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Optimal Design of Observable Multi-Agent Networks: A Structural System Approach

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***Abstract*— This paper introduces a method to design observable directed multi-agent networks, that are: 1) either minimal with respect to a communications-related cost function, or 2) idem, under possible failure of direct communication between two agents. An observable multi-agent network is characterized by agents that update their states using a neighboring rule based on directed communication graph topology in order to share information about their states; furthermore, each agent can infer the initial information shared by all the agents. Sufficient conditions to ensure that 1) is satisfied are obtained by reducing the original problem to the travelling salesman problem (TSP). For the case described in 2), sufficient conditions for the existence of a minimal network are shown to be equivalent to the existence of two disjoint solutions to the TSP. The results obtained are illustrated with an example from the area of cooperative path following of multiple networked vehicles by resorting to an approximate solution to the TSP.**

I. INTRODUCTION

The theory of multi-agent networks has witnessed significant theoretical advances recently. Its applications span a wide spectrum of activities that include field surveillance, environmental studies, and remote geophysical exploration; see [1], [2] and the references therein.

A central issue that arises in the operation of multiple networked agents is the parsimonious use of energy. This is particular important in a number of applications where the agents are required to move, sense, compute, and communicate among them to achieve a desired cooperative behaviour. In this paper, given a group of agents we focus on the problem of reducing a cost criterion that is related to the energy spent in transmitting information among them while ensuring that the multi-agent networked system remains observable.

By observability we mean the property satisfied by a network where each agent can infer pertinent information about all the other agents by sharing partial information via a directed graph communications topology (not necessarily bilateral). As explained in [2] for the case of consensus and cooperation in networked multi-agent systems, this property makes it possible to design decentralized control laws that yield better overall performance. The aforementioned problem is extremely challenging and several potential applications are known, for instance, *coordinated path following*. In this, each agent maneuvers to approach a pre-specified parameterized path and computes how far along it has progressed along that path. By proper choice of a communication topology, each agent will be able to recover the path parameters taken by all vehicles, allowing them to negotiate their speeds so that some form of consensus on the path parameters (with reflects into a desired formation pattern) is reached [2]. Therefore, by allowing for the transmission of parametrization variables among the different agents, such that each agent can recover the parametrization variables of the remaining ones, it becomes possible to design control laws that account and compensate for possible variations in the agents position and velocity.

This paper addresses the design of a communication topology dynamics that must be observable from each agent (i.e., each agent should be able recover the whole state of the system) and such that minimal overall transmission cost is achieved. In addition, we remark that such topologies are more general than the topologies associated with the consensus problems (see Remark 3). Formally, consider that communication updates are made according to a time linear dynamics using neighbor data, given by

$$\dot{\gamma}(t) = A\gamma(t), \tag{1}$$

where $\gamma \in \mathbb{R}^n$ is the collection of the scalar parameters of interest to the agents: for instance, in the path following problem γ_i parametrizes a desired path $p_{d_i}(\gamma_i) \in \mathbb{R}^3$ that each agent must follow (see Section II-A). Let $\mathcal{C}$ denote a communication cost matrix $\mathcal{C} \in (\mathbb{R}_0^+)^{n\times n}$ (i.e., each entry is a non-negative scalar) and $\mathcal{A} = \{M \in \mathbb{R}^{n\times n} : \ M_{ij} = 0$ if agent j cannot communicate with agent $i\}$ denotes the communication constraints between agents, i.e., it indicates which pairs of agents can communicate with each other. Here, we assume that each agent can communicate with itself. Ideally, we would like to solve the following problem:

$\mathcal{P}_\mathbf{1}$ Given a cost function matrix $\mathcal{C}$, determine A in (1) to be the solution of the following problem

$$\begin{aligned} A = \arg \min_{M\in\mathcal{A}} \quad & \mathbf{1}^T(\mathcal{C}\circ \bar{M})\mathbf{1} \\ \text{s.t.} \quad & (M, e_j) \text{ is observable}, \ j = 1,\dots,n \end{aligned}$$

where $\bar{M}_{ij} = 1$ if $M_{ij} \neq 0$ $(i,j = 1,\dots,n)$ and zero otherwise, $\mathbf{1}$ is the vector composed by ones with dimension n, $\circ$ denotes the Hadamard product and $e_j \in \{0,1\}^n$ is the j-th canonical vector. In addition, a system is observable if its state can be recovered entirely through the measurements within a non-empty window of time. ⋄

Notice that the communication between agent j and i

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does not need to occur bilaterally, i.e., if agent j transmits to agent i the opposite may not occur. The directed graph (digraph) representation of the matrix A (more precisely, its structure) represents a *communication digraph*, where each vertex denotes the agent and the directed edge between agents the existence of communication between both.

In what concerns the communication topology, all that matters is the structure of matrix A, which we denote by $\bar{A} \in \{0,1\}^{n\times n}$, where an entry in $\bar{A}$ is zero if the same entry in A is zero, and one otherwise. Since we are only considering the structure of A, bearing in mind that we aim to find an A such that the system is observable from each agent, we consider the notion of *structural observability*[1]. We are therefore interested in solving the following related problem:

$\bar{\mathcal{P}}_1$ Optimal cost problem: Given a cost function matrix $\mathcal{C}$, determine the communication topology $\bar{A} \in \{0,1\}^{n\times n}$, corresponding to the structure of A in (1), such that

$$\begin{aligned}\bar{A} = \arg \min_{\bar{M}\in\bar{\mathcal{A}}} \quad & \mathbf{1}^T(\mathcal{C}\circ\bar{M})\mathbf{1} \\ \text{s.t.} \quad & (\bar{M}, e_j) \text{ is structurally observable}, j = 1,\ldots,n\end{aligned}$$

where $\bar{\mathcal{A}} = \{\bar{M} \in \{0,1\}^{n\times n} : \bar{M}_{ij} = 0 \text{ if agent } j \text{ cannot communicate with agent } i\}$. ⋄

Thus, by definition of structural observability, it is possible to make a selection of numeric parameters $A \in \mathbb{R}^{n\times n}$, with the same structure of $\bar{A} \in \{0,1\}^{n\times n}$ that is the solution to $\mathcal{P}_1$. In this paper we show that the solution to $\bar{\mathcal{P}}_1$ reduces to that of finding the solution to a travelling salesman problem (TSP), i.e., the problem of finding a sequence of cities to visit ending in the initial one and such that the total cost associated with moving between cities is minimized, see Theorem 2 for a formal description.

Notice however that there is a natural trade-off between optimality and robustness (for example, with respect to communication link failures) that has to be accounted for. In fact, it is easy to see that a solution to $\bar{\mathcal{P}}_1$ is not necessarily robust; in other words, it is not a solution to the case where a possible communication link fails, as consequence of the necessary conditions enforced by Lemma 3. Therefore, the second problem addressed in this paper is that of finding the structure of $\bar{A}$ such that the system is structurally observable from any agent under a possible directed communication link failure between two agents.

$\bar{\mathcal{P}}_2$ Optimal robust (w.r.t. to direct communication link failure) problem: Given a cost function matrix $\mathcal{C}$, find the communication topology $\bar{A} \in \{0,1\}^{n\times n}$, corresponding to the structure of A in (1), such that

$$\begin{aligned}\bar{A} = \arg \min_{\bar{M}\in\bar{\mathcal{A}}} \quad & \mathbf{1}^T(\mathcal{C}\circ\bar{M})\mathbf{1} \\ \text{s.t.} \quad & (\bar{M}^{-(i,j)}, e_i) \text{ is structurally observable for any} \\ & \text{pair } (i,j) \text{ with } i \neq j,\ i,j = 1,\ldots,n\end{aligned}$$

where $\bar{M}^{-(i,j)}$ has the same sparseness as the matrix $\bar{M}$ except for the entry $\bar{M}_{ij}$, forced to be equal to zero. In other words, given a certain topology $\bar{A}$, under communication failure of the link (i,j), i.e., $\bar{M}^{-(i,j)}$ yields a structurally observable system from each agent. ⋄

Notice that by definition of structural observability, it is possible to obtain a numeric realization A satisfying the structure of $\bar{A}$ that is a solution to $\bar{\mathcal{P}}_2$ and yields an observable system from any agent under an arbitrary link failure.

In this paper, we make the following general assumptions:

A1 For simplicity (mainly to avoid cumbersome notation) we assume that the state of each agent is a real scalar;

A2 Each agent can access its state without incurring in additional cost (since no communication is required), therefore the diagonal entry of the cost matrix $\mathcal{C}$ is composed solely by zeros.

Finally, in this paper we illustrate the application of the aforementioned framework in the context of coordinated path following (CPF), more precisely, to achieve control of the formation of a number of moving vehicles.

Structural systems theory was previously used to address similar problems. For instance, in [4] the topology of a static network of sensors was designed in order to minimize the transmission cost among sensors and from the sensors to a central authority, allowing for field reconstruction by the single (central) identity. In [5] an alternative to the celebrated input-output decomposition method proposed by Siljak [6] was given. Namely, a multi-layer decomposition was provided, where given A in (1), the first layer obeys the constrain given in $\bar{\mathcal{P}}_1$ without any minimization cost under consideration. In the second layer, an output feedback matrix is sought to ensure full decentralized control for almost all realizations of the system matrices. In [7], some structural properties were sought to ensure the existence of stable estimators in a network; further design of a network communication topology is presented in [8]. In [8], necessary conditions were first presented and derived using structural systems theory, after which possible numeric realization of the non-zero entries of such structure was found using linear matrix inequalities. Finally, [9] provided a complete analysis of multi-agent formation using algebraic graph theory and well known control tools, but no cost constraints were addressed.

The main contributions of this paper are threefold: 1) we show that the problem in $\bar{\mathcal{P}}_1$ can be obtained by solving the travelling salesman problem; 2) given two disjoint solutions of $\bar{\mathcal{P}}_1$, a solution to the robust (w.r.t. transmission failure) optimal problem ($\bar{\mathcal{P}}_2$) can be easily obtained; and 3) an application of the previous results is given in the context of cooperative path following.

The rest of this paper is organized as follows. In section II we briefly introduce some results from structural systems, and provide a survey of some known results about the TSP. In section III, we provide the main results of our paper. Finally, in section IV, we provide an illustrative example where a suboptimal solution is found using an efficient approximation algorithm to the TSP to solve a coordinated path following problem.

[1]Given the structure of a dynamic matrix of a linear time invariant system $\bar{A}$ and the output matrix $\bar{C}$, a system is said to be structurally observable if and only if almost all pair of matrices (A', C') with real parameters and with the same structural pattern of $(\bar{A}, \bar{C})$ are observable [3].

II. PRELIMINARIES AND TERMINOLOGY

The following standard terminology and notions from graph theory can be found, for instance in [3]. Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the digraph representation of $\bar{A}$ in (1), where the vertex set $\mathcal{X}$ represents the set of state variables (also referred to as state vertices) and $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : \ A_{ji} \neq 0\}$ denotes the set of edges of the digraph. A digraph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$ with $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$ is called a *subgraph* of $\mathcal{D}$. If $\mathcal{V}_s = \mathcal{V}$, $\mathcal{D}_s$ is said to *span* $\mathcal{D}$. We denote by $\mathcal{D}_{\emptyset}$ the subgraph of $\mathcal{D}$ that consists of $\mathcal{D}$ without the self-loops, in particular, $\mathcal{D}_{\emptyset}(\bar{A}) = (\mathcal{X}, \mathcal{E}^{\emptyset}_{\mathcal{X},\mathcal{X}})$, where $\mathcal{E}^{\emptyset}_{\mathcal{X},\mathcal{X}} = \mathcal{E}_{\mathcal{X},\mathcal{X}} \backslash \{(x_i, x_i) : i = 1, \ldots, n\}$. A *direct path* consists of a sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$ where there vertices $v_i \in \mathcal{V}$ may be repeated. We say that we have a *directed closed path* if v_k coincides with v_1. If no vertex is used twice in a directed path we have *a direct elementary path* from v_1 to v_k. A *cycle* is a directed closed elementary path where no two vertices are the same. Now, we introduce the following characterization of digraphs: a digraph is said to be *balanced* if the number of incoming edges in each vertex equals the number of outgoing edges from that same vertex; and a digraph is *symmetric* if for every edge $(i, j) \in \mathcal{E}$ we have $(j, i) \in \mathcal{E}$.

In addition, we will require the following graph theoretic notions [10]: A digraph $\mathcal{D}$ is said to be strongly connected if there exists a directed path between any pair of vertices. A strongly connected component (SCC) is a maximal subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $v, w \in \mathcal{V}_S$ there exists a path from v to w and from w to v. A matrix A is said to be *irreducible* if its digraph representation $\mathcal{D}(A)$ is an SCC. Visualizing each SCC as a virtual node (or supernode), one may generate a *directed acyclic graph* (DAG), in which each node corresponds to a single SCC and a directed edge exists between two SCCs *if and only if* there exists a directed edge connecting the corresponding SCCs in the original digraph. In the DAG representation, we refer to an SCC that has no incoming edge from any state in a different SCC as a *non-top linked SCC* and, similarly, we have a *non-bottom linked SCC* if the SCC does not have an edge from its states to the states of another SCC.

For any two vertex sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$, we define the *bipartite graph* $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, to be a directed graph (bipartite), whose vertex set is given by $\mathcal{S}_1 \cup \mathcal{S}_2$ and the edge set $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ by $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} \ : \ s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2 \ \}$. Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, a matching M corresponds to a subset of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$ with $s_1, s_1' \in \mathcal{S}_1$ and $s_2, s_2' \in \mathcal{S}_2$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A maximum matching M^* is defined as a matching M that has the largest number of edges among all possible matchings. The maximum matching problem can be solved efficiently in $\mathcal{O}(\sqrt{|\mathcal{S}_1 \cup \mathcal{S}_2|}|\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}|)$ [10]. The vertices in $\mathcal{S}_1$ and $\mathcal{S}_2$ are *matched vertices* if they belong to an edge in the maximum matching M^*, otherwise, we designate the vertices as *unmatched vertices*. If there are no unmatched vertices, we say that we have a *perfect match*. It is to be noted that a maximum matching M^* may not be unique. For ease of referencing, in the sequel, the term *left-unmatched vertices* (w.r.t. $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ and a maximum matching M^*) correspond to those vertices in $\mathcal{S}_1$ that do not belong to a matched edge in M^*.

The following results will be required [11].

Lemma 1 ([11]): Given $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, a bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ has a perfect match if and only if $\mathcal{D}(\bar{A})$ is spanned by a disjoint union of cycles. ⋄

Definition 1 (Feasible dedicated output configuration): A subset $\mathcal{S}_y$ of state variables to which, by assigning dedicated outputs (measuring a single state variable) ensures structural observability of the system is referred to as *feasible dedicated output configuration.* ⋄

Remark 1: Notice that in both $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ a system $(\bar{A}, e_i)$, where e_i is a dedicated output measuring x_i, is structurally observable if and only if $\mathcal{S}_y = \{x_i\}$ is a feasible dedicated output configuration. Therefore, the solution to the problems $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ can be found in terms of a feasible dedicated output configuration comprising a single state variable. ⋄

A feasible dedicated output configuration is characterized in the following result.

Theorem 1 ([11]): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ denote the system digraph and the state bipartite graph $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, i.e., the bipartite representation of $\mathcal{D}(\bar{A})$. Let $S_y \subset \mathcal{X}$, then the following statements are equivalent:

1) The set $\mathcal{S}_y$ is a feasible dedicated output configuration;
2) There exists
    a) a subset $\mathcal{U}_L \subset \mathcal{S}_y$ corresponding to the set of left-unmatched vertices of some maximum matching of $\mathcal{B}$;
    b) a subset $\mathcal{A}_y \subset \mathcal{S}_y$ comprising one state variable from each non-bottom linked SCC of $\mathcal{D}(\bar{A})$. □

The traveling salesman problem (TSP) can be formulated as follows [10]. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with non-negative edge cost $\mathcal{C} : \mathcal{E} \to \mathbb{R}^+$ the TSP, which we denote by TSP($\mathcal{C}$), consists in finding the minimum cost directed closed path visiting every vertex in $\mathcal{V}$. Additionally, some efficient approximation solutions exist to the *metric* TSP (i.e., all edge cost are symmetric and satisfy the triangle inequality) using, for instance, the Christofides' algorithm which provides a worst case guaranty of a factor of $3/2$ from the optimal solution [10].

A. Cooperative Path Following

This section affords the reader a fast paced introduction to the problem of coordinated path following. The CPF architecture consists mainly of two interconnected subsystems:

Path-following controller: a dynamical system whose inputs are a path $\boldsymbol{p}_{d_i}(\gamma_i)$ (parameterized by $\gamma_i \in \mathbb{R}$), a desired speed profile v_r that is common to all agents, and the agent's position $\boldsymbol{p}_i$. Its output is the agent's input, computed so as to make it follow the path at the assigned speed. A path-following controller which achieves this objective is considered to solve the *path-following (PF) problem*. In preparation for the connection with the coordination controller, it accepts the time derivative of the path-following variable $\dot{\gamma}_i$ from the coordination controller.

Coordination controller: a dynamical system whose inputs are the coordination states of the neighbours $\gamma_j; j \in \mathcal{N}_i$, where $\mathcal{N}_i$ denotes the set of agents that agent i communicates with. Its outputs are the path variable γ_i, and its time derivative $\dot{\gamma}_i$. A coordination controller which achieves this objective is considered to solve the *coordination control (CC) problem*.

If both the path-following controllers and the coordination controllers achieve their objectives simultaneously they are said to solve the *coordinated path-following (CPF) problem*. See [12] for formal definitions of the PF, CC and CPF.

Remark 2: As shown in [12], the coordination controller may assume the form

$$\dot{\gamma}(t) = v_r(\gamma(t)) + f(\gamma(t)),$$

where $v_r(\gamma(t))$ is a vector of specified speed profile and $\lim_{t\to\infty} f(\gamma(t)) = 0$. In addition, the path following controller has to account for the mismatch of the γ parameters which is not globally available to each of the agents due to communication constraints. Therefore, an estimator is required to estimate $\gamma(t)$, i.e., $\hat{\gamma}(t)$, such that $\lim_{t\to\infty} \|\gamma(t) - \hat{\gamma}(t)\| = 0$. Commonly, the approach is to select $f(\gamma(t))$ to be a linear neighboring rule [13] where the dynamics is given by a consensus matrix, which subsequently implies that $\|\gamma(t) - \hat{\gamma}(t)\|$ will tend to zero as $t \to \infty$. However, alternative design schemes are possible, as we propose in the present paper. In particular, we show that $f(\gamma(t))$ can be a linear neighboring rule associated associated with a linear time invariant dynamics that is not necessarily a consensus matrix. Moreover, because such linear time invariant dynamics leads to an observable multi-agent network, an estimator can be design to infer $\gamma(t)$ within an arbitrary (non-empty) interval of time. ◇

III. MAIN RESULTS

In this section, we provide the main results of this paper. First, we reduce the problem $\bar{\mathcal{P}}_1$ to that of finding the TSP solution in a graph (see Theorem 2). Second, we show that given two disjoint solutions of $\bar{\mathcal{P}}_1$ we obtain a solution to $\bar{\mathcal{P}}_2$; see Theorem 3 for details.

We start by considering some intermediate results.

Lemma 2: If $\bar{A}$ is a solution to $\bar{\mathcal{P}}_1$ then $\bar{A}$ is irreducible. □

Notice that the reverse of Lemma 2 is not true. For example, suppose $\bar{A}$ is a full matrix, in particular, it is irreducible but it does not need to be a solution to $\bar{\mathcal{P}}_1$ (consider for instance the case where $\mathcal{C}$ has all off-diagonals equal to a common positive scalar).

From Lemma 2, we can reduce the set $\bar{\mathcal{A}}$ in $\bar{\mathcal{P}}_1$ to $\bar{\mathcal{A}}^\sharp$ given by $\bar{\mathcal{A}}^\sharp = \{M \in \{0,1\}^{n\times n} : M \text{ is irreducible}\} \cap \bar{\mathcal{A}}$, thus, hereafter we assume that $\emptyset \neq \bar{\mathcal{A}}^\sharp \subset \bar{\mathcal{A}}$. Notice that with $\bar{A} \in \bar{\mathcal{A}}^\sharp$ condition 2)-b) in Theorem 1 is always satisfied by an arbitrary subset of state variables $\mathcal{S}_y$ comprising a single state variable. In order to also ensure that $\mathcal{S}_y$ satisfies condition 2)-a) in Theorem 1 (i.e., to obtain a feasible dedicated output configuration, hence by Remark 1, a feasible solution to $\bar{\mathcal{P}}_1$), the subset $\mathcal{S}_y = \{x_i\}$ must contain the set of left-unmatched vertices with respect to some maximum matching of the state bipartite graph. This condition is readily met if we consider that the set of left-unmatched vertices is empty, which can be achieved by considering Lemma 1.

Now, consider the following result.

Lemma 3: If $\bar{A}^* \in \{0,1\}^{n\times n}$ is such that

$$\bar{A}^* = \arg\min_{\bar{M}\in\mathcal{A}^\sharp} \mathbf{1}^T(\mathcal{C} \circ \bar{M})\mathbf{1}, \quad (2)$$

then

1) $\mathcal{D}_\emptyset(\bar{A}^*)$ is a solution to TSP($\mathcal{C}$);
2) $\bar{A}' = \bar{A}^* \vee \mathbf{I}_{n\times n}$ is a solution to $\bar{\mathcal{P}}_1$, where $\mathbf{I}_{n\times n}$ is a $n \times n$ identity matrix and $\vee$ denotes the boolean logic OR operation performed entry-wise. □

Notice that requiring that $\bar{A}$ is a solution to (2) is only a necessary condition, but not sufficient to ensure structural observability from each agent. For instance, consider Figure 1-a), and notice that its digraph is not spanned by disjoint union of cycles, which by Lemma 1 means that there exists no perfect matching of the state bipartite graph. In particular, $\mathcal{S}_y = \{\gamma_1\}$ and $\mathcal{S}_y = \{\gamma_3\}$ are feasible dedicated output configurations because they correspond to the sets of left-unmatched vertices associated with a maximum matching of the state bipartite graph, but $\mathcal{S}_y = \{\gamma_2\}$ is not, hence condition 2-a) in Theorem 1 does not hold. In Figure 1-b) the solution obtained in Figure 1-a) can be transformed in a feasible solution to $\bar{\mathcal{P}}_1$ by using Lemma 3, where there exists a disjoint union of cycles that spans the digraph in Figure 1-b). In other words, a perfect matching exists (see Lemma 1) which implies that the set of left-unmatched vertices is empty, and condition 2-a) in Theorem 1 trivially holds. Nevertheless, notice that not all solutions to $\bar{\mathcal{P}}_1$ are required to have all diagonal entries different from zero. For instance, Figure 1-c) depicts the digraph of an irreducible matrix that is spanned by a disjoint union of cycles, hence a feasible solution to problem $\bar{\mathcal{P}}_1$.

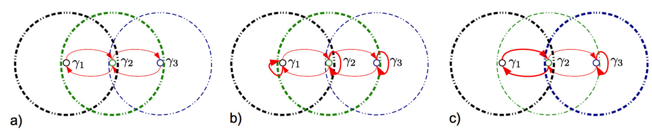

Fig. 1. In black/blue/green three different agents are presented with the dashed circles denoting the communication range, and the red edges represent the digraph communication constraints (represented by $\bar{\mathcal{A}}$), i.e., the possible communications between agents. In a) the directed closed elementary path that is a solution to the TSP is presented, although γ_2 does not correspond to the set of left-unmatched vertices with respect to a maximum matching of the state bipartite graph, hence it is not a solution to $\bar{\mathcal{P}}_1$. In b) a solution to the TSP together with self-loops in the agents represents a feasible solution to the problem $\bar{\mathcal{P}}_1$, since by Lemma 1 the set of left-unmatched vertices is empty; therefore, noticing that the solution is an SCC the result follows directly from Theorem 1. In c) an alternative solution to $\bar{\mathcal{P}}_1$ is provided where not all agents have self-loops, but the digraph is spanned by a disjoint union of cycles.

We now state one of the main results of the paper.

Theorem 2: Let $\bar{A} \in \{0,1\}^{n\times n}$. If $\mathcal{D}_\emptyset(\bar{A})$ is a solution to TSP($\mathcal{C}$) and $\mathcal{D}(\bar{A})$ is spanned by a disjoint union of cycles, then $\bar{A}$ is the solution of $\bar{\mathcal{P}}_1$. □

In particular, notice the following useful result.

Corollary 1: Let the cost matrix $\mathcal{C}$ and the communication graph to be symmetric. If $\bar{A}$ is a solution to $\bar{\mathcal{P}}_1$ then $\bar{A}^T$ is a solution to $\bar{\mathcal{P}}_1$. □

Next, we consider the trade-off between robustness and energy efficient communication topologies. Notice that, a solution to $\bar{\mathcal{P}}_1$ may have a digraph representation where the removal of a single edge (different from self-loops) leads to the loss of the SCC property required to ensure a feasible solution (see Lemma 3), in particular, it will be not a solution to $\bar{\mathcal{P}}_2$. In order to address this issue, we start by stating some necessary conditions to obtain robust solutions.

Lemma 4: If $\bar{A}$ is a solution to $\bar{\mathcal{P}}_2$ then there must exist

a) at least two incoming edges to each vertex of $\mathcal{D}(\bar{A})$;
b) at least two disjoint directed close paths in $\mathcal{D}(\bar{A})$. □

Next, we present a sufficient condition to obtain a solution to $\bar{\mathcal{P}}_2$.

Theorem 3: Let $\bar{A}, \bar{A}' \in \{0,1\}^{n\times n}$ be two solutions to $\bar{\mathcal{P}}_1$. If $\mathcal{D}(\bar{A})$ and $\mathcal{D}(\bar{A}')$ have disjoint directed closed paths with exactly n edges each, then $\bar{A}^* = \bar{A} \vee \bar{A}'$ is a solution to $\bar{\mathcal{P}}_2$. □

Often, communication graphs and communication cost matrices are assumed to be symmetric, then by recalling Corollary 1 we obtain the following corollary to Theorem 3.

Corollary 2: Let $\bar{A}, \bar{A}^T \in \{0,1\}^{n\times n}$ be two solutions to $\bar{\mathcal{P}}_1$. If $\mathcal{D}(\bar{A})$ and $\mathcal{D}(\bar{A}^T)$ have disjoint directed closed paths with exactly n edges each, then $\bar{A}^* = \bar{A} \vee \bar{A}^T$ is a solution to $\bar{\mathcal{P}}_2$. □

Corollary 2 also provides a computational insight. More precisely, suppose that a solution to the TSP has been found with a digraph representation with n edges, then no other solutions needs to be computed if the aforementioned conditions yield. In other words, we simply need to reverse the direction of the edges in the digraph. Thus, we obtain what can be understood as a bidirectional digraph. This digraph can be understood as an undirected graph, commonly used to model communication topology.

We now show that the previous results are applicable to the CPF problem, by providing the construction of the dynamic matrix A associated with the structural patterns of $\bar{A}$ that are solution to the problems $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ using the results in Theorem 2 and Theorem 3, respectively.

Cooperative Path Following

To ensure that the CPF strategy adopted in this paper (see Section II-A) meets the conditions for network observability, we need to ensure that a numerical realization of the matrices $\bar{A}$ (the solution to problems $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ as stated in Theorem 2 and Theorem 3) exist such that the following two conditions hold:

C1 it is marginally stable, i.e., all eigenvalues of A have non-positive real part and if they are zero then they correspond to simple roots;

C2 the vector $\mathbf{1}$ is the right-eigenvector associated with the zero eigenvalue.

Theorem 4: Given $\bar{A}$ that is a solution to either $\bar{\mathcal{P}}_1$ or $\bar{\mathcal{P}}_2$ as stated in Theorem 2 and Theorem 3, then there exists A with the same sparseness such that C1 and C2 hold. A constructive solution is as follows: first, assign positive real values to all entries $A_{i,j}$ whenever $\bar{A}_{i,j} \neq 0$ and $i \neq j$, $i, j = 1, \ldots, n$. Then, set $A_{i,i} = -\left(\sum_{j\neq i} |A_{i,j}|\right)$ for $i = 1, \ldots, n$. □

Remark 3: Notice that the matrix constructed as in Theorem 4 does not need to be a matrix that is associated with a consensus problem. More precisely, the matrix constructed as in Theorem 4 can have a digraph representation corresponding to an arbitrary strongly connected digraph, whereas the matrix associated with consensus problems are restricted to strongly connected digraph that are also balanced digraphs, see [2] for details. ◇

Remark 4: For general CPF strategies besides the one adopted in this paper, notice that C1 and C2 are not required to be satisfied to ensure CPF, since any numeric realization of $\bar{A}$ (a solution to either $\bar{\mathcal{P}}_1$ or $\bar{\mathcal{P}}_2$) that ensures the system to be observable and asymptotically stable, satisfies the conditions required, see Remark 2.

IV. Illustrative Example

In this section we illustrate the application of the main results of this paper, by designing the communication topology of the agents (in this case autonomous underwater vehicles (AUVs)), to achieve coordination while ensuring an observable multi-agent network. More precisely, we want to ensure a triangle formation with 6 agents, oriented according to the direction of motion.

Based on their relative positions in formation, the communication cost incurred is given by the quadratic relative distance between the agents, given by matrix $\mathcal{C}$ with $\mathcal{C}_{ij} = d_{ij}^2$, where d_{ij} is the distance between the nominal positions of agent i and agent j.

In order to solve problem $\bar{\mathcal{P}}_1$ we recall Theorem 2 and an approximate solution is determined using Christofides's algorithm that yields a solution that is, in the worst case scenario, $3/2$ of the optimal. The approximation obtained to the TSP($\mathcal{C}$) is given by $Agent1 \rightarrow Agent2 \rightarrow Agent3 \rightarrow Agent4 \rightarrow Agent2 \rightarrow Agent5 \rightarrow Agent6 \rightarrow Agent1$, yelding a cost of 6343, where $Agenti \rightarrow Agentj$ denotes a direct communication from agent i to agent j, in other words, $\bar{A}_{ji} \neq 0$. We notice that the optimal solution to the TSP($\mathcal{C}$) is given by $Agent1 \rightarrow Agent2 \rightarrow Agent3 \rightarrow Agent4 \rightarrow Agent5 \rightarrow Agent6 \rightarrow Agent1$, yielding a cost of 4229.

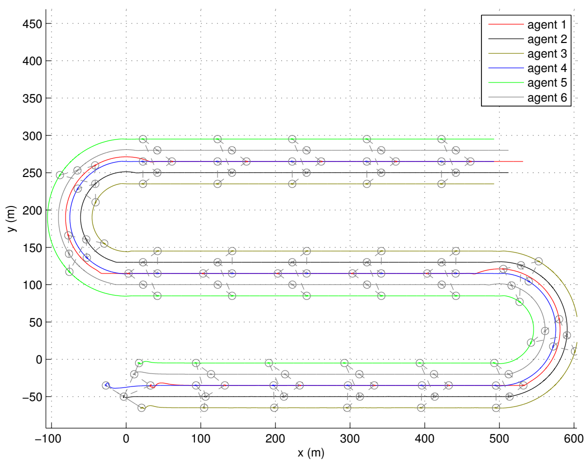

Fig. 2. Vehicle paths using A with the same structure as $\bar{A}$ that is a solution to $\bar{\mathcal{P}}_1$, generated as described in Theorem 4 where the off-diagonal entries are uniformly generated between 10^{-5} and 10^{-3}.

Denote by $\bar{A}$ the structure of matrix A associated with the approximate solution of the TSP($\mathcal{C}$) found, with the diagonal entries comprising solely non-zero entries (accordingly to Theorem 2). In addition, notice that $\bar{A}^T$ is also a solution to TSP($\mathcal{C}$) since the communication graph and the cost matrix are symmetric (see Corollary 1). Thus, since $\mathcal{D}(\bar{A})$ and $\mathcal{D}(\bar{A}^T)$ correspond to two disjoint solutions to $\bar{\mathcal{P}}_1$ we can obtain a robust solution to $\bar{P}_2$, as stated in Theorem 3 (more precisely, Corollary 2), which we denote by $\bar{A}' = \bar{A} \vee \bar{A}^T$.

Finally, we observe that exists a numerical realization A and A' with the same structure of $\bar{A}$ and $\bar{A}'$ respectively, by invoking Theorem 4, such that CPF holds. In particular, we consider the off-diagonal entries to be sampled from an uniform distribution between 10^{-5} and 10^{-3} and the diagonal entries to be the symmetric of the sum row (excluding the diagonal entry). We verified that the system is observable from each agent, as required. In fact, we emphasize that by definition of structural observability, by generating the entries of a matrix in an independent manner, we obtain an observable system from each agent with probability 1. Finally, we illustrate the use of such coordination matrices in simulation.

We used a Simulink model of an autonomous marine vehicle of the MEDUSA-$\mathcal{S}$ class, built at Instituto Superior Técnico, with the inner loop controller for heading and speed described in [14] and the path-following controller designed using Strategy I of [15] with the necessary adaptations.

The formation will follow straight trajectories with U-turns upon reaching $x = 500m$ and again upon reaching $x = 0m$, as depicted in Figure 2. The evolution of the path-following variables γ_i is determined by $v_r = 1.0$ and this reflects on the vehicle paths with $\left|\frac{\partial \boldsymbol{p}_d(\gamma)}{\partial \gamma}\right| = 0.5m$ in the straight lines. The initial gammas $\gamma_i(0),\ i = 1, \ldots, 6$ are chosen to be related to the initial position by $\boldsymbol{p}_{d_i}(\gamma_i(0))$.

The simulated paths of the six AUVs can be seen in Figure 2. Furthermore, the evolution of the coordination states is shown in Figure 3 when using a numeric realization of $\bar{A}$ and in Figure 4 when using a numeric realization of $\bar{A}' = \bar{A} \vee \bar{A}^T$.

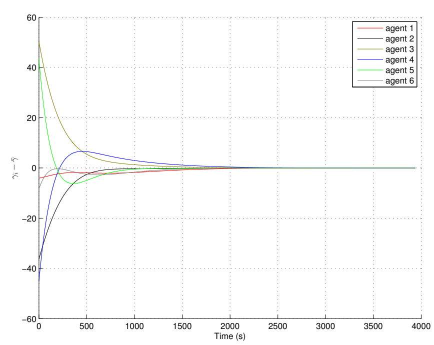

Fig. 3. Evolution of the path-following variables γ_i with a numeric realization A with the same sparseness of $\bar{A}$ that is a solution to $\bar{\mathcal{P}}_1$.

V. CONCLUSIONS AND FURTHER RESEARCH

We have introduced a method to design an observable directed multi-agent network that is minimal with respect to a communications-related cost function, or, idem under a possible communication failure. Sufficient conditions for the existence of such multi-agent network were derived by reducing the original problems to the travelling salesman problem. In addition, we illustrated the application of the results to the design of a coordinated path following algorithm using an approximation algorithm to the TSP that is efficient (polynomial in the number of state variables) and ensures to be (in the worst case scenario) $3/2$ of the optimal solution. Further research will include the analysis of necessary conditions to the proposed problems and the study of new conditions in the case where the cost matrix is time-varying.

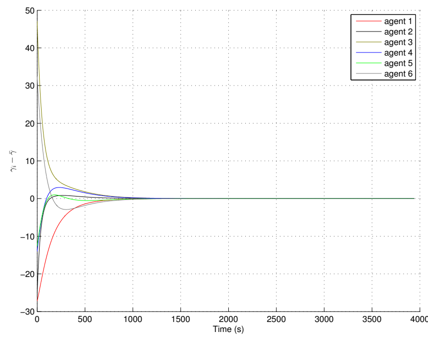

Fig. 4. Evolution of the path-following variables γ_i with a numeric realization A' with the same sparseness of $\bar{A}' = \bar{A} \vee \bar{A}^T$ that is a solution to $\bar{\mathcal{P}}_2$.

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