

EIGENFUNCTION EXPANSIONS OF ULTRADIFFERENTIABLE FUNCTIONS AND ULTRADISTRIBUTIONS

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ABSTRACT. In this paper we give a global characterisation of classes of ultradifferentiable functions and corresponding ultradistributions on a compact manifold X . The characterisation is given in terms of the eigenfunction expansion of an elliptic operator on X . This extends the result for analytic functions on compact manifold by Seeley [See69], and the characterisation of Gevrey functions and Gevrey ultradistributions on compact Lie groups and homogeneous spaces by the authors [DR14a].

1. INTRODUCTION

Let X be a compact analytic manifold and let E be an analytic, elliptic, positive differential operator of order ν . Let $\{\phi_j\}$ and $\{\lambda_j\}$ be respectively the eigenfunctions and eigenvalues of E , i.e. $E\phi_j = \lambda_j\phi_j$. Then acting by E on a smooth function $f = \sum_j f_j\phi_j$ we see that it is analytic if and only if that there is a constant $C > 0$ such that for all $k \geq 0$

$$\sum_j \lambda_j^{2k} |f_j|^2 \leq ((\nu k)!)^2 C^{2k+2}.$$

Consequently, Seeley has shown in [See69] that $f = \sum_j f_j\phi_j$ is analytic if and only if the sequence $\{A^{\lambda_j^{1/\nu}} f_j\}$ is bounded for some $A > 1$.

The aim of this paper is to extend Seeley's characterisation to classes more general than that of analytic functions. In particular, the characterisation we obtain will cover Gevrey spaces extending also the characterisation that was obtained previously by the authors in [DR14a] in the setting of compact Lie groups and compact homogeneous spaces. The characterisation will be given in terms of the behaviour of coefficients of the series expansion of functions with respect to the eigenfunctions of an elliptic positive pseudo-differential operator E on X , similar to Seeley's result in [See69], with a related construction in [See65]. Interestingly, our approach allows one to define analytic or Gevrey functions even if the manifold X is 'only' smooth.

Global characterisations as obtained in this paper have several applications. For example, the Gevrey spaces appear naturally when dealing with weakly hyperbolic problems, such as the wave equation for sums of squares of vector fields satisfying Hörmander's condition, also with the time-dependent propagation speed of low regularity. In the setting of compact Lie groups the global characterisation of Gevrey

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spaces that has been obtained by the authors in [DR14a] has been further applied to the well-posedness of Cauchy problems associated to sums of squares of vector fields in [GR14]. In this setting the Gevrey spaces appear already in \mathbb{R}^n and come up naturally in energy inequalities.

More generally, our argument will give a characterisation of functions in Komatsu type classes resembling but more general than those introduced by Komatsu in [Kom73, Kom77]. Consequently, we give a characterisation of the corresponding dual spaces of ultradistributions. We discuss Roumieu and Beurling type (or injective and projective limit, respectively) spaces as well as their duals and α -duals in the sense of Köthe [Köt69] which also turn out to be perfect spaces.

In the periodic setting (or in the setting of functions on the torus) different function spaces have been intensively studied in terms of their Fourier coefficients. Thus, periodic Gevrey functions have been discussed in terms of their Fourier coefficients by Taguchi [Tag87] (this was recently extended to general compact Lie groups and homogeneous spaces by the authors in [DR14a]), see also Delcroix, Hasler, Pilipović and Valmorin [DHPV04] for the periodic setting.

More general periodic Komatsu-type classes have been considered by Gorbachuk [Gor82], with tensor product structure and nuclearity properties analysed by Petzsche [Pet79]. See also Pilipović and Prangoski [PP14] for the relation to their convolution properties. Regularity properties of spaces of ultradistributions have been studied by Pilipović and Scarpalezos [PS01]. We can refer to a relatively recent book by Carmichael, Kamiński and Pilipović [CKP07] for more details on these and other properties of ultradistributions on \mathbb{R}^n and related references.

The paper is organised as follows. In Definition 2.1 we introduce the class $\Gamma_{\{M_k\}}(X)$ of ultradifferentiable functions on a manifold X , and in Theorem 2.3 we give several alternative reformulations and further properties of these spaces. In Theorem 2.4 we characterise these classes in terms of eigenvalues of a positive elliptic pseudo-differential operator E on X . Consequently, we show that since the spaces of Gevrey functions γ^s on X correspond to the choice $M_k = (k!)^s$, in Corollary 2.5 we obtain a global characterisation for spaces $\gamma^s(X)$. These results are proved in Section 3. Furthermore, in Theorem 2.7 we describe the eigenfunction expansions for the corresponding spaces of ultradistributions. This is achieved by first characterising the α -dual spaces in Section 4 and then relating the α -duals to topological duals in Section 5. In Theorem 2.7 we describe the counterparts of the obtained results for the Beurling-type spaces of ultradifferentiable functions and ultradistributions.

We will denote by C constants, taking different values sometimes even in the same formula. We also denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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2. FORMULATION OF RESULTS ON COMPACT MANIFOLDS

Let X be a compact C^∞ manifold of dimension n with a fixed measure. We fix E to be a positive elliptic pseudo-differential operator of an integer order $\nu \in \mathbb{N}$ on X , and we write $E \in \Psi_{+e}^\nu(X)$ in this case. The eigenvalues of E form a sequence $\{\lambda_j\}$, and for each eigenvalue λ_j we denote the corresponding eigenspace by H_j . We may assume that λ_j 's are ordered as $0 < \lambda_1 < \lambda_2 < \dots$. The space $H_j \subset L^2(X)$ consists

of smooth functions due to the ellipticity of E . We set

$$d_j := \dim H_j, \quad H_0 := \ker E, \quad \lambda_0 := 0, \quad d_0 := \dim H_0.$$

Since the operator E is elliptic, it is Fredholm, hence also $d_0 < \infty$. It can be shown (see [DR14c, Proposition 2.3]) that

$$d_j \leq C(1 + \lambda_j)^{\frac{n}{\nu}} \quad (2.1)$$

for all j , and that

$$\sum_{j=1}^{\infty} d_j (1 + \lambda_j)^{-q} < \infty \quad \text{if and only if} \quad q > \frac{n}{\nu}. \quad (2.2)$$

We denote by e_j^k , $1 \leq k \leq d_j$, an orthonormal basis in H_j . For $f \in L^2(X)$, we denote its ‘Fourier coefficients’ by

$$\widehat{f}(j, k) := (f, e_j^k)_{L^2}.$$

We will write

$$\widehat{f}(j) := \begin{pmatrix} \widehat{f}(j, 1) \\ \vdots \\ \widehat{f}(j, d_j) \end{pmatrix} \in \mathbb{C}^{d_j}, \quad (2.3)$$

for the whole Fourier coefficient corresponding to H_j , and we can then write

$$\|\widehat{f}(j)\|_{\text{HS}} = (\widehat{f}(j) \cdot \widehat{f}(j))^{1/2} = \left(\sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 \right)^{1/2}.$$

We note the Plancherel formula

$$\|f\|_{L^2(X)}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 = \sum_{j=0}^{\infty} \|\widehat{f}(j)\|_{\text{HS}}^2. \quad (2.4)$$

We refer to [DR14c], and for further developments in [DR14b] for a discussion of different properties of the associated Fourier analysis. Here we note that the ellipticity of E and the Plancherel formula imply the following characterisation of smooth functions in terms of their Fourier coefficients:

$$f \in C^\infty(X) \iff \forall N \exists C_N : |\widehat{f}(j, k)| \leq C_N \lambda_j^{-N} \text{ for all } j \geq 1, 1 \leq k \leq d_j. \quad (2.5)$$

If X and E are analytic, the result of Seeley [See69] can be reformulated (we will give a short argument for it in the proof of Corollary 2.5) as

$$f \text{ is analytic} \iff \exists L > 0 \exists C : |\widehat{f}(j, k)| \leq C e^{-L \lambda_j^{1/\nu}} \text{ for all } j \geq 1, 1 \leq k \leq d_j. \quad (2.6)$$

The first aim of this note is to provide a characterisation similar to (2.5) and (2.6) for classes of functions in between smooth and analytic functions, namely, for Gevrey functions, and for their dual spaces of (ultra)distributions. However, the proof works for more general classes than those of Gevrey functions, namely, for classes of functions considered by Komatsu [Kom73], as well as their extensions described below.

We now define an analogue of these classes in our setting. Let $\{M_k\}$ be a sequence of positive numbers such that

$$(M.0) \quad M_0 = 1$$

$$(M.1) \quad (\text{stability})$$

$$M_{k+1} \leq AH^k M_k, \quad k = 0, 1, 2, \dots$$

For our characterisation of functional spaces we will be also assuming condition

$$(M.2)$$

$$M_{2k} \leq AH^{2k} M_k^2, \quad k = 0, 1, 2, \dots$$

We note that conditions (M.1)+(M.2) are the weaker version of the condition assumed by Komatsu, namely, the condition

$$M_k \leq AH^k \min_{0 \leq q \leq k} M_q M_{k-q}, \quad k = 0, 1, 2, \dots,$$

which ensures the stability under the application of ultradifferential operators. The assumptions (M.0), (M.1) and (M.2) are weaker than those imposed by Komatsu in [Kom73, Kom77] who was also assuming

$$(M.3') \quad \sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

Thus, in [Kom73, Kom77, Kom82] Komatsu investigated classes of ultradifferentiable functions on \mathbb{R}^n associated to the sequences $\{M_k\}$, namely, the space of functions $\psi \in C^\infty(\mathbb{R}^n)$ such that for every compact $K \subset \mathbb{R}^n$ there exist $h > 0$ and a constant C such that

$$\sup_{x \in K} |\partial^\alpha \psi(x)| \leq Ch^{|\alpha|} M_{|\alpha|}. \quad (2.7)$$

Sometimes one also assumes logarithmic convexity, i.e. the condition

$$(\text{logarithmic convexity}) \quad M_k^2 \leq M_{k-1} M_{k+1}, \quad k = 1, 2, 3, \dots$$

This is useful but not restrictive, namely, one can always find an alternative collection of M_k 's defining the same class but satisfying the logarithmic convexity condition, see e.g. Rudin [Rud74, 19.6]. We also refer to Rudin [Rud74] and to Komatsu [Kom73] for examples of different classes satisfying (2.7). These include analytic and Gevrey functions, quasi-analytic and non-quasi-analytic functions (characterised by the Denjoy-Carleman theorem), and many others.

Given a space of ultradifferentiable functions satisfying (2.7) we can define a space of ultradistributions as its dual. Then, among other things, in [Kom77] Komatsu showed that under the assumptions (M.0), logarithmic convexity, (M.1) and (M.3'), f is an ultradistribution supported in $K \subset \mathbb{R}^n$ if and only if there exist L and C such that

$$|\tilde{f}(\xi)| \leq C \exp(M(L\xi)), \quad \xi \in \mathbb{R}^n, \quad (2.8)$$

and in addition for each $\epsilon > 0$ there is a constant C_ϵ such that

$$|\tilde{f}(\zeta)| \leq C_\epsilon \exp(H_K(\zeta) + \epsilon|\zeta|), \quad \zeta \in \mathbb{C}^n, \quad (2.9)$$

where $\tilde{f}(\zeta) = \langle e^{-i\zeta x}, f(x) \rangle$ is the Fourier-Laplace transform of f , $M(r) = \sup_{k \in \mathbb{N}} \log \frac{r^k}{M_k}$ and $H_K(\zeta) = \sup_{x \in K} \text{Im} \langle x, \zeta \rangle$. There are other versions of these estimates given by, for example, Roumieu [Rou63] or Neymark [Ney69]. Moreover, by further strengthening assumptions (M.1), (M.2) and (M.3') one can prove a version of these conditions without the term $\epsilon|\zeta|$ in (2.9), see [Kom77, Theorem 1.1] for the precise formulation.

We now give an analogue of Komatsu's definition on a compact C^∞ manifold X .

Definition 2.1. *The class $\Gamma_{\{M_k\}}(X)$ is the space of C^∞ functions ϕ on X such that there exist $h > 0$ and $C > 0$ such that we have*

$$\|E^k \phi\|_{L^2(X)} \leq Ch^{\nu k} M_{\nu k}, \quad k = 0, 1, 2, \dots \quad (2.10)$$

We can make several remarks concerning this definition.

Remark 2.2. (1) *Taking L^2 -norms in (2.10) is convenient for a number of reasons. But we can already note that by embedding theorems and properties of the sequence $\{M_k\}$ this is equivalent to taking L^∞ -norms, or to evaluating the corresponding action of a frame of vector fields (instead of the action of powers of a single operator E) on functions, see Theorem 2.3.*

(2) *This is also equivalent to classes of functions belonging to the corresponding function spaces in local coordinate charts, see Theorem 2.3, (v). In order to ensure that we cover the cases of analytic and Gevrey functions we will be assuming that X and E are analytic.*

(3) *The advantage of Definition 2.1 is that we do not refer to local coordinates to introduce the class $\Gamma_{\{M_k\}}(X)$. This allows for definition of analogues of analytic or Gevrey functions even if the manifold X is 'only' smooth. For example, by taking $M_k = k!$, we obtain the class $\Gamma_{\{k!\}}(X)$ of functions satisfying the condition*

$$\|E^k \phi\|_{L^2(X)} \leq Ch^{\nu k} M_{\nu k}, \quad k = 0, 1, 2, \dots \quad (2.11)$$

If X and E are analytic, we will show in Corollary 2.5 that this is precisely the class of analytic functions¹ on X . However, if X is 'only' smooth, the space of locally analytic functions does not make sense while definition given by (2.11) still does. We also note that such space $\Gamma_{\{k!\}}(X)$ is still meaningful, for example it contains constants (if E is a differential operator), as well as the eigenfunctions of the operator E .

We summarise properties of the space $\Gamma_{\{M_k\}}(X)$ as follows:

Theorem 2.3. *We have the following properties:*

- (i) *The space $\Gamma_{\{M_k\}}(X)$ is independent of the choice of an operator $E \in \Psi_e^\nu(X)$, i.e. $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if (2.10) holds for one (and hence for all) elliptic pseudo-differential operators $E \in \Psi_e^\nu(X)$.*
- (ii) *We have $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist constants $h > 0$ and $C > 0$ such that*

$$\|E^k \phi\|_{L^\infty(X)} \leq Ch^{\nu k} M_{\nu k}, \quad k = 0, 1, 2, \dots \quad (2.12)$$

- (iii) *Let $\partial_1, \dots, \partial_N$ be a frame of smooth vector fields on X (so that $\sum_{j=1}^N \partial_j^2$ is elliptic). Then $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist $h > 0$ and $C > 0$ such that*

$$\|\partial^\alpha \phi\|_{L^\infty(X)} \leq Ch^{|\alpha|} M_{|\alpha|}, \quad (2.13)$$

for all multi-indices α , where $\partial^\alpha = \partial_{j_1}^{\alpha_1} \dots \partial_{j_K}^{\alpha_K}$ with $1 \leq j_1, \dots, j_K \leq N$ and $|\alpha| = \alpha_1 + \dots + \alpha_K$.

¹ An alternative argument could be to use Theorem 2.4 for the characterisation of this class in terms of the eigenvalues of E , and then use Seeley's result [See69] showing that this is the analytic class.

(iv) We have $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist $h > 0$ and $C > 0$ such that

$$\|\partial^\alpha \phi\|_{L^2(X)} \leq Ch^{|\alpha|} M_{|\alpha|}, \quad (2.14)$$

for all multi-indices α as in (iii).

(v) Assume that X and E are analytic and that there is a constant $C > 0$ such that $M_k \geq Ck!$ for all k . Then the class $\Gamma_{\{M_k\}}(X)$ is preserved by analytic changes of variables, and hence is well-defined on X . Moreover, in every local coordinate chart, it consists of functions locally belonging to the class $\Gamma_{\{M_k\}}(\mathbb{R}^n)$.

The important example of the situation in Theorem 2.3, Part (v), is that of Gevrey classes $\gamma^s(X)$ of ultradifferentiable functions, when we have $\gamma^s(X) = \Gamma_{\{M_k\}}(X)$ with the constants $M_k = (k!)^s$ for $s \geq 1$. By Theorem 2.3, Part (v), this is the space of Gevrey functions on X , i.e. functions which belong to the Gevrey classes $\gamma^s(\mathbb{R}^n)$ in all local coordinate charts, i.e. such that there exist $h > 0$ and $C > 0$, such that

$$\|\partial^\alpha \psi\|_{L^\infty(\mathbb{R}^n)} \leq Ch^{|\alpha|} (|\alpha|!)^s,$$

for all localisations ψ of the function ϕ on X , and for all multi-indices α .

If $s = 1$, this is the space of analytic functions.

For the sequence $\{M_k\}$, we define the *associated function* as

$$M(r) := \sup_{k \in \mathbb{N}} \log \frac{r^{\nu k}}{M_{\nu k}}, \quad r > 0,$$

and we may set $M(0) := 0$. We briefly establish a simple property of eigenvalues λ_l useful in the sequel, namely that for every $q, L > 0$ and $\delta > 0$ there exists $C > 0$ such that we have

$$\lambda_l^q e^{-\delta M(L\lambda_l^{1/\nu})} \leq C \quad \text{uniformly in } l \geq 1. \quad (2.15)$$

Indeed, from the definition of the function M it follows that

$$\lambda_l^q e^{-\delta M(L\lambda_l^{1/\nu})} \leq \lambda_l^q \frac{M_{\nu p}^\delta}{L^{\nu p \delta} \lambda_l^{p\delta}},$$

for every $p \in \mathbb{N}$. In particular, using this with p such that $p\delta = q + 1$, we obtain

$$\lambda_l^q e^{-\delta M(L\lambda_l^{1/\nu})} \leq \frac{M_{\nu p}^\delta}{L^{\nu(q+1)} \lambda_l} \leq C$$

uniformly in $l \geq 1$, implying (2.15).

Now we characterise the class $\Gamma_{\{M_k\}}(X)$ of ultradifferentiable functions in terms of eigenvalues of operator E . Unless stated explicitly, we usually assume that X and E are only smooth (i.e. not necessarily analytic).

Theorem 2.4. *Assume conditions (M.0), (M.1), (M.2). Then $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist constants $C > 0, L > 0$ such that*

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C \exp\{-M(L\lambda_l^{1/\nu})\} \quad \text{for all } l \geq 1,$$

where $M(r) = \sup_k \log \frac{r^{\nu k}}{M_{\nu k}}$.

For the Gevrey class $\gamma^s(X) = \Gamma_{\{(k!)^s\}}(X)$ of (Gevrey-Roumieu) ultradifferentiable functions we have $M(r) \simeq r^{1/s}$. Indeed, using the inequality $\frac{t^N}{N!} \leq e^t$ we get

$$M(r) = \sup_k \log \frac{r^{\nu k}}{((\nu k)!)^s} \leq \sup_\ell \log \left[\frac{(r^{1/s})^\ell}{\ell!} \right]^s \leq \sup_\ell \log \left[e^{sr^{1/s}} \right] = sr^{1/s}.$$

On the other hand, first we note the inequality

$$\inf_{p \in \mathbb{N}} (2p)^{2ps} r^{-2p} \leq \exp \left(-\frac{s}{8e} r^{1/s} \right),$$

see [DR14a, Formula (3.20)] for the proof. Using the inequality $(k + \nu)^{k+\nu} \leq (4\nu)^k k^k$ for $k \geq \nu$, an analogous proof yields the inequality

$$\inf_{p \in \mathbb{N}} (\nu p)^{\nu ps} r^{-\nu p} \leq \exp \left(-\frac{s}{4\nu e} r^{1/s} \right).$$

This and the inequality $p! \leq p^p$ imply

$$\begin{aligned} \exp(M(r)) &= \sup_k \frac{r^{\nu k}}{((\nu k)!)^s} = \frac{1}{\inf_k \{(r^{1/s})^{-\nu k} ((\nu k)!)^s\}} \\ &\geq \frac{1}{\inf_k \{(r^{1/s})^{-\nu k} (\nu k)^{\nu k}\}^s} \geq \frac{1}{\exp\{-\frac{s}{4\nu e} r^{1/s}\}} = \exp \left(\frac{s}{4\nu e} r^{1/s} \right). \end{aligned}$$

Combining both inequalities we get

$$\frac{s}{4\nu e} r^{1/s} \leq M(r) \leq sr^{1/s}. \quad (2.16)$$

Consequently, we obtain

Corollary 2.5. *Let X and E be analytic  let $s \geq 1$. We have $\phi \in \gamma^s(X)$ if and only if there exist constants $C > 0$, $L > 0$ such that*

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C \exp(-L\lambda_l^{\frac{1}{s\nu}}) \quad \text{for all } l \geq 0.$$

In particular, for $s = 1$, we recover the characterisation of analytic functions in (2.6).

We now turn to the eigenfunction expansions of the corresponding spaces of ultradistributions.

Definition 2.6. *The space $\Gamma'_{\{M_k\}}(X)$ is the set of all linear forms u on $\Gamma_{\{M_k\}}(X)$ such that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$|u(\phi)| \leq C_\epsilon \sup_\alpha \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \sup_{x \in X} |E^{|\alpha|} \phi(x)|$$

holds for all $\phi \in \Gamma_{\{M_k\}}(X)$.

We can define the Fourier coefficients of such u by

$$\widehat{u}(e_l^k) := u(\overline{e_l^k}) \quad \text{and} \quad \widehat{u}(l) := \widehat{u}(e_l) := \left[u(\overline{e_l^k}) \right]_{k=1}^{d_l}.$$

Theorem 2.7. *Assume conditions (M.0), (M.1), (M.2). We have $u \in \Gamma'_{\{M_k\}}(X)$ if and only if for every $L > 0$ there exists $K = K_L > 0$ such that*

$$\|\widehat{u}(l)\|_{\text{HS}} \leq K \exp \left(M(L\lambda_l^{1/\nu}) \right)$$

holds for all $l \in \mathbb{N}$.

The spaces of ultradifferentiable functions in Definition 2.1 can be viewed as the spaces of Roumieu type. With natural modifications the results remain true for spaces of Beurling type. We summarise them below. We will not give complete proofs but can refer to [DR14a] for details of such modifications in the context of compact Lie groups.

The class $\Gamma_{(M_k)}(X)$ is the space of C^∞ functions ϕ on X such that for every $h > 0$ there exists $C_h > 0$ such that we have

$$\|E^k \phi\|_{L^2(X)} \leq C_h h^{\nu k} M_{\nu k}, \quad k = 0, 1, 2, \dots \quad (2.17)$$

The counterpart of Theorem 2.3 holds for this class as well, and we have

Theorem 2.8. *Assume conditions (M.0), (M.1), (M.2). We have $\phi \in \Gamma_{(M_k)}(X)$ if and only if for every $L > 0$ there exists $C_L > 0$ such that*

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C_L \exp\{-M(L\lambda_l^{1/\nu})\} \quad \text{for all } l \geq 1.$$

For the dual space and for the α -dual, the following statements are equivalent

- (i) $v \in \Gamma'_{(M_k)}(X)$;
- (ii) $v \in [\Gamma_{(M_k)}(X)]^\wedge$;
- (iii) *there exists $L > 0$ such that we have*

$$\sum_{l=1}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} < \infty;$$

- (iv) *there exist $L > 0$ and $K > 0$ such that*

$$\|v_l\|_{\text{HS}} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right)$$

holds for all $l \in \mathbb{N}$.

The proof of Theorem 2.8 is similar to the proof of the corresponding results for the spaces $\Gamma_{\{M_k\}}(X)$, and we omit the repetition. The only difference is that we need to use the Köthe theory of sequence spaces at one point, but this can be done analogous to [DR14a], so we may omit the details. Finally, we note that given the characterisation of α -duals, one can readily prove that they are *perfect*, namely, that

$$[\Gamma_{\{M_k\}}(X)]^\wedge = ([\Gamma_{\{M_k\}}(X)]^\wedge)^\wedge \quad \text{and} \quad [\Gamma_{(M_k)}(X)]^\wedge = ([\Gamma_{(M_k)}(X)]^\wedge)^\wedge, \quad (2.18)$$

see Definition 4.1 and condition (4.1) for their definition. Again, once we have, for example, Theorem 2.4 and Theorem 4.2, the proof of (2.18) is purely functional analytic and can be done almost identically to that in [DR14a], therefore we will omit it.

3. PROOFS

First we prove Theorem 2.3 clarifying the definition of the class $\Gamma_{\{M_k\}}(X)$. In the proof as well as in further proofs the following estimate will be useful:

$$\|e_l^j\|_{L^\infty(X)} \leq C \lambda_l^{\frac{n-1}{2\nu}} \quad \text{for all } l \geq 1. \quad (3.1)$$

This estimate follows, for example, from the local Weyl law [Hör68, Theorem 5.1], see also [DR14b, Lemma 8.5].

Proof of Theorem 2.3. (i) The statement follows from the observation that if $E_1, E_2 \in \Psi_e^\nu(X)$, then there is a constant $A > 0$ such that

$$\|E_1^k \phi\|_{L^2(X)} \leq A^k \|\phi\|_{L^2(X)} \quad (3.2)$$

holds for all $k \in \mathbb{N}_0$ and all $\phi \in C^\infty(X)$. In turn, (3.2) follows from the fact that the pseudo-differential operator $E_1^k \circ E_2^{-k} \in \Psi_e^0(X)$ is bounded on $L^2(X)$ (with E_2^{-k} denoting the parametrix for E_2^k), and by the Calderon-Vaillancourt theorem its operator norm can be estimated by A^k for some constant A depending only on finitely many derivatives of symbols of E_1 and E_2 .

(ii) The equivalence between (2.12) and (2.10) follows by embedding theorems but we give a short argument for it in order to keep a more precise track of the appearing constants. First we note that (2.12) implies (2.10) with a uniform constant in view of the continuous embedding $L^\infty(X) \hookrightarrow L^2(X)$. Conversely, suppose we have (2.10). Let $\phi \in \Gamma_{\{M_k\}}(X)$. Then using (3.1) we can estimate

$$\begin{aligned} \|\phi\|_{L^\infty(X)} &= \left\| \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{\phi}(j, k) e_j^k \right\|_{L^\infty(X)} \\ &\leq \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{\phi}(j, k)| \|e_j^k\|_{L^\infty(X)} \\ &\leq C \|\widehat{\phi}(0)\|_{\text{HS}} + C \sum_{j=1}^{\infty} \sum_{k=1}^{d_j} |\widehat{\phi}(j, k)| \lambda_j^{\frac{n-1}{2\nu}} \\ &\leq C \|\phi\|_{L^2(X)} + C \left(\sum_{j=1}^{\infty} \sum_{k=1}^{d_j} |\widehat{\phi}(j, k)| \lambda_j^{2\ell} \right)^{1/2} \left(\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \lambda_j^{\frac{n-1}{\nu} - 2\ell} \right)^{1/2}, \end{aligned}$$

where we take ℓ large enough so that the very last sum converges, see (2.2). This implies

$$\begin{aligned} \|\phi\|_{L^\infty(X)} &\leq C \|\phi\|_{L^2(X)} + C' \left(\sum_{j=1}^{\infty} \sum_{k=1}^{d_j} |\widehat{\phi}(j, k)| \lambda_j^{2\ell} \right)^{1/2} \\ &\leq C' (\|\phi\|_{L^2(X)} + \|E^\ell \phi\|_{L^2(X)}) \quad (3.3) \end{aligned}$$

by Plancherel's formula. We note that (3.3) follows, in principle, also from the local Sobolev embedding due to the ellipticity of E , however the proof above provides us with a uniform constant. Using (3.3) and (M.1) we can estimate

$$\begin{aligned} \|E^m \phi\|_{L^\infty(X)} &\leq C \|E^m \phi\|_{L^2(X)} + C' \|E^{\ell+m} \phi\|_{L^2(X)} \\ &\leq C h^{\nu m} M_{\nu m} + C'' h^{\nu(\ell+m)} M_{\nu(\ell+m)} \\ &\leq C''' h^{\nu(\ell+m)} h H^{\nu(\ell+m)-1} M_{\nu(\ell+m)-1} \\ &\leq \dots \\ &\leq C'''' h^{\nu(\ell+m)} h^{\nu\ell} H^{B_\ell + \nu\ell m} M_{\nu m} \\ &\leq C_\ell A^{\nu m} M_{\nu m} \end{aligned}$$

for some A independent of m , yielding (2.12).

(iv) We note that the proof as in (ii) also shows the equivalence of (2.13) and (2.14). Moreover, once we have condition (2.13), the statement (v) follows by using $M_k \geq Ck!$ and the chain rule.

(iii) Given properties (ii) and (iv), we need to show that (2.10) or (2.12) are equivalent to (2.13) or to (2.14). Using property (i), we can take $E = \sum_{j=1}^N \partial_j^2$.

To prove that (2.13) implies (2.12) we use the multinomial theorem², with the notation for multi-indices as in (iii). With $Y_j \in \{\partial_1, \dots, \partial_N\}$, $1 \leq j \leq |\alpha|$, and $\nu = 2$, we can estimate

$$\begin{aligned} \left| \left(\sum_{j=1}^N \partial_j^2 \right)^k \phi(x) \right| &\leq C \sum_{|\alpha|=k} \frac{k!}{\alpha!} |Y_1^2 \dots Y_{|\alpha|}^2 \phi(x)| \\ &\leq C \sum_{|\alpha|=k} \frac{k!}{\alpha!} A^{2|\alpha|} M_{2|\alpha|} \\ &\leq CA^{2k} M_{2k} \sum_{|\alpha|=k} \frac{k! N^{|\alpha|}}{|\alpha|!} \\ &\leq C_1 A^{2k} M_{2k} N^k 2^k \\ &\leq C_2 A_1^{2k} M_{2k}, \end{aligned}$$

with $A_1 = 2NA$, implying (2.12).

Conversely, we argue that (2.10) implies (2.14). We write $\partial^\alpha = P_\alpha \circ E^k$ with $P_\alpha = \partial^\alpha \circ E^{-k}$. Here and below, in order to use precise calculus of pseudo-differential operators we may assume that we work on the space $L^2(X) \setminus H_0$. The argument similar to that of (i) implies that there is a constant $A > 0$ such that $\|P_\alpha \phi\|_{L^2(X)} \leq A^k \|\phi\|_{L^2(X)}$ for all $|\alpha| \leq \nu k$. Therefore, we get

$$\|\partial^\alpha \phi\|_{L^2} = \|P_\alpha \circ E^k \phi\|_{L^2} \leq CA^k \|E^k \phi\|_{L^2} \leq C' A^k h^{\nu k} M_{\nu k} \leq C' A_1^{\nu k} M_{\nu k},$$

where we have used the assumption (2.10), and with C' and $A_1 = A^{1/\nu} h$ independent of k and α . This completes the proof of (iii) and of the theorem. \square

Proof of Theorem 2.4. “Only if” part. Let $\phi \in \Gamma_{\{M_k\}}(X)$. By the Plancherel formula (2.4) we have

$$\|E^m \phi\|_{L^2(X)}^2 = \sum_j \|\widehat{E^m \phi}(j)\|_{\text{HS}}^2 = \sum_j \|\lambda_j^m \widehat{\phi}(j)\|_{\text{HS}}^2 = \sum_j \lambda_j^{2m} \|\widehat{\phi}(j)\|_{\text{HS}}^2. \quad (3.4)$$

Now since $\|E^m \phi\|_{L^2(X)} \leq Ch^{\nu m} M_{\nu m}$, from (3.4) we get

$$\lambda_j^m \|\widehat{\phi}(j)\|_{\text{HS}} \leq Ch^{\nu m} M_{\nu m}$$

which implies

$$\|\widehat{\phi}(j)\|_{\text{HS}} \leq Ch^{\nu m} M_{\nu m} \lambda_j^{-m} \quad \text{for all } j \geq 1. \quad (3.5)$$

²But we use it in the form adapted to the noncommutativity of vector fields, namely, although the coefficients are all equal to one in the noncommutative form, the multinomial coefficient appears once we make a choice for $\alpha = (\alpha_1, \dots, \alpha_N)$.

Now, from the definition of $M(r)$ it follows that

$$\inf_{k \in \mathbb{N}} r^{-\nu k} M_{\nu k} = \exp(-M(r)), \quad r > 0. \quad (3.6)$$

Indeed, this identity follows by writing

$$\exp(M(r)) = \exp\left(\sup_k \log \frac{r^{\nu k}}{M_{\nu k}}\right) = \sup_k \left(\exp \log \frac{r^{\nu k}}{M_{\nu k}}\right) = \sup_k \left(\frac{r^{\nu k}}{M_{\nu k}}\right)$$

and using the identity $\inf_k r^{-k} = \frac{1}{\sup_k r^k}$. Setting $r = \frac{\lambda_j^{1/\nu}}{h}$, from (3.5) and (3.6) we can estimate

$$\begin{aligned} \|\widehat{\phi}(j)\|_{\text{HS}} &\leq C \inf_{m \geq 1} \left\{ \frac{h^{\nu m}}{\lambda_j^m} M_{\nu m} \right\} = C \inf_{m \geq 1} r^{-\nu m} M_{\nu m} \\ &= C \exp\left(-M\left(\frac{\lambda_j^{1/\nu}}{h}\right)\right) = C \exp\left(-M\left(L\lambda_j^{1/\nu}\right)\right), \end{aligned}$$

where $L = h^{-1}$.

“If” Part. Let $\phi \in C^\infty(X)$ be such that

$$\|\widehat{\phi}(j)\|_{\text{HS}} \leq C \exp\left(-M\left(L\lambda_j^{1/\nu}\right)\right)$$

holds for all $j \geq 1$. Then by Plancherel’s formula we have

$$\begin{aligned} \|E^m \phi\|_{L^2(X)}^2 &= \sum_{j=0}^{\infty} \lambda_j^{2m} \|\widehat{\phi}(j)\|_{\text{HS}}^2 \\ &\leq C \sum_{j=1}^{\infty} \lambda_j^{2m} \exp\left(-M(L\lambda_j^{1/\nu})\right) \exp\left(-M(L\lambda_j^{1/\nu})\right). \end{aligned} \quad (3.7)$$

Now we observe that

$$\lambda_j^{2m} \exp\left(-M(L\lambda_j^{1/\nu})\right) \leq \frac{\lambda_j^{2m}}{\sup_{p \in \mathbb{N}} \frac{L^{\nu p} \lambda_j^p}{M_{\nu p}}} \leq \frac{\lambda_j^{2m}}{L^{\nu p} \lambda_j^p} M_{\nu p} \quad (3.8)$$

for any $p \in \mathbb{N}$. Using this with $p = 2m + 1$, we get

$$\lambda_j^{2m} \exp\left(-M(L\lambda_j^{1/\nu})\right) \leq \frac{1}{L^{\nu(2m+1)}} \left(\frac{\lambda_j^{2m}}{\lambda_j^{2m+1}}\right) M_{\nu(2m+1)}.$$

Then, by this and property (M.1) of the sequence $\{M_k\}$, we get

$$\begin{aligned} \|E^m \phi\|_{L^2(X)}^2 &\leq C \sum_{j=1}^{\infty} \frac{1}{L^{\nu(2m+1)}} \left(\frac{\lambda_j^{2m}}{\lambda_j^{2m+1}}\right) M_{\nu(2m+1)} \exp\left(-M(L\lambda_j^{1/\nu})\right) \\ &\leq C_1 A H^{2\nu m} M_{2\nu m} \sum_{j=1}^{\infty} \lambda_j^{-1} \exp\left(-M(L\lambda_j^{1/\nu})\right), \end{aligned} \quad (3.9)$$

for some $A, H > 0$. Now we note that for all $j \geq 1$ we have

$$\lambda_j^{-1} \exp\left(-M(L\lambda_j^{1/\nu})\right) \leq \frac{\lambda_j^{-1} M_{\nu p}}{\lambda_j^p L^{\nu p}} = \frac{M_{\nu p}}{L^{\nu p}} \frac{1}{\lambda_j^{p+1}}. \quad (3.10)$$

In particular, in view of (2.2), for p such that $p+1 > n/\nu$ we obtain

$$\sum_{j=1}^{\infty} \lambda_j^{-1} \exp\left(-M(L\lambda_j^{1/\nu})\right) \leq \frac{M_{\nu p}}{L^{\nu p}} \sum_{j=1}^{\infty} \frac{1}{\lambda_j^{p+1}} < \infty.$$

This, (M.2) and (3.9) imply

$$\|E^m \phi\|_{L^2(X)} \leq \tilde{A} \tilde{H}^{\nu m} M_{\nu m}$$

and hence $\phi \in \Gamma_{\{M_k\}}(X)$. \square

Now we will check that Seeley's characterisation for analytic functions in [See69] follows from our theorem.

Proof of Corollary 2.5. The first part of the statement is a direct consequence of Theorem 2.4 and (2.16), so we only have to prove (2.6). Let X be a compact manifold and E an analytic, elliptic, positive differential operator of order ν . Let $\{\phi_k\}$ and $\{\lambda_k\}$ be respectively the eigenfunctions and eigenvalues of E , i.e. $E\phi_k = \lambda_k \phi_k$. As mentioned in the introduction, Seeley showed in [See69] that a C^∞ function $f = \sum_j f_j \phi_j$ is analytic if and only if there is a constant $C > 0$ such that for all $k \geq 0$ we have

$$\sum_j \lambda_j^{2k} |f_j|^2 \leq ((\nu k)!)^2 C^{2k+2}.$$

By Plancherel's formula this means that

$$\|E^k f\|_{L^2(X)}^2 = \sum_j \lambda_j^{2k} |f_j|^2 \leq ((\nu k)!)^2 C^{2k+2}.$$

For the class of analytic functions we can take $M_k = k!$ in Definition 2.1, and then by Theorem 2.4 we conclude that f is analytic if and only if

$$\|\widehat{f}(j)\|_{\text{HS}} \leq C \exp(-L\lambda_j^{1/\nu})$$

or to

$$|f_j| \leq C' \exp(-L'\lambda_j^{1/\nu}),$$

with $M(r) = \sup_p \log \frac{r^{\nu p}}{(\nu p)!} \simeq r$ in view of (2.16). This implies (2.6) and hence also Seeley's result [See69] that $f = \sum_j f_j \phi_j$ is analytic if and only if the sequence $\{A\lambda_j^{1/\nu} f_j\}$ is bounded for some $A > 1$. \square

4. α -DUALS

In this section we characterise the α -dual of the space $\Gamma_{\{M_k\}}(X)$. This will be instrumental in proving the characterisation for spaces of ultradistributions in Theorem 2.7.

Definition 4.1. The α -dual of the space $\Gamma_{\{M_k\}}(X)$ of ultradifferentiable functions, denoted by $[\Gamma_{\{M_k\}}(X)]^\wedge$, is defined as

$$\left\{ v = (v_l)_{l \in \mathbb{N}_0} : \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} |(v_l)_j| |\widehat{\phi}(l, j)| < \infty, v_l \in \mathbb{C}^{d_l}, \text{ for all } \phi \in \Gamma_{\{M_k\}}(X) \right\}.$$

We will also write $v(l, j) = (v_l)_j$ and $\|v_l\|_{\text{HS}} = (\sum_{j=1}^{d_l} |v(l, j)|^2)^{1/2}$.

It will be useful to have the definition of the second dual $([\Gamma_{\{M_k\}}(X)]^\wedge)^\wedge$ as the space of $w = (w_l)_{l \in \mathbb{N}_0}$, $w_l \in \mathbb{C}^{d_l}$ such that

$$\sum_{l=0}^{\infty} \sum_{j=1}^{d_l} |(w_l)_j| |(v_l)_j| < \infty \quad \text{for all } v \in [\Gamma_{\{M_k\}}(X)]^\wedge. \quad (4.1)$$

We have the following characterisations of the α -duals.

Theorem 4.2. Assume conditions (M.0), (M.1) and (M.2). The following statements are equivalent.

- (i) $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$;
- (ii) for every $L > 0$ we have

$$\sum_{l=1}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} < \infty;$$

- (iii) for every $L > 0$ there exists $K = K_L > 0$ such that

$$\|v_l\|_{\text{HS}} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right)$$

holds for all $l \in \mathbb{N}$.

Proof. (i) \implies (ii). Let $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$, $L > 0$, and let $\phi \in C^\infty(X)$ be such that

$$\widehat{\phi}(l, j) = e^{-M(L\lambda_l^{1/\nu})}.$$

We claim that $\phi \in \Gamma_{\{M_k\}}(X)$. First, using (2.1), for some q we have

$$\|\widehat{\phi}(l)\|_{\text{HS}} = d_l^{1/2} e^{-M(L\lambda_l^{1/\nu})} \leq C \lambda_l^q e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} \quad (4.2)$$

for all $l \geq 1$. Estimates (2.15) and (4.2) imply that $\lambda_l^q e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} \leq C$ and hence

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C' e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})}$$

holds for all $l \geq 1$. The claim would follow if we can show that

$$e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} \leq e^{-M(L_2\lambda_l^{1/\nu})} \quad \text{holds for } L_2 = \frac{L}{\sqrt{AH}}, \quad (4.3)$$

where A and H are constants in the condition (M.2). Now, substituting $p = 2q$, we note that

$$e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} = \inf_{p \in \mathbb{N}} \frac{M_{\nu p}^{1/2}}{L^{\nu p/2} \lambda_l^{p/2}} \leq \inf_{q \in \mathbb{N}} \frac{M_{2\nu q}^{1/2}}{L^{\nu q} \lambda_l^q}. \quad (4.4)$$

Using property (M.2) we can estimate

$$M_{2\nu q} \leq AH^{2\nu k} M_{\nu q}^2.$$

This and (4.4) imply

$$e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} \leq \frac{M_{\nu q}}{L_2^{\nu q} \lambda_l^q},$$

where $L_2 = \frac{L}{\sqrt{AH}}$. Taking infimum in $q \in \mathbb{N}$, we obtain

$$e^{-\frac{1}{2}M(L\lambda_l^{1/\nu})} \leq \inf_{q \in \mathbb{N}} \frac{M_{\nu q}}{L_2^{\nu q} \lambda_l^q} = e^{-M(L_2\lambda_l^{1/\nu})}.$$

Therefore, we get the estimate

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C' \exp\left(-M(L_2\lambda_l^{1/\nu})\right),$$

which means that $\phi \in \Gamma_{\{M_k\}}(X)$ by Theorem 2.4. Finally, this implies that

$$\begin{aligned} \sum_l e^{-M(L\lambda_l^{1/\nu})} \|v_l\|_{\text{HS}} &\leq \sum_l \sum_{j=1}^{d_l} e^{-M(L\lambda_l^{1/\nu})} |v_l(j)| \\ &= \sum_l \sum_{j=1}^{d_l} |\widehat{\phi}(l, j)| |v(l, j)| < \infty \end{aligned}$$

is finite by property (i), implying (ii).

(ii) \implies (i). Let $\phi \in \Gamma_{\{M_k\}}(X)$. Then by Theorem 2.4 there exists $L > 0$ such that

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C \exp\left(-M(L\lambda_l^{1/\nu})\right).$$

Then we can estimate

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} |(v_l)_j| |\widehat{\phi}(l, j)| &\leq \sum_{l=0}^{\infty} \|v_l\|_{\text{HS}} \|\widehat{\phi}(l)\|_{\text{HS}} \\ &\leq C \sum_{l=0}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} < \infty \end{aligned}$$

is finite by the assumption (ii). This implies $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$.

(ii) \implies (iii). We know that for every $L > 0$ we have

$$\sum_l \exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} < \infty.$$

Consequently, there exists K_L such that

$$\exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} \leq K_L$$

holds for all l , which implies (iii).

(iii) \implies (ii). Let $L > 0$. Let us define L_2 as in (4.3). If v satisfies (iii) this means, in particular, that there exists $K = K_{L_2} > 0$ such that

$$\|v_l\|_{\text{HS}} \leq K \exp\left(M(L_2\lambda_l^{1/\nu})\right). \quad (4.5)$$

We also note that by (4.4) we have

$$\exp\left(-\frac{1}{2}M(L\lambda_l^{1/\nu})\right) \leq \frac{M_{\nu p}^{1/2}}{L^{\nu p/2}\lambda_l^{p/2}} \quad \text{for all } p \in \mathbb{N}.$$

From this, (4.3) and (4.5) we conclude

$$\begin{aligned} & \sum_{l=0}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} \\ & \leq \sum_{l=0}^{\infty} \exp\left(-\frac{1}{2}M(L\lambda_l^{1/\nu})\right) \exp\left(-\frac{1}{2}M(L\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} \\ & \leq \sum_{l=0}^{\infty} \exp\left(-\frac{1}{2}M(L\lambda_l^{1/\nu})\right) \exp\left(-M(L_2\lambda_l^{1/\nu})\right) \|v_l\|_{\text{HS}} \\ & \leq K \sum_{l=0}^{\infty} \exp\left(-\frac{1}{2}M(L\lambda_l^{1/\nu})\right) \\ & \leq K + K \sum_{l=1}^{\infty} \frac{M_{\nu p}^{1/2}}{L^{\nu p/2}\lambda_l^{p/2}} \\ & \leq K + C_p \sum_{l=1}^{\infty} \frac{1}{\lambda_l^{p/2}} < \infty \end{aligned}$$

is finite provided we take p large enough in view of (2.2). \square

5. ULTRADISTRIBUTIONS

In this section we prove that the spaces of ultradistributions and α -duals coincide. Together with Theorem 4.2 this implies Theorem 2.7.

Theorem 5.1. *Assume conditions (M.0), (M.1) and (M.2). We have $v \in \Gamma'_{\{M_k\}}(X)$ if and only if $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$.*

Proof. “If” Part. Let $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$. For any $\phi \in \Gamma_{\{M_k\}}(X)$ let us define

$$v(\phi) := \sum_{l=0}^{\infty} \widehat{\phi}(l) \cdot v_l = \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} \widehat{\phi}(l, j) v_l(j).$$

Given $\phi \in \Gamma_{\{M_k\}}(X)$, by Theorem 2.4 there exist $C > 0$ and $L > 0$ such that

$$\|\widehat{\phi}(l)\|_{\text{HS}} \leq C e^{-M(L\lambda_l^{1/\nu})}.$$

Consequently, by Theorem 4.2, we get

$$|v(\phi)| \leq \sum_l \|\widehat{\phi}(l)\|_{\text{HS}} \|v_l\|_{\text{HS}} \leq C \sum_l e^{-M(L\lambda_l^{1/\nu})} \|v_l\|_{\text{HS}} < \infty,$$

which means that $v(\phi)$ is a well-defined linear functional. Next we check that v is continuous. Suppose $\phi_j \rightarrow \phi$ in $\Gamma_{\{M_k\}}(X)$ as $j \rightarrow \infty$, which means that

$$\sup_{\alpha} \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \sup_{x \in X} |E^{|\alpha|}(\phi_j(x) - \phi(x))| \rightarrow 0 \text{ as } j \rightarrow \infty.$$

It follows that

$$\|E^{|\alpha|}(\phi_j(x) - \phi(x))\|_{L^\infty(X)} \leq C_j A^{|\alpha|} M_{\nu|\alpha|},$$

where $C_j \rightarrow 0$ as $j \rightarrow \infty$. From the proof of Theorem 2.4 it follows that

$$\|\widehat{\phi}_j(l) - \widehat{\phi}(l)\|_{\text{HS}} \leq C'_j e^{-M(L\lambda_l^{1/\nu})}$$

with $C'_j \rightarrow 0$. Hence

$$|v(\phi_j - \phi)| \leq \sum_l \|\widehat{\phi}_j(l) - \widehat{\phi}(l)\|_{\text{HS}} \|v_l\|_{\text{HS}} \leq C'_j \sum_l e^{-M(L\lambda_l^{1/\nu})} \|v_l\|_{\text{HS}} \rightarrow 0$$

as $j \rightarrow \infty$. This implies $v \in \Gamma'_{\{M_k\}}(X)$.

“Only if” Part. Let $v \in \Gamma'_{\{M_k\}}(X)$. By definition, this implies

$$|v(\phi)| \leq C_\epsilon \sup_\alpha \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \sup_{x \in X} |E^{|\alpha|} \phi(x)|$$

for all $\phi \in \Gamma_{\{M_k\}}(X)$. In particular,

$$\begin{aligned} |v(\overline{e_l^j})| &\leq C_\epsilon \sup_\alpha \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \sup_{x \in X} |E^{|\alpha|} \overline{e_l^j}(x)| \\ &\leq C_\epsilon \sup_\alpha \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \lambda_l^{|\alpha|} \sup_{x \in X} |e_l^j(x)| \\ &\leq C_\epsilon \sup_\alpha \frac{\epsilon^{|\alpha|} \lambda_l^{|\alpha| + \frac{n-1}{2\nu}}}{M_{\nu|\alpha|}}. \end{aligned}$$

Here in the last line we used the estimate (3.1). Consequently, we get

$$|v(\overline{e_l^j})| \leq C_\epsilon \lambda_l^{\frac{n-1}{2\nu}} \sup_\alpha \frac{\epsilon^{|\alpha|} \lambda_l^{|\alpha|}}{M_{\nu|\alpha|}} \leq C'_\epsilon \sup_\alpha \frac{\epsilon^{|\alpha|} \lambda_l^{|\alpha|+k}}{M_{\nu|\alpha|}} \quad (5.1)$$

with $k := [\frac{n-1}{2\nu}] + 1$. By property (M.1) of the sequence $\{M_k\}$ we can estimate

$$\begin{aligned} M_{\nu(|\alpha|+k)} &\leq A H^{\nu(|\alpha|+k)-1} M_{\nu(|\alpha|+k)-1} \\ &\leq A^2 H^{2\nu(|\alpha|+k)-1-2} M_{\nu(|\alpha|+k)-2} \\ &\vdots \\ &\leq A^{\nu k} H^{\nu k|\alpha|} H^{f(k)} M_{\nu|\alpha|}, \end{aligned}$$

for some $f(k) = f(\nu, k)$ independent of α . This implies

$$M_{\nu|\alpha|}^{-1} \leq A^{\nu k} H^{f(k)} H^{\nu k|\alpha|} M_{\nu(|\alpha|+k)}^{-1}.$$

This and (5.1) imply

$$\begin{aligned} |v(\overline{e_l^j})| &\leq C'_\epsilon \epsilon^{-k} A^{\nu k} H^{f(k)} \sup_\alpha \frac{\epsilon^{|\alpha|+k} (H^{\nu k})^{|\alpha|} \lambda_l^{|\alpha|+k}}{M_{\nu(|\alpha|+k)}} \\ &\leq C'_{\epsilon, k, A} \sup_\alpha \frac{(\epsilon^{1/\nu} H^k)^{\nu(|\alpha|+k)} \lambda_l^{|\alpha|+k}}{M_{\nu(|\alpha|+k)}} \leq C'_{\epsilon, k, A} e^{M(L\lambda_l^{1/\nu})}, \quad (5.2) \end{aligned}$$

with $L = \epsilon^{1/\nu} H^k$. At the same time, it follows from (4.3) that

$$e^{M(L\lambda_l^{1/\nu})} \leq e^{\frac{1}{2}M(L_3\lambda_l^{1/\nu})} \text{ holds for } L = \frac{L_3}{\sqrt{AH}}.$$

This and (2.1) for some q , and then (2.15) imply

$$\|\widehat{v}(e_l)\|_{\text{HS}} \leq C d_l^{1/2} e^{M(L\lambda_l^{1/\nu})} \leq C \lambda_l^q e^{\frac{1}{2}M(L_3\lambda_l^{1/\nu})} \leq C e^{M(L_3\lambda_l^{1/\nu})},$$

that is $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$ by Theorem 4.2. \square

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