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Actions of amenable equivalence relations on CAT(0) fields

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Abstract. We present the general notion of Borel fields of metric spaces and show some properties of such fields. Then we make the study specific to the Borel fields of proper CAT(0) spaces and we show that the standard tools we need behave in a Borel way. We also introduce the notion of the action of an equivalence relation on Borel fields of metric spaces and we obtain a rigidity result for the action of an amenable equivalence relation on a Borel field of proper finite dimensional CAT(0) spaces. This main theorem is inspired by the result obtained by Adams and Ballmann regarding the action of an amenable group on a proper CAT(0) space.

1. *Introduction*

1.1. *Overview of the results.* One of the first links between amenability and negative curvature is the result of Avez [**6**], which states that a compact Riemannian manifold with non-positive sectional curvature is flat if and only if its fundamental group is of polynomial growth. The amenability here is implicit and it was Gromov [**26**] who pointed out that in this case the growth of the fundamental group is polynomial if and only if it is an amenable group. After several generalizations obtained by Zimmer [**39**] and Burger and Schroeder [**9**], Adams and Ballmann proved the following theorem.

THEOREM 1.1. [**1**, p. 184] *Let X be a proper CAT(0) space. If $G \subseteq \mathrm{Isom}(X)$ is an amenable group, then at least one of the following two assertions holds.*
(i) *There exists $\xi \in \partial X$ which is fixed by G.*
(ii) *The space X contains a G-invariant flat.*
A flat is a closed and convex subspace of X which is isometric to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. In particular, a point $x \in X$ is a flat of dimension zero.

It is interesting with respect to our work to mention that in his article Zimmer also proved the following theorem.

THEOREM 1.2. [**39**, p. 1012] *Let $\mathcal{F}$ be a Riemannian measurable foliation with a transversally (i.e., holonomy) invariant measure and finite total volume. Assume that almost every leaf is a complete simply connected manifold of non-positive sectional curvature. If $\mathcal{F}$ is amenable, then almost every leaf is flat.*

Without going into all the details, a Riemannian measurable foliation has to be understood as an equivalence relation on a measure space such that each equivalence class (*leaf*) is a smooth manifold endowed with a Riemannian structure that varies in a Borel way. Amenability of the foliation is defined as the amenability of the induced relation on a *transversal* (a Borel subset of the probability space that meets almost every leaf only countably many times).

In this paper, we study an object close to one of Riemannian measurable foliation in the context of CAT(0) spaces (or more generally of metric spaces), namely a Borel field of metric spaces. Suppose that a Borel space Ω is given and that to each ω we assign a metric space X_ω. The definition sets what it means for such an assignment to be Borel. This has been studied by some authors (see, e.g., [**10, 13, 14**] or [**28**]), often in the particular case when all the X_ω are a subspace of a given separable metric space X. This notion seems to be the natural one to define the action of an equivalence relation (or more generally of a groupoid). By adapting the techniques of [**1**] to the context of equivalence relations and Borel fields of CAT(0) spaces we managed to prove the following theorem; see the following for a precise meaning of the terminology.

THEOREM 1.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and $\mathcal{R} \subseteq \Omega^2$ be an amenable ergodic Borel equivalence relation which quasi preserves the measure. Assume that $\mathcal{R}$ acts by isometries on a Borel field $(\Omega, X_\bullet)$ of proper CAT(0) spaces of finite covering dimension. Then at least one of the following assertions is verified.*

(i) *There exists an invariant Borel section of points at infinity $[\xi_\bullet] \in L(\Omega, \partial X_\bullet)$.*

(ii) *There exists an invariant Borel subfield $(\Omega, A_\bullet)$, such that $A_\omega \simeq \mathbb{R}^n$ for almost every $\omega \in \Omega$.*

Our result is a generalization of the result by Adams and Ballmann, just as the one by Zimmer generalized the one by Avez. Recently, Caprace and Lytchak [**12**] proved a parallel version of Adams and Ballmann's result by replacing the locally compactness assumption by the one of finite telescopic dimension. Inspired by this result and by the tools we developed, Duchesne [**18**] managed to prove a version of the last theorem for a Borel field of such spaces.

In order to prove our main result, we recall and develop the theory of Borel fields of metric spaces. In doing so, we obtain in §2 some new results, especially Theorems 2.1 and 2.3, and Proposition 2.3. In §3, we give the first analysis of Borel fields of proper CAT(0) spaces. The next section is devoted to the action of an equivalence relation on a Borel field of metric spaces. In particular, we reformulate the notion of amenability for a relation in this context. The last section contains the proof of the main theorem.

1.2. *Basic definitions and notations.* Let (X, d) be a metric space. If $x \in X$ and r is a real number, we use the notation

$$B(x, r) = \{y \in X \mid d(y, x) < r\}, \quad \overline{B}(x, r) = \{y \mid d(x, y) \leq r\} \quad \text{and}$$
$$S(x, r) = \{y \in X \mid d(x, y) = r\}$$

to denote respectively the open ball, the closed ball and the sphere centered at x of radius r. Sometimes we also need to use the closure of the open ball, which is written $\overline{B(x, r)}$.

A metric space is called *proper* if all closed balls are compact.

A map $\gamma : I \to X$ from an interval of real numbers I to the space X is a *geodesic* if it is isometric. We say that it is a geodesic *segment* if I is compact and a geodesic *ray* if $I = [0, \infty[$. The space X is *geodesic* if every pair of points can be joined by a geodesic.

For $x, y \in X$, we denote the image of a geodesic $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$ by $[x, y] \subseteq X$. A *geodesic triangle* with vertex $x, y, z \in X$ is $\triangle(x, y, z) := [x, y] \cup [x, z] \cup [y, z]$. A *comparison triangle* for $x, y, z \in X$ is a Euclidean triangle $\triangle(\overline{x}, \overline{y}, \overline{z}) \subseteq \mathbb{R}^2$ such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(x, z) = d(\overline{x}, \overline{z})$, $d(y, z) = d(\overline{y}, \overline{z})$. It is unique up to isometry. If $q \in [x, y]$, we denote by $\overline{q}$ the point in $[\overline{x}, \overline{y}]$ such that $d(x, q) = d(\overline{x}, \overline{q})$.

Definition 1.1. A geodesic metric space X is CAT(0) if for every geodesic triangle $\Delta(x, y, z)$ and every point $q \in [x, y]$, the following inequality holds:

$$d(z, q) \leq d(\overline{z}, \overline{q}).$$

The general background reference concerning CAT(0) spaces is [**8**] (see also [**7**]). We will introduce the various objects and definitions associated with CAT(0) spaces that we need, such as the boundary, the projection on a convex subspace, and the angles, in §3, where we will prove that these notions behave 'well' in the context of Borel fields of CAT(0) spaces.

Besides CAT(0) spaces, other basic objects that we will consider in this paper are *Borel equivalence relations*. By a *Borel space* we mean a set Ω equipped with a σ-algebra $\mathcal{A}$ and we denote it by $(\Omega, \mathcal{A})$. If Ω is a completely metrizable separable topological space and $\mathcal{A}$ is the σ-algebra generated by the open subsets, then $(\Omega, \mathcal{A})$ is called a *standard Borel space*. A theorem of Kuratowski states that such spaces are all Borel isomorphic, provided they are uncountable (see, e.g., [**31**]). A standard Borel space together with a probability measure is called a *standard probability space*.

Definition 1.2. Let $(\Omega, \mathcal{A})$ be a standard Borel space. A Borel equivalence relation $\mathcal{R}$ is a Borel subset $\mathcal{R} \subseteq \Omega^2$ (where Ω^2 is endowed with the product σ-algebra) which satisfies the following conditions.

(i) For every $\omega \in \Omega$, the set $\mathcal{R}[\omega] := \{\omega' \in \Omega \mid (\omega, \omega') \in \mathcal{R}\}$, called the class of ω, is finite or countably infinite.

(ii) The set $\mathcal{R}$ contains the diagonal $\Delta_\Omega := \{(\omega, \omega) \mid \omega \in \Omega\}$, is symmetric in the sense that $\mathcal{R} = \mathcal{R}^{-1}$ (if for $S \subseteq \Omega^2$ we define $S^{-1} := \{(\omega', \omega) \mid (\omega, \omega') \in S\}$) and satisfies the following transitivity property: if $(\omega, \omega'), (\omega', \omega'') \in \mathcal{R}$, then $(\omega, \omega'') \in \mathcal{R}$.

Standard references for equivalence relations include [**16, 21, 22, 29, 30**] or [**32**]. For a Borel subset $A \subseteq \Omega$, we denote by $\mathcal{R}[A] := \bigcup_{\omega \in A} \mathcal{R}[\omega]$ the *saturation of* A, which is a Borel set, and we say that A is *invariant* if $\mathcal{R}[A] = A$. If a countable group G acts in a Borel way on Ω, then its action defines naturally the Borel equivalence relation $\mathcal{R}_G$, where $(\omega, \omega') \in \mathcal{R}_G$ if and only if there exists $g \in G$ such that $g\omega = \omega'$. Reciprocally, a now classical result due to Feldman and Moore [**21**] states that each Borel equivalence relation may be obtained by such an action.

If $(\Omega, \mathcal{A}, \mu)$ is now a standard probability space, we say that $\mathcal{R}$ *quasi preserves the measure* μ if for every $A \in \mathcal{A}$ such that $\mu(A) = 0$, we have $\mu(\mathcal{R}[A]) = 0$. This is equivalent to the requirement that for each group G such that $\mathcal{R} = \mathcal{R}_G$, the image measures $g_*(\mu)$, where $g \in G$, are equivalent to μ (i.e., the measure $g_*(\mu)$ is absolutely continuous with respect to μ, and conversely). In the case where the measure μ is invariant by G, we say that $\mathcal{R}$ *preserves the measure*. The relation $\mathcal{R}$ is *ergodic* if each Borel saturated set is such that $\mu(A) = 0$ or $\mu(A) = 1$.

2. *Borel fields of metric spaces*

2.1. *Definitions and first results.* A *field of metric spaces* on a set Ω is a family of metric spaces $\{(X_\omega, d_\omega)\}_{\omega\in\Omega}$ indexed by the elements of Ω. The set Ω is called the *base* of the field, and we denote the field by $(\Omega, (X_\bullet, d_\bullet))$ or $(\Omega, X_\bullet)$, or even just $X_\bullet$ when the base is implicit. A *section* of the field is the choice of an element of X_ω for each $\omega \in \Omega$, so it can be thought of as an element of the product $\prod_{\omega\in\Omega} X_\omega$. We write a section $x_\bullet$, and for each $\omega \in \Omega$, x_ω is used to denote the given element of X_ω. We denote by $\mathcal{S}(\Omega, X_\bullet)$ the set of all sections of the field $(\Omega, X_\bullet)$. Given two sections $x_\bullet, y_\bullet \in \mathcal{S}(\Omega, X_\bullet)$, we introduce the following *distance function*:

$$\begin{aligned} d_\bullet(x_\bullet, y_\bullet) : \quad \Omega &\mapsto [0, \infty[\\ \omega &\mapsto d_\omega(x_\omega, y_\omega). \end{aligned}$$

We are interested in fields of metric spaces on a Borel space.

Definition 2.1. Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ be a field of metric spaces on Ω. A Borel structure on $(\Omega, X_\bullet)$ is a subset $\mathcal{L}(\Omega, X_\bullet) \subseteq \mathcal{S}(\Omega, X_\bullet)$ such that:

(i) (compatibility) for all $x_\bullet, y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, the function $d_\bullet(x_\bullet, y_\bullet)$ is Borel;

(ii) (maximality) if $y_\bullet \in \mathcal{S}(\Omega, X_\bullet)$ is such that $d_\bullet(x_\bullet, y_\bullet)$ is Borel for all $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, then $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$; and

(iii) (separability) there exists a countable family $\mathcal{D} := \{x_\bullet^n\}_{n\geq 1} \subseteq \mathcal{L}(\Omega, X_\bullet)$ such that $\overline{\{x_\omega^n\}_{n\geq 1}} = X_\omega$ for all $\omega \in \Omega$. Sometimes we will write $\mathcal{D}_\omega := \{x_\omega^n\}_{n\geq 1}$.

If there exists such a $\mathcal{L}(\Omega, X_\bullet)$, we say that $(\Omega, X_\bullet)$ is a Borel field of metric spaces and $\mathcal{L}(\Omega, X_\bullet)$ is called the Borel structure of the field. The elements of $\mathcal{L}(\Omega, X_\bullet)$ are called the Borel sections. A set $\mathcal{D}$ satisfying condition (iii) is called a fundamental family of the Borel structure $\mathcal{L}(\Omega, X_\bullet)$.

Remark 2.1.

(i) Observe that Condition 2.1(iii) forces all the metric spaces X_ω to be separable.

(ii) It follows from Lemmas 2.1 and 2.2 that if a field is *trivial* (i.e., all the X_ω are the same separable metric space X), then the set of all Borel functions from Ω to X is naturally a Borel structure on this field. This observation should reinforce the intuition of thinking about the Borel sections as a replacement of Borel functions which cannot be defined when the field is not trivial.

(iii) If $\Omega' \subseteq \Omega$ is a Borel subset, then $\mathcal{L}(\Omega', X_\bullet) := \{(x_\bullet |_{\Omega'}) \mid x_\bullet \in \mathcal{L}(\Omega, X_\bullet)\}$ is a Borel structure on the field $(\Omega', X_\bullet)$, where $x_\bullet |_{\Omega'}$ denotes the section of $\mathcal{S}(\Omega', X_\bullet)$ obtained by restricting the section $x_\bullet$ to the subset Ω'.

(iv) Borel fields of metric spaces can also be presented as bundles (see, e.g., [**14**]).

We now describe two constructions that we will use to give a useful reformulation of the maximality condition of Definition 2.1. Let $\{x^n_\bullet\}_{n\geq 1}$ be a sequence of elements of $\mathcal{S}(\Omega, X_\bullet)$ such that $\{x^n_\omega\}_{n\geq 1}$ is a converging sequence in X_ω for all $\omega \in \Omega$. Then we can define a new section $x_\bullet \in \mathcal{S}(\Omega, X_\bullet)$ by $x_\omega := \lim_{n\to\infty} x^n_\omega$ for all $\omega \in \Omega$. This section is called the *pointwise limit* of the sequence $\{x^n_\bullet\}_{n\geq 1}$ and it is written $\lim_{n\to\infty} x^n_\bullet$. Let $\{x^n_\bullet\}_{n\geq 1}$ be a sequence of elements of $\mathcal{S}(\Omega, X_\bullet)$ and $\Omega = \bigsqcup_{n\geq 1} \Omega_n$ be a countable Borel partition of Ω. Define a new section $x_\bullet$ by setting $x_\bullet |_{\Omega_n} := x^n_\bullet |_{\Omega_n}$ for all $n \geq 1$. This new section is called a *countable Borel gluing* of the sequence with respect to the partition. If the partition is finite, we call the section a *finite Borel gluing*.

LEMMA 2.1. *Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ be a field of metric spaces. Suppose that the set $\mathcal{L}(\Omega, X_\bullet) \subseteq \mathcal{S}(\Omega, X_\bullet)$ is such that conditions (i) and (iii) of Definition 2.1 are satisfied. Then condition (ii) of the same definition is equivalent to saying:*

(ii)$'$ $\mathcal{L}(\Omega, X_\bullet)$ *is closed under pointwise limits and countable Borel gluings,*

or to saying:

(ii)$''$ $\mathcal{L}(\Omega, X_\bullet)$ *is closed under pointwise limits and finite Borel gluings.*

Proof. We will prove (ii) $\Rightarrow$ (ii)$''$ $\Rightarrow$ (ii)$'$ $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (ii)$''$: this assertion follows easily by applying to the distance functions the facts that a limit of a pointwise converging sequence of Borel functions is still Borel, and that a countable Borel gluing of Borel functions is again a Borel function.

(ii)$''$ $\Rightarrow$ (ii)$'$: assume that $x_\bullet \in \mathcal{S}(\Omega, X_\bullet)$ is the gluing of the sequence $\{y^n_\bullet\}_{n\geq 1} \subseteq \mathcal{L}(\Omega, X_\bullet)$ relative to the decomposition $\Omega = \bigsqcup_{n\geq 1} \Omega_n$. For all $n \geq 1$, we introduce the section $\widetilde{y}_\bullet^{\,n}$, defined by

$$\widetilde{y}_\bullet^{\,n} |_{\Omega_j} := \begin{cases} y^j_\bullet |_{\Omega_j} & \text{if } 1 \leq j \leq n, \\ y^1_\bullet |_{\Omega_j} & \text{if } j \geq n+1. \end{cases}$$

By hypothesis, $\widetilde{y}_\bullet^{\,n} \in \mathcal{L}(\Omega, X_\bullet)$ for all $n \geq 1$, and we have $\lim_{n\to\infty} \widetilde{y}_\bullet^{\,n} = x_\bullet$, so that $\mathcal{L}(\Omega, X_\bullet)$ is closed under countable Borel gluings.

(ii)$'$ $\Rightarrow$ (ii): suppose that $y_\bullet \in \mathcal{S}(\Omega, X_\bullet)$ is such that $d_\bullet(x_\bullet, y_\bullet)$ is Borel for all $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. Fix a fundamental family $\mathcal{D} := \{x^n_\bullet\}_{n\geq 1}$ and define, for all $k \geq 1$, a function $n^k_\bullet : \Omega \to \mathbb{N}$ by

$$n^k_\omega := \min\{n \in \mathbb{N} \mid d_\omega(x^n_\omega, y_\omega) \leq 1/k\}.$$

These functions are well-defined (because $\mathcal{D}_\omega$ is dense) and Borel because

$$\{\omega \in \Omega \mid n^k_\omega \leq N\} = \bigcup_{j=1}^{N} (d_\bullet(x^j_\bullet, y_\bullet))^{-1}([0, 1/k]) \in \mathcal{A} \quad \text{for all } N \geq 1.$$

For all $k \geq 1$, we can define a section $x_\bullet^{n_\bullet^k} \in \mathcal{S}(\Omega, X_\bullet)$ by gluing the sequence $\{x_\bullet^n\}_{n\geq 1}$ in this way:

$$x_\bullet^{n_\bullet^k} |_{(n_\bullet^k)^{-1}(\{j\})} := x_\bullet^j |_{(n_\bullet^k)^{-1}(\{j\})} \quad \text{for all } j \geq 1.$$

By hypothesis and construction, $\{x_\bullet^{n_\bullet^k}\}_{k\geq 1} \subseteq \mathcal{L}(\Omega, X_\bullet)$ and $\lim_{k\to\infty} x_\bullet^{n_\bullet^k} = y_\bullet$, so we can conclude that $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. □

The following lemma gives two characterizations of the Borel sections, knowing only a fundamental family.

LEMMA 2.2. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of metric spaces of Borel structure* $\mathcal{L}(\Omega, X_\bullet)$ *and* $\mathcal{D}$ *be a fundamental family. Then*

$$\begin{aligned}\mathcal{L}(\Omega, X_\bullet) &= \{y_\bullet \in \mathcal{S}(\Omega, X_\bullet) \mid d_\bullet(y_\bullet, z_\bullet) \textit{ is Borel for every } z_\bullet \in \mathcal{D}\} \\ &= \{y_\bullet \in \mathcal{S}(\Omega, X_\bullet) \mid y_\bullet \textit{ is a pointwise limit of countable Borel} \\ &\quad\ \textit{gluings of elements of } \mathcal{D}\}.\end{aligned}$$

Proof. Let us prove the first equality. The inclusion [⊆] is obvious. For the reverse, suppose that $y_\bullet$ is in the right-hand set. Since $\mathcal{D}$ is a fundamental family, the equality

$$d_\bullet(x_\bullet, y_\bullet) = \sup_{z_\bullet \in \mathcal{D}} |d_\bullet(x_\bullet, z_\bullet) - d_\bullet(z_\bullet, y_\bullet)|$$

holds for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. Therefore, $d_\bullet(x_\bullet, y_\bullet)$ is a Borel function and thus $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. Note that the second equality was already verified in the proof of Lemma 2.1. □

Example 2.1.

(i) As already said, a trivial field is a natural example of a Borel field of metric spaces. It is important to keep in mind that, even in the trivial case, many different Borel structures may exist on the same field.

(ii) A standard bundle (in the sense of Gaboriau and Alvarez, see, e.g., [**3**]) is a standard Borel space X with a Borel projection $\pi : X \to \Omega$ such that the fibers are countable. The field $\{(\pi^{-1}(\omega), d_\omega)\}_{\omega\in\Omega}$, where d_ω is the discrete distance, is a Borel field when endowed with the structure $\{f_\bullet : \Omega \to X \mid f_\bullet \text{ is Borel and } \pi(f_\omega) = \omega\}$. (A selection theorem can be used to construct a fundamental family, see [**3**].)

(iii) A Borel equivalence relation on Ω is a particular example of a standard bundle. This example can be turned into a more interesting one if the relation is graphed, so that we can consider on each equivalence class the metric induced by the graph structure instead of the discrete one (see, e.g., [**24**] for a definition of a graphed equivalence relation).

(iv) A Borel field of Hilbert spaces as defined in [**15**] or a Borel field of Banach spaces as defined in [**4**] are examples of Borel fields of metric spaces.

(v) Suppose that there exists a countable family $\mathcal{D} = \{x_\bullet^n\}_{n\geq 1} \subseteq \mathcal{S}(\Omega, X_\bullet)$ such that $d_\bullet(x_\bullet^n, x_\bullet^m)$ is Borel for every $n, m \geq 1$ and $\{x_\omega^n\}_{n\geq 1}$ is dense in X_ω for every $\omega \in \Omega$. Then it is easy to adapt the proof of Lemma 2.2 to show that

$$\mathcal{L}_{\mathcal{D}}(\Omega, X_\bullet) := \{y_\bullet \in \mathcal{S}(\Omega, X_\bullet) \mid d_\bullet(x_\bullet, z_\bullet) \text{ is Borel for every } z_\bullet \in \mathcal{D}\}$$

is a Borel structure on $(\Omega, X_\bullet)$.

Definition 2.2. Let $(\Omega, \mathcal{A})$ be a Borel space, and $(\Omega, X_\bullet)$ and $(\Omega, Y_\bullet)$ be two Borel fields of respective Borel structures $\mathcal{L}(\Omega, X_\bullet)$ and $\mathcal{L}(\Omega, Y_\bullet)$.

A morphism between these two Borel fields is a family of maps $\varphi_\bullet = \{\varphi_\omega : X_\omega \to Y_\omega\}_{\omega\in\Omega}$ such that for all $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, the section defined by $\varphi_\bullet(x_\bullet) : \omega \mapsto \varphi_\omega(x_\omega)$ is in $\mathcal{L}(\Omega, Y_\bullet)$. Sometimes we will simply write $\varphi_\bullet : (\Omega, Y_\bullet) \to (\Omega, Y_\bullet)$ or $\varphi_\bullet : X_\bullet \to Y_\bullet$. We denote by $\widetilde{\mathcal{L}}(\Omega, \mathcal{F}(X_\bullet, Y_\bullet))$ the set of morphisms from $(\Omega, X_\bullet)$ to $(\Omega, Y_\bullet)$.

A morphism $\varphi_\bullet$ is called continuous (isometric, injective, surjective, bijective) if φ_ω is continuous (isometric, injective, surjective, bijective) for every $\omega \in \Omega$. We will write $\widetilde{\mathcal{L}}(\Omega, \mathcal{C}(X_\bullet, Y_\bullet))$ for the set of continuous morphisms and $\widetilde{\mathcal{L}}(\Omega, \mathcal{I}(X_\bullet, Y_\bullet))$ for the set of isometric ones.

A continuous morphism $\varphi_\bullet \in \widetilde{\mathcal{L}}(\Omega, \mathcal{C}(X_\bullet, Y_\bullet))$ is invertible if φ_ω is a homeomorphism for every $\omega \in \Omega$ and if $\varphi_\bullet^{-1}$ (defined in the obvious way) is also a morphism.

Remark 2.2.

(i) Obviously, we could define a morphism for fields with different bases, but this is not relevant in our context.

(ii) To verify that a family of continuous maps $\{\varphi_\omega : X_\omega \to Y_\omega\}$ is a morphism, it is enough to check that $\varphi_\bullet(\mathcal{D}) \subseteq \mathcal{L}(\Omega, Y_\bullet)$ for a fundamental family $\mathcal{D}$ of $\mathcal{L}(\Omega, X_\bullet)$. Indeed, this is an easy consequence of Lemma 2.2. In the same spirit, we can verify that $\varphi_\bullet \in \widetilde{\mathcal{L}}(\Omega, \mathcal{C}(X_\bullet, Y_\bullet))$ is invertible if and only if φ_ω is a homeomorphism for every $\omega \in \Omega$.

Suppose now that we choose for every $\omega \in \Omega$ a subset (possibly empty) A_ω of X_ω. Such a choice is called a *subfield* and we would like to define when it is Borel. A natural way of doing this is to suppose that $\Omega' := \{\omega \in \Omega \mid A_\omega \neq \emptyset\}$ is a Borel subset of Ω and that $\mathcal{L}(\Omega', A_\bullet) := \mathcal{L}(\Omega', X_\bullet) \cap \mathcal{S}(\Omega', A_\bullet)$ is a Borel structure on the field $(\Omega', A_\bullet)$; observe that $(\Omega, A_\bullet)$ is *not* in general a field of metric spaces. As $\mathcal{L}(\Omega', X_\bullet) \cap \mathcal{S}(\Omega', A_\bullet)$ is closed under Borel gluings and pointwise limits†, and as condition (i) is obviously satisfied, Lemma 2.1 naturally leads to the following definition.

Definition 2.3. Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ a Borel field of metric spaces. A subfield $(\Omega, A_\bullet)$ is called Borel if:

(i) $\Omega' := \{\omega \in \Omega \mid A_\omega \neq \emptyset\} \in \mathcal{A}$; and

(ii) there exists a countable family of sections $\mathcal{D}' = \{y_\bullet^n\}_{n\geq 1} \subseteq \mathcal{L}(\Omega', A_\bullet)$ such that $A_\omega \subseteq \overline{\{y_\omega^n\}}_{n\geq 1}$ for every $\omega \in \Omega'$.

The set Ω' is called the base of the subfield and $\mathcal{D}'$ is called a fundamental family of the subfield.

Remark 2.3.

(i) $(\Omega', A_\bullet)$ is a Borel field of metric spaces.

(ii) In condition (ii) of Definition 2.3, the closure $\overline{\{y_\omega^n\}}_{n\geq 1}$ is taken in X_ω, which is why we used $\subseteq$ and not an equality. An obvious way to construct a Borel subfield is to take a countable family $\{x_\bullet^n\}_{n\geq 1} \subseteq \mathcal{L}(\Omega, X_\bullet)$ and to choose a subfield $A_\bullet$ such

† In this context, we think about $(\Omega', A_\bullet)$ as a field, so that we are only interested in pointwise limits that are in $\mathcal{S}(\Omega', A_\bullet)$. This does not mean that the set we consider is closed in $\mathcal{S}(\Omega', X_\bullet)$ or in $\mathcal{L}(\Omega', X_\bullet)$

that $\{x^n_\omega\}_{n\geq 1} \subseteq A_\omega \subseteq \overline{\{x^n_\omega\}}_{n\geq 1}$ for every $\omega \in \Omega$. In particular, a Borel section is an obvious example of a Borel subfield.

(iii) The previous construction can be generalized in the following way: if $A_\bullet$ is a Borel subfield of $X_\bullet$ and $B_\bullet$ is a subfield such that $A_\omega \subseteq B_\omega \subseteq \overline{A_\omega}$ for every $\omega \in \Omega$, then $B_\bullet$ is also a Borel subfield of $X_\bullet$.

(iv) Suppose that $\{A^n_\bullet\}_{n\geq 1}$ is a family of Borel subfields. Then the subfield $A_\bullet := \bigcup_{n\geq 1} A^n_\bullet$, defined by $A_\omega := \bigcup_{n\geq 1} A^n_\omega$, is also a Borel subfield. This can be shown in three steps. First, we observe that the base Ω' of $A_\bullet$ is $\bigcup_{n\geq 1} \Omega_n \in \mathcal{A}$, where Ω_n denotes the base of $A^n_\bullet$. Then we construct a section $z_\bullet \in \mathcal{L}(\Omega', A_\bullet)$: pick sections $z^1_\bullet \in \mathcal{L}(\Omega_1, A^1_\bullet)$, $z^2_\bullet \in \mathcal{L}(\Omega_2\backslash\Omega_1, A^2_\bullet)$, $z^3_\bullet \in \mathcal{L}(\Omega_3\backslash(\Omega_1 \cup \Omega_2), A^3_\bullet)$ and so on; gluing them together gives the desired section. Finally, we choose, for each $n \geq 1$, a fundamental family $\mathcal{D}^n$ of $A^n_\bullet$ and we modify each of its elements by gluing it with $z_\bullet|_{\Omega'\backslash\Omega_n}$ to obtain a subset $\widetilde{\mathcal{D}}^n$ of $\mathcal{L}(\Omega', A_\bullet)$. By construction, $\mathcal{D}' := \bigcup_{n\geq 1} \widetilde{\mathcal{D}}^n$ is a fundamental family of $A_\bullet$.

(v) Observe the following obvious property of transitivity for Borel subfields. Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_\bullet)$ be a Borel field of metric spaces, $(\Omega, A_\bullet)$ be a Borel subfield of $(\Omega, X_\bullet)$ of base Ω' and $(\Omega, B_\bullet)$ be a subfield of $(\Omega, X_\bullet)$ such that $B_\omega \subseteq A_\omega$ for all $\omega \in \Omega$. Then $(\Omega, B_\bullet)$ is a Borel subfield of $(\Omega, X_\bullet)$ if and only if $(\Omega', B_\bullet)$ is a Borel subfield of $(\Omega', A_\bullet)$.

2.2. *Equivalence classes.* Suppose now that a Borel probability measure μ on $(\Omega, \mathcal{A})$ is given. In this context, we can define the equivalence relation of equality almost everywhere on the set of Borel sections and more generally on the set of Borel subfields. Two sections (or two Borel subfields) are equal almost everywhere if the set where they differ is of measure 0. We write $x_\bullet =_{\mu\text{-a.e.}} y_\bullet$ (or $A_\bullet =_{\mu\text{-a.e.}} B_\bullet$) when two sections are equivalent, and we denote by $[x_\bullet]$ (or $[A_\bullet]$) the equivalence class of $x_\bullet$ (respectively of $A_\bullet$). We write $L(\Omega, X_\bullet)$ for the set of equivalence classes of Borel sections.

2.3. *Borel subfields of open subsets.* In the case of a subfield such that all the subsets are open, there is an easy sufficient criterion to verify that it is Borel.

LEMMA 2.3. *Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_\bullet)$ be a Borel field of metric spaces and $U_\bullet$ be a subfield such that U_ω is open for every $\omega \in \Omega$. If $\{\omega \in \Omega \mid x_\omega \in U_\omega\} \in \mathcal{A}$ is Borel for all sections $x_\bullet$ in a fundamental family of the Borel structure $\mathcal{L}(\Omega, X_\bullet)$, then $U_\bullet$ is a Borel subfield.*

Proof. Write $\mathcal{D} := \{x^n_\bullet\}_{n\geq 1}$, a fundamental family, and $\Omega_n := \{\omega \in \Omega \mid x^n_\omega \in U_\omega\} \in \mathcal{A}$. By density of $\{x^n_\omega\}_{n\geq 1}$, and since U_ω is open for all $\omega \in \Omega$, we have that $\Omega' := \{\omega \in \Omega \mid U_\omega \neq \emptyset\} = \bigcup_{n\geq 1} \Omega_n$ is Borel. For all $n \geq 1$, we can define a Borel subfield $(\Omega, A^n_\bullet)$ by

$$A^n_\omega := \begin{cases} \{x^n_\omega\} & \text{if } \omega \in \Omega_n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, by construction, $\bigcup_{n\geq 1} A^n_\bullet \subseteq U_\bullet \subseteq \overline{\bigcup_{n\geq 1} A^n_\bullet}$, and so $U_\bullet$ is a Borel subfield by Remarks 2.3(iii) and (iv). □

Example 2.2. If $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ and $r_\bullet$ is a Borel non-negative function, then the subfield of open balls $B(x_\bullet, r_\bullet)$ defined by assigning to each ω the set $B(x_\omega, r_\omega)$ is Borel. In fact, if $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, then

$$\{\omega \in \Omega \mid y_\omega \in B(x_\omega, r_\omega)\} = (d_\bullet(x_\bullet, y_\bullet))^{-1}([0, r_\bullet[) \in \mathcal{A}.$$

The field $\overline{B(x_\bullet, r_\bullet)}$ of the closure of the open balls is also Borel because of Remark 2.3(iii).

2.4. *Borel subfields of closed subsets.* If every metric space X_ω is complete, then there is a sufficient and necessary criterion for a subfield of closed subsets to be Borel. We choose the convention that the distance from a point to the empty set is infinite.

PROPOSITION 2.1. *Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_\bullet)$ be a Borel field of complete metric spaces and $F_\bullet$ be a subfield of closed subsets. Then the following assertions are equivalent.*

(i) *$F_\bullet$ is a Borel subfield.*

(ii) *$d_\bullet(x_\bullet, F_\bullet) : \Omega \to \mathbb{R}_+ \cup \{\infty\}$, $\omega \mapsto d_\omega(x_\omega, F_\omega)$ is a Borel map for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$.*

(iii) *$d_\bullet(x_\bullet, F_\bullet) : \Omega \to \mathbb{R}_+ \cup \{\infty\}$, $\omega \mapsto d_\omega(x_\omega, F_\omega)$ is a Borel map for every $x_\bullet \in \mathcal{D}$, where $\mathcal{D}$ is a fundamental family of $\mathcal{L}(\Omega, X_\bullet)$.*

Proof (sketch). (iii) $\Rightarrow$ (ii): Since the distance to a set is a continuous function, this assertion is a consequence of Lemma 2.2.

Implication (iii) $\Longleftrightarrow$ (i) was proved in the particular case of trivial fields by Castaing and Valadier [**13**]. Notice that $\{\omega \in \Omega \mid F_\omega \neq \emptyset\} = (d_\bullet(x_\bullet, F_\bullet))^{-1}(\mathbb{R}_+)$ for any $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, so we can suppose, without lost of generality, that this set is equal to Ω. The trivialization theorem due to Valadier (see [**36**] and the remark below), the implication (ii) $\Longleftrightarrow$ (iii), the completeness assumption and the property of transitivity of Borel subfields (see Remark 2.3(v)) can therefore be combined to extend the result to the general case. □

Remark 2.4. The trivialization theorem due to Valadier [**36**] states that for every Borel field of metric spaces $(\Omega, X_\bullet)$, there exists an isometric morphism $\varphi_\bullet : (\Omega, X_\bullet) \to (\Omega, \mathbf{U})$, where $\mathbf{U}$ is the universal separable metric space constructed by Urysohn [**35**]. In his paper, Valadier checks that the isometric embedding of a separable metric space X in $\mathbf{U}$ can be done in a Borel way.

Example 2.3. Suppose that every X_ω is geodesic. If $x_\bullet$ is a Borel section and $r_\bullet : \Omega \to \mathbb{R}_+$ is a Borel function, then the field of closed balls $\overline{B}(x_\bullet, r_\bullet)$ defined in the same way as in Example 2.2, is Borel. Indeed, if $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, then

$$d_\bullet(y_\bullet, \overline{B}(x_\bullet, r_\bullet)) = \lceil d_\bullet(y_\bullet, x_\bullet) - r_\bullet \rceil^0,$$

where if $\alpha \in \mathbb{R}$, then

$$\lceil \alpha \rceil^0 = \begin{cases} \alpha & \text{if } \alpha \geq 0, \\ 0 & \text{if } \alpha < 0. \end{cases}$$

In the measure case, we can show that the set of equivalence classes of a Borel subfield of closed sets can be turned into a complete lattice (i.e., every subset has an *infimum* and a *supremum*) when it is endowed with the following order: if $F_\bullet$ and $G_\bullet$ are Borel subfields of closed sets, then $[F_\bullet] \le [G_\bullet]$ if $F_\omega \subseteq G_\omega$ for almost every $\omega \in \Omega$.

We will need the following proposition, which can be deduced from [**28**] (Theorem 3.5 and the explanation at the beginning of §4) as Proposition 2.1 has been deduced from [**13**].

PROPOSITION 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and $X_\bullet$ be a Borel field of complete metric spaces. Let $\{F_\bullet^n\}_{n\ge1}$ be a family of Borel subfields of closed subsets. Then there exists a Borel subset Ω_0 of measure one such that if $\bigcap_{n\ge1} F_\bullet^n$ is defined by assigning to each ω the set $(\bigcap_{n\ge1} F_\bullet^n)_\omega := \bigcap_{n\ge1} F_\omega^n$, then $(\Omega_0, \bigcap_{n\ge1} F_\bullet^n)$ is a Borel subfield of $(\Omega_0, X_\bullet)$.*

THEOREM 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and let $X_\bullet$ be a Borel field of complete metric spaces. Then the set of equivalence classes of Borel subfields of closed subsets with the order of inclusion almost everywhere is a complete lattice.*

More precisely, if $\{[F_\bullet^\beta]\}_{\beta\in\mathcal{B}}$ is a family of equivalence classes of Borel subfields of closed subsets, then there exists a sequence of indices $\{\beta_n\}_{n\ge1} \subseteq \mathcal{B}$ such that

$$\left[\bigcap_{n\ge1} F_\bullet^{\beta_n}\right] \quad \text{and} \quad \left[\overline{\bigcup_{n\ge1} F_\bullet^{\beta_n}}\right]$$

are respectively the infimum and the supremum of the family $\{[F_\bullet^\beta]\}_{\beta\in\mathcal{B}}$.

Before proving this theorem, we recall the notion of an essential supremum of a family of Borel functions, whose existence is guaranteed by the following theorem.

THEOREM 2.2. [**17**, p. 71] *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space such that μ is σ-finite. Let $\{f_i : \Omega \to \mathbb{R} \cup \{\pm\infty\}\}_{i\in\mathcal{I}}$ be a family of Borel functions. Then there exists a Borel function $g : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ such that:*

(i) *for all $i \in \mathcal{I}$, we have $g(\omega) \ge f_i(\omega)$ for almost every $\omega \in \Omega$;*

(ii) *if $h : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is a Borel function satisfying (i), then $h(\omega) \ge g(\omega)$ for almost every $\omega \in \Omega$.*

The function g is uniquely determined up to null sets and all functions in its class satisfy (i) and (ii). Moreover, there exists a countable family of elements of I such that its supremum satisfies (i) and (ii).

We call g an essential supremum of the family $\{f_i\}_{i\in I}$ and we write $g = \sup \mathrm{ess}_{i\in I}\{f_i\}$.

Proof of Theorem 2.1. First, observe the following criterion for a closed subset to be included in another one. If X is a metric space, $D \subseteq X$ is a dense subset and $F_1, F_2 \subseteq X$ are two closed subsets, then

$$F_2 \subseteq F_1 \iff d(x, F_2) \ge d(x, F_1) \quad \text{for all } x \in D. \tag{1}$$

Now let $\{F_\bullet^\beta\}_{\beta\in\mathcal{B}}$ be a family of Borel subfields of closed subsets and fix $\mathcal{D} = \{x_\bullet^i\}_{i\ge1}$, a fundamental family of the Borel structure $\mathcal{L}(\Omega, X_\bullet)$. By Theorem 2.2, there exists, for each $i \ge 1$, a sequence of indices $\{\beta_n^i\}_{n\ge1}$ such that $\sup_{n\ge1} d_\bullet(x_\bullet^i, F_\bullet^{\beta_n^i})$ is an essential *supremum* of the family of functions $\{d_\bullet(x_\bullet^i, F_\bullet^\beta)\}_{\beta\in\mathcal{B}}$. If we set $\mathcal{B}' := \{\beta_n^i\}_{n,i\ge1}$, then we can simultaneously construct an essential *supremum* for each family $\{d_\bullet(x_\bullet^i, F_\bullet^\beta)\}_{\beta\in\mathcal{B}}$ by taking $\sup_{\beta\in\mathcal{B}'} d_\bullet(x_\bullet^i, F_\bullet^\beta)$. By Proposition 2.2, there exists a Borel subfield $F_\bullet$ of closed subsets such that $F_\omega = \bigcap_{\beta\in\mathcal{B}'} F^\beta_\omega$ for almost every $\omega \in \Omega$, and we will show that it is the *infimum* of the family $\{F^\beta_\bullet\}_{\beta\in\mathcal{B}}$. Let $\beta_0 \in \mathcal{B}$ be fixed. Then, for every $i \geq 1$, we have

$$d_\bullet(x^i_\bullet, F_\bullet) =_{a.e.} d_\bullet\Big(x^i_\bullet, \bigcap_{\beta\in\mathcal{B}'} F^\beta_\bullet\Big) \overset{(1)}{\geq} \sup_{\beta\in\mathcal{B}'} d_\bullet(x^i_\bullet, F^\beta_\bullet) \geq_{a.e.} d_\bullet(x^i_\bullet, F^{\beta_0}_\bullet),$$

which shows, by the preliminary observation, that $F_\omega \subseteq F^{\beta_0}_\omega$ for almost every $\omega \in \Omega$. Thus, $[F_\bullet]$ is a minorant and it is obvious from its definition that it is the biggest one.

The same argument can be made for the *supremum* by considering the essential *infimum* of the families $\{d_\bullet(x^i_\bullet, F^\beta_\bullet)\}_{\beta\in\mathcal{B}}$, realized as the *infimum* taken over a subset $\mathcal{B}'' \subseteq \mathcal{B}$. Then $[\overline{\bigcup_{\beta\in\mathcal{B}''} F^\beta}]$ will be the *supremum*. To have the exact formulation of the conclusion of the theorem, we only have to order the countable set $\mathcal{B}' \cup \mathcal{B}'' = \{\beta_n\}_{n\geq 1}$. □

2.5. *Borel fields of proper metric spaces.* In this section, $X_\bullet$ will denote a Borel field of proper metric spaces. We will show that the field assigning to each ω the space of continuous functions on X_ω is a Borel field of metric spaces. To do so, we will need the following lemma.

LEMMA 2.4. *Let X be a proper metric space and $x_0 \in X$ a fixed base point. Then the following function is a metric on $\mathcal{C}(X)$ that induces the topology of uniform convergence on compact sets:*

$$\begin{array}{rccl} \delta: & \mathcal{C}(X)\times\mathcal{C}(X) & \to & \mathbb{R} \\ & (f, g) & \mapsto & \inf\{\varepsilon > 0 \mid \sup_{x\in B(x_0,1/\varepsilon)} |f(x) - g(x)| < \varepsilon\}. \end{array}$$

Moreover, if $D \subseteq X$ is a dense countable subset, then the $\mathbb{Q}$-algebra generated by the functions $\{d_x\}_{x\in D}$ and the constant function **1** *is a dense countable subset of $\mathcal{C}(X)$.*

Proof (sketch). The proof of the first part is an easy exercise. The second part can be proven by applying for each $R \geq 1$ the Stone–Weierstrass theorem (see, e.g., [**25**, p. 198]) to the space $X \cap \overline{B(x_0, R)}$ and the set of functions $\{d_x\}_{x\in\overline{B(x_0,R)}\cap D}$. □

Let $(\Omega, \mathcal{A})$ be a Borel space and $X_\bullet$ a Borel field of proper metric spaces. We fix $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ and we consider the family of metrics $\delta_\bullet = \{\delta_\omega\}_{\omega\in\Omega}$ on $\mathcal{C}(X_\bullet) = \{\mathcal{C}(X_\omega)\}_{\omega\in\Omega}$ given by Lemma 2.4.

THEOREM 2.3. *The set*

$$\mathcal{L}(\Omega, \mathcal{C}(X_\bullet)) := \{f_\bullet \in \mathcal{S}(\Omega, \mathcal{C}(X_\bullet)) \mid f_\bullet(x_\bullet) \text{ is Borel for every } x_\bullet \in \mathcal{L}(\Omega, X_\bullet)\}$$

is a Borel structure on $(\mathcal{C}(X_\bullet), \delta_\bullet)$. Moreover, the subfield $\mathcal{C}_0(X_\bullet)$, where

$$\mathcal{C}_0(X_\omega) = \{f \in \mathcal{C}(X_\omega) \mid f(x^0_\omega) = 0\},$$

is Borel.

Proof. This set is clearly closed under countable Borel gluings and pointwise limits, so, by Lemma 2.1, we only have to check points (i) and (iii) of Definition 2.1.

Fix $\varepsilon > 0$ and choose $\mathcal{D}$, a fundamental family of the Borel subfield $B(x_\bullet^0, 1/\varepsilon)$ (see Example 2.2). Pick $f_\bullet, g_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$. Then,

$$\delta_\omega(f_\omega, g_\omega) \le \varepsilon \iff \sup_{x_\bullet \in \mathcal{D}} |f_\omega(x_\omega) - g_\omega(x_\omega)| \le \varepsilon,$$

and so $\mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ is compatible with the family of metrics $\delta_\bullet$ and point (i) of the definition is verified.

Now observe that $\mathcal{S}(\Omega, \mathcal{C}(X_\bullet))$ is naturally an algebra: if $f_\bullet, g_\bullet \in \mathcal{S}(\Omega, \mathcal{C}(X_\bullet))$ and $\lambda \in \mathbb{R}$, we can define

$$(f_\bullet g_\bullet)_\omega := f_\omega g_\omega, \quad (f_\bullet + g_\bullet)_\omega = f_\omega + g_\omega \quad \text{and} \quad (\lambda f_\bullet)_\omega := \lambda f_\omega.$$

It is clear from its definition that the subset $\mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ is a subalgebra of $\mathcal{S}(\Omega, \mathcal{C}(X_\bullet))$. We now fix $\mathcal{D} = \{x_\bullet^n\}_{n\ge 1}$, a fundamental family of the Borel field $X_\bullet$, and we define the following elements of $\mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$:

$$d_{x_\bullet^n} : \begin{array}{ccl} \Omega & \to & \mathcal{C}(X_\bullet) \\ \omega & \mapsto & d_{x_\omega^n} : \begin{array}{ccc} X_\omega & \to & \mathbb{R} \\ x & \mapsto & d_\omega(x_\omega^n, x) \end{array} \end{array} \quad \text{and}$$

$$\mathbf{1}_\bullet : \begin{array}{ccl} \Omega & \to & \mathcal{C}(X_\bullet) \\ \omega & \mapsto & \mathbf{1}_\omega : \begin{array}{ccc} X_\omega & \to & \mathbb{R} \\ x & \mapsto & 1. \end{array} \end{array}$$

We write $\mathcal{A}_\mathbb{Q}$, the countable $\mathbb{Q}$-subalgebra of $\mathcal{S}(\Omega, \mathcal{C}(X_\bullet))$ generated by $\{d_{x_\bullet}\}_{x_\bullet \in \mathcal{D}} \cup \{\mathbf{1}_\bullet\}$. Then $\mathcal{A}_\mathbb{Q}$ is contained in $\mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ because this last set is an algebra that contains the generators of $\mathcal{A}_\mathbb{Q}$; moreover, $\{f_\omega \mid f_\bullet \in \mathcal{A}_\mathbb{Q}\}$ is dense in $\mathcal{C}(X_\omega)$ for every $\omega \in \Omega$ by Lemma 2.4, so Condition 2.1(iii) is satisfied.

To prove that the subfield $\mathcal{C}_0(X_\bullet)$ is Borel, it is enough to realize that if $\mathcal{D}$ is a fundamental family of the field $\mathcal{C}(X_\bullet)$, then $\widetilde{\mathcal{D}} := \{f_\bullet - f_\bullet(x_\bullet^0) \mid f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))\}$ is obviously a fundamental family of the subfield. □

We will show now that the intersection behaves better in proper spaces than in complete ones (see Proposition 2.2).

PROPOSITION 2.3. *Let $(\Omega, \mathcal{A})$ be a Borel space and let $X_\bullet$ be a Borel field of proper metric spaces. Let $\{F_\bullet^n\}_{n\ge 1}$ be a family of Borel subfields of closed sets. Then the subfield $\bigcap_{n\ge 1} F_\bullet^n$ is a Borel subfield.*

We will need the following lemma, whose proof is straightforward.

LEMMA 2.5. *Let X be a proper metric space. Then the following assertions are true.*

(i) *Let F be a closed subset of X. Then, for every $x \in X$, the distance $d(x, F)$ is realized.*

(ii) *Let $\{F_n\}_{n\ge 1}$ be a decreasing sequence of closed subsets of X. Then, for every $x \in X$, we have $d(x, \bigcap_{n\ge 1} F_n) = \lim_{n\to\infty} d(x, F_n) = \sup_n d(x, F_n)$, where $d(x, \emptyset) = \infty$, by convention.*

Proof of Proposition 2.3. The proof proceeds in three steps.

(i) Proposition 2.1, applied twice, implies that the conclusion of the theorem is true in the particular case when $F_\omega^{n+1} \subseteq F_\omega^n$ for every $\omega \in \Omega$ and $n \geq 1$, since $d_\bullet(x_\bullet, \bigcap_{n\geq 1} F_\bullet^n) = \lim_{n\to\infty} d_\bullet(x_\bullet, F_\bullet^n)$, by Lemma 2.5.

(ii) Let $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ and $a_\bullet, b_\bullet \in \mathcal{L}(\Omega, \mathbb{R})$. Let $F_\bullet := f_\bullet^{-1}([a_\bullet, b_\bullet])$ be the subfield defined by $F_\omega = f_\omega^{-1}([a_\omega, b_\omega])$. We will show that $F_\bullet$ is a Borel subfield. For every integer $n \geq 1$, we set $U_\bullet^n := f_\bullet^{-1}(]a_\bullet - 1/n, b_\bullet + 1/n[)$ for the subfield defined in a similar way to $F_\bullet$. It is a Borel subfield by Lemma 2.3 because if $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, then

$$\{\omega \in \Omega \mid x_\omega \in U_\omega^n\} = \{\omega \in \Omega \mid a_\omega - 1/n \leq f_\omega(x_\omega) \leq b_\omega + 1/n\},$$

and the latter set is Borel by the definition of $\mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$. By Remark 2.3, $\overline{U_\bullet^n}$ is a Borel subfield, so the sequence $\{\overline{U_\bullet^n}\}_{n\geq 1}$ is a decreasing sequence of Borel subfields of closed subsets such that $\bigcap_{n\geq 1} \overline{U_\bullet^n} = F_\bullet$ (this equality is satisfied because every f_ω is continuous). Thus, $F_\bullet$ is a Borel subfield by the first step.

(iii) We will show now that if $F_\bullet$ and $G_\bullet$ are Borel subfields of closed subsets, then so is $F_\bullet \cap G_\bullet$ (and the conclusion of the proposition will then follow by applying this fact recursively and using step (i)). By Proposition 2.1, $d_{F_\bullet}, d_{G_\bullet} \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$, so $F_\bullet \cap G_\bullet = (d_{F_\bullet} + d_{G_\bullet})^{-1}(0)$ is a Borel subfield by step (ii). □

3. *Borel fields of CAT(0) spaces*

3.1. *Basic properties.* First, we recall some notation and terminology. A subset $C \subseteq X$ is *convex* if it contains any geodesic segment joining any two of its points. For such a closed convex subset in a complete CAT(0) space, we denote by $\pi_C(x)$ the unique point which satisfies $d(x, \pi_C(x)) = d(x, C) := \inf_{y\in C} d(x, y)$ [**8**, Proposition II.2.4]. This is the *projection* of x on C and the *projection map* $\pi_C : X \to C$ does not increase distances. The *circumradius* of a non-empty bounded set $A \subseteq X$ is $r(A) := \inf\{r > 0 \mid \exists x \in X, A \subseteq \overline{B}(x, r)\}$. This infimum is achieved and there exists a unique point $c_A \in X$ such that $A \subseteq \overline{B}(c_A, r(A))$ [**8**, Proposition II.2.7]. This point is called the *circumcenter* of A.

If $(\Omega, X_\bullet)$ is a field of metric spaces such that X_ω is a CAT(0) space for all $\omega \in \Omega$, we call it a *field of CAT(0) spaces*. Monod was the first to consider such fields in [**34**].

Definition 3.1. Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces, $x_\bullet, y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ be two sections and $(\Omega, C_\bullet)$ be a Borel subfield of non-empty closed convex sets.

We define

$$\begin{aligned} \gamma_{x_\bullet,y_\bullet} : \quad [0, 1] &\to \mathcal{S}(\Omega, X_\bullet) \\ t &\mapsto [\gamma_{x_\bullet,y_\bullet}(t) : \omega \mapsto \gamma_{x_\omega,y_\omega}(t)], \end{aligned}$$

where $\gamma_{x_\omega,y_\omega} : [0, 1] \to X_\omega$ is the unique geodesic with constant speed such that $\gamma_{x_\omega,y_\omega}(0) = x_\omega$ and $\gamma_{x_\omega,y_\omega}(1) = y_\omega$ for all $\omega \in \Omega$.

We also define $\pi_{C_\bullet}(x_\bullet) \in \mathcal{S}(\Omega, X_\bullet)$, where $\pi_{C_\omega}(x_\omega)$ is the projection of x_ω on C_ω for all $\omega \in \Omega$.

We know that the field $\overline{B}(x_\bullet, r_\bullet)$ is Borel if $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ and $r_\bullet$ is a non-negative Borel function. Since we have on the one hand $\gamma_{x_\bullet,y_\bullet}(t) = \overline{B}(x_\bullet, t d_\bullet(x_\bullet, y_\bullet)) \cap \overline{B}(y_\bullet, (1 - t) d_\bullet(x_\bullet, y_\bullet))$ for all $t \in [0, 1]$, and on the other hand $\pi_{C_\bullet}(x_\bullet) = C_\bullet \cap \overline{B}(x_\bullet, d_\bullet(x_\bullet, C_\bullet))$, we conclude, using Proposition 2.3, that the sections introduced in Definition 3.1 are Borel. In the same spirit, we can define a circumradius function and a circumcenter section whenever a Borel subfield of bounded sets is given.

LEMMA 3.1. *Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces and $(\Omega, B_\bullet)$ be a Borel subfield of bounded sets of $(\Omega, X_\bullet)$.*

(i) *The circumradius function $r(B_\bullet)$ is Borel.*

(ii) *The section of circumcenters is Borel, i.e., $c_{B_\bullet} \in \mathcal{L}(\Omega, X_\bullet)$.*

Proof. (i) Recall that if $B \subseteq X$ is a bounded subset in a proper CAT(0) space, then we have the equality $r(B) = \inf_{x \in X}\{\sup_{y \in B} d(x, y)\}$. So, define $f_\omega : X_\omega \to \mathbb{R}$ by $f_\omega(x_\omega) = \sup_{y_\omega \in B_\omega} d(x_\omega, y_\omega)$ for all $\omega \in \Omega$. We have $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ because if $\mathcal{D} \subseteq \mathcal{L}(\Omega, X_\bullet)$, $\mathcal{D}' \subseteq \mathcal{L}(\Omega, B_\bullet)$ are fundamental families, then $f_\bullet(x_\bullet) = \sup_{y_\bullet \in \mathcal{D}'} d(x_\bullet, y_\bullet)$ for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. Consequently, we deduce that the function $r(B_\bullet) = \inf f_\bullet = \inf_{x_\bullet \in \mathcal{D}} f_\bullet(x_\bullet)$ is Borel.

(ii) Observe that $c_{B_\bullet} = f_\bullet^{-1}(r(B_\bullet))$ is Borel (see the proof of Proposition 2.3, step (ii)). □

Other Borel functions appear naturally on Borel fields of CAT(0) spaces.

First, recall that for CAT(0) spaces it is possible to define several notions of angles. The *comparison angle* at p between x, y, denoted by $\overline{\angle}_p(x, y)$, is the corresponding angle in a comparison triangle. This allows us to define an infinitesimal notion of angle: if $p, x, y \in X$ and $c, c' : [0, b], [0, b'] \to X$ are two geodesic segments such that $c(0) = c'(0) = p$ and $c(b) = x$, $c'(b') = y$, then the *Alexandrov angle* at p between x and y is defined by $\angle_p(x, y) := \limsup_{t,t' \to 0} \overline{\angle}_p(c(t), c'(t))$, where the CAT(0) hypothesis ensures the existence of this limit.

LEMMA 3.2. *Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces.*

(i) *If $x_\bullet, y_\bullet, p_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ are such that $x_\omega \neq p_\omega \neq y_\omega$ for all $\omega \in \Omega$, then the comparison angle function $\overline{\angle}_{p_\bullet}(x_\bullet, y_\bullet)$ is Borel.*

(ii) *If we replace in (i) the comparison angle by the Alexandrov angle, then the function obtained is also Borel.*

Proof. (i) This assertion follows directly from the law of cosines, which can be used to write the angle in terms of the distances.

(ii) For each $n \in \mathbb{N}$, let $\Omega_n := \{\omega \in \Omega \mid \min\{d_\omega(p_\omega, x_\omega), d_\omega(p_\omega, y_\omega)\} \geq 1/n\} \in \mathcal{A}$, and define two sections in $\mathcal{L}(\Omega, X_\bullet)$ by

$$c_\bullet^n|_{\Omega_n} := \overline{B}(p_\bullet, 1/n) \cap \overline{B}(x_\bullet, d_\bullet(p_\bullet, x_\bullet) - 1/n) \quad \text{and}$$
$$\widetilde{c}_\bullet^n|_{\Omega_n} := \overline{B}(p_\bullet, 1/n) \cap \overline{B}(y_\bullet, d_\bullet(p_\bullet, y_\bullet) - 1/n),$$

which are extended in an arbitrarily Borel way to $\Omega \backslash \Omega_n$. Since $x_\omega \neq p_\omega \neq y_\omega$ for all $\omega \in \Omega$, we have $\Omega = \bigcup_{n \geq 1} \Omega_n$, and thus for every $\omega \in \Omega$, there exists $n_\omega \in \mathbb{N}$ such that $c_\omega^n = c_\omega(1/n)$ and $\widetilde{c}_\omega^n = \widetilde{c}_\omega(1/n)$ for all $n \geq n_\omega$, where $c_\omega : [0, d_\omega(p_\omega, x_\omega)] \to X_\omega$ (respectively $\widetilde{c}_\omega : [0, d_\omega(p_\omega, y_\omega)] \to X_\omega$) is the geodesic from p_ω to x_ω (respectively y_ω).

By [**8**, Proposition II.3.1], we have

$$\angle_{p_\bullet}(x_\bullet, y_\bullet) = \lim_{n\to\infty} 2 \arcsin\left(\frac{n}{2}\cdot d_\bullet(c^n_\bullet, \tilde{c}^n_\bullet)\right),$$

and this shows that the function is Borel. □

Recall [**8**, §II.8] that if X is a proper CAT(0) space, the boundary at infinity ∂X can be defined as the set of equivalence classes of geodesic rays in X, where two rays are equivalent (*asymptotic*) if they remain at a bounded distance from each other. Often we write $c(\infty)$ for the equivalence class of the geodesic ray c, and a typical point of ∂X is written ξ. Fixing a base point $x_0 \in X$ leads to a bijection between $\xi \in \partial X$ and the unique geodesic ray $c_{x_0,\xi}$, starting at x_0 and such that $c(\infty) = \xi$. This identification can be used to define the *conic topology* (which turns out to be independent of the choice of x_0): $\xi_n \to \xi$ if $c_{x_0,\xi_n}(t) \to c_{x_0,\xi}(t)$ for all $t \geq 0$. Another equivalent construction uses the map $i : X \to \mathcal{C}_0(X)$, defined by $x \mapsto d_x - d(x, x_0)$, where $\mathcal{C}_0(X)$ is endowed with the topology of uniform convergence on compact sets. In general, for an arbitrary proper metric space, this map is *not* a homeomorphism onto its image, but it is if the space is geodesic [**7**]. It can be shown that ∂X is homeomorphic to $\overline{i(X)}\backslash i(X)$: to $\xi \in \partial X$ and we can associate the Busemann function $b_{x_0,\xi} : X \to \mathbb{R}$, $x \mapsto \lim_{t\to\infty} d(x, c_{x_0,\xi}(t)) - d(x_0, c_{x_0,\xi}(t))$. These functions satisfy

$$b_{z,\xi}(y) = b_{x,\xi}(y) - b_{x,\xi}(z), \qquad \xi \in \partial X, x, y, z \in X. \tag{2}$$

For all $\omega \in \Omega$, we now define the map $i_\omega : X_\omega \longrightarrow \mathcal{C}_0(X_\omega)$ by setting $x_\omega \mapsto d_{x_\omega} - d(x_\omega, x^0_\omega)$, where $x^0_\bullet \in \mathcal{S}(\Omega, X_\bullet)$ is a fixed section. This will enable us to deal with the Borel structure on the fields of boundaries.

THEOREM 3.1. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper CAT(0) spaces and* $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$.

(i) *We have* $i_\bullet \in \widetilde{\mathcal{L}}(\Omega, \mathcal{C}(X_\bullet, \mathcal{C}_0(X_\bullet)))$*, and the subfields* $i_\bullet(X_\bullet)$, $\overline{i_\bullet(X_\bullet)}$ *are Borel, where* $\mathcal{C}_0(X_\bullet)$ *is endowed with the Borel structure inherited from* $(\mathcal{C}(X_\bullet), \delta_\bullet)$ *(see Theorem 2.3).*

(ii) *The field* $\partial X_\bullet$ *is a Borel subfield of closed sets of* $\overline{i_\bullet(X_\bullet)}$.

Proof. (i) Since $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, we observe that $i_\bullet(x_\bullet)(y_\bullet) = d_\bullet(x_\bullet, y_\bullet) - d_\bullet(x_\bullet, x^0_\bullet)$ is Borel for every $y_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. So, $i_\bullet(x_\bullet) \in \mathcal{L}(\Omega, \mathcal{C}_0(X_\bullet))$, and $i_\bullet$ is a morphism which is obviously continuous. Consequently, $i_\bullet(X_\bullet)$ is a Borel subfield of $\mathcal{C}_0(X_\bullet)$ as well as $\overline{i_\bullet(X_\bullet)}$.

(ii) Observe that if X is a proper CAT(0) space and $x_0 \in X$ is a fixed base point, we have the equality $\partial X = \bigcap_{n\in\mathbb{N}} \overline{i(X)\backslash i(\overline{B(x_0, n)})}$. We use this trick to show the assertion by using Proposition 2.3. Since $i_\bullet(X_\bullet)\backslash i_\bullet(\overline{B(x^0_\bullet, n)})$ is a Borel subfield of open sets of $i_\bullet(X_\bullet)$ for all $n \in \mathbb{N}$, we obtain that $\partial X_\bullet = \bigcap_{n\in\mathbb{N}} \overline{i_\bullet(X_\bullet)\backslash i_\bullet(\overline{B(x^0_\bullet, n)})}$ is also Borel. □

Remark 3.1. In particular, Theorem 3.1 describes the sections of $\mathcal{L}(\Omega', \partial X_\bullet)$, where Ω' is the base of the subfield $\partial X_\bullet$, which is equal to the set $\{\omega \in \Omega \mid X_\omega \text{ is unbounded}\}$. By definition of the Borel structure on $(\Omega', \mathcal{C}_0(X_\bullet))$, the section $\xi_\bullet \in \mathcal{S}(\Omega', \partial X_\bullet)$ is Borel if and only if the function $b_{x^0_\bullet,\xi_\bullet}(x_\bullet)$ is Borel for every $x_\bullet \in \mathcal{L}(\Omega', X_\bullet)$. Observe that this condition does not depend on the choice of $x_\bullet^0 \in \mathcal{L}(\Omega', X_\bullet)$ because if $y_\bullet^0 \in \mathcal{L}(\Omega', X_\bullet)$, we have $b_{y_\bullet^0,\xi_\bullet}(x_\bullet) = b_{x_\bullet^0,\xi_\bullet}(x_\bullet) - b_{x_\bullet^0,\xi_\bullet}(y_\bullet^0)$ by (2). Therefore, $\xi_\bullet$ is Borel if and only if $b_{y_\bullet,\xi_\bullet}(x_\bullet)$ is Borel for every $x_\bullet, y_\bullet \in \mathcal{L}(\Omega', X_\bullet)$.

The Borel structure of the field of boundaries is such that the natural sections and functions associated are Borel.

LEMMA 3.3. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper unbounded CAT(0) spaces and two sections* $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$*,* $\xi_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$*. Define*

$$\begin{array}{rccl} c_{x_\bullet,\xi_\bullet}: & [0,\infty[& \to & \mathcal{S}(\Omega, X_\bullet) \\ & t & \mapsto & c_{x_\bullet,\xi_\bullet}(t): \omega \mapsto c_{x_\omega,\xi_\omega}(t), \end{array}$$

where $c_{x_\omega,\xi_\omega}: [0,\infty[\to X_\omega$ *is the unique geodesic ray such that* $c_{x_\omega,\xi_\omega}(0) = x_\omega$ *and* $c_{x_\omega,\xi_\omega}(\infty) = \xi_\omega$*. Then we have* $c_{x_\bullet,\xi_\bullet}(t) \in \mathcal{L}(\Omega, X_\bullet)$ *for every* $t \in \mathbb{R}_+$*.*

Proof. Let $\mathcal{D} \subseteq \mathcal{L}(\Omega, X_\bullet)$ be a fundamental family. Since $i_\bullet(\mathcal{D})$ is also a fundamental family for the structure $\mathcal{L}(\Omega, \overline{i_\bullet(X_\bullet)})$, by Lemma 2.2, each Borel section in this set is a pointwise limit of countable Borel gluings of elements of $i_\bullet(\mathcal{D})$. In particular, there exists a sequence $\{x_\bullet^n\}_{n\geq 1} \subseteq \mathcal{D}$ such that $\lim_{n\to\infty} i_\bullet(x_\bullet^n) = \xi_\bullet$, and we suppose that $d_\omega(x_\omega^n, x_\omega) \geq n$ for all $\omega \in \Omega$. Consequently, we can define $\gamma_{x_\bullet,x_\bullet^n}(t) \in \mathcal{L}(\Omega, X_\bullet)$ at least for each $t \in [0, n]$. Now fix $t \in \mathbb{R}_+$. Since, by [**8**, Proposition II.8.19], $\lim_{n\to\infty} \gamma_{x_\omega,x_\omega^n}(t) = c_{x_\omega,\xi_\omega}(t)$ for each $\omega \in \Omega$, we deduce $c_{x_\bullet,\xi_\bullet}(t) = \lim_{n\to\infty} \gamma_{x_\bullet,x_\bullet^n}(t) \in \mathcal{L}(\Omega, X_\bullet)$. □

Recall that for every $\eta, \xi \in \partial X$ and $x \in X$, the Alexandrov angle between ξ and η in x is defined by $\angle_x(\xi, \eta) = \angle_x(c_{x,\xi}(1), c_{x,\eta}(1))$ and the Tits angle between ξ and η by $\angle(\xi, \eta) = \sup_{x\in X} \angle_x(\xi, \eta)$. The Tits angle defines a metric on the boundary, called the *angular metric*.

LEMMA 3.4. *Let* $(\Omega, \mathcal{A})$ *be a Borel space and* $(\Omega, X_\bullet)$ *be a Borel field of proper CAT(0) spaces. Consider a section* $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ *and two sections* $\xi_\bullet, \eta_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$*.*

(i) *The Alexandrov angle function* $\angle_{x_\bullet}(\xi_\bullet, \eta_\bullet)$ *is Borel.*

(ii) *The Tits angle function* $\angle_\bullet(\xi_\bullet, \eta_\bullet)$ *is Borel.*

Proof. By definition, we have $\angle_{x_\bullet}(\xi_\bullet, \eta_\bullet) = \angle_{x_\bullet}(c_{x_\bullet,\xi_\bullet}(1), c_{x_\bullet,\eta_\bullet}(1))$, and by [**8**, Proposition II.9.8(4)], $\angle_\bullet(\xi_\bullet, \eta_\bullet) = 2\arcsin(\lim_{t\to\infty}(1/2t) \cdot d(c_{x_\bullet,\xi_\bullet}(t), c_{x_\bullet,\eta_\bullet}(t)))$. So, we deduce from Lemmas 3.2 and 3.3 that these functions are Borel. □

We turn now to some subfields of the field of metric spaces $(\Omega, (\partial X_\bullet, \angle_\bullet))$. Notice that the latter is not always a Borel field of metric spaces because the topology induced by the angular metric may be not separable. Despite this problem, we prove the following theorem.

THEOREM 3.2. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper unbounded CAT(0) spaces and* $(\Omega, A_\bullet)$ *be a Borel subfield of non-empty closed sets, with respect to the conic topology, of* $(\Omega, \partial X_\bullet)$*.*

(i) *The circumradius function* $r(A_\bullet)$*, with respect to the angular metric, is Borel.*

Moreover, suppose that $r(A_\omega) < \pi/2$ *for all* $\omega \in \Omega$*.*

(ii) *The section of circumcenters is Borel, i.e.,* $c_{A_\bullet} \in \mathcal{L}(\Omega, \partial X_\bullet)$*.*

Proof. The proof of (i) proceeds in three steps.

(a) For a CAT(0) space X and $x_0 \in X$, define for each integer $n \geq 1$ the function

$$\begin{array}{rccl} \angle^n : & \partial X \times \partial X & \to & [0, \pi] \\ & (\xi, \eta) & \mapsto & \angle^n(\xi, \eta) := \sup_{t \in [1,n]} \overline{\angle}_{x_0}(c_{x_0,\xi}(t), c_{x_0,\eta}(t)). \end{array}$$

This increasing sequence of functions verifies $\angle(\xi, \eta) = \lim_{n\to\infty} \angle^n(\xi, \eta) = \sup_{n\geq 1} \angle^n(\xi, \eta)$ by [**8**, Proposition II.9.8(1)]. If ∂X is endowed with the conic topology, then $\angle^n$ is a continuous function. The argument is as follows. The function $f_n : [1, n] \times (\partial X)^2 \to [0, \pi]$, $(t, \xi, \eta) \mapsto \overline{\angle}_{x_0}(c_{x_0,\xi}(t), c_{x_0,\eta}(t))$, is continuous, hence uniformly continuous. It is easy then to check that the function $(\xi, \eta) \mapsto \angle^n(\xi, \eta) = \sup_{t\in[1,n]} f^n(t, \xi, \eta)$ is continuous.

(b) We will now prove that if $A \subseteq \partial X$ is a non-empty closed subset, then we have the equality

$$r(A) = \lim_{n\to\infty} \min_{\xi\in\partial X} \{\max_{\eta\in A} \angle^n(\xi, \eta)\}. \tag{3}$$

Indeed, we have

$$\begin{aligned} r(A) &\overset{\text{def.}}{=} \inf_{\xi\in\partial X} \{\sup_{\eta\in A}\{\sup_{x\in X} \angle_x(\xi, \eta)\}\} = \inf_{\xi\in\partial X} \{\sup_{\eta\in A}\{\sup_{n\geq 1} \angle^n(\xi, \eta)\}\} \\ &= \inf_{\xi\in\partial X} \{\sup_{n\geq 1}\{\sup_{\eta\in A} \angle^n(\xi, \eta)\}\}, \end{aligned}$$

and by (a) and compactness, $r(A) = \inf_{\xi\in\partial X}\{\sup_{n\geq 1}\{\max_{\eta\in A} \angle^n(\xi, \eta)\}\}$. Now consider the restriction $\angle^n |_{\partial X\times A}$, which is uniformly continuous by (i), and also the function $\angle^n_{\max} : \partial X \to [0, \pi]$, $\xi \mapsto \max_{\eta\in A}\{\angle^n(\xi, \eta)\}$, which is continuous. We will use the following observation, whose proof is not a very difficult exercise.

Observation: Let Y be a compact metrizable space and $\{g_n : Y \to [a, b]\}_{n\geq 1}$ be an increasing sequence of continuous functions. Taking a pointwise limit gives a function g which is obviously lower semi-continuous. Then $\min g = \lim_{n\to\infty} \min g_n$. Moreover, if x_n is such that $g_n(x_n) = \min g_n$, then any accumulation point x satisfies $g(x) = \min g$.

We can now easily deduce the formula (3) by applying the first assertion of the observation to the sequence of functions $g_n(\xi) := \angle^n_{\max}(\xi)$, since $r(A) = \min g$.

(c) Finally, if $\xi_\bullet, \eta_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$, then the function $\angle^n_\bullet(\xi_\bullet, \eta_\bullet)$ is Borel for every $n \geq 1$ because, on the one hand, we have, by continuity,

$$\angle^n_\bullet(\xi_\bullet, \eta_\bullet) = \sup_{t\in[1,n]\cap\mathbb{Q}} \overline{\angle}_{x^0_\bullet}(c_{x^0_\bullet,\xi_\bullet}(t), c_{x^0_\bullet,\eta_\bullet}(t)),$$

and, on the other hand, since $c_{x^0_\bullet,\xi_\bullet}(t), c_{x^0_\bullet,\eta_\bullet}(t) \in \mathcal{L}(\Omega, X_\bullet)$, the function $\overline{\angle}_{x^0_\bullet}(c_{x^0_\bullet,\xi_\bullet}(t), c_{x^0_\bullet,\eta_\bullet}(t))$ is Borel by Lemma 3.2. Consequently, if $\mathcal{D} \subseteq \mathcal{L}(\Omega, \partial X_\bullet)$ and $\mathcal{D}' \subseteq \mathcal{L}(\Omega, A_\bullet)$ are fundamental families, then we have

$$\text{rad}(A_\bullet) = \lim_{n\to\infty} \min_{\xi_\bullet\in\mathcal{D}} \{\max_{\eta_\bullet\in\mathcal{D}'} \angle^n_\bullet(\xi_\bullet, \eta_\bullet)\},$$

and this shows that the function is Borel.

We now undertake the proof of (ii). We will also perform this proof in three steps.

(a) By hypothesis, for each $\omega \in \Omega$, the set A_ω has a unique circumcenter c_{A_ω} [**8**, Theorem II.9.13 and Proposition II.2.7]. Therefore, the section $c_{A_\bullet} \in \mathcal{S}(\Omega, \partial X_\bullet)$ is well defined.

(b) Observe the following general fact. If $(\Omega, Y_\bullet)$ is a Borel field of compact spaces and $\{f_\bullet^n\}_{n\geq 1} \subseteq \mathcal{L}(\Omega, \mathcal{C}(Y_\bullet))$ is a sequence of continuous morphisms which is increasing (i.e., for each $\omega \in \Omega$, $n \geq 1$, it satisfies $f_\omega^{n+1} \geq f_\omega^n$), bounded (i.e., for each $\omega \in \Omega$ it satisfies $\sup_{n\geq 1} f_\omega^n < \infty$), and such that $f_\bullet := \lim_{n\to\infty} f_\bullet^n$ satisfies $|f_\omega^{-1}(\{\min f_\omega\})| = 1$ for all $\omega \in \Omega$, then $f_\bullet^{-1}(\{\min f_\bullet\}) \in \mathcal{L}(\Omega, Y_\bullet)$. Indeed, if $\mathcal{D} \subseteq \mathcal{L}(\Omega, Y_\bullet)$ is a fundamental family, we have $\min f_\bullet^n = \inf_{x_\bullet \in \mathcal{D}} f_\bullet(x_\bullet) \in \mathcal{L}(\Omega, \mathbb{R})$, and therefore $(f_\bullet^n)^{-1}(\{\min f_\bullet^n\})$ is a Borel field of closed subsets (see step (ii) of the proof of Proposition 2.3). Consequently, we can pick a Borel section $x_\bullet^n$ in it and, by the observation made in the second step of (i), $f_\bullet^{-1}(\{\min f_\bullet\}) = \lim_{n\to\infty} x_\bullet^n$ is Borel.

(c) Let $\mathcal{D} \subseteq \mathcal{L}(\Omega, \partial X_\bullet)$ and $\mathcal{D}' \subseteq \mathcal{L}(\Omega, A_\bullet)$ be fundamental families. We have seen that

$$\operatorname{rad}(A_\bullet) = \lim_{n\to\infty} \{\min_{\xi_\bullet \in \mathcal{D}} \{\max_{\eta_\bullet \in \mathcal{D}'} \angle_\bullet^n(\xi_\bullet, \eta_\bullet)\}\} = \min_{\xi_\bullet \in \mathcal{D}} \{\lim_{n\to\infty} \{\max_{\eta_\bullet \in \mathcal{D}'} \angle_\bullet^n(\xi_\bullet, \eta_\bullet)\}\}.$$

Define $g_\bullet^n(\xi_\bullet) := \max_{\eta_\bullet \in \mathcal{D}'} \angle_\bullet^n(\xi_\bullet, \eta_\bullet)$. We have $g_\bullet^n \in \mathcal{L}(\Omega, \mathcal{C}(\partial X_\bullet))$, and observe that the sequence of morphisms $\{g_\bullet^n\}_{n\geq 1}$ is increasing, bounded, and that

$$g_\bullet := \lim_{n\to\infty} g_\bullet^n \in \widetilde{\mathcal{L}}(\Omega, \mathcal{F}(\partial X_\bullet, \mathbb{R}))$$

is such that

$$g_\omega^{-1}(\{\min(g_\omega)\}) = \{c_{A_\omega}\} \quad \text{for each } \omega \in \Omega$$

because $\min g_\omega = r(A_\omega)$ and c_{A_ω} is unique. Therefore, by step (b), we obtain $c_{A_\bullet} \in \mathcal{L}(\Omega, \partial X_\bullet)$. □

3.2. *Limit set at infinity.* The goal of this section is to associate a canonical Borel section $\xi_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$ with a decreasing sequence $\{C_\bullet^n\}_{n\geq 1}$ of Borel subfields of convex, closed, non-empty subsets in a field of proper CAT(0) spaces which satisfies the hypothesis of 'finite covering dimension'. The section we are looking for is obtained by considering the circumcenter of the Borel field of limit sets at infinity.

Definition 3.2. Let X be a proper CAT(0) space and $\{C_n\}_{n\geq 1}$ be a decreasing sequence of convex, closed, non-empty subsets such that $\bigcap_{n\geq 1} C_n = \emptyset$. Since the space is proper, this assumption is equivalent to the fact that $\lim_{n\to\infty} d(x, \pi_{C_n}(x)) = \infty$ for every $x \in X$. For $x \in X$, we consider

$$L := \overline{\{\pi_{C_n}(x)\}_{n\geq 1}} \cap \partial X = \{\text{accumulation points of } \{\pi_{C_n}(x)\}_{n\geq 1}\},$$

where the closure is taken relative to the conic topology on $\overline{X}$. Since the projection on a convex set does not increase the distance, this set is independent of the chosen point x. We call this set L the limit set at infinity of the given sequence of subfields.

First, we show that this definition is independent of the choice of $x \in X$.

LEMMA 3.5. *Let X be a proper CAT(0) space and $\{C_n\}_{n\geq 1}$ as above. Then we have* $\operatorname{diam} L_x \leq \pi/2$ *with respect to the angular metric.*

Proof. By [**8**, Proposition II.2.4 (3)], we have

$$\angle_{\pi_{C_n}(x)}(x, \pi_{C_m}(x)) \geq \pi/2$$

if $C_m \subsetneq C_n$ and n is large enough so that $x \notin C_n$. In particular, we have $\overline{\angle}_{\pi_{C_n}(x)}(x, \pi_{C_m}(x)) \geq \pi/2$, and thus

$$\overline{\angle}_x(\pi_{C_n}(x), \pi_{C_m}(x)) \leq \pi/2$$

if $m > n$ are large enough so that $x \notin C_n$ and $C_m \subsetneq C_n$. Consider now any $\xi, \zeta \in L_x \subseteq \partial X$ as well as two subsequences $\{\pi_{C_{n_k}}(x)\}_{k\geq 1}$ and $\{\pi_{C_{m_k}}(x)\}_{k\geq 1}$ such that $\pi_{C_{n_k}}(x) \to \xi$ and $\pi_{C_{m_k}}(x) \to \zeta$ for $k \to \infty$. By [**8**, Proposition II.9.16], we conclude that

$$\angle(\xi, \zeta) \leq \liminf_{k\to\infty} \overline{\angle}_x(\pi_{C_{n_k}}(x), \pi_{C_{m_k}}(x)) \leq \pi/2$$

if $m_k > n_k$ are as above. □

The topological condition on X needed to ensure the uniqueness of the circumcenter of a limit set at infinity is the following.

Definition 3.3. The order of a family $\mathcal{E}$ of subsets of a set X is the largest integer n such that the family $\mathcal{E}$ contains $n + 1$ subsets with a non-empty intersection, or ∞ if no such integer exists. If X is a metrizable space, it is possible to define the covering dimension (also called the Čech–Lebesgue dimension) $\dim(X)$ by the following three steps.

(i) $\dim(X) \leq n$ if every finite open cover of X has a finite open refinement of order $\leq n$.
(ii) $\dim(X) = n$ if $\dim(X) \leq n$ and the inequality $\dim(X) \leq n - 1$ does not hold.
(iii) $\dim(X) = \infty$ if the inequality $\dim(X) \leq n$ does not hold for any n.

We also define $\dim_C(X) := \sup\{\dim(K) \mid K \subseteq X \text{ compact}\}$ and refer to [**19**, Ch. 7] for the properties of a covering dimension and some equivalent definitions.

Remark 3.2. Some authors refer to $\dim_C(X)$ as the geometric dimension of X (see, e.g., [**12**]). Note that a CAT(0) space X such that $\dim_C(X) = 0$ is a singleton, and if it satisfies $\dim_C(X) = 1$, it is an $\mathbb{R}$-tree.

THEOREM 3.3. [**23**, Theorem 1.7 and Proposition 1.8]

(i) *If X is a proper CAT(0) space, then the inequality* $\dim_C(\partial X, \angle) \leq \dim(X) - 1$ *holds.*
(ii) *If Y is a complete CAT(1) space such that* $\dim_C(Y) < \infty$ *and* $\operatorname{diam}(Y) \leq \pi/2$, *then there exists a constant* $\delta > 0$ *which only depends on* $\dim_C(Y)$, *and such that the inequality* $\operatorname{rad}(Y) \leq \pi/2 - \delta < \pi/2$ *holds. In particular, there exists a unique circumcenter* c_Y *for Y* [**8**, Proposition II.2.7].

Consequently, the limit set at infinity of a decreasing sequence $\{C_n\}_{n\geq 1}$ as above has a unique circumcenter. Indeed, if $L \subseteq \partial X$ is the limit set at infinity of such a sequence, we have $\operatorname{diam}(L) \leq \pi/2$ by Lemma 3.5. Since $(\partial X, \angle)$ is a complete CAT(1) space [**8**, Theorem II.9.13], the convex hull of L is such that $\operatorname{diam}(\overline{\operatorname{co}(L)}) = \operatorname{diam}(L) \leq \pi/2$ [**33**, Lemma 4.1]. By hypothesis and Theorem 3.3(i), we have $\dim_C(\partial X) < \infty$, and thus $\dim_C(\overline{\operatorname{co}(L)}) < \infty$. This allows us to apply Theorem 3.3(ii) to the complete CAT(1) space $\overline{\operatorname{co}(L)}$ and to conclude that $\operatorname{rad}(L) \leq \operatorname{rad}(\overline{\operatorname{co}(L)}) < \pi/2$ and that L has a unique circumcenter.

PROPOSITION 3.1. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper CAT(0) spaces with finite covering dimension, and* $\{C^n_\bullet\}_{n\geq 1}$ *be a sequence of Borel subfields of convex, closed, non-empty subsets which satisfies, for every* $\omega \in \Omega$, $C^n_\omega \supseteq C^{n+1}_\omega$ *for every* $n \geq 1$ *and* $\bigcap_{n\geq 1} C^n_\omega = \emptyset$.

(i) *The subfield $L_\bullet$ of $\partial X_\bullet$, where, for every $\omega \in \Omega$, L_ω is the limit set at infinity of the sequence $\{C^n_\omega\}_{n\geq 1}$, is Borel.*
(ii) *The section $\xi_\bullet := c_{L_\bullet}$ is Borel.*

Proof. (i) Fix a section $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. By definition, we have

$$L_\omega = \overline{\{\pi_{C^n_\omega}(x^0_\omega)\}_{n\geq 1}} \cap \partial X_\omega \quad \text{for every } \omega \in \Omega,$$

but $\overline{\{\pi_{C^n_\bullet}(x^0_\bullet)\}_{n\geq 1}}$ is a Borel subfield of $\overline{X}_\bullet$ since $\pi_{C^n_\bullet}(x^0_\bullet) \in \mathcal{L}(\Omega, X_\bullet)$ for every $n \geq 1$. Consequently, since $\overline{X}_\bullet$ is compact, the intersection of this Borel subfield with $\partial X_\bullet$ is Borel by Proposition 2.3. We conclude that $L_\bullet$ is a Borel subfield of $\partial X_\bullet$, using Remark 2.3(v).

(ii) This follows directly from (i) and the fact that $\mathrm{rad}(L_\omega) < \pi/2$ for every $\omega \in \Omega$, using Theorem 3.2. □

Remark 3.3. The previous results also hold for a generalized sequence $\{C^\alpha_\bullet\}_{\alpha\in\mathbb{R}}$ indexed by $\mathbb{R}$, provided we have the following condition: $C^\beta_\omega = \bigcap_{\alpha<\beta} C^\alpha_\omega$ for every $\omega \in \Omega$ and $\beta \in \mathbb{R}$. The limit set at infinity is in this case given by $L_\omega = \overline{\{\pi_{C^\alpha_\omega}(x)\}_{\alpha\in\mathbb{R}}} \cap \partial X_\omega$, and the 'continuity' condition ensures that if D is a dense subset of $\mathbb{R}$, then

$$L_\omega = \overline{\{\pi_{C^\alpha_\omega}(x)\}_{\alpha\in D}}.$$

This is used to prove that $L_\bullet$ is a Borel subfield.

3.3. *Adams–Ballmann decomposition.* We now turn our attention to the Adams–Ballmann decomposition of a proper CAT(0) space. First, we recall the following key definition.

Definition 3.4. Let X be a proper CAT(0) space. A point $\xi \in \partial X$ is called a flat point if the associated Busemann function b_ξ is an affine function. Note that the set of flat points, denoted by F, is $\mathrm{Isom}(X)$-invariant.

The boundary of a product $X \times Y$ is isometric (when endowed with the angular metric) to the spherical join of the boundaries, i.e., $\partial X * \partial Y$ (see [**8**, Definition I.5.13] for the definition of the spherical join of two metric spaces and [**8**, Corollary II.9.11] for the proof of the result). The following theorem states the existence of what we will call the Adams–Ballmann decomposition of a CAT(0) space.

THEOREM 3.4. [**1**, p. 188] *Let X be a proper CAT(0) space. Then there exists a real Hilbert space E, a complete CAT(0) space Y and an isometric map $i : X \to Z = Y \times E$ such that:*
(i) *$i(F) = \partial E \cap \partial(i(X)) \subseteq \partial Y * \partial E \simeq \partial Z$ and the set of directions $\{v(i(\xi)) \mid \xi \in F\}$ generates H as a real Hilbert space;*
(ii) *the set $Y' := \pi_Y(i(X))$ is convex and dense in Y; and*
(iii) *any isometry $\gamma : X \to X$ extends uniquely in $\widetilde{\gamma} : Z \to Z$ and $\widetilde{\gamma} = (\widetilde{\gamma}_Y, \widetilde{\gamma}_E)$.*

It follows from this theorem that the angular and the conic topology on F coincide, that the geometry on F is spherical and that F is closed and π-convex in ∂X. In order to adapt this result in the context of Borel fields of proper CAT(0) spaces, we have to observe the following.

Remark 3.4.

(i) Let X be a proper CAT(0) space. If D is a dense subset of F, then E is generated by $\{v(i(\xi)) \mid \xi \in D\}$. If $F = \emptyset$, then the decomposition is trivial with $E = \{*\}$ and $Y = X$.

(ii) A careful analysis of the proof of Theorem 3.4 shows that one can construct the decomposition such that the origin of E is $\pi_E(i(x_0))$, where $x_0 \in X$ is any chosen point.

LEMMA 3.6. *Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces. Then the subfield $F_\bullet$ of $\partial X_\bullet$, defined by F_ω, the set of flat points of ∂X_ω for every $\omega \in \Omega$, is a Borel subfield of closed subsets.*

Proof. We start by considering a proper CAT(0) space X and x_0 a base point of X. For every positive integer R, we introduce the function $\Delta^R : \mathcal{C}(X) \to \mathbb{R}$, defined by

$$\Delta^R(f) := \sup_{z,z' \in B(x_0,R)} \sup_{t\in[0,1]} |f(\gamma_{z,z'}(t)) - (1-t)f(z) - tf(z')|,$$

where we recall that $\gamma_{z,z'}$ is the geodesic from z to z'. It is straightforward to check that for every positive integer R, the function Δ^R is continuous, when $\mathcal{C}(X)$ is endowed with the uniform convergence on compact sets. Note that if $D \subseteq X$ is a dense subset and $f \in \mathcal{C}(X)$, we will obtain the same value for $\Delta^R(f)$ by taking the supremum on $B(x_0, R) \cap D$ and $[0, 1] \cap \mathbb{Q}$ because in a CAT(0) space a geodesic varies continuously with its endpoints (see [**8**, Proposition II.1.4]). We will use the functions Δ^R, which measure the lack of affinity of functions in $\mathcal{C}(X)$ on the balls $B(x_0, R)$, to show that $(\Omega, F_\bullet)$ is a Borel subfield. We fix $x_\bullet^0 \in \mathcal{L}(\Omega, X_\bullet)$ and we define $\overline{\Delta_\bullet}^R \in \mathcal{S}(\Omega, \mathcal{C}(\partial X_\bullet))$ by

$$\begin{array}{rccc} \overline{\Delta_\omega}^R : & \partial X_\omega & \to & \mathbb{R} \\ & \xi & \mapsto & \Delta^R(b_{\xi,x_\omega^0}). \end{array}$$

Indeed, $\overline{\Delta_\omega}^R$ is continuous because it is the restriction of Δ_ω^R to ∂X_ω. By definition of $F_\bullet$, we have that $F_\bullet = \bigcap_{R\geq 1}(\overline{\Delta_\bullet}^R)^{-1}(\{0\})$, so by Proposition 2.2, it remains to show that $(\overline{\Delta_\bullet}^R)^{-1}(\{0\})$ is a Borel subfield. By the second step of the proof of this same proposition, it is enough to show that

$$\overline{\Delta_\bullet}^R \in \mathcal{L}(\Omega, \mathcal{C}(\partial X_\bullet)).$$

So, we only have to check that $\overline{\Delta_\bullet}^R(\xi_\bullet)$ is a Borel function whenever $\xi_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$.

For every $R \geq 1$, we pick a fundamental family $\mathcal{D}^R$ of the Borel subfield $B(x_\bullet^0, R)$ of $X_\bullet$. Note that

$$\overline{\Delta_\bullet}^R(\xi_\bullet) = \sup_{z_\bullet, z'_\bullet \in \mathcal{D}^R} \sup_{t\in[0,1]\cap\mathbb{Q}} |b_{\xi_\bullet,x_\bullet^0}(\gamma_{z_\bullet,z'_\bullet}(t)) - (1-t)b_{\xi_\bullet,x_\bullet^0}(z_\bullet) - tb_{\xi_\bullet,x_\bullet^0}(z'_\bullet)|,$$

which is therefore Borel because the evaluation $b_{\xi_\bullet,x_\bullet^0}$ on Borel sections of $X_\bullet$ is Borel by definition, and since $\gamma_{z_\bullet,z'_\bullet}(t) \in \mathcal{L}(\Omega, X_\bullet)$ for every $t \in [0, 1]$ (see the comment after Definition 3.1). □

PROPOSITION 3.2. *Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces. For every $\omega \in \Omega$, we consider the Adams–Ballmann decomposition $i_\omega : X_\omega \to Z_\omega = Y_\omega \times E_\omega$. Then there exist Borel structures $\mathcal{L}(\Omega, Y_\bullet)$ and $\mathcal{L}(\Omega, E_\bullet)$ on* *the fields* $(\Omega, Y_\bullet)$ *and* $(\Omega, E_\bullet)$ *such that* $i_\bullet : (\Omega, X_\bullet) \to (\Omega, Z_\bullet)$ *is an isometric morphism, where the Borel structure on* $(\Omega, Z_\bullet)$ *is given by* $\mathcal{L}(\Omega, Y_\bullet) \times \mathcal{L}(\Omega, E_\bullet)$. *It can easily be seen to be a Borel structure on* $(\Omega, Z_\bullet)$.

Proof. We start by defining the Borel structure on $E_\bullet$. Fix $x_\bullet^0 \in \mathcal{L}(\Omega, X_\bullet)$. By Remark 3.4(ii), we can choose the decomposition such that the origin of E_ω is $\pi_{E_\omega}(i_\omega(x_\omega^0))$ for every $\omega \in \Omega$. We pick $\mathcal{D} := \{\xi_\bullet^n\}_{n\geq 1} \subseteq \mathcal{L}(\Omega, \partial X_\bullet)$, a fundamental family of the Borel subfield $F_\bullet$. By Remark 3.4(i), the sets $\{v(i_\omega(\xi_\omega^n))\}_{n\geq 1}$ are total in E_ω for every $\omega \in \Omega$. Moreover, if we denote by $\langle\cdot,\cdot\rangle_\omega$ the scalar product on E_ω, we have that the map $\Omega \to \mathbb{R}$, $\omega \mapsto \langle v(i_\omega(x_\omega i^n)), v(i_\omega(\xi_\omega^m))\rangle_\omega = \cos(\angle(\xi_\omega^n, \xi_\omega^m))$, is Borel, since $\angle(\xi_\bullet, \eta_\bullet)$ is Borel for every $\xi_\bullet, \eta_\bullet \in \mathcal{L}(\Omega, F_\bullet) \subseteq \mathcal{L}(\Omega, \partial X_\bullet)$ (see Lemma 3.4). So, the family $\{\xi_\bullet^n\}_{n\geq 1} \subseteq \mathcal{S}(\Omega, E_\bullet)$ is a fundamental family in the sense of Dixmier [**15**, p. 145], and so it generates a Borel structure

$$\mathcal{L}(\Omega, E_\bullet) := \{e_\bullet \in \mathcal{S}(\Omega, E_\bullet) \mid \langle e_\bullet, v(i_\bullet(\xi_\bullet^n))\rangle_\bullet \text{ is a Borel function for all } n \geq 1\}.$$

This is a structure in a Hilbert sense, but it is easy to see that it is a particular case of a Borel structure on a field of metric spaces. We claim that $\pi_{E_\bullet}(i_\bullet(x_\bullet)) \in \mathcal{L}(\Omega, E_\bullet)$ for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. We know that $i_\omega(\xi_\omega^n) \in \partial E_\omega \subseteq \partial Y_\omega * \partial E_\omega = \partial Z_\omega$ for every $n \geq 1$ and $\omega \in \Omega$. Thus, by the classical enumeration of Busemann functions in a product and in a Hilbert space (keep in mind here that $i_\omega(x_\omega^0)$ is the origin of the Hilbert space E_ω), we have

$$\langle \pi_{E_\bullet}(i_\bullet(x_\bullet)), v(i_\bullet(\xi_\bullet^n))\rangle_\bullet = -b_{i_\bullet(x_\bullet^0), i_\bullet(\xi_\bullet^n)}(i_\bullet(x_\bullet)) = -b_{x_\bullet^0, \xi_\bullet^n}(x_\bullet).$$

The last function being Borel (see Remark 3.1), we have therefore proven the claim.

Now we can deal with the structure on $Y_\bullet$. Let $\mathcal{D} = \{x_\bullet^n\}_{n\geq 1}$ be a fundamental family of the field $X_\bullet$. Then $\{\pi_{Y_\omega}(i_\omega(x_\omega^n))\}_{n\geq 1}$ is dense in Y_ω for every $\omega \in \Omega$. Moreover,

$$d_{Y_\bullet}(\pi_{Y_\bullet}(i_\bullet(x_\bullet^n)), \pi_{Y_\bullet}(i_\bullet(x_\bullet^m))) = \sqrt{d_{X_\bullet}(x_\bullet^n, x_\bullet^m)^2 - d_{Y_\bullet}(\pi_{Y_\bullet}(i_\bullet(x_\bullet^n)), \pi_{Y_\bullet}(i_\bullet(x_\bullet^m)))^2}$$

is a Borel function for every $n, m \geq 1$. By Example 2.1(v), the family $\{\pi_{Y_\bullet}(i_\bullet(x_\bullet^n))\}_{n\geq 1}$ defines a Borel structure $\mathcal{L}(\Omega, Y_\bullet)$ on $(\Omega, Y_\bullet)$. As before, we can easily show that if $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, then $d_{Y_\bullet}(\pi_{Y_\bullet}(i_\bullet(x_\bullet^n)), \pi_{Y_\bullet}(i_\bullet(x_\bullet)))$ is Borel for every $n \geq 1$, i.e., $\pi_{Y_\bullet}(i_\bullet(x_\bullet)) \in \mathcal{L}(\Omega, Y_\bullet)$.

Therefore, there exist Borel structures $\mathcal{L}(\Omega, Y_\bullet)$ and $\mathcal{L}(\Omega, Z_\bullet)$ such that $i_\bullet(x) \in \mathcal{L}(\Omega, Y_\bullet) \times \mathcal{L}(\Omega, Z_\bullet)$ for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, i.e., $i_\bullet$ is a morphism. □

There are two important subsets of F that are used in the proof of Theorem 1.1: the subset $A := \{\xi \in F \mid -\xi \in F\}$, which is well defined since the geometry of F is spherical, and $P := \{\xi \in F \mid \angle(\xi, A) = \pi/2\}$. Observe that these subsets are closed and π-convex, that if we decompose X with respect to A then we have $X \simeq Y \times \mathbb{R}^n$ and that $P = \emptyset$ if and only if $A = F$.

LEMMA 3.7. *Let* $(\Omega, \mathcal{A})$ *be a Borel space and* $(\Omega, X_\bullet)$ *be a Borel field of proper CAT(0) spaces. Then the subfields* $A_\bullet$ *and* $P_\bullet$ *of* $F_\bullet$ *are Borel.*

Proof. Recall that $F_\omega \subseteq \partial E_\omega$ and observe that ∂E_ω can be interpreted (topologically) as the unit sphere of $E_\bullet$. From Proposition 3.2, we have that $F_\bullet$ is a Borel subfield of $\partial E_\bullet$.

Since E_ω is a Hilbert space, we can consider $-F_\omega \subseteq \partial E_\omega \subseteq E_\omega$, and it is easy to be convinced that $F_\omega \cap (-F_\omega) = A_\omega$. As $-F_\bullet$ is obviously a Borel subfield of compact subsets of $E_\bullet$, then $A_\bullet$ is a Borel subfield of $F_\bullet$ by Proposition 2.3. Thus, $\angle_\bullet(\cdot, A_\bullet) \in \mathcal{L}(\Omega, \mathcal{C}(F_\bullet))$. Therefore, by the second step of the proof of this same proposition, $P_\bullet = \angle_\bullet(\cdot, A_\bullet)^{-1}(\{\pi/2\})$ is also a Borel field of closed subsets of $F_\bullet$. □

4. *Actions of equivalence relations*

4.1. *Definition.* In the following, $[\mathcal{R}]$ denotes the full group of $\mathcal{R}$, i.e., the group of Borel automorphisms of Ω whose graphs are contained in $\mathcal{R}$.

Definition 4.1. Let $(\Omega, \mathcal{A})$ be a standard Borel space, $(\Omega, X_\bullet)$ be a Borel field of metric spaces and $\mathcal{R} \subseteq \Omega^2$ be a Borel equivalence relation. An action of $\mathcal{R}$ on $(\Omega, X_\bullet)$ is given by a family of bijective maps indexed by $\mathcal{R}$, denoted by $\{\alpha(\omega, \omega') : X_\omega \to X_{\omega'}\}_{(\omega,\omega')\in\mathcal{R}}$, with the following conditions.

(i) (Cocycle rule). For every $(\omega, \omega'), (\omega', \omega'') \in \mathcal{R}$, $\alpha(\omega', \omega'') \circ \alpha(\omega, \omega') = \alpha(\omega, \omega'')$.

Observe that condition (i) implies the existence of a natural action of $[\mathcal{R}]$ on $\mathcal{S}(\Omega, X_\bullet)$: if $g \in [\mathcal{R}]$ and $x_\bullet \in \mathcal{S}(\Omega, X_\bullet)$, we can define a new section $gx_\bullet$ by $(gx_\bullet)_\omega = \alpha(g^{-1}\omega, \omega)x_{g^{-1}\omega}$. As we are in the Borel context, we will also require the following.

(ii) The set $\mathcal{L}(\Omega, X_\bullet) \subseteq \mathcal{S}(\Omega, X_\bullet)$ is invariant under the action of $[\mathcal{R}]$.

We denote such an action by $\alpha : \mathcal{R} \curvearrowright (\Omega, X_\bullet)$.

If, moreover, for all $(\omega, \omega') \in \mathcal{R}$, the map $\alpha(\omega, \omega')$ is continuous (respectively isometric or linear), we say that $\mathcal{R}$ acts by homeomorphisms (respectively by isometries or linearly).

4.2. *Basic properties.* If $\mathcal{R}$ acts by homeomorphisms, it is straightforward to see, using Lemma 2.2, that condition (ii) is equivalent to $g(\mathcal{D}) \subseteq \mathcal{L}(\Omega, X_\bullet)$ for all $g \in [\mathcal{R}]$, for any fundamental family $\mathcal{D} \subseteq \mathcal{L}(\Omega, X_\bullet)$. By using the classical techniques of decompositions and gluings, it is also possible to prove that it is enough to check Condition 4.1(ii) for every element of a countable group $G \subseteq [\mathcal{R}]$ such that $\mathcal{R} = \mathcal{R}_G$. Observe also that $[\mathcal{R}]$ acts on the set of subfields of $(\Omega, X_\bullet)$ in total analogy with its action on sections.

PROPOSITION 4.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space, $(\Omega, X_\bullet)$ be a Borel field of metric spaces and $\mathcal{R} \subseteq \Omega^2$ be a Borel equivalence relation which acts on $(\Omega, X_\bullet)$ by homeomorphisms. Then $[\mathcal{R}]$ acts on the set of Borel subfields of $(\Omega, X_\bullet)$. Moreover, if μ is quasi-invariant under $\mathcal{R}$, then $[\mathcal{R}]$ acts on $L(\Omega, X_\bullet)$ and, more generally, it acts on the equivalence classes of Borel subfields of $(\Omega, X_\bullet)$.*

Proof. Let $A_\bullet$ be a Borel field, $\Omega' := \{\omega \in \Omega \mid A_\omega \neq \emptyset\}$ its base and $g \in [\mathcal{R}]$. To show that the field $gA_\bullet$ defined by $(gA_\bullet)_\omega := \alpha(g^{-1}\omega, \omega)A_{g^{-1}\omega}$ is Borel, we observe that $\{\omega \in \Omega \mid (gA_\bullet)_\omega \neq \emptyset\} = g\Omega' \in \mathcal{A}$, and that if $\mathcal{D} \subseteq \mathcal{L}(\Omega', A_\bullet)$ is a fundamental family of $A_\bullet$, then $g(\mathcal{D}) \subseteq \mathcal{L}(g\Omega', gA_\bullet)$ satisfies Condition 2.3(ii). Under the hypothesis of quasi-invariance of μ, if $A_\bullet =_{\mu\text{-a.e.}} B_\bullet$, then $gA_\bullet =_{\mu\text{-a.e.}} gB_\bullet$, since the set $\{\omega \in \Omega \mid (gA_\bullet)_\omega = (gB_\bullet)_\omega\} = g\{\omega \in \Omega \mid A_\omega = B_\omega\}$ is of null measure. □

Definition 4.2. Under the hypothesis of Proposition 4.1, a section $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ is called $\mathcal{R}$-invariant (or simply invariant when the relation can be clearly identified) if $\alpha(\omega, \omega')x_\omega = x_{\omega'}$ for all $(\omega, \omega') \in \mathcal{R}$. We say that a section $x_\bullet$ is almost $\mathcal{R}$-invariant if there exists a Borel set $A \in \mathcal{A}$ with $\mu(A) = 1$ such that the equality holds for all $(\omega, \omega') \in \mathcal{R} \cap A^2$. *Mutatis mutandis*, we define $\mathcal{R}$-invariant and almost $\mathcal{R}$-invariant Borel subfields. We will also use the terminology (almost) α-invariant whenever it is necessary to be more precise.

Remark 4.1. It is straightforward to check that a Borel section $x_\bullet$ (or a Borel subfield) is (almost) $\mathcal{R}$-invariant if and only if it is (almost) $[\mathcal{R}]$-invariant. Therefore, the almost invariance of a Borel section $x_\bullet$ (or of a Borel subfield) is equivalent to the invariance of its class $[x_\bullet] \in L(\Omega, X_\bullet)$. The existence of a countable group generating $\mathcal{R}$ [**21**] allows us to always assume that the set $A \in \mathcal{A}$ in Definition 4.2 is invariant. On the other hand, if $(\Omega, Y_\bullet)$ is an invariant Borel subfield, then the action on $(\Omega, X_\bullet)$ induces an action on $(\Omega, Y_\bullet)$. Putting this all together shows that, in the measure context, we can always assume, without losing generality, that an almost-invariant Borel subfield is invariant.

We now show that an action of $\mathcal{R}$ on a Borel field of proper metric spaces $(\Omega, X_\bullet)$ naturally gives rise to an action on the previously constructed Borel field $(\Omega, \mathcal{C}(X_\bullet))$.

LEMMA 4.1. *Let* $(\Omega, \mathcal{A})$ *be a standard Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper metric spaces and* $\mathcal{R} \subseteq \Omega^2$ *be a Borel equivalence relation. Suppose that an action* $\alpha : \mathcal{R} \curvearrowright (\Omega, X_\bullet)$ *by homeomorphisms is given. Then there exists an induced action* $\widetilde{\alpha} : \mathcal{R} \curvearrowright (\Omega, \mathcal{C}(X_\bullet))$ *by linear homeomorphisms.*

Proof. The natural way to define the action is to write, for $(\omega, \omega') \in \mathcal{R}$,

$$\begin{aligned} \widetilde{\alpha}(\omega, \omega') : \quad \mathcal{C}(X_\omega) &\to \mathcal{C}(X_{\omega'}) \\ f_\omega &\mapsto f_\omega \circ \alpha(\omega', \omega). \end{aligned}$$

It is clear that $\widetilde{\alpha}(\omega, \omega')$ is a homeomorphism (with respect to the topology of uniform convergence on compact sets) because a homeomorphism between Hausdorff spaces preserves compact sets. The cocycle rule is obviously satisfied, so it only remains to check condition (ii) of Definition 4.1, i.e., that if $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ and $g \in [\mathcal{R}]$, then $gf_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$. To do this, we fix $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, and we observe that

$$(gf_\bullet)_\omega(x_\omega) = f_{g^{-1}\omega}(\alpha(\omega, g^{-1}\omega)(x_\omega)) = f_{g^{-1}\omega}(g^{-1}(x_\bullet)_{g^{-1}\omega}).$$

Consequently, we have $gf_\bullet(x_\bullet) = (f_\bullet(g^{-1}x_\bullet)) \circ g^{-1}$, and this shows that the evaluation $(gf_\bullet)(x_\bullet)$ is Borel because $g^{-1}x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$ (since α is an action) and $f_\bullet(g^{-1}x_\bullet)$ is Borel by definition of the Borel structure of the Borel field $(\Omega, \mathcal{C}(X_\bullet))$. So, we can conclude that $gf_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$. □

Moreover, if the field is a field of CAT(0) spaces, then the action extends to $(\Omega, \partial X_\bullet)$.

LEMMA 4.2. *Let* $(\Omega, \mathcal{A})$ *be a Borel space,* $(\Omega, X_\bullet)$ *be a Borel field of proper unbounded CAT(0) spaces,* $\mathcal{R} \subseteq \Omega^2$ *be a Borel equivalence relation and* $\alpha : \mathcal{R} \curvearrowright (\Omega, X_\bullet)$ *an action by isometries. Then there exists an induced action* $\widetilde{\alpha} : \mathcal{R} \curvearrowright (\Omega, \overline{X}_\bullet)$ *by homeomorphisms and* $(\Omega, \partial X_\bullet)$ *is invariant.*

Proof. Let X_1 and X_2 be two proper unbounded CAT(0) spaces and $\gamma : X_1 \to X_2$ an isometry. If we think of the boundary as the quotient of the geodesic rays, then the extension $\widetilde{\gamma} : \overline{X}_1 \to \overline{X}_2$ is purely geometric and is a homeomorphism such that $\widetilde{\gamma}(\partial X_1) = \partial X_2$ [**8**, Corollary II.8.9]. As we used the notion of Busemann functions to define the Borel structure of the field of boundaries, we need to transpose the situation to this context. If $x_i \in X_i$ are base points of X_i for $i = 1, 2$, then the map

$$\begin{array}{rccl} \widetilde{\gamma}_0 : & \mathcal{C}_0(X_1) & \to & \mathcal{C}_0(X_2) \\ & f & \mapsto & \widetilde{\gamma}_0(f) : y \mapsto f(\gamma^{-1}y) - f(\gamma^{-1}x_2) \end{array} \tag{4}$$

is a homeomorphism and it is such that the diagram

$$\begin{array}{ccc} \mathcal{C}_0(X_1) & \xrightarrow{\widetilde{\gamma}_0} & \mathcal{C}_0(X_2) \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ X_1 & \xrightarrow{\gamma} & X_2 \end{array}$$

commutes. It is easy to check that the two extensions coincide and that if $\xi \in \partial X_1$, then $\widetilde{\gamma}_0(b_{x_1,\xi}) = b_{x_2,\widetilde{\gamma}(\xi)}$.

Now let us turn to the case of fields. Given a fixed section $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, we use (i) to define, for each $(\omega, \omega') \in \mathcal{R}$,

$$\begin{array}{rccl} \widetilde{\alpha}(\omega, \omega') : & \mathcal{C}_0(X_\omega) & \to & \mathcal{C}_0(X_{\omega'}) \\ & f & \mapsto & \alpha(\omega, \omega')(f) : x \mapsto f(\alpha(\omega', \omega)x) - f(\alpha(\omega', \omega)x^0_{\omega'}). \end{array}$$

This formula defines an action by homeomorphisms. We proceed as in the proof of Lemma 4.1: the verification of the cocycle rule is straightforward, and an easy computation shows that for every $x_\bullet \in \mathcal{L}(\Omega, X_\bullet)$, $g \in [\mathcal{R}]$ and $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}_0(X_\bullet))$, we have

$$(gf_\bullet)(x_\bullet) = (f_\bullet(g^{-1}x_\bullet) - f_\bullet(g^{-1}x^0_\bullet)) \circ g^{-1}.$$

Thus, $gf_\bullet$ is Borel. □

4.3. *Amenability.* In this section, we define the amenability for an equivalence relation in terms of actions on Borel fields of Banach spaces, and we also show that our definition is equivalent to the one given originally by Zimmer [**37**].

Definition 4.3. Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, B_\bullet)$ be a field of Banach spaces on Ω. Such a field is called Borel if there exists a Borel structure on $(\Omega, B_\bullet)$, which is defined in the same way as in Definition 2.1 with the additional assumption that $\mathcal{L}(\Omega, B_\bullet) \subseteq \mathcal{S}(\Omega, B_\bullet)$ is a vector space. This definition can be easily seen to be equivalent to the ones given in [**20**, I, p. 77] or [**4**, p. 177], essentially by using Lemma 2.1.

We can consider for each $\omega \in \Omega$ the topological dual B^*_ω, which is not separable in general. Thus, the field $(\Omega, B^*_\bullet)$ is unable to satisfy Definition 4.3, but we still have the following result.

LEMMA 4.3. *Let $(\Omega, \mathcal{A})$ be a Borel space and $(\Omega, B_\bullet)$ be a Borel field of Banach spaces. Define*

$$\widetilde{\mathcal{L}}(\Omega, B^*_\bullet) := \{\varphi_\bullet \in \mathcal{S}(\Omega, B^*_\bullet) \mid \omega \mapsto \langle \varphi_\omega, x_\omega \rangle := \varphi_\omega(x_\omega) \textit{ is Borel for every } x_\bullet \in \mathcal{L}(\Omega, B_\bullet)\}.$$

Then $\widetilde{\mathcal{L}}(\Omega, B^*_\bullet)$ is a vector space which satisfies the following properties.

(i) *For every* $\varphi_\bullet \in \widetilde{\mathcal{L}}(\Omega, B_\bullet^*)$, *the function* $\omega \mapsto \|\varphi_\omega\|_\omega$ *is Borel.*
(ii) *The space* $\widetilde{\mathcal{L}}(\Omega, B_\bullet^*)$ *is closed under pointwise limits and Borel gluings.*
Moreover, if $\mathcal{R} \subseteq \Omega^2$ *is an equivalence relation and* $\alpha : \mathcal{R} \curvearrowright (\Omega, B_\bullet)$ *is a linear isometric action, then the fiberwise adjoint maps given by* $\alpha_*(\omega, \omega') := (\alpha(\omega', \omega))^* : B_\omega^* \to B_{\omega'}^*$ *satisfy:*
(iii) *the cocycle rule* $\alpha_*(\omega', \omega'') \circ \alpha_*(\omega, \omega') = \alpha_*(\omega, \omega'')$, *for every* (ω, ω'), $(\omega', \omega'') \in \mathcal{R}$; *and*
(iv) $g(\widetilde{\mathcal{L}}(\Omega, B_\bullet^*)) \subseteq \widetilde{\mathcal{L}}(\Omega, B_\bullet^*)$ *for every* $g \in [\mathcal{R}]$.
Thus, α_* *is an action in the sense of Definition 4.1 if the field is a Borel field of metric spaces.*

Proof. Properties (i) and (ii) are proved in [**4**, Definition A.3.6 and Lemma A.3.7, p. 179], property (iii) is obvious and property (iv) can be proved exactly as in Lemma 4.1. □

Definition 4.4. Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and $(\Omega, B_\bullet)$ be a Borel field of Banach spaces.
(i) Define

$$\mathcal{L}^1(\Omega, B_\bullet) := \left\{ x_\bullet \in \mathcal{L}(\Omega, B_\bullet) \;\middle|\; \|x_\bullet\|_1 := \int_\Omega \|x_\omega\|_\omega \, d\mu(\omega) < \infty \right\},$$

and $L^1(\Omega, B_\bullet) := \mathcal{L}^1(\Omega, B_\bullet)/=_{\mu\text{-a.e.}}$. Then $L^1(\Omega, B_\bullet)$ endowed with the norm $\|\cdot\|_1$ is a separable Banach space.
(ii) Define

$$\widetilde{\mathcal{L}}^\infty(\Omega, B_\bullet^*) := \{\varphi_\bullet \in \widetilde{\mathcal{L}}(\Omega, B_\bullet^*) \mid \|\varphi_\bullet\|_\bullet : \omega \mapsto \|\varphi_\omega\|_\omega \in \mathcal{L}^\infty(\Omega, \mathbb{R})\},$$

and $\widetilde{L}^\infty(\Omega, B_\bullet^*) := \widetilde{\mathcal{L}}^\infty(\Omega, B_\bullet^*)/=_{\mu\text{-a.e.}}$. Then $\widetilde{L}^\infty(\Omega, B_\bullet^*)$ is a Banach space when it is endowed with the norm $\|\cdot\|_\infty$, where $\|\varphi_\bullet\|_\infty$ is the ∞-norm of the function $\|\varphi_\bullet\|_\bullet$.
For a detailed proof of these two assertions, which are close to that of the trivial field case, we refer to [**5**] or [**27**].

In this context, the following result holds.

PROPOSITION 4.2. [**4**, Proposition A.3.9, pp. 179–180] *Let* $(\Omega, \mathcal{A}, \mu)$ *be a standard probability space and* $(\Omega, B_\bullet)$ *be a Borel field of Banach spaces. Then there exists an isometric isomorphism between* $\widetilde{L}^\infty(\Omega, B_\bullet^*)$ *and* $(L^1(\Omega, B_\bullet))^*$ *given by*

$$\begin{array}{ccccccc} \widetilde{L}^\infty(\Omega, B_\bullet^*) & \to & (L^1(\Omega, B_\bullet))^* & & & & \\ [\varphi_\bullet] & \mapsto & \langle [\varphi_\bullet], \cdot \rangle & : & L^1(\Omega, B_\bullet) & \to & \mathbb{R} \\ & & & & [x_\bullet] & \mapsto & \langle [\varphi_\bullet], [x_\bullet] \rangle = \int_\Omega \langle \varphi_\omega, x_\omega \rangle \, d\mu(\omega). \end{array}$$

When B is a Banach space, we use $B_{\leq 1}$ (respectively $B_{=1}$) to denote the closed ball (respectively sphere) of radius 1. If A is a subset of B^*, then $\overline{A}^{w*}$ is the closure of A with respect to the weak-$*$ topology.

Definition 4.5. [**4**, Definition 4.2.1, p. 97] Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and $(\Omega, B_\bullet)$ be a Borel field of Banach spaces. We say that $(\Omega, C_\bullet)$ is a Borel subfield of convex, weak-$*$ compact subsets of the field of closed balls of radius 1 in the duals $(\Omega, B^*_{\bullet,\leq 1})$ if there exists a family of sections $\{\varphi^n_\bullet\}_{n\geq 1} \subseteq (\widetilde{L}^\infty(\Omega, B^*_\bullet))_{\leq 1}$ such that

$$C_\omega = \overline{\mathrm{co}(\{\varphi^n_\omega\}_{n\geq 1})}^{w*} \quad \text{for } \mu\text{-almost every } \omega \in \Omega,$$

where the closure of the convex hull is taken relatively to the weak-$*$ topology. Note that the set $\widetilde{L}(\Omega, C_\bullet) := \{[\varphi_\bullet] \in \widetilde{L}^\infty(\Omega, B^*_\bullet) \mid \varphi_\omega \in C_\omega$ for μ-almost every $\omega \in \Omega\}$ is a convex weak-$*$ closed subset of the unit ball $(\widetilde{L}^\infty(\Omega, B^*_\bullet))_{\leq 1}$ [**4**, Proposition 4.2.2, p. 97].

Remark 4.2. It can be shown [**5**] that there exists $\Omega' \in \mathcal{A}$ of full measure and a family of metrics $\{d^*_\omega\}_{\omega\in\Omega'}$ such that $\widetilde{\mathcal{L}}^\infty(\Omega', B^*_\bullet)$ is a Borel structure on the field $(\Omega', (B^*_{\bullet,\leq 1}, d^*_\bullet))$. A Borel subfield of convex, weak-$*$ compact subsets is then a Borel subfield in the sense of Definition 2.3.

Historically, Zimmer was the first to introduce the notion of amenability for an equivalence relation. His definition can be formulated as a particular case of the following definition; it corresponds to the case of a trivial field. To show that Zimmer's definition corresponds to the case of a trivial field of Banach spaces, the only thing to check is the equivalence between an action of an equivalence relation on a trivial field (Ω, B) and a Borel cocycle from $\mathcal{R}$ to $\mathrm{Isom}(B)$. This can easily be done using [**38**, Corollary 1.2].

Definition 4.6. Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space and $\mathcal{R} \subseteq \Omega^2$ an equivalence relation which preserves the class of μ.

We say that $\mathcal{R}$ is amenable if for every action α of $\mathcal{R}$ by linear isometries on a Borel field of Banach spaces $(\Omega, B_\bullet)$, and almost α_*-invariant Borel subfield of convex, weak-$*$ compact subsets $(\Omega, C_\bullet)$ of $(\Omega, B^*_{\bullet,\leq 1})$, there exists a Borel section $\varphi_\bullet \in \widetilde{\mathcal{L}}(\Omega, B^*_\bullet)$ which is almost $\mathcal{R}$-invariant and such that $\varphi_\omega \in C_\omega$ for almost every $\omega \in \Omega$.

We will now prove that hyperfiniteness μ-almost everywhere implies amenability in the sense of Definition 4.6. Since Zimmer's amenability is equivalent to almost hyperfiniteness (see [**11, 2**]), this would show the equivalence between all definitions. The reader familiar with the notion of measurable groupoids could also consult [**4**, Theorem 4.2.7].

Since we are in the measure theoretic context, we can suppose by hypothesis that there exists a Borel action $\mathbb{Z} \curvearrowright \Omega$ such that $\mathcal{R}_\mathbb{Z} = \mathcal{R}$. By analogy with [**38**, Theorem 2.1], we can define an isometric representation of $\mathbb{Z}$ on $L^1(\Omega, B_\bullet)$ by

$$(T(g)x_\bullet)_\omega = \frac{dg_*(\mu)}{d\mu}(\omega) \cdot \alpha(g^{-1}\omega, \omega)x_{g^{-1}\omega} \quad \text{for every } g \in \mathbb{Z},\ x_\bullet \in L^1(\Omega, B_\bullet),$$

where $dg_*(\mu)/d\mu \in L^1(\Omega, \mathbb{R})$ is the Radon–Nikodym derivative. The adjoint representation $T_* := (T^{-1})^*$ acts on $(L^1(\Omega, B_\bullet))^* \simeq \widetilde{L}^\infty(\Omega, B^*_\bullet)$. Given $\varphi_\bullet \in \widetilde{L}^\infty(\Omega, B^*_\bullet)$, it is straightforward to see that

$$(T_*(g)\varphi_\bullet)_\omega = \alpha_*(g^{-1}\omega, \omega)\varphi_{g^{-1}\omega}.$$

This means that the adjoint representation is given by the fiberwise adjoint action. Consequently, $\mathbb{Z}$ acts by homeomorphisms on $\widetilde{L}^\infty(\Omega, B^*_\bullet)$ with respect to the weak-$*$ topology, and thus on $\widetilde{L}(\Omega, C_\bullet)$, because $(\Omega, C_\bullet)$ is supposed to be almost invariant.

Since $\mathbb{Z}$ is amenable, there exists a fixed point $[\varphi_\bullet]$, and any representant of the class has the desired property.

In the following, we will use the amenability in the following particular context.

Example 4.1. Let K be a compact metric space and consider the space $\mathcal{C}(K)^*_{\leq 1}$ endowed with the weak-$*$ topology. The map

$$\begin{array}{rccl} \delta: & K & \rightarrow & \mathcal{C}(K)^*_{\leq 1} \\ & x & \mapsto & [\delta_x : f \mapsto f(x)] \end{array}$$

is a homeomorphism onto its image such that $\delta_x \in \mathcal{C}(K)^*_{=1}$. Since $\mathcal{C}(K)^*_{\leq 1}$ is convex and weak-$*$ compact by the Banach–Alaoglu theorem, we have the equality $\mathcal{C}(K)^*_{\leq 1} = \overline{\mathrm{co}(\{\pm\delta_x\}_{x\in K})}^{w*}$ by the Krein–Milman theorem because the extremal points are $\{\pm\delta_x\}_{x\in K}$. Moreover, the bijection

$$\mathrm{Prob}(K) \simeq \{\varphi \in \mathcal{C}(K)^*_{\leq 1} \mid \varphi(\mathbf{1}) = 1 \text{ and for all } f \in \mathcal{C}(K)\ (f \geq 0 \Rightarrow \varphi(f) \geq 0)\},$$

given by the Riesz's representation theorem, allows us to consider the weak-$*$ compact set of probabilities in the dual, and it is well known that $\mathrm{Prob}(K) = \overline{\mathrm{co}(\{+\delta_x\}_{x\in K})}^{w*}$.

Now, if $D \subseteq K$ is a dense subset, then, using the continuity of δ, we obtain $\mathcal{C}(K)^*_{\leq 1} = \overline{\mathrm{co}(\{\pm\delta_x\}_{x\in D})}^{w*}$ and $\mathrm{Prob}(K) = \overline{\mathrm{co}(\{+\delta_x\}_{x\in D})}^{w*}$.

If $(\Omega, \mathcal{A}, \mu)$ is a standard probability space and $(\Omega, K_\bullet)$ is a Borel field of compact metric spaces, then the preceding results and the introduction of the Borel sections $\{\delta_{x_\bullet}\}_{x_\bullet \in \mathcal{D}} \subseteq \mathcal{L}^\infty(\Omega, \mathcal{C}(K_\bullet)^*)_{\leq 1}$ for a given fundamental family $\mathcal{D} \subseteq \mathcal{L}(\Omega, K_\bullet)$ show that the fields $(\Omega, \mathcal{C}(K_\bullet)^*_{\leq 1})$ and $(\Omega, \mathrm{Prob}(K_\bullet))$ are Borel fields of convex, weak-$*$ compact sets in the sense of Definition 4.5. A section $\pi_\bullet$ is Borel if the corresponding section $\varphi_{\pi_\bullet} \in \mathcal{L}(\Omega, \mathcal{C}(K_\bullet)^*)$, i.e., $\varphi_{\pi_\bullet}(f_\bullet)$ is Borel for every $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(K_\bullet))$.

Suppose now that an action by homeomorphisms $\alpha : \mathcal{R} \curvearrowright (\Omega, K_\bullet)$ is given. Lemmas 4.1 and 4.3 allow us to define $\alpha^*(\omega', \omega) := (\widetilde{\alpha}(\omega', \omega)^{-1})^*$.

The field $(\Omega, \mathrm{Prob}(K_\bullet))$ is obviously α^*-invariant since if $\mu_{\omega'} \in \mathrm{Prob}(K_{\omega'})$, then $\alpha^*(\omega', \omega)\mu_{\omega'}$ is the image measure $\alpha(\omega', \omega)_*(\mu_{\omega'})$. In particular, if $\mathcal{R}$ is amenable, then, by definition, there exists a Borel section $[\mu_\bullet] \in \widetilde{L}(\Omega, \mathrm{Prob}(K_\bullet))$ which is α^*-invariant.

5. *Proof of the main theorem*

5.1. *Fields of convex sets and invariant sections at infinity.* Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space. Given a Borel field $(\Omega, X_\bullet)$ of CAT(0) spaces, we introduce the following notations:

$$\mathcal{S} := \{[C_\bullet] \mid [C_\bullet] \text{ is an invariant class of Borel subfields of non-empty closed convex subsets}\},$$

$$\mathcal{M} := \{[C_\bullet] \in \mathcal{S} \mid (\Omega, C_\bullet) \text{ is minimal for } \leq\}.$$

LEMMA 5.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space, $\mathcal{R}$ an equivalence relation which quasi-preserves μ, $(\Omega, X_\bullet)$ a Borel field of proper metric spaces and an isometric action $\alpha : \mathcal{R} \curvearrowright (\Omega, X_\bullet)$.*

If $\{[C^\beta_\bullet]\}_{\beta\in\mathcal{B}}$ is a totally ordered family (i.e., a chain) of $\mathcal{S}$, then there exists a countable family of indices $\{\beta_n\}_{n\geq 1} \subseteq \mathcal{B}$ such that $[C^{\beta_{n+1}}_\bullet] \leq [C^{\beta_n}_\bullet]$ for each $n \geq 1$, and such that $C_\bullet := \bigcap_{n\geq 1} C^{\beta_n}_\bullet$ satisfies $[C_\bullet] \in \mathcal{S}$ and $[C_\bullet] \leq [C^\beta_\bullet]$ for all $\beta \in \mathcal{B}$.

Proof. By Theorem 2.1, there exists a countable family of indices $\{b_n\}_{n\geq 1} \subseteq \mathcal{B}$ such that the Borel subfield

$$C_\bullet := \bigcap_{n\geq 1} C_\bullet^{b_n}$$

is such that $[C_\bullet] \leq [C_\bullet^\beta]$, $\beta \in \mathcal{B}$. By setting $\beta_n := \min\{b_1, \ldots, b_n\}$ (where the minimum is taken for the induced order by the chain), it follows that $C_\bullet = \bigcap_{n\geq 1} C_\bullet^{\beta_n}$ satisfies our conditions (the invariance of the class follows from Remark 4.1). □

THEOREM 5.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space, $\mathcal{R}$ an ergodic equivalence relation which quasi-preserves μ and $(\Omega, X_\bullet)$ a Borel field of proper unbounded CAT(0) spaces with finite covering dimension.*

If $\alpha : \mathcal{R} \curvearrowright (\Omega, X_\bullet)$ is an isometric action such that $\mathcal{M} = \emptyset$, then there exists an almost-invariant section $\xi_\bullet \in \mathcal{L}(\Omega, \partial X_\bullet)$.

Proof. The proof proceeds in two steps.

(i) First, we show that under the hypothesis, there exists a sequence $\{[C_\bullet^{\beta_n}]\}_{n\geq 1} \subseteq [\mathcal{S}]$ such that the following conditions hold.

(a) $[C_\bullet^{\beta_{n+1}}] \leq [C_\bullet^{\beta_n}]$ for each $n \geq 1$.

(b) $[\bigcap_{n\geq 1} C_\bullet^{\beta_n}]$ is the class of the empty subfield, i.e., the field $A_\bullet$ defined by $A_\omega = \emptyset$, for every $\omega \in \Omega$.

By the transposition of Zorn's lemma, there exists a chain $\{[C_\bullet^\beta]\}_{\beta\in\mathcal{B}} \in \mathcal{S}$ without a lower bound in $\mathcal{S}$. So, by Lemma 5.1 applied to this chain, we conclude that the Borel subfield $[C_\bullet] := [\bigcap_{n\geq 1} C_\bullet^{\beta_n}]$ is not in $\mathcal{S}$. This means that $\mu(\{\omega \in \Omega \mid C_\omega \neq \emptyset\}) \neq 1$. We then show that this set is of null measure. By ergodicity, the invariant set $\{\omega \in \Omega \mid C_\omega \neq \emptyset\}$ is of measure one or null (see Remark 4.1). Since the first possibility is impossible, we conclude that $[C_\bullet]$ is the class of the empty field.

(ii) Consider the subchain $\{C_\bullet^{\beta_n}\}_{n\geq 1}$ and the subfield $C_\bullet$ given in (i). There exists $\Omega' \in \mathcal{A}$ of measure one such that the inclusion $C_\omega^{\beta_n} \supseteq C_\omega^{\beta_{n+1}}$ holds and that $C_\omega = \emptyset$ for every $\omega \in \Omega'$. So, by Proposition 3.1, we can consider the Borel field $(\Omega', L_\bullet) \leq (\Omega', \partial X_\bullet)$ of limit sets at infinity of the subchain, and this field has a unique Borel section of circumcenters

$$\xi_\bullet := c_{L_\bullet} \in \mathcal{L}(\Omega', \partial X_\bullet).$$

To prove the invariance of $[\xi_\bullet]$, it is enough to show that $(\Omega', L_\bullet)$ is almost invariant. Recall that for every $\omega \in \Omega'$ and $x \in X_\omega$,

$$L_\omega := \overline{i_\omega(\{\pi_{C_\omega^{\beta_n}}(x)\}_{n\geq 1})} \cap \partial X_\omega.$$

So, we have

$$\begin{aligned}
\widetilde{\alpha}(\omega, \omega')L_\omega &\overset{\widetilde{\alpha}\text{ homeo.}}{=} \overline{\widetilde{\alpha}(\omega, \omega')i_\omega(\{\pi_{C_\omega^{\beta_n}}(x)\}_{n\geq 1})} \cap \partial X_{\omega'} \\
&\overset{\widetilde{\alpha}\text{ ext. of }\alpha}{=} \overline{i_{\omega'}(\alpha(\omega, \omega')(\{\pi_{C_\omega^{\beta_n}}(x)\}_{n\geq 1}))} \cap \partial X_{\omega'} \\
&\overset{\alpha\text{ isom.}}{=} \overline{i_{\omega'}(\{\pi_{\alpha(\omega,\omega')(C_\omega^{\beta_n})}(\alpha(\omega, \omega')x)\}_{n\geq 1})} \cap \partial X_{\omega'} \\
&\overset{\text{inv.}}{=} \overline{i_{\omega'}(\{\pi_{C_{\omega'}^{\beta_n}}(\alpha(\omega, \omega')x)\}_{n\geq 1})} \cap \partial X_{\omega'} = L_{\omega'}.
\end{aligned}$$

□

5.2. *$\mathcal{R}$-quasi-invariant sections*

Definition 5.1. Let $(\Omega, \mathcal{A})$ be a standard Borel space, $\mathcal{R} \subseteq \Omega^2$ be a Borel equivalence relation and $(\Omega, X_\bullet)$ be a Borel field of metric spaces. Assume that the relation $\mathcal{R}$ acts on $(\Omega, X_\bullet)$ by isometries. We say that a section $f_\bullet \in \mathcal{S}(\Omega, \mathcal{F}(X_\bullet))$ is $\mathcal{R}$-quasi-invariant (or simply invariant when the relation can clearly be identified) if there exists $c : \mathcal{R} \to \mathbb{R}$ such that

$$\widetilde{\alpha}(\omega, \omega') f_\omega := f_\omega \circ \alpha(\omega', \omega) = f_{\omega'} + c(\omega, \omega') \quad \text{for every } (\omega, \omega') \in \mathcal{R}. \tag{5}$$

It can easily be checked that c is a cocycle, i.e., c satisfies $c(\omega, \omega'') = c(\omega, \omega') + c(\omega', \omega'')$.

LEMMA 5.2. *Let $(\Omega, \mathcal{A})$ be a standard Borel space, $\mathcal{R} \subseteq \Omega^2$ be a Borel equivalence relation and $(\Omega, X_\bullet)$ be a Borel field of proper CAT(0) spaces. Assume that $\mathcal{R}$ acts on $(\Omega, X_\bullet)$ by isometries. If $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ is a quasi-invariant section such that f_ω is convex for every $\omega \in \Omega$, then the following assertions are verified.*

(i) *The sets*

$$\begin{aligned} \Omega_{\inf=-\infty} &:= \{\omega \in \Omega \mid \inf f_\omega = -\infty\}, \\ \Omega_{\inf>-\infty} &:= \{\omega \in \Omega \mid \inf f_\omega > -\infty \text{ and is not attained}\}, \\ \Omega_{\min} &:= \{\omega \in \Omega \mid \inf f_\omega \text{ is attained}\} \end{aligned}$$

are Borel and invariant.

(ii) *The subfield $(f_\bullet|_{\Omega_{\min}} - \min(f_\bullet|_{\Omega_{\min}}))^{-1}(\{0\})$ is a Borel subfield of non-empty closed and convex sets of $(\Omega_{\min}, X_\bullet)$, which is $\mathcal{R}|_{\Omega_{\min}}$-invariant.*

(iii) *If we assume that X_ω is of finite covering dimension for every $\omega \in \Omega_{\inf}$, and we set $\Omega_{\inf} := \Omega_{\inf=-\infty} \cup \Omega_{\inf>-\infty}$, then there exists a section $\xi_\bullet \in \mathcal{L}(\Omega_{\inf}, \partial X_\bullet)$ which is $\mathcal{R}|_{\Omega_{\inf}}$-invariant.*

Proof. (i) The convexity assumption is not necessary to prove this assertion. The section $\inf f_\bullet$ is Borel since, by continuity, $\inf f_\bullet = \inf_{x_\bullet \in \mathcal{D}} f_\bullet(x_\bullet)$, where $\mathcal{D}$ is a fundamental family and thus $\Omega_{\inf=-\infty} = (\inf f_\bullet)^{-1}(\{-\infty\})$. Now we fix $x^0_\bullet \in \mathcal{L}(\Omega, X_\bullet)$. Because $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$, we have

$$\Omega_{\min} = \bigcup_{R \in \mathbb{N}} (\inf f_\bullet|_{B(x^0_\bullet, R)} - \inf f_\bullet)^{-1}(\{0\}) \in \mathcal{A},$$

and thus $\Omega_{\inf>-\infty} = \Omega \setminus (\Omega_{\inf=-\infty} \cup \Omega_{\min}) \in \mathcal{A}$. The invariance of these sets follows directly from equality (5).

(ii) Since $f_\bullet$ is quasi-invariant, we have $\min f_\omega = \min f_{\omega'} + c(\omega, \omega')$, and thus

$$\widetilde{\alpha}(\omega, \omega')(f_\omega - \min f_\omega) = f_{\omega'} + c(\omega, \omega') - \min f_\omega = f_{\omega'} - \min f_{\omega'}.$$

Consequently, the section $\widetilde{f}_\bullet \in \widetilde{\mathcal{L}}(\Omega_{\min}, \mathcal{C}(X_\bullet))$, defined by $\widetilde{f}_\omega := f_\omega - \min f_\omega$, for every $\omega \in \Omega_{\min}$, is $\mathcal{R}|_{\Omega_{\min}}$-invariant and such that $(\widetilde{f}_\bullet)^{-1}(\{0\})$ has the required properties.

(iii) If $\omega \in \Omega_{\inf>-\infty}$, we define $\widetilde{f}_\omega := f_\omega - \inf f_\omega$ and $C^n_\omega := (\widetilde{f}_\omega)^{-1}([0, 1/n])$. The sequence $\{C^n_\bullet\}_{n \geq 1}$ satisfies the hypothesis of Proposition 3.1, and thus the section of circumcenters of the limit sets at infinity is $\mathcal{R}|_{\Omega_{\inf>-\infty}}$-invariant. If $\omega \in \Omega_{\inf=-\infty}$, we consider the generalized sequence $\{C^{\beta}_{\bullet}\}_{\beta\in\mathbb{R}}$, defined by $C^{\beta}_{\omega} := f_{\omega}^{-1}(]-\infty, -\beta])$, and construct its field of limit sets at infinity, $(\Omega_{\inf=-\infty}, L_{\bullet})$. It is invariant since

$$\begin{aligned}\alpha(\omega,\omega')C^{\beta}_{\omega} &= \{\alpha(\omega,\omega')x \in X_{\omega'} \mid f_{\omega}(x) \le -\beta\}\\ &= \{y \in X_{\omega'} \mid f_{\omega}(\alpha(\omega',\omega)y) \le -\beta\}\\ &= \{y \in X_{\omega'} \mid f_{\omega'}(y) + c(\omega,\omega') \le -\beta\} = C^{\beta+c(\omega,\omega')}_{\omega'}.\end{aligned}$$

Consequently, the section of circumcenters is also $\mathcal{R}|_{\Omega_{\inf=-\infty}}$-invariant. We conclude the proof by gluing the two sections together. □

The following proposition is an adaptation of [**1**, Lemma 2.5] and is a key step in the proof of the main theorem.

PROPOSITION 5.1. *Let $(\Omega, \mathcal{A})$ be a Borel space, $(\Omega, X_{\bullet})$ be a Borel field of proper CAT(0) spaces and let $x^0_{\bullet} \in \mathcal{L}(\Omega, X_{\bullet})$ and $(\Omega, B_{\bullet}) \le (\Omega, \partial X_{\bullet})$ be a Borel subfield of closed sets.*

(i) *Assume that $\pi_{\bullet} \in \mathcal{L}(\Omega, \mathrm{Prob}(B_{\bullet}))$ is fixed. For every $\omega \in \Omega$ and $x^0_{\omega} \in X_{\omega}$, we define*

$$\begin{aligned} b_{\omega}: \ X_{\omega} &\to \mathbb{R}\\ x_{\omega} &\mapsto \int_{B_{\omega}} b_{x_{\omega}}(\xi)\, d\pi_{\omega}(\xi),\end{aligned}$$

where, if $x_{\omega} \in X_{\omega}$,

$$\begin{aligned} b_{x_{\omega}}: \ B_{\omega} &\to \mathbb{R}\\ \xi_{\omega} &\mapsto b_{x^0_{\omega},\xi_{\omega}}(x_{\omega}).\end{aligned}$$

Then $b_{\bullet} \in \mathcal{L}(\Omega, \mathcal{C}(X_{\bullet}))$ and b_{ω} is convex for every $\omega \in \Omega$.

(ii) *If $\alpha : \mathcal{R} \curvearrowright (\Omega, X_{\bullet})$ acts by isometries, such that $(\Omega, B_{\bullet})$ and $\pi_{\bullet}$ are invariant, then the section $b_{\bullet}$ is quasi-invariant.*

Proof. (i) A Busemann function is convex, 1-Lipschitz, and so are integrals of such functions. In particular, b_{ω} is convex and continuous for every $\omega \in \Omega$. Moreover, by Riesz's representation theorem, used to define the Borel structure of the field of probabilities $(\Omega, \mathrm{Prob}(B_{\bullet}))$ (see Example 4.1), the evaluation

$$b_{\bullet}(x_{\bullet}) = \int_{B_{\bullet}} b_{x_{\bullet}}(\xi)\, d\pi_{\bullet}(\xi) = \varphi_{\pi_{\bullet}}(b_{x_{\bullet}})$$

is Borel since $b_{x_{\bullet}} \in \mathcal{L}(\Omega, \mathcal{C}(B_{\bullet}))$ for every $x_{\bullet} \in \mathcal{L}(\Omega, X_{\bullet})$ (see Remark 3.1).

(ii) This assertion is proved by the following calculation.

$$\begin{aligned}(\widetilde{\alpha}(\omega,\omega')b_{\omega})(x_{\omega'}) &= b_{\omega}(\alpha(\omega',\omega)x_{\omega'}) = \int_{B_{\omega}} b_{x^0_{\omega},\xi}(\alpha(\omega',\omega)x_{\omega'})\, d\pi_{\omega}(\xi)\\ &\overset{(2)}{=} \int_{B_{\omega}} (b_{\alpha(\omega',\omega)x^0_{\omega'},\xi}(\alpha(\omega',\omega)x_{\omega'}) - b_{\alpha(\omega',\omega)x^0_{\omega'},\xi}(x^0_{\omega}))\, d\pi_{\omega}(\xi)\\ &\overset{\alpha\text{ isom.}}{=} \int_{B_{\omega}} b_{x^0_{\omega'},\alpha(\omega,\omega')\xi}(x_{\omega'})\, d\pi_{\omega}(\xi)\\ &\quad - \int_{B_{\omega}} b_{x^0_{\omega'},\alpha(\omega,\omega')\xi}(\alpha(\omega,\omega')x^0_{\omega})\, d\pi_{\omega}(\xi)\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_\omega} b_{x_{\omega'}}(\alpha(\omega,\omega')\xi)\,d\pi_\omega(\xi) \\
&\quad - \int_{B_\omega} b_{\alpha(\omega,\omega')x^0_\omega}(\alpha(\omega,\omega')\xi)\,d\pi_\omega(\xi) \\
&\overset{\pi_\bullet\ \text{inv.}}{=} \int_{B_{\omega'}} b_{x_{\omega'}}(\xi')\,d\pi_{\omega'}(\xi') - \int_{B_{\omega'}} b_{\alpha(\omega,\omega')x^0_\omega}(\xi')\,d\pi_{\omega'}(\xi') \\
&= b_{\omega'}(x_{\omega'}) - b_{\omega'}(\alpha(\omega,\omega')x^0_\omega).
\end{aligned}
$$

So, the function b_ω is quasi-invariant with $c(\omega,\omega') = b_{\omega'}(\alpha(\omega,\omega')x^0_\omega)$. □

5.3. *Final proof.* We now have the material to undertake the proof of the main theorem.

Proof of Theorem 1.3. Assume that assertion (i) is not satisfied. Theorem 5.1 implies the existence of an almost-invariant Borel subfield $C_\bullet$ of closed convex non-empty subsets which is minimal for these properties. Without lost of generality, we can assume that this field is invariant (see Remark 4.1). Let $(\Omega, F_\bullet)$ be the Borel subfield of flat points of $(\Omega, \partial C_\bullet)$ and $C_\bullet \hookrightarrow E_\bullet \times Y_\bullet$ the Adams–Ballmann decomposition (see Lemma 3.6 and Proposition 3.2). Let $P_\bullet$ be the Borel subfield of $\partial C_\bullet$ introduced before Lemma 3.7. Define

$$\Omega' := \{\omega \in \Omega \mid P_\omega \neq \emptyset\},$$

which is a Borel and invariant subset of Ω. Since $\mathcal{R}$ is ergodic, this set is either of full or null measure. In the first case, there exists an invariant section

$$\xi_\bullet \in \mathcal{L}(\Omega, P_\bullet) \subseteq \mathcal{L}(\Omega, \partial C_\bullet) \subseteq \mathcal{L}(\Omega, \partial X).$$

Indeed, it is proven in [**1**] that $\operatorname{rad}(P_\omega) < \pi/2$ when $P_\omega \neq \emptyset$, and therefore the section of the circumcenters is Borel (see Theorem 3.2) and invariant. This contradicts the assumption made at the beginning of the proof.

We can therefore assume that $\Omega \backslash \Omega'$ is of full measure and we will not lose generality if we assume that $\Omega' = \emptyset$. Then $C_\omega = E_\omega \times Y_\omega$, where E_ω is a finite dimensional space and ∂Y_ω does not contain flat points for all $\omega \in \Omega$. The Borel field $\partial Y_\bullet$ is invariant by property (iii) of the Adams–Ballmann decomposition in Theorem 3.4. We set

$$\Omega'' := \{\omega \in \Omega \mid \partial Y_\omega = \emptyset\},$$

which is an invariant and Borel subset of Ω and therefore of full or null measure. Assume first that it is of full measure. Then $(\Omega'', Y_\bullet)$ is a Borel field of bounded CAT(0) spaces and therefore the section of the circumcenters $c_{Y_\bullet} \in \mathcal{L}(\Omega'', Y_\bullet)$ is Borel and invariant (see Lemma 3.1). Thus, $(\Omega'', E_\bullet \times \{c_{Y_\bullet}\})$ is a Borel subfield of flats of $C_\bullet$ (and thus of $X_\bullet$) which is invariant. Ergodicity obviously implies that the dimension is essentially constant.

So, we can assume that $\Omega'' = \emptyset$. Since the relation is amenable, there exists an invariant Borel section of probabilities $\pi_\bullet \in \mathcal{L}(\Omega, \operatorname{Prob}(\partial Y_\bullet))$. By Proposition 5.1, the section of convex functions

$$b_\bullet \in \mathcal{S}(\Omega, \mathcal{F}(X_\bullet)),$$

defined, for $x^0_\bullet \in \mathcal{L}(\Omega, C_\bullet)$ fixed, by

$$b_\omega(x) = \int_{\partial Y_\omega} b_{\xi, x^0_\omega}(x)\,d\pi_\omega(\xi),$$

is such that $b_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(X_\bullet))$ and that b_ω is convex for every $\omega \in \Omega$. Moreover, it is quasi-invariant. If we define f_ω to be the restriction of b_ω to C_ω, then $f_\bullet \in \mathcal{L}(\Omega, \mathcal{C}(C_\bullet))$ is again quasi-invariant. Thus, by Lemma 5.2, the subsets

$$\Omega_{\text{inf}} := \{\omega \in \Omega \mid \inf f_\omega \text{ is not attained}\},$$
$$\Omega_{\text{min}} := \{\omega \in \Omega \mid \inf f_\omega \text{ is attained}\}$$

are invariant Borel subsets of Ω. By ergodicity, one of it has to be of full measure. Let us prove that $\mu(\Omega_{\text{min}}) = 1$ is not possible. In that case, $B_\bullet := (b_\bullet|_{\Omega_{\text{min}}} - \min(b_\bullet|_{\Omega_{\text{min}}}))^{-1}$ $(\{0\})$ would be an invariant Borel subfield of closed convex subsets and (by minimality of $C_\bullet$) $B_\omega = C_\omega$ would hold for almost every $\omega \in \Omega$. This would mean that b_ω is a constant function and that the points of ∂Y_ω in the support of π_ω are flats. This contradicts the construction of Y_ω. Therefore, Ω_{inf} has to be of full measure, but we have shown in Lemma 5.2 that in this case we can construct an invariant section $\xi_\bullet \in \mathcal{L}(\Omega_{\text{inf}}, \partial Y_\bullet)$, and this contradicts the original assumption of the proof. □

We note in closing that it is very likely that our result might hold for any amenable Borel groupoid; at least it has been checked for amenable G-spaces (see [**5**] or [**18**]).

Our result will be used in two forthcoming articles, one from each author.

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REFERENCES

[1] S. Adams and W. Ballmann. Amenable isometry groups of Hadamard spaces. *Math. Ann.* **312** (1998), 183–195.

[2] S. Adams and R. Lyons. Amenability, Kazhdan's property and percolation for trees, groups and equivalence relations. *Israel J. Math.* **75** (1991), 341–370.

[3] A. Alvarez. Une théorie de Bass–Serre pour les relations d'équivalence et les groupoïdes boréliens. *Thèse*, Université de Lyon–École Normale Supérieure de Lyon, 2008.

[4] C. Anantharaman-Delaroche and J. Renault. Amenable groupoids. *Monographie de l'enseignement mathématique*. Genève, 2000.

[5] M. Anderegg. Moyennabilité et courbure: G-espaces boréliens et espaces CAT(0). *Thèse no 4790*, École polytechnique fédérale de Lausanne, 2010.

[6] A. Avez. Variétés riemanniennes sans points focaux. *C. R. Acad. Sci. Paris Sér. A–B* **270** (1970), 188–191.

[7] W. Ballmann. *Lectures on Spaces of Nonpositive Curvature*. Birkhäuser, Basel, 1995.

[8] M. R. Bridson and A. Heafliger. *Metrics Spaces of Non-Positive Curvature*. Springer, Berlin, 1999.

[9] M. Burger and V. Schroeder. Amenable groups and stabilizers of measures on the boundary of a Hadamard manifold. *Math. Ann.* **276**(3) (1987), 505–514.

[10] C. Castaing. Sur les multi-applications mesurables. *Rev. Française Informat. Recherche Opérationnelle* **1** (1967), 91–126.

[11] A. Connes, J. Feldman and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergod. Th. & Dynam. Sys.* **1** (1981), 431–450.

[12] P.-E. Caprace and A. Lytchak. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.* **346**(1) (2010), 1–21.

[13] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions (Lecture Notes in Mathematics, 580)*. Springer, Berlin, 1977.
[14] C. Delode, O. Arino and J.-P. Penot. Champs mesurables et multisections. *Ann. Inst. H. Poincaré Sect. B* **12**(1) (1976), 11–42.
[15] J. Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de Von Neumann)*. Gauthier-Villars, Paris, 1969.
[16] R. Dougherty, S. Jackson and A. S. Kechris. The structure of hyperfinite borel equivalence relations. *Trans. Amer. Math. Soc.* **341**(1) (1994), 193–225.
[17] J. L. Doob. *Measure Theory*. Springer, New York, 1994.
[18] B. Duchesne. Infinite dimensional non-positively curved symmetric spaces of finite rank. *Preprint*, 2011, arXiv:1109.0441. *Int. Math. Res. Not.* http://imrn.oxfordjournals.org/content/early/2012/03/06/imrn.rns093.full.pdf?keytype=ref&ijkey=2f9cf5LGkTS8w9R.
[19] R. Engelking. *General Topology*. Heldermann, Berlin, 1989.
[20] J. M. G. Fell and R. S. Doran. *Representations of ∗-Algebras, Locally Compact Groups and Banach ∗-Algebraic Bundles (two volumes)*. Academic Press, London, 1988.
[21] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology and Von Neumann algebras, part I. *Trans. Amer. Math. Soc.* **234**(2) (1977), 289–324.
[22] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology and Von Neumann algebras, part II. *Trans. Amer. Math. Soc.* **234**(2) (1977), 325–359.
[23] K. Fujiwara, K. Nagano and T. Shioya. Fixed point sets of parabolic isometries of CAT(0)-spaces. *Comment. Math. Helv.* **81**(2) (2006), 305–335.
[24] D. Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.* **139**(1) (2000), 41–98.
[25] R. J. Giles. *Introduction to the Analysis of Metric Spaces*. Cambridge University Press, Cambridge, 1987.
[26] M. Gromov. *Structures métriques pour les variétés riemanniennes (Textes Mathématiques, 1)*. Eds. J. Lafontaine and P. Pansu. CEDIC, Paris, 1981, pp. 1–120.
[27] Ph. Henry. Actions de relations d'équivalence sur les champs d'espaces métriques CAT(0). *Thèse no. 4825*, École polytechnique fédérale de Lausanne, 2010.
[28] C. J. Himmelberg. Measurable relations. *Fund. Math.* **87** (1975), 53–72.
[29] S. Jackson, A. S. Kechris and A. Louveau. Countable Borel equivalence relations. *J. Math. Log.* **2**(1) (2002), 1–80.
[30] V. G. Kanovei. *Borel Equivalence Relations: Structure and Classification*. American Mathematical Society, Providence, 2008.
[31] A. S. Kechris. *Classical Descriptive Set Theory*. Springer, New York, 1995.
[32] A. S. Kechris and B. D. Miller. *Topics in Orbit Equivalence (Lecture Notes in Mathematics, 1852)*. Springer, Berlin, 2004.
[33] U. Lang and V. Schroeder. Kirszbraun's theorem and metric spaces of bounded curvature. *Geom. Funct. Anal.* **7**(3) (1997), 535–560.
[34] N. Monod. Superrigidity for irreductible lattices and geometric splitting. *J. Amer. Math. Soc.* **19**(4) (2006), 781–814.
[35] P. Urysohn. Sur un espace métrique universel. *Bull. Sci. Math.* **51** (1927), 43–64 & 74–90.
[36] M. Valadier. Sur le plongement d'un champ mesurable d'espaces métriques dans un champ trivial. *Ann. Inst. H. Poincaré Sect. B* **14**(2) (1978), 165–168.
[37] R. J. Zimmer. Hyperfinite factors and amenable ergodic actions. *Invent. Math.* **41** (1977), 23–31.
[38] R. J. Zimmer. Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Funct. Anal.* **27** (1978), 350–372.
[39] R. J. Zimmer. Curvature of leaves in amenable foliations. *Amer. J. Math.* **105**(4) (1983), 1011–1022.