FULL BLOW-UP RANGE FOR CO-ROTAIONAL WAVE MAPS TO SURFACES OF REVOLUTION

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Abstract. We construct blow-up solutions of the energy critical wave map equation on $\mathbb{R}^{2+1} \to \mathcal{N}$ with polynomial blow-up rate ($t^{-1-\nu}$ for blow-up at $t = 0$) in the case when $\mathcal{N}$ is a surface of revolution. Here we extend the blow-up range found by Carstea ($\nu > \frac{1}{2}$) based on the work by Krieger, Schlag and Tataru to $\nu > 0$. This work relies on and generalizes the recent result of Krieger and the author where the target manifold is chosen as the standard sphere.

1. Introduction

A *wave map* is a map u from $n + 1$ dimensional Minkowski space-time with signature $(-1, 1, ..., 1)$ to a Riemannian Manifold $\mathcal{N}$. It is defined as a critical point of the action functional, which is the following Lagrangian

$$\mathcal{L}(u) := \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha u, \partial^\alpha u \rangle_{\mathcal{N}} \, d\sigma, \; \partial^\alpha = m^{\alpha\beta} \partial_\beta$$

where $\alpha = 0, 1, ..., n$, and $m^{\alpha\beta}$ is the Minkowski metric.

The wave map $u : \mathbb{R}^{3+1} \to S^3$ has application to *nonlinear sigma model*[4] from quantum field theory in modern physics, so it is very interesting to study the cases when target manifolds are spheres. The case $u : \mathbb{R}^{2+1} \to H^2$ is a model problem arising from the study of *Einstein's equation*[2]. The curvature of the target manifold plays an important role in the global well-posedness properties of the corresponding equation. In the energy critical case (we will explain below what is energy critical) global well-posedness fails for the S^2 target, while it holds for H^2 (see below theorem 1.1 and see[6, 7] and references therein). Another important observation is wave maps are the natural hyperbolic analogues of the much studied *harmonic map* heat flow, which in local coordinates is described by

$$\partial_t u^i = \Delta u^i + \sum_{\alpha=1}^{n} \Gamma^i_{jk} \partial_\alpha u^j \partial^\alpha u^k$$

Consider the following model equation

$$\Box u = N(u, \nabla u), \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \tag{1.1}$$

for some smooth N(., .). Wave maps in local coordinates fall into this category. Major studies of this problem fall into the following directions: i) local existence theory(strong local well-posedness); ii) small data global existence theory(weak global well posed-ness); iii) approaching the large data problem in the critical dimension

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n=2 and hyperbolic target; iv) imposing symmetry: radial and equivariant wave maps in the case n=2; v) singularity formation in the critical dimension. For details of the main results in those directions, we refer the reader to a very well-written survey paper on wave maps by Krieger [5] and the references therein.

In this paper, we study the blow-up solutions of energy critical co-rotational wave map equation on $\mathbb{R}^{2+1} \to \mathcal{N}$ with polynomial blow-up rate in the case when $\mathcal{N}$ is a surface of revolution. Before we move further, we shall explain first about energy critical and definition of co-rotational.

Scaling constraints. Assume that the set of solutions $u(t,x)$ of (1.1) is invariant under the scaling transformation $u(t,x) \to \lambda^{\alpha} u(\lambda t, \lambda x)$.Then one introduces the *critical Sobolev index* $s_c = \frac{n}{2} - \alpha$. Observe that the norm

$$\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c - 1}}$$

is left invariant under the re-scaling. Note that

$$s_c = \frac{n}{2}$$

for wave maps in the local coordinate formulation.

Energy constraints. A quantity

$$E[u] \gtrsim \|u\|_{H^{s_0}} + \|u_t\|_{H^{s_0 - 1}}$$

which is preserved under the flow. Then one distinguishes between: i) energy subcritical $s_c < s_0$: one expects global well-posedness, provided strong local well-posedness in the full subcritical range, or also just for some $s_c < s < s_0$; ii) energy critical $s_c = s_0$: global well-posedness hinges on fine structure of equation; iii) energy supercritical $s_c > s_0$: no global well-posedness for generic large data expected.

Note that when the background is $2+1$-dimensional, wave maps are *energy critical.* This means explicitly the following quantity

$$\mathcal{E}(u) := \int_{\mathbb{R}^2} \left[|u_t|^2 + |\nabla_x u|^2 | \right] dx \tag{1.2}$$

is invariant under the intrinsic scaling (recall that $s_c = n/2$ in the local coordinate formulation)

$$u(t,x) \to u(\lambda t, \lambda x)$$

Co-rotational wave maps. A wave map $u : \mathbb{R}^{2+1} \to M$ is called *equivariant* provided we have

$$u(t, \omega x) = \rho(\omega) u(t,x), \forall \omega \in S^1$$

Here $\rho(\omega)$ acts as an isometry on M and $\omega \in S^1$ acts on $\mathbb{R}^2$ in the canonical fashion as rotations. For global well-posedness of equivariant wave maps we have the following important results by Shatah, Tahvildar-Zadeh [10]

Theorem 1.1 (Shatah, Tahvildar-Zadeh)**.** *Let the target (M,g) be a warped product manifold satisfying a suitable geodesic convexity condition. Then equivariant wave maps $u : \mathbb{R}^{2+1} \to M$ with smooth data stay globally regular.*

However, the case $u : \mathbb{R}^{2+1} \to S^2$ does not satisfy the hypotheses of the preceding theorem. Thus the discovery of the singularity for this case is very crucial. We let S^1 act on S^2 by means of rotations around the z-axis via $\rho(\omega) = k\omega, k \in Z/\{0\}$,

$\omega \in S^1$. Fixing a k, the wave map is then determined in terms of the polar angle, and becomes a scalar equation on $\mathbb{R}^{1+1}$ as follows:

$$-u_{tt} + u_{rr} + \frac{1}{r}u_r = k^2 \frac{\sin(2u)}{2r^2} \tag{1.3}$$

The case k = 1 in particular is called *co-rotational*.

M. Struwe's fundamental work [11] on the structure of singularities of co-rotational Wave maps shows that

Theorem 1.2 (Struwe)**.** *Let u be a smooth co-rotational wave map which cannot be smoothly extended past time T, there exists $t_i \to T$, $\lambda_i \to +\infty$ s.t. on each fixed time slice $t = t_i$, we can write*

$$u(t_i, x) = Q(\lambda(t_i)x) + \varepsilon(t_i, x)$$

where Q is ground state (harmonic map) $Q : \mathbb{R}^2 \to S^2$, while the local energy of ε converges to 0.

Furthermore, Struwe established an upper bound on the blow up rate

$$\lim_{i\to\infty} \lambda(t_i)(T - t_i) = +\infty \tag{1.4}$$

The approach we take starts from [9], where the authors demonstrated a method of building finite time blow-up solutions for critical wave maps by adding corrections to an ansatz generated by rescaling the ground-state harmonic map to form an approximate solution and controlling the errors to zero. The blow-up rate from their paper is $\lambda(t) = t^{-1-\nu}$, with a blow-up range $\nu > \frac{1}{2}$. According to the work [11] by M. Struwe (see above), this result is not optimal (Replace $\lambda(t) = t^{-1-\nu}$ in (1.4), one can see that the optimal range for ν shall be $(0, \infty)$).

In a joint work by the author and Krieger in [3], the blow-up range is extended to the full range $\nu > 0$ which is optimal. It is also interesting to consider the same problem in a more general situation when the target manifold is a surface of revolution. A work on this case which is parallel of [9] was due to Cârstea [1]. However, as in [9], the blow-up range in [1] is not optimal. In this paper, we will indicate how to combine the techniques of [1, 3] to obtain the optimal blow-up range in this setting. For more detailed references concerning the blow-up dynamic of wave maps one can refer to [3].

Let $\mathcal{N}$ be a surface of revolution equipped with a Riemannian metric

$$ds^2 = d\rho^2 + g(\rho)^2 d\theta$$

for $\mathcal{N}$ being produced by rotating the graph of a function $y = f(z)$ around the z-axis.

Remark 1.3*.* A detailed discussion of what properties g shall satisfy can be found in [1]. Those properties will give the relevant properties of the ground state (harmonic map) which one needs to use when proving some intermediate conclusions when building the approximate solutions. What this paper will focus on is the main difference and changes raised because of the new setting of target manifold we have. However, no changes are required according to the parts of proofs relevant to g. Thus, we refer the reader to [1] for the details about what properties g need to satisfy.

In the case of surfaces of revolution, the equation for co-rotational wave maps takes a form similar to (1.3). A simple computation (see [1]) gives

$$-\partial_t^2 u + \partial_r^2 u + \frac{1}{r}\partial_r u = \frac{f(u)}{r^2}, \quad f(u) = g(u)g'(u). \tag{1.5}$$

Pick a stationary solution with finite energy for (1.5) as was shown in [1]. We state our result

Theorem 1.4. *For any $\nu > 0$, there exist $T > 0$ and co-rotational initial data (f, g) with*

$$(f - \pi, g) \in H^{1+\frac{\nu}{2}-}_{\mathbb{R}^2} \times H^{\frac{\nu}{2}-}_{\mathbb{R}^2}$$

a solution $u(t, r)$, $t \in (0, T]$ which blows up at time $t = 0$ and has the following representation:*

$$u(t, r) = Q(\lambda(t)r) + \varepsilon(t, r)$$

where $\lambda(t) = t^{-1-\nu}$, and such that the function

$$(\theta, r) \longrightarrow \big(e^{i\theta}\varepsilon(t, r), e^{i\theta}\varepsilon_t(t, r)\big) \in H^{1+\nu-}(\mathbb{R}^2) \times H^{\nu-}(\mathbb{R}^2)$$

uniformly in t. Also, we have the asymptotic as $t \to 0$

$$\mathcal{E}_{loc}\big(\varepsilon(t, \cdot)\big) \lesssim t^{\nu} \log^2 t$$

2. A overview of the proof for theorem 1.4

In the work on co-rotational wave maps to S^2 target by Krieger, Schlag, and Tataru [9], it was found that solutions exist with the blow-up rate $\lambda(t) = t^{-1-\nu}$, for the continnum of blow-up rates of any $\nu > 1/2$. In a joint work of the author and Krieger [3], this range was extended to $\nu > 0$. Since the construction to be described in this paper is based heavily on that of the previously mentioned works, we recall for the convenience of the readers the basic scheme.

The method of construction relies on building approximate solutions starting from the initial guess $u(t, r) \approx Q(\lambda(t)r)$ where $Q(r)$ is the stationary ground state. If one naively plugs in $Q(\lambda(t)r)$ into the equation, the error term generated is $(r\lambda'(t))^2 Q''(\lambda(t)r) + r\lambda''(t)Q'(\lambda(t)r)$, which turns out to be "large". Thus one cannot directly use perturbative techniques to find the solution. Instead, we first correct the error (within the past light cone from the singularity) using an iterative scheme, until the error becomes sufficiently small. In the following we will using the notation $R = \lambda(t)r$."

Theorem 2.1. *Assume $k \in \mathbf{N}$. There exists an approximate solution $u_{2k-1}(R)$ within the backwards light cone from the singularity for* (1.5) *which can be written as*

$$u_{2k-1}(t, r) = Q(R) + \frac{c_k}{(t\lambda)^2} R\log(1 + R^2) + \frac{\widetilde{c}_k}{(t\lambda)^2} R + O\Big(\frac{(\log(1 + R^2))^2}{(t\lambda)^2}\Big)$$

with a corresponding error of size

$$\begin{aligned} e_{2k-1} :&= \Big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\Big)u_{2k-1} - \frac{f(u_{2k-1})}{2r^2} \\ &= (1 - \frac{R}{\lambda t})^{-\frac{1}{2}+\nu} O\Big(\frac{R(\log(1 + R^2))^2}{(t\lambda)^{2k}}\Big) \end{aligned}$$

*Here we use the identification of the wave map with a function $u(t, r)$ as before.

Here the implied constant in the $O(\dots)$ symbols are uniform in $t \in (0, \delta]$ for some $\delta = \delta(k) > 0$ sufficiently small.

This is proved by means of an iterative scheme (see section 4) that improves the error at each double step. Actually at each step we approximately solve the wave equation first close to $r = 0$ then close to the light cone $r = t$. In both cases it will reduce to solve an ODE (a Sturm-Louville equation). It is important to observe here that the restriction $\nu > \frac{1}{2}$ imposed in [1] does not come in at this stage; in fact, any $\nu > 0$ will suffice. For the sake of readability, only theorem 2.1 as well as the finer representation of the errors as specified in (4.8) will be used in the final proof of the main theorem (the exact solution) in section 3. The reader can treat section 4 as a black box if desired only up to these statements.

In section 3, we complete the approximate solution to the exact one by adding correction via the ansatz $u(t, r) = u_{2k-1}(t, r) + \varepsilon(t, r)$. Before giving the relevant PDE of such term ε. We first renormalize the time t into $\tau := \nu^{-1} t^{-\nu}$, note that with respect to this time, we get

$$\lambda(\tau) := \lambda(t(\tau)) = (\nu\tau)^{\frac{1+\nu}{\nu}}$$

We also have the re-scaled variable $R = \lambda(\tau) r$ respectively. We shall assume that

$$|e_{2k-1}(t, r)| \lesssim \tau^{-N},\ r \leq t$$

for some sufficiently large N, which is possible if we choose k large enough. We shall also assume the fine structure of e_{2k-1} as in section 4, and more specifically as in (4.8). We can complete the approximate solution u_{2k-1} to an exact solution $u = u_{2k-1} + \varepsilon$. , where ε solves the following equation:

$$\begin{aligned} &- \Big[\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)^2 + \frac{\lambda_\tau}{\lambda}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)\Big]\varepsilon + \Big(\partial_R^2 + \frac{1}{R}\partial_R - \frac{f'(Q(R))}{R^2}\Big)\varepsilon \\ &= -\frac{1}{\lambda^2}[e_{2k-1} + N_{2k-1}(\varepsilon)], \end{aligned} \tag{2.1}$$

where

$$N_{2k-1}(\varepsilon) = \frac{1}{r^2}[f'(u_0)\varepsilon - f(u_{2k-2} + \varepsilon) + f(u_{2k-2})]. \tag{2.2}$$

After changing of function $\widetilde{\varepsilon}(\tau, R) = R^{1/2}\varepsilon(\tau, R)$, (2.1) becomes

$$\big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)\widetilde{\varepsilon} - \mathcal{L}\widetilde{\varepsilon} = \lambda^{-2} R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) + e_{2k-1}\big) \tag{2.3}$$

The strategy is to formulate this equation in terms of the Fourier coefficients of $\widetilde{\varepsilon}$ with respect to the generalized Fourier basis associated with $\mathcal{L}$ given by

$$\mathcal{L} = -\partial_R^2 + \frac{3}{4R^2} + V(R), \quad V(R) = -\frac{1}{R^2}[1 - f'(Q(R))]$$

with $Q(R)$ the ground state. Dealing with (2.3), one needs to develop some rather sophisticated spectral theory. The spectral theory of $\mathcal{L}$ follows from [1] (more exactly [9]), we refer the reader to [9] to see a detailed discussion. To find $\widetilde{\varepsilon}$, one employes a fixed point argument in suitable Banach spaces, and it is here, in the treatment of the nonlinear terms with singular weights, that the restriction on ν comes in (see [1, 9]). More precisely (see lemma 7.2 in [1]), this condition is needed there to make sufficient embedding between suitable function spaces to control the nonlinear terms.

In [3], the authors overcome this restriction (in the case while target manifold is sphere). We will employ this method in our problem (while target manifold is surfaces of revolution) in section 3 which is as following:

Firstly, by a more closely analysis of the zeroth iterate (to be explained below) for $\widetilde{\varepsilon}$. We show that one can split this into the sum of two terms, one of which has a regularity gain which lands us in the regime in [9] is applicable, the other of which does not gain regularity but satisfies an a priori L^∞ bound near the symmetry axis $R = 0$. So the relevant terms with a singular weight $R^{-3/2}$ at $R = 0$, such as $R^{-3/2}\widetilde{\varepsilon}^2$ (see section 3) can be estimated without adding any conditions for the regularity. The reason why they can control the part of the zeroth iterate near $R = 0$ comes from the fact that the singular behavior of the approximate solution from the first part of the construction and the error it generates is localized to the boundary of the light cone. Then, by writing the equation for the distorted Fourier transform of $\widetilde{\varepsilon}$ we will show that the higher iterates all differ from the zeroth iterate by terms with a smoothness gain. This will then suffice to show the desired convergence.

Remark 2.2. The proof of Theorem 1.4, unsurprisingly, has large overlap with the constructions of [1, 3]. For brevity we will only indicate in this note the modifications necessary, and will refer the reader to [1, 3] for the proofs of many intermediate steps.

Remark 2.3. In the new situation, the main difficulty for proof of Theorem 1.4 is that we can not write the nonlinear term explicitly. Thus in the relevant step (see step 3 below) when constructing the approximate solutions and in the second part where the 'perturbative scheme' is introduced for the exact solutions, one needs to redo or adjust the proofs for the new nonlinear source term. In [3], the authors correct the inaccuracies in [9] according to the approximate solution step such as the omission of some logarithm factors in the algebra of the special function spaces. In out paper here, the different function spaces are used correspondingly to fix such inaccuracies in [1]. So some part of the arguments need to be restated during the construction of the approximate solutions.

3. Construction of the exact solutions

This is the very end of the proof of the main theorem. However this is where the 'key structure' is introduced following [3] to make it possible to relax the constraint on ν. For the readers who are interested in the construction of the approximate solutions, we give the proof in section 4.

On the base that an approximate solution has been constructed with a corresponding error term which decays rapidly in the renormalized time $\tau := \nu^{-1}t^{-\nu}$, we can complete the approximate solution u_{2k-1} to an exact solution $u = u_{2k-1} + \varepsilon$. After changing of function (which gives us a new relevant $\widetilde{\varepsilon}$, see section 2) and

applying a *distorted Fourier transform** to the equation of $\widetilde{\varepsilon}$ ((2.3) in section 2):
(3.1)

$$\big(-(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2+\frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2+\frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\big)\widetilde{\varepsilon}-\mathcal{L}\widetilde{\varepsilon}=\lambda^{-2}R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon})+e_{2k-1}\big)$$

One shall get a equation of the Fourier coefficients, which we call the *transport equation.*

The main difficulty is caused by the operator $R\partial_R$ which is not diagonal in the Fourier basis. To deal with this, we replace the distorted Fourier transform of $R\partial_R u$ with $2\xi\partial_\xi$ modulo an error which will be treated perturbatively. We define the error operator $\mathcal{K}$ by

$$\widehat{R\partial_R u}=-2\xi\partial_\xi\widehat{u}+\mathcal{K}\widehat{u}$$

where $\widehat{f}=\mathcal{F}f$ is the distorted Fourier transform.

To proceed further, we have to precisely understand the structure of the 'transference operator' $\mathcal{K}$. Make the

Definition 3.1. *We call an operator $\widetilde{\mathcal{K}}$ to be 'smoothing', provided it enjoys the mapping property*

$$\widetilde{\mathcal{K}}:L^{2,\alpha}_\rho\longrightarrow L^{2,\alpha+\frac{1}{2}}_\rho\ \ \forall\alpha$$

For the definition of a weighted L^2-space $L^{2,\alpha}_\rho$, we have

$$\|u\|_{L^{2,\alpha}_\rho}:=\big(\int_0^\infty|u(\xi)|^2\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi\big)^{\frac{1}{2}}$$

If we put the terms with a 'smooth' property to the right hand side of the equality in the transport equation. Then the Fourier coefficients (we call them $x(\tau,\xi)$) of $\widetilde{\varepsilon}$ with respect to the generalized Fourier basis satisfy

$$\mathcal{D}_\tau^2x+\xi x=f(x,\widetilde{\varepsilon}),\tag{3.2}$$

where we have the operator

$$\mathcal{D}_\tau:=\partial_\tau-\frac{\lambda_\tau}{\lambda}[2\xi\partial_\xi+\frac{3}{2}+\frac{\rho'(\xi)\xi}{\rho(\xi)}]$$

and

$$\begin{aligned}-f=&2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x+(\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2-(\mathcal{K}-\mathcal{K}_0)^2-2[\xi\partial_\xi,\mathcal{K}_0]\big]x\\&+\partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0x+\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon})+e_{2k-1}\big)\big]-c\tau^{-2}x\end{aligned}\tag{3.3}$$

For $\mathcal{K}_0$, according to [1] we give it as (see theorem 5.1[1])

$$\mathcal{K}=-\Big(\frac{3}{2}+\frac{\eta\rho'(\eta)}{\rho(\eta)}\Big)\delta_0(\xi-\eta)+\mathcal{K}_0.$$

*Here the distorted Fourier transform is defined via combining one function $\phi(r,z)$ from the fundamental system for $\mathcal{L}-z$ and its inverse is given using the density function $\rho(\xi)$ of the spectral measure of $\mathcal{L}$, where $\mathcal{L}$ is a key operator raised from the exact solution's equation and $z\in\mathbb{C}$. More precisely, the distorted Fourier transform is

$$\mathcal{F}:\quad\widehat{h}(\xi):=\int_0^\infty\phi(r,\xi)h(r)dr$$

when the inverse is

$$\mathcal{F}^{-1}:\quad h(r):=\int_0^\infty\phi(r,\xi)\widehat{h}(\xi)\rho(\xi)d\xi.$$

The detailed explanation for $\phi(r,z)$ and $\rho(\xi)$ is in [3, 9].

Remark 3.2. Although the problem dealt in [3] is different than ours, the process at this stage is very close. We refer the readers to [3] for those technical details we omit here when deducing the final transport equation (mainly the straightforward computation) and below for brevity.

The explicit solution of (3.2) is given as:

Lemma 3.3 ([3]). *The equation* (3.2) *is formally solved by the following parametrix*

$$x(\tau,\xi)=\int_\tau^\infty \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}S(\tau,\sigma,\lambda^2(\tau)\xi)f(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma=:(Uf)(\tau,\xi) \tag{3.4}$$

One key fact from [3] is we have the following mapping property of the parametrix with respect to suitable Banach spaces:

Lemma 3.4 (lemma 5.6, [3]). *Introducing the norm*

$$\|f\|_{L^{2,\alpha;N}_\rho}:=\sup_{\tau>\tau_0}\tau^N\|f(\tau,\cdot)\|_{L^{2,\alpha}_\rho},$$

we have

$$\|Uf\|_{L^{2,\alpha+\frac{1}{2};N-2}_\rho}\lesssim\|f\|_{L^{2,\alpha;N}_\rho}$$

provided N is sufficiently large.

For the future reference, we will use the following norm:

$$\|h\|_{H^\alpha_\rho}:=\Big(\int_0^\infty x^2(\xi)\langle\xi\rangle^{2\alpha}\,\rho(\xi)\,d\xi\Big)^{\frac{1}{2}}$$

where

$$h(R)=\int_0^\infty\phi(R,\xi)x(\xi)\rho(\xi)\,d\xi.$$

3.1. **Zeroth, first and higher iterative schemes.** After formulating (3.2) as an integral equation, we need to find a suitable fixed point, which will be the desired $x(\tau,\xi)$. We construct these via

$$x(\tau,\xi)=(Uf)(\tau,\xi) \tag{3.5}$$

with $f(x,\widetilde{\varepsilon})$ as in (3.4). To find such a fixed point, we use the iterative scheme

$$x_j(\tau,\xi)=(Uf_{j-1})(\tau,\xi),\ j\geq 1$$

The function f_j is given as

$$\begin{aligned}-f_j=&2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0\big(\partial_\tau-\frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)x_j+(\frac{\lambda_\tau}{\lambda})^2\big[\mathcal{K}^2-(\mathcal{K}-\mathcal{K}_0)^2-2[\xi\partial_\xi,\mathcal{K}_0]\big]x_j\\&+\partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0x_j+\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}_j)+\widetilde{e}_{2k-1}\big)\big]-c\tau^{-2}x_j\end{aligned} \tag{3.6}$$

The zeroth iterate in turn is defined via

$$x_0(\tau,\xi)=(U\lambda^{-2}\mathcal{F}\big[R^{\frac{1}{2}}\big(e_{2k-1}\big)\big])(\tau,\xi);$$

We have the following proposition proved in [3]

Proposition 3.5 (proposition 5.7, [3])**.** *Replacing* e_{2k-1} *with* $\widetilde{e}_{2k-1} \in H^{\frac{\nu}{2}-}_{RdR}$ *where* $\widetilde{e}_{2k-1}|_{r\leq t} = e_{2k-1}$*, we can write*

$$x_0 = x_0^{(1)} + x_0^{(2)}$$

where

$$x^{(1)} \in \tau^{-N} L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}, \quad x^{(2)} \in \tau^{-N} L^{2,1+\frac{\nu}{2}-}_{\rho}$$

and also

$$\chi_{R<1}\widehat{\varepsilon}^{(1)}_0(\tau,R) = \chi_{R<1}\int_0^\infty \phi(R,\xi)x_0^{(1)}(\tau,\xi)\rho(\xi)d\xi \in \tau^{-N}R^{\frac{3}{2}}L^\infty, \quad \chi_{R\geq 1}|\widehat{\varepsilon}^{(1)}_0| \lesssim \tau^{-N}$$

We can rephrase it as following, which is identical to Corollary 5.9 in [3].

Proposition 3.6. *Denote by* P_λ *the frequency localizers*

$$\mathcal{F}\big(P_{<\lambda}f\big)(\xi) = \chi_{<\lambda}(\xi)\big(\mathcal{F}f\big)(\xi)$$

where $\chi_{<\lambda}(\xi)$ *is a smooth cutoff function localizing to* $\xi \lesssim \lambda$*, as in* [9]*; here* λ *is a dyadic number. Then we have*

$$\chi_{R<1}P_{<\lambda}\widehat{\varepsilon}^{(1)}_0 \in \tau^{-N}R^{\frac{3}{2}}L^\infty$$

uniformly in $\lambda > 1$*. Furthermore, for any integer* $l \geq 0$*, we have*

$$\nabla^l_R R^{-\frac{3}{2}}P_{<\lambda}\widehat{\varepsilon}^{(1)}_0 = O(\tau^{-N})$$

uniformly in $\lambda > 1$*.*

Remark 3.7*.* This is the key structure from [3], with which the we are able to invoke lemma 3.11 to control the nonlinear term and prove (3.7) (see below).

Based on lemma 3.4, we know

$$\|Uf_{j-1}\|_{L^{2,1+\frac{\nu}{2}-}_{\rho}} \lesssim \|f_{j-1}\|_{L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}}$$

For the first iterate, the estimate for the most terms in (3.6) follows the same arguments in [3]. We list the unchanged results (see [3] for proof) as following

$$(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi)x_0 \in \tau^{-N-1}L^{2,\frac{\nu}{2}-}_{\rho}$$
$$2\frac{\lambda_\tau}{\lambda}\mathcal{K}_0(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi)x_0 \in \tau^{-N-2}L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}$$
$$(\frac{\lambda_\tau}{\lambda})^2[\mathcal{K}^2 - (\mathcal{K}-\mathcal{K}_0)^2 - 2[\xi\partial_\xi,\mathcal{K}_0]]x_0 \in \tau^{N-2}L^{2,1+\frac{\nu}{2}-}_{\rho}$$
$$\partial_\tau(\frac{\lambda_\tau}{\lambda})\mathcal{K}_0x_0 - c\tau^{-2}x_0 \in \tau^{-N-2}L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}$$

For the nonlinear term, which is the key of the whole argument, we will prove the following in the next section (according to Lemma 3.4)

$$\lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\widehat{\varepsilon}) \in \tau^{-N-2}L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho} \tag{3.7}$$

Let us for now accept the facts above and conclude here the key conclusion in this step

$$\big\|x_1(\tau,\cdot) - x_0(\tau,\cdot)\big\|_{L^{2,1+\frac{\nu}{2}-}_{\rho}} \lesssim N^{-1}\tau^{-N},$$

$$\big\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\big(x_1(\tau,\cdot) - x_0(\tau,\cdot)\big)\big\|_{L^{2,\frac{1}{2}+\frac{\nu}{2}-}_{\rho}} \lesssim N^{-1}\tau^{-N-1}$$

Then we define

$$\widetilde{\varepsilon}_1 = \int_0^\infty \phi(R,\xi)\big(x_1(\tau,\cdot) - x_0(\tau,\cdot)\big)\rho(\xi)d\xi + \int_0^\infty \phi(R,\xi)x_0(\tau,\xi)\rho(\xi)d\xi$$

which will allow us to write

$$\widetilde{\varepsilon}_1 = \widetilde{\varepsilon}^{(1)}(\tau,\cdot) + \widetilde{\varepsilon}^{(2)}(\tau,\cdot)$$

$\widetilde{\varepsilon}^{(1)}(\tau,\cdot)$ and $\widetilde{\varepsilon}^{(2)}(\tau,\cdot)$ satisfy exactly the kind of structure we need to invoke the bound for nonlinear source term in lemma 3.11. Continuing running the iterate scheme will give us the bounds

$$\big\|x_j(\tau,\cdot) - x_{j-1}(\tau,\cdot)\big\|_{L_\rho^{2,1+\frac{\nu}{2}-}} \lesssim N^{-j}\tau^{-N},$$

$$\big\|\big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\big)\big(x_j(\tau,\cdot) - x_{j-1}(\tau,\cdot)\big)\big\|_{L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}} \lesssim N^{-j}\tau^{-N-1}$$

This will close the fix point argument which proves we have

$$x_{\tau,\xi} \in H^{\frac{1}{2}+\frac{\nu}{2}-}, \quad \partial_\tau x_{\tau,\xi} \in H^{\frac{\nu}{2}-}.$$

Through lemma 7.1 in [1] (it was proven in [9]):

Lemma 3.8. *Assume $|\alpha| < \frac{\nu}{2} + \frac{3}{4}$, $g \in IS(1,\mathcal{Q})$. Then we have*

$$\|gf\|_{H_\rho^\alpha} \lesssim \|f\|_{H_\rho^\alpha}$$

It indicates the existence of the exact solution $\varepsilon(\tau,\cdot) \in \tau^{-N}H_{\mathbb{R}^2}^{1+\nu-}$, as well as $\partial_\tau\varepsilon(\tau,\cdot) \in \tau^{-N-1}H_{\mathbb{R}^2}^{\nu-}$.

3.2. **The nonlinear source terms.** We will give an analysis to the new nonlinear source term to complete our work in this section. We recall the following formula for the main source term:

(3.8)

$$\lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\widetilde{\varepsilon}) = \frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f(u_{2k-2} + R^{-\frac{1}{2}}\widetilde{\varepsilon})R^{\frac{1}{2}} + f(u_{2k-2})R^{\frac{1}{2}}\big]$$

$$= \frac{1}{R^2}\big[f'(u_0) - f'(u_{2k-2})\big]\widetilde{\varepsilon} - \frac{1}{R^{\frac{3}{2}}}\sum_{l\geq 2}\frac{1}{l!}f^{(l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^l \tag{3.9}$$

According to the preceding proposition, we have

$$x_0 \in \tau^{-N}L_\rho^{2,\frac{1}{2}+\frac{\nu}{2}-}$$

whence

$$\widetilde{\varepsilon}_0(\tau,\cdot) \in \tau^{-N}H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}$$

This means that for the source terms, we need at least $H_\rho^{\frac{\nu}{2}-}$-regularity. In fact, we can do much better for the term $\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]$. Recall that

$$u_{2k-2} = u_0 + \sum_{j=1}^{2k-2} v_j$$

where we have

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}}IS^3\big(R(\log R)^{2k-1},\mathcal{Q}_{k-1}\big), \quad v_{2k} \in \frac{1}{(t\lambda)^{2k+2}}IS^3\big(R^3(\log R)^{2k-1},\mathcal{Q}_k\big)$$

which implies

$$u_{2k-2} - u_0 \in \frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q})$$

Moreover, we recall some useful results in [1, 9].

Lemma 3.9 (lemma 3.9-10, [1]). *$f^{(2k)}(u_0) \in IS^1(R^{-1})$ and $f^{(2k+1)}(u_0) \in IS^0(1)$. Moreover, if*

$$z \in \frac{1}{(t\lambda)^2} IS^1(R(\log R), \mathcal{Q}),$$

then

$$f^{(2k)}(u_0 + z(R)) \in \frac{1}{(t\lambda)^2} IS^1(R(\log R), \mathcal{Q})$$

and

$$f^{(2k+1)}(u_0 + z(R)) \in IS^0(1, \mathcal{Q}).$$

Thus for $\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]$, we have

$$\frac{f'(u_0) - f'(u_{2k-2})}{R^2} = \frac{\sum_{l\geq 2} \frac{1}{l!} f^{(l)}(u_0)(u_{2k-2} - u_0)^l}{R^2} \in \frac{1}{(t\lambda)^2} IS(1, \mathcal{Q})$$

and lemma 3.8 will give us the following bound

$$\|\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim (t\lambda)^{-2} \|\widetilde{\varepsilon}\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}} \tag{3.10}$$

To deal with the rest 'truly' nonlinear terms, we first split them into two parts

$$\frac{1}{R^{\frac{3}{2}}} \sum_{l\geq 2} \frac{1}{l!} f^{(l)}(u_{2k-2}) \big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^l =$$

$$\frac{1}{R^{\frac{3}{2}}} \sum_{l\geq 1} \frac{1}{l!} f^{(2l)}(u_{2k-2}) \big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l} \tag{3.11}$$

$$+ \frac{1}{R^{\frac{3}{2}}} \sum_{l\geq 1} \frac{1}{l!} f^{(2l+1)}(u_{2k-2}) \big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l+1} \tag{3.12}$$

We can write (3.11) in the form

$$R^{-\frac{3}{2}} \widetilde{\varepsilon}^2 \sum_{l\geq 1} \frac{1}{l!} \frac{f^{(2l)}(u_0 + u_{2k-2} - u_0)}{R} \big(R^{-1}\widetilde{\varepsilon}^2\big)^{l-1}$$

and meanwhile write (3.12) as

$$R^{-3} \widetilde{\varepsilon}^3 \sum_{l\geq 1} \frac{1}{l!} f^{(2l+1)}(u_0 + u_{2k-2} - u_0) \big(R^{-1}\widetilde{\varepsilon}^2\big)^{l-1}$$

According to Lemma 3.9, we observe that

$$\frac{f^{(2l)}(u_0 + u_{2k-2} - u_0)}{R}, \quad f^{(2l+1)}(u_0 + u_{2k-2} - u_0) \in IS^0(1, Q).$$

Thus via Lemma (3.8), we can estimate the H^α_ρ norm of (3.11) and (3.12) by the H^α_ρ norm of

$$R^{-\frac{3}{2}} \widetilde{\varepsilon}^2 q(R^{-1}\widetilde{\varepsilon}^2), \quad R^{-3} \widetilde{\varepsilon}^3 q(R^{-1}\widetilde{\varepsilon}^2)$$

where α here is $\frac{1}{2} + \frac{\nu}{2}-$ and $q(\cdot)$ is a real analytic function.

We recall a very technical and crucial lemma proved in [3]

Lemma 3.10 (lemma 5.12, [3])**.** *Assume that all of f, g, h are either in $H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}}L^\infty$ as well as with their frequency localized constituents $P_{<\lambda}(\cdot) \in \log\lambda R^{\frac{3}{2}}L^\infty$ and $\chi_{R<1}\nabla_R^l\big(R^{-\frac{3}{2}}P_{<\lambda}(\cdot)\big) \in L^\infty$, $l \geq 0$, uniformly in $\lambda > 1$, or in $H_\rho^{1+\frac{\nu}{2}-}$. Then we have*

$$R^{-3}fgh \in H^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}}L^\infty,\ P_{<\lambda}(R^{-3}fgh) \in \log\lambda R^{\frac{3}{2}}L^\infty,\ P_{<\lambda}(R^{-3}fgh) \in RL^\infty$$

with the latter two inclusions uniformly in $\lambda > 1$. Also, if $h_j \in H_\rho^{\frac{1}{2}+\frac{\nu}{2}-} \cap R^{\frac{3}{2}}L^\infty$ and further $P_{<\lambda}h_j \in RL^\infty$ as well as $\chi_{R<1}\nabla_R^l\big(R^{-1}P_{<\lambda}h_j\big) \in L^\infty$, $l \geq 0$, uniformly in λ, or else $h_j \in H_\rho^{1+\frac{\nu}{2}-}$, for $j = 1, 2, \ldots, 2N$, then we have

$$R^{-3}fgh\prod_{j=1}^N(\frac{1}{R}h_{2j}h_{2j-1}) \in H^{\frac{1}{2}+\frac{\nu}{2}-}$$

We also get

$$R^{-\frac{3}{2}}fg\prod_{j=1}^N(\frac{1}{R}h_{2j}h_{2j-1}) \in H^{\frac{1}{2}+\frac{\nu}{2}-}$$

Invoke the conclusion from lemma 3.10, one can prove:

Lemma 3.11. *Providing*

$$\|\widetilde{\varepsilon}\|_{H^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty} \lesssim 1, \quad \|R^{-\frac{3}{2}}P_{<\lambda}\widetilde{\varepsilon}\|_{L^\infty} \lesssim 1, \quad \|\chi_{R<1}\nabla_R^l(R^{-\frac{3}{2}}P_{<\lambda}\widetilde{\varepsilon})\|_{L^\infty} \lesssim 1$$

uniformly in $\lambda > 1$ $l \geq 0$, we have

$$\begin{aligned}
&\|\frac{1}{R^2}\big[f'(u_0)\widetilde{\varepsilon} - f'(u_{2k-2})\widetilde{\varepsilon}\big]\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim (t\lambda)^{-2}\|\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}}\\
&\|\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq1}\frac{1}{l!}f^{(2l)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \|\widetilde{\varepsilon}\|^2_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}\\
&\|\frac{1}{R^{\frac{3}{2}}}\sum_{l\geq1}\frac{1}{l!}f^{(2l+1)}(u_{2k-2})\big(R^{-\frac{1}{2}}\widetilde{\varepsilon}\big)^{2l+1}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim \|\widetilde{\varepsilon}\|^3_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}\cap R^{\frac{3}{2}}L^\infty}
\end{aligned}$$

The last two estimates' right hand side space can be replaced by $H^{\frac{1}{2}+\frac{\nu}{2}-}$ with a change of the bound of $\widetilde{\varepsilon}$ by

$$\|\widetilde{\varepsilon}\|_{H_\rho^{\frac{1}{2}+\frac{\nu}{2}-}} \lesssim 1.$$

4. The construction of the approximate solutions

To build the approximate solution as in theorem 2.1, we follow the scheme in [9]. We start from the stationary harmonic map* $Q(R)$. Setting $R = \lambda(t)r$ we take $u_0(t, x) = Q(\lambda(t)x)$ for $\lambda(t) = t^{-1-\nu}$ and then add corrections v_k iteratively $u_k = u_0 + \sum_{j=1}^k v_k$. In a first approximation we linearize the equation for the correction $\varepsilon = u - u_k$ around $\varepsilon = 0$ and substitute u_k by u_0. Then we have the linear approximate equation

$$\big(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\big)\varepsilon - \frac{1}{r^2}f'(u_0)\varepsilon \approx -e_k$$

* The properties of ground state are needed to prove the spectral theory of $\mathcal{L}$. Since we will employ the same spectral theory as it is in [1], we refer the reader to section 2 [1] for the discussion of properties of such ground states,

From here we split into two different cases: considering the case $r \ll t$ when we expect the time derivative to play a lesser role thus we neglect it (where (4.1) below comes from); considering the case $r \approx t$ when the time and spatial derivative have the same strength. We can identify another principal variable, namely $a = r/t$ and think of ε as a function of $\varepsilon(t, a)$ so we can reduce this case to a Strum-Liouville problem in a which becomes singular at $a = 1$ (where(4.2) comes from). After each step of adding the correction, we also estimate the size of the errors. This makes each round of the scheme with four steps to go. For odd and even steps, we have different equations for the corrections v_k:

$$\big(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} f'(u_0)\big) v_{2k+1} = -e_{2k}^0 \tag{4.1}$$

$$\big(- \partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\big) v_{2k+2} = -e_{2k+1}^0 \tag{4.2}$$

with Cauchy zero data* at $r = 0$, and† where

$$e_k = \big(- \partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\big) u_k - \frac{1}{r^2} f(u_k) \tag{4.3}$$

$$e_{2k+1} = e_{2k}^1 - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1}), \quad e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k}) \tag{4.4}$$

$$N_{2k-1}(v) = \frac{1}{r^2}[f'(u_0)v - f(u_{2k-2} + v) + f(u_{2k-2})] \tag{4.5}$$

$$N_{2k}(v) = \frac{v}{r^2} - \frac{1}{r^2}[f(u_{2k-1} + v) - f(u_{2k-1})] \tag{4.6}$$

Remark 4.1. Note here a technical detail is we split e_k into $e_k = e_k^0 + e_k^1$ where e_k^0 is the so-called principle part and the rest e_k^1, the so-called higher order part, will be left and merge into the next step while analyzing the error v_{k+1} (will be precise below in step 1 and 3). Also we will switch to the principle variable 'a' for equation (4.2) in step 3 as already mentioned in the above section.

To formalize this scheme we need to define suitable function spaces in the light-cone

$$\mathcal{C}_0 = \{(t, r) : 0 \le r < t, 0 < t < t_0\}$$

to put our successive corrections and errors. They are following closely from those in [3].‡

Definition 4.2. *For $i \in \mathbf{N}$, let $j(i) = i$ if ν is irrational, respectively $j(i) = 2i^2$ if ν is rational. Then*

- *$\mathcal{Q}$ is the algebra of continuous functions $q : [0, 1] \to \mathbb{R}$ with the following properties:*
  (i) q is analytic in $[0, 1)$ with even expansion around $a = 0$.

---

*The coefficients are singular at $r = 0$, therefore this has to be given a suitable interpretation below (see remark 4.6).

†There is a typo in [1] for the sign of the term $f(u_{2k-2})$. This does not influence the result in [1] but it matters for our analysis for the nonlinear source terms in later section.

‡One shall note that those definitions are very natural according to a direct computation for the first round of the iterative scheme (see [3] for the case when target manifold is sphere).

(ii) near $a = 1$ we have an absolutely convergent expansion of the form

$$\begin{aligned} q(a) =& q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)+\frac{1}{2}} \sum_{j=0}^{j(i)} q_{i,j}(a)\big(\log(1-a)\big)^j \\ &+ \sum_{i=1}^{\infty} (1-a)^{\widetilde{\beta}(i)+\frac{1}{2}} \sum_{j=0}^{j(i)} \widetilde{q}_{i,j}(a)\big(\log(1-a)\big)^j \end{aligned}$$

with analytic coefficients $q_0, q_{i,j}$, and $\beta(i) = i\nu$, $\widetilde{\beta}(i) = \nu i + \frac{1}{2}$.

- *$\mathcal{Q}_n$ is the algebra which is defined similarly, but also requiring $q_{i,j}(1) = 0$ if $i \geq 2n+1$.*

We also define the space of functions obtained by differentiating $\mathcal{Q}_n$:

Definition 4.3. *Define $\mathcal{Q}'$ as in the preceding definition but replacing $\beta(i)$ by $\beta'(i) := \beta(i) - 1$, and similarly for $\mathcal{Q}'_n$.*

Definition 4.4. *$S^n(R^k(\log R)^l)$ is the class of analytic functions $v : [0,\infty) \to \mathbb{R}$ with the following properties:*
(i) v vanishes of order n at $R = 0$.
(ii) v has a convergent expansion near $R = \infty$

$$v = \sum_{\substack{0 \leq j \leq l+i \\ i \geq 0}} c_{ij} R^{k-i} (\log R)^j$$

The final function space $S^m(R^k(\log R)^l, Q_n)$ is defined slightly different than Definition 3.5 in [3] where we add an extra 'b' into it. This is simply for applying the results from [1] later. We state it here precisely.

Definition 4.5. *(Definition 3.5, [3]) Introduce the symbols*

$$b = \frac{\big(\log(1+R^2)\big)^2}{(t\lambda)^2},\ b_1 = \frac{\big(\log(1+R^2)\big)}{(t\lambda)^2},\ b_2 = \frac{1}{(t\lambda)^2}$$

Pick t sufficiently small such that all b, b_1, b_2, when restricted to the light cone $r \leq t$ are of size at most b_0.

- *$S^m(R^k(\log R)^l, \mathcal{Q}_n)$ is the class of analytic functions $v : [0,\infty) \times [0,1) \times [0,b_0]^3 \to \mathbb{R}$ so that*
  (i) v is analytic as a function of R, b, b_1, b_2,
  $$v : [0,\infty) \times [0,b_0]^3 \to \mathcal{Q}_n$$
  (ii) v vanishes to order m at $R = 0$.
  (iii) v admits a convergent expansion at $R = \infty$,
  $$v(R,\cdot,b,b_1,b_2) = \sum_{\substack{0 \leq j \leq l+i \\ i \geq 0}} c_{ij}(\cdot,b,b_1,b_2) R^{k-i} (\log R)^j$$
  where the coefficients $c_{ij} : [0,b_0]^3 \to \mathcal{Q}_n$ are analytic with respect to $b, b_{1,2}$.
- *$IS^m(R^k(\log R)^l, \mathcal{Q}_n)$ is the class of analytic functions w inside the cone $r < t$ which can be represented as*
  $$w(t,r) = v(R,a,b,b_1,b_2),\ v \in S^m(R^k(\log R)^l, \mathcal{Q}_n)$$
  and $t > 0$ sufficiently small.

Remark 4.6. The functional spaces $S^m(R^k(\log R)^l, Q_n)$ satisfy some good asymptotic behaviors (for example, they vanish in order m at $R = 0$) so the existence of the solutions to equation (4.1) and (4.2) will make sense in those spaces although the coefficients are singular at $R = 0$ in general.

Following the method in [9], the idea for proving theorem 2.1 is to inductively show that we can choose the corrections v_k to be in relevant function spaces:

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^3\big(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}\big) \tag{4.7}$$

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1}\big) \tag{4.8}$$

$$v_{2k} \in \frac{1}{(t\lambda)^{2k+2}} IS^3\big(R^3(\log R)^{2k-1}, \mathcal{Q}_k\big) \tag{4.9}$$

$$t^2 e_{2k} \in \frac{1}{(t\lambda)^{2k}} \big[IS^1\big(R^{-1}(\log R)^{2k}, \mathcal{Q}_k\big) + \langle b, b_1, b_2\rangle [IS^1\big(R(\log R)^{2k-1}, \mathcal{Q}'_k\big)\big] \tag{4.10}$$

and the starting error e_0 satisfying

$$e_0 \in IS^1(R^{-1})$$

Here we denote by $\langle b, b_1, b_2\rangle$ the ideal generated by b, b_1, b_2 inside the algebra generated by b, b_1, b_2. Now we give a brief outline of the proof for 1.4:

Proof. First one shall check $e_0 \in IS^1(R^{-1})$, this can be done by a direct computation (see step 0 in [1]). Then assuming $(4.7 - 4.10)$ hold up to $k - 1$, the first task would be proving (4.7) for k.

Step 1: *For $e_{2k-2}, k \geq 1$, proves v_{2k-1} satisfies* (4.7).

For this one first needs to choose the right 'principal part' of e_{2k-2} which we call e^0_{2k-2}. This is done by throwing away the 'higher order parts', which we call e^1_{2k-2} and which belong to the same space as e_{2k-1}. The way to do it is as following: when $k = 1$ we let $e^0_0 := e_0$, if $k > 1$, we let $e^0_{2k-2} := e_{2k-2}(R, a, 0)$ with the setting $b, b_1, b_2 = 0$. By changing into variable R, equation (4.1) becomes:

$$(t\lambda)^2 L v_{2k-1} = -t^2 e^0_{2k-2}.$$

Here the operator L is

$$L := \partial_R^2 + \frac{1}{R}\partial_R - \frac{f'(u_0)}{R^2}$$

To get the desired result, one needs to prove the following lemma:

Lemma 4.7. *The solution of $Lv = \varphi \in S^1(R^{-1}(\log R)^{2k-2})$, with $v(0) = v'(0) = 0$, has the regularity*

$$v \in S^3(R(\log R)^{2k-1}).$$

This is already proven as Lemma 3.11 in [1], so we conclude (4.7).

Step 2: *Choose v_{2k-1} as in* (4.7) *with error e_{2k-1} satisfying* (4.8).

According to the definition of e_{2k-1} above, we have

$$t^2 e_{2k-1} = t^2 e^1_{2k-2} - t^2\partial_t^2 v_{2k-1} + t^2 N_{2k-1}(v_{2k-1})$$

Since in the former step we treat a as a parameter and now we will defreeze it, some extra terms will show up while calculating the error e_{2k-1}. To be more precise, the amended term $t^2 e_{2k-1}$ we need to deal with is as following (note that $t^2 e^1_{2k-2}$ is proved automatically thanks to the assumptions)

$$t^2 e_{2k-1} = t^2 N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

where $E^2 v_{2k-1}$ is the term in $\partial_t^2 v_{2k-1}$ with no derivation on the a variable, and the term $E^a v_{2k-1}$ is the terms in $(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r)v_{2k-1}$ where derivative hits the a variable (the extra terms from defreezing of a are included here). To prove all those terms in 4.8, we refer the reader to step 2 in [1].

Step 3: *Given e_{2k-1} as in (4.8), construct v_{2k} as in (4.9)*

Here we have to diverge slightly from [1], since our definition of the algebra $S^m(R^k \log R^l)$ is different (we follow the definition in [3]). Since the equation (4.2) for v_{2k} is identical with equation (3.2) for v_{2k} in [3]. We follow the same arguments of step 2 in [3].

Assume

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1},\ \mathcal{Q}'_{k-1})$$

is given. We begin by isolating the leading component e^0_{2k-1} which includes the terms of top degree in R as well as those of one degree less (the rest will merge into e_{2k}, see step 4 below). Thus we write

$$t^2 e^0_{2k-1} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} a q_j(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{q}_j(a)(\log R)^j$$

Consider the following equation

$$t^2 \widetilde{L}(v_{2k}) = t^2 e^0_{2k-1}$$

where $\widetilde{L}$ is

$$\widetilde{L} := -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}$$

Homogeneity considerations suggest that we should look for a solution v_{2k} which has the form (notice here we already switched into R)

$$v_{2k} = \frac{1}{(t\lambda)^{2k-1}} \sum_{j=0}^{2k-1} W^j_{2k}(a)(\log R)^j + \frac{1}{(t\lambda)^{2k}} \sum_{j=0}^{2k} \widetilde{W}^j_{2k}(a)(\log R)^j$$

The one-dimensional equations for W^j_{2k}, $\widetilde{W}^j_{2k}$ are obtained by matching the powers of $\log R$. Then we conjugate out the power of t and rewrite the systems in the a variable, we get (see step 2 in [3] for details)

$$\mathcal{L}_{(2k-1)\nu} W^j_{2k} = a q_j(a) - F_j(a)$$
$$\mathcal{L}_{2k\nu} \widetilde{W}^i_{2k} = \widetilde{q}_i(a) - \widetilde{F}_i(a)$$

the definition of $\mathcal{L}_\beta$ is following [3]. Solving this system with Cauchy data at $a=0$ yields solutions which satisfy

$$\begin{aligned} &W^j_{2k}(a) \in a^3\mathcal{Q}_k, \quad j=\overline{0,2k-1} \\ &\widetilde{W}^i_{2k} \in a^2\mathcal{Q}_k, \quad i=\overline{0,2k} \end{aligned}$$

This is guaranteed by lemma 3.9 from [9]

To finish this step, we need to make a adjustment for v_{2k} because of the singularity of $\log R$ at $R=0$. Also, we need to make sure that v_{2k} has order 3 vanishing at $R=0$. Thus we define v_{2k} as

$$v_{2k} :=$$

$$\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1} W^j_{2k}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^j + \frac{1}{(t\lambda)^{2k}}\frac{R}{(1+R^2)^{\frac{1}{2}}}\sum_{j=0}^{2k}\widetilde{W}^j_{2k}(a)\Big(\frac{1}{2}\log(1+R^2)\Big)^j$$

We will get a large error near $R=0$, but it is not very important since the purpose of the correction is to improve the error near large R. Since $a=R/t\lambda$, it's easy to pull out a a^3 factor from W's and a^2 from $\widetilde{W}$'s to see that we have (4.9).

Step 4: *Show that the error e_{2k} generated by $u_{2k}=u_{2k-1}+v_{2k}$ satisfies* (4.10).

Write

$$t^2 e_{2k} = t^2(e_{2k-1}-e^0_{2k-1}) + t^2\big(e^0_{2k-1} - (-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r-\frac{1}{r^2})(v_{2k})\big) + t^2 N_{2k}(v_{2k})$$

where we recall that except the nonlinear term $t^2N_{2k}(v_{2k})$ the rest is proved satisfying (4.10) following the same arguments as step 3 in [3]. For the term $t^2N_{2k}(v_{2k})$, the main method here is to split the nonlinear term in three parts

$$\begin{aligned} -t^2N_{2k}(v_{2k}) = I + II + III = &a^{-2}\Big[\big(f(u_{2k-1}+v_{2k}) - f(u_{2k-2}) - f'(u_{2k-1})\big)v_{2k}\Big] \\ &+a^{-2}\Big[\big(f'(u_{2k-1})-f'(u_0)\big)v_{2k}\Big] + a^{-2}\Big[\big(f'(u_0)-1\big)v_{2k}\Big] \end{aligned}$$

and prove each of them lies in a sub-space of what we need in (4.10)

$$\begin{aligned} &I \in a^6\frac{1}{(t\lambda)^{2k}}\sum_{\beta=b,b_{1,2}}\beta IS^1\Big(R(\log R)^{2k-1},\mathcal{Q}'_k\Big) \\ &II \in a^2\frac{1}{(t\lambda)^{2k}}\sum_{\beta=b,b_{1,2}}\beta IS^1\Big(R(\log R)^{2k-1},\mathcal{Q}'_k\Big) \\ &III \in a^2\frac{1}{(t\lambda)^{2k}}IS^3\Big(R^{-1}(\log R)^{2k},\mathcal{Q}_k\Big) \end{aligned}$$

The arguments to prove those mimic section 3.8.3 in [1].

Remark 4.8. One might have doubts since the function space $IS^k(R^m(\log R)^l)$ we are using here is different than [1]. To verify this, one just needs to see that the function spaces defined in [1] are the subspaces of our new defined function space in [3]. Thus the argument in [1] applies to our case.

Iteration of **Step 1 - Step 4** immediately furnishes the proof of Theorem 2.1 .

□

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