



Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

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On minimal trajectories for mobile sampling of bandlimited fields

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ARTICLE INFO

Article history:

Received 30 December 2013

Received in revised form 27 October 2014

Accepted 23 November 2014

Available online xxxx

Communicated by Patrick Flandrin

MSC:

94A20

94A12

Keywords:

Spatial field sampling

Bandlimited field sampling

Mobile sensing

Sensor trajectories

Path density

Convex spectrum

Beurling density

ABSTRACT

We study the design of sampling trajectories for stable sampling and the reconstruction of bandlimited spatial fields using mobile sensors. The spectrum is assumed to be a symmetric convex set. As a performance metric we use the path density of the set of sampling trajectories that is defined as the total distance traveled by the moving sensors per unit spatial volume of the spatial region being monitored. Focusing first on parallel lines, we identify the set of parallel lines with minimal path density that contains a set of stable sampling for fields bandlimited to a known set. We then show that the problem becomes ill-posed when the optimization is performed over all trajectories by demonstrating a feasible trajectory set with arbitrarily low path density. However, the problem becomes well-posed if we explicitly specify the stability margins. We demonstrate this by obtaining a non-trivial lower bound on the path density of an arbitrary set of trajectories that contain a sampling set with explicitly specified stability bounds.

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1. Introduction

The reconstruction of a function from given measurements is a fundamental task in data processing and occupies numerous directions of research in mathematics and engineering. A typical problem requires the reconstruction or approximation of a physical field from pointwise measurements. A field may be a distribution of temperatures or water pollution or a solution to a diffusion equation, in mathematical terminology a field is simply a smooth function of several variables. The standard assumption on the smoothness is that the field is bandlimited to a compact spectrum. If the spectrum is a fundamental

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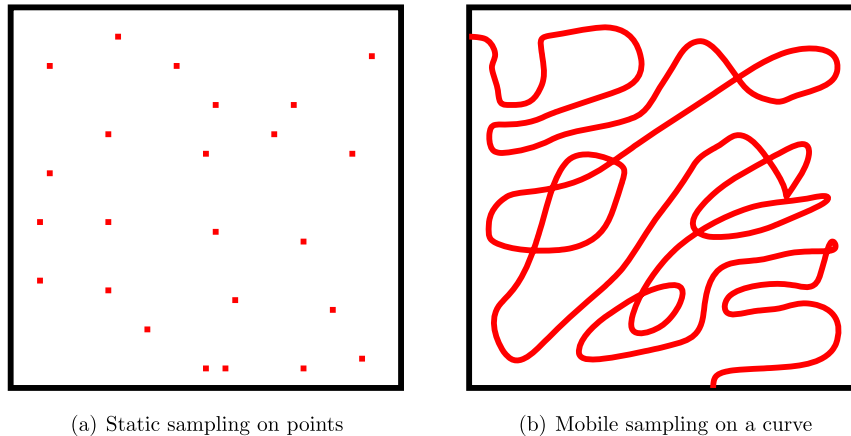


Fig. 1. Two approaches for sampling a field in \mathbb{R}^2 .

domain of a lattice in \mathbb{R}^d or a symmetric convex polygon in \mathbb{R}^2 , then there exist precise reconstruction formulas from sufficiently many samples in analogy to the Shannon–Whittaker–Kotelnikov sampling theorem [15,21].

Let

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(r) e^{-2\pi i \langle \omega, r \rangle} dr, \quad \omega \in \mathbb{R}^d, \quad (1)$$

be the Fourier transform¹ of $f \in L^1(\mathbb{R}^d)$ or $f \in L^2(\mathbb{R}^d)$, where i denotes the imaginary unit and $\langle \omega, r \rangle$ denotes the scalar product between vectors ω and r in \mathbb{R}^d . We say that f is bandlimited to the closed set $\Omega \subset \mathbb{R}^d$, if its Fourier transform \widehat{f} is supported on Ω . In this case we write

$$\mathcal{B}_\Omega := \{f \in L^2(\mathbb{R}^d) : \widehat{f}(\omega) = 0 \text{ for almost every } \omega \notin \Omega\} \quad (2)$$

for the space of fields with finite energy bandlimited to the spectrum Ω . In the context of field estimation we always assume that the spectrum is a compact, symmetric, convex set.

The classical theory of sampling and reconstructing of such high-dimensional bandlimited fields dates back to Petersen and Middleton [21] in signal analysis and to Beurling [5] in harmonic analysis. Both identified conditions for reconstructing such fields from their point measurements in \mathbb{R}^d . Further research on non-uniform sampling generated more results on conditions for perfect reconstruction from samples taken at non-uniformly distributed spatial locations. See [9,10] and the survey [1]. Previous work deals primarily with the problem of reconstructing the field from measurements taken by a collection of static sensors distributed in space, like that shown in Fig. 1(a). In this case the performance metric for quantifying the efficiency of a sampling scheme is the spatial density of samples. This is the average number of sensors per unit volume required for the stable sampling of the monitored region.

In this paper we investigate a different method for the acquisition of the samples, which we call *mobile sampling*. The samples are taken by a mobile sensor that moves along a continuous path, as is shown in Fig. 1(b). In such a case it is often relatively inexpensive to increase the spatial sampling rate along the sensor's path while the main cost of the sampling scheme comes from the total distance that needs to be traveled by the moving sensor. Hence it is reasonable to assume that the sensor can record the field values at an arbitrarily high but finite resolution on its path.

¹ Note that in [23] and [26] the Fourier transform was defined without the 2π in the exponent.

The new method for the acquisition of samples changes the mathematical nature of the problem completely. When using samples from static sensors, we need to establish a sampling inequality with evaluations of the form

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{B}_\Omega, \quad (3)$$

for constants $A, B > 0$ independent of f .

For mobile sampling, we need to establish a “continuous” sampling inequality of the form

$$A\|f\|^2 \leq \int_{\mathbb{R}^d} |f(r)|^2 d\mu(r) \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{B}_\Omega, \quad (4)$$

where μ is the sum of line integrals along the paths.

Again, the performance metric should reflect the cost required for the data acquisition. For (3) the appropriate metric is the average number of sensors, i.e., samples, per unit volume. For (4) some of us have argued in [23] and in [26] that the relevant metric should be the average path length traveled by the sensors per unit volume (or area, if $d = 2$). We call this metric the *path density*. Such a metric is directly relevant in applications like environmental monitoring using moving sensors [25,22]. In retrospect this metric is also useful in designing k -space trajectories for Magnetic Resonance Imaging (MRI) [4], where the path density can be used as a proxy for the total scanning time per unit area in k -space.

The continuous sampling inequality (4) raises many interesting questions both for engineers and for mathematicians. On the mathematical side are the abstract construction of continuous frames in the sense of [2, Chaps. 3 and 5] or [8] or the analysis of sampling measures and their properties, see [14,20] for a theory of sampling measures for Fock spaces and for Bergmann spaces. On the engineering side, we need to design concrete, realizable trajectories with a small path density for bandlimited fields with convex spectrum. This problem was introduced by some of us in [23] and [26] and answered for the special case of trajectory sets that consist of a union of uniformly spaced lines.

The contribution in this article is twofold. First, we study arbitrary trajectory sets of parallel lines and derive a necessary condition for the minimal path density in the style of Landau’s famous result in [13]. Extending the results in [23,26,27] we show, in [Theorem 3.2](#), that the minimal path density achievable by sampling along trajectories of arbitrary parallel lines is exactly the area of the maximal hyperplane section of the spectrum. We work under the standard assumption that the spectrum of the signals is convex and symmetric (although some results holds for more general spectra, see [Section 5](#)).

At first glance, the sampling along parallel lines seems to be an easy generalization of point sampling, because it can be reduced to the sampling problem in smaller dimensions. However, even this case offers some interesting and challenging problems that we did not envision before. For instance, in [Section 3](#) we will use the existence of universal samplings sets as established by Olevsky and Ulanovsky [19] and by Matei and Meyer [16] in order to prove that the frame bounds are uniform for sections of convex sets. In addition, this result enriches our knowledge about the properties of universal sampling sets. For another crucial argument we need the Brunn–Minkowski inequality [11].

Of course, the mathematician’s immediate instinct is to study more general sets of trajectories and try to prove a result analogous to Landau’s necessary condition for the path density. We show in [Proposition 4.1](#) that such a result cannot hold by constructing stable trajectory sets with arbitrarily small path density. Thus in a sense there is no optimal configuration of paths and the problem of optimizing the path density is ill-posed. This answers a question raised in [26]. However, as soon as we minimize over trajectory sets with given stability parameters A, B (uniform frame bounds) the optimization problem becomes well-posed. Our main density result ([Theorem 4.8](#)) shows that the path density for a stable set of trajectories is bounded below by an expression involving the stability parameters and the geometry of the spectrum.

This is a report on a successful and fertile collaboration between engineers and mathematicians. We, the mathematicians, are intrigued by the questions that motivate mobile sensing. Although the mathematical literature has investigated generalizations of sampling (the theorems of Sereda–Logvinenko and the theory of sampling measures) for the sake of generalization, we would never have dreamt of the particular conditions on the paths that are imposed by practical considerations (see condition **(C2)**). We, the engineers, are intrigued by the mathematical subtleties that popped up at every corner and subsequently led to an extended theory of path sampling.

The paper is organized as follows. In Section 2 we describe the formal problem statement. Then, in Section 3, we characterize the minimal density of sampling trajectories consisting of parallel lines. Section 4 treats the problem of optimizing over arbitrary trajectories, and Section 5 presents some conclusions. The proofs of some technical lemmas needed throughout the article are postponed to Section 6, so as not to obstruct the flow of the article.

Notation. We use $\langle \cdot, \cdot \rangle$ to denote the canonical inner product on \mathbb{R}^d and $L^2(\mathbb{R}^d)$, and e_k to denote the unit vector along the k -th coordinate axis. For $u \in \mathbb{R}^d$ we denote the hyperplane orthogonal to u through the origin by $u^\perp = \{x \in \mathbb{R}^d : \langle x, u \rangle = 0\}$, and $\mathcal{P}_{u^\perp} S$ denotes the orthogonal projection of a set $S \subset \mathbb{R}^d$ onto the hyperplane u^\perp . For a set $S \subset \mathbb{R}^d$ we use $|S|$ to denote the volume of S with respect to Lebesgue measure. By $B_a^d(x)$ we denote the closed Euclidean ball of radius a centered at $x \in \mathbb{R}^d$ and $B_a^d = B_a^d(0)$, and by $Q_a(x) = x + [-a, a]^d$ we denote the cube of width $2a$ centered at x . The cardinality of a finite set A is denoted by $\#A$.

Let $I = [a, b]$ be a bounded interval and $\gamma : I \rightarrow \mathbb{R}^d$ is a curve in \mathbb{R}^d . We say that γ is rectifiable, if $\ell(\gamma) = \sup \sum_{k=0}^{n-1} \|\gamma(t_{k+1}) - \gamma(t_k)\|$ is finite, where the supremum is taken over all finite partitions $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$. In this case $\ell(\gamma)$ is called the arc length of γ . Every piecewise differentiable curve is rectifiable.

If quantities X and Y satisfy the condition that there exist $A, B > 0$ with

$$AY \leq X \leq BY,$$

we write $X \asymp Y$. We also use the notation $X \lesssim Y$ to indicate that there exists $B > 0$ such that $X \leq BY$. To compare the size of functions $g, h : \mathbb{R} \rightarrow \mathbb{R}^+$, we use the Landau notation O and o . The symbol $h = O(g(x))$ means that there exist $k > 0$ and y such that for all $x > y$, we have $|h(x)| \leq kg(x)$, and $h = o(g(x))$ if $\lim_{x \rightarrow \infty} \frac{h(x)}{g(x)} = 0$.

We say that a set of points $A \subset \mathbb{R}^d$ is *uniformly discrete* or *separated* if $\inf\{\|x - y\| : x, y \in A, x \neq y\} > 0$, i.e., there exists $r > 0$ such that for any two distinct points $x, y \in A$ we have $\|x - y\| > r$. For example, a lattice in \mathbb{R}^d is uniformly discrete, but a sequence in \mathbb{R}^d converging to a point in \mathbb{R}^d is not. The lower and upper Beurling densities of $A \subseteq \mathbb{R}^d$ are

$$D^-(A) := \liminf_{a \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(A \cap B_a^d(x))}{|B_a^d|},$$

$$D^+(A) := \limsup_{a \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(A \cap B_a^d(x))}{|B_a^d|}.$$

For every compact set $K \subseteq \mathbb{R}^d$ with non-empty interior and whose boundary has measure zero, the lower density can be also calculated as:

$$D^-(A) = \liminf_{a \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(A \cap (aK + x))}{a^n |K|},$$

and a similar statement holds for the upper density [13, Lemma 4].

The *covering constant* of a set $\Lambda \subseteq \mathbb{R}^d$ is

$$\nu = \nu(\Lambda) = \sup_{x \in \mathbb{R}^d} \#(\Lambda \cap Q_{1/2}(x)).$$

A set is called *relatively separated* if it has a finite covering constant, which holds if and only if it has finite upper Beurling density.

A set $E \subseteq \mathbb{R}^d$ is called a *convex body* if it is convex, compact and has non-empty interior. A convex body is called *centered* if $0 \in E^\circ$ and symmetric if $E = -E$. The following fact will be frequently used in approximation arguments.

Lemma 1.1 (*Dilation of centered convex bodies*). *Let $E \subseteq \mathbb{R}^d$ be a centered convex body. Let $\delta \in (0, 1)$, then $E \subseteq (1 + \delta)E^\circ$ and $(1 - \delta)E \subseteq E^\circ$.*

2. Trajectory sets and sampling

A *trajectory* p in \mathbb{R}^d is the image of curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$, i.e., $p = \gamma(\mathbb{R})$ such that the restriction of γ to any finite interval is rectifiable. A *trajectory set* P is defined as a countable collection of trajectories:

$$P = \{p_i : i \in \mathbb{I}\}, \quad (5)$$

where \mathbb{I} is a countable set of indices and every p_i is a trajectory. In analogy to the Beurling density we define the lower and upper *path density* of a trajectory set P as follows:

Definition 2.1. Let $\mathcal{M}^P(a, x)$ be the total arc-length of the trajectories in $P \cap B_a^d(x)$. Then the *lower path density* and the upper path density are

$$\ell^-(P) := \liminf_{a \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\mathcal{M}^P(a, x)}{|B_a^d|}, \quad (6)$$

$$\ell^+(P) := \limsup_{a \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \frac{\mathcal{M}^P(a, x)}{|B_a^d|}. \quad (7)$$

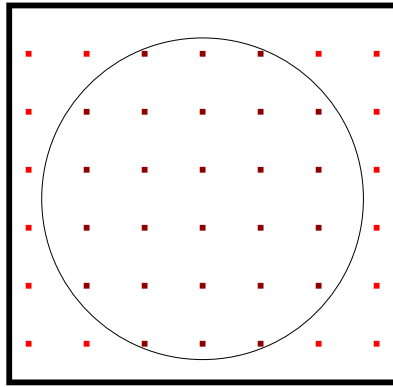
If $\ell^-(P) = \ell^+(P)$, then P is said to possess the homogeneous path density $\ell(P) = \ell^\pm(P)$.

An illustration comparing Beurling and path densities is provided in Fig. 2. As with Beurling's density, the path density does not depend on the particular choice of the Euclidean ball. More precisely, we have the following result.

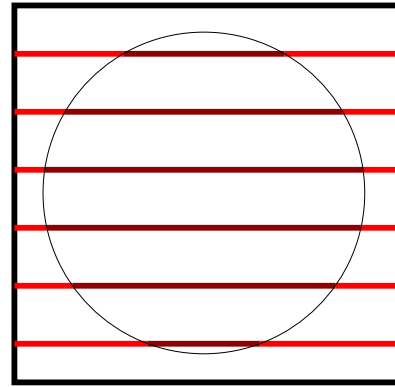
Lemma 2.1. *Let $K \subset \mathbb{R}^d$ be a compact set with non-empty interior and with a boundary of measure zero and let $\mathcal{M}^P(a, x, K)$ be the total arc-length of trajectories from P located in $x + aK$. Then $\ell^-(P) := \liminf_{a \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\mathcal{M}^P(a, x, K)}{a^n |K|}$ and $\ell^+(P) := \limsup_{a \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \frac{\mathcal{M}^P(a, x, K)}{a^n |K|}$.*

Lemma 2.1 can be proved by following Landau's proof of the analogous result for Beurling's density [13, Lemma 4]. We refer the reader to that article.

The simplest example of a trajectory set in \mathbb{R}^2 is a sequence of equispaced parallel lines in \mathbb{R}^2 (e.g., see Fig. 3(a)). We call such a trajectory set a *uniform set in \mathbb{R}^2* . Such a uniform set has a path density equal to $\frac{1}{\Delta}$, where Δ is the spacing between the lines (see [26, Lemma 2.2]). Similarly a *uniform set in \mathbb{R}^d* is defined as a collection of parallel lines in \mathbb{R}^d such that the cross-section forms a $(d - 1)$ -dimensional lattice, see Fig. 3(b). Recall that for static sampling with fixed sensors the appropriate notion of stability

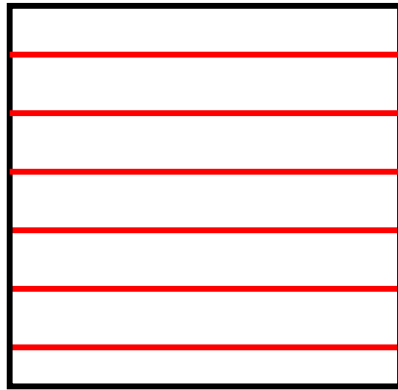


(a) The Beurling density of a lattice used in classical sampling theory quantifies the number of samples per unit area (or per volume in higher dimensions).

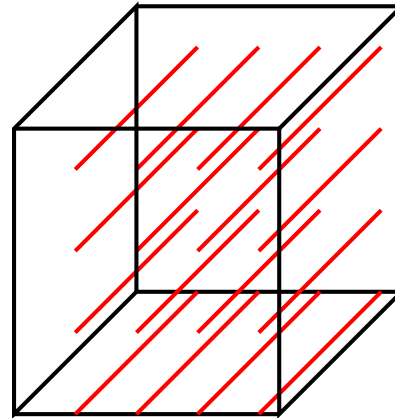


(b) The path density of a trajectory set used in mobile sampling quantifies the total length of the paths per unit area (or per volume in higher dimensions).

Fig. 2. Illustration of Beurling and path densities for classical sampling on a lattice and mobile sampling on a set of equispaced parallel lines (uniform set) in \mathbb{R}^2 .



(a) Uniform set in \mathbb{R}^2



(b) Uniform set in \mathbb{R}^3

Fig. 3. Examples of uniform sets in \mathbb{R}^2 and \mathbb{R}^3 .

was the sampling inequality (3). For mobile sampling along trajectory sets we require similar conditions for the stability and are led to the following definition.

Definition 2.2. A trajectory set P of the form (5) is called a *stable Nyquist trajectory set* for \mathcal{B}_Ω if P satisfies the following conditions:

- (C1) [Nyquist] There exists a uniformly discrete set Λ of points on the trajectories in P , $\Lambda \subset P$, such that Λ forms a set of stable sampling for \mathcal{B}_Ω .
- (C2) [Non-degeneracy] There exists a function $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\delta(a) = o(a^d)$ with the following property: For every $x \in \mathbb{R}^d$ and every $a \gg 1$, there is a rectifiable curve $\alpha : [0, 1] \rightarrow \mathbb{R}^d$, (depending on x and a), such that (i) $\ell(\alpha) = \mathcal{M}^P(a, x) + \delta(a)$, and (ii) $P \cap B_a^d(x) \subset \alpha([0, 1])$, i.e., the curve α contains the portion of the trajectory set P that is located within $B_a^d(x)$.

For brevity, we will denote the collection of all stable Nyquist trajectory sets for Ω by Nyq_Ω .

Condition (C2) is a regularity condition motivated by the model of mobile sensors. It ensures that for all $x \in \mathbb{R}^d$ a single sensor moving along a rectifiable curve with total length $\mathcal{M}^P(a, x) + \delta(a)$ can cover

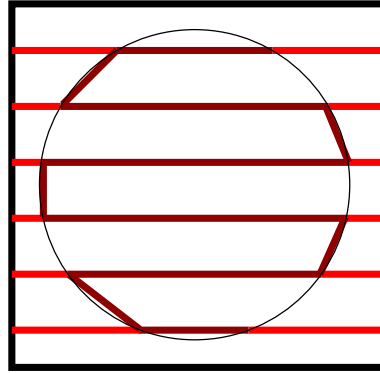


Fig. 4. An illustration of a curve (shown in dark) that satisfies condition (C2) for the trajectory set in Fig. 2(b).

the portions of the trajectories in the ball $B_a^d(x)$. An illustration of such a curve for the trajectory set in Fig. 2(b) is shown in Fig. 4. Thus although there may be a countable collection of paths in P , a single sensor can be used to cover the portions of P inside $B_a^d(x)$, without affecting the total distance traveled per unit area. This means, in particular, that the path density does indeed capture the total distance per unit area covered by a single moving sensor using the trajectories in P .

Definition 2.2 is related to the concept of sampling measures. A (positive Radon) measure μ on \mathbb{R}^d is called a sampling measure for \mathcal{B}_Ω , if there exist $A, B > 0$ such that

$$A\|f\|_2^2 \leq \int_{\mathbb{R}^d} |f(r)|^2 d\mu(r) \leq B\|f\|_2^2 \quad \text{for all } f \in \mathcal{B}_\Omega.$$

If μ is a sum of point measures, $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$, then one recovers (3). For a trajectory set P one can define a natural measure, namely the sum of line integrals along the trajectories p_i . Precisely, if each p_i is parametrized by $\gamma_i : I_i \rightarrow \mathbb{R}^d$ for some (finite or infinite) interval I_i , then the corresponding path sampling measure is defined as

$$\int_{\mathbb{R}^d} f(r) d\mu(r) = \sum_{i \in \mathbb{I}} \int_{I_i} f(\gamma_i(t)) \gamma_i'(t) dt$$

(for C^1 -curves p_i , otherwise we use the Riemann–Stieltjes integrals $d\gamma_i(t)$).

An interesting line of research in complex analysis investigates sampling measures and their characterizations for spaces of complex functions, such as Fock space, Bergmann spaces, and also spaces of bandlimited functions. See [14,20] for a representative list of contributions. Roughly speaking, if μ is a sampling measure, then its support contains a set of sampling, although the precise formulations are much more delicate and technical.

We prefer Definition 2.2 to the abstract definition of sampling measures, because our definition models faithfully the acquisition of data by mobile sampling: the samples are taken along a path, possibly with high density, whence condition (C1). The requirement of a realistic motion of the sensors in \mathbb{R}^d leads to the regularity condition (C2).

For a more quantitative version of stable trajectory sets we restrict the range of the stability parameters A, B .

Definition 2.3. A trajectory set P of the form (5) is called a *stable Nyquist trajectory set* for \mathcal{B}_Ω with stability parameters A and B if P satisfies condition (C2) and the modified condition

(C1^{A,B}) [Nyquist] There exists a uniformly discrete set Λ of points on the trajectories in P , $\Lambda \subset P$, such that Λ forms a set of stable sampling for \mathcal{B}_Ω with fixed stability parameters A and B .

We denote the collection of all stable Nyquist trajectory sets for Ω with stability parameters A, B by $Nyq_\Omega^{A,B}$. Then $Nyq_\Omega = \bigcup_{0 < A, B < \infty} Nyq_\Omega^{A,B}$.

The sampling theory for mobile sensing is primarily concerned with identifying suitable trajectory sets in Nyq_Ω and $Nyq_\Omega^{A,B}$. The key optimization problem is the identification and description of trajectory sets with minimal path density from these classes:

$$\inf_{P \in Nyq_\Omega} \ell^+(P), \quad (8)$$

and

$$\inf_{P \in Nyq_\Omega^{A,B}} \ell^+(P). \quad (9)$$

In [26] and [24] we identified various examples of trajectory sets in Nyq_Ω , and obtained partial solutions to (8) for restricted classes of trajectories, for instance uniform sets and unions of uniform sets. In this paper we derive a lower bound of the path density for the entire class of trajectories consisting of arbitrary parallel lines, and we study the well-posedness of the optimization problem in both Nyq_Ω and $Nyq_\Omega^{A,B}$.

3. Optimal stable sampling sets composed of parallel lines

In this section we consider a trajectory set composed of parallel lines in \mathbb{R}^d . For these trajectories, the path density coincides with the Beurling density of a cross-section.

Lemma 3.1. *Let P be a trajectory set consisting of lines parallel to a vector $q \in \mathbb{R}^d \setminus \{0\}$ and let $\Lambda := P \cap q^\perp$ be the intersection of P with the hyperplane orthogonal to q . Then $D^-(\Lambda) = \ell^-(P)$ and $D^+(\Lambda) = \ell^+(P)$.*

In particular P is homogeneous if and only if $D^-(\Lambda) = D^+(\Lambda)$ and in this case

$$\ell(P) = D^-(\Lambda) = D^+(\Lambda) = \lim_{a \rightarrow +\infty} \frac{\#(\Lambda \cap B_a^{d-1}(x))}{|B_a^{d-1}|}, \quad \text{for all } x \in q^\perp \cong \mathbb{R}^{d-1}.$$

Proof. The lemma is clear if in the definition of Beurling and path density we use cubes with sides aligned to q^\perp instead of Euclidean balls. Lemma 2.1 allows us to make this choice. \square

Most practically useful parallel trajectory sets such as uniform sets, approximately uniform sets (e.g., with bounded offsets) and their finite unions are homogeneous. To formalize the optimization problem we introduce some classes of trajectories. For $\Omega \subset \mathbb{R}^d$ we define:

- Par_Ω^q : the class of all Nyquist trajectories consisting of lines parallel to q , ($q \in \mathbb{R}^d \setminus \{0\}$ a direction parameter).
- Hom_Ω^q : the class of all homogeneous Nyquist trajectories consisting of lines parallel to q , ($q \in \mathbb{R}^d \setminus \{0\}$ a direction parameter).
- Par_Ω : the union of the classes $Par_\Omega^q, q \neq 0$; that is, the collection of all trajectories consisting of parallel lines.
- Hom_Ω : the union of the classes $Hom_\Omega^q, q \neq 0$; that is, the collection of all homogeneous trajectories consisting of parallel lines.

The following is our main result about sampling along parallel lines.

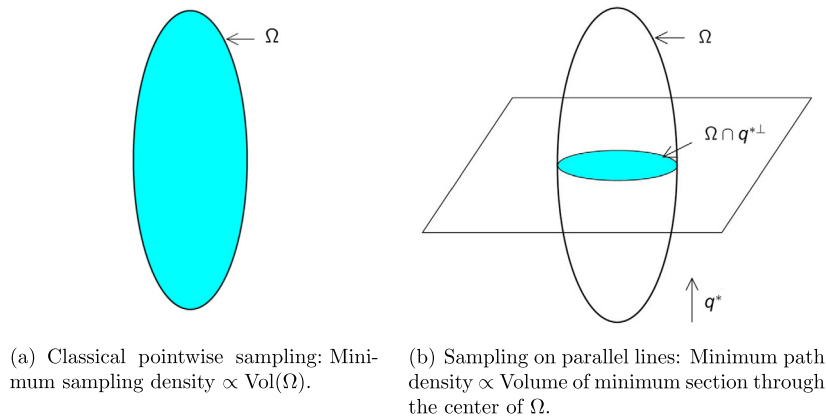


Fig. 5. Fundamental sampling limits for a convex symmetric body Ω .

Theorem 3.2. Let $\Omega \subset \mathbb{R}^d$ be a centered symmetric convex body. Then

$$\inf_{P \in \text{Par}_{\Omega}^q} \ell^-(P) = \inf_{P \in \text{Hom}_{\Omega}^q} \ell(P) = |\Omega \cap q^{\perp}|. \quad (10)$$

In particular, by optimizing over all $q \in \mathbb{R}^d \setminus \{0\}$, it follows that

$$\inf_{P \in \text{Par}_{\Omega}} \ell^-(P) = \inf_{P \in \text{Hom}_{\Omega}} \ell(P) = \min_{q \in \mathbb{R}^d \setminus \{0\}} |\Omega \cap q^{\perp}|. \quad (11)$$

This result shows that the lowest path density of a set of parallel trajectories that admits stable sampling of a field bandlimited to a convex, compact and symmetric set Ω is given by the volume of the smallest section of Ω through the origin. Furthermore, this density can be almost attained by a homogeneous trajectory set. This result is in the spirit of Landau's result [13] on the minimum sampling density for stable pointwise sampling, as illustrated in Fig. 5. In the rest of this section we present arguments that build up to this result.

Sampling along parallel lines has been studied early on as an extension of non-uniform sampling theorems from 1-D to 2-D under the name of *line sampling* [7,10]. In particular, in [10] sampling sets of the form (x_i, y_{ik}) are studied with non-uniformly spaced lines at x_i and non-uniformly spaced samples y_{ik} along each line. Let us emphasize that our objective is rather different, as we try to understand the relation between the path density and the sampling pattern consisting of parallel lines. This is not about a particular set of parallel lines (as in the literature on line sampling), but about all sets of parallel lines. Note that in Theorem 3.2 we characterize the optimal direction in which the sensors have to move. This is an entirely new aspect of the sampling problem.

3.1. Regularity of paths of parallel lines

The first step towards the proof of Theorem 3.2 is showing that the trajectory sets consisting of parallel lines based on a set with finite upper Beurling density do satisfy the regularity condition (C2). To this end we need the following lemma on the length of the shortest path that passes through a given set of points (for a proof see [3]).

Lemma 3.3. For every $d \geq 2$ there exists a constant $C_d > 0$ with the following property: let $a > 1$, and $x_1, \dots, x_n \in B_a^d(0)$. Then there exists a continuous curve $\alpha : [0, 1] \rightarrow B_a^d(0)$ consisting of $n - 1$ concatenated line segments that contains each point x_i , $i = 1, \dots, n$, and

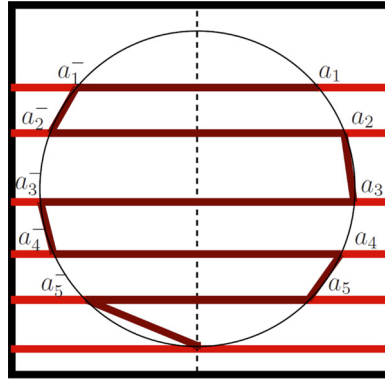


Fig. 6. The intersection of P and ball is prolonged to a rectifiable path with negligible additional length.

$$\ell(\alpha) \leq C_d a n^{\frac{d-1}{d}}.$$

We can now prove the following.

Lemma 3.4. *Let P be a trajectory set consisting of lines parallel to a vector $q \in \mathbb{R}^d \setminus \{0\}$ and let $\Lambda := P \cap q^\perp$ be the intersection of P with the hyperplane orthogonal to q . Assume that $D^+(\Lambda) < \infty$. Then the trajectory set P satisfies condition (C2).*

Proof. For every ball $B_a^d(x) \subset \mathbb{R}^d$, we need to construct a single path α containing all line segments of $P \cap B_a^d(x)$, but without increasing the path length significantly. For this we need to connect the points of intersection of $P \cap \partial B_a^d(x)$ on each hemisphere by a short path. Such a choice in dimension $d = 2$ is plotted in Fig. 6. In higher dimensions, we resort to Lemma 3.3.

For a rigorous argument, we may assume without loss of generality that the lines in P are parallel to $e_d = (0, \dots, 0, 1)$. Let $x \in \mathbb{R}^d$ be arbitrary. Let H^+ and H^- denote the half-spaces

$$H^+ = \{y \in \mathbb{R}^d : y_d > x_d\} \quad \text{and} \quad H^- = \{y \in \mathbb{R}^d : y_d < x_d\},$$

and H the hyperplane

$$H = \{y \in \mathbb{R}^d : y_d = x_d\}.$$

Let $A^+ := P \cap \partial B_a^d(x) \cap H^+$, $A^- := P \cap \partial B_a^d(x) \cap H^-$ and $A := P \cap \partial B_a^d(x) \cap H$. To each point $y = (y_1, y_2, \dots, y_d) \in A^+$ corresponds a symmetric point $y^- = (y_1, y_2, \dots, y_{d-1}, 2x_d - y_d) \in A^-$. Let us further denote $N := \#A^+ = \#A^-$ and $M := \#A$. Since $D^+(\Lambda) < +\infty$, it follows that

$$N, M = O(a^{d-1}). \quad (12)$$

By Lemma 3.3, there exists a path Q contained in $B_a^d(x)$ consisting of $N - 1$ line segments, that passes through all the points in A^+ , and such that $\ell(Q) = O(a)N^{\frac{d-1}{d}} \asymp aa^{\frac{(d-1)^2}{d}} = a^{d+\frac{1}{d}-1} = o(a^d)$.

Let the sequence a_1, a_2, \dots, a_N denote the order in which points in A^+ appear in Q . By symmetry, the sequence of line segments connecting the points $a_1^-, a_2^-, \dots, a_N^-$ is a path contained in $B_a^d(x)$ that connects all points in A^- and has length $O(a)N^{\frac{d-1}{d}} = o(a^d)$.

We construct a rectifiable curve α containing $P \cap B_a^d(x)$ as follows. Let β denote the curve comprising the sequence of line segments connecting the points

$$\begin{aligned} a_1, a_1^-, a_2^-, a_2, a_3, a_3^-, a_4^-, a_4, a_5, \dots, a_N^-, a_N, & \quad \text{if } N \text{ is even,} \\ a_1, a_1^-, a_2^-, a_2, a_3, a_3^-, a_4^-, a_4, a_5, \dots, a_N, a_N^-, & \quad \text{if } N \text{ is odd.} \end{aligned}$$

Since for all $1 \leq i \leq N$, the curve β contains the line segment connecting a_i and a_i^- exactly once, it follows that β contains $(P \cap B_a^d(x)) \setminus H$. Furthermore for all $1 \leq i \leq N-1$, the curve β contains either the line segment connecting a_i and a_{i+1} or that connecting a_i^- and a_{i+1}^- . Thus counting all line segments in β we obtain

$$\ell(\beta) = \ell(Q) + \mathcal{M}^P(a, x). \quad (13)$$

Invoking again [Lemma 3.3](#) we obtain a curve β' contained in $B_a^d(x)$ that goes through each point in point A and has length $O(a)M^{\frac{d-1}{d}} \asymp a^{d+\frac{1}{d}-1} = o(a^d)$. Finally we form α by linking β' to β by means of a line segment contained in $B_a^d(x)$ (of length at most a).

The curve α is completely contained in $B_a^d(x)$, it contains $P \cap B_a^d(x)$ and it is rectifiable since it consists of a finite number of line segments. In addition, from the length estimates above we conclude that $\ell(\alpha) = \mathcal{M}^P(a, x) + o(a^d)$, as desired. (Note that in all the estimates, the implicit constants depend on the set A but not on the center of the ball x .) \square

Remark 3.5. For the proof of [Lemma 3.4](#) we do not need the full strength of [Lemma 3.3](#). If we accept (without proof) that in condition **(C2)** instead of balls $B_a^d(x)$ one may use cubes Q_a with side length $2a$ and aligned parallel to lines in P , then [Lemma 3.3](#) can be replaced by the following, more elementary argument. If a cube Q_a is parallel to P , then $P \cap \partial Q_a$ contains two copies of $P \cap q^\perp$. As $\Lambda = P \cap q^\perp$ is relatively separated, it can be approximated by a finite union of lattices isomorphic to \mathbb{Z}^{d-1} (with asymptotically small error). It is now elementary to connect the lattice points in $\mathbb{Z}^{d-1} \cap [-a, a]^{d-1}$ by a path of length at most $(2a)^{d-1}$. The proof of [Lemma 3.4](#) remains unchanged.

3.2. Lower bounds for the path density

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^d$ be a convex centered symmetric body. Let $P \in \text{Par}_\Omega$ be a Nyquist trajectory set composed of lines parallel to $q \in \mathbb{R}^d \setminus \{0\}$. Then $\ell^-(P) \geq |\Omega \cap q^\perp|$.*

Proof. After a rotation, we may assume without loss of generality that $q = e_d = (0, 0, \dots, 0, 1)$. Denote $\Omega \cap q^\perp = \Omega_0 \times \{0\}$ with $\Omega_0 \subseteq \mathbb{R}^{d-1}$. Let $\delta \in (0, 1)$ and consider the set $(1 - \delta)\Omega$. By [Lemma 1.1](#), $(1 - \delta)\Omega \subseteq \Omega^\circ$. Hence $(1 - \delta)\Omega$ and $\partial\Omega$ are two disjoint compact sets and consequently

$$\varepsilon := d((1 - \delta)\Omega, \partial\Omega) > 0.$$

This implies that

$$(1 - \delta)\Omega_0 \subseteq \Omega_0^\varepsilon := \{x \in \Omega_0 \mid d((x, 0), \partial\Omega) \geq \varepsilon\} \subseteq \mathbb{R}^{d-1},$$

where $d((x, 0), \partial\Omega)$ denotes the Euclidean distance from $(x, 0)$ to the set $\partial\Omega$. Let $\Lambda \subseteq \mathbb{R}^{d-1}$ be the set at which the lines in P intersect the hyperplane e_d^\perp , i.e., $P = \{(\lambda, t) : t \in \mathbb{R}\} = \Lambda \times \mathbb{R}$. Since P is a Nyquist trajectory set, assumption **(C1)** implies the existence of a sampling set $\Gamma \subseteq \mathbb{R}^d$ for \mathcal{B}_Ω whose points belong to the trajectories in P . Hence $\Gamma \subseteq \Lambda \times \mathbb{R}$. For each $\lambda \in \Lambda$, let $I_\lambda := \{t \in \mathbb{R} \mid (\lambda, t) \in \Gamma\}$.

Let $\nu_d(\Gamma) = \max_{x \in \mathbb{R}^d} \#(\Gamma \cap Q_{1/2}(x))$ be the covering constant of $\Gamma \subset \mathbb{R}^d$. Since Γ is a set of stable sampling for \mathcal{B}_Ω , its upper Beurling density $D^+(\Gamma)$ is finite and consequently $\nu_d(\Gamma) < +\infty$. Hence, for all $\lambda \in \Lambda$,

$$\nu_1(I_\lambda) = \nu_d(\{\lambda\} \times I_\lambda) \leq \nu_d(\Gamma) < \infty.$$

Let $g \in L^2(\mathbb{R}^{d-1})$ be bandlimited on Ω_0^ε and set $f(x) = g(x_1, x_2, \dots, x_{d-1})\text{sinc}(\varepsilon x_d)$ with $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ as usual. Since $\Omega_0^\varepsilon \times [-\varepsilon, \varepsilon] \subset \Omega$, we have $f \in \mathcal{B}_\Omega$. Using the fact that Γ is a sampling set for \mathcal{B}_Ω we have

$$\begin{aligned} \varepsilon^{-1} \|g\|_2^2 &= \|f\|_2^2 \lesssim \sum_{\gamma \in \Gamma} |f(\gamma)|^2 = \sum_{\lambda \in \Lambda} |g(\lambda)|^2 \sum_{t \in I_\lambda} |\text{sinc}(\varepsilon t)|^2 \\ &\lesssim \varepsilon^{-1} \nu_d(\Gamma) \sum_{\lambda \in \Lambda} |g(\lambda)|^2. \end{aligned}$$

Hence $\|g\|_2^2 \lesssim \sum_{\lambda \in \Lambda} |g(\lambda)|^2$ for every g bandlimited to Ω_0 . By Landau's result on necessary density conditions for sampling [13] we deduce that $D^-(\Lambda) \geq |\Omega_0^\varepsilon|$. Thus, by Lemma 3.1, it follows that

$$\ell^-(P) = D^-(\Lambda) \geq |\Omega_0^\varepsilon| \geq |(1 - \delta)\Omega_0| = (1 - \delta)^{d-1} |\Omega_0|.$$

The conclusion follows because $\delta > 0$ was arbitrary and $|\Omega_0| = |\Omega \cap q^\perp|$. \square

3.3. Reduction to sampling in each section

To prove that equality holds in (10) we must show that there are Nyquist trajectories with path density arbitrarily close to the volume of the section of Ω through the origin. The following proposition shows that this problem can be reduced to finding sampling sets for each section of Ω with uniform bounds. Precisely, for $t \in \mathbb{R}$ let

$$\Omega_t = \{x \in \mathbb{R}^{d-1} : (x, t) \in \Omega\} \subset \mathbb{R}^{d-1}$$

be the section of Ω at height t . Then $\Omega = \bigcup_{t \in \mathbb{R}} (\Omega_t \times \{t\})$.

Proposition 3.7. *Let $\Omega \subseteq \mathbb{R}^d$ be a closed set. Assume that $\Lambda = \{\lambda_k : k \geq 1\} \subset \mathbb{R}^{d-1}$ is a set of stable sampling for $\mathcal{B}_{\Omega_t} \subseteq L^2(\mathbb{R}^{d-1})$ with uniform bounds $0 < A \leq B < \infty$ for all $t \in \mathbb{R}$, then for every $f \in L^2(\mathbb{R}^d)$ with $\text{supp}(\hat{f}) \subseteq \Omega$*

$$\|f\|_2^2 \asymp \sum_{k \geq 1} \int_{\mathbb{R}} |f(\lambda_k, t)|^2 dt. \quad (14)$$

If in addition Ω is compact, then there exists a lattice $\Gamma \subseteq \mathbb{R}$, such that

$$\|f\|_2^2 \asymp \sum_{k \geq 1} \sum_{\gamma \in \Gamma} |f(\lambda_k, \gamma)|^2. \quad (15)$$

Proof. Set $g = \hat{f}$ and $g_t(x') = g(x', t)$ for $x = (x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Then $\text{supp}(g) \subseteq \Omega$ and $\text{supp}(g_t) \subset \Omega_t$. We further define the partial Fourier transform

$$G_k(t) := \int_{\mathbb{R}^{d-1}} g(x', t) e^{2\pi i \langle \lambda_k, x' \rangle} dx' = \hat{g}_t(\lambda_k).$$

Since

$$\int_{\mathbb{R}} G_k(t) e^{2\pi i w t} dt = \int_{\mathbb{R}^d} g(x', t) e^{2\pi i \langle \lambda_k, x' \rangle} e^{2\pi i w t} dx' dt = f(\lambda_k, w),$$

Plancherel's theorem yields

$$\int_{\mathbb{R}} |G_k(t)|^2 dt = \int_{\mathbb{R}} |f(\lambda_k, w)|^2 dw.$$

Using the support property $\text{supp}(g_t) \subseteq \Omega_t$, we obtain for almost all $t \in \mathbb{R}$ that

$$\begin{aligned} \sum_k |G_k(t)|^2 &= \sum_k |\widehat{g}_t(-\lambda_k)|^2 \asymp \int_{\mathbb{R}^{d-1}} |g_t(x')|^2 dx' \\ &= \int_{\mathbb{R}^{d-1}} |g(x', t)|^2 dx', \end{aligned}$$

with constants independent of t by assumption. Finally,

$$\begin{aligned} \sum_{k \geq 1} \int_{\mathbb{R}^{d-1}} |f(\lambda_k, t)|^2 dt &= \sum_{k \geq 1} \int_{\mathbb{R}} |G_k(t)|^2 dt = \int_{\mathbb{R}} \sum_{k \geq 1} |G_k(t)|^2 dt \\ &\asymp \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} |g(x', t)|^2 dx' dt = \|g\|_2^2 = \|f\|_2^2. \end{aligned}$$

If in addition Ω is compact, then f is bandlimited to a compact set and the integrals involving f can be replaced by sums over a suitably dense lattice. \square

Remark 3.8. Proposition 3.7 applies to spectra of the form

$$\Omega = \{(x', t) \in \mathbb{R} \times \mathbb{R} : g_1(x') \leq t \leq g_2(x')\}$$

with two continuous functions g_1, g_2 . This set can have a very large projection onto the last coordinate while $|g_2(x') - g_1(x')|$ remains small.

3.4. Universal sampling sets

In order to prove Theorem 3.2, we need to find, for each centered symmetric convex body Ω and each direction $q \neq 0$, a stable Nyquist trajectory for \mathcal{B}_Ω consisting of lines parallel to q and with a path-density close to the measure of the central section of Ω by q^\perp . After a rotation, we may assume that $q = e_d$ and analyze the horizontal sections $\Omega_t := \{(x_1, \dots, x_{d-1}) : (x_1, \dots, x_{d-1}, t) \in \Omega\} \subseteq \mathbb{R}^{d-1}$ of Ω . According to Proposition 3.7, we need to find a set $\Lambda \subseteq \mathbb{R}^{d-1}$, such that (a) its Beurling density is close to $|\Omega_0|$ and (b) Λ is simultaneously a sampling set for all spaces of functions bandlimited \mathcal{B}_{Ω_t} for all t with uniform sampling bounds.

In the special case when Ω is contained in an “oblique” cylinder, i.e., $\Omega_t \subseteq tv + \Omega_0$ for some vector $v \in \mathbb{R}^{d-1}$ and all t (Fig. 5(b)), it suffices to find a sampling set only for \mathcal{B}_{Ω_0} with density close to the critical one. This problem was already solved in [17].

In general, the horizontal sections Ω_t are *not* contained in translates of the central section Ω_0 . As a simple example we mention the regular octahedron and two sections perpendicular to $(1, 1, 1)$. The octahedron fits into a cylinder with a cross-section that is strictly larger than the central *minimal* cross-section (see Fig. 7). Therefore the simple argument sketched above does not work. To solve the general case, we need the concept of universal sampling sets, as introduced in [19, 16].

Given $\eta > 0$, a η -universal sampling set Λ is a set with uniform density η that is a sampling set for \mathcal{B}_Ω , for all compact spectra $\Omega \subseteq \mathbb{R}^d$ with $|\Omega| < \eta$. It is known that for all $\eta > 0$ there exist universal sampling sets [19, 16]. For example, in dimension $d = 1$ the set $\{n + \{\sqrt{2}n\} : n \in \mathbb{Z}\}$ is a universal sampling set with

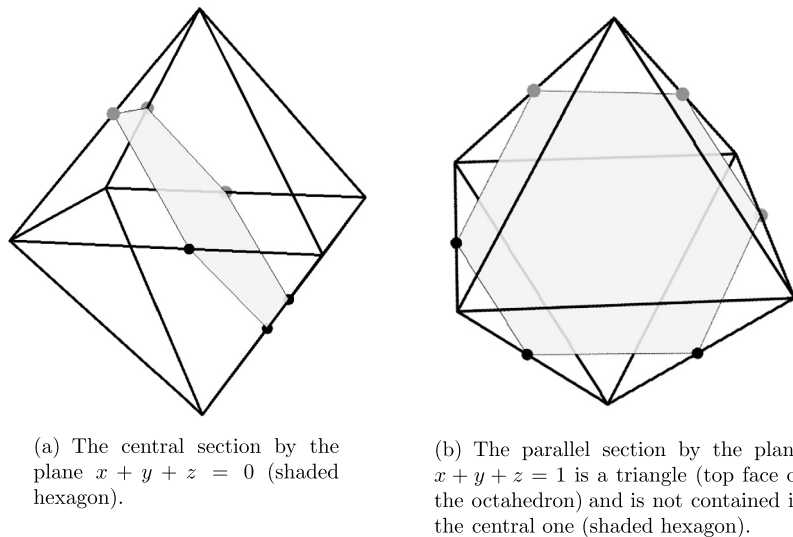


Fig. 7. The regular octahedron $\{(x, y, z) \in \mathbb{R}^3 : |x| + |y| + |z| \leq 1\}$ and two parallel sections.

density $\eta = 1$ (with $\{x\} = x - \lfloor x \rfloor$ denoting the fractional part of x). On the other hand, if the requirement that Ω be compact is dropped, universal sampling sets do not exist [19].

A universal η -sampling set is a set of stable sampling for all compact spectra Ω with $|\Omega| < \eta$, but the frame bounds may depend on Ω . We now argue that when the spectra consist of sections of a compact convex body, then these bounds can be chosen to be uniform. We need the following technical lemma, whose proof is deferred to Section 6.3.

Lemma 3.9 (*Continuity of the sections*). *Let $\Omega \subseteq \mathbb{R}^d$ be a convex and compact set, $t \in \mathbb{R}$, and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $s \in (t - \delta, t + \delta)$*

$$\Omega_s \subseteq \Omega_t + B_\varepsilon^{d-1}.$$

We now show that the sections of a convex compact set admit a universal sampling set with uniform stability bounds.

Proposition 3.10. *Let $\Omega \subseteq \mathbb{R}^d$ be a convex and compact set and let*

$$\eta > \max_{t \in \mathbb{R}} |\Omega_t|.$$

Let Λ be an η -universal sampling set. Then Λ is a sampling set for all Ω_t , $t \in \mathbb{R}$, with sampling bounds uniform in t .

Proof. Let $I \subseteq \mathbb{R}$ be compact interval such that $\Omega \subseteq \mathbb{R}^{d-1} \times I$. Let $t \in I$. Since Ω_t is closed, there exists $\varepsilon_t > 0$ such that

$$|\Omega_t + B_{\varepsilon_t}| < \eta.$$

We let $\tilde{\Omega}_t := \Omega_t + B_{\varepsilon_t}$ denote the slightly enlarged section.

With this notation, by Lemma 3.9, there exists $\delta_t > 0$ such that

$$\Omega_s \subseteq \tilde{\Omega}_t, \quad \text{if } s \in (t - \delta_t, t + \delta_t). \quad (16)$$

The family of intervals $\{(t - \delta_t, t + \delta_t) : t \in I\}$ is an open cover of I . Then, by compactness, $I \subseteq \bigcup_{k=1}^N (t_k - \delta_{t_k}, t_k + \delta_{t_k})$ for finitely many $t_k \in \mathbb{R}$. Hence, for every $s \in I$, there exists $k \in \{1, \dots, N\}$ such that

$$\Omega_s \subseteq \tilde{\Omega}_{t_k}. \quad (17)$$

Since $|\tilde{\Omega}_{t_k}| < \eta$ for $k = 1, \dots, N$, the universal sampling property implies that Λ is a sampling set for $\mathcal{B}_{\tilde{\Omega}_{t_k}}$ with bounds $0 < A_k \leq B_k < \infty$. Let

$$A := \min\{A_1, \dots, A_N\},$$

$$B := \max\{B_1, \dots, B_N\}.$$

Hence, Λ is a sampling set for $\mathcal{B}_{\tilde{\Omega}_{t_k}}$ with bounds A, B for all $k = 1, \dots, N$. Since, according to (17), every section Ω_s is contained in some set $\tilde{\Omega}_{t_k}$, it follows that Λ is a sampling set with bounds A, B for all \mathcal{B}_{Ω_s} with $s \in I$. Note finally that $\Omega_s = \emptyset$, for $s \notin I$. This completes the proof. \square

3.5. Upper path density bounds

With Proposition 3.10 we can now show the estimates (10) for the necessary path density for convex spectra.

Proof of Theorem 3.2. From Proposition 3.6 it follows that

$$\inf_{P \in \text{Hom}_{\Omega}^q} \ell(P) \geq \inf_{P \in \text{Par}_{\Omega}^q} \ell^-(P) \geq |\Omega \cap q^{\perp}|.$$

Let us show that all these inequalities are actually equalities. Assume without loss of generality that $q = e_d = (0, \dots, 0, 1)$ and note that since Ω is convex and symmetric the section through the origin is the one with maximal area. This is a consequence of the Brunn–Minkowski inequality, see for example [11]. Given a number η satisfying

$$\eta > |\Omega \cap q^{\perp}|,$$

let $\Lambda \subseteq \mathbb{R}^{d-1}$ be a η -universal sampling set and let P be a set of lines parallel to q that go through Λ . Since Λ possesses finite (uniform) density, P satisfies condition (C2) by Lemma 3.4. In addition, the fact that Λ possesses a uniform density and Lemma 3.1 imply that $\ell(P) = D(\Lambda) = \eta$ and that P is homogeneous. Propositions 3.10 and 3.7 imply that P is a Nyquist trajectory set. This shows that $\inf_{P \in \text{Hom}_{\Omega}^q} \ell(P) \leq \eta$. The conclusion follows by letting η tend to $|\Omega \cap q^{\perp}|$. \square

4. Optimizing over arbitrary trajectory sets

We now consider the problem of designing trajectory sets without requiring the trajectories to be straight lines.

4.1. Ill-posedness of the unconstrained problem

In the following proposition we show that the optimization problem (8) is ill-posed by constructing a sequence of trajectory sets in Nyq_{Ω} with arbitrarily small path density.

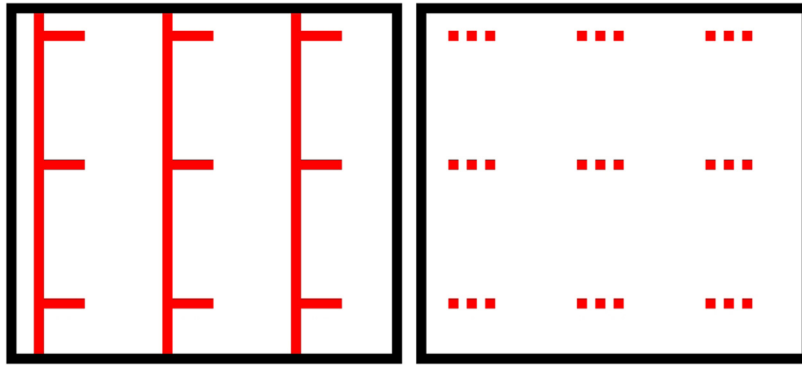


Fig. 8. Left: the path trajectory P_n . Right: a set of stable sampling contained in the trajectory set.

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a compact set. For every $\epsilon > 0$ there exists a trajectory set $P \in \text{Nyq}_\Omega$, such that $\ell^+(P) < \epsilon$. Thus,*

$$\inf_{P \in \text{Nyq}_\Omega} \ell^+(P) = 0.$$

Proof. By enlarging Ω if necessary, we can assume that it is a cube. Since the statement to be proved is invariant under dilations we further assume that $\Omega = [-1/2, 1/2]^2$. For each $n \geq 1$ we construct a trajectory set P_n , in such a way that $\ell^+(P_n) \rightarrow 0$, as $n \rightarrow \infty$.

The counterexample is given by the path P_n resulting from the set

$$(n\mathbb{Z} \times \mathbb{R}) \cup \left((n\mathbb{Z} + [0, \tfrac{1}{n}]) \times \mathbb{Z} \right), \quad (18)$$

which is the union of vertical lines with spacing n and small horizontal segments emerging at the point (nj, k) , $j, k \in \mathbb{Z}$. See Fig. 8. This construction ensures that $\ell^+(P_n) \lesssim 1/n$. Clearly P_n satisfies condition (C2). It remains to show that P_n contains a sampling set for \mathcal{B}_Ω .

Let $F_n \subseteq [0, 1/n]$ be a finite set of cardinality $2n$ and $\Gamma_n := \{nk + t : k \in \mathbb{Z}, t \in F_n\}$ its periodization with period n . Then $\Lambda_n = \Gamma_n \times \mathbb{Z}$ is separated and contained in P_n . Since $D^-(\Gamma_n) = 2 > 1$, it follows that Γ_n is a sampling set for $\mathcal{B}_{[-1/2, 1/2]}$, and consequently Λ_n is a sampling set for \mathcal{B}_Ω . \square

Remark 4.2. A similar example can be constructed in dimension d .

Remark 4.3. The path density of a Nyquist trajectory $P \in \text{Nyq}_\Omega$ is always strictly positive, thus the infimum in Proposition 4.1 is never attained. To see this, choose a uniformly discrete subset $\Lambda \subseteq P$ that is a set of sampling for \mathcal{B}_Ω (by condition (C1)). Let $\delta := \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'\} > 0$ be the separation of Λ . Since Λ is a set of sampling for \mathcal{B}_Ω , Landau's density result asserts that $D^-(\Lambda) \geq |\Omega|$ [13]. This means that, for fixed η , $0 < \eta < |\Omega|$, and sufficiently large $a > 0$, we have

$$\#(\Lambda \cap B_a^d(x)) \geq \eta |B_a^d(x)| \quad \forall x \in \mathbb{R}^d.$$

For a sufficiently large and $x \in \mathbb{R}^d$, let $\alpha : [0, 1] \rightarrow \mathbb{R}^d$ be the curve granted by condition (C2) in Definition 2.2. Then $\alpha([0, 1]) \supset P \cap B_a^d(x) \supset \Lambda \cap B_a^d(x)$. Since the minimum distance between points in Λ is at least δ , it follows that

$$\mathcal{M}^P(a, x) + o(a^d) = \ell(\alpha) \geq \delta \#(\Lambda \cap B_a^d(x)) \geq \delta \eta |B_a^d(x)|.$$

Hence,

$$\ell^-(P) = \liminf_{a \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\mathcal{M}^P(a, x)}{|B_a^d(x)|} \geq \delta\eta > 0.$$

We conclude from [Proposition 4.1](#) that the optimization problem (8) which was first posed in [26] has a trivial solution. In other words, for every compact set Ω it is possible to design a stable Nyquist trajectory set for \mathcal{B}_Ω with arbitrarily small path density. Although at first glance this result may look counter-intuitive, a closer look at the sequence of trajectory sets in the counter-example reveals that the condition number $\frac{B}{A}$ of the sampling set from (3) diverges to ∞ . Thus although we have a stable trajectory set, the stability margin may be arbitrarily bad.

4.2. Trajectory sets with given stability parameters

One way to address the ill-posedness of this problem is to restrict the optimization to trajectory sets that contain stable sampling sets with given stability parameters A and B . In this section we show that this problem is indeed well-posed by identifying a non-zero lower bound on the path density for every trajectory set in $Nyq_\Omega^{A,B}$.

In order to obtain a lower bound on the path density we exploit the key fact that the size of the largest hole of a sampling set is determined by the condition number B/A [12].

Proposition 4.4.

- (a) Let $\Omega \subseteq \mathbb{R}^d$ be a compact set with a smooth boundary and surface measure $\sigma(\partial\Omega)$. Let $\Lambda \subset \mathbb{R}^d$ be a sampling set for \mathcal{B}_Ω with stability bounds A, B . Then Λ intersects every cube $x + [-R, R]^d$, where

$$R = C \frac{B\sigma(\partial\Omega)}{A|\Omega|},$$

and C is a constant that depends only on d .

- (b) If $\Omega = [-1/2, 1/2]^d$, then R may be chosen explicitly as

$$R = \frac{1}{2} + \frac{d\pi^{2d-2}}{2^{2d-1}} \frac{B}{A}.$$

Proof. Part (a) is a simplified version of the main result in [12]. The result in [12] is more general and also covers the case of spectra with fractal boundaries.

Part (b) follows from explicit estimates. In Section 6.2 we give a full argument based on [18]. \square

For a compact set $E \subseteq \mathbb{R}^d$ we define a quantity Δ_E by

$$\Delta_E := \sup_{q \in \mathbb{R}^d \setminus \{0\}} |\mathcal{P}_{q^\perp} E|. \quad (19)$$

This quantity is the volume in \mathbb{R}^{d-1} of the maximal projection of E onto a hyperplane. It satisfies the following invariance properties:

$$\Delta_{E+x} = \Delta_E \quad \forall x \in \mathbb{R}^d, \quad (20)$$

$$\Delta_{(1+\delta)E} = (1+\delta)^{d-1} \Delta_E \quad \forall \delta > 0. \quad (21)$$

The following technical lemma uses Δ_E to bound the volume covered by the translates of a convex set along a smooth curve. The proof can be found in Section 6.4.

Lemma 4.5. Let $E \subseteq \mathbb{R}^d$ be a compact and convex set and let $\alpha : [0, L] \rightarrow \mathbb{R}^d$ be a rectifiable curve. Let $F \subseteq [0, L]$ be a finite set and consider the set

$$E_F := \bigcup_{t \in F} E + \alpha(t). \quad (22)$$

Then $|E_F| \leq |E| + \ell(\alpha)\Delta_E$.

We now prove the main proposition that relates gaps and the path density.

Proposition 4.6. Let E be a convex compact set $E \subseteq \mathbb{R}^d$ with $0 \in E^\circ$ and let $P = \{p_i : i \in \mathbb{I}\}$ be a trajectory set satisfying condition **(C2)**. If the translates of E along the trajectories in P cover \mathbb{R}^d , i.e.,

$$P + E = \bigcup_{\substack{t \in \mathbb{R} \\ i \in \mathbb{I}}} E + p_i(t) = \mathbb{R}^d, \quad (23)$$

then

$$\ell^-(P) \geq \frac{1}{\Delta_E}.$$

Proof. Since E is compact, there exists $R > 0$ such that $E \subseteq B_R^d$. Let $a \geq R$ and $x \in \mathbb{R}^d$ be arbitrary. Since P satisfies condition **(C2)**, there exists a continuous rectifiable curve $\alpha : [0, 1] \rightarrow \mathbb{R}^d$ such that (i) α contains the entire portion of P inside $B_a^d(x)$ and (ii) $\ell(\alpha) = \mathcal{M}^P(a, x) + o(a^d)$.

Let us consider the set

$$S = E + \alpha([0, 1]).$$

We estimate $|S|$ in two different ways. Firstly, if $p_i(t) \notin B_a^d(x)$, then $B_R^d(p_i(t)) \cap B_{a-R}^d(x) = \emptyset$. In view of (23) we have

$$B_{a-R}^d(x) \subset \bigcup_{p_i(t) \in B_a^d(x)} E + p_i(t) \subseteq S.$$

Consequently,

$$|B_{a-R}^d(x)| = (a-R)^d |B_1^d| = a^d |B_1^d| - O(a^{d-1}) = |B_a^d(x)| - O(a^{d-1}) \leq |S|. \quad (24)$$

Secondly, since S is a sum of two compact sets, S is compact. Let $\delta \in (0, 1)$ and consider the (open) set $(1+\delta)E^\circ$. By Lemma 1.1, $E \subseteq (1+\delta)E^\circ$. Consequently,

$$\{(1+\delta)E^\circ + \alpha(t) : t \in [0, 1]\}$$

is an open cover of S , and there exists a finite set $F \subseteq [0, 1]$ such that

$$S \subseteq \bigcup_{t \in F} (1+\delta)E + \alpha(t).$$

Using Lemma 4.5 and (20), (21) it follows that

$$\begin{aligned}
|S| &\leq \left| \bigcup_{t \in F} (1 + \delta)E + \alpha(t) \right| \leq |(1 + \delta)E| + \ell(\alpha)\Delta_{(1+\delta)E} \\
&= (1 + \delta)^d |E| + \ell(\alpha)(1 + \delta)^{d-1} \Delta_E.
\end{aligned}$$

Combining this estimate with (24) we deduce that

$$|B_a^d(x)| \leq O(a^{d-1}) + (1 + \delta)^d |E| + \ell(\alpha)(1 + \delta)^{d-1} \Delta_E, \quad \delta \in (0, 1).$$

Since this inequality holds for all $\delta > 0$ and $o(a^d)$ is independent of δ by assumption (C2), we obtain

$$|B_a^d(x)| \leq O(a^{d-1}) + |E| + \ell(\alpha)\Delta_E, \quad a \geq R, \quad x \in \mathbb{R}^d.$$

Recalling that $\ell(\alpha) = \mathcal{M}^P(a, x) + o(a^d)$ we obtain

$$|B_a^d(x)| \leq O(a^{d-1}) + |E| + \mathcal{M}^P(a, x)\Delta_E + o(a^d)\Delta_E, \quad a \geq R, \quad x \in \mathbb{R}^d.$$

Therefore,

$$\ell^-(P) = \liminf_{a \rightarrow \infty} \frac{\inf_{x \in \mathbb{R}^d} \mathcal{M}^P(a, x)}{|B_a^d(x)|} \geq \frac{1}{\Delta_E},$$

as claimed. \square

Remark 4.7. In [6], Beurling gave sufficient conditions for a non-uniform collection of points $\Lambda \subseteq \mathbb{R}^d$ to form a stable sampling set for the class of bandlimited functions in high dimensions. These are expressed in terms of a covering condition: $\Lambda + E = \mathbb{R}^d$ for a certain convex set E associated with the spectrum support of the signals. On the other hand, Proposition 4.6 gives a condition on a trajectory that is necessary for it to contain a sampling set Λ satisfying $\Lambda + E = \mathbb{R}^d$.

We finally prove the main estimate on the density of paths that contain sampling sets with given stability parameters.

Theorem 4.8. *Let $\Omega \subseteq \mathbb{R}^d$ be a compact set with smooth boundary. Then*

$$\inf_{P \in \text{Nyq}_{\Omega}^{A,B}} \ell^-(P) \geq C_d \left(\frac{A|\Omega|}{B\sigma(\partial\Omega)} \right)^{d-1},$$

where C_d is a constant that depends only on d .

If $\Omega = [-1/2, 1/2]^2$, then explicitly

$$\inf_{P \in \text{Nyq}_{\Omega}^{A,B}} \ell^-(P) \geq \frac{A}{\pi^2 \sqrt{2} B}.$$

Proof. Since P contains a sampling set with stability parameters $A, B > 0$, Proposition 4.4 implies that $\Lambda \cap Q_R(x) \neq \emptyset$ for all $x \in \mathbb{R}^d$ and with $R = C \frac{B\sigma(\partial\Omega)}{A|\Omega|}$. Then $\Lambda + Q_R = \mathbb{R}^d$ and thus also $P + Q_R = \mathbb{R}^d$. By Proposition 4.6 we obtain that $\ell^-(P) \geq 1/\Delta_{Q_R}$. Since by (21)

$$\Delta_{Q_R} = (2R)^{d-1} \Delta_{[-1/2, 1/2]^d} = 2^{d-1} \Delta_{[-1/2, 1/2]^d} C^{d-1} \left(\frac{B\sigma(\partial\Omega)}{A|\Omega|} \right)^{d-1}, \quad (25)$$

the conclusion follows. For the case $\Omega = [-1/2, 1/2]^2$ we use the exact value $\Delta_{[-1/2, 1/2]^2} = \sqrt{2}$ and the explicit estimate for R from [Proposition 4.4](#):

$$R = \frac{1}{2} + \frac{\pi^2 B}{4 A} \leq \frac{\pi^2 B}{2 A}. \quad \square$$

5. Conclusion

We have studied the problem of designing trajectories for sampling bandlimited spatial fields using mobile sensors. We have identified trajectory sets composed of parallel lines that (i) possess minimal path density and (ii) admit the stable reconstruction of bandlimited fields from measurements taken on these trajectories. We also have shown that the problem of minimizing the path density is ill-posed if we allow arbitrary trajectory sets that admit stable reconstruction. As a positive result we have shown that the problem is well-posed if we restrict the trajectory sets to contain a stable sampling set with given stability parameters.

We point out that, for the results presented here, the assumption that the spectrum of the signals is convex is not essential, but a matter of convenience. Indeed, in most results the convexity of Ω can be replaced by a suitable assumption on the regularity of its boundary (e.g. [Lemma 3.9](#)). In [Theorem 3.2](#) the convexity of Ω is used to guarantee that the maximal area of the cross-sections by hyperplanes is attained by a hyperplane that goes through the origin. For non-convex spectra, a characterization analogous to the one in [Theorem 3.2](#) should consider cross-sections by arbitrary hyperplanes.

This work opens up several possible research directions. One question is whether we can solve the problem [\(9\)](#) exactly. This would require a tight lower bound on the path density of every trajectory set in $Nyq_{\Omega}^{A,B}$. Another interesting variation concerns trajectory sets consisting of arbitrary, not necessarily parallel lines and the necessary path density.

6. Some technical tools and proofs

6.1. Translations and projections of convex sets

Lemma 6.1. *Let $E \subseteq \mathbb{R}^d$ be a compact convex set and $q \in \mathbb{R}^d \setminus \{0\}$. Then*

$$|(E + q) \setminus E| \leq |\mathcal{P}_{q^\perp} E| \|q\|_2.$$

Proof. By applying a suitable rotation, we may assume without loss of generality that $q = \alpha e_d = (0, \dots, 0, \alpha)$ for some $\alpha > 0$. Then the projection of E onto the hyperplane determined by q is simply

$$P_{q^\perp} E = \{(x', 0) \in \mathbb{R}^{d-1} \times \mathbb{R} : (x', t) \in E\}.$$

For $x' \in P_{q^\perp} E$ we set $\tau_-(x') = \min\{t : (x', t) \in E\}$ and $\tau_+(x') = \max\{t : (x', t) \in E\}$. Since E is compact, the minima and maxima exist; and since E is convex, the line segments $\{(x', t) : \tau_-(x') \leq t \leq \tau_+(x')\}$ are contained in E , so that

$$E = \{(x', t) \in \mathbb{R}^d : x' \in P_{q^\perp} E, \tau_-(x') \leq t \leq \tau_+(x')\}.$$

Consequently

$$\begin{aligned} (E + q) \setminus E &= (E + \alpha e_d) \setminus E \\ &= \{(x', t) \in \mathbb{R}^d : x' \in P_{q^\perp} E, t \in [\tau_-(x') + \alpha, \tau_+(x') + \alpha] \setminus [\tau_-(x'), \tau_+(x')]\}, \end{aligned}$$

and each fibre over x' has length $\leq \alpha = \|q\|_2$. Now using Fubini's theorem, we obtain that

$$\begin{aligned} |(E+q) \setminus E| &= \int_{\mathbb{R}^d} 1_{(E+q) \setminus E}(x', t) dx' dt \\ &= \int_{P_{q^\perp} E} \int_{\mathbb{R}} 1_{[\tau_-(x')+\alpha, \tau_+(x')+\alpha] \setminus [\tau_-(x'), \tau_+(x')]}(t) dt dx' \\ &\leq \alpha \int_{P_{q^\perp} E} 1 dx' = \alpha |P_{q^\perp} E| = |P_{q^\perp} E| \|q\|_2, \end{aligned}$$

as claimed. \square

6.2. Spectral gaps for the square. Proof of Proposition 4.4(b)

The following proposition – that is part (b) of Proposition 4.4, restated for convenience – gives an explicit estimate for the gap of sampling sets for the spectrum $[-1/2, 1/2]^d$. Its proof is inspired by the simple proof of Landau's necessary conditions for sampling and interpolation given in [18].

Proposition. Let $\Omega := [-1/2, 1/2]^d$ and assume that Λ is a sampling set for \mathcal{B}_Ω with bounds A, B . Then Λ intersects every cube $Q_R(x) = [-R, R]^d + x$, where

$$R = \frac{1}{2} + \frac{2d}{\pi^2} \frac{B}{A} \left(\frac{\pi^2}{4} \right)^d. \quad (26)$$

Proof. Since every translation of Λ is also a sampling set for \mathcal{B}_Ω with bounds A, B , it suffices to show that Λ intersects $[-R, R]^d$, where R is given by (26). Let $h(x) := \text{sinc}(x) = \prod_{k=1}^d \frac{\sin(\pi x_k)}{\pi x_k}$, so $\hat{h} = 1_\Omega$. We start by noting some facts.

Claim 1.

$$A \leq \sum_{\lambda \in \Lambda} |h(\cdot - \lambda)|^2 \leq B.$$

Proof of the claim. Note that

$$\sum_{\lambda \in \Lambda} |h(x - \lambda)|^2 = \sum_{\lambda \in \Lambda} |\langle h(\cdot - \lambda), h(\cdot - x) \rangle|^2.$$

Since $\{h(\cdot - \lambda) : \lambda \in \Lambda\}$ is a frame with bounds A, B and $\|h\|_2 = 1$, the conclusion follows. \square

Claim 2.

$$\#(\Lambda \cap Q_{1/2}(x)) \leq \left(\frac{\pi^2}{4} \right)^d B.$$

Proof of the claim. Since $\frac{\sin \pi t}{\pi t} \geq 2/\pi$ for $|t| \leq 1/2$, we have $h(x) \geq (2/\pi)^d$ for $x \in [-1/2, 1/2]^d = Q_{1/2}(0)$. Therefore we obtain

$$\left(\frac{4}{\pi^2} \right)^d \#(\Lambda \cap Q_{1/2}(x)) \leq \sum_{\lambda \in \Lambda} |h(\lambda - x)|^2 \leq B \|h(\cdot - x)\|_2^2 = B. \quad \square$$

Claim 3.

$$\int_{\mathbb{R}^d \setminus Q_r(0)} |h(x)|^2 dx \leq \frac{2d}{\pi^2 r}, \quad \forall r > 0.$$

Proof of the claim. Since $\text{sinc}(x) = \text{sinc}(x_1) \dots \text{sinc}(x_d)$ and each one-dimensional sinc is normalized in L^2 , we estimate

$$\begin{aligned} \int_{x \in \mathbb{R}^d, |x|_\infty > r} |\text{sinc}(x)|^2 dx &\leq \sum_{k=1}^d \int_{x \in \mathbb{R}^d, |x_k| > r} |\text{sinc}(x)|^2 dx \\ &= \sum_{k=1}^d \int_{t \in \mathbb{R}: |t| > r} |\text{sinc}(t)|^2 dt \\ &\leq 2d \int_r^\infty \frac{1}{(\pi t)^2} dt = \frac{2d}{\pi^2 r}. \quad \square \end{aligned}$$

Combining the claims we get

$$\begin{aligned} A = A|_{Q_{1/2}(0)} &\leq \int_{Q_{1/2}(0)} \sum_{\lambda \in \Lambda} |h(x - \lambda)|^2 dx = \int_{\mathbb{R}^d} |h(x)|^2 \sum_{\lambda \in \Lambda} 1_{Q_{1/2}(\lambda)}(x) dx \\ &= \int_{\mathbb{R}^d} |h(x)|^2 \#(\Lambda \cap Q_{1/2}(-x)) dx \\ &\leq \left(\frac{\pi^2}{4}\right)^d B \int_{\bigcup_{\lambda \in \Lambda} Q_{1/2}(\lambda)} |h(x)|^2 dx. \end{aligned}$$

Now assume that $\Lambda \cap Q_R(0) = \emptyset$. Then, for every $\lambda \in \Lambda$, $Q_{1/2}(\lambda) \cap Q_{R-1/2}(0) = \emptyset$. Therefore,

$$A \leq \left(\frac{\pi^2}{4}\right)^d B \int_{\mathbb{R}^d \setminus Q_{R-1/2}(0)} |h(x)|^2 dx \leq \frac{2d}{\pi^2} B \left(\frac{\pi^2}{4}\right)^d (R - 1/2)^{-1}.$$

Hence, $R \leq \frac{1}{2} + \frac{d\pi^{2d-2}}{2^{2d-1}} \frac{B}{A}$.

This means that Λ must intersect $[-R, R]^d$ if $R > \frac{1}{2} + \frac{d\pi^{2d-2}}{2^{2d-1}} \frac{B}{A}$, as desired. (Since Λ is closed, it also follows that Λ intersects $[-R, R]^d$ for $R = \frac{1}{2} + \frac{d\pi^{2d-2}}{2^{2d-1}} \frac{B}{A}$.) \square

6.3. Continuity of sections of convex sets. Proof of Lemma 3.9

Proof of Lemma 3.9. Without loss of generality let us assume that $\Omega_t \neq \emptyset$. Suppose that the conclusion does not hold. Then there exists a sequence of real numbers $\{t_n : n \geq 1\}$ such that $t_n \rightarrow t$ and

$$\Omega_{t_n} \not\subseteq \Omega_t + B_\varepsilon.$$

Hence there exist points $x_n \in \Omega_{t_n}$ such that

$$r_n := d(x_n, \Omega_t) = \inf\{|x_n - y| : y \in \Omega_t\} \geq \varepsilon. \quad (27)$$

Since Ω_t is closed, there exists $y_n \in \Omega_t$ such that $|x_n - y_n| = r_n$.

Consider the sequences $\{(x_n, t_n) : n \geq 1\}, \{(y_n, t) : n \geq 1\} \subseteq \Omega$. By passing to subsequences we may assume that both of them are convergent:

$$\begin{aligned} (x_n, t_n) &\longrightarrow (x, t), \\ (y_n, t) &\longrightarrow (y, t). \end{aligned}$$

Hence, $x, y \in \Omega_t$. In addition, by (27), $r := |x - y| = \lim_n |x_n - y_n| = \lim_n r_n \geq \varepsilon > 0$.

Since Ω is convex, so is Ω_t . Consequently, $z = (x + y)/2 \in \Omega_t$. Let us estimate

$$|x_n - z| \longrightarrow |x - z| = r/2 < r = \lim_n r_n.$$

Therefore, there exist $n \in \mathbb{N}$ such that $|x_n - z| < r_n$. Since $z \in \Omega_t$, this contradicts the fact that $r_n = d(x_n, \Omega_t)$. \square

6.4. Sliding convex sets. Proof of Lemma 4.5

Proof of Lemma 4.5. Let us enumerate the points of F as $0 \leq t_0 < \dots < t_N \leq L$. Without loss of generality we further assume that $\alpha(t_k) \neq \alpha(t_j)$, for $k \neq j$ (Indeed, if $\alpha(t_k) = \alpha(t_j)$, for some $k \neq j$, then we may remove t_j from the set F without altering the set E_F). Let us consider the sets $E_k := E + \alpha(t_k)$.

For $1 \leq k \leq N$, let $q_k := \alpha(t_{k-1}) - \alpha(t_k)$. By Lemma 6.1, it follows that

$$|E_k \setminus E_{k-1}| \leq |\mathcal{P}_{q_k^\perp}(E)| \|\alpha(t_k) - \alpha(t_{k-1})\|_2.$$

Since $\alpha(t_{k-1}) \neq \alpha(t_k)$, $q_k \neq 0$, for all k . Hence, considering the vectors $q'_k := \|q_k\|^{-1} q_k$ we see that

$$|\mathcal{P}_{q_k^\perp}(E)| = |\mathcal{P}_{(q'_k)^\perp}(E)| \leq \Delta_E.$$

Therefore,

$$|E_k \setminus E_{k-1}| \leq \Delta_E \|\alpha(t_k) - \alpha(t_{k-1})\|_2, \quad 1 \leq k \leq N.$$

Let us decompose E_F as

$$E_F := E_0 \cup \bigcup_{k=1}^N (E_k \setminus E_{k-1}).$$

Hence,

$$\begin{aligned} |E_F| &\leq |E_0| + \sum_{k=1}^N |E_k \setminus E_{k-1}| \\ &\leq |E| + \sum_{k=1}^N \Delta_E \|\alpha(t_k) - \alpha(t_{k-1})\|_2 \leq |E| + \Delta_E \ell(\alpha). \end{aligned}$$

as claimed. \square

Acknowledgments

K. Gröchenig was partially supported by National Research Network S106 SISE and by the project P 26273-N25 of the Austrian Science Fund (FWF). J.L. Romero gratefully acknowledges support from the project M1586-N25 of the Austrian Science Fund (FWF) and from an individual Marie Curie Fellowship, within the 7th European Community Framework program, under grant PIIF-GA-2012-327063. J. Unnikrishnan and M. Vetterli were supported by ERC Advanced Investigators Grant: Sparse Sampling: Theory, Algorithms and Applications – SPARSAM – No. 247006.

References

- [1] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.* 43 (4) (Apr. 2001) 585–620.
- [2] S.T. Ali, J.-P. Antoine, J.-P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer-Verlag, New York, 2000.
- [3] J. Beardwood, J.H. Halton, J.M. Hammersley, The shortest path through many points, *Proc. Cambridge Philos. Soc.* 55 (4) (1959) 299–327.
- [4] J.J. Benedetto, H.-C. Wu, Nonuniform sampling and spiral MRI reconstruction, in: A. Aldroubi, A. Laine, M. Unser (Eds.), *Proc. SPIE Symp. Wavelets Applications in Signal and Image Processing VIII*, vol. 4119, June 2000, pp. 130–141.
- [5] A. Beurling, Local harmonic analysis with some applications to differential operators, in: *Some Recent Advances in the Basic Sciences*, vol. 1, *Proc. Annual Sci. Conf.*, Belfer Grad. School Sci., Yeshiva Univ., New York, 1962–1964, Belfer Graduate School of Science, Yeshiva Univ., New York, 1966, pp. 109–125.
- [6] A. Beurling, On balayage of measures in Fourier transforms (seminar, inst. for advanced studies, 1959–60, unpublished), in: L. Carleson, P. Malliavin, J. Neuberger, J. Wermer (Eds.), *Collected Works of Arne Beurling*, Birkhauser, Boston, 1989.
- [7] P.L. Butzer, G. Hinsen, Two-dimensional nonuniform sampling expansions—an iterative approach. II. Reconstruction formulae and applications, *Appl. Anal.* 32 (1) (1989) 69–85.
- [8] M. Fornasier, H. Rauhut, Continuous frames, function spaces, and the discretization problem, *J. Fourier Anal. Appl.* 11 (3) (2005) 245–287.
- [9] K. Gröchenig, Reconstruction algorithms in irregular sampling, *Math. Comp.* 59 (199) (1992) 181–194.
- [10] K. Gröchenig, T. Strohmer, Numerical and theoretical aspects of non-uniform sampling of band-limited images, in: F. Marvasti (Ed.), *Nonuniform Sampling: Theory and Applications*, Kluwer, 2001, pp. 283–324, chapter 6.
- [11] R. Gardner, The Brunn–Minkowski inequality, *Bull. Amer. Math. Soc.* 39 (3) (2002) 355–405.
- [12] A. Iosevich, S. Pedersen, How large are the spectral gaps?, *Pacific J. Math.* 192 (2) (Feb. 2000) 307–314.
- [13] H. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.* 117 (1) (July 1967) 37–52.
- [14] D.H. Luecking, Sampling measures for Bergman spaces on the unit disk, *Math. Ann.* 316 (4) (2000) 659–679.
- [15] Y.I. Lyubarskii, A. Rashkovskii, Complete interpolating sequences for Fourier transforms supported by convex symmetric polygons, *Ark. Mat.* 38 (1) (2000) 139–170.
- [16] B. Matei, Y. Meyer, Simple quasicrystals are sets of stable sampling, *Complex Var. Elliptic Equ.* 55 (8–10) (2010) 947–964.
- [17] J. Marzo, Riesz basis of exponentials for a union of cubes in \mathbb{R}^d , arXiv:math/0601288, 2006.
- [18] S. Nitzan, A. Oleviskii, Revisiting Landau’s density theorems for Paley–Wiener spaces, *C. R. Math. Acad. Sci. Paris* 350 (9–10) (2012) 509–512.
- [19] A. Oleviskii, A. Ulanovskii, Universal sampling and interpolation of band-limited signals, *Geom. Funct. Anal.* 18 (3) (2008) 1029–1052.
- [20] J. Ortega-Cerdà, Sampling measures, *Publ. Mat.* 42 (2) (1998) 559–566.
- [21] D.P. Petersen, D. Middleton, Sampling and reconstruction of wave-number-limited functions in N -dimensional Euclidean spaces, *Inform. Control* 5 (1962) 279–323.
- [22] A. Singh, R. Nowak, P. Ramanathan, Active learning for adaptive mobile sensing networks, in: *Proceedings of Information Processing in Sensor Networks*, IPSN, 2006, pp. 60–68.
- [23] J. Unnikrishnan, M. Vetterli, Sampling trajectories for mobile sensing, in: *Proc. 2011 49th Annual Allerton Conference on Communication, Control, and Computing* (Allerton), Allerton House, UIUC, Illinois, USA, Sept. 2011, pp. 1230–1237.
- [24] J. Unnikrishnan, M. Vetterli, On sampling a high-dimensional bandlimited field on a union of shifted lattices, in: *2012 IEEE International Symposium on Information Theory Proceedings, ISIT*, July 2012, pp. 1468–1472.
- [25] J. Unnikrishnan, M. Vetterli, Sampling and reconstruction of spatial fields using mobile sensors, *IEEE Trans. Signal Process.* 61 (9) (2013) 2328–2340.
- [26] J. Unnikrishnan, M. Vetterli, Sampling high-dimensional bandlimited fields on low-dimensional manifolds, *IEEE Trans. Inform. Theory* 59 (4) (2013) 2103–2127.
- [27] J. Unnikrishnan, M. Vetterli, On optimal sampling trajectories for mobile sensing, in: *Proceedings of the 10th International Conference on Sampling Theory and Applications*, SampTA 2013, July 2013, pp. 352–355.