

# Torsion homology of arithmetic lattices and $K_2$ of imaginary fields

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**Abstract** Let  $X = G/K$  be a symmetric space of noncompact type. A result of Gelander provides exponential upper bounds in terms of the volume for the torsion homology of the noncompact arithmetic locally symmetric spaces  $\Gamma \backslash X$ . We show that under suitable assumptions on  $X$  this result can be extended to the case of nonuniform arithmetic lattices  $\Gamma \subset G$  that may contain torsion. Using recent work of Calegari and Venkatesh we deduce from this upper bounds (in terms of the discriminant) for  $K_2$  of the ring of integers of totally imaginary number fields  $F$ . More generally, we obtain such bounds for rings of  $S$ -integers in  $F$ .

## 1 Introduction

Let  $\mathbf{G}$  be a connected semisimple real algebraic group such that  $\mathbf{G}(\mathbb{R})$  has trivial center and no compact factor. We denote by  $X$  the associated symmetric space, i.e., the homogeneous space  $X = \mathbf{G}(\mathbb{R})/K$  where  $K$  is a maximal compact subgroup  $K \subset \mathbf{G}(\mathbb{R})$ . Many properties of torsion-free lattices  $\Gamma \subset \mathbf{G}(\mathbb{R})$  can be studied with the help of geometry, by analyzing the corresponding locally symmetric spaces  $\Gamma \backslash X$ . For example, a theorem of Gromov shows that the Betti numbers of  $\Gamma$  are bounded linearly in the covolume, i.e., there exists a constant  $C_{\mathbf{G}}$  such that

$$\dim H_j(\Gamma, \mathbb{Q}) \leq C_{\mathbf{G}} \operatorname{vol}(\Gamma \backslash X). \quad (1.1)$$

Recently, Samet has extended this result to the case of lattices that may contain torsion [11]. Note that these results are valid in a more general context than symmetric spaces.

Lately, much interest in the torsion part in the integral homology of lattices—especially arithmetic lattices—has arisen, due to connection with number theory (cf. [2, 7]). We will

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denote by  $H_j(\Gamma)$  the homology with integral coefficients, and by  $H_j(\Gamma)_{\text{tors}}$  its torsion part. In [9] Gelander proved that each noncompact arithmetic manifold has the homotopy type of a simplicial complex whose size is linearly bounded by the volume. This shows (cf. Lemma 2.2) that for nonuniform torsion-free arithmetic lattices,  $\log |H_j(\Gamma)_{\text{tors}}|$  is linearly bounded by  $\text{vol}(\Gamma \backslash X)$ . Note that  $\log |\cdot|$  for finite sets is analogue to the dimension and thus this result provides a version of (1.1) for the torsion. The corresponding result for compact manifolds would follow quite easily from Lehmer’s conjecture on Mahler measure of integral polynomials (cf. [9, §10]).

In this article we show that Gelander’s result can also be used—at least for suitable  $\mathbf{G}$ —to bound the torsion homology of nonuniform arithmetic lattices  $\Gamma \subset \mathbf{G}(\mathbb{R})$ , without the restriction that  $\Gamma$  is torsion-free. Namely, in Sect. 2 we prove the following theorem. Recall that a lattice  $\Gamma \subset \mathbf{G}(\mathbb{R})$  is called irreducible if it is dense in each nontrivial direct factor of  $\mathbf{G}(\mathbb{R})^\circ$ . If  $\mathbf{G}(\mathbb{R})$  is simple then all its lattices are irreducible.

**Theorem 1.1** *Let  $\mathbf{G}$  be as above such that for all irreducible lattices  $\Gamma \subset \mathbf{G}(\mathbb{R})$  we have  $H_q(\Gamma, \mathbb{Q}) = 0$  for  $q = 1, \dots, j$ . Then, there exists a constant  $C_{\mathbf{G}} > 0$  such that for each irreducible nonuniform arithmetic lattice  $\Gamma \subset \mathbf{G}(\mathbb{R})$  the following bound on torsion homology holds:*

$$\log |H_j(\Gamma)| \leq C_{\mathbf{G}} \text{vol}(\Gamma \backslash X).$$

Superrigidity of lattices (which holds for all  $\mathbf{G}(\mathbb{R})$  except  $\text{PO}(n, 1)$  and  $\text{PU}(n, 1)$ ) implies at once arithmeticity and the vanishing of the first Betti number. Thus, for the first homology our theorem reads:

**Corollary 1.2** *If  $\mathbf{G}(\mathbb{R})$  is not locally isomorphic to  $\text{PO}(n, 1)$  or  $\text{PU}(n, 1)$  then there exists a constant  $C_{\mathbf{G}} > 0$  such that for each irreducible nonuniform lattice  $\Gamma \subset \mathbf{G}(\mathbb{R})$  we have*

$$\log |H_1(\Gamma)| \leq C_{\mathbf{G}} \text{vol}(\Gamma \backslash X).$$

It is possible that the condition  $H_q(\Gamma, \mathbb{Q}) = 0$  in Theorem 1.1 is not necessary. However, it is not clear from our proof how to remove it (see Remark 2 at the end of Sect. 2).

One motivation for Theorem 1.1 is of arithmetic nature: it concerns the  $K$ -theory of number fields. Let  $F$  be a number field with ring of integers  $\mathcal{O}_F$ . We denote by  $D_F$  the discriminant of  $F$  and by  $w_F$  the number of roots of unity in  $F$ . The group  $K_2(\mathcal{O}_F)$  is known to be finite and to injects as a subgroup – the tame kernel – of  $K_2(F)$  (see [17, §5.2]). By a theorem of Suslin (cf. [7, Theorem 4.12]),  $K_2(F)$  corresponds to the second homology of  $\text{PGL}_2(F)$ . Using this as a starting point, in [7, §4.5] Calegari and Venkatesh have been able to relate in turn  $K_2(\mathcal{O}_F)$  to the second homology of  $\text{PGL}_2(\mathcal{O}_F)$ . From their results and Theorem 1.1, we will show that the following holds.

**Theorem 1.3** *Let  $d \geq 2$  be an even integer. There exists a constant  $C(d) > 0$  such that for each totally imaginary field  $F$  of degree  $d$  we have:*

$$\log |K_2(\mathcal{O}_F) \otimes R| \leq C(d) |D_F|^2 (\log |D_F|)^{d-1}, \tag{1.2}$$

where  $R = \mathbb{Z}[\frac{1}{6w_F}]$ .

For  $d \geq 6$  the theorem follows immediately from the mentioned results by noticing that the Betti numbers of  $\text{PGL}_2(\mathcal{O}_F)$  in degree  $j = 1, 2$  vanish in this case. This fact is not only needed to satisfy the conditions of Theorem 1.1 but also to apply the work of Calegari and Venkatesh, which in particular requires the congruence subgroup property (it fails for  $d = 2$ ).

This also explains the appearance in the statement of the value  $w_F$ , which corresponds to the order of the congruence kernel.

If  $S$  is a finite set of places of  $F$ , for the ring  $\mathcal{O}_F(S)$  of  $S$ -integers of  $F$  we have  $K_2(\mathcal{O}_F) \subset K_2(\mathcal{O}_F(S))$  (cf. [14, Théorème 1]). To deal with the cases  $d = 2, 4$  we will then consider rings of  $S$ -integers. As far as we know, there is no known counterpart to Gelander’s result [9] for  $S$ -arithmetic groups. However, for groups of type  $A_1$  (e.g.,  $SL_2(\mathcal{O}_F(S))$ ) we can use their action on Bruhat-Tits trees to write them as amalgamated products and obtain this way upper bounds for their torsion homology (see Sect. 5). As a corollary we obtain in Theorem 6.2 a generalization of Theorem 1.3 for rings of  $S$ -integers. There again, the work of Calegari and Venkatesh is essential in our proof.

General upper bounds for the  $K$ -theory of rings of integers have been obtained by Soulé in [16], without restriction on the signature of  $F$ . His method was later improved by Bayer and Houriet (their results are contained in the second part of Houriet’s thesis, see [10, Theorem 4.3]). For  $K_2$  of totally imaginary fields, Theorem 1.3 improves considerably these known bounds – at least asymptotically (since we do not provide an explicit value for  $C(d)$ ). Still, our bounds might be very far from being sharp. Explicit computations for  $K_2(\mathcal{O}_F)$  are very difficult but some were obtained by Belabas and Gangl in [1], mostly for  $F$  imaginary quadratic. Their result tends to show that  $K_2(\mathcal{O}_F)$  has a quite slow growth rate.

Recall that for the order of the torsion of  $K_0$ , that is, the class number  $h_F$ , we have the following:

$$h_F \leq C(d)|D_F|^{1/2}(\log |D_F|)^{d-1}, \tag{1.3}$$

for some constant  $C(d)$ . For higher degrees  $m$ , Soulé has conjectured (see [16, §5]) that the torsion part of  $K_m(\mathcal{O}_F)$  can be bounded as follows:

$$\log |K_m(\mathcal{O}_F)_{\text{tors}}| \leq C(m, d) \log |D_F|, \tag{1.4}$$

for some constant  $C(m, d)$  and  $F$  of degree  $d$  (and arbitrary signature). Such a bound is out of reach with the method presented in this article, and some new ideas will be needed to further improve the situation.

## 2 Bounds for torsion homology

Let  $\mathbf{G}$  and  $X$  be as in the introduction. For a torsion-free discrete subgroup  $\Gamma \subset \mathbf{G}(\mathbb{R})$  the quotient  $M = \Gamma \backslash X$  is a manifold locally isometric to  $X$ . We call such an  $M$  a  $X$ -manifold.  $M$  is called arithmetic if  $\Gamma$  is an arithmetic subgroup of  $\mathbf{G}(\mathbb{R})$ . In this case  $M$  has finite volume. The following result, proved in [9], gives a strong quantitative relation between the geometry and topology of noncompact arithmetic manifolds.

**Theorem 2.1** (Gelander) *There exists a constant  $\beta = \beta(X)$  such that any noncompact arithmetic  $X$ -manifold  $M$  is homotopically equivalent to a simplicial complex  $\mathcal{X}$  whose numbers of  $q$ -cells is at most  $\beta \text{vol}(M)$  for each  $q \leq \dim(X)$ .*

In particular,  $\beta \text{vol}(M)$  is an upper bound for the Betti numbers, a result already known from the work of Gromov in a larger context. But Theorem 2.1 also allows us to study the torsion part in the homology. For this we will use the following result, whose proof can be found in [15, §2.1]. For an abelian group  $A$ , we denote by  $A_{\text{tors}}$  its subgroup of torsion elements.

**Lemma 2.2** (Gabber) *Let  $A = \mathbb{Z}^a$  with the standard basis  $(e_i)_{i=1,\dots,a}$  and  $B = \mathbb{Z}^b$ , so that  $B \otimes \mathbb{R}$  is equipped with the standard Euclidean norm  $\|\cdot\|$ . Let  $\phi : A \rightarrow B$  be a  $\mathbb{Z}$ -linear map such that  $\|\phi(e_i)\| \leq \alpha$  for each  $i = 1, \dots, a$ . If we denote by  $Q$  the cokernel of  $\phi$ , then*

$$|Q_{\text{tors}}| \leq \alpha^{\min\{a,b\}}.$$

This lemma applies in particular to simplicial complexes, where the boundary map on a  $q$ -simplex is a sum of  $(q + 1)$  basis elements. From Theorem 2.1 we obtain a bound

$$H_q(M)_{\text{tors}} \leq \alpha^{\text{vol}(M)}, \tag{2.1}$$

for some  $\alpha$  that depends only on  $X$ . We want to extend this result to lattices  $\Gamma \subset \mathbf{G}(\mathbb{R})$  that may contain torsion. Note that there exists a bound  $\gamma = \gamma(\mathbf{G})$  such that each nonuniform arithmetic  $\Gamma \subset \mathbf{G}(\mathbb{R})$  contains a torsion-free normal subgroup  $\Gamma_0$  of index  $[\Gamma : \Gamma_0] \leq \gamma$ . This is a well-known fact that follows from the existence of a  $\mathbb{Q}$ -structure on  $\mathbf{G}$  such that  $\Gamma \subset \mathbf{G}(\mathbb{Q})$  (cf. [9, Lemmas 5.2 and 13.1]). This proves:

**Proposition 2.3** *There exist  $\alpha, \gamma > 0$  depending only on  $\mathbf{G}$  such that for each nonuniform arithmetic lattice  $\Gamma \subset \mathbf{G}(\mathbb{R})$  there is a  $X$ -manifold  $M$  that is a normal cover of  $\Gamma \backslash X$ , with Galois group  $G$ , such that:*

- (1) *the order of  $G$  is bounded by  $\gamma$ ;*
- (2) *the order of  $H_q(M)_{\text{tors}}$  is bounded by  $\alpha^{\text{vol}(\Gamma \backslash X)}$  for each  $q$ .*

For a given nonuniform arithmetic lattice  $\Gamma \subset \mathbf{G}(\mathbb{R})$ , let the manifold  $M$  and the group  $G$  be as in Proposition 2.3. Thus we have the exact sequence

$$1 \rightarrow \pi_1(M) \rightarrow \Gamma \rightarrow G \rightarrow 1, \tag{2.2}$$

and the homology of  $\Gamma$  can be studied with help of the Lyndon-Hochschild-Serre spectral sequence (see [6, §VII.6]):

$$E_{pq}^2 = H_p(G, H_q(M)) \Rightarrow H_{p+q}(\Gamma). \tag{2.3}$$

In particular, the order of  $H_j(\Gamma)_{\text{tors}}$  is equal to  $\prod_{p+q=j} |(E_{pq}^\infty)_{\text{tors}}|$ . Since the homology of the finite group  $G$  is torsion outside the degree 0, the second page  $E_{pq}^2$  has no infinite factor outside the vertical line  $p = 0$ . It follows that the successive (diagonal) differentials on the spectral sequence (2.3) do not add any torsion, so that the torsion in  $E_{pq}^\infty$  is bounded by the torsion in  $E_{pq}^2$ . Thus,

$$|H_j(\Gamma)_{\text{tors}}| \leq \prod_{p+q=j} |H_p(G, H_q(M))_{\text{tors}}|. \tag{2.4}$$

We can now conclude the proof of Theorem 1.1. The factor  $|H_j(G, \mathbb{Z})_{\text{tors}}|$  in (2.4) depends only on  $j$  and  $G$ , and since there exist only a finite number of groups of order less than  $\gamma$ , we have a uniform bound for it. For  $q = 1, \dots, j$ , our hypothesis on the Betti numbers of lattices in  $\mathbf{G}(\mathbb{R})$  implies that the abelian group  $A = H_q(M)$  is finite. It follows that  $H_p(G, A)_{\text{tors}}$  is equal to  $H_p(G, A)$ . This homology group can be computed by the standard bar complex (see [6, §III.1]), which in degree  $p$  corresponds to finite sums of symbols  $a \otimes [g_1 | \dots | g_p]$ , with  $a \in A$  and  $g_i \in G$ . In particular, this shows that the order of  $H_p(G, A)$  is at most  $|A|^{|G|^p}$ , which by Proposition 2.3 is bounded by  $\alpha^{\gamma^p \text{vol}(\Gamma \backslash X)}$ . This concludes the proof of Theorem 1.1.

*Remark 1* If one forgets about the small primes, the proof can be simplified. Let  $N$  be the product of all primes bounded by  $\gamma$  (it depends on  $\mathbf{G}$  only). The transfer map for homology (cf. [6, §III.9–10]) shows that  $H_j(\Gamma, \mathbb{Z}[1/N])$  is given by the module of co-invariants  $H_j(M, \mathbb{Z}[1/N])_G$ , which is isomorphic to the submodule of invariants  $H_j(M, \mathbb{Z}[1/N])^G$  (since  $|G|$  is invertible in  $\mathbb{Z}[1/N]$ ). So we may use  $|H_j(M, \mathbb{Z}[1/N])|$  as an upper bound, and the latter is bounded by Proposition 2.3. In particular this argument does not need the vanishing of the Betti numbers.

*Remark 2* To avoid the hypothesis  $H_q(M, \mathbb{Q}) = 0$  in our proof we would need to obtain a bound for the torsion in  $H_p(G, A_{\text{free}})$ , where  $A_{\text{free}}$  is the free part of the  $G$ -module  $A = H_q(M)$ . The boundary map on a basis element in the bar complex for  $H_p(G, A_{\text{free}})$  is given by

$$\partial(a \otimes [g_1] \cdots [g_p]) = ag_1 \otimes [g_2] \cdots [g_p] - a \otimes [g_1]g_3 \cdots [g_p] + \cdots \tag{2.5}$$

The problem that prevents us to apply Lemma 2.2 in this context to obtain a good bound lies in the first term: even though the group  $G$  is finite of order uniformly bounded by  $\gamma$ , we cannot bound the size of  $ag_1$  uniformly. The reason is that in general (indecomposable) integral representations of a finite group can be arbitrarily large (see [8, Theorem (81.18)]).

### 3 Bounds for $K_2$ of imaginary number fields. The case $d \geq 6$

In this section we prove Theorem 1.3 for fields  $F$  of degree  $d \geq 6$ . The cases  $d = 2$  and  $d = 4$  are dealt with as a special case of Theorem 6.2, which will require more preparation.

Let  $F$  be a number field with class number  $h_F$ . We denote by  $V_f$  (resp.  $V_\infty$ ) the set of finite (resp. archimedean) places of  $F$  and by  $\mathbb{A}_f$  its ring of finite adèles. Let  $G$  be the algebraic group  $\text{PGL}_2$  defined over  $F$ , and let  $\mathbf{G}(\mathbb{R}) = \text{PGL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$  be the associated real Lie group (which depends only on the signature of  $F$ ) and  $X$  its associated symmetric space (which is a product of hyperbolic spaces of dimensions 2 and 3). In [7] Calegari and Venkatesh consider the following double cosets space:

$$Y_0(1) = G(F) \backslash (X \times G(\mathbb{A}_f) / K), \tag{3.1}$$

where  $X$  is identified as the quotient of  $\mathbf{G}(\mathbb{R})$  by a maximal compact subgroup, and  $K \subset G(\mathbb{A}_f)$  is the compact open subgroup  $K = \prod_{v \in V_f} \text{PGL}_2(\mathcal{O}_v)$ . Then,  $Y_0(1)$  consists of a disjoint union of copies of the quotient  $\text{PGL}_2(\mathcal{O}_F) \backslash X$ , and the number of connected components is indexed by the finite set

$$G(F) \backslash G(\mathbb{A}_f) / K. \tag{3.2}$$

In our case  $G = \text{PGL}_2$ , the size of this set corresponds to the order of the class group  $\text{Cl}(F)$  modulo its square elements. In particular, we can use the bound  $|\pi_0(Y_0(1))| \leq h_F$ . Moreover, if  $h_F$  is odd then  $Y_0(1)$  is connected.

We follow the convention used in [7] that the homology is understood in the orbifold sense. For example, if  $h_F$  is odd then  $H_\bullet(Y_0(1))$  corresponds exactly to the group homology of  $\text{PGL}_2(\mathcal{O}_F)$ . For  $F$  with at least two archimedean places, Calegari and Venkatesh prove (cf. [7, Theorem 4.5.1]) the existence of a surjective map

$$H_2(Y_0(1), R) \rightarrow K_2(\mathcal{O}_F) \otimes R, \tag{3.3}$$

where  $R = \mathbb{Z}[\frac{1}{6w_F}]$ .

Suppose that  $F$  is totally imaginary of degree  $d \geq 6$ . Then,  $\mathbf{G}(\mathbb{R})$  has real rank at least 3, and from the discussion in [4, §XIV.2.2] we know that for each irreducible arithmetic subgroup  $\Gamma \subset \mathbf{G}(\mathbb{R})$  there is in degree  $j = 1, 2$  a surjective homomorphism

$$H^j(X_u, \mathbb{C}) \rightarrow H^j(\Gamma, \mathbb{C}), \tag{3.4}$$

where  $X_u$  denotes the compact dual of  $X$ . But since  $F$  is totally imaginary,  $X$  is a product of hyperbolic 3-spaces and thus  $X_u$  is a product of 3-spheres. It follows from the Künneth formula that  $H^j(\Gamma, \mathbb{C}) = 0$  for  $j = 1, 2$ . This shows that Theorem 1.1 applies to the groups  $\mathrm{PGL}_2(\mathcal{O}_F)$ . This also shows that the surjective map (3.3) provides in this case the bound

$$\begin{aligned} |K_2(\mathcal{O}_F) \otimes R| &\leq |H_2(Y_0(1), R)| \\ &\leq |H_2(\mathrm{PGL}_2(\mathcal{O}_F))|^{h_F}. \end{aligned} \tag{3.5}$$

The covolume of  $\mathrm{PGL}_2(\mathcal{O}_F)$  is well known and has been computed for instance in [3]. We can bound it above by  $|D_F|^{3/2}$ . The claim in Theorem 1.3 then follows from Theorem 1.1, and the bounds (3.5) and (1.3).

*Remark 3* Note that the proof shows that we have the better bound

$$\log |K_2(\mathcal{O}_F) \otimes R| \leq C(d)|D_F|^{3/2} \tag{3.6}$$

for the totally imaginary fields  $F$  of degree  $d \geq 6$  and odd class number.

### 4 Two lemmas about torsion

We state here two lemmas that will be needed in Sect. 5. If  $A$  is a module, we denote by  $A_{\mathrm{free}}$  its free part:  $A_{\mathrm{free}} = A/A_{\mathrm{tors}}$ .

**Lemma 4.1** *Let  $A \xrightarrow{\phi} B \rightarrow C \rightarrow D$  be an exact sequence of finitely generated  $\mathbb{Z}$ -module, and denote by  $Q$  the cokernel of the map  $\phi_{\mathrm{free}} : A_{\mathrm{free}} \rightarrow B_{\mathrm{free}}$  induced by  $\phi$ . Then,*

$$|C_{\mathrm{tors}}| \leq |Q_{\mathrm{tors}}| \cdot |B_{\mathrm{tors}}| \cdot |D_{\mathrm{tors}}|.$$

*Proof* Let us use the notation  $\beta : B \rightarrow C$  and  $\gamma : C \rightarrow D$ . Since  $C/\mathrm{im}(\beta) \cong \mathrm{im}(\gamma) \subset D$ , by restricting to the torsion elements we find

$$|C_{\mathrm{tors}}| \leq |\mathrm{im}(\beta)_{\mathrm{tors}}| \cdot |D_{\mathrm{tors}}|.$$

We can decompose each element of  $c \in \mathrm{im}(\beta)_{\mathrm{tors}}$  into  $c = \beta(b_{\mathrm{free}}) + \beta(b_{\mathrm{tors}})$ , where  $b_{\mathrm{tors}}$  is a torsion element and  $b_{\mathrm{free}}$  is not. Then,  $|B_{\mathrm{tors}}|$  is obviously an upper bound for the number of elements of the form  $\beta(b_{\mathrm{tors}})$ . By exactness of the sequence, the elements in the coset  $b_{\mathrm{free}} + \phi(A_{\mathrm{free}})$  all have same image  $\beta(b_{\mathrm{free}}) \in C$ , so that there are at most  $|Q_{\mathrm{tors}}|$  elements  $\beta(b_{\mathrm{free}}) \in C_{\mathrm{tors}}$ . □

**Lemma 4.2** *Let  $\Gamma$  be a group and  $\Gamma_0 \subset \Gamma$  a normal subgroup of finite index. We consider the map  $\phi : H_j(\Gamma_0)_{\mathrm{free}} \rightarrow H_j(\Gamma)_{\mathrm{free}}$  induced by the inclusion on the free part of the homology. Then, the order of the cokernel of  $\phi$  is bounded above by  $[\Gamma : \Gamma_0]^{b_j(\Gamma)}$ .*

*Proof* The result follows from the existence of the transfer map  $\tau : H_j(\Gamma) \rightarrow H_j(\Gamma_0)$ , for which  $(\phi \circ \tau)(z) = [\Gamma : \Gamma_0] \cdot z$  for any  $z \in H_j(\Gamma)$  (cf. [6, §III.9]): if  $M = H_j(\Gamma)_{\mathrm{free}}$  then we have

$$(\phi \circ \tau)(M) \subset \phi(H_j(\Gamma_0)_{\mathrm{free}}) \subset M,$$

with  $[M : (\phi \circ \tau)(M)] = [\Gamma : \Gamma_0]^{b_j(\Gamma)}$ . □

### 5 Torsion homology of $S$ -arithmetic groups of type $A_1$

To finish the proof of Theorem 1.3 we will need to consider  $S$ -arithmetic groups. Let  $F$  be a number field. For a finite set  $S$  of places of  $F$  containing all the archimedean, we denote by  $\mathcal{O}_F(S) = \mathcal{O}_F[S^{-1}]$  the ring of  $S$ -integers in  $F$ . We write  $S_f$  for the subset  $S_f \subset S$  of finite places. We define the *norm* of  $S$  in  $F$  by

$$N_{F/\mathbb{Q}}(S) = \prod_{\mathfrak{p} \in S_f} N_{F/\mathbb{Q}}(\mathfrak{p}), \tag{5.1}$$

where  $N_{F/\mathbb{Q}}(\mathfrak{p})$  is the norm of the ideal  $\mathfrak{p}$ .

For a given  $S$  we consider the  $S$ -arithmetic group  $SL_2(\mathcal{O}_F(S))$  and its subgroups of finite index. They act on the space  $X_S = X \times X_{S_f}$ , where  $X = X_\infty$  is the symmetric space associated with  $SL_2(F \otimes_{\mathbb{Q}} \mathbb{R})$  and

$$X_{S_f} = \prod_{\mathfrak{p} \in S_f} X_{\mathfrak{p}}, \tag{5.2}$$

where  $X_{\mathfrak{p}}$  is the Bruhat-Tits tree associated with the  $\mathfrak{p}$ -adic group  $SL_2(F_{\mathfrak{p}})$ .

**Proposition 5.1** *Let  $d, N > 0$  be two integers. There exists a constant  $C(d, N)$  such that for any number field  $F$  of degree  $d$  and any torsion-free congruence subgroup  $\Gamma \subset SL_2(\mathcal{O}_F(S))$  with  $N_{F/\mathbb{Q}}(S) \leq N$  we have that for any  $j > 0$  the Betti number  $b_j(\Gamma)$  and  $\log |H_j(\Gamma)_{\text{tors}}|$  are bounded above by*

$$C(d, N) \cdot [SL_2(\mathcal{O}_F(S)) : \Gamma] \cdot |D_F|^{3/2}.$$

*Proof* For a fixed degree  $d$  we proceed by induction on  $N$ . If  $N = 1$ , we have that  $S_f$  is empty and so  $\Gamma$  is a genuine arithmetic group in  $SL_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ . Moreover, since  $\Gamma$  is torsion-free, it is isomorphic to its image in  $PGL_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ . The result in this case follows then from the bound (2.1) explained in Sect. 2, together with the volume formula for  $SL_2(\mathcal{O}_F)$  (see [3]).

Suppose that the result holds for any set  $S'$  of places of norm  $N_{F/\mathbb{Q}}(S')$  less than  $N$ , and let  $\Gamma$  be an  $S$ -arithmetic group as in the statement, and with  $N_{F/\mathbb{Q}}(S) = N > 1$ . The cohomological dimension of  $\Gamma$  is then bounded (see [12, Proposition 21]) and thus it suffices to prove the induction step for a fixed  $j$  (the constant  $C(d, N)$  can be constructed as the maximum of the constants  $C(d, N, j)$ ). For some  $\mathfrak{q} \in S_f$ , let  $S' = S \setminus \{\mathfrak{q}\}$ . Since  $\Gamma$  is a congruence subgroup of  $SL_2(\mathcal{O}_F(S))$ , it follows from the strong approximation property that  $\Gamma$  is dense in  $SL_2(F_{\mathfrak{q}})$ . Then, by letting  $\Gamma$  act on the tree  $X_{\mathfrak{q}}$  (see [13, §II.1.4]), it can be written as an amalgamated product

$$\Gamma = \Gamma' *_{\Gamma'_0(\mathfrak{q})} \Gamma', \tag{5.3}$$

where  $\Gamma' = \Gamma \cap SL_2(\mathcal{O}_F(S'))$  and  $\Gamma'_0(\mathfrak{q})$  is its congruence subgroup of level  $\mathfrak{q}$ , that is,

$$\Gamma'_0(\mathfrak{q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \mid c \equiv 0 \pmod{\mathfrak{q}} \right\}. \tag{5.4}$$

In particular,  $\Gamma'$  and  $\Gamma'_0(\mathfrak{q})$  are  $S'$ -arithmetic, with  $N_{F/\mathbb{Q}}(S') < N$ . Since the norm of  $\mathfrak{q}$  is bounded by  $N$ , the index  $[\Gamma' : \Gamma'_0(\mathfrak{q})]$  is uniformly bounded, say by  $\gamma(d, N)$ . Then, we have

$$\begin{aligned} [SL_2(\mathcal{O}_F(S')) : \Gamma'_0(\mathfrak{q})] &\leq \gamma(d, N) \cdot [SL_2(\mathcal{O}_F(S')) : \Gamma'] \\ &\leq \gamma(d, N) \cdot [SL_2(\mathcal{O}_F(S)) : \Gamma]. \end{aligned} \tag{5.5}$$

From (5.3) we obtain the following exact sequence in homology (see [6, §II.7]):

$$H_j(\Gamma'_0(q)) \rightarrow H_j(\Gamma') \oplus H_j(\Gamma') \rightarrow H_j(\Gamma) \rightarrow H_{j-1}(\Gamma'_0(q)), \tag{5.6}$$

where the first map comes from the diagonal inclusion  $\Gamma'_0(q) \rightarrow \Gamma' \times \Gamma'$ . By the recurrence hypothesis together with (5.5), the rank of  $H_{j-1}(\Gamma'_0(q))$  is bounded by

$$C(d, N - 1)\gamma(d, N) \cdot [\text{SL}_2(\mathcal{O}_F(S)) : \Gamma] \cdot |D_F|^{3/2}, \tag{5.7}$$

and so is also the rank  $b_j(\Gamma')$  of  $H_j(\Gamma')$ . It follows from the exact sequence (5.6) that  $b_j(\Gamma)$  can be bounded as wanted.

It remains to bound the torsion homology. Applying Lemma 4.2 we see that the torsion in  $H_j(\Gamma')$  that comes from the free part of  $H_j(\Gamma'_0(q))$  is bounded by  $[\Gamma : \Gamma_0]^{b_j(\Gamma')}$ . Since by induction the expression (5.7) also serves as a bound for  $\log |H_j(\Gamma')_{\text{tors}}|$  and  $\log |H_{j-1}(\Gamma'_0(q))_{\text{tors}}|$ , by applying Lemma 4.1 on (5.6) we obtain the needed bound for  $\log |H_j(\Gamma)_{\text{tors}}|$ . This shows that both  $b_j(\Gamma)$  and  $\log |H_j(\Gamma)_{\text{tors}}|$  can be bounded by

$$C(d, N) \cdot [\text{SL}_2(\mathcal{O}_F) : \Gamma] \cdot |D_F|^{3/2}$$

for some constant  $C(d, N)$  constructed from  $C(d, N - 1)$  and  $\gamma(d, N)$ . □

As a corollary of Proposition 5.1, we can bound the low degree torsion homology of the  $S$ -arithmetic groups  $\text{PSL}_2(\mathcal{O}_F(S))$  for totally imaginary fields.

**Theorem 5.2** *Let  $S$  be a set of places of the totally imaginary number field  $F$  of degree  $d$ , containing all the archimedean. Suppose that  $N_{F/\mathbb{Q}}(S) \leq N$ , and  $|S_f| \geq 3$ . Then, for  $j = 1, 2$  we have*

$$\log |H_j(\text{PSL}_2(\mathcal{O}_F(S)))| \leq C(d, N) \cdot |D_F|^{3/2}$$

for some constant  $C(d, N)$ .

*Proof* Let  $\Gamma_S = \text{SL}_2(\mathcal{O}_F(S))$ . We can choose a finite prime  $q$  outside of  $S$ , large enough so that the principal congruence subgroup  $\Gamma_S(q) \subset \Gamma_S$  is torsion-free. Moreover,  $q$  can be chosen of norm bounded above by some constant depending only on  $d$  and  $N$ . In particular, there is some upper bound  $\gamma(d, N)$  for the index  $[\Gamma_S : \Gamma_S(q)]$ . It is known (cf. for instance [5, proof of Theorem 2.1]) that for  $q < |S_f|$  the cohomology groups  $H^q(\Gamma_S(q), \mathbb{R})$  correspond to the continuous cohomology of  $\text{PGL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ , which equals the cohomology  $H^q(X_u, \mathbb{R})$  of the compact dual  $X_u$  of  $X_{\infty}$ . Since  $F$  is totally imaginary, we find (cf. Sect. 3) that the Betti numbers  $b_q(\Gamma_S(q))$  vanishes for  $q = 1$  and  $2$ .

Since  $\Gamma_S(q)$  is torsion-free, it is isomorphic to its image in  $\text{PSL}_2(\mathcal{O}_F(S))$ , which we will denote by the same symbol. Then  $\Gamma_S(q)$  is viewed as a normal subgroup of  $\text{PSL}_2(\mathcal{O}_F(S))$ , whose index is bounded by  $\gamma(d, N)$ . Let  $G$  be the quotient of  $\text{PSL}_2(\mathcal{O}_F(S))$  by  $\Gamma_S(q)$ . Then, as was done in Sect. 2 for the proof of Theorem 1.1, we can bound the torsion of  $H_j(\text{PSL}_2(\mathcal{O}_F(S)))$  in terms of  $|H_q(\Gamma_S(q))|$ , using the spectral sequence

$$H_p(G, H_q(\Gamma_S(q))) \Rightarrow H_{p+q}(\text{PSL}_2(\mathcal{O}_F(S))).$$

The conclusion then follows from the bound for the torsion homology of  $\Gamma_S(q)$  provided by Proposition 5.1. □



### 6 $K_2$ of rings of $S$ -integers in imaginary fields

Let us write again  $G$  for the algebraic group  $\mathrm{PGL}_2$  over the number field  $F$ . We denote by  $\mathbb{A}^S$  the adèle ring of  $F$  omitting the places  $v \in S$  (where as usual we suppose  $S$  finite with  $V_\infty \subset S$ ). For the open compact subgroup  $K^S = \prod_{v \notin S} \mathrm{PGL}_2(\mathcal{O}_v) \subset G(\mathbb{A}^S)$ , following [7, §4.4], we define

$$Y[S^{-1}] = G(F) \backslash (X_S \times G(\mathbb{A}^S) / K^S). \tag{6.1}$$

This generalizes the construction of the space  $Y_0(1)$  used in Sect. 3, which corresponds to the case  $S = V_\infty$ . Similarly as for  $Y_0(1)$ , we have that  $Y[S^{-1}]$  consists of at most  $h_F$  copies of the quotient  $\mathrm{PGL}_2(\mathcal{O}_F(S)) \backslash X_S$ .

Although it is not stated explicitly in the work of Calegari and Venkatesh, the result that follows is obtained by the same spectral sequence argument used to prove [7, Theorem 4.5.1]. See in particular [7, §4.5.7.5], where the case  $S = V_\infty \cup \{q\}$  is used.

**Theorem 6.1** (Calegari-Venkatesh) *Let  $F$  be a number field and  $S$  be a finite set of places of  $F$  containing the archimedean and of size  $|S| > 1$ . Let  $R = \mathbb{Z}[\frac{1}{6w_F}]$ . Then, there exists a surjective map*

$$H_2(Y[S^{-1}], R) \rightarrow K_2(\mathcal{O}_F(S)) \otimes R.$$

The case  $N = 1$  of the following theorem corresponds to Theorem 1.3, and in particular it supplies the proof of the latter for the cases  $d = 2$  and  $d = 4$ , which remained unproved in Sect. 3.

**Theorem 6.2** *Let  $d, N \geq 1$  be two integers with  $d$  even. There exists a constant  $C(d, N)$  such that for any totally imaginary number field  $F$  of degree  $d$  and any set of places  $S$  of  $F$  of norm  $N_{F/\mathbb{Q}}(S) \leq N$ , we have*

$$\log |K_2(\mathcal{O}_F(S)) \otimes R| \leq C(d, N) |D_F|^2 (\log |D_F|)^{d-1},$$

where  $R = \mathbb{Z}[\frac{1}{6w_F}]$ .

*Proof* Since for  $S \subset S'$  we have  $K_2(\mathcal{O}_F(S)) \subset K_2(\mathcal{O}_F(S'))$ , by adding at most two places (of bounded norm) to  $S$  we may assume that  $|S_F| \geq 3$ , so that Theorem 5.2 applies. Now, the quotient of  $\mathrm{PGL}_2(\mathcal{O}_F(S))$  by  $\mathrm{PSL}_2(\mathcal{O}_F(S))$  is given by

$$\mathcal{O}_F(S)^\times / (\mathcal{O}_F(S)^\times)^2,$$

whose order is invertible in  $R$ . We deduce that

$$|H_2(\mathrm{PGL}_2(\mathcal{O}_F(S)), R)| \leq |H_2(\mathrm{PSL}_2(\mathcal{O}_F(S)), R)|. \tag{6.2}$$

From Theorem 6.1, Theorem 5.2 and (1.3), we have

$$\begin{aligned} \log |K_2(\mathcal{O}_F(S)) \otimes R| &\leq h_F \cdot \log |H_2(\mathrm{PGL}_2(\mathcal{O}_F(S)), R)| \\ &\leq C(d, N) \cdot |D_F|^2 (\log |D_F|)^{d-1}, \end{aligned}$$

for some constant  $C(d, N)$ . □

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## References

1. Belabas, K., Gangl, H.: Generators and relations for  $K_2 O_F$ . *J. K-Theory* **31**(3), 195–231 (2004)
2. Bergeron, N., Venkatesh, A.: The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu* **12**(2), 391–447 (2013)
3. Borel, A.: Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **8**(1), 1–33 (1981)
4. Borel, A., Wallach, N.: Continuous cohomology, discrete subgroups, and representations of reductive groups. *Math. Surveys Monogr.*, vol. 67, American Mathematical Society (2000)
5. Borel, A., Yang, J.: The rank conjecture for number fields. *Math. Res. Lett.* **1**, 689–699 (1994)
6. Brown, K.S.: Cohomology of groups, Graduate texts in mathematics, vol. 87. Springer, Berlin (1982)
7. Calegari, F., Venkatesh, A.: A torsion Jacquet-Langlands correspondence, preprint [arXiv:1212.3847v1](https://arxiv.org/abs/1212.3847v1)
8. Curtis, C.W., Reiner, I.: Representation Theory of Finite Groups and Associative Algebras, Pure and Applied Mathematics, vol. 11. Wiley, NY (1962)
9. Gelander, Tsachik: Homotopy type and volume of locally symmetric manifolds. *Duke Math. J.* **124**(3), 459–515 (2004)
10. Houriet, J.: Arithmetic bounds—Lenstra’s constant and torsion of  $K$ -groups. Ph.D. thesis, EPF Lausanne (2010)
11. Samet, I.: Betti numbers of finite volume orbifolds. *Geom. Topol.* **17**(2), 1113–1147 (2013)
12. Serre, J.-P.: Cohomologie des groupes discrets. *Prospects in mathematics. Ann. Math. Stud.*, vol. 70. Princeton University Press, pp. 77–169 (1971)
13. Serre, J.-P.: *Trees*, Springer (1980)
14. Soulé, C.: Groupes de Chow et  $K$ -théorie de variétés sur un corps fini. *Math. Ann.* **268**, 317–346 (1984)
15. Soulé, C.: Perfect forms and the Vandiver conjecture. *J. Reine Angew. Math.* **517**, 209–221 (1999)
16. Soulé, C.: A bound for the torsion in the  $K$ -theory of algebraic integers. *Doc. Math. Extra*, vol. Kato, pp. 761–788 (2003)
17. Weibel, C.: Algebraic  $K$ -theory of rings of integers in local and global fields. In: Friedlander, E.M., Grayson, D.R. (eds.) *Handbook of  $K$ -theory*, vol. 1. Springer (2005)