Fixed-order Controller Design of Linear Systems*

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I. Introduction

The problem of fixed-order dynamic output feedback control of systems subject to polytopic uncertainties is a challenging issue in the community of robust control theory. Various LMI-based methods have been developed since the last decade. In this report, we show that most of slack-matrix based methods in the literature implicitly/explicitly rely on the concept of Strictly Positive Realness (SPRness) of transfer functions presented by KYP Lemma. In fact, using the SPR property, a fixed-order controller is designed via a solution of a set of Linear Matrix Inequalities (LMIs), thanks to KYP Lemma.

II. Problem Formulation and Preliminaries

Consider a linear time-invariant system described by the following dynamical equations:

$$\begin{aligned} \delta[x_g(t)] &= A_g x_g(t) + B_g u(t) + B_w w(t) \\ z(t) &= C_z x_g(t) + D_{zu} u(t) \\ y(t) &= C_g x_g(t) \end{aligned} \tag{1}$$

where $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^{n_o}$, and $z \in \mathbb{R}^s$ are the state, the control input, the exogenous input, the measured output, and the controlled output, respectively. The symbol $\delta[.]$ presents the derivative term for continuous-time and the forward operator for discrete-time systems. It is assumed that polytopic uncertainties affect all the state space matrices as follows:

$$\begin{aligned} \Omega &= \{(A_g(\lambda), B_g(\lambda), B_w(\lambda), C_z(\lambda), C_g(\lambda), D_{zu}(\lambda)) \\ &= \sum_{i=1}^{q} \lambda_i (A_{g_i}, B_{g_i}, B_{w_i}, C_{z_i}, C_{g_i}, D_{zu_i})\} \end{aligned} \tag{2}$$

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where $\lambda = [\lambda_1, \ldots, \lambda_q]$ belongs to the following unit simplex Λ_q:

$$\Lambda_q = \left\{ \lambda_1, \ldots, \lambda_q \;\middle|\; \sum_{i=1}^{q} \lambda_i = 1, \quad \lambda_i \geq 0 \right\} \tag{3}$$

and matrices A_{g_i}, B_{g_i}, B_{w_i}, C_{z_i}, C_{g_i}, and D_{zu_i} are the i-th vertex of the polytope.

The main objective of this report is to design a robust fixed-order stabilizing controller for the polytopic system given by:

$$\begin{aligned} \delta[x_c(t)] &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{4}$$

where $A_c \in \mathbb{R}^{m \times m}$ and B_c, C_c, and D_c are of appropriate dimensions. The problem of dynamic output-feedback controller synthesis can be equivalently transformed to a static output feedback one by creating an augmented system as follows [1]:

$$\begin{aligned} \delta[\bar{x}_g(t)] &= \bar{A}_g(\lambda)\bar{x}_g(t) + \bar{B}_g(\lambda)u(t) + \bar{B}_w(\lambda)w(t) \\ z(t) &= \bar{C}_z(\lambda)\bar{x}_g(t) + \bar{D}_{zu}(\lambda)u(t) \\ y(t) &= \bar{C}_g(\lambda)\bar{x}_g(t) \end{aligned} \tag{5}$$

where

$$\bar{A}_g(\lambda) = \begin{bmatrix} A_g(\lambda) & 0 \\ 0 & 0_m \end{bmatrix}, \quad \bar{B}_g(\lambda) = \begin{bmatrix} 0 & B_g(\lambda) \\ I_m & 0 \end{bmatrix}$$

$$\bar{B}_w(\lambda) = \begin{bmatrix} B_w(\lambda) \\ 0 \end{bmatrix}, \quad \bar{C}_g(\lambda) = \begin{bmatrix} 0 & I_m \\ C_g(\lambda) & 0 \end{bmatrix}$$

$$\bar{C}_z(\lambda) = \begin{bmatrix} C_z(\lambda) & 0 \end{bmatrix}, \quad \bar{D}_{zu}(\lambda) = \begin{bmatrix} 0 & D_{zu}(\lambda) \end{bmatrix}$$

Closed-loop system $H_{zw}(\lambda)$, transfer function from w to z, can be described in state space framework as follows:

$$\begin{aligned} \delta[x(t)] &= A(\lambda)x(t) + B(\lambda)w(t) \\ z(t) &= C(\lambda)x(t) \end{aligned} \tag{6}$$

where $x = \bar{x}_g$ and

$$K = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{7}$$

and

$$\begin{aligned} A(\lambda) &= \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K\bar{C}_g(\lambda) \\ B(\lambda) &= \bar{B}_w(\lambda) \\ C(\lambda) &= \bar{C}_z(\lambda) + \bar{D}_{zu}(\lambda)K\bar{C}_g(\lambda) \end{aligned} \tag{8}$$

The remains of this section provide basic lemmas which are used throughout this report.

Lemma 1: (Kalman-Yakubovich-Popov (KYP) Lemma [2]) A transfer matrix $H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SPR if and only if there exists a symmetric matrix $P = P^T > 0$ such that:

- For continuous-time systems:

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} < 0 \tag{9}$$

- For discrete-time systems:

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} < 0 \tag{10}$$

Lemma 2: The following statements are equivalent [3] and [4]:

1) $H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SPR with Lyapunov matrix P.

2) $H^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$ is SPR with Lyapunov matrix P.

As a result, the following inequalities are equivalent:

- For continuous-time systems:

$$\begin{gathered} \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} < 0 \\ \begin{bmatrix} (A - BD^{-1}C)^TP + P(A - BD^{-1}C) & -PBD^{-1} - C^TD^{-T} \\ -D^{-T}B^TP - D^{-1}C & -D^{-1} - D^{-T} \end{bmatrix} < 0 \end{gathered} \tag{11}$$

- For discrete-time systems:

$$\begin{gathered} \begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} < 0 \\ \begin{bmatrix} (A - BD^{-1}C)^TP(A - BD^{-1}C) - P & -(A - BD^{-1}C)^TPBD^{-1} - C^TD^{-T} \\ -D^{-T}B^TP(A - BD^{-1}C) - D^{-1}C & D^{-T}B^TPBD^{-1} - D^{-1} - D^{-T} \end{bmatrix} < 0 \end{gathered} \tag{12}$$

III. Application of Strictly Positive Realness to Fixed-order Controller Design

Strictly Positive Realness (SPRness) plays an important role in fixed-order control design. Recently, some methods for fixed-order control design of LTI systems have been developed in [3], [5]–[13] (for continuous-time systems) and in [4], [7], [8], [14]–[18] (for discrete-time case). The proposed methods can be generally classified into two main frameworks: polynomial and state space form.

A. Polynomial-based Approaches

The main idea behind fixed-order controller design in the polynomial approaches [5], [7], [8] is based on the strictly positive realness (SPRness) of some transfer functions. The idea is presented as follows: Suppose that $c_i(s)$ is the closed-loop characteristic polynomial at the i-th vertex, for $i = 1, 2, \ldots, q$, then the polytopic system is stable if the transfer function $c_i(s)/d(s)$ for $i = 1, 2, \ldots, q$ is a strictly positive real (SPR) transfer function, where $d(s)$ is a given stable polynomial called central polynomial. Therefore, in order to check the stability of the polynomial

$c_i(s)$, it is sufficient to test the SPRness of the transfer functions $c_i(s)/d(s)$; where, the central polynomial $d(s)$ is given a priori. The SPR transfer functions $c_i(s)/d(s)$ with a fixed denominator have the controllable canonical realization with known matrices A and B. Therefore, the SPRness of $c_i(s)/d(s)$ in state space can be parameterized by some LMIs thanks to KYP Lemma.

The choice of the central polynomial is very important because it affects the control performance as well as the conservatism of the approach. The main drawback of these approaches is that they are limited to SISO polytopic systems.

B. State Space-based Approaches

Most of the existing methods available in the literature for fixed-order control design are implicitly based on the concept of SPRness, where state matrix A is fixed by introducing a central matrix M determined by different methods, e.g. initial output feedback(s), state feedback controllers, etc. Therefore, the proposed methods can be generally categorized in three parts:

- Methods initialized by output feedback controllers
- Methods initialized by state feedback controllers
- One-step approaches

In this subsection, we study the relationship of some available methods in [3], [4], [6], [9]–[18]. For the simplicity of the presentation, only the results of stabilizing controller design are discussed.

1) Fixed-order Controller Design Approaches Initialized by Output Feedback Controllers: In this part, the relation among the two-stages approaches initialized by output feedback controllers [3], [4], [12], [13] are considered. The proposed method in [3] (or in [4] for the discrete-time systems) uses the SPRness of the following transfer matrix:

$$H_1(s) = \left[\begin{array}{c|c} M & I \\ \hline M - T^{-1}A(\lambda)T & I \end{array}\right] \tag{13}$$

where the central state matrix M and the similarity transform T are chosen from a set of initial stabilizing output feedback controllers designed for each vertex of the polytopic system. The proposed method of [12] is the special case of the above results where T is considered as an identity matrix. In other words, in [12], the SPRness of the following transfer matrix is considered:

$$H_2(s) = \left[\begin{array}{c|c} M & I \\ \hline M - A(\lambda) & I \end{array}\right] \tag{14}$$

which is a special case of (13) with $T = I$.

The proposed fixed-order controller design method in [13] is based on the SPRness of the following transfer matrix:

$$H_3(s) = \left[\begin{array}{c|c} M & \tau I \\ \hline M - A(\lambda) & \tau I \end{array}\right]; \quad \tau > 0 \tag{15}$$

It can be easily shown that the scalar parameter τ can be removed. In other words, the SPRness of $H_3(s)$ is equivalent to the one of $H_2(s)$.

2) Fixed-order Controller Design Approaches Initialized by State Feedback Controllers: In this part, we study the relation among the two-stages approaches initialized by state feedback controllers [6], [10], [15], [18]. The results are given in the following Lemmas.

Lemma 3: Let $K_{sf}(\lambda)$ be a stabilizing parameter-dependent state feedback controller for the continuous-time system described by (1) and (2). Then, the following statements are equivalent:

(a) If there exist two matrices X and L such that the following transfer function is SPR:

$$H(s) = \left[\begin{array}{c|c} \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda) & \bar{B}_g(\lambda) \\ \hline XK_{sf}(\lambda) - L\bar{C}_g(\lambda) & X \end{array}\right] \tag{16}$$

(b) If there exist a Lyapunov matrix $P(\lambda) > 0$, and some matrices X and L such that the following inequality holds:

$$\begin{bmatrix} M^T(\lambda)P(\lambda) + P(\lambda)M(\lambda) & \star \\ \bar{B}_g^T(\lambda)P(\lambda) - (XK_{sf}(\lambda) - L\bar{C}_g(\lambda)) & -X - X^T \end{bmatrix} < 0 \tag{17}$$

where $M(\lambda) = \bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda)$.

(c) (Theorem 2 in [10]) If there are a Lyapunov matrix $P(\lambda)$ and some matrices $F(\lambda)$, $V(\lambda)$, X, and L such that:

$$\begin{bmatrix} (\bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda))^T F^T(\lambda) + F(\lambda)(\bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda)) & \star & \star \\ P(\lambda) - F^T(\lambda) + V(\lambda)(\bar{A}_g(\lambda) + \bar{B}_g(\lambda)K_{sf}(\lambda)) & -V(\lambda) - V^T(\lambda) & \star \\ \bar{B}_g^T(\lambda)F^T(\lambda) + L\bar{C}_g(\lambda) - XK_{sf}(\lambda) & \bar{B}_g^T(\lambda)V^T(\lambda) & -X - X^T \end{bmatrix} < 0 \tag{18}$$

(d) (Corollary 1 in [6]) If there exist a Lyapunov matrix $P > 0$ and two matrices X and L such that:

$$\begin{bmatrix} \bar{A}_g^T P + P\bar{A}_g & \star \\ \bar{B}_g P & 0 \end{bmatrix} + \text{He}\left(\begin{bmatrix} K_{sf}^T \\ -I \end{bmatrix} \begin{bmatrix} L\bar{C}_g & -X \end{bmatrix} \right) < 0 \tag{19}$$

Then $K = X^{-1}L$ is a robust dynamic output feedback controller which stabilizes the continuous-time system given in (1) and (2).

Proof: The statements (a) and (b) can directly result from KYP Lemma. Therefore, it is enough to show that (18) is equivalent to (17). Post-multiplying (18) by U and pre-multiplying by U^T, the inequality given in (17) achieves.

$$U = \begin{bmatrix} I & 0 \\ \bar{A}_g + \bar{B}_g K_{sf} & \bar{B}_g \\ 0 & I \end{bmatrix} \tag{20}$$

To prove the statement (d), the inequality given in (17) is obtained by pre- and post-multiplication of (19) by the following matrix:

$$\begin{bmatrix} I & K_{sf}^T \\ 0 & I \end{bmatrix} \tag{21}$$

■

Lemma 4: Let $K_{sf}(\lambda)$ be a stabilizing parameter-dependent state feedback controller for the discrete-time system given described by (1) and (2). Then, the following statements are equivalent:

(a) If there exist two matrices X and L such that the following transfer function is SPR:

$$H(z)=\left[\begin{array}{c|c}\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda) & \bar{B}_g(\lambda)\\ \hline XK_{sf}(\lambda)-L\bar{C}_g(\lambda) & X\end{array}\right] \tag{22}$$

(b) If there exist a Lyapunov matrix $P(\lambda)>0$, and some matrices X and L such that:

$$\begin{bmatrix} M^T(\lambda)P(\lambda)M(\lambda)-P(\lambda) & \star \\ \bar{B}_g^T(\lambda)P(\lambda)M(\lambda)-(XK_{sf}(\lambda)-L\bar{C}_g(\lambda)) & \bar{B}_g^T(\lambda)P(\lambda)\bar{B}_g(\lambda)-X-X^T\end{bmatrix}<0 \tag{23}$$

where $M(\lambda)=\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)$.

(c) ([18]) If there are a Lyapunov matrix $P(\lambda)$ and some matrices $F(\lambda)$, X, and L such that:

$$\begin{bmatrix} -P(\lambda) & \star & \star \\ F^T(\lambda)(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)) & P(\lambda)-F(\lambda)-F^T(\lambda) & \star \\ L\bar{C}_g(\lambda)-X^TK_{sf}(\lambda) & \bar{B}_g^T(\lambda)F(\lambda) & -X-X^T\end{bmatrix}<0 \tag{24}$$

(d) (Theorem 4.1 in [15]) If there exist a Lyapunov matrix $P(\lambda)$ and some matrices F_1, F_2, F_3, F_4, X, and L such that:

$$\begin{bmatrix} F_1(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))+(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))^TF_1^T-P(\lambda) & \star & \star & \star \\ F_2(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)) & -P(\lambda) & \star & \star \\ F_3(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))+\bar{B}_g^T(\lambda)F_1^T+L\bar{C}_g-XK_{sf}(\lambda) & \bar{B}_g^T(\lambda)F_2^T & F_3\bar{B}_g^T(\lambda)F_3^T-(X+X^T) & \star \\ F_4(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))-F_1^T & P(\lambda)-F_2^T & F_4\bar{B}_g(\lambda)-F_3^T & -F_4-F_4^T\end{bmatrix}<0 \tag{25}$$

Then $K=X^{-1}L$ is a robust dynamic output feedback controller which stabilizes the discrete-time-time system given in (1) and (2).

Proof: According to KYP Lemma, statements (a) and (b) are equivalent. Then, we show that inequalities given in (24) and (23) are equivalent. Inequality (24) can be represented as:

$$\begin{bmatrix} -P & \star & \star \\ 0 & P & \star \\ L\bar{C}_g-X^TK_{sf} & 0 & -X-X^T\end{bmatrix}+\text{He}\left(\begin{bmatrix}0\\ F^T\\ 0\end{bmatrix}\begin{bmatrix}\bar{A}_g+\bar{B}_gK_{sf} & -I & \bar{B}_g\end{bmatrix}\right)<0 \tag{26}$$

Then, the slack matrix F can be removed using Projection Lemma as follows:

$$\mathscr{N}(\begin{bmatrix}\bar{A}_g+\bar{B}_gK_{sf} & -I & \bar{B}_g\end{bmatrix})^T\begin{bmatrix} -P & \star & \star \\ 0 & P & \star \\ L\bar{C}_g-X^TK_{sf} & 0 & -X-X^T\end{bmatrix}\mathscr{N}(\begin{bmatrix}\bar{A}_g+\bar{B}_gK_{sf} & -I & \bar{B}_g\end{bmatrix})<0 \tag{27}$$

By choosing

$$\mathscr{N}(\begin{bmatrix}\bar{A}_g+\bar{B}_gK_{sf} & -I & \bar{B}_g\end{bmatrix})=\begin{bmatrix} I & 0 \\ \bar{A}_g+\bar{B}_gK_{sf} & \bar{B}_g \\ 0 & I\end{bmatrix} \tag{28}$$

we obtain

$$\begin{bmatrix} I & 0 \\ \bar{A}_g+\bar{B}_gK_{sf} & \bar{B}_g \\ 0 & I\end{bmatrix}^T\begin{bmatrix} -P & \star & \star \\ 0 & P & \star \\ L\bar{C}_g-X^TK_{sf} & 0 & -X-X^T\end{bmatrix}\begin{bmatrix} I & 0 \\ \bar{A}_g+\bar{B}_gK_{sf} & \bar{B}_g \\ 0 & I\end{bmatrix}<0 \tag{29}$$

or equivalently

$$\begin{bmatrix} (\bar{A}_g+\bar{B}_gK_{sf})^TP(\bar{A}_g+\bar{B}_gK_{sf})-P & \star \\ \bar{B}_g^TP(\bar{A}_g+\bar{B}_gK_{sf})+L\bar{C}_g-X^TK_{sf} & \bar{B}_g^TP\bar{B}_g-X-X^T \end{bmatrix}<0 \tag{30}$$

A similar procedure can be applied to show that (23) and (25) are equivalent. The statement (d) can be rewritten as:

$$\begin{bmatrix} -P & \star & \star & \star \\ 0 & -P & \star & \star \\ L\bar{C}_g-XK_{sf} & 0 & -(X+X^T) & \star \\ 0 & P & 0 & 0 \end{bmatrix}+\text{He}\{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}\begin{bmatrix} \bar{A}_g+\bar{B}_gK_{sf} & 0 & \bar{B}_g & -I \end{bmatrix}\}<0 \tag{31}$$

By considering

$$\mathscr{N}(\begin{bmatrix} \bar{A}_g+\bar{B}_gK_{sf} & 0 & \bar{B}_g & -I \end{bmatrix})=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \bar{A}_g+\bar{B}_gK_{sf} & 0 & \bar{B}_g \end{bmatrix} \tag{32}$$

we have

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \bar{A}_g+\bar{B}_gK_{sf} & 0 & \bar{B}_g \end{bmatrix}^T\begin{bmatrix} -P & \star & \star & \star \\ 0 & -P & \star & \star \\ L\bar{C}_g-XK_{sf} & 0 & -(X+X^T) & \star \\ 0 & P & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \bar{A}_g+\bar{B}_gK_{sf} & 0 & \bar{B}_g \end{bmatrix}<0 \tag{33}$$

which is equal to the following inequality:

$$\begin{bmatrix} -P(\lambda) & \star & \star \\ P(\lambda)(\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda)) & -P(\lambda) & \star \\ L\bar{C}_g(\lambda)-XK_{sf}(\lambda) & \bar{B}_g^T(\lambda)P(\lambda) & -X-X^T \end{bmatrix}<0 \tag{34}$$

By pre- and post-multiplication of the above inequality by the following matrix, (23) is derived. Thus, the proof is complete.

$$\begin{bmatrix} I & (\bar{A}_g(\lambda)+\bar{B}_g(\lambda)K_{sf}(\lambda))^T & 0 \\ 0 & \bar{B}_g(\lambda) & I \end{bmatrix} \tag{35}$$

■

3) One-step Approaches to Fixed-order Control Design: In this part, the one-step LMI-based methods for fixed-order dynamical output feedback controller design are considered. First, the results of the existing methods in [9], [11], [14], [16], [17] are given. Then, we show that these approaches are based on the SPR-ness of some transfer functions where matrix A is fixed by different approaches.

Lemma 5: The following two statements are equivalent.

(a) (Theorem 4 in [11]) If there exist $P^{-1}(\lambda) > 0$, matrices G and L, and a positive scalar $\tau > 0$ such that:

$$\begin{bmatrix} \mathrm{He}\{\bar{A}_g(\lambda)P^{-1}(\lambda) + \bar{B}_g(\lambda)LT\bar{C}_g(\lambda)\} & \star \\ \bar{C}_g(\lambda)P^{-1}(\lambda) - GT\bar{C}_g(\lambda) + \tau L^T \bar{B}_g^T(\lambda) & -\tau G - \tau G^T \end{bmatrix} < 0 \tag{36}$$

where

$$T = \begin{cases} (\bar{C}_{g_i}\bar{C}_{g_i}^T)^{-1} & \exists \quad i_0 \in \{1, 2, \ldots, N\} \text{ s.t. } \bar{C}_{g_i} \text{ is full row rank} \\ I & \text{o.w.} \end{cases} \tag{37}$$

(b) If there are matrices G and L, and a positive scalar $\tau > 0$ such that the following transfer matrix is SPR:

$$H_4(s) = \left[\begin{array}{c|c} M(\lambda) & -(\tau^{-1}\bar{C}_g(\lambda)T + \bar{B}_g(\lambda)LG^{-1}) \\ \hline \tau^{-1}G^{-T}\bar{C}_g(\lambda) & \tau^{-1}G^{-T} \end{array}\right] \tag{38}$$

where $M(\lambda) = \bar{A}_g(\lambda) - \tau^{-1}\bar{C}_g^T(\lambda)T\bar{C}_g(\lambda)$.

Then, the static output feedback $K = LG^{-1}$ stabilizes the continuous-time augmented system in (1) and (2).

Proof: Inequality (36) is equivalent to the following one:

$$\begin{aligned} &\begin{bmatrix} PA + A^T P & \bar{C}_g^T - P(\bar{C}_g^T T^T + \tau \bar{B}_g K)G^T \\ \star & -\tau G - \tau G^T \end{bmatrix} = \\ &\begin{bmatrix} P & P\bar{B}_g K \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathrm{He}\{\bar{A}_g P^{-1} + \bar{B}_g LT\bar{C}_g\} & \star \\ \bar{C}_g P^{-1} - GT\bar{C}_g + \tau L^T \bar{B}_g^T & -\tau G - \tau G^T \end{bmatrix} \begin{bmatrix} P & P\bar{B}_g K \\ 0 & I \end{bmatrix}^T < 0 \end{aligned} \tag{39}$$

The above inequality indicates that the following transfer matrix is SPR.

$$H_4^{-1}(s) = \left[\begin{array}{c|c} A & (\bar{C}_g^T T^T + \tau \bar{B}_g K)G^T \\ \hline \bar{C}_g & \tau G^T \end{array}\right] \tag{40}$$

According to Lemma 2, $H_4(s)$ in (42) is also an SPR transfer matrix. Thus, the proof is complete. ■

Lemma 6: Assume that matrices C_{g_i} in (2) are full-row rank for $i = 1, \ldots, q$. The following statements are equivalent.

(a) (Lemma 4.2 in [9]) If there exist a positive scalar α, matrices $\tilde{Y}_j > 0$ and structured matrices $Z_i = \begin{bmatrix} Z_1 & 0_{n_o \times (n-n_o)} \\ Z_{2_i} & Z_{3_i} \end{bmatrix}$ and $L = \begin{bmatrix} L_1 & 0_{n_i \times (n-n_o)} \end{bmatrix}$ such that:

$$\begin{bmatrix} \alpha \mathrm{He}\{\tilde{A}_{g_i} Z_i + \tilde{B}_{g_{ij}} L\} & \star \\ (T_i \tilde{Y}_j T_i^T + \tilde{A}_{g_i} Z_i + \tilde{B}_{g_{ij}} L)^T - \alpha Z_i & -Z_i - Z_i^T \end{bmatrix} < 0 \tag{41}$$

where T_i is a non-singular similarity transformation matrix for each vertex of the polytope satisfying $\tilde{C}_{g_i} = C_{g_i} T_i^{-1} = [I_{n_0 \times n_o} \quad 0_{n_o \times (n-n_o)}]$, $\tilde{A}_{g_i} = T_i A_{g_i} T_i^{-1}$ and $\tilde{B}_{g_{ij}} = T_i B_{g_j}$.

(b) If there exist a positive scalar α, matrices $\tilde{Y}_j > 0$ and structured matrices $Z_i = \begin{bmatrix} Z_1 & 0_{n_o \times (n-n_o)} \\ Z_{2_i} & Z_{3_i} \end{bmatrix}$ and $L = \begin{bmatrix} L_1 & 0_{n_i \times (n-n_o)} \end{bmatrix}$ such that the following transfer matrix is SPR:

$$H_5(s) = \left[\begin{array}{c|c} M & -\tilde{Z}^{-1}(\lambda) \\ \hline \alpha I + A^T(\lambda) & \tilde{Z}^{-1}(\lambda) \end{array}\right] \tag{42}$$

where $M = -\alpha I$, $\tilde{Z}^{-1}(\lambda) = \sum_{i=1}^{q} \lambda_i \tilde{Z}_i^{-1}$, and $\tilde{Z}_i = T_i^{-1} Z_i T_i^{-T}$.

Then, the continuous-time system in (1) is robustly stable by a static output feedback controller $K = L_1 Z_1^{-1}$.

Proof: By using the fact that $LT_i^{-T} = KC_{g_i}\tilde{Z}_i$, the inequality given in (41) is equivalent to:

$$\begin{bmatrix} \alpha \mathrm{He}\{T_i(A_{g_i} + B_{u_j}KC_{g_i})\tilde{Z}_i T_i^T\} & \star \\ T_i(\tilde{Y}_j + \tilde{Z}_i^T(A_{g_i} + B_{u_j}KC_{g_i})^T - \alpha\tilde{Z}_i)T_i^T & -T_i(\tilde{Z}_i + \tilde{Z}_i^T)T_i^T \end{bmatrix} < 0 \tag{43}$$

Pre- and post-multiplying of the above inequality by $\begin{bmatrix} T_i^{-1} & -\alpha T_i^{-1} \\ 0 & \tilde{Z}_i^{-T} T_i^{-1} \end{bmatrix}$ and its transpose lead to the following set of inequalities:

$$\begin{bmatrix} -2\alpha\tilde{Y}_j & \star \\ \tilde{Z}_i^{-T}\tilde{Y}_j^T + (A_{g_i} + B_{g_j}KC_{g_i})^T + \alpha I & -\tilde{Z}_i^{-1} - \tilde{Z}_i^{-T} \end{bmatrix} < 0 \tag{44}$$

Multiplying (44) by $\lambda_i\lambda_j$ and summing them, the following inequality is obtained:

$$\begin{bmatrix} -2\alpha\tilde{Y}(\lambda) & \star \\ \tilde{Z}^{-T}(\lambda)\tilde{Y}^T(\lambda) + (A_g(\lambda) + B_g(\lambda)KC_g(\lambda))^T + \alpha I & -\tilde{Z}^{-1}(\lambda) - \tilde{Z}^{-T}(\lambda) \end{bmatrix} < 0 \tag{45}$$

where $\tilde{Y}(\lambda) = \sum_{i=1}^{q} \lambda_i \tilde{Y}_i$. Based on Lemma 1, the above inequality is equivalent to the SPRness of the transfer matrix $H_5(s)$ in (42).

■

Lemma 7: The following statements are equivalent.

(a) ([14]) If there exist matrices P, N, and U such that:

$$\begin{aligned} &\begin{bmatrix} -P & \star \\ P\bar{A}_g + \bar{B}_g N \bar{C}_g & -P \end{bmatrix} < 0 \\ &P\bar{B}_g = \bar{B}_g U \end{aligned} \tag{46}$$

(b) If there exist matrices $P > 0$, N, and U such that the following transfer matrix is SPR,

$$H_6(z) = \left[\begin{array}{c|c} 0 & I \\ \hline -P\bar{A}_g - \bar{B}_g N \bar{C}_g & P \end{array}\right] \tag{47}$$

and $P\bar{B}_g = \bar{B}_g U$.

Then, the closed-loop stability of the discrete-time system in (5) is guaranteed with $K = U^{-1}N$.

Proof: It should be noted that

$$P\bar{A}_g + \bar{B}_g N \bar{C}_g = PA \tag{48}$$

Therefore, (46) is equivalent to the following inequality indicating that $H_6(z)$ in (47) is SPR.

$$\begin{bmatrix} -P & \star \\ PA & -P \end{bmatrix} < 0 \tag{49}$$

■

Lemma 8: Without loss of generality, we assume that there exists a matrix T such that $T\bar{B}_g = \begin{bmatrix} I \\ 0 \end{bmatrix}$. The following statements are equivalent.

(a) ([16]) If there exist matrices $\bar{P}$, $\bar{S} = \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix}$, and $\bar{L}$ such that:

$$\begin{bmatrix} -\bar{P} & \star \\ \bar{S}T\bar{A}_g T^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} \bar{C}_g T^{-1} & \bar{P} - \bar{S} - \bar{S}^T \end{bmatrix} < 0 \tag{50}$$

(b) If there exist matrices $\bar{S} = \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix}$, and $\bar{L}$ such that the following transfer matrix is SPR,

$$H_7(z) = \left[\begin{array}{c|c} 0 & I \\ \hline -(T^T\bar{S}T)A & T^T\bar{S}T \end{array}\right] \tag{51}$$

Then, $K = \bar{S}_1^{-1}L$ stabilizes the discrete-time system in (5).

Proof: It should be noted that

$$\begin{aligned} \begin{bmatrix} L \\ 0 \end{bmatrix} &= \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{S}_1 \bar{S}_1^{-1} L \\ &= \begin{bmatrix} \bar{S}_1 \\ 0 \end{bmatrix} \bar{S}_1^{-1} L \\ &= \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} K \\ &= \bar{S}T\bar{B}_g K \end{aligned} \tag{52}$$

Hence, (50) is equivalent to the following inequality:

$$\begin{bmatrix} -\bar{P} & \star \\ \bar{S}T(\bar{A}_g + \bar{B}_g K \bar{C}_g)T^{-1} & \bar{P} - \bar{S} - \bar{S}^T \end{bmatrix} < 0 \tag{53}$$

Multiply the above inequality on the left by $\begin{bmatrix} T^T & 0 \\ 0 & T^T \end{bmatrix}$ and on the right by $\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$. Then, we obtain:

$$\begin{bmatrix} -T^T\bar{P}T & \star \\ T^T\bar{S}T(\bar{A}_g + \bar{B}_g K \bar{C}_g) & T^T\bar{P}T - T^T\bar{S}T - T^T\bar{S}^T T \end{bmatrix} < 0 \tag{54}$$

Inequality (54) indicates that $H_7(z)$ in (51) is an SPR transfer function. ■

Lemma 9: Without loss of generality, assume that there exists a matrix T such that $\bar{C}_g T = [I \quad 0]$. The following statements are equivalent.

(a) ([17]) If there exist a scalar λ, matrices $P > 0$, $G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, $F = \begin{bmatrix} \lambda G_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$, and $Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix}$ such that:

$$\begin{bmatrix} P - TG - G^T T^T & \star \\ \bar{A}_g TG + \bar{B}_g Y - F^T T^T & -P + \text{He}\{\bar{A}_g TF + \lambda \bar{B}_g Y\} \end{bmatrix} < 0 \tag{55}$$

(b) If there exist a scalar λ, matrices $G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, $F = \begin{bmatrix} \lambda G_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$, and $Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix}$ such that the following transfer matrix is SPR,

$$H_8(z) = \left[\begin{array}{c|c} -G^{-1}F & -G^{-1}T^{-1} \\ \hline G^{-1}F + A^T & G^{-1}T^{-1} \end{array}\right] \tag{56}$$

Then, $K = Y_1 G_{11}^{-1}$ stabilizes the discrete-time system in (5).

Proof: Based on the structure of matrices Y, G, and F, we have:

$$\begin{aligned} Y &= K\bar{C}_g TG \\ &= \lambda^{-1} K \bar{C}_g TF \end{aligned} \tag{57}$$

Therefore, inequality (55) is equivalent to:

$$\begin{bmatrix} P - TG - G^T T^T & \star \\ ATG - F^T T^T & -P + \text{He}\{ATF\} \end{bmatrix} < 0 \tag{58}$$

The above inequality shows that the following transfer function is SPR.

$$H_8^{-1}(z) = \left[\begin{array}{c|c} A^T & I \\ \hline T(F + GA^T) & TG \end{array}\right] \tag{59}$$

According to Lemma 2, $H_8(z)$ in (56) is SPR. The proof is complete. ■

Lemma 10: Without loss of generality, assume that there exists a matrix T such that $\bar{C}_g T = [I \quad 0]$. The following statements are equivalent.

(a) ([17]) If there exist a scalar λ, matrices $P > 0$, $G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, $F = \begin{bmatrix} \lambda G_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$, and $Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix}$ such that:

$$\begin{bmatrix} P - TGT^T - TG^T T^T & \star \\ \bar{A}_g TGT^T + \bar{B}_g Y T^T - TF^T T^T & -P + \text{He}\{\bar{A}_g TFT^T + \lambda \bar{B}_g Y T^T\} \end{bmatrix} < 0 \tag{60}$$

(b) If there exist a scalar λ, matrices $G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, $F = \begin{bmatrix} \lambda G_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$, and $Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix}$ such that the following transfer matrix is SPR:

$$H_9(z) = \left[\begin{array}{c|c} -\bar{G}^{-1}\bar{F} & -\bar{G}^{-1} \\ \hline A^T + \bar{G}^{-1}\bar{F} & \bar{G}^{-1} \end{array}\right] \tag{61}$$

where $\bar{G} = TGT^T$ and $\bar{F} = TFT^T$.

Then, $K = Y_1 G_{11}^{-1}$ stabilizes the discrete-time system in (5).

Proof: Similar to the proof of Lemma (9), we can show that the following transfer function is SPR.

$$H_9^{-1}(z) = \left[\begin{array}{c|c} A^T & I \\ \hline TFT^T + TGT^T A^T) & TGT^T \end{array}\right] \tag{62}$$

As a result, $H_9(z)$ in (61) is an SPR transfer function. ■

Lemma 11: If the conditions of Theorem 1 presented in [19] hold, the following transfer function is SPR.

$$H_{10}(z) = \left[\begin{array}{c|c} 0 & \alpha I \\ \hline -GA & \alpha G \end{array}\right] \tag{63}$$

IV. Conclusions

This report reviews the recent slack variable-based methods for design of fixed-order controllers. It is shown that the main relation among these approaches is the concept of Strictly Positive Realness (SPRness) of transfer function matrices represented by KYP Lemma.

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