^{k-\frac{1}{2}-\frac{\nu}{2}+}\|P_{\lambda_2}g\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}$$

which suffices to square-sum over σ. On the other hand, including a smooth cutoff $\chi_{R\ge1}$, and assuming $\lambda_2 \geq \lambda_1$ as we may, we get

$$\begin{aligned}\|\mathscr{L}^k\big(\chi_{R\geq 1}R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{L^2} &\lesssim \sum_{m+l\leq k}\|\nabla_R^m P_{\lambda_1}f\|_{L^\infty}\|\nabla_R^l P_{\lambda_2}g\|_{L^2}\\ &\lesssim \sum_{m+l\leq k}\lambda_1^{-0+}\|\nabla_R^m P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|\nabla_R^l P_{\lambda_2}g\|_{L^2}\\ &\lesssim \lambda_1^{-0+}\lambda_2^{-\frac{1}{2}-\frac{\nu}{2}+}\sum_{m+l\leq k}\lambda_1^m\lambda_2^l\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|P_{\lambda_2}g\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\end{aligned}$$

whence

$$\|P_\sigma\big(\chi_{R\geq 1}R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \lambda_1^{-0+}\big(\frac{\lambda_2}{\sigma}\big)^{k-\frac{1}{2}-\frac{\nu}{2}+}\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|P_{\lambda_2}g\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}.$$

This again suffices to square-sum over σ and l^1-sum over λ_1. If $g \in H^{1+\frac{\nu}{2}-2\delta-}$, we note that the argument for lemma 8.5 in [17] furnishes the bound

$$\|\mathscr{L}^k\big(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{L^2} \lesssim \lambda_1^{\frac{1}{2}}\lambda_2^k\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}}}\|P_{\lambda_2}g\|_{L_2^\rho},$$

and so we get

$$\|\mathscr{L}^kP_\sigma\big(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \lambda_1^{\frac{1}{2}}\lambda_2^k\sigma^{\frac{1}{2}+\frac{\nu}{2}-}\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}}}\|P_{\lambda_2}g\|_{L_2^\rho}$$

The duality argument in [17] then yields (provided $\sigma > \lambda_2 \geq \lambda_1$)

$$\|P_\sigma\big(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \big(\frac{\lambda_2}{\sigma}\big)^{\frac{1}{2}}\lambda_1^{-\frac{\nu}{2}+2\delta}\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|P_{\lambda_2}g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}$$

which suffices for the case $\lambda_1 \leq \lambda_2 < \sigma$, and the necessary summations. For the case $\lambda_1 \geq \lambda_2$, one instead uses that

$$\|\mathscr{L}^k\big(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{L^2} \lesssim \lambda_1^k\lambda_2\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2},$$

which implies that

$$\|P_\sigma\big(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \big(\frac{\lambda_1}{\sigma}\big)^{k-\frac{1}{2}-\frac{\nu}{2}+}\lambda_2^{-\frac{\nu}{2}+2\delta}\|P_{\lambda_1}f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\|P_{\lambda_2}g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}$$

This is again enough to sum over all dyadic frequencies. Finally, to obtain the inclusion

$R^{-\frac{3}{2}}fg \in R^{\frac{3}{2}}L^\infty$, we observe that

$$
\begin{aligned}
&|g(R)| = \\
&|\int_0^\infty \phi(R,\xi)x(\xi)\rho(\xi)\,d\xi| \lesssim R^{\frac{3}{2}}\Big(\int_0^\infty x^2(\xi)\langle\xi\rangle^{2+\nu-2\delta-}\rho(\xi)\,d\xi\Big)^{\frac{1}{2}}\Big(\int_0^\infty \langle\xi\rangle^{-2-\nu+2\delta+}\rho(\xi)\,d\xi\Big)^{\frac{1}{2}} \\
&\lesssim \|g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}
\end{aligned}
$$

whence $|g(R)| \lesssim R^{\frac{3}{2}}\|g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}$. This implies

$$
\|R^{-3}fg\|_{L^\infty} \lesssim \|f\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}\|g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}+R^{\frac{3}{2}}L^\infty}
$$

We also need to control $\|R^{-\frac{3}{2}}P_{<\lambda}\big(R^{-\frac{3}{2}}fg\big)\|_{L_x^\infty}$ for arbitrary dyadic $\lambda > 1$. Write

$$
\begin{aligned}
&R^{-\frac{3}{2}}P_{<\lambda}\big(R^{-\frac{3}{2}}fg\big) \\
&= R^{-\frac{3}{2}}P_{<\lambda}\big(\chi_{R\sim\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}fg\big) && (1.3.24)\\
&+ R^{-\frac{3}{2}}P_{<\lambda}\big(\chi_{R\ll\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}fg\big) && (1.3.25)\\
&+ R^{-\frac{3}{2}}P_{<\lambda}\big(\chi_{R\gg\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}fg\big) && (1.3.26)
\end{aligned}
$$

for smooth cutoffs $\chi_{R\sim\widetilde{R}}$ etc. To bound the first term on the right, we use that the operator $P_{<\lambda}$ is given by integration against the kernel

$$
K_{<\lambda}(R,\widetilde{R}) = \chi_{R\sim\widetilde{R}}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\,d\xi \tag{1.3.27}
$$

for a smooth kernel function $\chi_{\xi<\lambda}$. We claim that this kernel maps L^∞ continuously into L^∞. Taking this for granted, we obtain for the term (1.3.24) the bound

$$
\begin{aligned}
\|R^{-\frac{3}{2}}P_{<\lambda}\big(\chi_{R\sim\widetilde{R}}R^{-\frac{3}{2}}fg\big)\|_{L^\infty} &\lesssim \sup_{\widetilde{R}\sim 2^j}\|P_{<\lambda}\big(\chi_{\widetilde{R}}\widetilde{R}^{-3}fg\big)\|_{L^\infty}\\
&\lesssim \|f\|_{R^{\frac{3}{2}}L^\infty}\|g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}+R^{\frac{3}{2}}L^\infty}
\end{aligned}
$$

To get the L^∞-boundedness of (1.3.27), write

$$
\begin{aligned}
&\chi_{R\sim\widetilde{R}}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\,d\xi\\
&= \sum_{N\,\text{dyadic}}\chi_{R\sim\widetilde{R}\sim N}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{\xi<\min\{\lambda,N^{-2}\}}\phi(\widetilde{R},\xi)\,d\xi\\
&+ \sum_{N\,\text{dyadic}}\chi_{R\sim\widetilde{R}\sim N}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{N^{-2}\leq\xi<\lambda}\phi(\widetilde{R},\xi)\,d\xi
\end{aligned}
$$

Using theorem 1.2.3, one infers for the first term on the right the bound

$$|\sum_{N\,\text{dyadic}}\chi_{R\sim\widetilde{R}\sim N}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{\xi<\min\{\lambda,N^{-2}\}}\phi(\widetilde{R},\xi)\,d\xi|\lesssim\frac{\chi_{R\sim\widetilde{R}}}{R},$$

and this kind of kernel is easily seen to act boundedly on L^∞. For the oscillatory integral kernel above, write schematically, using theorem 1.2.4, theorem 1.2.5

$$\begin{aligned}
&\chi_{R\sim\widetilde{R}\sim N}\int_0^\infty \rho(\xi)\phi(R,\xi)\chi_{N^{-2}\le\xi<\lambda}\phi(\widetilde{R},\xi)\,d\xi\\
&=\chi_{R\sim\widetilde{R}\sim N}\int_0^\infty \rho(\xi)a(\xi)^2\xi^{-\frac{1}{2}}e^{\pm iR\xi^{\frac{1}{2}}\pm i\widetilde{R}\xi^{\frac{1}{2}}}\big(1+O(\frac{1}{R\xi^{\frac{1}{2}}})\big)^2\chi_{N^{-2}<\xi<\lambda}\,d\xi\\
&=\chi_{R\sim\widetilde{R}\sim N}\big[-N\widehat{\chi_1}(N(\pm R\pm\widetilde{R}))+\lambda\widehat{\chi_1}(\lambda(\pm R\pm\widetilde{R}))\big]+O(|\log(\frac{R\pm\widetilde{R}}{R})|\frac{\chi_{R\sim\widetilde{R}\sim N}}{R}),
\end{aligned}$$

for a suitable smooth and compactly supported function χ_1, and the L^∞-boundedness of the (sum over dyadic N of) these operators follows easily. This concludes the estimate for (1.3.24). To bound the term (1.3.25), we break it into a number of constituents, using theorem 1.2.2 - theorem 1.2.5. Write

$$\begin{aligned}
&R^{-\frac{3}{2}}P_{<\lambda}\big(\chi_{R\ll\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}fg\big)\\
&=R^{-\frac{3}{2}}\int_0^\infty\int_0^\infty \chi_{R\ll\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}f(\widetilde{R})g(\widetilde{R})\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\phi(R,\xi)\rho(\xi)\,d\xi d\widetilde{R}
\end{aligned}$$

with smooth cutoffs $\chi_{R\ll\widetilde{R}}, \chi_{\xi<\lambda}$. We further split this as

$$R^{-\frac{3}{2}}\int_0^\infty\int_0^\infty \chi_{R\ll\widetilde{R}}\widetilde{R}^{-\frac{3}{2}}f(\widetilde{R})g(\widetilde{R})\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\phi(R,\xi)\rho(\xi)\,d\xi d\widetilde{R}$$

$$=R^{-\frac{3}{2}}\int_0^\infty\int_0^\infty \chi_{R\ll\widetilde{R}}\chi_{R^2\xi\ge1}\widetilde{R}^{-\frac{3}{2}}f(\widetilde{R})g(\widetilde{R})\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\phi(R,\xi)\rho(\xi)\,d\xi d\widetilde{R} \tag{1.3.28}$$

$$+R^{-\frac{3}{2}}\int_0^\infty\int_0^\infty \chi_{R\ll\widetilde{R}}\chi_{R^{-2}>\xi\ge\widetilde{R}^{-2}}\widetilde{R}^{-\frac{3}{2}}f(\widetilde{R})g(\widetilde{R})\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\phi(R,\xi)\rho(\xi)\,d\xi d\widetilde{R} \tag{1.3.29}$$

$$+R^{-\frac{3}{2}}\int_0^\infty\int_0^\infty \chi_{R\ll\widetilde{R}}\chi_{\widetilde{R}^2\xi<1}\widetilde{R}^{-\frac{3}{2}}f(\widetilde{R})g(\widetilde{R})\chi_{\xi<\lambda}\phi(\widetilde{R},\xi)\phi(R,\xi)\rho(\xi)\,d\xi d\widetilde{R} \tag{1.3.30}$$

For the first term on the right, (1.3.28), both functions $\phi(R,\xi)$, $\phi(\widetilde{R},\xi)$, are in the oscillatory regime, and can thus be written schematically as

$$\phi(R,\xi)=a(\xi)\xi^{-\frac{1}{4}}e^{\pm iR\xi^{\frac{1}{2}}}\big(1+O(\frac{1}{R\xi^{\frac{1}{2}}})\big),\ \phi(\widetilde{R},\xi)=a(\xi)\xi^{-\frac{1}{4}}e^{\pm i\widetilde{R}\xi^{\frac{1}{2}}}\big(1+O(\frac{1}{\widetilde{R}\xi^{\frac{1}{2}}})\big).$$

By applying integration by parts with respect to the variable $\xi^{\frac{1}{2}}$, we find

$$|(1.3.28)|\lesssim\int_0^\infty \chi_{R\ll\widetilde{R}}(\frac{R}{\widetilde{R}})^N\widetilde{R}^{-4}|f(\widetilde{R})||g(\widetilde{R})|\,d\widetilde{R}$$

and from here we get

$$\|(1.3.28)\|_{L^\infty} \lesssim \|f\|_{R^{\frac{3}{2}}L^\infty}\|g\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}+R^{\frac{3}{2}}L^\infty}$$

For the intermediate term (1.3.29), one uses the expansions

$$\phi(R,\xi) = \phi_0(R) + \phi_0(R)O(R\xi^2),\ \phi(\tilde{R},\xi) = a(\xi)\xi^{-\frac{1}{4}}e^{\pm i\tilde{R}\xi^{\frac{1}{2}}}\big(1 + O(\frac{1}{\tilde{R}\xi^{\frac{1}{2}}})\big),$$

and then uses again integration by parts with respect to $\xi^{\frac{1}{2}}$, obtaining bounds just as in the preceding case. Finally, for the remaining integral (1.3.30), using the expansions

$$\phi(R,\xi) = \phi_0(R) + \phi_0(R)O(R\xi^2),\ \phi(\tilde{R},\xi) = \phi_0(\tilde{R}) + \phi_0(\tilde{R})O(\tilde{R}\xi^2),$$

we find

$$\begin{aligned}|(1.3.30)| &\lesssim \big(\int_0^\lambda \rho(\xi)\langle\xi\rangle^{-2}\,d\xi\big)\|\frac{f}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty}\|\frac{g}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty}\\ &\lesssim \log\lambda\|\frac{f}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty}\|\frac{g}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty}\end{aligned}$$

If we replace here the outer factor $R^{-\frac{3}{2}}$ by R^{-1}, one instead gets the bound

$$\lesssim \big(\int_0^\lambda \rho(\xi)\langle\xi\rangle^{-\frac{9}{4}}\,d\xi\big)\|\frac{f}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty}\|\frac{g}{\tilde{R}^{\frac{3}{2}}}\|_{L^\infty},$$

and so we no longer get a logarithmic correction for $\|R^{-1}P_{<\lambda}(R^{-\frac{3}{2}}fg)\|_{L^\infty}$.

Observe that in order to bound $\|R^{-1}P_{<\lambda}R^{-1}fg\|_{L^\infty}$, and under the assumption $f \in RL^\infty, g \in RL^\infty$, proceeding just as before, we encounter instead of (1.3.30) a similar expression with the factors $R^{-\frac{3}{2}}, \tilde{R}^{-\frac{3}{2}}$ replaced by $R^{-1}, \tilde{R}^{-1}$. This we can then bound by

$$\lesssim \|\frac{f}{R}\|_{L^\infty}\|\frac{g}{R}\|_{L^\infty}\int_{R\ll\tilde{R}}\tilde{R}^{\frac{5}{2}}R^{\frac{1}{2}}\tilde{R}^{-4}\,d\tilde{R} \lesssim \|\frac{f}{R}\|_{L^\infty}\|\frac{g}{R}\|_{L^\infty},$$

thus without logarithmic correction. It is clear that the remaining cases occuring in the bound for (1.3.25), as well as for (1.3.24), are easier for the expression $\|R^{-1}P_{<\lambda}(R^{-1}fg)\|_{L^\infty}$, and hence omitted. The bound for (1.3.26) is more of the same. This completes the proof of the lemma. □

Lemma 1.3.12. *Assume that all of f, g, h are either in $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}}L^\infty$ as well as with their frequency localized constituents $P_{<\lambda}(\cdot) \in \log\lambda R^{\frac{3}{2}}L^\infty$ and $\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}(\cdot)\big) \in L^\infty$, $l \geq 0$, uniformly in $\lambda > 1$, or in $H_\rho^{1+\frac{\nu}{2}-2\delta-}$. Then we have*

$$R^{-3}fgh \in H^{\frac{1}{2}+\frac{\nu}{2}-2\delta-} \cap R^{\frac{3}{2}}L^\infty,\ P_{<\lambda}(R^{-3}fgh) \in \log\lambda R^{\frac{3}{2}}L^\infty,\ P_{<\lambda}(R^{-3}fgh) \in RL^\infty$$

with the latter two inclusions uniformly in $\lambda > 1$*. Also, if* $h_j \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}} L^\infty$ *and further* $P_{<\lambda} h_j \in RL^\infty$ *as well as* $\chi_{R<1} \nabla_R^l \big(R^{-1} P_{<\lambda} h_j\big) \in L^\infty$*,* $l \geq 0$*, uniformly in* λ*, or else* $h_j \in H_\rho^{1+\frac{\nu}{2}-2\delta-}$*, for* $j = 1, 2, \ldots, 2N$*, then we have*

$$R^{-3} f g h \prod_{j=1}^{N} (\frac{1}{R} h_{2j} h_{2j-1}) \in H^{\frac{1}{2}+\frac{\nu}{2}-2\delta-}$$

We also get

$$R^{-\frac{3}{2}} f g \prod_{j=1}^{N} (\frac{1}{R} h_{2j} h_{2j-1}) \in H^{\frac{1}{2}+\frac{\nu}{2}-\delta-}$$

For the proof of this, one notes that by the preceding lemma,

$$R^{-\frac{3}{2}} f g \in H^{\frac{1}{2}+\frac{\nu}{2}-\delta-} \cap R^{\frac{3}{2}} L^\infty, \, R^{-\frac{3}{2}} P_{<\lambda}(R^{-\frac{3}{2}} f g) \in \log \lambda L^\infty$$

uniformly in $\lambda > 1$. Also, we have

$$R^{-1} P_{<\lambda}(R^{-\frac{3}{2}} f g) \in L^\infty$$

uniformly in $\lambda > 1$. By another application of the preceding Lemma, we obtain the conclusions concerning $R^{-3} f g h$. The conclusion concerning

$$R^{-3} f g h \prod_{j=1}^{N} (\frac{1}{R} h_{2j} h_{2j-1})$$

then follows by further iterative application of the preceding lemma. The last statement of the lemma follows similarly.

We can now complete the estimate for the remaining two nonlinear source terms. Observe that we can write the first of these, (1.3.20) in the form

$$\begin{aligned}
\frac{\sin(2u_{2k})}{2R^{\frac{3}{2}}} \big(1 - \cos(2R^{-\frac{1}{2}} \widetilde{\varepsilon})\big) &= \frac{\sin(2u_{2k})}{2R^{\frac{3}{2}}} \big(R^{-\frac{1}{2}} \widetilde{\varepsilon}\big)^2 q(R^{-1} \widetilde{\varepsilon}^2) \\
&= \frac{\sin(2u_{2k})}{2R} R^{-\frac{3}{2}} \widetilde{\varepsilon}^2 q(R^{-1} \widetilde{\varepsilon}^2)
\end{aligned}$$

where $q(\cdot)$ is real analytic. By combining Lemma 1.3.12 and Lemma 1.3.10 (with $\alpha = \frac{1}{2} + \frac{\nu}{2}$) and using

$$\frac{\sin(2u_{2k})}{2R} \in IS(1, \mathcal{Q}),$$

we find

Lemma 1.3.13. *We have the source term bound*

$$\|\frac{\sin(2u_{2k})}{2R^{\frac{3}{2}}}\big(1-\cos(2R^{-\frac{1}{2}}\tilde{\varepsilon})\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-\delta-}}\lesssim\|\tilde{\varepsilon}\|^2_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}$$

provided we have

$$\|\tilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}\lesssim 1,\ \|R^{-\frac{3}{2}}P_{<\lambda}\tilde{\varepsilon}\|_{L^\infty}\lesssim 1,\ \|\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}\tilde{\varepsilon}\big)\|_{L^\infty}\lesssim 1,\ l\geq 0 \tag{1.3.31}$$

uniformly in $\lambda>1$. *The same bound obtains with the space* $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty$ *on the right replaced by* $H_\rho^{1+\frac{\nu}{2}-2\delta-}$*, and the bounds* (1.3.31) *replaced by*

$$\|\tilde{\varepsilon}\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}\lesssim 1.$$

To deal with the last source term (1.3.21), we write

$$\frac{\cos(2u_{2k})}{2R^{\frac{3}{2}}}\big(2R^{-\frac{1}{2}}\tilde{\varepsilon}-\sin(2R^{-\frac{1}{2}}\tilde{\varepsilon})\big)=\cos(2u_{2k})\frac{\tilde{\varepsilon}^3}{R^3}q(R^{-1}\tilde{\varepsilon}^2)$$

where again $q(\cdot)$ is real analytic. Combining Lemma 1.3.12, and Lemma 1.3.10, we infer

Lemma 1.3.14. *We have the source term bound*

$$\|\frac{\cos(2u_{2k})}{2R^{\frac{3}{2}}}\big(2R^{-\frac{1}{2}}\tilde{\varepsilon}-\sin(2R^{-\frac{1}{2}}\tilde{\varepsilon})\big)\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-2\delta-}}\lesssim\|\tilde{\varepsilon}\|^3_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}$$

provided we have

$$\|\tilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}\lesssim 1,\ \|R^{-\frac{3}{2}}P_{<\lambda}\tilde{\varepsilon}\|_{L^\infty}\lesssim 1,\ \|\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}\tilde{\varepsilon}\big)\|_{L^\infty}\lesssim 1,\ l\geq 0 \tag{1.3.32}$$

uniformly in $\lambda>1$. *The same bound obtains with the space* $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty$ *on the right replaced by* $H_\rho^{1+\frac{\nu}{2}-2\delta-}$*, and* (1.3.32) *replaced by*

$$\|\tilde{\varepsilon}\|_{H_\rho^{1+\frac{\nu}{2}-2\delta-}}\lesssim 1.$$

1.3.3 The first iterate

Recall that we have constructed the zeroth iterate via

$$x_0(\tau,\xi)=(U\lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(\tilde{e}_{2k-1}\big)\big])(\tau,\xi),$$

so that Proposition 1.3.7 applies. Now we construct the *first iterate* via

$$x_1(\tau,\xi)=(Uf_0)(\tau,\xi),$$

where we have

$$\begin{aligned}-f_0 =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_0 + (\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi, \mathcal{K}_0]\big]x_0\\ &+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x_0 + \lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_0) + \widetilde{e}_{2k-1}\big)\big] - c\tau^{-2}x_0\end{aligned}$$

Observe that we have

$$\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_0 \in \tau^{-N-1}L_\rho^{2,\frac{\nu}{2}-}$$

Due to the smoothing property of $\mathcal{K}_0$, we conclude that

$$2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_0 \in \tau^{-N-2}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}$$

Further, we get the even better bounds (which however we won't fully exploit)

$$(\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi, \mathcal{K}_0]\big]x_0 \in \tau^{-N-2}L_\rho^{2,1+\frac{\nu}{2}-}$$

$$\partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x_0 - c\tau^{-2}x_0 \in \tau^{-N-2}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-},$$

while from Lemma 1.3.13, Lemma 1.3.14 as well as (1.3.22), we infer

$$\|\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_0)\|_{L_\rho^{\frac{1}{2}+\frac{\nu}{2}-2\delta-}} \lesssim \tau^{-N-2}$$

The key conclusion of all this is then the following

Lemma 1.3.15. *The difference* $\Delta x_1 := x_1 - x_0$ *satisfies the bound*

$$\|\Delta x_1(\tau,\cdot)\|_{L_\rho^{2,1+\frac{\nu}{2}-2\delta-}} \lesssim N^{-1}\tau^{-N},$$

$$\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\Delta x_1(\tau,\cdot)\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-2\delta-}} \lesssim N^{-1}\tau^{-N-1}$$

The implicit constant is independent of N, whence picking N large enough makes the overall constant on the right $\ll 1$.

Note that the key aspect here is the gain of one derivative (which translates to a $1/2$ weight in terms of ξ). This is essential in order to replicate the reasoning used above for the new source term

$$\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_1)\big]$$

where we define the first iterate on the physical side via

$$\widetilde{\varepsilon}_1(\tau, R) = \int_0^\infty \phi(R,\xi) x_1(\tau,\xi)\rho(\xi)\,d\xi = \int_0^\infty \phi(R,\xi)\Delta x_1(\tau,\xi)\rho(\xi)\,d\xi + \int_0^\infty \phi(R,\xi) x_0(\tau,\xi)\rho(\xi)\,d\xi$$

Thus from Proposition 1.3.7, the remark following it, as well as Corollary 1.3.9 and the preceding lemma, we infer that we can write

$$\widetilde{\varepsilon}_1(\tau,\cdot) = \widetilde{\varepsilon}_1^{(1)}(\tau,\cdot) + \widetilde{\varepsilon}_1^{(2)}(\tau,\cdot),$$

where we have

$$\widetilde{\varepsilon}_1^{(1)}(\tau,\cdot) \in \tau^{-N}\big(H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}} L^\infty\big), \nabla_R^l\big(R^{-\frac{3}{2}} P_{<\lambda}\widetilde{\varepsilon}_1^{(1)}(\tau,\cdot)\big) \in \tau^{-N} L^\infty,\ l \geq 0,$$

the latter inclusion uniformly in $\lambda > 1$, while we have

$$\widetilde{\varepsilon}_1^{(2)}(\tau,\cdot) \in \tau^{-N} H_\rho^{1+\frac{\nu}{2}-2\delta-}$$

This is precisely the kind of structure necessary to invoke the bound (1.3.22) as well as Lemma 1.3.13, Lemma 1.3.14.

1.3.4 Higher iterates

Here we have

$$x_j(\tau,\xi) = (U f_{j-1})(\tau,\xi),\ j \geq 2,$$

and we have

$$\begin{aligned} -f_{j-1} =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_{j-1} + (\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi, \mathcal{K}_0]\big]x_{j-1} \\ &+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x_{j-1} + \lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_{j-1}) + \widetilde{e}_{2k-1}\big)\big] - c\tau^{-2}x_{j-1} \end{aligned}$$

Then using induction on j and exactly the same bounds as in the preceding subsection, one infers with

$$\Delta x_j = x_j - x_{j-1}$$

the bounds

$$\|\Delta x_j(\tau,\cdot)\|_{L_\rho^{2,1+\frac{\nu}{2}-2\delta-}} \lesssim N^{-j}\tau^{-N},$$

$$\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi\partial_\xi\big)\Delta x_j(\tau,\cdot)\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-2\delta-}} \lesssim N^{-j}\tau^{-N-1}$$

The desired fixed point of (1.3.4) is now obtained via

$$x(\tau,\xi) = x_0(\tau,\xi) + \sum_{j=1}^{\infty} \Delta x_j(\tau,\xi)$$

and is a function in $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$, such that $\partial_\tau x(\tau,\cdot) \in H_\rho^{\frac{\nu}{2}-}$. Due to Lemma 9.1 of [17], the corresponding

$$\varepsilon(\tau,R) := R^{-\frac{1}{2}} \int_0^\infty \phi(R,\xi) x(\tau,\xi)\rho(\xi)\, d\xi$$

satisfies $\varepsilon(\tau,\cdot) \in \tau^{-N} H_{\mathbb{R}^2}^{1+\nu-}$, as well as $\partial_\tau \varepsilon(\tau,\cdot) \in \tau^{-N-1} H_{\mathbb{R}^2}^{\nu-}$. This is the desired solution.

2 Full blow-up range for co-rotaional wave maps to surfaces of revolution

It is also interesting to consider the same problem in a more general situation when the target manifold is a surface of revolution. A work on this case which is parallel of [17] was due to Cârstea [3]. However, as in [17], the blow-up range in [3] is not optimal. In this paper, we will indicate how to combine the techniques of [3, 13] to obtain the optimal blow-up range in this setting. For more detailed references concerning the blow-up dynamic of wave maps one can refer to [13].

Let $\mathscr{N}$ be a surface of revolution equipped with a Riemannian metric

$$ds^2 = d\rho^2 + g(\rho)^2 d\theta$$

for $\mathscr{N}$ being produced by rotating the graph of a function $y = f(z)$ around the z-axis.

Remark 2.0.16. A detailed discussion of what properties g shall satisfy can be found in [3]. Those properties will give the relevant properties of the ground state (harmonic map) which one needs to use when proving some intermediate conclusions when building the approximate solutions. What this paper will focus on is the main difference and changes raised because of the new setting of target manifold we have. However, no changes are required according to the parts of proofs relevant to g. Thus, we refer the reader to [3] for the details about what properties g need to satisfy.

In the case of surfaces of revolution, the equation for co-rotational wave maps takes a form similar to (0.0.3). A simple computation (see [3]) gives

$$-\partial_t^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u = \frac{f(u)}{r^2}, \quad f(u) = g(u)g'(u). \tag{2.0.1}$$

Pick a stationary solution with finite energy for (2.0.1) as was shown in [3]. We state our result

Theorem 2.0.17. *For any $\nu > 0$, there exist $T > 0$ and co-rotational initial data (f, g) with*

$$(f - \pi, g) \in H^{1+\frac{\nu}{2}-}_{\mathbb{R}^2} \times H^{\frac{\nu}{2}-}_{\mathbb{R}^2}$$

a[1] *solution* $u(t,r)$, $t \in (0,T]$ *which blows up at time* $t=0$ *and has the following representation:*

$$u(t,r) = Q(\lambda(t)r) + \varepsilon(t,r)$$

where $\lambda(t) = t^{-1-\nu}$*, and such that the function*

$$(\theta, r) \longrightarrow \big(e^{i\theta}\varepsilon(t,r), e^{i\theta}\varepsilon_t(t,r)\big) \in H^{1+\nu-}(\mathbb{R}^2) \times H^{\nu-}(\mathbb{R}^2)$$

uniformly in t*. Also, we have the asymptotic as* $t \to 0$

$$\mathscr{E}_{loc}\big(\varepsilon(t,\cdot)\big) \lesssim t^{\nu}\log^2 t$$

2.1 An overview of the proof for theorem 2.0.17

In the work on co-rotational wave maps to S^2 target by Krieger, Schlag, and Tataru [17], it was found that solutions exist with the blow-up rate $\lambda(t) = t^{-1-\nu}$, for the continnum of blow-up rates of any $\nu > 1/2$. In a joint work of the author and Krieger [13], this range was extended to $\nu > 0$. Since the construction to be described in this paper is based heavily on that of the previously mentioned works, we recall for the convenience of the readers the basic scheme.

The method of construction relies on building approximate solutions starting from the initial guess $u(t,r) \approx Q(\lambda(t)r)$ where $Q(r)$ is the stationary ground state. If one naively plugs in $Q(\lambda(t)r)$ into the equation, the error term generated is $(r\lambda'(t))^2 Q''(\lambda(t)r) + r\lambda''(t)Q'(\lambda(t)r)$, which turns out to be "large". Thus one cannot directly use perturbative techniques to find the solution. Instead, we first correct the error (within the past light cone from the singularity) using an iterative scheme, until the error becomes sufficiently small. In the following we will using the notation $R = \lambda(t)r$."

Theorem 2.1.1. *Assume* $k \in \mathbf{N}$*. There exists an approximate solution* $u_{2k-1}(R)$ *within the backwards light cone from the singularity for* (2.0.1) *which can be written as*

$$u_{2k-1}(t,r) = Q(R) + \frac{c_k}{(t\lambda)^2} R\log(1+R^2) + \frac{\widetilde{c}_k}{(t\lambda)^2} R + O\Big(\frac{(\log(1+R^2))^2}{(t\lambda)^2}\Big)$$

with a corresponding error of size

$$\begin{aligned} e_{2k-1} &:= \Big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\Big) u_{2k-1} - \frac{f(u_{2k-1})}{2r^2} \\ &= \big(1 - \frac{R}{\lambda t}\big)^{-\frac{1}{2}+\nu} O\Big(\frac{R(\log(1+R^2))^2}{(t\lambda)^{2k}}\Big) \end{aligned}$$

Here the implied constant in the $O(\dots)$ *symbols are uniform in* $t \in (0,\delta]$ *for some* $\delta = \delta(k) > 0$ *sufficiently small.*

[1] Here we use the identification of the wave map with a function $u(t,r)$ as before.

This is proved by means of an iterative scheme (see section 2.3) that improves the error at each double step. Actually at each step we approximately solve the wave equation first close to $r = 0$ then close to the light cone $r = t$. In both cases it will reduce to solve an ODE (a Sturm-Louville equation). It is important to observe here that the restriction $\nu > \frac{1}{2}$ imposed in [3] does not come in at this stage; in fact, any $\nu > 0$ will suffice. For the sake of readability, only theorem 2.1.1 as well as the finer representation of the errors as specified in (2.3.8) will be used in the final proof of the main theorem (the exact solution) in section 2.2. The reader can treat section 2.3 as a black box if desired only up to these statements.

In section 2.2, we complete the approximate solution to the exact one by adding correction via the ansatz $u(t,r) = u_{2k-1}(t,r) + \varepsilon(t,r)$. Before giving the relevant PDE of such term ε. We first renormalize the time t into $\tau := \nu^{-1} t^{-\nu}$, note that with respect to this time, we get

$$\lambda(\tau) := \lambda(t(\tau)) = (\nu\tau)^{\frac{1+\nu}{\nu}}$$

We also have the re-scaled variable $R = \lambda(\tau) r$ respectively. We shall assume that

$$|e_{2k-1}(t,r)| \lesssim \tau^{-N}, \; r \leq t$$

for some sufficiently large N, which is possible if we choose k large enough. We shall also assume the fine structure of e_{2k-1} as in section 2.3, and more specifically as in (2.3.8). We can complete the approximate solution u_{2k-1} to an exact solution $u = u_{2k-1} + \varepsilon$. , where ε solves the following equation:

$$\begin{aligned}
&-\Big[\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)^2 + \frac{\lambda_\tau}{\lambda}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)\Big]\varepsilon + \Big(\partial_R^2 + \frac{1}{R}\partial_R - \frac{f'(Q(R))}{R^2}\Big)\varepsilon \\
&= -\frac{1}{\lambda^2}[e_{2k-1} + N_{2k-1}(\varepsilon)],
\end{aligned} \tag{2.1.1}$$

where

$$N_{2k-1}(\varepsilon) = \frac{1}{r^2}[f'(u_0)\varepsilon - f(u_{2k-2} + \varepsilon) + f(u_{2k-2})]. \tag{2.1.2}$$

After change of function $\widetilde{\varepsilon}(\tau,R) = R^{1/2}\varepsilon(\tau,R)$, (2.1.1) becomes

$$\big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)\widetilde{\varepsilon} - \mathscr{L}\widetilde{\varepsilon} = \lambda^{-2} R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big) \tag{2.1.3}$$

The strategy is to formulate this equation in terms of the Fourier coefficients of $\widetilde{\varepsilon}$ with respect to the generalized Fourier basis associated with $\mathscr{L}$ given by

$$\mathscr{L} = -\partial_R^2 + \frac{3}{4R^2} + V(R), \quad V(R) = -\frac{1}{R^2}[1 - f'(Q(R))]$$

with $Q(R)$ the ground state. Dealing with (2.1.3), one needs to develop some rather sophisticated spectral theory. The spectral theory of $\mathscr{L}$ follows from [3] (more exactly [17]), we refer the reader to [17] to see a detailed discussion. To find $\widetilde{\varepsilon}$, one employes a fixed point argument in suitable Banach spaces, and it is here, in the treatment of the nonlinear terms with singular weights, that the restriction on ν comes in (see [3, 17]). More precisely (see lemma 7.2 in [3]), this condition is needed there to make sufficient embedding between suitable function spaces to control the nonlinear terms.

In [13], the authors overcome this restriction (in the case while target manifold is sphere). We will employ this method in our problem (while target manifold is surfaces of revolution) in section 2.2 which is as following:

Firstly, by a more closely analysis of the 'zeroth iterate' (to be explained below) for $\widetilde{\varepsilon}$. We show that one can split this into the sum of two terms, one of which has a regularity gain which lands us in the regime in [17] is applicable, the other of which does not gain regularity but satisfies an a priori L^∞ bound near the symmetry axis $R = 0$. So the relevant terms with a singular weight $R^{-3/2}$ at $R = 0$, such as $R^{-3/2}\widetilde{\varepsilon}^2$ (see section 2.2) can be estimated without adding any conditions for the regularity. The reason why they can control the part of the zeroth iterate near $R = 0$ comes from the fact that the singular behavior of the approximate solution from the first part of the construction and the error it generates is localized to the boundary of the light cone. Then, by writing the equation for the distorted Fourier transform of $\widetilde{\varepsilon}$ we will show that the higher iterates all differ from the zeroth iterate by terms with a smoothness gain. This will then suffice to show the desired convergence.

Remark 2.1.2. The proof of Theorem 2.0.17, unsurprisingly, has large overlap with the constructions of [3, 13]. For brevity we will only indicate in this note the modifications necessary, and will refer the reader to [3, 13] for the proofs of many intermediate steps.

Remark 2.1.3. In the new situation, the main difficulty for proof of Theorem 2.0.17 is that we can not write the nonlinear term explicitly. Thus in the relevant step (see step 3 below) when constructing the approximate solutions and in the second part where the 'perturbative scheme' is introduced for the exact solutions, one needs to redo or adjust the proofs for the new nonlinear source term. In [13], the authors correct the inaccuracies in [17] according to the approximate solution step such as the omission of some logarithm factors in the algebra of the special function spaces. In out paper here, the different function spaces are used correspondingly to fix such inaccuracies in [3]. So some part of the arguments need to be restated during the construction of the approximate solutions.

2.2 Construction of the exact solutions

This is the very end of the proof of the main theorem. However this is where the 'key structure' is introduced following [13] to make it possible to relax the constraint on ν. For the readers who are interested in the construction of the approximate solutions, we give the proof in section 2.3.

On the base that an approximate solution has been constructed with a corresponding error term which decays rapidly in the renormalized time $\tau := \nu^{-1} t^{-\nu}$, we can complete the approximate solution u_{2k-1} to an exact solution $u = u_{2k-1} + \varepsilon$. After changing of function (which gives us a new relevant $\widetilde{\varepsilon}$, see section 2) and applying a *distorted Fourier transform*[2] to the equation of $\widetilde{\varepsilon}$ ((2.1.3) in section 2):

$$\big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)\widetilde{\varepsilon} - \mathscr{L}\widetilde{\varepsilon} = \lambda^{-2} R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big) \qquad (2.2.1)$$

One shall get a equation of the Fourier coefficients, which we call the *transport equation.*

The main difficulty is caused by the operator $R\partial_R$ which is not diagonal in the Fourier basis. To deal with this, we replace the distorted Fourier transform of $R\partial_R u$ with $2\xi\partial_\xi$ modulo an error which will be treated perturbatively. We define the error operator $\mathscr{K}$ by

$$\widehat{R\partial_R u} = -2\xi\partial_\xi \widehat{u} + \mathscr{K}\widehat{u}$$

where $\widehat{f} = \mathscr{F} f$ is the distorted Fourier transform.

To proceed further, we have to precisely understand the structure of the 'transference operator' $\mathscr{K}$. Make the

Definition 2.2.1. *We call an operator $\widetilde{\mathscr{K}}$ to be 'smoothing', provided it enjoys the mapping property*

$$\widetilde{\mathscr{K}} : L^{2,\alpha}_\rho \longrightarrow L^{2,\alpha+\frac{1}{2}}_\rho \quad \forall \alpha$$

For the definition of a weighted L^2-space $L^{2,\alpha}_\rho$, we have

$$\|u\|_{L^{2,\alpha}_\rho} := \Big(\int_0^\infty |u(\xi)|^2 \langle\xi\rangle^{2\alpha} \rho(\xi)\, d\xi\Big)^{\frac{1}{2}}$$

If we put the terms with a 'smooth' property to the right hand side of the equality in the transport equation. Then the Fourier coefficients (we call them $x(\tau,\xi)$) of $\widetilde{\varepsilon}$ with respect to the generalized Fourier basis satisfy

$$\mathscr{D}_\tau^2 x + \xi x = f(x, \widetilde{\varepsilon}), \qquad (2.2.2)$$

[2]Here the distorted Fourier transform is defined via combining one function $\phi(r,z)$ from the fundamental system for $\mathscr{L} - z$ and its inverse is given using the density function $\rho(\xi)$ of the spectral measure of $\mathscr{L}$, where $\mathscr{L}$ is a key operator raised from the exact solution's equation and $z \in \mathbb{C}$.
More precisely, the distorted Fourier transform is

$$\mathscr{F}: \quad \widehat{h}(\xi) := \int_0^\infty \phi(r,\xi) h(r) dr$$

when the inverse is

$$\mathscr{F}^{-1}: \quad h(r) := \int_0^\infty \phi(r,\xi)\widehat{h}(\xi)\rho(\xi) d\xi.$$

The detailed explanation for $\phi(r,z)$ and $\rho(\xi)$ is in [13, 17].

where we have the operator

$$\mathscr{D}_\tau := \partial_\tau - \frac{\lambda_\tau}{\lambda}[2\xi\partial_\xi + \frac{3}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}]$$

and

$$\begin{aligned} -f =& 2\frac{\lambda_\tau}{\lambda}\mathscr{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x + (\frac{\lambda_\tau}{\lambda})^2\big[\mathscr{K}^2 - (\mathscr{K}-\mathscr{K}_0)^2 - 2[\xi\partial_\xi, \mathscr{K}_0]\big]x \\ &+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathscr{K}_0 x + \lambda^{-2}\mathscr{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\tilde{\varepsilon}) + e_{2k-1}\big)\big] - c\tau^{-2}x \end{aligned} \tag{2.2.3}$$

For $\mathscr{K}_0$, according to [3] we give it as (see theorem 5.1[3])

$$\mathscr{K} = -\Big(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho(\eta)}\Big)\delta_0(\xi-\eta) + \mathscr{K}_0.$$

Remark 2.2.2. Although the problem dealt in [13] is different than ours, the process at this stage is very close. We refer the readers to [13] for those technical details we omit here when deducing the final transport equation (mainly the straightforward computation) and below for brevity.

The explicit solution of (2.2.2) is given as:

Lemma 2.2.3 ([13]). *The equation* (2.2.2) *is formally solved by the following parametrix*

$$x(\tau,\xi) = \int_\tau^\infty \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}S(\tau,\sigma,\lambda^2(\tau)\xi)f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma =: (Uf)(\tau,\xi) \tag{2.2.4}$$

One key fact from [13] is we have the following mapping property of the parametrix with respect to suitable Banach spaces:

Lemma 2.2.4 (lemma 5.6, [13]). *Introducing the norm*

$$\|f\|_{L^{2,\alpha;N}_\rho} := \sup_{\tau>\tau_0}\tau^N\|f(\tau,\cdot)\|_{L^{2,\alpha}_\rho},$$

we have

$$\|Uf\|_{L^{2,\alpha+\frac{1}{2};N-2}_\rho} \lesssim \|f\|_{L^{2,\alpha;N}_\rho}$$

provided N is sufficiently large.

For the future reference, we will use the following norm:

$$\|h\|_{H^\alpha_\rho} := \Big(\int_0^\infty x^2(\xi)\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi\Big)^{\frac{1}{2}}$$

where
$$h(R) = \int_0^\infty \phi(R,\xi)x(\xi)\rho(\xi)\,d\xi.$$

2.2.1 Zeroth, first and higher iterative schemes

After formulating (2.2.2) as an integral equation, we need to find a suitable fixed point, which will be the desired $x(\tau,\xi)$. We construct these via

$$x(\tau,\xi) = (Uf)(\tau,\xi) \tag{2.2.5}$$

with $f(x,\widetilde{\varepsilon})$ as in (2.2.4). To find such a fixed point, we use the iterative scheme

$$x_j(\tau,\xi) = (Uf_{j-1})(\tau,\xi),\ j \geq 1$$

The function f_j is given as

$$\begin{aligned} -f_j =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_j + (\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi,\mathcal{K}_0]\big]x_j \\ &+ \partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0 x_j + \lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_j) + \widetilde{e}_{2k-1}\big)\big] - c\tau^{-2}x_j \end{aligned} \tag{2.2.6}$$

The zeroth iterate in turn is defined via

$$x_0(\tau,\xi) = (U\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(e_{2k-1}\big)\big])(\tau,\xi);$$

We have the following proposition proved in [13]

Proposition 2.2.5 (proposition 5.7, [13]). *Replacing e_{2k-1} with $\widetilde{e}_{2k-1} \in H^{\frac{\nu}{2}-}_{RdR}$ where $\widetilde{e}_{2k-1}|_{r\leq t} = e_{2k-1}$, we can write*
$$x_0 = x_0^{(1)} + x_0^{(2)}$$
where
$$x^{(1)} \in \tau^{-N}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-},\quad x^{(2)} \in \tau^{-N}L_\rho^{2,1+\frac{\nu}{2}-}$$
and also
$$\chi_{R<1}\widetilde{\varepsilon}_0^{(1)}(\tau,R) = \chi_{R<1}\int_0^\infty \phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)d\xi \in \tau^{-N}R^{\frac{3}{2}}L^\infty,\quad \chi_{R\geq1}|\widetilde{\varepsilon}_0^{(1)}| \lesssim \tau^{-N}$$

We can rephrase it as following, which is identical to Corollary 5.9 in [13].

Proposition 2.2.6. *Denote by P_λ the frequency localizers*

$$\mathcal{F}\big(P_{<\lambda}f\big)(\xi) = \chi_{<\lambda}(\xi)\big(\mathcal{F}f\big)(\xi)$$

where $\chi_{<\lambda}(\xi)$ is a smooth cutoff function localizing to $\xi \lesssim \lambda$, as in [17]; here λ is a dyadic number.

Then we have

$$\chi_{R<1} P_{<\lambda} \widetilde{\varepsilon}_0^{(1)} \in \tau^{-N} R^{\frac{3}{2}} L^{\infty}$$

uniformly in $\lambda > 1$. *Furthermore, for any integer* $l \geq 0$, *we have*

$$\nabla_R^l R^{-\frac{3}{2}} P_{<\lambda} \widetilde{\varepsilon}_0^{(1)} = O(\tau^{-N})$$

uniformly in $\lambda > 1$.

Remark 2.2.7. This is the key structure from [13], with which the we are able to invoke lemma 2.2.11 to control the nonlinear term and prove (2.2.7) (see below).

Based on lemma 2.2.4, we know

$$\|U f_{j-1}\|_{L_\rho^{2,1+\frac{\nu}{2}-}} \lesssim \|f_{j-1}\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}}$$

For the first iterate, the estimate for the most terms in (2.2.6) follows the same arguments in [13]. We list the unchanged results (see [13] for proof) as following

$$\begin{aligned}
&(\partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi\partial_\xi) x_0 \in \tau^{-N-1} L_\rho^{2,\frac{\nu}{2}-} \\
&2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0(\partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi\partial_\xi) x_0 \in \tau^{-N-2} L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-} \\
&(\frac{\lambda_\tau}{\lambda})^2 [\mathcal{K}^2 - (\mathcal{K} - \mathcal{K}_0)^2 - 2[\xi\partial_\xi, \mathcal{K}_0]] x_0 \in \tau^{N-2} L_\rho^{2,1+\frac{\nu}{2}-} \\
&\partial_\tau (\frac{\lambda_\tau}{\lambda}) \mathcal{K}_0 x_0 - c\tau^{-2} x_0 \in \tau^{-N-2} L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}
\end{aligned}$$

For the nonlinear term, which is the key of the whole argument, we will prove the following in the next section (according to Lemma 3.4)

$$\lambda^{-2} R^{\frac{1}{2}} N_{2k-1}(R^{-\frac{1}{2}} \widetilde{\varepsilon}) \in \tau^{-N-2} L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-} \tag{2.2.7}$$

Let us for now accept the facts above and conclude here the key conclusion in this step

$$\|x_1(\tau,\cdot) - x_0(\tau,\cdot)\|_{L_\rho^{2,1+\frac{\nu}{2}-}} \lesssim N^{-1}\tau^{-N},$$

$$\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi\partial_\xi\big)\big(x_1(\tau,\cdot) - x_0(\tau,\cdot)\big)\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}} \lesssim N^{-1}\tau^{-N-1}$$

Then we define

$$\widetilde{\varepsilon}_1 = \int_0^\infty \phi(R,\xi)\big(x_1(\tau,\cdot) - x_0(\tau,\cdot)\big)\rho(\xi)d\xi + \int_0^\infty \phi(R,\xi) x_0(\tau,\xi)\rho(\xi)d\xi$$

which will allow us to write

$$\widetilde{\varepsilon}_1 = \widetilde{\varepsilon}^{(1)}(\tau,\cdot) + \widetilde{\varepsilon}^{(2)}(\tau,\cdot)$$

$\widetilde{\varepsilon}^{(1)}(\tau,\cdot)$ and $\widetilde{\varepsilon}^{(2)}(\tau,\cdot)$ satisfy exactly the kind of structure we need to invoke the bound for nonlinear source term in lemma 2.2.11. Continuing running the iterate scheme will give us the bounds

$$\|x_j(\tau,\cdot) - x_{j-1}(\tau,\cdot)\|_{L_\rho^{2,1+\frac{\nu}{2}-}} \lesssim N^{-j}\tau^{-N},$$

$$\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\big(x_j(\tau,\cdot) - x_{j-1}(\tau,\cdot)\big)\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}} \lesssim N^{-j}\tau^{-N-1}$$

This will close the fix point argument which proves we have

$$x_{\tau,\xi} \in H^{\frac{1}{2}+\frac{\nu}{2}-}, \quad \partial_\tau x_{\tau,\xi} \in H^{\frac{\nu}{2}-}.$$

Through lemma 7.1 in [3] (it was proven in [17]):

Lemma 2.2.8. *Assume $|\alpha| < \frac{\nu}{2} + \frac{3}{4}$, $g \in IS(1,\mathcal{Q})$. Then we have*

$$\|gf\|_{H_\rho^\alpha} \lesssim \|f\|_{H_\rho^\alpha}$$

It indicates the existence of the exact solution $\varepsilon(\tau,\cdot) \in \tau^{-N}H_{\mathbb{R}^2}^{1+\nu-}$, as well as $\partial_\tau\varepsilon(\tau,\cdot) \in \tau^{-N-1}H_{\mathbb{R}^2}^{\nu-}$.

2.2.2 The nonlinear source terms

We will give an analysis to the new nonlinear source term to complete our work in this section. We recall the following formula for the main source term:

$$\lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) = \frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f(u_{2k-2} + R^{-\frac{1}{2}}\widetilde{\varepsilon})R^{\frac{1}{2}} + f(u_{2k-2})R^{\frac{1}{2}}\big] \tag{2.2.8}$$

$$= \frac{1}{R^2}\big[f'(u_0) - f'(u_{2k-2})\big]\widetilde{\varepsilon} - \frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 2}\frac{1}{l!}f^{(l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^l \tag{2.2.9}$$

According to the preceding proposition, we have

$$x_0 \in \tau^{-N}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}$$

whence

$$\widetilde{\varepsilon}_0(\tau,\cdot) \in \tau^{-N}H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$$

This means that for the source terms, we need at least $H_\rho^{\frac{\nu}{2}-}$-regularity. In fact, we can do much better for the term $\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]$. Recall that

$$u_{2k-2} = u_0 + \sum_{j=1}^{2k-2} v_j$$

where we have

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^3\big(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}\big), \quad v_{2k} \in \frac{1}{(t\lambda)^{2k+2}} IS^3\big(R^3(\log R)^{2k-1}, \mathcal{Q}_k\big)$$

which implies

$$u_{2k-2} - u_0 \in \frac{1}{(t\lambda)^2} IS^3(R\log R, \mathcal{Q})$$

Moreover, we recall some useful results in [3, 17].

Lemma 2.2.9 (lemma 3.9-10, [3])**.** *$f^{(2k)}(u_0) \in IS^1(R^{-1})$ and $f^{(2k+1)}(u_0) \in IS^0(1)$. Moreover, if*

$$z \in \frac{1}{(t\lambda)^2} IS^1(R(\log R), \mathcal{Q}),$$

then

$$f^{(2k)}(u_0 + z(R)) \in \frac{1}{(t\lambda)^2} IS^1(R(\log R), \mathcal{Q})$$

and

$$f^{(2k+1)}(u_0 + z(R)) \in IS^0(1, \mathcal{Q}).$$

Thus for $\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]$, we have

$$\frac{f'(u_0) - f'(u_{2k-2})}{R^2} = \frac{\sum_{l\geq 2} \frac{1}{l!} f^{(l)}(u_0)(u_{2k-2} - u_0)^l}{R^2} \in \frac{1}{(t\lambda)^2} IS(1, \mathcal{Q})$$

and lemma 2.2.8 will give us the following bound

$$\|\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim (t\lambda)^{-2}\|\widetilde{\varepsilon}\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}} \tag{2.2.10}$$

To deal with the rest 'truly' nonlinear terms, we first split them into two parts

$$\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 2}\frac{1}{l!} f^{(l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^l =$$

$$\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 1}\frac{1}{l!} f^{(2l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l} \tag{2.2.11}$$

$$+\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 1}\frac{1}{l!} f^{(2l+1)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l+1} \tag{2.2.12}$$

We can write (2.2.11) in the form

$$R^{-\frac{3}{2}}\widetilde{\varepsilon}^2\sum_{l\geq 1}\frac{1}{l!}\frac{f^{(2l)}(u_0+u_{2k-2}-u_0)}{R}\big(R^{-1}\widetilde{\varepsilon}^2\big)^{l-1}$$

and meanwhile write (2.2.12) as

$$R^{-3}\widetilde{\varepsilon}^3\sum_{l\geq 1}\frac{1}{l!}f^{(2l+1)}(u_0+u_{2k-2}-u_0)\big(R^{-1}\widetilde{\varepsilon}^2\big)^{l-1}$$

According to Lemma 2.2.9, we observe that

$$\frac{f^{(2l)}(u_0+u_{2k-2}-u_0)}{R},\quad f^{(2l+1)}(u_0+u_{2k-2}-u_0)\in IS^0(1,Q).$$

Thus via Lemma (2.2.8), we can estimate the H^{α}_{ρ} norm of (2.2.11) and (2.2.12) by the H^{α}_{ρ} norm of

$$R^{-\frac{3}{2}}\widetilde{\varepsilon}^2q(R^{-1}\widetilde{\varepsilon}^2),\quad R^{-3}\widetilde{\varepsilon}^3q(R^{-1}\widetilde{\varepsilon}^2)$$

where α here is $\frac{1}{2}+\frac{\nu}{2}-$ and $q(\cdot)$ is a real analytic function.

We recall a very technical and crucial lemma proved in [13]

Lemma 2.2.10 (lemma 5.12, [13])**.** *Assume that all of f,g,h are either in $H_{\rho}^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^{\infty}$ as well as with their frequency localized constituents $P_{<\lambda}(\cdot)\in\log\lambda R^{\frac{3}{2}}L^{\infty}$ and $\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}(\cdot)\big)\in L^{\infty}$, $l\geq 0$, uniformly in $\lambda>1$, or in $H_{\rho}^{1+\frac{\nu}{2}-}$. Then we have*

$$R^{-3}fgh\in H^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^{\infty},\ P_{<\lambda}(R^{-3}fgh)\in\log\lambda R^{\frac{3}{2}}L^{\infty},\ P_{<\lambda}(R^{-3}fgh)\in RL^{\infty}$$

with the latter two inclusions uniformly in $\lambda>1$. Also, if $h_j\in H_{\rho}^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^{\infty}$ and further $P_{<\lambda}h_j\in RL^{\infty}$ as well as $\chi_{R<1}\nabla_R^l\big(R^{-1}P_{<\lambda}h_j\big)\in L^{\infty}$, $l\geq 0$, uniformly in λ, or else $h_j\in H_{\rho}^{1+\frac{\nu}{2}-}$, for $j=1,2,\ldots,2N$, then we have

$$R^{-3}fgh\prod_{j=1}^{N}(\frac{1}{R}h_{2j}h_{2j-1})\in H^{\frac{1}{2}+\frac{\nu}{2}-}$$

We also get

$$R^{-\frac{3}{2}}fg\prod_{j=1}^{N}(\frac{1}{R}h_{2j}h_{2j-1})\in H^{\frac{1}{2}+\frac{\nu}{2}-}$$

Invoke the conclusion from lemma 2.2.10, one can prove:

Lemma 2.2.11. *Providing*

$$\|\widetilde{\varepsilon}\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^{\infty}}\lesssim 1,\quad \|R^{-\frac{3}{2}}P_{<\lambda}\widetilde{\varepsilon}\|_{L^{\infty}}\lesssim 1,\quad \|\chi_{R<1}\nabla_R^l(R^{-\frac{3}{2}}P_{<\lambda}\widetilde{\varepsilon})\|_{L^{\infty}}\lesssim 1$$

uniformly in $\lambda > 1$ $l \geq 0$*, we have*

$$\|\frac{1}{R^2}\left[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\right]\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim (t\lambda)^{-2}\|\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}$$
$$\|\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 1}\frac{1}{l!}f^{(2l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \|\widetilde{\varepsilon}\|^2_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}$$
$$\|\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 1}\frac{1}{l!}f^{(2l+1)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l+1}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \|\widetilde{\varepsilon}\|^3_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}$$

The last two estimates' right hand side space can be replaced by $H^{\frac{1}{2}+\frac{\nu}{2}-}$ *with a change of the bound of* $\widetilde{\varepsilon}$ *by*

$$\|\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim 1.$$

2.3 The construction of the approximate solutions

To build the approximate solution as in theorem 2.1.1, we follow the scheme in [17]. We start from the stationary harmonic map[3] $Q(R)$. Setting $R = \lambda(t)r$ we take $u_0(t,x) = Q(\lambda(t)x)$ for $\lambda(t) = t^{-1-\nu}$ and then add corrections v_k iteratively $u_k = u_0 + \sum_{j=1}^k v_k$. In a first approximation we linearize the equation for the correction $\varepsilon = u - u_k$ around $\varepsilon = 0$ and substitute u_k by u_0. Then we have the linear approximate equation

$$\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\big)\varepsilon - \frac{1}{r^2}f'(u_0)\varepsilon \approx -e_k$$

From here we split into two different cases: considering the case $r \ll t$ when we expect the time derivative to play a lesser role thus we neglect it (where (2.3.1) below comes from); considering the case $r \approx t$ when the time and spatial derivative have the same strength. We can identify another principal variable, namely $a = r/t$ and think of ε as a function of $\varepsilon(t,a)$ so we can reduce this case to a Strum-Liouville problem in a which becomes singular at $a = 1$ (where(2.3.2) comes from). After each step of adding the correction, we also estimate the size of the errors. This makes each round of the scheme with four steps to go. For odd and even steps, we have different equations for the corrections v_k:

$$\big(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}f'(u_0)\big)v_{2k+1} = -e^0_{2k} \tag{2.3.1}$$
$$\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)v_{2k+2} = -e^0_{2k+1} \tag{2.3.2}$$

[3] The properties of ground state are needed to prove the spectral theory of $\mathscr{L}$. Since we will employ the same spectral theory as it is in [3], we refer the reader to section 2 [3] for the discussion of properties of such ground states,

with Cauchy zero data[4] at $r = 0$, and[5] where

$$e_k = \big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\big)u_k - \frac{1}{r^2}f(u_k) \tag{2.3.3}$$

$$e_{2k+1} = e_{2k}^1 - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1}), \quad e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k}) \tag{2.3.4}$$

$$N_{2k-1}(v) = \frac{1}{r^2}[f'(u_0)v - f(u_{2k-2}+v) + f(u_{2k-2})] \tag{2.3.5}$$

$$N_{2k}(v) = \frac{v}{r^2} - \frac{1}{r^2}[f(u_{2k-1}+v) - f(u_{2k-1})] \tag{2.3.6}$$

Remark 2.3.1. Note here a technical detail is we split e_k into $e_k = e_k^0 + e_k^1$ where e_k^0 is the so-called principle part and the rest e_k^1, the so-called higher order part, will be left and merge into the next step while analyzing the error v_{k+1} (will be precise below in step 1 and 3). Also we will switch to the principle variable 'a' for equation (2.3.2) in step 3 as already mentioned in the above section.

To formalize this scheme we need to define suitable function spaces in the light-cone

$$\mathscr{C}_0 = \{(t,r) : 0 \le r < t, 0 < t < t_0\}$$

to put our successive corrections and errors. They are following closely from those in [13].[6]

Definition 2.3.2. *For $i \in \mathbf{N}$, let $j(i) = i$ if ν is irrational, respectively $j(i) = 2i^2$ if ν is rational. Then*

- *$\mathcal{Q}$ is the algebra of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:*
  (i) q is analytic in $[0,1)$ with even expansion around $a = 0$.
  (ii) near $a = 1$ we have an absolutely convergent expansion of the form

$$\begin{aligned} q(a) = & q_0(a) + \sum_{i=1}^{\infty}(1-a)^{\beta(i)+\frac{1}{2}}\sum_{j=0}^{j(i)} q_{i,j}(a)\big(\log(1-a)\big)^j \\ & + \sum_{i=1}^{\infty}(1-a)^{\widetilde{\beta}(i)+\frac{1}{2}}\sum_{j=0}^{j(i)} \widetilde{q}_{i,j}(a)\big(\log(1-a)\big)^j \end{aligned}$$

  with analytic coefficients $q_0, q_{i,j}$, and $\beta(i) = i\nu$, $\widetilde{\beta}(i) = \nu i + \frac{1}{2}$.

- *$\mathcal{Q}_n$ is the algebra which is defined similarly, but also requiring $q_{i,j}(1) = 0$ if $i \ge 2n+1$.*

We also define the space of functions obtained by differentiating $\mathcal{Q}_n$:

[4] The coefficients are singular at $r = 0$, therefore this has to be given a suitable interpretation below (see remark 2.3.6).

[5] There is a typo in [3] for the sign of the term $f(u_{2k-2})$. This does not influence the result in [3] but it matters for our analysis for the nonlinear source terms in later section.

[6] One shall note that those definitions are very natural according to a direct computation for the first round of the iterative scheme (see [13] for the case when target manifold is sphere).

Definition 2.3.3. *Define $\mathcal{Q}'$ as in the preceding definition but replacing $\beta(i)$ by $\beta'(i) := \beta(i) - 1$, and similarly for $\mathcal{Q}'_n$.*

Definition 2.3.4. *$S^n(R^k(\log R)^l)$ is the class of analytic functions $v : [0,\infty) \to \mathbb{R}$ with the following properties:*
(i) v vanishes of order n at $R = 0$.
(ii) v has a convergent expansion near $R = \infty$

$$v = \sum_{\substack{0\le j\le l+i\\ i\ge 0}} c_{ij} R^{k-i} (\log R)^j$$

The final function space $S^m(R^k(\log R)^l, Q_n)$ is defined slightly different than Definition 3.5 in [13] where we add an extra 'b' into it. This is simply for applying the results from [3] later. We state it here precisely.

Definition 2.3.5. *(Definition 3.5, [13]) Introduce the symbols*

$$b = \frac{\left(\log(1+R^2)\right)^2}{(t\lambda)^2},\ b_1 = \frac{\left(\log(1+R^2)\right)}{(t\lambda)^2},\ b_2 = \frac{1}{(t\lambda)^2}$$

Pick t sufficiently small such that all b, b_1, b_2, when restricted to the light cone $r \le t$ are of size at most b_0.

- *$S^m(R^k(\log R)^l, \mathcal{Q}_n)$ is the class of analytic functions $v : [0,\infty) \times [0,1) \times [0, b_0]^3 \to \mathbb{R}$ so that*
  (i) v is analytic as a function of R, b, b_1, b_2,

  $$v : [0,\infty) \times [0, b_0]^3 \to \mathcal{Q}_n$$

  (ii) v vanishes to order m at $R = 0$.
  (iii) v admits a convergent expansion at $R = \infty$,

  $$v(R, \cdot, b, b_1, b_2) = \sum_{\substack{0\le j\le l+i\\ i\ge 0}} c_{ij}(\cdot, b, b_1, b_2) R^{k-i} (\log R)^j$$

  where the coefficients $c_{ij} : [0, b_0]^3 \to \mathcal{Q}_n$ are analytic with respect to $b, b_{1,2}$.

- *$IS^m(R^k(\log R)^l, \mathcal{Q}_n)$ is the class of analytic functions w inside the cone $r < t$ which can be represented as*

  $$w(t, r) = v(R, a, b, b_1, b_2),\ v \in S^m(R^k(\log R)^l, \mathcal{Q}_n)$$

  and $t > 0$ sufficiently small.

Remark 2.3.6. The functional spaces $S^m(R^k(\log R)^l, Q_n)$ satisfy some good asymptotic behaviors (for example, they vanish in order m at $R = 0$) so the existence of the solutions to equation

(2.3.1) and (2.3.2) will make sense in those spaces although the coefficients are singular at $R = 0$ in general.

Following the method in [17], the idea for proving theorem 2.1.1 is to inductively show that we can choose the corrections v_k to be in relevant function spaces:

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^3\big(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}\big) \tag{2.3.7}$$

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1}\big) \tag{2.3.8}$$

$$v_{2k} \in \frac{1}{(t\lambda)^{2k+2}} IS^3\big(R^3(\log R)^{2k-1}, \mathcal{Q}_k\big) \tag{2.3.9}$$

$$t^2 e_{2k} \in \frac{1}{(t\lambda)^{2k}} [IS^1\big(R^{-1}(\log R)^{2k}, \mathcal{Q}_k\big) + \langle b, b_1, b_2\rangle [IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_k\big)] \tag{2.3.10}$$

and the starting error e_0 satisfying

$$e_0 \in IS^1(R^{-1})$$

Here we denote by $\langle b, b_1, b_2\rangle$ the ideal generated by b, b_1, b_2 inside the algebra generated by b, b_1, b_2. Now we give a brief outline of the proof for 2.0.17:

Proof. First one shall check $e_0 \in IS^1(R^{-1})$, this can be done by a direct computation (see step 0 in [3]). Then assuming (2.3.7 – 2.3.10) hold up to $k-1$, the first task would be proving (2.3.7) for k.

Step 1: *For* $e_{2k-2}, k \geq 1$, *proves* v_{2k-1} *satisfies* (2.3.7).

For this one first needs to choose the right 'principal part' of e_{2k-2} which we call e^0_{2k-2}. This is done by throwing away the 'higher order parts', which we call e^1_{2k-2} and which belong to the same space as e_{2k-1}. The way to do it is as following: when $k = 1$ we let $e^0_0 := e_0$, if $k > 1$, we let $e^0_{2k-2} := e_{2k-2}(R, a, 0)$ with the setting $b, b_1, b_2 = 0$. By changing into variable R, equation (2.3.1) becomes:

$$(t\lambda)^2 L v_{2k-1} = -t^2 e^0_{2k-2}.$$

Here the operator L is

$$L := \partial_R^2 + \frac{1}{R}\partial_R - \frac{f'(u_0)}{R^2}$$

To get the desired result, one needs to prove the following lemma:

Lemma 2.3.7. *The solution of $Lv = \varphi \in S^1(R^{-1}(\log R)^{2k-2})$, with $v(0) = v'(0) = 0$, has the regularity*

$$v \in S^3(R(\log R)^{2k-1}).$$

This is already proven as Lemma 3.11 in [3], so we conclude (2.3.7).

Step 2: *Choose v_{2k-1} as in* (2.3.7) *with error e_{2k-1} satisfying* (2.3.8).

According to the definition of e_{2k-1} above, we have

$$t^2 e_{2k-1} = t^2 e^1_{2k-2} - t^2 \partial_t^2 v_{2k-1} + t^2 N_{2k-1}(v_{2k-1})$$

Since in the former step we treat a as a parameter and now we will defreeze it, some extra terms will show up while calculating the error e_{2k-1}. To be more precise, the amended term $t^2 e_{2k-1}$ we need to deal with is as following (note that $t^2 e^1_{2k-2}$ is proved automatically thanks to the assumptions)

$$t^2 e_{2k-1} = t^2 N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

where $E^2 v_{2k-1}$ is the term in $\partial_t^2 v_{2k-1}$ with no derivation on the a variable, and the term $E^a v_{2k-1}$ is the terms in $(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r) v_{2k-1}$ where derivative hits the a variable (the extra terms from defreezing of a are included here). To prove all those terms in 2.3.8, we refer the reader to step 2 in [3].

Step 3: *Given e_{2k-1} as in* (2.3.8), *construct v_{2k} as in* (2.3.9)

Here we have to diverge slightly from [3], since our definition of the algebra $S^m(R^k \log R^l)$ is different (we follow the definition in [13]). Since the equation (2.3.2) for v_{2k} is identical with equation (3.2) for v_{2k} in [13]. We follow the same arguments of step 2 in [13].

Assume

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1})$$

is given. We begin by isolating the leading component e^0_{2k-1} which includes the terms of top degree in R as well as those of one degree less (the rest will merge into e_{2k}, see step 4 below).

Thus we write

$$t^2 e^0_{2k-1} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} a q_j(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{q}_j(a)(\log R)^j$$

Consider the following equation

$$t^2 \widetilde{L}(v_{2k}) = t^2 e^0_{2k-1}$$

where $\widetilde{L}$ is

$$\widetilde{L} := -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}$$

Homogeneity considerations suggest that we should look for a solution v_{2k} which has the form (notice here we already switched into R)

$$v_{2k} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} W^j_{2k}(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{W}^j_{2k}(a)(\log R)^j$$

The one-dimensional equations for W^j_{2k}, $\widetilde{W}^j_{2k}$ are obtained by matching the powers of $\log R$. Then we conjugate out the power of t and rewrite the systems in the a variable, we get (see step 2 in [13] for details)

$$\mathscr{L}_{(2k-1)\nu} W^j_{2k} = a q_j(a) - F_j(a)$$
$$\mathscr{L}_{2k\nu} \widetilde{W}^i_{2k} = \widetilde{q}_i(a) - \widetilde{F}_i(a)$$

the definition of $\mathscr{L}_\beta$ is following [13]. Solving this system with Cauchy data at $a = 0$ yields solutions which satisfy

$$W^j_{2k}(a) \in a^3 \mathscr{Q}_k, \quad j = \overline{0, 2k-1}$$
$$\widetilde{W}^i_{2k} \in a^2 \mathscr{Q}_k, \quad i = \overline{0, 2k}$$

This is guaranteed by lemma 3.9 from [17]

To finish this step, we need to make a adjustment for v_{2k} because of the singularity of $\log R$ at $R = 0$. Also, we need to make sure that v_{2k} has order 3 vanishing at $R = 0$. Thus we define v_{2k} as

$$v_{2k} :=$$
$$\frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} W^j_{2k}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^j + \frac{1}{(t\lambda)^{2k}} \frac{R}{(1+R^2)^{\frac{1}{2}}} \sum_{j=0}^{2k} \widetilde{W}^j_{2k}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^j$$

We will get a large error near $R = 0$, but it is not very important since the purpose of the correction is to improve the error near large R. Since $a = R/t\lambda$, it's easy to pull out a a^3 factor from W's and a^2 from $\widetilde{W}$'s to see that we have (2.3.9).

Step 4: *Show that the error e_{2k} generated by $u_{2k} = u_{2k-1} + v_{2k}$ satisfies* (2.3.10).

Write

$$t^2 e_{2k} = t^2(e_{2k-1} - e^0_{2k-1}) + t^2\left(e^0_{2k-1} - (-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})(v_{2k})\right) + t^2 N_{2k}(v_{2k})$$

where we recall that except the nonlinear term $t^2 N_{2k}(v_{2k})$ the rest is proved satisfying (2.3.10) following the same arguments as step 3 in [13]. For the term $t^2 N_{2k}(v_{2k})$, the main method here is to split the nonlinear term in three parts

$$\begin{aligned}-t^2 N_{2k}(v_{2k}) = I + II + III =& a^{-2}\left[\left(f(u_{2k-1} + v_{2k}) - f(u_{2k-2}) - f'(u_{2k-1})\right)v_{2k}\right] \\ &+ a^{-2}\left[\left(f'(u_{2k-1}) - f'(u_0)\right)v_{2k}\right] + a^{-2}\left[\left(f'(u_0) - 1\right)v_{2k}\right]\end{aligned}$$

and prove each of them lies in a sub-space of what we need in (2.3.10)

$$\begin{aligned}
I &\in a^6 \frac{1}{(t\lambda)^{2k}} \sum_{\beta=b,b_{1,2}} \beta IS^1\left(R(\log R)^{2k-1}, \mathcal{Q}'_k\right) \\
II &\in a^2 \frac{1}{(t\lambda)^{2k}} \sum_{\beta=b,b_{1,2}} \beta IS^1\left(R(\log R)^{2k-1}, \mathcal{Q}'_k\right) \\
III &\in a^2 \frac{1}{(t\lambda)^{2k}} IS^3\left(R^{-1}(\log R)^{2k}, \mathcal{Q}_k\right)
\end{aligned}$$

The arguments to prove those mimic section 3.8.3 in [3].

Remark 2.3.8. One might have doubts since the function space $IS^k(R^m(\log R)^l)$ we are using here is different than [3]. To verify this, one just needs to see that the function spaces defined in [3] are the subspaces of our new defined function space in [13]. Thus the argument in [3] applies to our case.

Iteration of **Step 1** - **Step 4** immediately furnishes the proof of Theorem 2.1.1 .

□

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Education

Imperial College(IC), London, United Kingdom 2014.10-
M.Sc. in Quantum fields and fundamental forces

École Polytechnique Fédérale de Lausanne(EPFL), Switzerland 2011.09-2014.10
Ph.D. in Mathematics, (master skipped)
Thesis topic: Optimal blow-up range for critical wave maps
Advisor: Prof. Joachim Krieger.

China Agriculture University(CAU), Beijing, China 2007.09-2011.07
B.Sc. in pure and applied Mathematics
Thesis topic: Generalized dressing method for toda lattice hierarchy
Advisor: Prof. Xiaojun Liu.

Experience

Doctoral Researcher in Chair of PDE at EPFL, Switzerland 2011.09-2014.10
• Work on two cooperate projects and one independent project of wave maps under the supervision of Prof. Krieger
• Finish and submit three papers relevant to different projects

Teaching Assistant at EPFL and Lausanne Uni., Switzerland 2011.09-2014.10
• Teach the exercise courses on: Analysis I-III, Algebra, Geometry, General Mathematics
• Monitor exams and grade the papers (class size: 150-300)

Undergrad Research Assistant at CAU, Beijing, China 2008.09-2011.05
• Work on three different projects with three different professors from CAU on integrable system and soliton waves; elliptic PDE; bifurcation and chaos
• Publish one of the results with Prof. Liu which is my undergraduate thesis.

Internship in Agricultural Bank of China, Yangzhou, China 2009.08
Department of sales:
• Revise and confirm sales data and learn the operation of counter accounting, balance and personal business
• Learn and practice the rules, daily operation and management in accounting balance and individual/enterprise customers' products

Internship in Agricultural Development Bank of China, Taizhou, China 2008.08
Vice president office:
• Analyze data of loans for business in summertime grain and oil harvest and give suggestions for risk management
• Assist vice president's daily operation management and investigate the main enterprise customers' financial performance

Publications

C. Gao, J. Krieger and A. Dasgupta, On global regularity for systems of nonlinear wave equations with the null-condition, 2014, submitted.

C. Gao, Full blow-up range for co-rotaional wave maps to surfaces of revolution, preprint 2014, arXiv:1409.0672, submitted.

C. Gao and J. Krieger, Optimal polynomial blow up range for critical wave maps, preprint 2014, arXiv:1403.7356, accepted by CPAA.

X. Liu and C. Gao, Generalized dressing method for the extended 2-D Toda lattice hierarchy and its reductions, Science China Mathematics, 2011(54):365-380.

Awards

Funding for doctoral student in the form of annual salary, EPFL	2011-2014
Scholarship of Scientific Innovation, CAU	2010
Scholarship of Academic Excellence, CAU	2008-2010
Awards for scientific olympic competition in China	2001-2007

Skills

IT: Microsoft Office, Photoshop, LaTeX, SAS, SPSS, C, R, Python, Matlab, Octave and Mathematica.
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