Discrete stochastic heat equation driven by fractional noise: Feynman-Kac representation and asymptotic behavior

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To my parents…

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Abstract

We consider the parabolic Anderson model on $\mathbb{Z}^d$ driven by fractional noise. We prove that it has a mild solution given by Feynman-Kac representation which coincides with the partition function of a directed polymer in a fractional Brownian environment. Our argument works in a unified way for every Hurst parameter in $(0,1)$.
We also study the asymptotic time evolution of this solution. We show that for $H \leq 1/2$, almost surely, it converges asymptotically to $e^{\lambda t}$ for some deterministic strictly positive constant 'λ'. Our argument is robust for every jump rate and non-pathological spatial covariance structures. For $H > 1/2$ on one hand, we demonstrate that the solution grows asymptotically no faster than $e^{kt\sqrt{\log t}}$, for some positive deterministic constant 'k'. On the other hand, the asymptotic growth is lower-bounded by e^{ct} for some positive deterministic constant 'c'.
Invoking Malliavin calculus seems inevitable for our results.

Key words: Parabolic Anderson model, stochastic heat equation, fractional Brownian motion, Feynman-Kac formula, Lyapunov exponents, Malliavin calculus

Résumé

Nous considérons le modèle parabolique d'Anderson sur $\mathbb{Z}^d$ sous l'environnement aléatoire du bruit fractionnaire. On prouve qu'il a une solution faible donée par la formule de Feynman-Kac qui coïncide avec la fonction de partition d'un polymère dirigé en milieu aléatoire du mouvement Brownien fractionnaire. Notre argumentation marche d'une manière unifiée pour tout paramètre de Hurst dans $(0,1)$.
Ensuite, nous étudions l'évolution temporelle asymptotique de cette solution. Nous montrons que pour $H \leq 1/2$, presque sûrement elle converge asymptotiquement vers $e^{\lambda t}$, 'λ' étant une constante déterministe et strictement positive. Notre argument est solide pour tous les taux de saut et toute structure de covariance spatiale non pathologique.
Pour $H > 1/2$, d'une part, nous démontrons que la solution se développe asymptotiquement pas plus vite que $e^{kt\sqrt{\log t}}$, pour une constante positive et déterministe 'k'. D'autre part, il est facilement montré que sa croissance asymptotique a pour une borne inférieure la fonction exponentiele e^{ct}, 'c' étant une constante déterministe et strictement positif.
Le calcul de Malliavin semble inévitable pour nos résultats.

Mots clefs : Modèle parabolique d'Anderson, équation stochastique de chaleur, Mouvement Brownien fractionnaire, Formule de Feynman-Kac, Exposant de Lyapunov, Calcul de Malliavin

Contents

Introduction

The parabolic Anderson model (PAM) is the parabolic partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = \kappa \boldsymbol{\Delta} u(t,x) + \xi(t,x)\, u(t,x), \qquad x \in \mathbb{Z}^d\,,\ t \geq 0\,, \tag{1}$$

where $\kappa > 0$ is a diffusion constant and $\boldsymbol{\Delta}$ is the discrete Laplacian defined by $\boldsymbol{\Delta} f(x) := \frac{1}{2d}\sum_{|y-x|=1}[f(y) - f(x)]$. The potential $\{\xi(t,x)\}_{t,x}$ can be a random or deterministic field or even a Schwartz distribution.

The parabolic Anderson model, named after Philip Warren Anderson, the American physicist and Nobel laureate, has applications and connections to problems in chemical kinetics, magnetic fields with random flow and the spectrum of random Schrödinger operators, to mention a few. The solution $u(t,x)$ of (1) has also a population dynamics interpretation as the average number of particles at site x and time t conditioned on a realization of the medium ξ where the particles perform branching random walks in random media. In this case, the first right-hand-side term of (1) signifies the diffusion and the second term represents the birth/death of the particles. For more details, we refer to [17] and [3].

The parabolic Anderson model has been extensively studied, particularly in the last twenty years. We refer to the classical work of Carmona and Molchanov [3], the survey by Gärtner and König [17] and to the very recent survey [26]. Many variants of PAM have been studied, such as the cases where the potential is white Gaussian noise [3, 7], Lévy noise [8], a family of independent random walks [14], exclusion process and Voter model [15, 16], to mention a few. It should be noted that the former case is different from the rest, as the white noise is not a real valued function but a distribution. PAM has also been considered for the case of continuous space $\mathbb{R}^d$, for example in [6, 4].

The Feynman-Kac formula, named after the American theoretical physicist Richard Feynman and the Polish mathematician Mark Kac, establishes a probabilistic solution to certain parabolic partial differential equations, particularly the heat equation. This closed-form solution has been proved to be an extremely useful tool in the investigation of these partial differential equations. So it is natural to expect some Feynman-Kac representation for the PAM which is a stochastic heat equation. The general form of the Feynman-Kac representation for the solution of the PAM is

$$u(t,x) = \mathbb{E}^x\Big[u_o(X(t))\exp\int_0^t \xi\big(s, X(t-s)\big)\mathrm{d}s\Big],$$

where $u_o(\cdot) := u(0,\cdot)$ is the initial value at time $t = 0$, $X(\cdot)$ is a simple random walk of jump rate κ started at $x \in \mathbb{Z}^d$ and independent of ξ, and $\mathbb{E}^x$ is expectation with respect to this random walk.

Carmona and Molchanov in [3] proved that for a deterministic potential ξ such that $\xi(\cdot, x)$ is locally integrable in t for every x, the Feynman-Kac formula is a solution to PAM if it is finite for every x and t. They also showed that the Feynman-Kac representation is valid when the potential is white Gaussian noise.

Fractional Brownian motion (fBM) which is a generalization of Brownian motion, is a suitable process to incorporate long-range spatial and temporal correlations. Many phenomena in physics, biology, economy and telecommunications show long range memory [38, 18, 24].

The PAM driven by fractional noise has not been much studied yet. The Feynman-Kac representation of the solution to continuous sate-space PAM driven by fractional noise has been proved for $H > 1/2$ in [21] and for $H > 1/4$ in [22]. The asymptotic behavior of the discrete PAM driven by Riemann-Liouville fractional noise has been considered in [50].

The results of this thesis are in two directions. Firstly, in establishing the Feynman-Kac representation for the discrete PAM driven by fractional noise in chapter 2. We were able to extend the results of [22] and [21] to every $H \in (0,1)$ for the case of discrete space $\mathbb{Z}^d$. Then in chapter 3, we will study the asymptotic behavior of the Feynman-Kac formula. There we extend the results of [50] in several ways.

In [50] the following expression over a compact space χ is considered

$$u(t,x) = \mathbb{E}^x\Big[e^{\int_0^t B_s^{X(s)}\mathrm{d}s}\Big] \qquad ; x \in \chi\,,\ t > 0\,,$$

where $\{B_\cdot^x\}_{x\in\chi}$ a family of Riemann-Liouville fractional Brownian motions of Hurst parameter H, and $X(\cdot)$ is a simple random walk on χ with jump rate κ, and $\mathbb{E}^x$ is expectation with respect to the random walk.

They show that $\mathbb{E}\log u(t,x)$, where $\mathbb{E}$ is the expectation with respect to the random environment, i.e. the fBM field, is almost super-additive (although their proof seems to have some problems) and hence $\frac{1}{t}\mathbb{E}\log u(t,x)$ converges to some non-negative extended-real number λ. Using some Malliavin concentration inequalities, they show that $\{\frac{1}{n}\mathbb{E}\log u(n,x)\}_{n\in\mathbb{N}}$ and $\{\frac{1}{n}\log u(n,x)\}_{n\in\mathbb{N}}$ have the same asymptotic behavior and hence $\frac{1}{n}\mathbb{E}\log u(n,x)$ converges over the natural numbers to the same deterministic limit λ. Then for $H < 1/2$ where the finiteness of λ is easy to show, its positivity is proved under strong conditions on κ, H and the spatial covariance. For $H > 1/2$ they try to show that λ is ∞ and hence $\log u(n,x)$ grows faster than any linear function. In fact they try to show that $\log u(t,x)$ grows at least faster than $\frac{t^{2H}}{\log t}$.

We extend their results and also modify them as follows:

- We consider an unbounded non-compact space, namely $\mathbb{Z}^d$.
- We prove an approximate super-additivity of $\mathbb{E}\log u(t,x)$ which would suffice for our conclusions.
- We show that the limit behavior of $\{\frac{1}{t}\log u(t,x)\}_{t\in\mathbb{R}^+}$ is the same as $\{\frac{1}{n}\log u(n,x)\}_{n\in\mathbb{N}}$, hence filling the gap between discrete and continuous time.
- We prove the strict positivity of λ for any $H \in (0,1)$ and without any restriction on κ.
- For $H \le 1/2$ it is easily shown that λ is finite hence completely settling this case.
- For $H > 1/2$, although we haven't been able to establish the finiteness of λ, we prove that $\log u(t,x)$ grows no faster than $Ct\sqrt{\log t}$, for some positive constant C.

1 Preliminaries

1.1 Fractional Brownian Motion

A Gaussian random process $\{B_t\}_{t\in\mathbb{R}}$ is called a fractional Brownian motion (fBM) of Hurst parameter $H \in (0,1)$ if it has continuous sample paths and its covariance function is of the following form:

$$\mathbb{E}\big(B_t B_s\big) = R_H(t,s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

The non-negative definiteness of this function was first proved by Schoenberg [43] in a more general setting. For a proof we refer to [41] for example.

This process was first introduced by Kolmogorov in [25], but the term "Fractional Brownian motion" was coined by Mandelbrot and Van Ness in [31].

fBM is a self-similar process in the sense that for any $\alpha > 0$, the process $\{\alpha^{-H} B_{\alpha t}\,;\, t > 0\}$ has the same distribution as $\{B_t\,;\, t > 0\}$. Like Brownian motion, fBM has stationary increments and its sample are almost all nowhere differentiable. Unlike the Brownian motion, fBM doesn't have independent increments, is neither a Markov process nor a semi-martingale [34].

A fractional Brownian motion $\{B_t\}_t$, of Hurst parameter $H \in (0,1)$, can be represented as a Voterra process [34]

$$B_t = \int_0^t K_H(t,s)\mathrm{d}W_s\,, \tag{1.1}$$

where W_s is a standard Brownian motion and $K_H(t,s)$ is a square integrable kernel. Here the stochastic integration is in Itō sense (for Itō theory we refer to e.g. [39, 27]). For the other representations of the fractional Brownian motion see e.g. [41, 31, 34].

This integral representation can be used to define stochastic integrations with respect to fractional Brownian motion as in [34]. It is also useful for our analysis as Itō integrals are straightforward and easy to work with. For Itō integrals we refer to for example [27] or [39].

The value of $K_H(t,s)$ for $H > 1/2$ is given by

$$K_H(t,s) := c_H \int_s^t (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \mathrm{d}u,$$

and for $H \le 1/2$ is given by

$$K_H(t,s) := c'_H\Big((\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\mathrm{d}u\Big),$$

where c_H and c'_H are positive constants that depend only on H.

For $H < 1/2$ we have

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\big(\frac{t}{s}\big)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}},$$

where $c_H := c'_H(H-\frac{1}{2})$.
Although $\frac{\partial K_H}{\partial t}$ is not properly integrable, in fact one can easily show that $K_H(t,s)$ is the Cauchy principle value integral [20, 51] of $\frac{\partial K_H}{\partial t}$, i.e.

$$K_H(t,s) = \lim_{\alpha \downarrow s}\int_\alpha^t \frac{\partial K_H}{\partial t}(u,s)\mathrm{d}u + c'_H\big(\frac{\alpha}{s}\big)^{H-\frac{1}{2}}(\alpha-s)^{H-\frac{1}{2}}.$$

This shows that for any $0 < t_1 < t_2$ and any $H \in (0,1)$ we have

$$\begin{aligned}K_H(t_2,s) - K_H(t_1,s) &= \int_{t_1}^{t_2} \frac{\partial K_H}{\partial t}(u,s)\mathrm{d}u \\ &= c_H\int_{t_1}^{t_2}(u-s)^{H-\frac{3}{2}}(\frac{u}{s})^{H-\frac{1}{2}}\mathrm{d}u.\end{aligned}$$

We will frequently use this equality in chapter 3.

A related process is Riemann-Liouville fractional Brownian motion which has a simpler integral representation and hence easier to handle than the fBM. A Riemann-Liouville fractional Brownian motion of Hurst parameter $0 < H < 1$ is the process defined by

$$\bar{B}_t = \int_0^t \bar{K}_H(t,s)\mathrm{d}W_s, \tag{1.2}$$

with

$$\bar{K}_H(t,s) = \sqrt{2H}(t-s)^{H-\frac{1}{2}}.$$

It is a well-known fact that the increments of a fractional Brownian motion with Hurst parameters larger than half are positively correlated and those of a fBM with $H < 1/2$ are negatively correlated. Indeed, for disjoint intervals (t_1,T_1) and (t_2,T_2) with lengths L_1 and L_2 respectively and distance A, i.e. with $T_1 = t_1 + L_1$, $t_2 = T_1 + A$ and $T_2 = t_2 + L_2$, where $A \leq 0$, we have

$$\begin{aligned}
&\mathbb{E}[(B_{T_1} - B_{t_1})(B_{T_2} - B_{t_2})] = \\
&\qquad = (T_2 - t_1)^{2H} + (t_2 - T_1)^{2H} - (t_2 - t_1)^{2H} - (T_2 - T_1)^{2H} \\
&\qquad = (L_1 + L_2 + A)^{2H} + (A)^{2H} - (L_1 + A)^{2H} - (L_2 + A)^{2H} \\
&\qquad = 2H(2H-1)\int_0^{L_2}\int_0^{L_1} (x + y + A)^{2H-2}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

So for $H > 1/2$ the correlation is positive and for $H > 1/2$ it is negative. In fact this equation shows other important properties of fBM. First it shows that the correlation depends only on the interval distances and their lengths so it is translation invariant, which is nothing other than stationarity of a fBM. Secondly, as $2H - 2$ is always negative, the integrand is a decreasing function of A, which means the correlation is a decreasing function of A for H larger than half and an increasing function of A when H is less than half.

Now let $A \subseteq [0, T]$ be the union of disjoint intervals $\{(t_i, T_i)\}_{i=1}^n$ of lengths $\{L_i\}_i$. Define $L := \sum_i L_i$, the total length of A, and let

$$S := \int_0^T \mathbf{1}_A(s)\mathrm{d}B_s = \sum_i (B_{T_i} - B_{t_i}).$$

When $H > 1/2$, as the increments are positively correlated, we have

$$var(S) = \mathbb{E}(S^2) \geq \sum_i \mathbb{E}[(B_{T_i} - B_{t_i})^2] = \sum_i (T_i - t_i)^{2H}.$$

When $H < 1/2$, the increments are negatively correlated, so

$$var(S) = \mathbb{E}(S^2) \leq \sum_i \mathbb{E}[(B_{T_i} - B_{t_i})^2] = \sum_i (T_i - t_i)^{2H}.$$

It is also useful to have an upper bound on the variance of S when $H > 1/2$ and a lower bound on it for the case $H < 1/2$.
We construct from A a new set A' by simply gluing the adjacent intervals together while keeping their orders. So A' can be written as $\bigcup_i (t'_i, T'_i)$ with $T'_i = t'_{i+1}$. We have

$$S' := \int_0^T \mathbf{1}_{A'}(s)\mathrm{d}B_s = \sum_i (B_{T'_i} - B_{t'_i}) = B_{T'_n} - B_{t'_1},$$

hence

$$var(S') = \mathbb{E}[(B_{T'_n} - B_{t'_1})^2] = (\sum_i L_i)^{2H}$$

As the correlation of disjoint intervals are translation invariant, and that it is a decreasing (increasing) function of the distance between the intervals for $H > 1/2$ ($H < 1/2$), we have $var(S) \leq var(S')$ for $H > 1/2$ and $var(S) \geq var(S')$ for $H > 1/2$.

So in summary we have proved that for $H > 1/2$

$$\sum_i L_i^{2H} \leq var(S) \leq (\sum_i L_i)^{2H}, \tag{1.3}$$

and for $H < 1/2$

$$(\sum_i L_i)^{2H} \leq var(S) \leq \sum_i L_i^{2H}, \tag{1.4}$$

The positivity of the correlations is also true for the Riemann-Liouville fBm with H larger than half. Indeed, for disjoint intervals $[t_1, T_1]$ and $[t_2, T_2]$ we have

$$\bar{B}_{T_1} - \bar{B}_{t_1} = \int_0^{t_1} [\bar{K}_H(T_1, s) - \bar{K}_H(t_1, s)] \mathrm{d}W_s + \int_{t_1}^{T_1} \bar{K}_H(T_1, s) \mathrm{d}W_s,$$

and

$$\bar{B}_{T_2} - \bar{B}_{t_2} = \int_0^{t_2} [\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)] \mathrm{d}W_s + \int_{t_2}^{T_2} [\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)] \mathrm{d}W_s.$$

As the Itō integrals over disjoint intervals are independent, using the Itō isometry we obtain

$$\begin{aligned}
&\mathbb{E}\big[(\bar{B}_{T_1} - \bar{B}_{t_1})(\bar{B}_{T_2} - \bar{B}_{t_2})\big] \\
&\quad = \mathbb{E}\Big(\int_0^{t_1} [\bar{K}_H(T_1, s) - \bar{K}_H(t_1, s)] \mathrm{d}W_s \int_0^{t_1} [\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)] \mathrm{d}W_s\Big) \\
&\qquad + \mathbb{E}\Big(\int_{t_1}^{T_1} \bar{K}_H(T_1, s) \mathrm{d}W_s \int_{t_1}^{T_1} [\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)] \mathrm{d}W_s\Big) \\
&\quad = \int_0^{t_1} \big[\bar{K}_H(T_1, s) - \bar{K}_H(t_1, s)\big]\big[\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)\big] \mathrm{d}s \\
&\qquad + \int_{t_1}^{T_1} \bar{K}_H(T_1, s)\big[\bar{K}_H(T_2, s) - \bar{K}_H(t_2, s)\big] \mathrm{d}s.
\end{aligned}$$

As $\bar{K}_H$ is an increasing function of its first argument, it is clear that the integrands are all positive and hence we obtain the positivity of the correlation.

Now let $A \subseteq [0, T]$ be again the union of disjoint intervals $\{(t_i, T_i)\}_{i=1}^n$ of lengths $\{L_i\}_i$ with $L := \sum_i L_i$, the total length of A, and let

$$S := \int_0^T \mathbf{1}_A(s) \mathrm{d}\bar{B}_s = \sum_i (\bar{B}_{T_i} - \bar{B}_{t_i}).$$

We would like to show that for H larger than half the variance of S is bounded (up to a positive multiplicative constant) by L^{2H}.
Let's first look at the integral over a single interval (t_i, T_i). We have

$$\bar{B}_{T_i} - \bar{B}_{t_i} = \int_0^{t_i} (\bar{K}_H(T_i, s) - \bar{K}_H(t_i, s)) \,\mathrm{d}W_s + \int_{t_i}^{T_i} \bar{K}_H(T_i, s) \,\mathrm{d}W_s$$

Defining $f(u,s) := \frac{\partial}{\partial u}\bar{K}_H(u,s) = (H-\frac{1}{2})\sqrt{2H}(u-s)^{H-\frac{3}{2}}$, we have

$$\begin{aligned}\bar{B}_{T_i} - \bar{B}_{t_i} &= \int_0^{t_i}\int_{t_i}^{T_i} f(u,s)\,\mathrm{d}u\,\mathrm{d}W_s + \int_{t_i}^{T_i}\int_s^{T_i} f(u,s)\,\mathrm{d}u\,\mathrm{d}W_s\\ &= \int_0^T\int_s^T \mathbf{1}_{(t_i,T_i)}(u) f(u,s)\,\mathrm{d}u\,\mathrm{d}W_s.\end{aligned}$$

So for $A = \bigcup_{i=1}^n (t_i, T_i)$, we have

$$S = \int_0^T \mathbf{1}_A(s)\mathrm{d}\bar{B}_s = \int_0^T\int_s^T \mathbf{1}_A(u) f(u,s)\,\mathrm{d}u\,\mathrm{d}W_s.$$

Using the Itō calculus, and then Hölder inequality with exponent $p = \frac{1}{H}$ we have

$$\begin{aligned}var(S) &= \int_0^T \Big(\int_s^T \mathbf{1}_A(u) f(u,s)\,\mathrm{d}u\Big)^2 \mathrm{d}s.\\ &\leq \int_0^T \bigg(\Big(\int_s^T \mathbf{1}_A^{\frac{1}{H}}(u)\,\mathrm{d}u\Big)^H \Big(\int_s^T f^{\frac{1}{1-H}}(u,s)\,\mathrm{d}u\Big)^{1-H}\bigg)^2 \mathrm{d}s\\ &\leq L^{2H}\int_0^T \Big(\int_s^T f(u,s)^{\frac{1}{1-H}}\,\mathrm{d}u\Big)^{2-2H}\mathrm{d}s.\end{aligned}$$

It remains to show that last integral is a constant. Indeed, with the change of variables $s' := \frac{s}{T}$ and $u' := \frac{u}{T}$, we get

$$\int_0^T \Big(\int_s^T f(u,s)^{\frac{1}{1-H}}\,\mathrm{d}u\Big)^{2-2H}\mathrm{d}s = \int_0^1 \Big(\int_s^1 f(u',s')^{\frac{1}{1-H}}\,\mathrm{d}u'\Big)^{2-2H}\mathrm{d}s' < \infty.$$

1.2 Malliavin Calculus

The Malliavin calculus, named after Paul Malliavin [45, 30], extends the calculus of variations from functions to stochastic processes, hence alternatively called the stochastic calculus of variations. In particular, it allows a differential calculus on the space of random variables. Malliavin's motivation to initiate the theory was to provide a probabilistic proof of *Hörmander's sum of squares* theorem. Since then the theory has been successfully developed to investigate the existence and smoothness of a density for the solution of a stochastic differential equation. See for example [42, 35, 23].

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathbf{G}$ a Gaussian linear space on it. Let also $\mathbf{H}$ be a Hilbert space with the isometry $\mathbf{W}: \mathbf{H} \to \mathbf{G}$. Define $\mathscr{S}$ as the space of random variables F of the form:

$$F = f\big(\mathbf{W}(\varphi_1), \ldots, \mathbf{W}(\varphi_n)\big),$$

where $\varphi_i \in \mathbf{H}, f \in C^\infty(\mathbb{R}^n)$, f and all its partial derivatives have polynomial growth. The Malliavin derivative of F, ∇F, is defined (see e.g. [21, 23, 35, 42]) as an $\mathbf{H}$-valued random variable given by

$$\nabla F := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{W}(\varphi_1),\ldots,\mathbf{W}(\varphi_n)).\varphi_i$$

The operator ∇ is closable from $L^2(\Omega)$ into $L^2(\Omega;\mathbf{H})$ and one defines the Sobolev space $\mathbb{D}^{1,2}$ as the closure of $\mathscr{S}$ with respect to the following norm [21, 23]:

$$||F||_{1,2} = \sqrt{\mathbb{E}(F^2) + \mathbb{E}(||\nabla F||^2_{\mathbf{H}})}.$$

The divergence operator $\boldsymbol{\delta}$, is the adjoint of the derivative operator ∇, determined by the duality relationship[21, 23]

$$\mathbb{E}(\boldsymbol{\delta}(u)F) = \mathbb{E}(\langle \nabla F, u\rangle_{\mathbf{H}}) \quad \text{for every } F \in \mathbb{D}^{1,2}.$$

The space of $\mathbf{H}$-valued Malliavin derivable $\mathscr{L}^2$ random variables with $\mathscr{L}^2$ derivatives, denoted by $\mathbb{D}^{1,2}(\mathbf{H})$, is contained in the domain of $\boldsymbol{\delta}$, and moreover for any $u \in \mathbb{D}^{1,2}(\mathbf{H})$, we have

$$\mathbb{E}\big(\boldsymbol{\delta}(u)^2\big) \le \mathbb{E}\big(\|u\|^2_{\mathbf{H}}\big) + \mathbb{E}\big(\|\nabla u\|^2_{\mathbf{H}\otimes\mathbf{H}}\big). \tag{1.5}$$

For any random variable $F \in \mathbb{D}^{1,2}$ and $\varphi \in \mathbf{H}$ there holds the following equality called the change of variable formula [21, 23]:

$$F\mathbf{B}(\varphi) = \boldsymbol{\delta}(F\varphi) + \langle \nabla F, \varphi\rangle_{\mathbf{H}}. \tag{1.6}$$

For more on Malliavin calculus we refer to [23, 35].

Let $\{B(t,x)\,;\, t\in\mathbb{R}\}_{x\in\mathbb{Z}^d}$ be a family of independent fractional Brownian motions indexed by $x\in\mathbb{Z}^d$ all with Hurst parameter H.
Following [21], let $\mathscr{H}$ be the Hilbert space defined by the completion of the linear span of indicator functions $\mathbf{1}_{[0,t]\times\{x\}}$ for $t\in\mathbb{R}$ and $x\in\mathbb{Z}^d$ under the scalar product

$$\langle \mathbf{1}_{[0,t]\times\{x\}}, \mathbf{1}_{[0,s]\times\{y\}}\rangle_{\mathscr{H}} = R_H(t,s)\,\delta_x(y)\,,$$

where δ is the Kronecker delta. For negative t we assume the convention $\mathbf{1}_{[0,t]\times\{x\}} := -\mathbf{1}_{[t,0]\times\{x\}}$. The mapping $\mathbf{B}(\mathbf{1}_{[0,t]\times\{x\}}) := B(t,x)$ can be extended to a linear isometry from $\mathscr{H}$ onto the Gaussian space spanned by $\{B(t,x)\,;\, t\in\mathbb{R}, x\in\mathbb{Z}^d\}$. This is the only setting to which we will apply Malliavin calculus in the following chapters.

1.3 Some useful theorems

In this section we assemble some basic results that we will need in the succeeding chapters.

The following lemma allows interchanging integration with a continuous linear operator.

Lemma 1.3.1. *Let $(M, \mathcal{M}, \mu)$ be a measure space and B, B' be Banach spaces. Let also $\Lambda : B \to B'$ be a continuous linear operator and $f : M \to B$ a separably-valued measurable function, i.e. there exists a separable subspace B_1 of B such that $f \in B_1$ almost surely. If $\int ||f||_B \mathrm{d}\mu < \infty$ then*

$$\Lambda \int f \mathrm{d}\mu = \int \Lambda f \mathrm{d}\mu.$$

Proof. As f is separably-valued, there exists [23, 12, 29] a sequence of simple functions $\{u_n\}_n$ of the form $\sum_i \mathbf{1}_{A_i} h_i$ with $A_i \in \mathcal{M}$ and $h_i \in B$ with the property that

$$\int ||u_n - f||_B \mathrm{d}\mu \longrightarrow 0 \qquad as \qquad n \to \infty.$$

As Λ is linear, it commutes with integration on $\{u_n\}_n$. As Λ is continuous we have $||\Lambda(x)||_B \leq C||x||_{B'}$ for some positive constant C. So

$$\int ||\Lambda(u_n - f)||_{B'} \mathrm{d}\mu \leq C \int ||(u_n - f)||_B \mathrm{d}\mu$$

and also

$$\begin{aligned} ||\Lambda \int (u_n - f) \mathrm{d}\mu||_{B'} &\leq C|| \int (u_n - f) \mathrm{d}\mu||_B \\ &\leq C \int ||u_n - f||_B \mathrm{d}\mu. \end{aligned}$$

Hence Λ commutes with integration for f too. $\square$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{H}$ be a Gaussian Hilbert space on it and $\mathcal{F}(\mathcal{H})$ be the sigma algebra generated by $\mathcal{H}$. The following theorem [48] shows that the distribution of a Malliavin derivable random variable with bounded derivative has exponentially decaying tails. We will this theorem in section 3.6 for establishing the quenched limits.

Theorem 1.3.2 (B.8.1 in [48])**.** *Suppose that $\varphi \in \mathbb{D}^{1,p}$ for some $p > 1$ with $\nabla\phi \in \mathcal{L}^\infty(\Omega; \mathcal{H})$, i. e. $||\nabla\varphi||_{\mathcal{H}}$ is almost surely bounded. Then we have the following tail probability estimate:*

$$P\{\omega\,;\, |\varphi(\omega) - \mathbb{E}[\varphi]| > c\} \leq 2\, exp\{\frac{-c^2}{2\,||\nabla\varphi||^2_{\mathcal{L}^\infty(\Omega;\mathcal{H})}}\} \tag{1.7}$$

The same way Fubini's theorem allows the interchange of classical (deterministic) integrals, stochastic Fubini theorem [49, 37] allows the interchange of a classical integral with an Itō integral. The following theorem gives two sufficient conditions that imply the possibility of the interchange. The first one is quite classical [37], and is basically a special case of theorem 1.3.1. The second sufficient condition is a recent one due to Veraar [49].

Theorem 1.3.3. *Let $W(.)$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and T a positive number possibly $+\infty$. Suppose ψ : $X \times [0, T] \times \Omega \to \mathbb{R}$ is jointly measurable and adapted, in the sense that for all $x \in X$, the process*

$\psi(x,\cdot,\cdot)$ *is adapted. If either*

$$\int_X \Big(\mathbb{E}\int_0^T |\psi(x,t)|^2 \mathrm{d}t\Big)^{1/2} \mathrm{d}\mu(x) < \infty$$

or

$$\int_X \Big(\int_0^T |\psi(x,t)|^2 \mathrm{d}t\Big)^{1/2} \mathrm{d}\mu(x) < \infty \quad \textit{almost surely,}$$

then the following integrals exist and are equal [49, 37]

$$\int_X \int_0^T \psi(x,t)\mathrm{d}W_t \, \mathrm{d}\mu(x) = \int_0^T \int_X \psi(x,t)\mathrm{d}\mu(x)\,\mathrm{d}W_t\,.$$

Separability is a property that enables us to deal with a random process basically as if it has a countable domain. We need this concept for the two succeeding theorems.

Definition 1.3.1. A random process $\{X(t)\}_{t\in T}$ on an arbitrary topological space T, is called *separable* if T has a dense countable subset D such that almost surely

$$\forall t \in T : \exists \{t_n\}_{n\in\mathbb{N}} \subseteq D \quad ; \quad \lim_{n\to\infty} t_n = t \quad \text{and} \quad \lim_{n\to\infty} X(t_n) = X(t)$$

Dudley's theorem or Dudley's entropy bound [46, 29] is a strong tool for bounding the expectation of the supremum of a family of Gaussian random variables. Although it was Dudley who defined the metric entropy integral (as an equivalent sum in [10], then explicitly in [11]), it was Pisier [36] who actually proved the inequality. The proof uses a chaining argument [46].

Theorem 1.3.4 (Dudley). *Let* $\{X_t\}_{t\in T}$ *be a family of centered Gaussian random variables indexed by some set* T *and* ρ *be the pseudo-metric on* T *defined by* $\rho(s,t) := \sqrt{\mathbb{E}(X_t - X_s)^2}$. *Then for any finite subset* $F \subseteq T$ *we have*

$$\mathbb{E}(\sup_{t\in F} X_t) \le K \int_0^\infty \sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon\,, \tag{1.8}$$

where $N(\varepsilon)$ *is the minimum number of* ρ*-balls of radius* ε *required to cover* T*, and* K *is a universal positive constant.*

Remark 1.3.1. Inequality (1.8) holds also for any countable subset $F \subseteq T$. Indeed F being countable, can be expressed as $\bigcup_n F_n$ for some finite increasing sets $\{F_n\}$. Using Fatou's lemma

$$\mathbb{E}(\sup_{t\in F} X_t) = \mathbb{E}(\lim_{n\to\infty} \sup_{t\in F_n} X_t) = \mathbb{E}(\liminf_{n\to\infty} \sup_{t\in F_n} X_t) \le \liminf_{n\to\infty} \mathbb{E}(\sup_{t\in F_n} X_t)$$

Remark 1.3.2. When T has a topological structure and $X(.)$ is separable, Dudley's theorem can be expressed in the following stronger form

$$\mathbb{E}(\sup_{t\in T} X_t) \le K \int_0^\infty \sqrt{\log N(\varepsilon)}\,\mathrm{d}\varepsilon\,. \tag{1.9}$$

The reason is that in this case $\sup_{t\in T} X_t = \sup_{t\in D} X_t$ and the statement is established using remark 1.3.1.

Borell's inequality [28] shows that under some reasonably weak conditions, the supremum of a family of Gaussian random variables concentrates only around its mean and its probability tails away from its mean, decay exponentially.

Theorem 1.3.5 (**Borell's inequality**)**.** *Let T be a countable set and $\{X_t\}_{t\in T}$ be a family of centered Gaussian random variables indexed by T with $\sup_{t\in T} X_t < \infty$ almost surely. Then [28] the expectation $\mathbb{E}(\sup_{t\in T} X_t)$ is finite and for any $c > 0$*

$$\mathbb{P}\Big(|\sup_{t\in T} X_t - \mathbb{E}(\sup_{t\in T} X_t)| \geq \lambda\Big) \leq 2e^{-\frac{\lambda^2}{2\sigma_T^2}},$$

where $\sigma_T^2 := \sup_{t\in T} \mathbb{E}(X_t^2)$.

This theorem can also be formulated using the median of supremum instead of its mean [28, 1]. In fact the original result of Borell [2] which is in a much more general and abstract setting, uses the median.

Remark 1.3.3. For T uncountable, the Borell's inequality still holds true provided that T is equipped with a topological structure and $\{X_t\}_{t\in T}$ is separable with respect to that topology.

The classical well-known Stirling formula gives the asymptotic value of the factorial function. The following stronger version [40, 13] which gives tight lower and upper bounds on $n!$, although not really necessary for our proofs, makes some of our proofs simpler in saving us an unspecified multiplicative constant everywhere.

Theorem 1.3.6 (Stirling)**.** *For any $n \in \mathbb{N}$ we have [40, 13]*

$$(n/e)^n \sqrt{2\pi n}\, e^{\frac{1}{12n+1}} \;\leq\; n! \;\leq\; (n/e)^n \sqrt{2\pi n}\, e^{\frac{1}{12n}}. \tag{1.10}$$

In particular

$$(n/e)^n \sqrt{2\pi n} \;\leq\; n! \;\leq\; e\,(n/e)^n \sqrt{2\pi n}. \tag{1.11}$$

2 Feynman-Kac representation

2.1 Introduction

Consider the following parabolic Anderson model(PAM) on $\mathbb{Z}^d$

$$\frac{\partial}{\partial t}u(t,x) = \kappa\boldsymbol{\Delta}u(t,x) + u(t,x)\frac{\partial}{\partial t}B(t,x) \qquad x \in \mathbb{Z}^d\,,\ t \geq 0\,,$$

where $\kappa > 0$ is a diffusion constant, $\boldsymbol{\Delta}$ is the discrete Laplacian defined by $\boldsymbol{\Delta}f(x) := \frac{1}{2d}\sum_{|y-x|=1}[f(y)-f(x)]$ and $\{B(\cdot,x)\}_{x\in\mathbb{Z}^d}$ is a family of independent fractional Brownian motions(fBM) of Hurst parameter H, indexed by $\mathbb{Z}^d$.

As the paths of fBM are like Brownian motion paths, almost surely nowhere differentiable, this equation doesn't make sense in the classical sense and hence it should be reformulated in the following mild sense

$$\begin{cases} u(t,x) - u(0,x) = \displaystyle\int_0^t \boldsymbol{\Delta}u(s,x)\,\mathrm{d}s + \int_0^t u(s,x)\,B(\mathrm{d}s,x) \\ u(0,x) = u_o(x) \end{cases}, \tag{2.1}$$

where the stochastic integral is Stratonovich type in the sense that the fractional Brownian motion is approximated by a sequence of smooth processes $\{B^\varepsilon\}_\varepsilon$ and the integral $\int u\,\mathrm{d}B$ is given as the limit of the sequence $\{\int u\,\mathrm{d}B^\varepsilon\}_\varepsilon$. We assume that $u_o(\cdot)$ is a bounded measurable function. It should be noted that unlike the Brownian motion for which there are basically two standard integral types namely Itō and Stratonovich, which are easily related to each other by an additive 'correction' term, for the fractional Brownian motion there are several competing approaches whose relation to each other has not been fully established yet. We refer to [32] and [5].

We will show that the following Feynman-Kac representation gives a solution to (2.1):

$$u(t,x) = \mathbb{E}^x\Big[u_o(X(t))\exp\int_0^t B\big(\mathrm{d}s, X(t-s)\big)\Big], \tag{2.2}$$

where $X(t)$ is a simple random walk with jump rate κ, started at $x \in \mathbb{Z}^d$ and independent of the family $\{B(\cdot,x)\}_{x\in\mathbb{Z}^d}$. Here the stochastic integral is nothing other than a summation. Indeed, suppose $\{t_i\}_{i=1}^n$ be the jump times of the time-reversed random walk $\{X(t-s)\ ,\ s\in[0,t]\}$, and $\{x_i\}_{i=0}^n$ be the value of $\{X(t-\cdot)\}$ at time interval $[t_i, t_{i+1})$. Then we have

$$\int_0^t B\big(\mathrm{d}s, X(t-s)\big) = \sum_{i=0}^n \big(B(t_{i+1}, x_i) - B(t_i, x_i)\big).$$

Carmona and Molchanov in their memoir [3] prove that for bounded u_o and $H = 1/2$ i.e. standard Brownian motion, the Feynman-Kac representation (2.2) solves eqution (2.1). Nualart et al. proved this result for PAM on $\mathbb{R}^d$ driven by fractional noise of Hurst parameter $H \geq 1/2$ in [22] and for $H \geq 1/4$ in [21]. Our method is able to prove this property without any restriction on H due to the fact that in the discrete case one deals with locally constant random walk instead of Brownian motion which is only locally α-Hölder continuous for $\alpha < 1/2$.

In section 2.2 we explain the approximation scheme we are going to use. There we outline our methodology without delving much into technicalities. We show that the problem reduces to demonstrating the converge of three expressions u_ε, $V_{1,\varepsilon}$ and $V_{2,\varepsilon}$. In section 2.3, using only elementary probability we prove that the piecewise-constant integrals with respect to the approximation processes proposed in section 2.2, approach the integral with respect to fractional Brownian motion. The proposition 2.3.1 serves as the building block of our arguments.
The remaining chapters are devoted to the showing the convergence of u_ε, $V_{1,\varepsilon}$ and $V_{2,\varepsilon}$.

2.2 Setting

As explained in the previous section we aim to approximate the fractional Brownian motions with a family of smooth Gaussian processes. There are basically two natural ways to approximate a (fractional) Brownian motion: The so-called Wong-Zakai approximation scheme [47, 52] which is the piecewise linear approximation of (fractional) Brownian motion paths. The second natural scheme is as follows: The time derivative of a fractional Brownian motion does not exist in the classical sense but only in the distributional sense. The idea is to approximate the 'derivative' of the fractional Brownian motion and then integrate it. Indeed we define the approximate derivative of $B(\cdot,x)$ as $\dot{B}_\varepsilon(\cdot,x)$

$$\dot{B}_\varepsilon(t,x) := \frac{1}{2\varepsilon}\big(B(t+\varepsilon,x) - B(t-\varepsilon,x)\big). \tag{2.3}$$

Proposition 2.3.1 shows in particular that the integral of this family of Gaussian processes converges to fractional Brownian motion.

While the first scheme doesn't seem to be easy to work with, the second one has been proved to be very suitable in our setting where we use the Wiener space technics and Malliavin calculus

[21].

Now let first replace in equation (2.1), the fBM family $\{B(\cdot,x)\}_{x\in\mathbb{Z}^d}$ by a family of absolutely continuous functions $\{\Xi(\cdot,x)\}_{x\in\mathbb{Z}^d}$, or equivalently replace the family of fractional noises $\{\frac{\partial}{\partial t}B(\cdot,x)\}_{x\in\mathbb{Z}^d}$ by a family of locally integrable functions $\{\xi(\cdot,x)\}_{x\in\mathbb{Z}^d}$ where $\Xi(t,x)=\int_0^t \xi(s,x)\mathrm{d}s$ for every x and t. Carmona and Molchanov in [3] showed that the Feynman-Kac formula

$$\mathscr{F}(\Xi) := \mathbb{E}^x\Big[u_o(X(t))\exp\int_0^t \Xi\big(\mathrm{d}s, X(t-s)\big)\Big] = \mathbb{E}^x\Big[u_o(X(t))\exp\int_0^t \xi(s,X(t-s)\mathrm{d}s\Big]$$

solves the PAM driven by the potential $\{\xi(\cdot,x)\}_{x\in\mathbb{Z}^d}$ if this expression is finite for every x and t.

If we approximate the fractional Brownian motions by the sequence of families $\{B^\varepsilon(\cdot,x)\}_{x\in\mathbb{Z}^d}$ where every $B^\varepsilon(\cdot,x)$ is a random process with absolutely continuous sample paths that converges to $B(\cdot,x)$, we expect that $\mathscr{F}(B^\varepsilon)$ should also converge $\mathscr{F}(B)$. On the other hand, if we denote by u^ε the solution of equation (2.1) with B replaced by B^ε, we also expect that u^ε should converge to the solution of (2.1) with the integral understood in the Stratonovich sense. The reason is that for the stochastic differential equations with Brownian motion or more generally semi-martingale terms, if the Brownian motions are approximated by a sequence of absolutely continuous processes, the sequence of solutions converge to the Stratonovich solution of the original differential equation [44, 37]. Note that for each path of an absolutely continuous processes, a solution in the classical sense exists due to the its differentiability.

The above intuitive explanation suggests that if this Feynman-Kac representation is possible only if the integration is in Stratonovich sense.

So we consider the approximation scheme of equation (2.3). In the rest of this chapter without any loss of generality, we assume $\kappa = 1$.

Let

$$u_\varepsilon(t,x) := \mathbb{E}^x\Big[u_o(X(t))\exp\int_0^t \dot{B}_\varepsilon\big(s,X(t-s)\big)\mathrm{d}s\Big], \tag{2.4}$$

where $\dot{B}_\varepsilon$ is defined in (2.3).

By lemma 2.4.4, we have $\mathbb{E}|u_\varepsilon(t,x)|<\infty$ for every x and t. So almost surely, $u_\varepsilon(t,x)$ is finite for every x and t. On the other hand, the sample paths of $\dot{B}_\varepsilon$ are locally integrable. So by the above mentioned theorem of Carmona and Molchanov [3] the field $\{u_\varepsilon(t,x)\}_{x,t}$ solves the following equation

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \boldsymbol{\Delta} u_\varepsilon + u_\varepsilon \dot{B}_\varepsilon \\ u_\varepsilon(0,x) = u_o(x)\,. \end{cases} \tag{2.5}$$

We aim to show that (2.2) gives a solution to (2.1) with the Stratonovich integral $\int_0^t u(s,x)B(\mathrm{d}s,x)$

defined in the following natural manner which was also used in [21].

Definition 2.2.1. For a random field $u = \{u(t,x); t \in \mathbb{R}, x \in \mathbb{Z}^d\}$, the Stratonovich integral

$$\int_0^t u(s,x)B(\mathrm{d}s,x)$$

is defined [21] as the following $\mathscr{L}^2$ limit (if it exists)

$$\lim_{\varepsilon\to 0}\int_0^t u(s,x)\dot{B}_\varepsilon\big(s,x\big)\mathrm{d}s\,.$$

Using the same methodology of [21] we will show that the Stratonovich integral of the Feynman-Kac formula (2.2) exists and moreover it satisfies (2.1).

Indeed equation (2.5) can be integrated to

$$u_\varepsilon(t,x) - u_o(x) = \int_0^t \boldsymbol{\Delta} u_\varepsilon(s,x)\mathrm{d}s + \int_0^t u_\varepsilon(s,x)\dot{B}_\varepsilon(s,x)\mathrm{d}s\,. \tag{2.6}$$

Once we show that u_ε (given by (2.4)) converges to u (given by (2.2)) in $\mathscr{L}^2$ sense and uniformly in $t \in [0,T]$ as ε goes down to zero, along with equation (2.6), it would imply the $\mathscr{L}^2$-convergence of $\int\big(u_\varepsilon\dot{B}_\varepsilon\big)$ to some random variable. If moreover one shows that $\int\big(u_\varepsilon\dot{B}_\varepsilon - u\dot{B}_\varepsilon\big)$ converges in $\mathscr{L}^2$ to zero, it would imply the convergence of $\int\big(u\dot{B}_\varepsilon\big)$ and hence the existence of the Stratonovich integral $\int u\,\mathrm{d}B$. But this means that u satisfies equation (2.1).

Let

$$g^\varepsilon_{s,x}(r,z) := \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\,\delta_x(z)\,. \tag{2.7}$$

It is easy to show that $g^\varepsilon_{s,x}(r,z)$ is in $\mathscr{H}$ defined in section 1.2, and moreover

$$\mathbf{B}(g^\varepsilon_{s,x}) = \dot{B}_\varepsilon(s,x)\,.$$

So by the change of variable formula (1.6) we have

$$\begin{aligned} u_\varepsilon(s,x)\dot{B}_\varepsilon(s,x) - u(s,x)\dot{B}_\varepsilon(s,x) &= \tilde{u}_\varepsilon(s,x)\mathbf{B}(g^\varepsilon_{s,x}) \\ &= \boldsymbol{\delta}(\tilde{u}_\varepsilon(s,x)g^\varepsilon_{s,x}) + \langle\nabla\tilde{u}_\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathscr{H}}\,, \end{aligned}$$

where $\tilde{u}_\varepsilon := u_\varepsilon - u$.

Hence it suffices to show that $V_{1,\varepsilon} := \int_0^t \boldsymbol{\delta}(\tilde{u}_\varepsilon(s,x)g^\varepsilon_{s,x})\mathrm{d}s$ and $V_{2,\varepsilon} := \int_0^t \langle\nabla\tilde{u}_\varepsilon(s,x), g^\varepsilon_{s,x}\rangle_{\mathscr{H}}\mathrm{d}s$ both converge to zero as ε goes to zero. In sections 2.4, 2.5 and 2.6 we will deal with the convergence of u_ε, $V_{1,\varepsilon}$ and $V_{2,\varepsilon}$.

2.3 Approximation rate

In this section we prove the following theorem that establishes the approximation of $B(\mathrm{d}s)$ by $\dot{B}_\varepsilon(s)\mathrm{d}s$. In the proof we will use some ideas of [21] as well as simple properties of random walk.

Proposition 2.3.1. *Let t, T, t_1, t_2, ..., t_N be some positive real numbers with $t_0 := 0 < t_1 < \dots < t_N < t_{N+1} := t \le T$ and $X(\cdot)$ a jump function on $[0,t]$ with values in $\mathbb{Z}^d$ and jump times $\{t_1,\dots,t_N\}$, i.e. $X(s) = x_i \in \mathbb{Z}^d$ for $s \in (t_i, t_{i+1}]$. Then*

$$\mathbb{E}\Big|\int_0^t \dot{B}_\varepsilon\big(s, X(s)\big)\mathrm{d}s - \int_0^t B\big(\mathrm{d}s, X(s)\big)\Big|^2 \le CN^2\varepsilon^{\min\{2H,1\}},$$

where C is a constant depending only on T and H and

$$\int_0^t B\big(\mathrm{d}s, X(s)\big) = \sum_{i=0}^N \big(B(t_{i+1}, x_i) - B(t_i, x_i)\big).$$

Proof. First we show that for every t_1 and t_2, $t_1 < t_2 \le T$, and any fractional Brownian motion $B(\cdot)$ with Hurst parameter $H \in (0,1)$ we have

$$\mathbb{E}\Big|B(t_2) - B(t_1) - \int_{t_1}^{t_2} \dot{B}_\varepsilon(\theta)\mathrm{d}\theta\Big|^2 \le C\varepsilon^{\min\{2H,1\}}, \tag{2.8}$$

where $\dot{B}_\varepsilon$ is the symmetric ε-derivative of W:

$$\dot{B}_\varepsilon(t) := \frac{1}{2\varepsilon}\big(B(t+\varepsilon) - B(t-\varepsilon)\big)$$

and C is some positive constant depending only on T and H. We have to calculate and bound

$$\begin{aligned}\mathbb{E}\Big|B(t_2) - B(t_1) - \int_{t_1}^{t_2} \dot{B}_\varepsilon(\theta)\mathrm{d}\theta\Big|^2 &= \mathbb{E}\Big|B(t_2) - B(t_1)\Big|^2 \\ &+ \int_{t_1}^{t_2}\int_{t_1}^{t_2} \mathbb{E}\Big[\dot{B}_\varepsilon(\theta)\dot{B}_\varepsilon(\eta)\Big]\mathrm{d}\theta\,\mathrm{d}\eta - 2\int_{t_1}^{t_2} \mathbb{E}\Big[\big(B(t_2) - B(t_1)\big)\dot{B}_\varepsilon(\theta)\Big]\mathrm{d}\theta.\end{aligned} \tag{2.9}$$

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the first and second terms on the right hand side of this equation and $\mathfrak{S}_3$ be the third term without its -2 factor.
Using the following equality

$$\mathbb{E}\Big[\big(B(a) - B(b)\big)\big(B(c) - B(d)\big)\Big] = \frac{1}{2}\Big[|a-d|^{2H} + |b-c|^{2H} - |a-c|^{2H} - |b-d|^{2H}\Big]$$

we have:

$$\mathfrak{S}_1 = |t_2 - t_1|^{2H},$$

$$\mathfrak{S}_2 = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \frac{1}{8\varepsilon^2}\Big[|s-\eta+2\varepsilon|^{2H} + |\eta - s + 2\varepsilon|^{2H} - 2|s-\eta|^{2H}\Big]\mathrm{d}\eta\,\mathrm{d}s$$

and

$$\mathfrak{S}_3 = \frac{1}{4\varepsilon}\int_{t_1}^{t_2}\Big[|t_2-\theta+\varepsilon|^{2H}+|\theta-t_1+\varepsilon|^{2H}-|t_2-\theta-\varepsilon|^{2H}-|\theta-t_1-\varepsilon|^{2H}\Big]\mathrm{d}\theta\,.$$

We will show that both $\mathfrak{S}_2$ and $\mathfrak{S}_3$ converge to $|t_2-t_1|^{2H}$.

Step I: Limiting behavior of $\mathfrak{S}_2$

By a change of variable we can replace the integration interval with $[0, t_2-t_1]$ with the integrand remaining intact. But as the integrand is symmetric in s and η, we may calculate the integral over a triangular surface hence getting:

$$\mathfrak{S}_2 = \frac{2}{8\varepsilon^2}\int_0^{t_2-t_1}\int_0^{s}\Big[|s-\eta+2\varepsilon|^{2H}+|\eta-s+2\varepsilon|^{2H}-2|s-\eta|^{2H}\Big]\mathrm{d}\eta\,\mathrm{d}s\,.$$

By a change of variable of $\gamma = s-\eta$ we get:

$$\mathfrak{S}_2 = \frac{1}{4\varepsilon^2}\int_0^{t_2-t_1}\int_0^{s}\Big[|\gamma+2\varepsilon|^{2H}+|\gamma-2\varepsilon|^{2H}-2|\gamma|^{2H}\Big]\mathrm{d}\gamma\,\mathrm{d}s\,. \tag{2.10}$$

We will show that $\mathfrak{S}_2$ converges to $|t_2-t_1|^{2H}$ with the following rate of convergence for $H<\frac{1}{2}$

$$\Big|\mathfrak{S}_2-|t_2-t_1|^{2H}\Big| \le 4(2\varepsilon)^{2H} \tag{2.11}$$

and

$$\Big|\mathfrak{S}_2-|t_2-t_1|^{2H}\Big| \le C\varepsilon \tag{2.12}$$

for $H>\frac{1}{2}$. Here C is some constant depending only on T and H. For the simplicity of notation let $t := t_2-t_1$. Defining $g(s) := \int_0^s |r|^{2H}\mathrm{d}r$, (2.10) can be written as:

$$\mathfrak{S}_2 = \frac{1}{4\varepsilon^2}\int_0^{t}\big[g(s+2\varepsilon)+g(s-2\varepsilon)-2g(s)\big]\,\mathrm{d}s\,. \tag{2.13}$$

As g' is continuous everywhere and $g''(r) = 2H\,\mathrm{sgn}(r)|r|^{2H-1}$ is continuous everywhere except for the origin when $H<\frac{1}{2}$ and everywhere when $H\ge\frac{1}{2}$, this equation can be written as:

$$\mathfrak{S}_2 = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\int_0^{t} g''(s+\xi\varepsilon+\eta\varepsilon)\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{2.14}$$

Let $\Delta := \xi\varepsilon+\eta\varepsilon$ and first suppose that $H<\frac{1}{2}$.

Case i) $\Delta\ge 0$:

$$\begin{aligned}\left|\int_0^t\big(g''(s+\Delta)-2Hs^{2H-1}\big)\mathrm{d}s\right| &= 2H\int_0^t\big(s^{2H-1}-(s+\Delta)^{2H-1}\big)\mathrm{d}s\\ &= \big[t^{2H}-(t+\Delta)^{2H}\big]+\Delta^{2H} \le \Delta^{2H}\,.\end{aligned}$$

Case ii) $-t < \Delta < 0$:

$$\begin{aligned}\int_0^t \left(g''(s+\Delta) - 2Hs^{2H-1}\right)\mathrm{d}s = &-2H\int_0^{-\Delta}\left((-s-\Delta)^{2H-1} + s^{2H-1}\right)\mathrm{d}s \\ &+2H\int_{-\Delta}^{t}\left((s+\Delta)^{2H-1} - s^{2H-1}\right)\mathrm{d}s.\end{aligned} \tag{2.15}$$

The first term equals $-2|\Delta|^{2H}$ and the second term equals $(t+\Delta)^{2H} - t^{2H} + \Delta^{2H}$ which is bounded by $2|\Delta|^{2H}$.
Case iii) $\Delta \le -t$:

$$\begin{aligned}\left|\int_0^t \left(g''(s+\Delta) - 2Hs^{2H-1}\right)\mathrm{d}s\right| &= 2H\int_0^{t}\left((-s-\Delta)^{2H-1} + s^{2H-1}\right)\mathrm{d}s \\ &\le 2H\int_0^{-\Delta}\left((-s-\Delta)^{2H-1} + s^{2H-1}\right)\mathrm{d}s = 2|\Delta|^{2H}.\end{aligned} \tag{2.16}$$

Noting that $|\Delta| < 2\varepsilon$, inequality (2.11) is proved.

Now we consider the case of $H \ge \frac{1}{2}$.
Case i) $\Delta \ge 0$:

$$\begin{aligned}\int_0^t \left(g''(s+\Delta) - 2Hs^{2H-1}\right)\mathrm{d}s &= 2H\int_0^{t}\left((s+\Delta)^{2H-1} - s^{2H-1}\right)\mathrm{d}s \\ &= 2H\int_0^t\int_0^\Delta (2H-1)(s+\alpha)^{2H-2}\mathrm{d}\alpha\,\mathrm{d}s \\ &= 2H\int_0^\Delta\left((t+\alpha)^{2H-1} - \alpha^{2H-1}\right)\mathrm{d}\alpha.\end{aligned} \tag{2.17}$$

As $2H-1 < 1$ we have $(t+\alpha)^{2H-1} - \alpha^{2H-1} \le t^{2H-1}$ which shows that the above integral is bounded by $2Ht^{2H-1}|\Delta|$ and hence by $2HT^{2H-1}|\Delta|$.
Case ii) $-t < \Delta < 0$: Equation (2.15) remains valid with its first term bounded by $2|\Delta|^{2H}$ which is smaller than $2|\Delta|$, assuming $|\Delta| < 1$. As $2H-1 > 0$, the absolute value of the second term equals:

$$\begin{aligned}2H\int_{-\Delta}^{t}\left(s^{2H-1} - (s+\Delta)^{2H-1}\right)\mathrm{d}s &= 2H\int_\Delta^0\int_{-\Delta}^t (s+\alpha)^{2H-2}(2H-1)\mathrm{d}s\,\mathrm{d}\alpha \\ &= 2H\int_\Delta^0\left[(\alpha+t)^{2H-1} - (-\Delta+\alpha)^{2H-1}\right]\mathrm{d}\alpha \\ &\le 2H\int_\Delta^0 (t+\Delta)^{2H-1} \le 2Ht^{2H-1}|\Delta| \le 2HT^{2H-1}|\Delta|.\end{aligned}$$

The last inequality is true because $2H-1 < 1$. So we get the bound $(2+2HT^{2H-1})|\Delta|$.
Case iii) $\Delta \le -t$: Equation (2.16) works without any change and we get the bound $2|\Delta|^{2H} \le 2|\Delta|$.

Noting $|\Delta| \le 2\varepsilon$ the proof of inequality (2.12) is complete with $C = 2^{2H}(2+2HT^{2H-1})$.

In the $H \ge \frac{1}{2}$ regime we can establish the following alternative bound which will be used in section 2.4

$$\left|\mathfrak{S}_2 - |t_2 - t_1|^{2H}\right| \le 2|t_2 - t_1|(2H+1)\varepsilon^{2H-1}. \tag{2.18}$$

It is shown case by case

- For case i), using the first equality in equation (2.17) and noting $(s+\Delta)^{2H-1} - s^{2H-1} \le \Delta^{2H-1}$ we have the bound $2Ht\Delta^{2H-1}$.
- For case ii), the second term on the right hand side in (2.15) can be bounded by $2H(t-|\Delta|)|\Delta|^{2H-1} \le 2Ht|\Delta|^{2H-1}$ and the first term by $2|\Delta|^{2H} \le 2t|\Delta|^{2H-1}$.
- In case iii), using the first equality in (2.16) it can be bounded by $4Ht|\Delta|^{2H-1}$.

So we have the bound $2t(2H+1)|\Delta|^{2H-1} \le 2t(2H+1)\varepsilon^{2H-1}$.
Step II: Limiting behavior of $\mathfrak{S}_3$

By setting $t := t_2 - t_1$ and two changes of variables, $\mathfrak{S}_3$ can be written as

$$\frac{2}{4\varepsilon}\int_0^t \left(|\theta+\varepsilon|^{2H} - |\theta-\varepsilon|^{2H}\right)\mathrm{d}\theta = \frac{1}{2\varepsilon}\int_0^t\int_{-\varepsilon}^{+\varepsilon} 2H|\theta+\alpha|^{2H-1}\mathrm{d}\alpha\,\mathrm{d}\theta .$$

So

$$(\mathfrak{S}_3 - t^{2H}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon}\int_0^t 2H\left(|\theta+\alpha|^{2H-1} - \theta^{2H-1}\right)\mathrm{d}\theta\,\mathrm{d}\alpha . \tag{2.19}$$

Let's first assume $\varepsilon \le t$. Let's break this integral into three sub-integrals:

$$\int_0^{+\varepsilon}\int_0^t \cdots + \int_{-\varepsilon}^0\int_0^{-\alpha} \cdots + \int_{-\varepsilon}^0\int_{-\alpha}^t \cdots$$

and call them A, B and C, respectively.
We bound these terms separately for $H \le \frac{1}{2}$ and $H > \frac{1}{2}$.

First suppose $H \le \frac{1}{2}$.

$$\begin{aligned} |A| &= \frac{1}{2\varepsilon}\int_0^{+\varepsilon}\int_0^t 2H\left[\theta^{2H-1} - (\theta+\alpha)^{2H-1}\right]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{1}{2\varepsilon}\int_0^{+\varepsilon}\left[\alpha^{2H} - (\alpha+t)^{2H} + t^{2H}\right]\mathrm{d}\alpha \\ &\le \frac{1}{2\varepsilon}\int_0^{+\varepsilon}\alpha^{2H}\mathrm{d}\alpha = \frac{1}{2(2H+1)}\varepsilon^{2H}. \end{aligned} \tag{2.20}$$

For the second term we have

$$\begin{aligned}|B| &\leq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{-\alpha} 2H\big[\theta^{2H-1} + (-\theta-\alpha)^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{1}{\varepsilon}\int_{-\varepsilon}^{0}(-\alpha)^{2H}\mathrm{d}\alpha = \frac{1}{2H+1}\varepsilon^{2H}.\end{aligned}$$

Finally:

$$\begin{aligned}|C| &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}\int_{-\alpha}^{t} 2H\big[(\theta+\alpha)^{2H-1} - \theta^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}\big[(t+\alpha)^{2H} - t^{2H} + (-\alpha)^{2H}\big]\mathrm{d}\alpha \\ &\leq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}(-\alpha)^{2H}\mathrm{d}\alpha = \frac{1}{2(2H+1)}\varepsilon^{2H}.\end{aligned}$$

So for $H \leq \frac{1}{2}$:

$$|\mathfrak{S}_3 - t^{2H}| \leq \frac{2}{2H+1}\varepsilon^{2H}.$$

Now for $H > \frac{1}{2}$: we again examine each of the terms:

$$\begin{aligned}|A| &= \frac{1}{2\varepsilon}\int_{0}^{+\varepsilon}\int_{0}^{t} 2H\big[(\theta+\alpha)^{2H-1} - \theta^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{H}{\varepsilon}\int_{0}^{+\varepsilon}\int_{0}^{t}\int_{0}^{\alpha}(2H-1)(\theta+\xi)^{2H-2}\mathrm{d}\xi\,\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{H}{\varepsilon}\int_{0}^{+\varepsilon}\int_{0}^{\alpha}\big[(t+\xi)^{2H-1} - \xi^{2H-1}\big]\mathrm{d}\xi\,\mathrm{d}\alpha \\ &\leq \frac{H}{\varepsilon}\int_{0}^{+\varepsilon}\int_{0}^{\alpha} t^{2H-1}\mathrm{d}\xi\,\mathrm{d}\alpha = \frac{1}{2}Ht^{2H-1}\varepsilon.\end{aligned} \tag{2.21}$$

As equation (2.3) remains valid for $H > \frac{1}{2}$, we have:

$$|B| \leq \frac{1}{2H+1}\varepsilon^{2H} \leq \frac{1}{2H+1}\varepsilon.$$

For $|C|$ we use the same trick as in (2.21):

$$\begin{aligned}|C| &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}\int_{-\alpha}^{t} 2H\big[\theta^{2H-1} - (\theta+\alpha)^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &= \frac{H}{\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{-\alpha}\int_{-\alpha}^{t}(2H-1)(\theta+\xi)^{2H-2}\,\mathrm{d}\theta\,\mathrm{d}\xi\,\mathrm{d}\alpha \\ &= \frac{H}{\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{-\alpha}\big[(t+\xi)^{2H-1} - (\xi-\alpha)^{2H-1}\big]\mathrm{d}\xi\,\mathrm{d}\alpha \\ &\leq \frac{H}{\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{-\alpha}(t+\alpha)^{2H-1}\mathrm{d}\xi\,\mathrm{d}\alpha \\ &\leq \frac{H}{\varepsilon}\int_{-\varepsilon}^{0}\int_{0}^{-\alpha} t^{2H-1}\mathrm{d}\xi\,\mathrm{d}\alpha = \frac{1}{2}Ht^{2H-1}\varepsilon.\end{aligned} \tag{2.22}$$

Now we address the case where $\varepsilon > t$. Here we need to break the integral in (2.19) into four sub-integrals:

$$\int_0^{+\varepsilon}\int_0^t \cdots + \int_{-t}^0\int_0^{-\alpha} \cdots + \int_{-t}^0\int_{-\alpha}^t \cdots + \int_{-\varepsilon}^{-t}\int_0^t \cdots$$

Let's call the terms as A', B', C', D', respectively.

One can check easily that the same procedures used for bounding A and C work for A' and C'. For B' and D' we have

$$|B'| \le \frac{1}{2\varepsilon}\int_{-t}^0\int_0^{-\alpha} 2H\big[\theta^{2H-1} + (-\theta-\alpha)^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha\,,$$

and

$$\begin{aligned}|D'| &\le \frac{1}{2\varepsilon}\int_{-\varepsilon}^{-t}\int_0^t 2H\big[\theta^{2H-1} + (-\theta-\alpha)^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha \\ &\le \frac{1}{2\varepsilon}\int_{-\varepsilon}^{-t}\int_0^{-\alpha} 2H\big[\theta^{2H-1} + (-\theta-\alpha)^{2H-1}\big]\mathrm{d}\theta\,\mathrm{d}\alpha\,.\end{aligned}$$

Hence

$$|B'| + |D'| \le |B|$$

So in brief the same bounds found above for $|\mathfrak{S}_3 - t^{2H}|$ for the case $\varepsilon \le t$ remain valid for the case $\varepsilon > t$ too. So inequality (2.8) is proved.

Now we turn back to the proof of proposition 2.3.1. we have:

$$\begin{aligned}&\mathbb{E}\Big|\int_0^t \dot{B}_\varepsilon\big(s, X(s)\big)\mathrm{d}s - \int_0^t B\big(\mathrm{d}s, X(s)\big)\Big|^2 \\ &\qquad \le \mathbb{E}\Big\{\Big(\sum_{i=0}^N\Big|B(t_{i+1}) - B(t_i) - \int_{t_i}^{t_{i+1}} \dot{B}_\varepsilon(\theta)\mathrm{d}\theta\Big|\Big)^2\Big\} \\ &\qquad \le C_1(N+1)^2\varepsilon^{\min\{2H,1\}} \le C_2 N^2 \varepsilon^{\min\{2H,1\}}\,.\end{aligned}$$

□

2.4 Convergence of u_ε

In this section, using simple random walk properties we prove that $\tilde{u}_\varepsilon$ and its Malliavin derivative both converge to zero in $\mathscr{L}^2$.

Proposition 2.4.1. *$\tilde{u}_\varepsilon := u_\varepsilon - u$ converges to* 0 *in* $\mathbb{D}^{1,2}$ *uniformly in* $[0,T]$*, i.e.*

$$\sup_{s\in[0,T]} \mathbb{E}\big[|\tilde{u}_\varepsilon(s,x)|^2 + \|\nabla\tilde{u}_\varepsilon(s,x)\|^2_{\mathscr{H}}\big] \longrightarrow 0 \qquad as \quad \epsilon \downarrow 0\,.$$

Let $X : [0,T] \to \mathbb{Z}^d$ be a piecewise constant function on the lattice $\mathbb{Z}^d$ with jump times

$t_1 < t_2 < \cdots < t_N$. Let also $t_0 := 0$ and $t_{N+1} := T$. For any given $\delta > 0$ we may chop up $[0, T]$ into calm periods and rough ones. A calm period is defined as an interval in which all the consecutive jumps are at least δ apart, and a rough period as one in which all the consecutive jumps are at most δ apart. We additionally require that these intervals begin with a jump and end with another.

We also define R as the number of jumps in $[0, T]$ that are within δ distance of their previous one. In other words, R is defined to be the cardinality of $\{i \mid t_i - t_{i-1} < \delta, t_i \leq T\}$

Lemma 2.4.2. *Consider a Poisson process with intensity λ and let $R(=R_T)$ be defined for any sample path of the Poisson process as above. Then for any given $\delta > 0$, we have*

$$\mathrm{P}(R \geq n) \leq (C\delta)^n,$$

where C is a constant that depends only on T and λ.

Proof. Let A be the event of having at least one jump in $[0, t]$ which is within δ of a previous one and B be the event of having at least one jump in $[0, \delta]$. Let also $N(t)$ be the number of jumps in $[0, t]$ and $t_0 := 0$. We have

$$\begin{aligned}
\mathrm{P}(A \cup B) &\leq \sum_{k=1}^{\infty} \mathrm{P}(t_k - t_{k-1} < \delta \text{ and } t_{k-1} < t) \\
&= \sum_{k=1}^{\infty} \mathrm{P}(t_k - t_{k-1} < \delta \mid t_{k-1} < t)\,\mathrm{P}(t_{k-1} < t) \\
&= (1 - e^{-\lambda\delta}) \sum_{k=1}^{\infty} \mathrm{P}(t_{k-1} < t) \\
&= (1 - e^{-\lambda\delta}) \sum_{k=0}^{\infty} \mathrm{P}(N(t) \geq k) \\
&= (1 - e^{-\lambda\delta})\big(\mathbb{E}(N(t)) + 1\big).
\end{aligned}$$

Using the fact that the expectation of $N(t)$ is λt and noting the inequality $1 - e^{-\lambda\delta} \leq \lambda\delta$, we get $\mathrm{P}(A \cup B) \leq C_t\delta$, where $C_t = \lambda\delta(1 + t\lambda)$. In particular C_t is increasing in t.

Now we define σ_1 as the first jump time that is within δ of the previous one, i.e. $\sigma_1 := \inf\{t_k > 0\,;\ t_k - t_{k-1} < \delta\}$. Having defined σ_n we define σ_{n+1} as the first jump time after σ_n that is within δ of the previous one, i.e. $\sigma_{n+1} := \inf\{t_k > \sigma_n\,;\ t_k - t_{k-1} < \delta\}$. We have

$$\mathrm{P}(\sigma_{i+1} < T \mid \sigma_i) \leq \begin{cases} 0 & \text{if } \sigma_i \geq T \\ C_{T-\sigma_i} & \text{if } \sigma_i < T. \end{cases}$$

As C_t is an increasing function in t we have the following uniform bound:

$$\mathrm{P}(\sigma_{i+1} < T \mid \sigma_i) \leq (C_T\,\delta)\mathbf{1}_{\{\sigma_i < T\}}.$$

So

$$P(\sigma_{i+1} < T) = \mathbb{E}\big[P(\sigma_{i+1} < T \mid \sigma_i)\big] \le (C_T\delta)P(\sigma_i < T).$$

So by induction

$$P(\sigma_k < T) \le (C_T\delta)^k.$$

Now noticing that $R \ge n$ implies $\sigma_n < T$ we get

$$P(R \ge n) \le P(\sigma_n < T) \le (C_T\delta)^n.$$

□

Lemma 2.4.3. *For a Poisson process of intensity λ and for any given $\delta > 0$, let L be the total length of its rough periods in $[0,T]$ and K be the number of rough periods in $[0,T]$. Then there exists a constant C depending only on T and λ such that*

$$P(K \ge n) \le (C\delta)^n$$

and

$$P(L \ge n\delta) \le (C\delta)^n$$

Proof. As $L < R\delta$ and $K \le R$, any of $L \ge n\delta$ or $K \ge n$ implies $R \ge n$. The result follows from the previous lemma. □

Now we are ready to prove the following lemma.

Lemma 2.4.4. *For any $p \ge 1$, there exists $M > 0$ such that $\mathbb{E}|u_\varepsilon(t,x)|^p$ is bounded uniformly in $(\varepsilon,t,x) \in (0,M] \times [0,T] \times \mathbb{Z}^d$. $\mathbb{E}|u(t,x)|^p$ is also bounded uniformly in $(t,x) \in [0,T] \times \mathbb{Z}^d$.*

Proof. First consider $\mathbb{E}|u(t,x)|^p$.

$$\begin{aligned}\mathbb{E}|u(t,x)|^p &\le \|u_o\|_\infty^p \mathbb{E}^x\, \mathbb{E}\exp\Big[p\int_0^t B\big(\mathrm{d}s, X(t-s)\big)\Big]\\ &= \|u_o\|_\infty^p \mathbb{E}^x \exp\Big(\frac{p^2}{2}\operatorname{var}\Big[\int_0^t B\big(\mathrm{d}s, X(t-s)\big)\Big]\Big).\end{aligned}$$

So it is enough to find a uniform bound on $\operatorname{var}\big[\int_0^t B\big(\mathrm{d}s, X(t-s)\big)\big]$. For any sample path $X(\cdot)$ of simple random walk on $\mathbb{Z}^d$ let $t_1 < t_2 < \cdots < t_N$ be the jump times of the reversed path $X(t-\cdot)$ and $x_1, x_2, \ldots, x_{N+1}$ be its values. Let also $t_0 := 0$ and $t_{N+1} := t$. We have

$$\begin{aligned}\operatorname{var}\Big[\int_0^t B\big(\mathrm{d}s, X(t-s)\big)\Big] &= \operatorname{var}\Big[\sum_{i=1}^{N+1}\int_{t_{i-1}}^{t_i} B\big(\mathrm{d}s, x_i\big)\Big]\\ &= \operatorname{var}\Big[\sum_{i=1}^{N+1} B(t_i,x_i) - B(t_{i-1},x_i)\Big].\end{aligned}$$

For $H \geq \frac{1}{2}$ we have

$$\begin{aligned}
&\operatorname{var}\Big[\sum_{i=1}^{N+1} B(t_i,x_i) - B(t_{i-1},x_i)\Big] \\
&\leq (N+1)\sum_{i=1}^{N+1} \operatorname{var}\big[B(t_i,x_i) - B(t_{i-1},x_i)\big] \\
&= (N+1)\sum_{i=1}^{N+1} (t_i - t_{i-1})^{2H} \leq (N+1)\,t^{2H}.
\end{aligned}$$

As N is a Poisson random variable, $\mathbb{E}\exp(CN)$ is finite for any constant C.

For $H \leq \frac{1}{2}$ we use the well-known property that disjoint increments of a fractional Brownian motion with Hurst parameter less than half are negatively correlated. So we have

$$\begin{aligned}
\operatorname{var}\Big[\sum_{i=1}^{N+1} B(t_i,x_i) - B(t_{i-1},x_i)\Big] &\leq \sum_{i=1}^{N+1} \operatorname{var}\big[B(t_i,x_i) - B(t_{i-1},x_i)\big] \\
&= \sum_{i=1}^{N+1} (t_i - t_{i-1})^{2H} \leq (N+1)^{1-2H}\,t^{2H}.
\end{aligned}$$

In the last inequality we have used the fact that for $H \leq \frac{1}{2}$, the expression $x_1^{2H} + x_2^{2H} + \cdots + x_m^{2H}$ achieves its maximum when all x_i's are equal and the maximum is hence $m^{1-2H}(\sum_i x_i)^{2H}$. Again as N is Poisson, $\mathbb{E}\exp(CN^\alpha)$ is finite for any constants C and $\alpha \leq 1$.

Now let us consider $\mathbb{E}|u_\varepsilon(t,x)|^p$

$$\begin{aligned}
\mathbb{E}|u_\varepsilon(t,x)|^p &\leq \|u_o\|_\infty^p \mathbb{E}^x\,\mathbb{E}\exp\Big[p\int_0^t \dot{B}_\varepsilon\big(s, X(t-s)\big)\mathrm{d}s\Big] \\
&= \|u_o\|_\infty^p \mathbb{E}^x \exp\Big(\frac{p^2}{2}\operatorname{var}\Big[\int_0^t \dot{B}_\varepsilon\big(s, X(t-s)\big)\mathrm{d}s\Big]\Big)
\end{aligned} \tag{2.23}$$

Again we need to distinguish between H larger and less than half.

When H is larger than a half, $\operatorname{var}\Big(\int_{t_1}^{t_2} \dot{B}_\varepsilon(s)\mathrm{d}s\Big)$ being equal to $\mathfrak{S}_2$ introduced in section 2.3, is bounded by $(t_2-t_1)^{2H} + 2(t_2-t_1)(2H+1)\varepsilon^{2H-1}$ by inequality (2.18). With the above notation

$$\begin{aligned}
\operatorname{var}\Big[\int_0^t \dot{B}_\varepsilon\big(s, X(t-s)\big)\mathrm{d}s\Big] &= \operatorname{var}\Big[\sum_{i=1}^{N+1}\int_{t_{i-1}}^{t_i} \dot{B}_\varepsilon(s,x_i)\mathrm{d}s\Big] \\
&\leq (N+1)\sum_{i=1}^{N+1}\operatorname{var}\Big(\int_{t_{i-1}}^{t_i} \dot{B}_\varepsilon(s,x_i)\mathrm{d}s\Big) \\
&\leq (N+1)\sum_{i=1}^{N+1}\Big((t_{i+1}-t_i)^{2H} + 2(t_{i+1}-t_i)(2H+1)\varepsilon^{2H-1}\Big) \\
&\leq (N+1)\Big(t^{2H} + 2(2H+1)\varepsilon^{2H-1}\,t\Big).
\end{aligned}$$

Again we get a multiple of N and hence a finite bound.

When $H \leq \frac{1}{2}$, the situation is more complicated. Let $\{t_i\}_{i=1}^N$ be the increasingly ordered jump times of $\{X(t-s)\,;\, s \in [0,t]\}$ with additional convention of $t_0 := 0$ and $t_{N+1} := t$. We decompose $[0,t]$ into calm and rough periods of $X(t-\cdot)$ with respect to $\delta = 2\varepsilon$. Let increasingly enumerate the set of indices $\{i\ ;\ t_i - t_{i-1} \geq \delta\}$ as $\{t_{i_k}\}_k$. In other words, we single out and enumerate those time intervals $[t_i - 1, t_i]$ whose length is larger than or equal to $\delta = 2\varepsilon$. It is evident that such intervals constitute the calm periods. Let also $\{Y_k\}_k$ be the integral of $\dot{W}_\varepsilon(\cdot, x_{i_k})$ over the time interval $[t_{i_k-1}, t_{i_k}]$, i.e. $Y_k := \int_{t_{i_k-1}}^{t_{i_k}} \dot{W}_\varepsilon(s, x_{i_k})\mathrm{d}s$. Let also Z be the sum of the integrals over all rough periods. Using equation (2.23), Cauchy-Schwartz and the simple inequality $\mathbb{E}(X+Y)^2 \leq 2\mathbb{E}X^2 + 2\mathbb{E}Y^2$, we have

$$\begin{aligned}
\mathbb{E}|u_\varepsilon(t,x)|^p &\leq \|u_o\|_\infty^p \mathbb{E}^x \exp\Big(\frac{p^2}{2}\mathbb{E}(Z + \sum_k Y_k)^2\Big) \\
&\leq \|u_o\|_\infty^p \big[\mathbb{E}^x \exp\big(2p^2\,\mathbb{E}(Z^2)\big)\big]^{1/2} \big[\mathbb{E}^x \exp\big(2p^2\,\mathbb{E}(\sum_k Y_k)^2\big)\big]^{1/2}.
\end{aligned}$$

Once again we will use the negativeness of the covariance of disjoint increments of a fractional Brownian motion with Hurst parameter less than half.

First we consider the integral over the rough periods, i.e. the first term above. Let I be the union of all the rough intervals in $[0,t]$.

We notice that for $\alpha, \beta \in [0,t]$, and a fractional Brownian motion $B(\cdot)$ of Hurst parameter $H \leq 1/2$ we have

$$\mathbb{E}\dot{B}_\varepsilon(\alpha)\dot{B}_\varepsilon(\beta) \leq 0 \qquad \text{for} \quad |\alpha - \beta| \geq 2\varepsilon\,,$$

which is nothing but the negative correlation of non-overlapping increments of a fBM, and

$$|\mathbb{E}\dot{B}_\varepsilon(\alpha)\dot{B}_\varepsilon(\beta)| \leq \frac{4(4\varepsilon)^{2H}}{(2\varepsilon)^2} \qquad \text{for} \quad |\alpha - \beta| < 2\varepsilon\,,$$

which is easily followed by a simple calculation.

This shows that for $\alpha, \beta \in [0,t]$, there are only two possibilities: either $\dot{B}_\varepsilon\big(\alpha, X(t-\alpha)\big)$ and $\dot{B}_\varepsilon\big(\beta, X(t-\beta)\big)$ have negative correlation or they are uncorrelated, depending on whether

$X(t-\alpha)$ is the same as $X(t-\beta)$ or not. So we have

$$\begin{aligned}
\mathbb{E}(Z^2) &= \mathbb{E}\Big[\int_I \dot{B}_\varepsilon\big(\alpha, X(t-\alpha)\big)\mathrm{d}\alpha \int_I \dot{B}_\varepsilon\big(\beta, X(t-\beta)\big)\mathrm{d}\beta\Big] \\
&= \int_{\alpha\in I}\int_{\beta\in I} \mathbb{E}\big[\dot{B}_\varepsilon\big(\alpha, X(t-\alpha)\big)\dot{B}_\varepsilon\big(\beta, X(t-\beta)\big)\big]\mathrm{d}\beta\mathrm{d}\alpha \\
&\le \int_{\alpha\in I}\int_{\beta\in I} \mathbb{E}\big[\dot{B}_\varepsilon\big(\alpha, X(t-\alpha)\big)\dot{B}_\varepsilon\big(\beta, X(t-\beta)\big)\big]\mathbf{1}_{|\alpha-\beta|<2\varepsilon}\mathrm{d}\beta\mathrm{d}\alpha \\
&\le \int_{\alpha\in I}\int_{\beta\in I} |\mathbb{E}\big(\dot{B}_\varepsilon(\alpha)\dot{B}_\varepsilon(\beta)\big)|\mathbf{1}_{|\alpha-\beta|<2\varepsilon}\mathrm{d}\beta\mathrm{d}\alpha \\
&\le \int_{\alpha\in I}\int_{\beta\in I} \frac{2\varepsilon^{2H}}{\varepsilon^2}\mathbf{1}_{|\alpha-\beta|<2\varepsilon}\mathrm{d}\beta\mathrm{d}\alpha \\
&= \frac{2\varepsilon^{2H}}{\varepsilon^2}\int_{\alpha\in I}(4\varepsilon)\mathrm{d}\alpha \le 8\varepsilon^{2H-1}L,
\end{aligned}$$

where L is the total length of rough periods, i.e. the length of I.
So

$$\mathbb{E}^x \exp\big(2p^2\,\mathbb{E}(Z^2)\big) \le \mathbb{E}^x \exp\big(16p^2\varepsilon^{2H}L/\varepsilon\big).$$

As L/ε has exponential tail by lemma 2.4.3, the above expectation is finite for ε small enough.

For the second term, $\mathbb{E}(\sum_k Y_k)^2$, observe that the length of each time interval $[t_{i_k-1}, t_{i_k}]$ is larger than 2ε which means the distance of every two non-neighboring such intervals is at least 2ε. But this means that only consecutive Y_k's can be positively correlated because for any two intervals I_1 and I_2 that are at least 2ε apart, the integrals $\int_{I_1}\dot{B}_\varepsilon(s)\mathrm{d}s$ and $\int_{I_2}\dot{B}_\varepsilon(s)\mathrm{d}s$ are negatively correlated which in turn is a consequence of the negative correlation of disjoint intervals of a fractional Brownian motion with $H \le \frac{1}{2}$. So

$$\begin{aligned}
\mathbb{E}\big[(\sum_k Y_k)^2\big] &\le \mathbb{E}(Y_1^2) + 2\mathbb{E}(Y_1Y_2) + \mathbb{E}(Y_2^2) + 2\mathbb{E}(Y_2Y_3) + \mathbb{E}(Y_3^2) + \dots \\
&\quad + 2\mathbb{E}(Y_{n-1}Y_n) + \mathbb{E}(Y_m^2) \\
&\le 2\mathbb{E}(Y_1^2) + 3\mathbb{E}(Y_2^2) + 3\mathbb{E}(Y_3^2) + \dots + 3\mathbb{E}(Y_{n-1}^2) + 2\mathbb{E}(Y_m^2) \\
&\le 3\sum_k \mathbb{E}(Y_k^2).
\end{aligned}$$

In the first inequality we have used the fact that for non-consecutive Y_i and Y_j, their covariance $\mathbb{E}(Y_iY_j)$ is negative and in the last inequality we have used $2\mathbb{E}(XY) \le \mathbb{E}(X^2) + \mathbb{E}(Y^2)$. Using equation (2.11) we have

$$\mathrm{var}\left[\int_{t_i}^{t_{i+1}} \dot{B}_\varepsilon(s)\mathrm{d}s\right] \le (t_{i+1}-t_i)^{2H} + 4(2\varepsilon)^{2H}.$$

So noting $m \le N$, where N denotes the number of jumps in $[0,t]$ and using the fact that $x_1^{2H} + x_2^{2H} + \cdots + x_m^{2H}$ is bounded by $m^{1-2H}(\sum_i x_i)^{2H}$ for $H \le \frac{1}{2}$ which is a consequence of concavity of $(\cdot)^{2H}$, we get

$$\begin{aligned}
\mathbb{E}\big[(\sum_k Y_k)^2\big] &\le 3\sum_{k=1}^{m}[(t_{i_k}-t_{i_k-1})^{2H}+4(2\varepsilon)^{2H}] \\
&\le 3m^{1-2H}[\sum_{k=1}^{m}(t_{i_k}-t_{i_k-1})]^{2H}+12m(2\varepsilon)^{2H} \\
&\le 3(N+1)^{1-2H}t^{2H}+12(N+1)(2\varepsilon)^{2H}.
\end{aligned}$$

□

PROOF OF PROPOSITION 2.4.1. We give the same argument used in [21].
Since u_o is bounded, for simplicity and without any loss of generality we drop it from now on.

Let $X(\cdot)$ be an arbitrary but fixed sample path of the simple random walk on $\mathbb{Z}^d$ started at x, following [21] we define:

$$g^{\varepsilon}_{s,x}(r,z) := \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\,\delta_x(z) \tag{2.24}$$

$$g^{X}_{s,x}(r,z) := \mathbf{1}_{[0,s]}(r)\,\delta_{X(s-r)}(z) \tag{2.25}$$

$$g^{\varepsilon,X}_{s,x}(r,z) := \int_0^s \frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)\,\delta_{X(s-\theta)}(z)\,\mathrm{d}\theta \tag{2.26}$$

It can be easily shown that $g^{\varepsilon}_{s,x}(r,z)$, $g^{X}_{s,x}$ and $g^{\varepsilon,X}_{s,x}$ are all in the Hilbert space $\mathscr{H}$ introduced in chapter **??**, and moreover

$$\mathbf{B}(g^{\varepsilon}_{s,x}) = \dot{B}_{\varepsilon}(s,x)$$

$$\mathbf{B}(g^{X}_{s,x}) = \int_0^s B\big(\mathrm{d}\theta, X(s-\theta)\big)$$

and

$$\mathbf{B}(g^{\varepsilon,X}_{s,x}) = \int_0^s \dot{B}_{\varepsilon}\big(\theta, X(s-\theta)\big)\mathrm{d}\theta.$$

For $p \ge 1$ arbitrary, using the inequalities $|e^a - e^b| \le (e^a + e^b)|a-b|$ and $(a+b)^n \le 2^{n-1}(a^n + b^n)$

and also Hölder's and Jensen's inequalities we get

$$
\begin{aligned}
&\mathbb{E}|u^\varepsilon(t,x)-u(t,x)|^p \\
&\quad = \mathbb{E}|\mathbb{E}^x\big(e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}-e^{\mathbf{B}(g^{X}_{t,x})}\big)|^p \\
&\quad \le \mathbb{E}^x\,\mathbb{E}|e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}-e^{\mathbf{B}(g^{X}_{t,x})}|^p \\
&\quad \le \mathbb{E}^x\Big(\mathbb{E}\big(e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}+e^{\mathbf{B}(g^{X}_{t,x})}\big)^{2p}\Big)^{1/2}\mathbb{E}^x\Big(\mathbb{E}|\mathbf{B}(g^{\varepsilon,X}_{t,x})-\mathbf{B}(g^{X}_{t,x})|^{2p}\Big)^{1/2} \\
&\quad \le C\Big(\mathbb{E}^x\,\mathbb{E}\big(e^{2p\mathbf{B}(g^{\varepsilon,X}_{t,x})}+e^{2p\mathbf{B}(g^{X}_{t,x})}\big)\Big)^{1/2}\mathbb{E}^x\,\mathbb{E}|\mathbf{B}(g^{\varepsilon,X}_{t,x})-\mathbf{B}(g^{X}_{t,x})|^2\,,
\end{aligned}
\tag{2.27}
$$

where in the second inequality we have used the fact that for Gaussian random variables all the n-norms are equivalent to 2-norm.
So by applying lemma 2.4.4 and proposition 2.3.1 we obtain

$$
\sup_{t\in[0,T]}\ \mathbb{E}|\tilde{u}_\varepsilon(t,x)|^2 \longrightarrow 0 \qquad \text{as} \quad \varepsilon\downarrow 0\,.
$$

For the convergence of $\nabla\widetilde{u}_\varepsilon$, we use the fact that for a separably-valued $\mathbb{D}^{1,2}$-valued random variable $f\in\mathscr{L}^1(\mathscr{X};\mathbb{D}^{1,2})$ with $\mathscr{X}$ a probability space independent of the underlying Gaussian space of $\mathbb{D}^{1,2}$, we have $\mathbb{E}\nabla f=\nabla\mathbb{E}f$ provided that $\mathbb{E}(\|f\|_{\mathbb{D}^{1,2}})<\infty$, where the expectations are taken with respect to $\mathscr{X}$. This follows from lemma 1.3.1.
So we have

$$
\begin{aligned}
\nabla u_\varepsilon(t,x) &= \mathbb{E}^x[g^{\varepsilon,X}_{t,x}\,e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}] \\
\nabla u(t,x) &= \mathbb{E}^x[g^{X}_{t,x}e^{\mathbf{B}(g^{X}_{t,x})}]\,.
\end{aligned}
$$

So

$$
\begin{aligned}
&\mathbb{E}\|\nabla u^\varepsilon(t,x)-\nabla u(t,x)\|^2_{\mathscr{H}} \\
&\quad = \mathbb{E}\|\mathbb{E}^x\big[g^{\varepsilon,X}_{t,x}\,e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}-g^{X}_{t,x}e^{\mathbf{B}(g^{X}_{t,x})}\big]\|^2_{\mathscr{H}} \\
&\quad \le 2\mathbb{E}\mathbb{E}^x\big[e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}\|g^{\varepsilon,X}_{t,x}-g^{X}_{t,x}\|^2_{\mathscr{H}}\big] \\
&\qquad + 2\mathbb{E}\mathbb{E}^x\big[|e^{\mathbf{B}(g^{\varepsilon,X}_{t,x})}-e^{\mathbf{B}(g^{X}_{t,x})}|^2\|g^{X}_{t,x}\|^2_{\mathscr{H}}\big]\,.
\end{aligned}
$$

If we apply the Schwartz inequality and note that $\|g^{\varepsilon,X}_{t,x}-g^{X}_{t,x}\|^2_{\mathscr{H}}=\mathbb{E}|\mathbf{B}(g^{\varepsilon,X}_{t,x})-\mathbf{B}(g^{X}_{t,x})|^2$, along with fact that for Gaussian random variables all norms are equivalent to the 2-norm, using equation (2.27), lemma 2.4.4 and proposition 2.3.1 we get

$$
\sup_{t\in[0,T]}\ \mathbb{E}\|\nabla\tilde{u}_\varepsilon(t,x)\|^2_{\mathscr{H}} \longrightarrow 0 \qquad \text{as} \quad \epsilon\downarrow 0\,.
$$

□

2.5 Convergence of $V_{1,\varepsilon}$

For $V_{1,\varepsilon}$ we use basically the same proof as in [21]. As one can easily show that

$$\int_0^t \|\tilde{u}_\varepsilon(s,x) g^\varepsilon_{s,x}\|_{\mathbb{D}^{1,2}(\mathscr{H})} \mathrm{d}s < \infty,$$

where $\mathbb{D}^{1,2}(\mathscr{H})$ denotes the Sobolev space of $\mathscr{H}$-valued $\mathscr{L}^2$ random variables with $\mathscr{L}^2$ Malliavin derivatives, we can apply lemma 1.3.1 to get:

$$V_{1,\varepsilon} = \boldsymbol{\delta}(\psi_\varepsilon),$$

where

$$\psi_\varepsilon := \int_0^t \tilde{u}_\varepsilon(s,x) g^\varepsilon_{s,x} \mathrm{d}s.$$

So using inequality (1.5), we have

$$\mathbb{E}\big(|V_{1,\varepsilon}|^2\big) = \mathbb{E}\big(\boldsymbol{\delta}(\psi_\varepsilon)^2\big) \le \mathbb{E}\big(\|\psi_\varepsilon\|^2_{\mathscr{H}}\big) + \mathbb{E}\big(\|\nabla\psi_\varepsilon\|^2_{\mathscr{H}\otimes\mathscr{H}}\big).$$

For the first right hand side term we have

$$\begin{aligned}
&\mathbb{E}\big(\|\psi_\varepsilon\|^2_{\mathscr{H}}\big)\\
&\quad= \int_0^t \int_0^t \mathbb{E}\big(\tilde{u}_\varepsilon(s_1,x)\tilde{u}_\varepsilon(s_2,x)\big)\langle g^\varepsilon_{s_1,x}, g^\varepsilon_{s_2,x}\rangle \mathrm{d}s_1 \mathrm{d}s_2\\
&\quad\le M_1 \int_0^t \int_0^t |\mathbb{E}\big(\dot{B}_\varepsilon(s_1,x)\dot{B}_\varepsilon(s_2,x)\big)| \mathrm{d}s_1 \mathrm{d}s_2,
\end{aligned}$$

where $M_1 = \sup_{s\in[0,t]} \mathbb{E}|\tilde{u}_\varepsilon(s,x)|^2$. Here taking the integration out of the inner product is justified by once more using lemma 1.3.1.
$\int_0^t \int_0^t |\mathbb{E}\big(\dot{B}_\varepsilon(s_1,x)\dot{B}_\varepsilon(s_2,x)\big)| \mathrm{d}s_1 \mathrm{d}s_2$ being the same as the term $\mathfrak{S}_2$ in equation (2.9), is uniformly upper-bounded using equations (2.11) and (2.12). On the other hand, M_1 goes to zero as $\varepsilon \downarrow 0$. So it follows that $\mathbb{E}\big(\|\psi_\varepsilon\|^2_{\mathscr{H}}\big)$ converges to zero.

For the second term, applying lemma 1.3.1 to the derivative operator and inner product we get

$$
\begin{aligned}
&\mathbb{E}\big(\|\nabla\psi_\varepsilon\|^2_{\mathcal{H}\otimes\mathcal{H}}\big)\\
&=\mathbb{E}\langle\nabla\int_0^t \tilde{u}_\varepsilon(s_1,x)g^\varepsilon_{s_1,x}\mathrm{d}s_1,\nabla\int_0^t \tilde{u}_\varepsilon(s_2,x)g^\varepsilon_{s_2,x}\mathrm{d}s_2\rangle\\
&=\mathbb{E}\langle\int_0^t \nabla\big(\tilde{u}_\varepsilon(s_1,x)\big)\otimes g^\varepsilon_{s_1,x}\mathrm{d}s_1,\int_0^t \nabla\big(\tilde{u}_\varepsilon(s_2,x)\big)\otimes g^\varepsilon_{s_2,x}\mathrm{d}s_2\rangle\\
&=\mathbb{E}\int_0^t\int_0^t \langle\nabla\big(\tilde{u}_\varepsilon(s_1,x)\big)\otimes g^\varepsilon_{s_1,x},\nabla\big(\tilde{u}_\varepsilon(s_2,x)\big)\otimes g^\varepsilon_{s_2,x}\rangle\mathrm{d}s_1\mathrm{d}s_2\\
&=\int_0^t\int_0^t \mathbb{E}\langle\nabla\big(\tilde{u}_\varepsilon(s_1,x)\big),\nabla\big(\tilde{u}_\varepsilon(s_2,x)\big)\rangle\langle g^\varepsilon_{s_1,x},g^\varepsilon_{s_2,x}\rangle\mathrm{d}s_1\mathrm{d}s_2\\
&\le M_2\int_0^t\int_0^t |\langle g^\varepsilon_{s_1,x},g^\varepsilon_{s_2,x}\rangle|\mathrm{d}s_1\mathrm{d}s_2\\
&=M_2\int_0^t\int_0^t |\mathbb{E}\big(\dot{B}_\varepsilon(s_1,x)\dot{B}_\varepsilon(s_2,x)\big)|,
\end{aligned}
$$

where $M_2=\sup_{s\in[0,t]}\mathbb{E}\|\nabla\tilde{u}_\varepsilon(s,x)\|^2_{\mathcal{H}}$.
The same argument given for the first term above shows that $\mathbb{E}\big(\|\nabla\psi_\varepsilon\|^2_{\mathcal{H}\otimes\mathcal{H}}\big)$ also converges to zero as ε goes down to zero.

2.6 Convergence of $V_{2,\varepsilon}$

Establishing the convergence of $V_{2,\varepsilon}$ is more involved. First applying lemma 1.3.1 to u and u_ε for the derivative operator we get

$$\nabla u_\varepsilon(s,x)=\mathbb{E}^x[u_o(X(s))\,e^{\mathbf{B}(g^{\varepsilon,X}_{s,x})}g^{\varepsilon,X}_{s,x}]$$

and

$$\nabla u_(s,x)=\mathbb{E}^x[u_o(X(s))\,e^{\mathbf{B}(g^{X}_{s,x})}g^{X}_{s,x}]\,.$$

Let

$$A^X(s,x):=u_o(X(t))\,e^{\mathbf{B}(g^X_{s,x})}$$

and

$$A^{\varepsilon,X}(s,x):=u_o(X(s))\,e^{\mathbf{B}(g^{\varepsilon,X}_{s,x})}\,.$$

Hence we have

$$
\begin{aligned}
V_{2,\varepsilon}&=\int_0^t\langle\nabla u_\varepsilon(s,x)-\nabla u(s,x),g^\varepsilon_{s,x}\rangle\mathrm{d}s\\
&=\int_0^t\mathbb{E}^x\big[\langle A^X(s,x)g^X_{s,x}-A^{\varepsilon,X}(s,x)g^{\varepsilon,X}_{s,x}\ ,\ g^\varepsilon_{s,x}\rangle\big]\mathrm{d}s\\
&=\int_0^t\mathbb{E}^x\big[\langle(A^X-A^{\varepsilon,X})g^{\varepsilon,X},g^\varepsilon\rangle+\langle A^X(g^X-g^{\varepsilon,X}),g^\varepsilon\rangle\big]\mathrm{d}s\\
&=\int_0^t\mathbb{E}^x[(A^X-A^{\varepsilon,X})\langle g^{\varepsilon,X},g^\varepsilon\rangle]+\int_0^t\mathbb{E}^x[A^X\langle g^X-g^{\varepsilon,X},g^\varepsilon\rangle]\mathrm{d}s\,.
\end{aligned}
$$

Let

$$\mathrm{P}_{1,\varepsilon} := \int_0^t \mathbb{E}^x[(A^X - A^{\varepsilon,X})\langle g^{\varepsilon,X}, g^\varepsilon\rangle]\mathrm{d}s$$

and

$$\mathrm{P}_{2,\varepsilon} := \int_0^t \mathbb{E}^x[A^X\langle g^X - g^{\varepsilon,X}, g^\varepsilon\rangle]\mathrm{d}s.$$

So we will show in two steps that each of these terms converge to zero in $\mathscr{L}^2$.
Step I: Convergence of $\mathrm{P}_{1,\varepsilon}$. For the first term, using Hölder inequality for $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\mathbb{E}^x|(A^X - A^{\varepsilon,X})\langle g^{\varepsilon,X}, g^\varepsilon\rangle| \le \big(\mathbb{E}^x|A^X - A^{\varepsilon,X}|^q\big)^{1/q}\big(\mathbb{E}^x|\langle g^{\varepsilon,X}, g^\varepsilon\rangle|^p\big)^{1/p}.$$

In fact equation (2.27) also proves that for any $p \ge 1$

$$\sup_{s\in[0,t]} \mathbb{E}\mathbb{E}^x|A^X(s,x) - A^{\varepsilon,X}(s,x)|^p \longrightarrow 0 \qquad \text{as} \quad \varepsilon \downarrow 0.$$

So if we can show that $\mathbb{E}^x|\langle g^{\varepsilon,X}, g^\varepsilon\rangle|^p$ is bounded by some constant which depends only on H and t we are done because then

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^t \mathbb{E}^x\big[(A^X - A^{\varepsilon,X})\langle g^{\varepsilon,X}, g^\varepsilon\rangle\big]\,\mathrm{d}s\Big)^2 \\
&\quad\le \mathbb{E}\Big(\int_0^t [\mathbb{E}^x|A^X - A^{\varepsilon,X}|^q]^{1/q}[\mathbb{E}^x|\langle g^{\varepsilon,X}, g^\varepsilon\rangle|^p]^{1/p}\,\mathrm{d}s\Big)^2 \\
&\quad\preccurlyeq \int_0^t \mathbb{E}\,[\mathbb{E}^x|A^X - A^{\varepsilon,X}|^q]^{2/q}\,\mathrm{d}s,
\end{aligned}$$

where $\preccurlyeq$ means *less than up to a positive constant.*
So either $q>2$, where we get $\int_0^t[\mathbb{E}\mathbb{E}^x|A^X - A^{\varepsilon,X}|^q]^{2/q}\,\mathrm{d}s$ as an upper bound or $q \le 2$, where we get the upper bound $\int_0^t \mathbb{E}\mathbb{E}^x|A^X - A^{\varepsilon,X}|^2\,\mathrm{d}s$.
Let $\{t_i\}_{i=1}^n$ be the jump times of the path $X(\cdot)$ up to time s, $t_0 := 0$ and $t_n := s$. Let then J be the set of indices j for which $X(\cdot)$ stays at site x in the time interval $[t_j, t_{j+1}]$. Now applying the definitions (2.24)-(2.26) we get

$$\begin{aligned}
\langle g^{\varepsilon,X}, g^\varepsilon\rangle &= \langle \sum_{i\in J}\int_{s-t_{i+1}}^{s-t_i} \frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}\mathrm{d}\theta\,,\ \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\rangle \\
&= \frac{1}{4\varepsilon^2}\sum_{i\in J}\int_{s-t_{i+1}}^{s-t_i}\langle \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}\,,\ \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\rangle \mathrm{d}\theta \\
&= \frac{1}{4\varepsilon^2}\sum_{i\in J}\int_{s-t_{i+1}}^{s-t_i}\mathbb{E}[(B_{\theta+\varepsilon}-B_{\theta-\varepsilon})(B_{s+\varepsilon}-B_{s-\varepsilon})]\mathrm{d}\theta \\
&= \frac{1}{8\varepsilon^2}\sum_{i\in J}\int_{t_i}^{t_{i+1}}\big[(\gamma+2\varepsilon)^{2H} + |\gamma-2\varepsilon|^{2H} - 2\gamma^{2H}\big]\mathrm{d}\gamma.
\end{aligned}$$

We split this expression into two terms

$$\Gamma_1 := \frac{1}{8\varepsilon^2}\int_0^{t_1}\left[(\gamma+2\varepsilon)^{2H}+|\gamma-2\varepsilon|^{2H}-2\gamma^{2H}\right]\mathrm{d}\gamma \tag{2.28}$$

and

$$\Gamma_2 := \frac{1}{8\varepsilon^2}\sum_{i\in J, i\geq 2}\int_{t_i}^{t_{i+1}}\left[(\gamma+2\varepsilon)^{2H}+|\gamma-2\varepsilon|^{2H}-2\gamma^{2H}\right]\mathrm{d}\gamma\,.$$

For the first term, using the same reasoning as in (2.13) and (2.14), we have

$$\Gamma_1 = \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1} f''(t_1+\xi\varepsilon+\eta\varepsilon)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \tag{2.29}$$

where $f(s) := \int_0^s |r|^{2H}\mathrm{d}r$ and hence $f''(r) = 2H\,\mathrm{sgn}(r)|r|^{2H-1}$.
Letting $\Delta := \xi\varepsilon+\eta\varepsilon$ and noting that t_1 is exponentially distributed, we have

$$\mathbb{E}^x|f''(t_1+\Delta)|^p \leq 2H\int_0^s |t_1+\Delta|^{(2H-1)p}\mathrm{d}t_1\,.$$

As we can restrict ourselves to $\varepsilon\leq 1$ and hence $|\Delta|\leq 1$ and as $0<s<t$, we have

$$\int_0^s |t_1+\Delta|^{(2H-1)p}\mathrm{d}t_1 \leq \int_{-1}^{t+1}|t_1|^{(2H-1)p}\mathrm{d}t_1\,.$$

So if we choose $p>1$ such that $(2H-1)p>-1$, we get a finite bound on $\mathbb{E}^x|f''(t_1+\Delta)|^p$ and hence a bound on $\mathbb{E}^x|\Gamma_1|^p$ that only depends on t and H.

Now for the second term, Γ_2, let

$$f^\varepsilon(\gamma) := \frac{1}{4\varepsilon^2}\left[(\gamma+2\varepsilon)^{2H}+|\gamma-2\varepsilon|^{2H}-2\gamma^{2H}\right]. \tag{2.30}$$

We have $|f^\varepsilon(\gamma)|\leq 18\gamma^{2H-2}$ because either $\gamma\leq 4\varepsilon$ which implies that $|\gamma-2\varepsilon|^{2H}\leq(2\varepsilon)^{2H}$ and $(\gamma+2\varepsilon)^{2H}\leq(6\varepsilon)^{2H}$ and hence $|f^\varepsilon|(\gamma)\leq 18\gamma^{2H-2}$ or $\gamma>4\varepsilon$ in which case we may write $f^\varepsilon(\gamma)$ as the following

$$f^\varepsilon(\gamma) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} 2H(2H-1)(\gamma+\xi\varepsilon+\eta\varepsilon)^{2H-2}\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{2.31}$$

Letting again $\Delta := \xi\varepsilon+\eta\varepsilon$, we have $|\Delta|\leq 2\varepsilon$ and so

$$(\gamma+\Delta)^{2H-2}\leq\gamma^{2H-2}(1+\Delta/\gamma)^{2H-2}\leq 2^{2-2H}\gamma^{2H-2}\,,$$

which gives $|f^\varepsilon(\gamma)|\leq 8\gamma^{2H-2}$.
So we have

$$\Gamma_2 \preccurlyeq \int_{t_1}^{s}|f^\varepsilon(\gamma)|\mathrm{d}\gamma \preccurlyeq \int_{t_1}^{s}\gamma^{2H-2}\,\mathrm{d}\gamma\,.$$

So Γ_2 is bounded (up to a constant) by either t_1^{2H-1} for $H<\frac{1}{2}$, or s^{2H-1} for $H>\frac{1}{2}$. The case

$H = \frac{1}{2}$ can also be treated easily using the inequality $ln(x) \preccurlyeq x^\alpha$ for any α positive. So as $(2H-1)p > -1$, $\mathbb{E}^x|\Gamma_2|^p$ can be bounded by a constant only dependant on t and H. So this competes the proof showing that $\mathbb{E}^x|\langle g^{\varepsilon,X}, g^\varepsilon\rangle|^p \le C$, for some $p > 1$ and C a constant only dependant on t and H.

Step II: Convergence of $\mathrm{P}_{2,\varepsilon}$. For establishing the convergence of $\mathrm{P}_{2,\varepsilon}$ we will use the dominated convergence theorem.

In 'step I' we showed that

$$\langle g^{\varepsilon,X}, g^\varepsilon\rangle = \frac{1}{2}\sum_{i\in J}\int_{t_i}^{t_{i+1}} f^\varepsilon(r)\,\mathrm{d}r\,,$$

where f^ε is defined in (2.30).

Now let $\{t_i\}_{i=0}^{n+1}$ and J be as in 'step I', i.e. $\{t_i\}_{i=1}^{n}$ be the jump times of the path $X(\cdot)$ up to time s, $t_0 := 0$ and $t_n := s$ and J the set of indices j for which $X(\cdot)$ stays at site x in the time interval $[t_j, t_{j+1}]$. So we have

$$\begin{aligned}
\langle g^X, g^\varepsilon\rangle &= \langle \mathbf{1}_{[0,s]}(r)\,\delta_{X(s-r)}(z)\,,\ \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)\,\delta_x(z)\rangle \\
&= \sum_{i\in J}\langle \mathbf{1}_{[s-t_{i+1}\,,\,s-t_i]}\,,\ \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon\,,\,s+\varepsilon]}\rangle \\
&= \sum_{i\in J}\frac{1}{4\varepsilon}\big[|t_{i+1}+\varepsilon|^{2H} - |t_i+\varepsilon|^{2H} + |t_i-\varepsilon|^{2H} - |t_{i+1}-\varepsilon|^{2H}\big] \\
&= \frac{1}{4\varepsilon}\big(|t_1+\varepsilon|^{2H} - |t_1-\varepsilon|^{2H}\big) + \frac{1}{2}\sum_{i\in J, i>1}\int_{t_i}^{t_{i+1}} h^\varepsilon(r)\,\mathrm{d}r\,,
\end{aligned} \tag{2.32}$$

where

$$h^\varepsilon(r) := \frac{2H}{2\varepsilon}\big[|r+\varepsilon|^{2H-1} - \mathrm{sgn}(r-\varepsilon)|r-\varepsilon|^{2H-1}\big]\,.$$

We will show that $\langle g^X, g^\varepsilon\rangle - \langle g^{\varepsilon,X}, g^\varepsilon\rangle$ converges to zero. For doing so we shall show that $[\frac{1}{4\varepsilon}\big(|t_1+\varepsilon|^{2H} - |t_1-\varepsilon|^{2H}\big) - \frac{1}{2}\int_0^{t_1} f^\varepsilon(r)\,\mathrm{d}r]$ converges to zero and that every $\int_{t_i}^{t_{i+1}}(h^\varepsilon - f^\varepsilon)(r)\,\mathrm{d}r$ also converges to zero.

By equations (2.28) and (2.29), we have

$$\int_0^{t_1} f^\varepsilon(r)\,\mathrm{d}r = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} 2H\,\mathrm{sgn}(r+\xi\varepsilon+\eta\varepsilon)|r+\xi\varepsilon+\eta\varepsilon|^{2H-1}\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

So for a fixed positive t_1 this converges to $2Ht_1^{2H-1}$. On the other hand $\frac{1}{4\varepsilon}\big(|t_1+\varepsilon|^{2H} - |t_1-\varepsilon|^{2H}\big)$ also converges to $\frac{1}{2}2Ht_1^{2H-1}$.

For $\int_{t_i}^{t_{i+1}}(h^\varepsilon - f^\varepsilon)(r)\,\mathrm{d}r$, we will show that $h^\varepsilon - f^\varepsilon$ converges to zero and then apply the dominated convergence to the integral.

Using (2.31) it can be easily shown that

$$\lim_{\varepsilon\downarrow 0} f^{\varepsilon}(r) = 2H(2H-1)r^{2H-2}.$$

By simply recognizing the definition of derivative we have

$$\lim_{\varepsilon\downarrow 0} h^{\varepsilon}(r) = 2H(2H-1)r^{2H-2}.$$

So it remains to find an integrable ε-independent upper bound. As shown in the paragraph following (2.30), $f^{\varepsilon}(r)$ is bounded by $18\gamma^{2H-2}$ and for $h^{\varepsilon}(r)$, restricting ε to be less than $\frac{t_{i_1}}{2}$, where i_1 is the first index in J after 1, we have for all $r \geq t_{i_1}$

$$h^{\varepsilon}(r) = \frac{1}{2}2H(2H-1)\int_{-1}^{1} |r+u\varepsilon|^{2H-2}\,\mathrm{d}u. \tag{2.33}$$

But then as $|r+u\varepsilon|^{2H-2} \leq (\frac{r}{2})^{2H-2}$ it gives $8r^{2H-2}$ as an upper bound on h^{ε}. This completes the proof for convergence to zero of $\langle g^X, g^{\varepsilon}\rangle - \langle g^{\varepsilon,X}, g^{\varepsilon}\rangle$.

Now, for applying the dominated convergence theorem to $\mathrm{P}_{2,\varepsilon}$ we only need to find an ε-independent upper bound G on $\langle g^X, g^{\varepsilon}\rangle - \langle g^{\varepsilon,X}, g^{\varepsilon}\rangle$ having the property that $\mathbb{E}\big(\int_0^t \mathbb{E}^x(G)\big)^2 < \infty$. For $\langle g^{\varepsilon,X}\rangle - \langle g^{\varepsilon,X}, g^{\varepsilon}\rangle$ such an upper bound has been established in step I above. It remains to find an upper bound on $\langle g^X, g^{\varepsilon}\rangle$.

For $2H-1 \geq 0$ the situation is quite trivial because using equation (2.32) we easily get

$$\langle g^X, g^{\varepsilon}\rangle = \frac{1}{2}\sum_{i\in J}\int_{t_i}^{t_{i+1}} h^{\varepsilon}(r)\,\mathrm{d}r.$$

When $2H-1 \geq 0$, equation (2.33) remains valid for any value of ε and r. As for any $\varepsilon \leq 1$ we have

$$\int_{-1}^{1} |r+u\varepsilon|^{2H-2}\,\mathrm{d}u \leq \int_{-1}^{t+1} |u|^{2H-2}\,\mathrm{d}u,$$

hence we get an upper bound dependant only on t and H.

So we consider now the case of $2H-1<0$. For $2H<1$ and any $r>0$ we have

$$\rho(r) := \frac{1}{4\varepsilon}\big(|r+\varepsilon|^{2H} - |r-\varepsilon|^{2H}\big) \leq 2r^{2H-1}.$$

This is true because either $r \leq 2\varepsilon$ in which case

$$\begin{aligned}\rho(r) &\leq \frac{1}{4\varepsilon}\big((3\varepsilon)^{2H} - \varepsilon^{2H}\big)\\ &\leq \varepsilon^{2H-1} \leq 2r^{2H-1},\end{aligned}$$

or $r > 2\varepsilon$, where we have

$$\rho(r) \le \frac{1}{4}\int_{-1}^{1} 2H(r+\varepsilon u)^{2H-1}\,\mathrm{d}r$$
$$\le \frac{1}{4}\int_{-1}^{1} (\frac{r}{2})^{2H-1}\,\mathrm{d}r \le r^{2H-1}\,.$$

So by (2.32) we have

$$|\langle g^X, g^\varepsilon\rangle| \le 2\sum_{i\in J}(t_i^{2H-1} + t_{i+1}^{2H-1}) \le 2N t_1^{2H-1}\,,$$

where N is the number of jumps in $[0,t]$.
Applying the Hölder inequality with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ we have

$$\mathbb{E}^x|A^X\langle g^X, g^\varepsilon\rangle| \preccurlyeq (\mathbb{E}^x|A^X|^q)^{1/q}(\mathbb{E}^x N^r)^{1/r}(\mathbb{E}^x\, t_1^{(2H-1)p})^{1/p}.$$

So we just need to pick a $p>1$ with $(2H-1)p+1>0$, in which case the exponential distribution of t_1 implies

$$\mathbb{E}^x\, t_1^{(2H-1)p} \le \int_0^s t_1^{(2H-1)p}\,\mathrm{d}t_1 = s^{(2H-1)p+1} \le t^{(2H-1)p+1}\,.$$

In fact the proof of lemma 2.4.4 also shows that for any $q\ge 1$, $\mathbb{E}\mathbb{E}^x|A^X|^q$ is uniformly bounded in $0\le s\le t$. As N has a Poisson distribution $\mathbb{E}^x N^r$ is also finite.

3 Asymptotic Behavior

3.1 Introduction

In this chapter we study the exponential behavior of the solution to parabolic Anderson model (PAM) driven by fractional noise.

Let $(\Omega^X, \mathscr{F}^X, (\mathscr{F}^X_t)_{t\geq 0}, \mathrm{P}^X)$ be a complete filtered probability space with P^X being the probability law of the simple (nearest-neighbor) symmetric random walk on $\mathbb{Z}^d$ indexed by $t \in \mathbb{R}^{\geq 0}$, started from the origin. We denote the jump rate of the random walk by κ , the corresponding expectation by $\mathbb{E}^X$ and a random walk sample path by $X(\cdot)$.

We consider

$$u(T) := \mathbb{E}^X\left[\exp\int_0^T \mathrm{d}B_t^{X(t)}\right], \tag{3.1}$$

where $\{B_t^x;\, t \geq 0\}_{x\in\mathbb{Z}^d}$ is a family of independent fractional Brownian motions (fBM) with Hurst parameter H indexed by $\mathbb{Z}^d$ and independent of the random walk. Here the stochastic integral is nothing other than a summation. Indeed, suppose $\{t_i\}_{i=1}^n$ are the jump times of the random walk $\{X(s)\ ,\ s\in[0,t]\}$, and for each i, $\{x_i\}_{i=0}^n$ is the value of $\{X(\cdot)\}$ at time interval $[t_i, t_{i+1})$. Then we have

$$\int_0^t B\big(\mathrm{d}s, X(s)\big) = \sum_{i=0}^n \big(B(t_{i+1}, x_i) - B(t_i, x_i)\big).$$

We also define

$$U(T) := \mathbb{E}\log\mathbb{E}^X\left[\exp\int_0^T \mathrm{d}B_t^{X(t)}\right], \tag{3.2}$$

where "$\mathbb{E}$" is expectation with respect to the fBM's.
Sometimes when there is no loss of generality and for the sake of simplicity we let $\kappa = 1$.

Our goal is to show that $u(t)$ behaves asymptotically as $e^{\lambda t}$ for some positive constant λ. For $H \leq 1/2$ we show this property in a very general setting. However, the situation for $H > 1/2$ is more complicated. Here we just managed to show that $u(t)$ grows asymptotically slower than $e^{\lambda_1 t\sqrt{\log t}}$ for some λ_1. This along with the fact that it grows faster than $e^{\lambda_2 t}$ for some positive constant λ_2, strengthen the conjecture that the asymptotic behavior is exactly as $e^{\lambda t}$, for some positive λ. This remains an open problem.

The case of Brownian motion, i.e. $H = 1/2$ was proved by Carmona and Molchanov in [3] using simple subadditivity properties and independent increments of the Brownian motion. These arguments do not apply to the general case of $H \in (0,1)$.

Viens and Zhang in [50], study the PAM driven by Riemann-Liouville fractional noise (1.2) with the space variable x running through a compact space χ. For $H \leq 1/2$, and under some strong conditions on H, κ and spatial covariance, they prove that $\{\frac{1}{n}\log u(n)\}_{n\in\mathbb{N}}$ converges to some deterministic positive number. For $H > 1/2$, they try to prove that $\log u(t)$ grows asymptotically faster than $\frac{t^{2H}}{\log t}$, which is in contrast with our results.

We consider the PAM driven by fractional noise over $\mathbb{Z}^d$. Although we assume that the fractional Brownian motions associated to different sites of $\mathbb{Z}^d$ are independent, our results remain valid for much more general spatial covariance structures.

In section 3.2, we demonstrate that the main contribution to $U(t)$ comes from those random walk occurrences that have restricted number of jumps over the time period $[0, t]$. This basically turns our setup to the compact setting. We denote by $\widehat{U}(t)$ the part of $U(t)$ that comes from this kind of random walk occurrences.

In section 3.3, we show that the asymptotic behavior of $\{\widehat{U}(t)\}_{t\in\mathbb{R}^+}$ is not different from its behavior over the positive integers, i.e. when $t \in \mathbb{Z}^+$. Hence we can confine our attention to this latter case.

In Section 3.4, we develop a Lipschitz inequality that will serve as a building block for all our subsequent arguments.

In section 3.5, we prove an approximate super-additivity for $\widehat{U}(\cdot)$. This would then imply the convergence of $\frac{1}{t}\widehat{U}(t)$ as t goes to infinity.

Section 3.6 is devoted to the quenched asymptotic behavior. In mathematical physics terminology the quenched statements are those statements that are formulated almost surely. Here we seek the almost sure behavior of $\log u(t)$ when t approaches infinity. In this section we show that $\log u(\cdot)$ has the same asymptotic behavior as $\widehat{U}(\cdot)$. In particular we obtain limits over the positive real t's instead of integers.

In section 3.7, we establish a strictly positive asymptotic lower bound on $\{\frac{1}{t}\widehat{U}(t)\}_t$, for any κ and $H \in (0, 1)$. Hence along with the super-additivity result, it shows that $\widehat{U}(t)$ grows in t at least as fast as λt for some strictly positive λ.

Section 3.8 deals with finding an asymptotic upper bound on $\{\frac{1}{t}\widehat{U}(t)\}_t$. Although for the case of $H \leq 1/2$ we easily find a finite asymptotic upper bound which settles the question for this case, we didn't manage to get such a finite upper bound for $H > 1/2$. In this latter case we instead, established for $\{\frac{1}{t}\widehat{U}(t)\}_t$, the asymptotic upper bound $Ct\sqrt{\log t}$ for some positive constant C.

3.2 Approximation via constraining the number of jumps

In this section we justify the approximation via restricting the random walk to have a limited number of jumps. We show that the greatest contribution comes from the random walk paths that have a restricted number of jumps.

For $T \geq 1$, let $\mathscr{A}$ be the event that the number of jumps of the random walk in the time interval $[0, T]$ is less than T^2 for $H > 1/2$, and less than $\beta\kappa T$ for $H \leq 1/2$, where $\beta := \max\{e^6, \kappa^{-1}\}$. Define $\widehat{U}(T)$ as follows

$$\widehat{U}(T) := \mathbb{E}\log\mathbb{E}^X\left[e^{\int_0^T \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{A}}\right].$$

Proposition 3.2.1. *For any real positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ that grows at least as fast as a linear function, we have*

$$\limsup_{t\to\infty}\frac{\widehat{U}(t)}{f(t)} = \limsup_{t\to\infty}\frac{U(t)}{f(t)},$$

and

$$\liminf_{t\to\infty}\frac{\widehat{U}(t)}{f(t)} = \liminf_{t\to\infty}\frac{U(t)}{f(t)}.$$

Proof. We would like to show that $U(T)$ is close to $\widehat{U}(T)$. We denote by $\mathscr{S}_X$ the integral $\int_0^T \mathrm{d}B_t^{X(t)}$. Using the inequality $log(1+a) \leq a$ and then Cauchy-Schwarz we have

$$\begin{aligned}
U(T) - \widehat{U}(T) &= \mathbb{E}\log\left(1 + \frac{\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}^c}\right]}{\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}}\right]}\right)\\
&\leq \mathbb{E}\left(\frac{\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}^c}\right]}{\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}}\right]}\right)\\
&\leq \sqrt{\mathbb{E}\left(\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}^c}\right]\right)^2}\sqrt{\mathbb{E}\left(\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}}\right]\right)^{-2}},
\end{aligned}$$

where $\mathscr{A}^c$ is the complement of $\mathscr{A}$.
As x^{-2} is convex, we have

$$\mathbb{E}\left(\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}}\right]\right)^{-2} \leq p_{\mathscr{A}}^{-3}\mathbb{E}\mathbb{E}^X\left[e^{-2\mathscr{S}_X}\mathbf{1}_{\mathscr{A}}\right] \leq p_{\mathscr{A}}^{-3}\mathbb{E}^X\left[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}}\right],$$

where $p_{\mathscr{A}}$ is the probability of $\mathscr{A}$.
For the other term, again by Cauchy-Schwarz we have

$$\mathbb{E}\left(\mathbb{E}^X\left[e^{\mathscr{S}_X}\mathbf{1}_{\mathscr{A}^c}\right]\right)^2 \leq p_{\mathscr{A}^c}\,\mathbb{E}\mathbb{E}^X\left[e^{2\mathscr{S}_X}\mathbf{1}_{\mathscr{A}^c}\right] \leq p_{\mathscr{A}^c}\,\mathbb{E}^X\left[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}^c}\right],$$

where $p_{\mathscr{A}^c}$ is the probability of $\mathscr{A}^c$.

i) For $H > 1/2$:
In this case we have

$$var(\mathscr{S}_X) \leq T^{2H}.$$

So

$$\mathbb{E}^X\big[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}}\big] \le p_{\mathscr{A}}e^{2T^{2H}} \qquad \text{and} \qquad \mathbb{E}^X\big[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}^c}\big] \le p_{\mathscr{A}^c}e^{2T^{2H}}$$

hence

$$U(T)-\widehat{U}(T) \le p_{\mathscr{A}}^{-1}p_{\mathscr{A}^c}e^{2T^{2H}}.$$

For a Poisson random variable N with mean λ we have the following tail probability bound [33]

$$P(N\ge n) \le e^{-\lambda}(\frac{e\lambda}{n})^n \qquad \text{for } n>\lambda. \tag{3.3}$$

Using this bound, for $T\ge\kappa e^2$ we have

$$p_{\mathscr{A}^c} \le e^{-\kappa T}(\frac{e\kappa T}{T^2})^{T^2} \le e^{-\kappa T}e^{-T^2},$$

which implies $p_{\mathscr{A}}\ge 1/2$. Hence

$$0\le U(T)-\widehat{U}(T) \le 2e^{-T^2}e^{2T^{2H}} \sim \mathscr{O}(e^{-T}). \tag{3.4}$$

ii) For $H\le 1/2$: In this case we have

$$var(\mathscr{S}_X) \le n(\frac{T}{n})^{2H},$$

where n is the number of jumps in [0,T]. So

$$\begin{aligned}\mathbb{E}^X\big[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}}\big] &\le \mathbb{E}^X\big[e^{2n^{1-2H}T^{2H}}\mathbf{1}_{\mathscr{A}}\big] \le \mathbb{E}^X\big[e^{2(\beta\kappa T)^{1-2H}T^{2H}}\mathbf{1}_{\mathscr{A}}\big]\\ &\le e^{2(\beta\kappa)^{1-2H}T}p_{\mathscr{A}},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}^X\big[e^{2var(\mathscr{S}_X)}\mathbf{1}_{\mathscr{A}^c}\big] &\le \mathbb{E}^X\big[e^{2n(\frac{T}{n})^{2H}}\mathbf{1}_{\mathscr{A}^c}\big] \le \mathbb{E}^X\big[e^{2n(\beta\kappa)^{-2H}}\mathbf{1}_{\mathscr{A}^c}\big]\\ &\le \mathbb{E}^X\big[e^{2n}\mathbf{1}_{\mathscr{A}^c}\big] = e^{-\kappa T}\sum_{n>\beta\kappa T}\frac{(\kappa T)^n}{n!}e^{2n} \le e^{-\kappa T}e^{e^2\kappa T},\end{aligned}$$

where we have used the fact that $\beta\kappa\ge 1$.
Finally using $\beta\ge e^6$ and Poisson tail probability bound (3.3) we have

$$p_{\mathscr{A}^c} \le e^{-\kappa T}(\frac{e\kappa T}{\beta\kappa T})^{\beta\kappa T} \le e^{-\kappa T}e^{-5\beta\kappa T},$$

which also implies $p_{\mathscr{A}}\ge 31/32$.
Hence

$$\begin{aligned}0\le U(T)-\widehat{U}(T) &\le (31/32)^{-1}\exp\{(\beta\kappa)^{1-2H}T-\kappa T/2+e^2\kappa T/2-5\beta\kappa T/2\}\\ &\sim \mathscr{O}(e^{-T}),\end{aligned} \tag{3.5}$$

where we have used $\beta \geq e^6$ and $\beta\kappa \geq 1$.

So in any case and using $\frac{1}{f(T)} \sim \mathcal{O}(1)$ we have

$$\frac{\widehat{U}(T)}{f(T)} \leq \frac{U(T)}{f(T)} \leq \frac{\widehat{U}(T)}{f(T)} + \mathcal{O}(e^{-T}).$$

The statement follows by taking lim inf and lim sup. □

3.3 Quantization

In this section we show that restricting the time to be integer valued does not affect the generality of the our results on the asymptotic behavior of $\widehat{U}(t)$ and hence of $U(t)$. Our super-additivity arguments in section 3.5 hold only for the discretized time.

Proposition 3.3.1. *For any real positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ that grows at least as fast as a linear function, we have*

$$\limsup_{t\to\infty} \frac{\widehat{U}(t)}{f(t)} = \limsup_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\widehat{U}(n)}{f(n)},$$

and

$$\liminf_{t\to\infty} \frac{\widehat{U}(t)}{f(t)} = \liminf_{\substack{n\to\infty\\ n\in\mathbb{N}}} \frac{\widehat{U}(n)}{f(n)}.$$

Proof. For $t \geq 1$, we define $\mathscr{A}_t$ to be the event that as in the last section, the random walk has at most $N := t^2$ or $N := \beta\kappa t$ jumps on the interval $[0, t]$ depending if $H > 1/2$ or $H \leq 1/2$ respectively, where $\beta := \max\{e^6, \kappa^{-1}\}$. For $0 < t_1 < t_2$ define $\mathscr{C}_{t_1,t_2}$ to be the event that the random walk has no jump on the interval $(t_1, t_2]$. Let $n \in \mathbb{N}$ be the largest integer not greater than t, i.e. $n := \lfloor t \rfloor$, and for any $x \in \mathbb{Z}^d$ denote $\Delta B^x_{n,t} := B^x_t - B^x_n$. We have

$$\begin{aligned}
\widehat{u}(t) := \mathbb{E}^X\left[e^{\int_0^t \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_t}\right] &\geq \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_n} \mathbf{1}_{\mathscr{C}_{n,t}} e^{\int_n^t \mathrm{d}B_s^{X(s)}}\right] \\
&= \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_n} \mathbf{1}_{\mathscr{C}_{n,t}} e^{\Delta B^{X(n)}_{n,t}}\right] \\
&\geq \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_n} \mathbf{1}_{\mathscr{C}_{n,t}} e^{\inf_{|x|\leq N} \Delta B^x_{n,t}}\right] \\
&= \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_n} \mathbf{1}_{\mathscr{C}_{n,t}}\right] e^{\inf_{|x|\leq N} \Delta B^x_{n,t}} \\
&= \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_n}\right] \mathrm{P}^X(\mathscr{C}_{n,t})\, e^{\inf_{|x|\leq N} \Delta B^x_{n,t}}.
\end{aligned}$$

So we have

$$\widehat{U}(t) = \mathbb{E}\log \widehat{u}(t) \geq \widehat{U}(n) - \kappa(t-n) + \mathbb{E} \inf_{|x|\leq N} \Delta B^x_{n,t}.$$

Now as

$$\mathbb{E} \inf_{|x|\leq N} \Delta B^x_{n,t} = -\mathbb{E} \sup_{|x|\leq N} \Delta B^x_{n,t}$$

and noticing that for $x, y \in \mathbb{Z}^d$, $x \neq y$

$$var(\Delta B^x_{n,t} - \Delta B^y_{n,t}) = 2var(\Delta B^x_{n,t}) = 2(t-n)^{2H},$$

So by Dudley's theorem we have

$$\begin{aligned}
\mathbb{E} \sup_{|x|\leq N} \Delta B^x_{n,t} &\leq K \int_0^{\sqrt{2}(t-n)^H} \sqrt{\log(2N+1)^d} \\
&= K(t-n)^H \sqrt{2d\log(2N+1)} \leq K' \sqrt{\log(n)}.
\end{aligned}$$

It should be noted that one can show by elementary probability tools that the expectation of the maximum of n Gaussian random variables is bounded by $K\sqrt{\log n}$ for some positive constant K, and the whole machinery of Dudley's theorem 1.3.4 is not needed at all. But we apply Dudley's theorem even for the finite case simply in order to have a single uniform argument for both finite and infinite supremums.

So we have

$$\widehat{U}(t) \geq \widehat{U}(n) - K\sqrt{\log(n)}.$$

We can similarly show that

$$\widehat{U}(n+1) \geq \widehat{U}(t) - K\sqrt{\log(t)}.$$

So we have

$$\widehat{U}(n) - K\sqrt{\log(n)} \leq \widehat{U}(t) \leq \widehat{U}(n+1) + K\sqrt{\log(t)},$$

and hence if $\{\frac{\widehat{U}(n)}{n}\}_{n\in\mathbb{N}}$ converges, $\frac{\widehat{U}(t)}{t}$ also converges to the same limit. $\square$

3.4 Lipschitz continuity of residues of fBM increments

In this section we consider the following stochastic process

$$Y_n(u) := \int_0^n (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \,\mathrm{d}W_s\,,$$

and establish its Lipschitz continuity. This will play a vital role in the succeeding sections. Indeed for $n \in \mathbb{N}^{\geq 1}$ and $n+1 \leq t_1 < t_2$ we have

$$B_{t_2} - B_{t_1} = \int_0^n \Big(K_H(t_2,s) - K_H(t_1,s) \Big) \mathrm{d}W_s + Z_{n,t_2}\,, \tag{3.6}$$

where Z_{n,t_2} is measurable with respect to the sigma field generated by $\{W_s - W_n\,;\, s \in [n,t_2]\}$.
Applying the stochastic Fubini theorem 1.3.3 to the first right hand side term of (3.6) we get

$$\begin{aligned}\int_0^n \Big(K_H(t_2,s) - K_H(t_1,s) \Big) \mathrm{d}W_s &= \int_0^n \int_{t_1}^{t_2} (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \,\mathrm{d}u\,\mathrm{d}W_s \\ &= \int_{t_1}^{t_2} Y_n(u)\,\mathrm{d}u\,.\end{aligned}$$

For $k,n \in \mathbb{N}^{\geq 1}$ and $u \in [n+k, n+k+1]$ we define the process $Y_{n,k}$ as $Y_{n,k}(u) := Y_n(u)$.
We denote by $\asymp$ and $\preccurlyeq$ respectively, equality and inequality up to a positive constant that only possibly depends on H.

Proposition 3.4.1. *Let $k,n \in \mathbb{N}^{\geq 1}$ and $u,v \in [n+k, n+k+1]$. Then*

$$\mathbb{E}\big[Y_{n,k}(u) - Y_{n,k}(v) \big]^2 \preccurlyeq (1+\frac{k}{n})^{2H-1} k^{2H-4} (u-v)^2, \tag{3.7}$$

and

$$\mathbb{E}\big(Y_{n,k}(u) \big)^2 \preccurlyeq (1+\frac{k}{n})^{2H-1} k^{2H-2}. \tag{3.8}$$

Proof. Without loss of generality we may assume that $u \leq v$. Using the Itō isometry for stochastic integrals we have

$$\begin{aligned}\mathbb{E}\big[Y_{n,k}(u) - Y_{n,k}(v) \big]^2 &= \int_0^n \Big((u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} - (v-s)^{H-\frac{3}{2}} (\frac{v}{s})^{H-\frac{1}{2}} \Big)^2 \mathrm{d}s \\ &\leq 2(I_1 + I_2)\,,\end{aligned}$$

where

$$I_1 := \int_0^n (\frac{u}{s})^{2H-1} \Big((u-s)^{H-\frac{3}{2}} - (v-s)^{H-\frac{3}{2}} \Big)^2 \mathrm{d}s\,,$$

and

$$I_2 := \int_0^n (v-s)^{2H-3} \Big((\frac{u}{s})^{H-\frac{1}{2}} - (\frac{v}{s})^{H-\frac{1}{2}} \Big)^2 \mathrm{d}s\,.$$

We use several times the following inequality which holds for any $0<\alpha<\beta$ and $H<1$

$$|\alpha^H-\beta^H| \asymp \int_\alpha^\beta \gamma^{H-1}\mathrm{d}\gamma \preccurlyeq |\beta-\alpha|\alpha^{H-1}. \tag{3.9}$$

We break I_1 and I_2 into integrals over $[0,\frac{n}{2}]$ and $[\frac{n}{2},n]$ so that $I_1=I_{1a}+I_{1b}$ and $I_2=I_{2a}+I_{2b}$ and will bound these terms.
Using inequality (3.9) we have

$$|(u-s)^{H-\frac{3}{2}}-(v-s)^{H-\frac{3}{2}}| \preccurlyeq \frac{|u-v|}{(u-s)^{\frac{5}{2}-H}}. \tag{3.10}$$

Applying (3.10) we get

$$\begin{aligned} I_{1b} &= \int_{\frac{n}{2}}^n (\frac{u}{s})^{2H-1}\Big((u-s)^{H-\frac{3}{2}}-(v-s)^{H-\frac{3}{2}}\Big)^2 \mathrm{d}s \\ &\preccurlyeq (u-v)^2 \int_{\frac{n}{2}}^n (\frac{u}{s})^{2H-1}(u-s)^{2H-5}\,\mathrm{d}s. \end{aligned}$$

But for $\frac{n}{2}<s$ and $u<n+k+1$, when $H>1/2$ we have

$$(\frac{u}{s})^{2H-1} \le (\frac{n+k+1}{n/2})^{2H-1} \preccurlyeq (1+\frac{k}{n})^{2H-1}$$

and when $H\le 1/2$ we have

$$(\frac{u}{s})^{2H-1} \le (\frac{n+k}{n})^{2H-1}.$$

So

$$\begin{aligned} I_{1b} &\preccurlyeq (u-v)^2\,(1+\frac{k}{n})^{2H-1}\int_{\frac{n}{2}}^n (u-s)^{2H-5}\,\mathrm{d}s \\ &\preccurlyeq (u-v)^2(1+\frac{k}{n})^{2H-1}k^{2H-4}. \end{aligned}$$

For I_{1a}, using the fact that $u^{2H-1}\preccurlyeq (n+k)^{2H-1}$, and that for $s<\frac{n}{2}$ we have $u-s\ge k+n/2 \succcurlyeq k+n$ and applying the inequality (3.10) we get

$$\begin{aligned} I_{1a} &= \int_0^{\frac{n}{2}} (\frac{u}{s})^{2H-1}\Big((u-s)^{H-\frac{3}{2}}-(v-s)^{H-\frac{3}{2}}\Big)^2 \mathrm{d}s \\ &\preccurlyeq (u-v)^2\,u^{2H-1}\int_0^{\frac{n}{2}} \frac{1}{s^{2H-1}(u-s)^{5-2H}}\,\mathrm{d}s \\ &\preccurlyeq (u-v)^2\,(n+k)^{2H-1}(n+k)^{2H-5}\int_0^{\frac{n}{2}} s^{1-2H}\,\mathrm{d}s \\ &= (u-v)^2\,(n+k)^{4H-6}\,n^{2-2H} \le (u-v)^2\,(n+k)^{2H-4} \\ &\le (u-v)^2\,k^{2H-4}. \end{aligned}$$

For I_2 we need the following inequality which is a special case of inequality (3.9)

$$|(u-s)^{H-\frac{1}{2}}-(v-s)^{H-\frac{1}{2}}| \preccurlyeq \frac{|u-v|}{(u-s)^{\frac{3}{2}-H}}. \tag{3.11}$$

For I_{2a}, as we have $v-s \geq k+n/2 \succcurlyeq k+n$ for $s \leq n/2$, using inequality (3.11) we get

$$\begin{aligned}
I_{2a} &= \int_0^{\frac{n}{2}} (v-s)^{2H-3}\Big((\frac{u}{s})^{H-\frac{1}{2}}-(\frac{v}{s})^{H-\frac{1}{2}}\Big)^2 \mathrm{d}s \\
&\preccurlyeq (n+k)^{2H-3}\int_0^{\frac{n}{2}} \Big((\frac{u}{s})^{H-\frac{1}{2}}-(\frac{v}{s})^{H-\frac{1}{2}}\Big)^2 \mathrm{d}s \\
&\preccurlyeq (n+k)^{2H-3}\int_0^{\frac{n}{2}} \frac{(u-v)^2}{s^2}(\frac{u}{s})^{2H-3}\mathrm{d}s \\
&\preccurlyeq (u-v)^2\,(n+k)^{2H-3}(n+k)^{2H-3}\int_0^{\frac{n}{2}} s^{1-2H}\,\mathrm{d}s \\
&\preccurlyeq (u-v)^2(n+k)^{4H-6}n^{2-2H} \\
&\leq (u-v)^2(n+k)^{4H-6}(n+k)^{2-2H} \\
&\leq (u-v)^2 k^{2H-4}.
\end{aligned}$$

For I_{2b}, applying (3.11) we have

$$\begin{aligned}
I_{2b} &= \int_{\frac{n}{2}}^{n} (v-s)^{2H-3}\Big((\frac{u}{s})^{H-\frac{1}{2}}-(\frac{v}{s})^{H-\frac{1}{2}}\Big)^2 \mathrm{d}s \\
&\preccurlyeq \int_{\frac{n}{2}}^{n} (v-s)^{2H-3}\,\frac{(u-v)^2}{s^2}(\frac{u}{s})^{2H-3}\mathrm{d}s \\
&\leq (u-v)^2(n+k)^{2H-3}\int_{\frac{n}{2}}^{n} s^{1-2H}(v-s)^{2H-3}\,\mathrm{d}s.
\end{aligned}$$

But as for $n/2 \leq s \leq n$ we have $s^{1-2H} \preccurlyeq n^{1-2H}$, we get

$$\begin{aligned}
I_{2b} &\preccurlyeq (u-v)^2(n+k)^{2H-3}n^{1-2H}\int_{\frac{n}{2}}^{n}(v-s)^{2H-3}\,\mathrm{d}s \\
&\preccurlyeq (u-v)^2(n+k)^{2H-3}n^{1-2H}k^{2H-2}.
\end{aligned}$$

So

$$I_{2b} \preccurlyeq (u-v)^2(1+\frac{k}{n})^{2H-1}k^{2H-4}$$

So this completes the proof of Hölder continuity.

Now for the variance bound using the similar technics used above we have

$$\mathbb{E}[(Y_{n,k}(u))^2] = \int_0^n (\frac{u}{s})^{2H-1}(u-s)^{2H-3}\,\mathrm{d}s = J_1 + J_2\,,$$

where

$$
\begin{aligned}
J_1 := \int_0^{n/2} (\frac{u}{s})^{2H-1}(u-s)^{2H-3}\,\mathrm{d}s &\preccurlyeq (n+k)^{2H-3}\int_0^{n/2}(\frac{u}{s})^{2H-1}\,\mathrm{d}s \\
&\preccurlyeq (n+k)^{2H-3}(n+k)^{2H-1}\int_0^{n/2} s^{1-2H}\,\mathrm{d}s \\
&\preccurlyeq (n+k)^{4H-4}n^{2-2H} \preccurlyeq (1+\frac{k}{n})^{2H-2}k^{2H-2}
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 := \int_{n/2}^{n} (\frac{u}{s})^{2H-1}(u-s)^{2H-3}\,\mathrm{d}s &\preccurlyeq (1+\frac{k}{n})^{2H-1}\int_{n/2}^{n}(u-s)^{2H-3}\,\mathrm{d}s \\
&\preccurlyeq (1+\frac{k}{n})^{2H-1}k^{2H-2}\,\mathrm{d}s,.
\end{aligned}
$$

□

3.5 Super-additivity

In this section we will show that $\{\widehat{U}(n)\}_{n\in\mathbb{N}}$, although not super-additive, has some super-additivity properties and this way we prove that $\{\frac{\widehat{U}(n)}{n}\}_{n\in\mathbb{N}}$ converges to some positive extended-real number λ.

Theorem 3.5.1. *The sequence* $\{\frac{\widehat{U}(n)}{n}\}_{n\in\mathbb{N}}$ *converges to some positive extended real number* $\lambda \in [0,+\infty]$.

While $\{\widehat{U}(n)\}_{n\in\mathbb{N}}$ is not super-additive in general as it is in the Brownian motion case, we seek some approximate super-additivity. Although the super-additivity arguments in Viens and Zhang [50] seem to have some problems, their idea of recognizing an approximate super-additivity is a major observation. We will build our argument by following some of their ideas.

Let $\{f(n)\}_{n\in\mathbb{N}}$ be a sequence of real numbers and $\{\epsilon(n)\}_{n\in\mathbb{N}}$ a sequence of non-negative numbers with the property that

$$\text{(i)}\ \lim_{n\to\infty}\frac{\epsilon(n)}{n}=0;\qquad \text{(ii)}\ \sum_{n=1}^{\infty}\frac{\epsilon(2^n)}{2^n}<\infty.$$

Then $\{f(n)\}_{n\in\mathbb{N}}$ is called *almost super-additive* relative to $\{\epsilon(n)\}_{n\in\mathbb{N}}$ if

$$f(n+m)\geq f(n)+f(m)-\epsilon(n+m)$$

for any $n,m\in\mathbb{N}$. We have the following theorem [50, 9]

Theorem 3.5.2. *Let* $\{f(n)\}_{n\in\mathbb{N}}$ *be almost super-additive relative to* $\{\epsilon(n)\}_{n\in\mathbb{N}}$ *as defined above.*
(1) If $\sup_n\frac{f(n)}{n}<+\infty$, *then* $\lim_{n\to\infty}\frac{f(n)}{n}$ *exists and is finite.*
(2) If $\sup_n\frac{f(n)}{n}=+\infty$, *then* $\{\frac{f(n)}{n}\}$ *diverges to* $+\infty$.

Lemma 3.5.3. *For any* $n,m\in\mathbb{N}^0$ *we have*

$$\widehat{U}(n+m+1)\geq\widehat{U}(n)+\widehat{U}(m)-c_{\kappa,H}(m+n)^H\sqrt{\log(m+n)},$$

Proof of Lemma. Take arbitrary $n,m\in\mathbb{N}^0$ and without loss of generality assume that $n\geq m$. Let $\mathscr{A}_n$ be the event that the random walk on the time interval $[0,n)$ has no more jumps than $\mathfrak{N}_n$

$$\mathfrak{N}_n:=\begin{cases} n^2 & \text{for } H>1/2\\ \beta\kappa n & \text{for } H\leq 1/2,\end{cases}$$

where $\beta:=\max\{e^6,\kappa^{-1}\}$ and similarly $\mathscr{B}_m$ be the event that the random walk has at most $\mathfrak{N}_m$ jumps on the interval $[n+1,n+m+1)$. Let also $\mathscr{C}$ be the event that the random walk has no jump on the interval $[n, n+1)$. We have

$$\widehat{U}(m+n+1) - \widehat{U}(n) \geq \mathbb{E}\log\mathbb{E}^X\Big(\frac{e^{\int_0^n \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{A}_n}}{\mathbb{E}^X[e^{\int_0^n \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{A}_n}]} e^{\int_n^{n+m+1} \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\Big). \tag{3.12}$$

Let $\mathscr{F}$ be the sigma field generated by the random walk up to time n. Then the right-hand-side of the above equation would be equal to

$$\mathbb{E}^X\Big(\frac{e^{\int_0^n \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{A}_n}}{\mathbb{E}^X[e^{\int_0^n \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{A}_n}]}\mathbb{E}^X\Big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}|\mathscr{F}\Big)\Big). \tag{3.13}$$

For any $t \geq n$, let $\widetilde{X}(t) := X(t) - X(n)$. By the Markov property of the random walk, and then the fact that $\{\widetilde{X}(t)\}_{t\geq n}$ is independent of $\mathscr{F}$ we have

$$\begin{aligned}
\mathbb{E}^X\big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}|\mathscr{F}\big) &= \mathbb{E}^X\big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{X(t)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}|X(n)\big)\\
&= \mathbb{E}^X\big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+X(n)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}|X(n)\big)\\
&= \mathbb{E}^{\widetilde{X}}\big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+X(n)}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\big)\\
&= \mathbb{E}^{\widetilde{X}}\big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y}}\mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\big),
\end{aligned}$$

where $Y := X(n)$.

Let $\{\widehat{W}^x\}_{x\in\mathbb{Z}^d}$ be a family of independent standard Brownian motions, which is independent of the random walks $X(\cdot)$ and $\widetilde{X}(\cdot)$, the fractional Brownian motions $\{B^x\}_{x\in\mathbb{Z}^d}$ and hence their corresponding Brownian motions $\{W^x\}_{x\in\mathbb{Z}^d}$ appearing in their integral representation. For any $x \in \mathbb{Z}^d$ define $\widetilde{W}_s^x$ as

$$\widetilde{W}_t^x := \begin{cases} \widehat{W}_t^x & \text{for } 0 \leq t \leq n \\ W_t^x - W_n^x + \widehat{W}_n^x & \text{for } t > n. \end{cases}$$

It is easily verified that $\widetilde{W}^x$ is itself a standard Brownian motion.

We define the following family of fractional Brownian motions indexed by $\mathbb{Z}^d$

$$\widetilde{B}_t^x := \int_0^t K_H(t,s)\mathrm{d}\widetilde{W}_s^x. \tag{3.14}$$

It is clear that for $t \geq n$

$$\widetilde{B}_t^x = \int_0^n K_H(t,s)\mathrm{d}\widehat{W}_s^x + \int_n^t K_H(t,s)\mathrm{d}W_s^x.$$

Let $\widehat{\mathscr{G}}_{[0,n]}$ be the sigma field generated by $\{\widehat{W}_s^x\,;\ s \in [0,n]\,,\, x \in \mathbb{Z}^d\}$ and $\mathscr{G}_{[n,\infty)}$ the sigma field generated by $\{W_s^x - W_n^x\,;\ s \in [n,\infty)\,,\, x \in \mathbb{Z}^d\}$. Also denote by $\mathscr{G}_o$ the sigma field generated by $\{W_s^x\,;\ s \in [0,n]\,,\, x \in \mathbb{Z}^d\}$. It is evident that for any $t \geq n$ the process $\widetilde{B}_t^x$ is measurable with respect to $\mathscr{G}_1 := \widehat{\mathscr{G}}_{[0,n]} \vee \mathscr{G}_{[n,\infty)}$ where $\vee$ denotes the smallest sigma field containing the both.

So $\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+y}$ is also measurable with respect to $\mathcal{G}_1$ which is independent of $\mathcal{G}_o$.
Now denote by $\mathbb{E}^Y$ the expectation with respect to the random variable Y with the following distribution

$$\mathrm{P}(Y=y) = \mathbb{E}^X\Big(\frac{e^{\int_0^n \mathrm{d}B_t^{X(t)}} \mathbf{1}_{\mathcal{A}_n}}{\mathbb{E}^X[e^{\int_0^n \mathrm{d}B_t^{X(t)}} \mathbf{1}_{\mathcal{A}_n}]} \mathbf{1}_{X(n)=y} \Big) \qquad : \qquad y \in \mathbb{Z}^d.$$

So equations (3.12) and (3.13) imply

$$\begin{aligned} \widehat{U}(m+n+1) - \widehat{U}(n) &\geq \mathbb{E} \log \mathbb{E}^Y \Big(\mathbb{E}^{\widetilde{X}} \big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathcal{B}_m \cap \mathcal{C}} \big) \Big) \\ &\geq \mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{\widetilde{X}} \big(e^{\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathcal{B}_m \cap \mathcal{C}} \big). \end{aligned} \tag{3.15}$$

Now let $\{t_i\}_i$, $t_i \geq n+1$, be the jump times of the random walk after time $t = n+1$, and for every i let x_i be the position of the random walk on the time interval $[t_i, t_{i+1})$. Then we have

$$\int_{n+1}^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y} = \int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y} + \Delta^X,$$

where

$$\begin{aligned} \Delta^X := &\sum_i \int_0^n \big(K_H(s, t_{i+1}) - K_H(s, t_i) \big) \mathrm{d}W_s^{x_i} \\ &- \sum_i \int_0^n \big(K_H(s, t_{i+1}) - K_H(s, t_i) \big) \mathrm{d}\widetilde{W}_s^{x_i}. \end{aligned}$$

By the definition of K_H and using the stochastic Fubini we have

$$\begin{aligned} \int_0^n \big(K_H(s, t_{i+1}) - K_H(s, t_i) \big) \mathrm{d}W_s^{x_i} &= c_H \int_0^n \int_{t_i}^{t_{i+1}} (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \,\mathrm{d}u \,\mathrm{d}W_s^{x_i} \\ &= c_H \int_{t_i}^{t_{i+1}} \int_0^n (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \,\mathrm{d}W_s^{x_i} \,\mathrm{d}u \\ &= c_H \int_{t_i}^{t_{i+1}} Y_n^{x_i}(u) \,\mathrm{d}u, \end{aligned}$$

and similarly

$$\int_0^n \big(K_H(s, t_{i+1}) - K_H(s, t_i) \big) \mathrm{d}\widetilde{W}_s^{x_i} = c_H \int_{t_i}^{t_{i+1}} \widetilde{Y}_n^{x_i}(u) \,\mathrm{d}u,$$

where

$$\widetilde{Y}_n^{x_i}(u) = \int_0^n (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \,\mathrm{d}\widetilde{W}_s^{x_i}.$$

So

$$\begin{aligned} \Delta^X &= c_H \int_{n+1}^{n+m+1} Y_n^{X(u)}(u) \,\mathrm{d}u - c_H \int_{n+1}^{n+m+1} \widetilde{Y}_n^{X(u)}(u) \,\mathrm{d}u \\ &\geq c_H \sum_{k=1}^m \inf_{\substack{|x| \leq \mathfrak{N}_n + \mathfrak{N}_n \\ u \in [n+k, n+k+1]}} Y_n^x(u) - c_H \sum_{k=1}^m \sup_{\substack{|x| \leq \mathfrak{N}_n + \mathfrak{N}_n \\ u \in [n+k, n+k+1]}} \widetilde{Y}_n^x(u). \end{aligned}$$

On the event $\mathscr{C}$ we also have

$$\begin{aligned}\int_n^{n+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y} &= B_{n+1}^Y - B_n^Y \\ &\geq \inf_{|y|\leq \mathfrak{N}_n} (B_{n+1}^y - B_n^y)\,.\end{aligned}$$

So on the event $\mathscr{B}_m \cap \mathscr{C}$ we have

$$\begin{aligned}\int_n^{n+m+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y} &= \int_n^{n+1} \mathrm{d}B_t^{\widetilde{X}(t)+Y} + \int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y} + \Delta^X \\ &\geq \int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y} + \inf_{|y|\leq \mathfrak{N}_n} (B_{n+1}^y - B_n^y) \\ &\quad + c_H \sum_{k=1}^m \inf_{\substack{|x|\leq \mathfrak{N}_n+\mathfrak{N}_n \\ u\in[n+k,n+k+1]}} Y_n^x(u) - c_H \sum_{k=1}^m \sup_{\substack{|x|\leq \mathfrak{N}_n+\mathfrak{N}_n \\ u\in[n+k,n+k+1]}} \widetilde{Y}_n^x(u)\,.\end{aligned}$$

Plugging this inequality into equation (3.15) we get

$$\begin{aligned}\widehat{U}(m+n+1) - \widehat{U}(n) \geq{}& \mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{\widetilde{X}}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\right) \\ &+ \mathbb{E} \inf_{|y|\leq \mathfrak{N}_n} (B_{n+1}^y - B_n^y) + c_H \sum_{k=1}^m \mathbb{E} \inf_{\substack{|x|\leq \mathfrak{N}_n+\mathfrak{N}_n \\ u\in[n+k,n+k+1]}} Y_n^x(u) \\ &- c_H \mathbb{E} \sum_{k=1}^m \sup_{\substack{|x|\leq \mathfrak{N}_n+\mathfrak{N}_n \\ u\in[n+k,n+k+1]}} \widetilde{Y}_n^x(u)\end{aligned}$$

For $t \geq n+1$, let $X'(t) := X(t) - X(n+1)$. Then we have

$$\begin{aligned}&\mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{\widetilde{X}}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\right) \\ &\quad = \mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{\widetilde{X}}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{X'(t)+Y}} \mathbf{1}_{\mathscr{B}_m\cap\mathscr{C}}\right) \\ &\quad = \mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{X'}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{X'(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right) + \log \mathbb{P}(\mathscr{C})\,.\end{aligned}$$

Let $\mathbb{E}_{\mathscr{G}_o} := \mathbb{E}(\cdot|\mathscr{G}_o)$ be the conditional expectation on the sigma field $\mathscr{G}_o$. As $\frac{e^{\int_0^n \mathrm{d}B_t^{X(t)}} \mathbf{1}_{\mathscr{A}_n}}{\mathbb{E}^X[e^{\int_0^n \mathrm{d}B_t^{X(t)}} \mathbf{1}_{\mathscr{A}_n}]}$ is measurable with respect to $\mathscr{G}_o$, the expectations $\mathbb{E}^Y$ and $\mathbb{E}_{\mathscr{G}_o}$ can be interchanged by Fubini's theorem. So

$$\begin{aligned}&\mathbb{E}\mathbb{E}^Y \log \mathbb{E}^{X'}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{X'(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right) \\ &\quad = \mathbb{E}\mathbb{E}_{\mathscr{G}_o} \mathbb{E}^Y \log \mathbb{E}^{X'}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{X'(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right) \\ &\quad = \mathbb{E}\mathbb{E}^Y \mathbb{E}_{\mathscr{G}_o} \log \mathbb{E}^{X'}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{X'(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right).\end{aligned}$$

But $\mathbb{E}^{X'}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right)$ has the same distribution as $\mathbb{E}^X\left(e^{\int_0^m \mathrm{d}B_t^{X(t)}} \mathbf{1}_{\mathscr{A}_m}\right)$. So we have

$$\mathbb{E}_{\mathscr{G}_o} \log \mathbb{E}^{\widetilde{X}}\left(e^{\int_{n+1}^{n+m+1} \mathrm{d}\widetilde{B}_t^{\widetilde{X}(t)+Y}} \mathbf{1}_{\mathscr{B}_m}\right) = \widehat{U}(m)\,.$$

Hence we get the following conclusion

$$\widehat{U}(m+n+1) - \widehat{U}(n) \geq \widehat{U}(m) - \widehat{\epsilon}(n,m),$$

where

$$\begin{aligned}\widehat{\epsilon}(n,m) &:= -\mathbb{E}\inf_{|y|\leq\mathfrak{N}_n}(B^y_{n+1}-B^y_n) - c_H\sum_{k=1}^m \mathbb{E}\inf_{\substack{|x|\leq\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+k,n+k+1]}} Y^x_n(u)\\ &\quad -\log\mathbb{P}(\mathscr{C}) + c_H\mathbb{E}\sum_{k=1}^m \sup_{\substack{|x|\leq\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+k,n+k+1]}} \widetilde{Y}^x_n(u)\\ &= \mathbb{E}\sup_{|y|\leq\mathfrak{N}_n}(B^y_{n+1}-B^y_n) + c_H\sum_{i=1}^m \mathbb{E}\sup_{\substack{|x|\leq\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+k,n+k+1]}} Y^x_n(u)\\ &\quad -\log\mathbb{P}(\mathscr{C}) + c_H\mathbb{E}\sum_{k=1}^m \sup_{\substack{|x|\leq\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+k,n+k+1]}} \widetilde{Y}^x_n(u).\end{aligned}$$

We are going to bound these terms applying Dudley's theorem 1.3.4.
For $y_1, y_2 \in \mathbb{Z}^d$ with $|y_1|, |y_2| \leq \mathfrak{N}_n$ and $y_1 \neq y_2$ we have

$$\mathbb{E}\big[\big(B^{y_1}_{n+1}-B^{y_1}_n\big) - \big(B^{y_2}_{n+1}-B^{y_2}_n\big)\big]^2 = \mathbb{E}\big(B^y_{n+1}-B^y_n\big)^2 + \mathbb{E}\big(B^y_{n+1}-B^y_n\big)^2 = 2.$$

So by Dudley's theorem 1.3.4 we have

$$\mathbb{E}\sup_{|y|\leq\mathfrak{N}_n}(B^y_{n+1}-B^y_n) \leq K\int_0^2 \sqrt{\log\mathfrak{N}_n}\,\mathrm{d}\varepsilon \leq c'_{\kappa,H}\sqrt{\log n},$$

where K is a universal constant and $c'_{\kappa,H}$ is some positive constant that can only possibly depend on κ and H.

For $l \in \mathbb{N}$, let $\{u_i\}_{i=1}^l$ be the l equally-spaced points on the interval $(n+k, n+k+1)$. Then for any $u \in [n+k, n+k+1]$ there exists a u_i with $|u-u_i| \leq \frac{1}{2l}$. Using the proposition 3.4.1 on the Hölder continuity of Y_n and noting that $k \leq m \leq n$, for every $x \in \mathbb{Z}^d$ we have

$$\mathbb{E}\big[Y^x_n(u) - Y_n(u_i)\big]^2 \leq c_H k^{2H-4}(u-u_i)^2 \leq c_H k^{2H-4}\frac{1}{(2l)^2}$$

and

$$\mathbb{E}\big(Y^x_n(u)\big)^2 \leq C_H k^{2H-2},$$

where c_H and C_H are some universal positive constants that can only possibly depend on H.
This means that for $0 < \varepsilon < c'_H k^{H-2}$, where $c'_H := \sqrt{c_H}/2$, we can cover
$\{Y^x_n(u)\,;\, u \in [n+k, n+k+1], x \in \mathbb{Z}^d, |x| \leq \mathfrak{N}_n + \mathfrak{N}_m\}$ by $(\mathfrak{N}_n + \mathfrak{N}_m)\frac{c'_H k^{H-2}}{\varepsilon}$ ε-balls.
For $c'_H k^{H-2} \leq \varepsilon < C'_H k^{H-1}$, where $C'_H := \sqrt{2C_H}$, this set can be covered by $\mathfrak{N}_n + \mathfrak{N}_m$ ε-balls.
And finally for $\varepsilon \geq C'_H k^{H-1}$, the whole set can be cover with one single ball. So once again by

Dudley's theorem 1.3.4 we have

$$\begin{aligned}\mathbb{E}\sup_{\substack{|x|\le\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+i,n+i+1]}} Y_n^x(u) &\le K\int_0^{c'_H k^{H-2}}\sqrt{\log\Big((\mathfrak{N}_n+\mathfrak{N}_m)\frac{c'_H k^{H-2}}{\varepsilon}\Big)}\,\mathrm{d}\varepsilon\\ &\quad + K\int_{c'_H k^{H-2}}^{C'_H k^{H-1}}\sqrt{\log(\mathfrak{N}_n+\mathfrak{N}_m)}\,\mathrm{d}\varepsilon\\ &\le k^{H-1}c''_{\kappa,H}\sqrt{\log(n+m)}\,.\end{aligned}$$

So

$$\begin{aligned}\sum_{k=1}^m \mathbb{E}\sup_{\substack{|x|\le\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+i,n+i+1]}} Y_n^x(u) &\le c''_{\kappa,H}\sqrt{\log(n+m)}\sum_{k=1}^m k^{H-1}\\ &\le c''_{\kappa,H}m^H\sqrt{\log(n+m)}\end{aligned}$$

In the same way we have

$$\sum_{k=1}^m \mathbb{E}\sup_{\substack{|x|\le\mathfrak{N}_n+\mathfrak{N}_n\\ u\in[n+i,n+i+1]}} \widetilde{Y}_n^x(u)\le c''_{\kappa,H}m^H\sqrt{\log(n+m)}\,.$$

As we additionally have $\mathbb{P}(\mathscr{C})=e^{-\kappa}$, we obtain

$$\widehat{\epsilon}(n,m)\le c_{\kappa,H}m^H\sqrt{\log(n+m)}\,.$$

□

Proof of Theorem 3.5.1. Applying the above lemma we can easily see that $\{\widehat{U}(n-1)\}_{n\in\mathbb{N}}$ is almost-super-additive with respect to $\epsilon(n):=c_{\kappa,H}n^H\sqrt{\log(n)}$. Then theorem 3.5.2 implies that $\{\frac{\widehat{U}(n-1)}{n}\}_{n\in\mathbb{N}}$ converges to some positive extended real number and hence so does $\{\frac{\widehat{U}(n)}{n}\}_{n\in\mathbb{N}}$. □

3.6 Quenched limits

In this section we consider the quenched limits.
We recall the notations

$$u(t) = \mathbb{E}^X\left[e^{\int_0^t \mathrm{d}B_s^{X(s)}}\right],$$

$$\widehat{u}(t) := \mathbb{E}^X\left[e^{\int_0^t \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_t}\right],$$

and

$$\widehat{U}(t) := \mathbb{E}\log\widehat{u}(t)$$

where $\mathscr{A}_t$, as in previous sections, denotes the event that the random walk has at most $\mathfrak{N}_t$ jumps in the time interval $[0,t]$.
In the first proposition we show that the convergence of $\{\frac{\widehat{U}(n)}{n}\}_{n\in\mathbb{N}}$ to strictly positive λ implies the convergence of $\{\frac{\log\widehat{u}(n)}{n}\}_{n\in\mathbb{N}}$ to λ. Then in the second proposition we show that this result in its turn implies that $\{\frac{\log u(t)}{t}\}_{t\in\mathbb{R}^+}$ converges to λ as t goes off to $+\infty$.

Proposition 3.6.1. *For any real positive function $f:\mathbb{R}^+\to\mathbb{R}^+$ that grows at least as fast as a linear function we have*

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}\left(\frac{\widehat{U}(n)}{f(n)}-\frac{\log\widehat{u}(n)}{f(n)}\right)=0.$$

Proof. We will apply theorem 1.3.2 which provides concentration bounds on Malliavin derivable random variables.
For $X(\cdot)$, an arbitrary but fixed sample path of the random walk and $t\in\mathbb{R}$, let $g_t^X:\mathbb{R}\times\mathbb{Z}^d\longrightarrow\mathbb{R}$ be the function defined as

$$g_t^X(s,x) := \mathbf{1}_{[0,t]}(s)\,\mathbf{1}_{X(s)}(x).$$

With the notions introduced in section 1.2 it can be easily seen that g_t^X is in $\mathscr{H}$ and moreover

$$\mathbf{B}(g_t^X) = \int_0^t \mathrm{d}B_s^{X(s)},$$

which shows that

$$\nabla\int_0^t \mathrm{d}B_s^{X(s)} = g_t^X.$$

Hence we have

$$\nabla\widehat{u}(n) = \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_n}\,g_n^X\right]$$

and

$$\nabla\Big(\log\widehat{u}(n)\Big) = \frac{1}{\widehat{u}(n)}\nabla\widehat{u}(n) = \frac{1}{\widehat{u}(n)}\mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_n}\,g_n^X\right].$$

For $X_1(\cdot)$ and $X_2(\cdot)$, independent random walks having the same law as $X(\cdot)$, we have

$$\begin{aligned}
||\nabla \widehat{u}(n)||^2_{\mathscr{H}} &= \Big\langle \mathbb{E}^X\big[e^{\int_0^n \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_n}\, g_n^X\big], \mathbb{E}^X\big[e^{\int_0^n \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_n}\, g_n^X\big]\Big\rangle_{\mathscr{H}} \\
&= \Big\langle \mathbb{E}^{X_1}\big[e^{\int_0^n \mathrm{d}B_s^{X_1(s)}}\mathbf{1}_{\mathscr{A}_n^1}\, g_n^{X_1}\big], \mathbb{E}^{X_2}\big[e^{\int_0^n \mathrm{d}B_s^{X_2(s)}}\mathbf{1}_{\mathscr{A}_n^2}\, g_n^{X_2}\big]\Big\rangle_{\mathscr{H}} \\
&= \mathbb{E}^{X_1}\mathbb{E}^{X_2}\Big[e^{\int_0^n \mathrm{d}B_s^{X_1(s)}}\mathbf{1}_{\mathscr{A}_n^1}\, e^{\int_0^n \mathrm{d}B_s^{X_2(s)}}\mathbf{1}_{\mathscr{A}_n^2}\, \langle g_n^{X_1}, g_n^{X_2}\rangle_{\mathscr{H}}\Big] \\
&\le \mathbb{E}^{X_1}\mathbb{E}^{X_2}\Big[e^{\int_0^n \mathrm{d}B_s^{X_1(s)}}\mathbf{1}_{\mathscr{A}_n^1}\, e^{\int_0^n \mathrm{d}B_s^{X_2(s)}}\mathbf{1}_{\mathscr{A}_n^2}\, ||g_n^{X_1}||_{\mathscr{H}}||g_n^{X_2}||_{\mathscr{H}}\Big] \\
&\le \Big(\mathbb{E}^X\Big(e^{\int_0^n \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_n}||g_n^X||_{\mathscr{H}}\Big)\Big)^2.
\end{aligned}$$

But we have

$$||g_n^X||^2_{\mathscr{H}} = \mathbb{E}\Big(\int_0^n \mathrm{d}B_s^{X(s)}\Big)^2.$$

So for $H > 1/2$ we have

$$||g_n^X||^2_{\mathscr{H}} \le n^{2H},$$

and for $H \le 1/2$ and under $\mathscr{A}_n$

$$||g_n^X||^2_{\mathscr{H}} \le \mathfrak{N}_n\Big(\frac{n}{\mathfrak{N}_n}\Big)^{2H} \le n(\beta\kappa)^{1-2H}.$$

The fact that $||g_n^X||_{\mathscr{H}}$ has an upper bound that doesn't depend on the random walk leads to the following bound

$$||\nabla\Big(\log \widehat{u}(n)\Big)||^2 \le ||g_n^X||^2_{\mathscr{H}}.$$

So by theorem 1.3.2 we have

$$\mathrm{P}\Big(|\log \widehat{u}(n) - \widehat{U}(n)| > 2n^H\sqrt{\log n}\Big) \le 2e^{-2\log n} = 2n^{-2}.$$

As the right-hind-side of this inequality is summable we can apply Borel-Cantelli lemma to conclude that almost surely there exists N such that for any $n \in \mathbb{N}$ with $n \ge N$ we have

$$|\log \widehat{u}(n) - \widehat{U}(n)| \le 2n^H\sqrt{\log n},$$

which along with the assumption on the growth rate of $f(\cdot)$ implies the almost sure limit

$$\lim_{n\to\infty} \frac{\log \widehat{u}(n)}{f(n)} - \frac{\widehat{U}(n)}{f(n)} = 0.$$

□

Proposition 3.6.2. *For any real positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ that grows at least as fast as a linear function we have*

$$\limsup_{t\to\infty} \frac{\log u(t)}{f(t)} = \limsup_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{\log \widehat{u}(n)}{f(n)},$$

and

$$\liminf_{t\to\infty}\frac{\log u(t)}{f(t)}=\liminf_{\substack{n\to\infty\\ n\in\mathbb{N}}}\frac{\log\widehat{u}(n)}{f(n)}.$$

Proof. For $l,n\in\mathbb{N}$, let $\{t_i\}_{i=1}^{l}$ be the l uniformly spaced points on the interval $(n-1,n)$. It is evident that for any $x\in\mathbb{Z}^d$ and for any $t\in[n-1,n]$, there exists a t_i with $|t-t_i|\le\frac{1}{2l}$. Then we have

$$\mathbb{E}\Big((B_t^x-B_n^x)-(B_{t_i}^x-B_n^x)\Big)^2=\mathbb{E}\Big(B_t^x-B_{t_i}^x\Big)^2=\frac{1}{(2l)^{2H}}.$$

So for $0<\varepsilon<2^{-H}$ we can cover the set $\{B_t^x-B_n^x\,;\ \ t\in[n-1,n]\}$ by $l=\frac{1}{2\varepsilon^{1/H}}$ ε-balls and for $2^{-H}\le\varepsilon$ the whole set can be covered by a single element. So by Dudley's theorem we have

$$\mathbb{E}\Big(\sup_{n-1\le t\le n}(B_t^x-B_n^x)\Big)\le K\int_0^{2^{-H}}\sqrt{\log\frac{1}{2\varepsilon^{1/H}}}=K_1,$$

where K and K_1 are some universal constants.
We also have $\mathbb{E}(B_t^x-B_n^x)^2\le1$ for every $t\in[n-1,n]$. So by Borell's inequality 1.3.5, for any $k\in\mathbb{N}_0$ and any n large enough we have

$$\begin{aligned}\mathrm{P}\Big(\sup_{n-1\le t\le n}&(B_t^x-B_n^x)\ge(k+2)(d+1)\log n\Big)\\&\le e^{-2(k+2)(d+1)\log n}=n^{-2(k+2)(d+1)}.\end{aligned}$$

So

$$\begin{aligned}\mathrm{P}\Big(\bigcup_{|x|\le\mathfrak{N}_n n^k}\{\sup_{n-1\le t\le n}&(B_t^x-B_n^x)\ge(k+2)(d+1)\log n\}\Big)\\&\le(2\mathfrak{N}_n n^k+1)^d n^{-(k+2)(d+1)}\le n^{-(k+2)},\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{P}\Big(\bigcup_{k\in\mathbb{N}_0}\bigcup_{|x|\le\mathfrak{N}_n n^k}\{\sup_{n-1\le t\le n}&(B_t^x-B_n^x)\ge(k+2)(d+1)\log n\}\Big)\\&\le\sum_k n^{-(k+2)}\le 2n^{-2}.\end{aligned}$$

By Borel-Cantelli lemma, almost surely there exists N_1 such that for any $n\ge N_1$ and for every $k\in\mathbb{N}_0$ we have

$$\sup_{|x|\le\mathfrak{N}_n n^k}\ \sup_{n-1\le t\le n}(B_t^x-B_n^x)\le(k+2)(d+1)\log n$$

which is equivalent to

$$\inf_{|x|\le\mathfrak{N}_n n^k}\ \inf_{n-1\le t\le n}(B_n^x-B_t^x)\ge-(k+2)(d+1)\log n$$

For any $t\in\mathbb{R}^+$ and $k\in\mathbb{N}_0$, let $\mathscr{A}_{t,k}$ be the event that the number of jumps of the random walk on $[0,t]$ is larger than $\mathfrak{N}_n n^k$ but less than $\mathfrak{N}_n n^{k+1}$, where $n:=\lceil t\rceil$ is the smallest integer not less than t. We use the following notations

$$\widehat{u}_k(t) := \mathbb{E}^X\left[e^{\int_0^t \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_{t,k}}\right],$$

and

$$\widehat{u}(t) := \mathbb{E}^X\left[e^{\int_0^t \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{A}_t}\right].$$

For $H > 1/2$ we have

$$\mathbb{E}\widehat{u}_k(n) = \mathbb{E}^X\left[\mathbf{1}_{\mathscr{A}_{n,k}}\mathbb{E}e^{\int_0^n \mathrm{d}B_s^{X(s)}}\right] \le \mathrm{P}(\mathscr{A}_{n,k})e^{\frac{1}{2}n^{2H}}$$

As in this case $\mathfrak{N}_n = n^2$, by the Poisson tail probability bound 3.3 we have

$$\mathrm{P}(\mathscr{A}_{n,k}) \le (\frac{e\kappa n}{n^{k+2}})^{n^{k+2}}.$$

For $H \le 1/2$, where $\mathfrak{N}_n = \beta\kappa n$, we have

$$\begin{aligned}\mathbb{E}\widehat{u}_k(n) = \mathbb{E}^X\left[\mathbf{1}_{\mathscr{A}_{n,k}}\mathbb{E}e^{\int_0^n \mathrm{d}B_s^{X(s)}}\right] &\le \mathbb{E}\left[\mathbf{1}_{\mathscr{A}_{n,k}}e^{\frac{1}{2}J(\frac{n}{J})^{2H}}\right]\\ &\le \mathrm{P}(\mathscr{A}_{n,k})e^{\frac{1}{2}\mathfrak{N}_n n^{k+1}(\frac{n}{\mathfrak{N}_n n^{k+1}})^{2H}},\end{aligned}$$

where J is the number of jumps of the random walk on $[0, n]$.
For this case again by the Poisson tail probability bound 3.3 we have

$$\mathrm{P}(\mathscr{A}_{n,k}) \le (\frac{e\kappa n}{\beta\kappa n^{k+1}})^{\beta\kappa n^{k+1}}.$$

So in both the cases, for n large enough and every $k \in \mathbb{N}_0$ we have

$$\mathbb{E}\widehat{u}_k(n) \le e^{-2n^{k+2}}.$$

So by Markov's inequality for n large enough and every $k \in \mathbb{N}_0$ we easily get

$$\mathrm{P}\Big(\widehat{u}_k(n) \ge e^{-n^{k+2}}e^{-(k+1)(d+1)\log n}\Big) \le n^{-(k+2)},$$

and hence

$$\mathrm{P}\Big(\bigcup_{k\in\mathbb{N}_0}\{\widehat{u}_k(n) \ge e^{-n^{k+2}}e^{-(k+1)(d+1)\log n}\}\Big) \le 2n^{-2}.$$

As the right hand side of this inequality is summable, Borel-Cantelli lemma implies that almost surely there exists N_2 such that for any $n \ge N_2$ and for any $k \in \mathbb{N}_0$ we have

$$\widehat{u}_k(n) \le e^{-n^{k+2}}e^{-(k+1)(d+1)\log n}.$$

Using the same technic as above we can easily show that almost surely there exists N_3 such that for any $n \ge N_3$ we have

$$\inf_{|x|\le\mathfrak{N}_n}\inf_{n-1\le t\le n}(B_t^x - B_{n-1}^x) \ge -\log n.$$

For $t_1, t_2 \in \mathbb{R}^+$ let $\mathscr{C}_{t_1,t_2}$ be the event that the random walk has no jump in the time interval $[t_1, t_2]$.
For any $k \in \mathbb{N}_0$ and any integer $n \geq \max\{N_1, N_2, N_3\}$ We have

$$\begin{aligned}
\widehat{u}_k(n) &\geq \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_{t,k}} \mathbf{1}_{\mathscr{C}_{t,n}}\right] \\
&\geq e^{\inf_{|x| \leq \mathfrak{R}_n n^k} \inf_{n-1 \leq t \leq n}(B_n^x - B_t^x)} \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_{t,k}} \mathbf{1}_{\mathscr{C}_{t,n}}\right] \\
&\geq e^{-(k+2)(d+1)\log n} \mathbb{E}^X\left[e^{\int_0^n \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_{t,k}} \mathbf{1}_{\mathscr{C}_{t,n}}\right] \\
&= e^{-(k+2)(d+1)\log n}\, \mathrm{P}(\mathscr{C}_{t,n})\, \widehat{u}_k(t)
\end{aligned}$$

hence

$$\widehat{u}_k(t) \leq e^{\kappa} e^{(k+2)(d+1)\log n}\, \widehat{u}_k(n) \quad \leq e^{-n^{k+2}} e^{\kappa} \leq e^{\kappa} e^{-n^2(k+1)}\,.$$

In the same way we have

$$\widehat{u}(t) \leq e^{\kappa} e^{(k+2)(d+1)\log n}\, \widehat{u}(n) = e^{\kappa} n^{(k+2)(d+1)}\, \widehat{u}(n)\,,$$

and

$$\widehat{u}(t) \geq e^{\kappa} e^{-\log n}\, \widehat{u}(n-1)\,.$$

So using the inequality $\log(\alpha+1) \leq \alpha$ we have

$$\begin{aligned}
\kappa - \log n + \log \widehat{u}(n-1) \leq \log u(t) &= \log\Big(\widehat{u}(t) + \sum_{k=0}^{\infty} \widehat{u}_k(t)\Big) \\
&\leq \log\Big(e^{\kappa} n^{(k+2)(d+1)}\, \widehat{u}(n) + e^{\kappa} \sum_{k=0}^{\infty} e^{-n^2(k+1)}\Big) \\
&\leq \log\Big(e^{\kappa} n^{(k+2)(d+1)}\, \widehat{u}(n) + e^{\kappa} e^{-n^2}\Big) \\
&\leq \log \widehat{u}(n) + \Delta_n
\end{aligned}$$

where

$$\Delta_n := \kappa + (k+2)(d+1)\log n + n^{-(k+2)(d+1)}\, \widehat{u}(n)^{-1} e^{-n^2}\,.$$

This, along with the fact that $\{\frac{\log \widehat{u}(n)}{n}\}_n$ converges to some strictly positive number (possibly $+\infty$ for $H > 1/2$), proves the assertion of the proposition for any positive function $f(t)$ growing at least as fast as the identity function. □

3.7 Lower Bound

In this section we prove the positivity of $\lambda = \lim \frac{\widehat{U}(n)}{n}$ for any $H \in (0,1)$ and any κ. This is a much stronger result than what has been proved in [50] where they prove the positivity of λ under quite strong conditions on the covariance structure of the fBM's and only when $H \in (H_o, 1/2]$ and $\kappa < \kappa_o$ for some H_o and κ_o. Although we assume the independence of the fractional Brownian motions associated to different sites of $\mathbb{Z}^d$, our argument for the proof of the following theorem holds true for much more general setting on the spatial covariance structure.

Theorem 3.7.1. $\lambda = \lim \frac{\widehat{U}(n)}{n}$ *is strictly positive for any* $H \in (0,1)$ *and any* κ.

The following well-known lemma (see for example [13, 19]), which is a corollary to the reflection principle, shows that the probability on a simple random walk started from the origin, of returning to the origin for the first at time $2m$, decays only polynomially in m, in contrast to the first impression that it would decay exponentially.

Lemma 3.7.2 (First return to the origin)**.** *Let* $\{S_n\}_n$ *be a discrete-time random walk on* $\mathbb{Z}$ *starting off the origin, i.e.* $S_n = \sum_{k=1}^n X_k$ *where* $X_i \in \{-1,+1\}$ *and* $S_0 = 0$. *Let* ν_{2m} *be number of different ways for the random walk to visit the origin for the first time at time* $2m$, *i.e.* $S_{2m} = 0$ *but* $S_k \neq 0$ *for any* $k \in \{1,\cdots,2m-1\}$. *We have*

$$\nu_{2m} = \frac{1}{2m-1}\binom{2m}{m}$$

Proof of Theorem 3.7.1. For the d-dimensional simple random walk $X(\cdot)$ on $\mathbb{Z}^d$, Let π_i be the projection to the i-th coordinate; In other words if $X = (x_i)_i$, then for each i we have $x_i := \pi_i \text{o} X$.

Let $T := 2md/\kappa$ with $m \in \mathbb{N}$. For any $k \in \mathbb{N}_0$, let $\mathscr{B}_k$ be the event that the random walk $X(\cdot)$ has the following property: for each $i \in \{1,\cdots,d\}$, the i-th projection, i.e. $\pi_i \text{o} X$ be zero at time kT, make $2m$ jumps in the time interval $\big(kT,(k+1)T\big)$ and at its $2m$-th jump returns to the origin for the first time. It is clear that then for each i, $\pi_i \text{o} X$ doesn't change sign in the time interval $\big(kT,(k+1)T\big)$.

We have

$$\frac{\widehat{U}(nT)}{nT} \geq \frac{1}{nT}\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_0^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=0}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big).$$

But

$$\begin{aligned}
\mathbb{E}^X\Big(e^{\int_0^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=0}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big) &= \mathrm{P}\big(X(T)=0\big)\mathbb{E}^X\Big(e^{\int_0^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=0}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big| X(T)=0\Big)\\
&= \mathrm{P}\big(X(T)=0\big)\mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big| X(T)=0\Big)\mathbb{E}^X\Big(e^{\int_T^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=1}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big| X(T)=0\Big)\\
&= \mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big)\mathbb{E}^X\Big(e^{\int_T^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=1}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big| X(T)=0\Big).
\end{aligned}$$

Continuing this procedure, by induction we have

$$\mathbb{E}^X\Big(e^{\int_0^{nT}\mathrm{d}B_s^{X(s)}}\prod_{k=0}^{n-1}\mathbf{1}_{\mathscr{B}_k}\Big)=\sum_{k=0}^{n-1}\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_{kT}^{(k+1)T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_k}\,\Big|\,X(kT)=0\Big).$$

So we have

$$\begin{aligned}\frac{\widehat{U}(nT)}{nT}&\geq\frac{1}{nT}\sum_{k=0}^{n-1}\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_{kT}^{(k+1)T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_k}\,\Big|\,X(kT)=0\Big)\\&=\frac{1}{T}\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big)\end{aligned}$$

where we have used the time invariance of the random walk and the random environment, i.e. the fBM's.
Taking the limit when n goes to ∞ we obtain

$$\lambda\geq\frac{1}{T}\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big).$$

So it suffices to show the positivity of the right-hand-side of this inequality.

Let $\mathfrak{D}$ be the set of all possible paths of a $2md$-step discrete-time random walk on $\mathbb{Z}^d$ started at the origin with the property that their projections over each coordinate make exactly $2m$ jumps and at its $2m$-th jump returns to the zero for the first time. As $\mathscr{B}_0$ is an event that concerns only the number of jumps and the positions of the random walk and not its jump times, conditional on the number of jumps it is independent of the jump times. Let $\mathbb{E}^{\mathbf{t}}$ be the expectation with respect to the jump times conditioned on the event that number of jumps are $2md$, i.e. the expectation with respect to the jump times $t_1,\cdots,t_{2md}$ which are uniformly distributed on $(0,T)$. Let also p_m be the probability that a simple random walk has $2md$ jumps in the time interval $[0,T]$.
We have

$$\mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big)=p_m\frac{1}{(2d)^{2md}}\sum_{j\in\mathfrak{D}}\mathbb{E}^{\mathbf{t}}\Big(e^{\int_0^{T}\mathrm{d}B_s^{X_j(s)}}\Big).$$

Where X_j represents the paths of the continuous-time random walk whose position path is the same as $j\in\mathfrak{D}$. For each path j in $\mathfrak{D}$ it is evident that $-j\in\mathfrak{D}$. So let $\mathfrak{D}/2$ be a subset of $\mathfrak{D}$ with the property that from each pair $(j,-j)$ contains only one; In other words it is the equivalence class of $\mathfrak{D}$ under the relation $j\sim i\iff j=\pm i$. Then we have

$$\mathbb{E}^X\Big(e^{\int_0^{T}\mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big)=p_m\frac{1}{(2d)^{2md}}\sum_{j\in\mathfrak{D}/2}\mathbb{E}^{\mathbf{t}}\Big(e^{\int_0^{T}\mathrm{d}B_s^{X_j(s)}}+e^{\int_0^{T}\mathrm{d}B_s^{-X_j(s)}}\Big),$$

hence

$$
\begin{aligned}
&\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_0^T \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big)\\
&\qquad = \log p_m + \mathbb{E}\log\frac{1}{(2d)^{2md}}\sum_{j\in\mathfrak{D}/2}\mathbb{E}^{\mathbf{t}}\Big(e^{\int_0^T \mathrm{d}B_s^{X_j(s)}} + e^{\int_0^T \mathrm{d}B_s^{-X_j(s)}}\Big)\\
&\qquad \geq \log p_m + \frac{2}{|\mathfrak{D}|}\sum_{j\in\mathfrak{D}/2}\mathbb{E}^{\mathbf{t}}\mathbb{E}\log\frac{|\mathfrak{D}|}{(2d)^{2md+1}}\Big(e^{\int_0^T \mathrm{d}B_s^{X_j(s)}} + e^{\int_0^T \mathrm{d}B_s^{-X_j(s)}}\Big)\\
&\qquad = \log p_m + \log\frac{|\mathfrak{D}|}{(2d)^{2md+1}} + \frac{2}{|\mathfrak{D}|}\sum_{j\in\mathfrak{D}/2}\mathbb{E}^{\mathbf{t}}\mathbb{E}\log\Big(e^{\int_0^T \mathrm{d}B_s^{X_j(s)}} + e^{\int_0^T \mathrm{d}B_s^{-X_j(s)}}\Big).
\end{aligned}
$$

If $Y_1 := \int_{t_1}^{t_{2md}} \mathrm{d}B_s^{X_j(s)}$ and $Y_2 := \int_{t_1}^{t_{2md}} \mathrm{d}B_s^{-X_j(s)}$ we have

$$
\begin{aligned}
e^{\int_0^T \mathrm{d}B_s^{X_j(s)}} + e^{\int_0^T \mathrm{d}B_s^{-X_j(s)}} &= e^{\int_0^{t_1} \mathrm{d}B_s^{X_j(s)} + \int_{t_{2md}}^T \mathrm{d}B_s^{X_j(s)}}\,(e^{Y_1}+e^{Y_2})\\
&\geq e^{\int_0^{t_1} \mathrm{d}B_s^{X_j(s)} + \int_{t_{2md}}^T \mathrm{d}B_s^{X_j(s)}}\,e^{\max\{Y_1,Y_2\}}.
\end{aligned}
$$

As Y_1 and Y_2 are independent identically distributed zero-mean normal random variables we have

$$
\mathbb{E}\max\{Y_1,Y_2\} = \mathbb{E}\Big(\frac{|Y_1-Y_2|+Y_1+Y_2}{2}\Big) = \frac{\sigma}{\sqrt{\pi}},
$$

where σ^2 is the variance of Y_1. So we have

$$
\mathbb{E}^{\mathbf{t}}\mathbb{E}\log\Big(e^{\int_0^T \mathrm{d}B_s^{X_j(s)}} + e^{\int_0^T \mathrm{d}B_s^{-X_j(s)}}\Big) \geq \mathbb{E}^{\mathbf{t}}(\sigma/\sqrt{\pi}).
$$

Let $\Delta := t_1 + (T - t_{2md})$, i.e. the total amount of time that the random walk spends at the origin during the time interval $[0, T]$. As $t_1, \cdots t_{2md}$ are uniformly distributed in $(0, T)$, it is clear that $\mathbb{E}^{\mathbf{t}}(\Delta) = 2\frac{T}{2md+1}$.
When $H \leq 1/2$, as the increments are negatively correlated, staying in a single site gives a lower bound on the variance, i.e. $\sigma^2 \geq (T-\Delta)^{2H}$. But for any $0 \leq \alpha \leq T$, we have $\alpha^H \geq \left(\frac{\alpha}{T}\right) T^H$. This shows that in this case we have $\sigma \geq \left(\frac{T-\Delta}{T}\right) T^H$ and hence

$$
\mathbb{E}^{\mathbf{t}}(\sigma) \geq \mathbb{E}^{\mathbf{t}}\Big(\frac{T-\Delta}{T}\Big)\,T^H = \frac{2md-1}{2md+1}\,T^H \asymp m^H.
$$

When $H > 1/2$, as the increments are positively correlated, visiting every site for no more than once gives a lower bound on the variance, i.e. $\sigma^2 \geq \sum_{i=2}^{2md}(t_i - t_{i-1})^{2H}$ and hence

$$\begin{aligned}
\mathbb{E}^{\mathbf{t}}(\sigma) &\geq \mathbb{E}^{\mathbf{t}}\sqrt{\sum_{i=2}^{2md}(t_i - t_{i-1})^{2H}} \\
&\geq \mathbb{E}^{\mathbf{t}}\sqrt{(2md-1)^{1-2H}\Big(\sum_{i=2}^{2md}(t_i - t_{i-1})\Big)^{2H}} \\
&= (2md-1)^{1/2-H}\,\mathbb{E}^{\mathbf{t}}\Big(\sum_{i=2}^{2md}(t_i - t_{i-1})\Big)^{H} \\
&\geq (2md-1)^{1/2-H}\,\mathbb{E}^{\mathbf{t}}\Big(\frac{\sum_{i=2}^{2md}(t_i - t_{i-1})}{T}\Big)\,T^H \\
&\geq (2md-1)^{1/2-H}\Big(\frac{2md-1}{2md+1}\Big)\,T^H \asymp \sqrt{m}.
\end{aligned}$$

where we have used $\alpha^H \geq \left(\frac{\alpha}{T}\right)T^H$ which is true for any $\alpha \geq 0$ and $0 < H < 1$, and also the following inequality that is easily proved by Hölder's inequality and holds for any $H \geq 1/2$ and $\alpha_i \geq 0$, $i = 1, \cdots, N$

$$\sum_{i=1}^{N}\alpha_i^{2H} \geq N\Big(\frac{1}{N}\sum_{i=1}^{N}\alpha_i\Big)^{2H}.$$

Hence we have shown that

$$\mathbb{E}\log\mathbb{E}^X\Big(e^{\int_0^T \mathrm{d}B_s^{X(s)}}\mathbf{1}_{\mathscr{B}_0}\Big) \geq \log p_m + \log\frac{|\mathfrak{D}|}{(2d)^{2md+1}} + Cm^{\gamma},$$

where C is some positive constant and $\gamma := 1/2$ for $H > 1/2$ and $\gamma := H$ for $H \leq 1/2$.
P_m is the probability that a Poisson random variable with average $\kappa T = 2md$ hast $2md$ jumps. So by Stirling formula (1.3.6) we have

$$p_m = e^{-2md}\frac{(2md)^{2md}}{(2md)!} \geq \frac{1}{2e\sqrt{\pi md}}$$

hence

$$\log p_m \asymp -\log m.$$

For determining $|\mathfrak{D}|$, first we notice that there are $\binom{2md}{2m\cdots 2m} = \frac{(2md)!}{(2m)!^d}$ different ways of distributing the $2md$ jumps uniformly between the d coordinates. For each $i = 1, \cdots, d$, there are ν_{2m} different possible excursions for $\pi_i oX$ such that it starts from zero, makes $2d$ jumps and at its $2d$-th jump returns to zero for the first time. So we have

$$|\mathfrak{D}| = \frac{(2md)!}{(2m)!^d}\nu_{2m}^d = \frac{(2md)!}{(2m)!^d}\frac{(2m)!^d}{(m)!^{2d}}\frac{1}{(2m-1)^d} = \frac{(2md)!}{(m)!^{2d}}\frac{1}{(2m-1)^d}.$$

By Stirling's formula we have

$$\frac{(2md)!}{(m)!^{2d}} \asymp \frac{(2d)^{2md}}{m^{2d-1/2}},$$

and hence

$$\log \frac{|\mathfrak{D}|}{(2d)^{2md+1}} \asymp -\log m\,.$$

This shows that

$$\mathbb{E} \log \mathbb{E}^X \left(e^{\int_0^T \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{B}_0} \right) \geq -C_1 \log m + C m^\gamma\,,$$

which guarantees the positivity of this expression for m large enough and hence completing proof. □

3.8 Upper Bounds

In this section we will establish upper bounds on $\widehat{U}(T)$. For $H \le 1/2$ this upper bound is linear in T which shows that λ is finite. For $H \ge 1/2$ the problem is much more complicated and what we have been able to show is that $\widehat{U}(T)$ and hence $U(T)$ grow slower than $T\sqrt{\log(T)}$. This is in contrast to a result of Viens and Zhang in [50] asserting that $U(T)$ grows faster than $\frac{T^{2H}}{\log T}$.

Our arguments hold true for much more general spatial covariance structures than independent fractional Brownian motions associated to each site of $\mathbb{Z}^d$.

Theorem 3.8.1. *For $H \le 1/2$, the limit* $\lim_{T\to\infty} \frac{\widehat{U}(T)}{T} = \lambda$ *is finite.*

Proof. By convexity of log and using Jensen's inequality and then by the negative correlation of the fBMs' increments we have

$$\begin{aligned}\widehat{U}(T) &\le \log \mathbb{E}^X \Big(\mathbb{E}\, e^{\int_0^T \mathrm{d}B_s^{X(s)}} \mathbf{1}_{\mathscr{A}_T} \Big)\\ &= \log \mathbb{E}^X \Big(e^{\frac{1}{2}\, var(\int_0^T \mathrm{d}B_s^{X(s)})} \mathbf{1}_{\mathscr{A}_T} \Big)\\ &\le \log \mathbb{E}^X \Big(e^{\frac{1}{2} \sum_{i=0}^n (t_{i+1}-t_i)^{2H}} \mathbf{1}_{\mathscr{A}_T} \Big),\end{aligned}$$

where $\{t_i\}_i$ are the jump times of the random walk $X(\cdot)$ in $(0,T)$, including the end points, and n is the number of jumps. Then as the function $(\cdot)^{2H}$ would be concave, by Jensen's inequality we have

$$\frac{1}{n+1}\sum_i (\triangle t_i)^{2H} \le \Big(\frac{\sum_i \triangle t_i}{n+1}\Big)^{2H} = \Big(\frac{T}{n+1}\Big)^{2H}.$$

But as under the event $\mathscr{A}_T$ the number of jumps is smaller than $\mathfrak{N}_T = \beta T$, we have

$$\begin{aligned}\widehat{U}(T) &\le \log \mathbb{E}^X \Big(e^{\frac{1}{2}(n+1)^{1-2H} T^{2H}} \mathbf{1}_{\mathscr{A}_T} \Big)\\ &\le \log \mathbb{E}^X \Big(e^{\frac{1}{2}(\beta T+1)} \mathbf{1}_{\mathscr{A}_T} \Big)\\ &\le \frac{1}{2}(\beta T + 1).\end{aligned}$$

This shows that $\lambda = \lim_{T\to\infty} \frac{\widehat{U}(T)}{T}$ is finite. □

When $H > 1/2$, we apply a more elaborate method.

Theorem 3.8.2. *For $H > 1/2$, we have $\widehat{U}(n) \preccurlyeq n\sqrt{\log n}$.*

Proof. We chop up the interval $[0,n]$ into n subintervals and decompose each integral $\int_l^{l+1} \mathrm{d}B_s^{X(s)}$ into two parts: the residue part, that comes from the Brownian motions up to time $l-1$ and the innovation part that comes from the Brownian motions in the interval $[l-1, l+1]$. We expect the innovation part to be the main contribution to the integral, and the residue part to be reasonably small.

We begin by the Volterra representation (1.1) of a fBM. For $l \in \mathbb{N}^{\geq 2}$ and $l \leq t_1 < t_2 \leq l+1$, we have

$$B_{t_2} - B_{t_1} = \int_0^{l-1} \Big(K_H(t_2, s) - K_H(t_1, s)\Big) \mathrm{d}W_s + Z_{t_2} - Z_{t_1}, \tag{3.16}$$

where

$$Z_t := \int_{l-1}^{t} K_H(t,s) \mathrm{d}W_s. \tag{3.17}$$

For $0 \leq t \leq 2$ we also define Z_t by

$$Z_t := \int_0^t K_H(t,s) \mathrm{d}W_s. \tag{3.18}$$

Applying the stochastic Fubini theorem 1.3.3 to the first right-hand-side term of (3.16) we have

$$\begin{aligned} \int_0^{l-1} \Big(K_H(t_2, s) - K_H(t_1, s)\Big) \mathrm{d}W_s &= c_H \int_0^{l-1} \int_{t_1}^{t_2} (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \, \mathrm{d}u \, \mathrm{d}W_s \\ &= \int_{t_1}^{t_2} Y_l(u) \, \mathrm{d}u, \end{aligned} \tag{3.19}$$

where

$$Y_l(u) := c_H \int_0^{l-1} (u-s)^{H-\frac{3}{2}} (\frac{u}{s})^{H-\frac{1}{2}} \, \mathrm{d}W_s. \tag{3.20}$$

Applying this procedure to the family $\{B^x\}_{x \in \mathbb{Z}^d}$, there exists a family of independent standard Brownian motions $\{W^x\}_{x \in \mathbb{Z}^d}$ such that

$$B^x(t) = \int_0^t K_H(t,s) \, W^x(\mathrm{d}s).$$

So for each site $x \in \mathbb{Z}^d$, the processes Y_l^x and Z^x can be defined as above.

Back to the integral $\int_0^n \mathrm{d}B_s^{X(s)}$, it can be easily verified that

$$\int_0^n \mathrm{d}B_t^{X(t)} = \int_0^n \mathrm{d}Z_t^{X(t)} + \sum_{l=2}^{n-1} \int_l^{l+1} Y_l^{X(t)}(t) \mathrm{d}t. \tag{3.21}$$

Our goal is to show that in some sense the first term grows linearly in n and the second term grows no faster than $n\sqrt{\log n}$.

By adding and subtracting a reasonably small artificial term to $\int_0^n \mathrm{d}Z_t^{X(t)}$ we may turn it into a summation of mostly independent terms and hence getting a linear upper bound.
Indeed, let $\{\widetilde{W}^{l,x}\}_{x \in \mathbb{Z}^d, \, l \in \mathbb{N}_0}$ be a family of independent standard Brownian motions, independent of any process introduced so far, in particular independent of the random walk $X(.)$,

the fractional Brownian motions $\{B^x\}_{x\in\mathbb{Z}^d}$ and hence their corresponding Brownian motions $\{W^x\}_{x\in\mathbb{Z}^d}$ appearing in their integral representation. For any $l\in\mathbb{N}^{\geq 2}$ and $x\in\mathbb{Z}^d$ define $\widehat{W}^{l,x}$ as

$$\widehat{W}^{l,x} := \begin{cases} \widetilde{W}^{l,x}_s & \text{for} \quad s\in[0,l-1] \\ W^x_s - W^x_{l-1} + \widetilde{W}^{l,x}_{l-1} & \text{for} \quad s\in(l-1,\infty)\,. \end{cases}$$

and for $l=0,1$, define $\widehat{W}^{l,x} := W^x$.

It is easily verified that $\widehat{W}^{l,x}$ is itself a standard Brownian motion and hence the following expression

$$\widehat{B}^{l,x}_t := \int_0^t K_H(t,s)\mathrm{d}\widehat{W}^{l,x}_s = \int_0^{l-1} K_H(t,s)\mathrm{d}\widetilde{W}^{l,x}_s + \int_{l-1}^t K_H(t,s)\mathrm{d}W^x_s\,, \tag{3.22}$$

is a fractional Brownian motion of Hurst parameter H.
Note also that for any $x\in\mathbb{Z}^d$ and $l\leq t<l+1$, we have

$$Z^x_t = \int_{l-1}^t K_H(t,s)\mathrm{d}W^x_s\,.$$

By the same procedure as in equations (3.16) through (3.20), for any $t\in[l,l+1)$ we have

$$\int_l^{l+1}\mathrm{d}Z^{X(t)}_t = \int_l^{l+1}\mathrm{d}\widehat{B}^{l,X(t)}_t - \int_l^{l+1}\widehat{Y}^{X(t)}_l(t)\mathrm{d}t\,,$$

where

$$\widehat{Y}^x_l(t) := c_H\int_0^{l-1}(u-s)^{H-\frac{3}{2}}(\frac{u}{s})^{H-\frac{1}{2}}\,\mathrm{d}\widetilde{W}^{l,x}_s \quad \text{for} \quad t\in[l,l+1)\,.$$

We therefore have

$$\int_0^n \mathrm{d}Z^{X(t)}_t = \sum_{l=0}^{n-1}\int_l^{l+1}\mathrm{d}\widehat{B}^{l,X(t)}_t - \sum_{l=2}^{n-1}\int_l^{l+1}\widehat{Y}^{X(t)}_l(t)\mathrm{d}t\,.$$

This along with (3.21) implies

$$\int_0^n \mathrm{d}B^{X(t)}_t = \sum_{l=0}^{n-1}\int_l^{l+1}\mathrm{d}\widehat{B}^{l,X(t)}_t - \sum_{l=2}^{n-1}\int_l^{l+1}\widehat{Y}^{X(t)}_l(t)\mathrm{d}t + \sum_{l=2}^{n-1}\int_l^{l+1}Y^{X(t)}_l(t)\mathrm{d}t\,.$$

So we have

$$\begin{aligned}\widehat{U}(n) &= \mathbb{E}\log\mathbb{E}\Big(e^{\int_0^n \mathrm{d}B^{X(t)}_t}\mathbf{1}_{\mathscr{A}_n}\Big) \\ &\leq \mathbb{E}\log\mathbb{E}e^{\sum_{l=0}^{n-1}\int_l^{l+1}\mathrm{d}\widehat{B}^{l,X(t)}_t} + \sum_{l=2}^{n-1}\mathbb{E}\Big(\sup_{\substack{|x|\leq n^2\\ l\leq u\leq l+1}}|\widehat{Y}^x_l(u)| + \sup_{\substack{|x|\leq n^2\\ l\leq u\leq l+1}}|Y^x_l(u)|\Big)\end{aligned} \tag{3.23}$$

First we find an upper bound on the first right-hand-side term. Here we need an easy observation. Let $\widetilde{\sigma}^l$ be the sigma field generated by $\{\widetilde{W}_s^{l,x}\ ;\ s \in [0, l-1]\ ,\ x \in \mathbb{Z}^d\}$ and σ^l be the sigma field generated by $\{W_s^x - W_{l-1}^x\ ;\ s \in (l-1, l+1]\ ,\ x \in \mathbb{Z}^d\}$. It is evident by (3.22) that for any $l \le t < l+1$ the process $\widehat{B}_t^{l,x}$ is measurable with respect to $\sigma^l \vee \widetilde{\sigma}^l$ where $\vee$ denotes the smallest sigma field containing the both. So $\int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}$ is also measurable with respect to $\sigma^l \vee \widetilde{\sigma}^l$. As $\sigma^l \vee \widetilde{\sigma}^l$ and $\sigma^k \vee \widetilde{\sigma}^k$ are independent for $|k-l| \ge 2$, this shows that $\int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}$ and $\int_k^{k+1} \mathrm{d}\widehat{B}_t^{l,X(t)}$ are independent for $|k-l| \ge 2$. Hence, using the inequality $\mathbb{E}XY \le \frac{1}{2}(\mathbb{E}X^2 + \mathbb{E}Y^2)$, we have

$$\operatorname{var} \sum_{l=0}^{n-1} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)} \le 3 \sum_{l=0}^{n-1} \operatorname{var} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}\,.$$

We also notice that

$$\operatorname{var} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)} \le 1\,,$$

which follows from the positive correlation of increments of fMB implying that the upper bound is obtained when the random walk stays in a single site on the whole time interval $[l, l+1)$. This can be equivalently deduced from (1.3) and the inequality $\sum_i \alpha_i^{2H} \le (\sum_i \alpha_i)^{2H}$ which is true for all $H \ge 1/2$ and $\alpha_i \ge 0$. Hence we have

$$\begin{aligned}
\mathbb{E}\Big(e^{\sum_{l=0}^{n-1} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}}\Big) &= e^{\frac{1}{2} \operatorname{var} \sum_{l=0}^{n-1} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}} \\
&\le e^{\frac{3}{2} \sum_{l=0}^{n-1} \operatorname{var} \int_l^{l+1} \mathrm{d}\widehat{B}_t^{l,X(t)}} \\
&\le e^{\frac{3}{2} n}\,,
\end{aligned}$$

Now turn to the second right-hand-side term of term equation (3.23).
Applying Dudley's theorem as stated in remark 1.3.2, for any $l \in \mathbb{N}^{\ge 2}$ we have

$$\mathbb{E}\Big(\sup_{\substack{|x| \le n^2 \\ l \le u \le l+1}} |Y_l^x(u)| \Big) \le K \int_0^\infty \sqrt{\log N(\varepsilon)}\, \mathrm{d}\varepsilon\,,$$

where K is a universal constant.
Using proposition 3.4.1, for any $u, v \in [l, l+1]$ we have

$$\mathbb{E}\big[Y_l(u) - Y_l(v)\big]^2 \preccurlyeq (u-v)^2.$$

Particularly the upper bound doesn't depend on l.
So with the same argument given in section 3.5, it follows that there are positive numbers M_1 and M_2 depending only on H, such that $N(\varepsilon) \preccurlyeq \frac{1}{\varepsilon}$ for $0 < \varepsilon \le M_1$, $N(\varepsilon) \asymp n^{2d}$ for $M_1 \le \varepsilon < M_2$ and finally $N(\varepsilon) = 1$ for $\varepsilon > M_2$ and. So there exists a positive constant M such that for any l

$$\mathbb{E}\Big(\sup_{\substack{|x| \le n^2 \\ l \le u \le l+1}} |Y_l^x(u)| \Big) \le K_1 \sqrt{\log n}.$$

The same is true for $\widehat{Y}_l^x$

$$\mathbb{E}\Big(\sup_{\substack{|x|\leq n^2\\ l\leq u\leq l+1}} |\widehat{Y}_l^x(u)|\Big) \leq K_2\sqrt{\log n}.$$

Hence we have

$$\widehat{U}(n) \leq 3/2n + Kn\sqrt{\log n},$$

where K is a positive constant that doesn't depend on anything other than H. □

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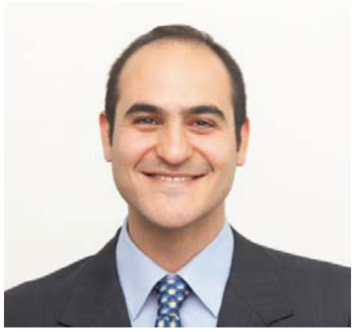

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