CONFORMAL INVARIANCE OF SPIN PATTERN PROBABILITIES IN THE PLANAR ISING MODEL

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Abstract. We study the 2-dimensional Ising model at critical temperature on a smooth simply-connected graph Ω_δ. We rigorously prove the conformal invariance of arbitrary spin-pattern probabilities centered at a point a and derive formulas to compute the probabilities as functions of the conformal map from Ω to the unit disk. Our methods extend those of [Hon10] and [CHI13] which proved conformal invariance of energy densities and spin correlations for points fixed far apart from each other. We use discrete complex analysis techniques and construct a discrete multipoint fermionic observable that takes values related to pattern probabilities in the planar Ising model. Refined analysis of the convergence of the discrete observable to a continuous, conformally covariant function completes the result.

Contents

1. Introduction

The 2D Ising model is one of the most studied models of equilibrium statistical mechanics. It consists of a random assignment of ± 1 spins σ_x to the faces of (subgraphs of) the square grid $\mathbb{Z}^2$; the spins tend to align with their neighbors; the probability of a configuration is proportional to $e^{-\beta H}$ where the energy H is $\sum_{i\sim j}(-\sigma_i\sigma_j)$, summing over pairs of adjacent faces ; alignment strength is controlled by the parameter $\beta > 0$, usually identified with the inverse of the temperature. See Section 1.1 for a more rigorous definition.

The 2D Ising model has found applications in many areas of science, from description of magnets to ecology and image processing. Due to its simplicity and emergent features, it is interesting both as a discrete probability and statistical field theory model. Of particular physical interest is the phase transition at critical value β_c such

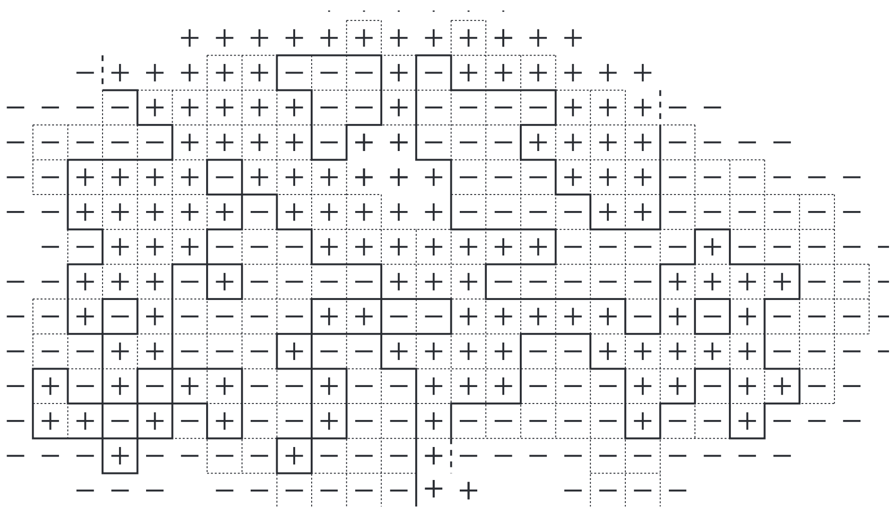

FIGURE 1.1.
A possible Ising model configuration at critical temperature along with the interfaces between plus and minus spins.

that for $\beta < \beta_c$ the system is disordered at large scales while for $\beta > \beta_c$ a long-range ferromagnetic order arises. In classical discrete probability, the phase transition can be described in terms of the infinite-volume limits: in the disordered phase $\beta < \beta_c$ only one Gibbs probability measure exists, while for $\beta > \beta_c$, measures are convex combinations of two pure measures. It has a continuous phase transition since only one Gibbs measure exists at $\beta = \beta_c$.

Critical lattice models at continuous phase transitions are expected to have universal scaling limits (independent of the choice of lattice and other details), as (non-rigorously) suggested by the Renormalization Group. The scaling limits of (a large class of) critical 2D models are expected to exhibit conformal symmetry. This can loosely be formulated as follows: for a conformal mapping φ acting on the domain $\Omega \subset \mathbb{C}$, we have

$$\varphi\,(\text{scaling limit of } \mathcal{M} \text{ on } \Omega) = \text{scaling limit of } \mathcal{M} \text{ on } \varphi\,(\Omega)\,.$$

There are two points of view describing the scaling limits of planar lattice models: curves and fields. The curves that arise in conformally invariant setups are Schramm-Loewner Evolution (SLE) curves: they describe the scaling limit of the interfaces between opposite spins. The fields on a discrete level, such as the ± 1-valued *spin field* formed by the spin values, can be described by Conformal Field Theory (CFT): their correlations in principle can be computed (non-rigorously), using representation-theoretic methods. For the 2D Ising model, most of the above program can be implemented rigorously: the interfaces of ± 1 spins can be shown to converge to SLE curves, and the correlations of the most natural fields can be shown to converge to the formulae predicted by CFT.

What makes it possible to mathematically analyze the model with great precision is its exactly solvable structure, revealed by Onsager [Ons44]. The exact solvability can be formulated in many different ways; in the recent years, the formulation in terms of discrete complex analysis has emerged as one the most powerful ways to understand rigorously the scaling limit of the model. Specifically the model's conformal symmetry becomes much more transparent in this context.

The results of [Hon10] and [CHI13] concerning the (asymptotic) conformal invariance of lattice fields can be formulated, in their simplest instance, as follows: consider the critical Ising model on discretization Ω_δ of a simply-connected domain Ω by a square grid of mesh size $\delta > 0$; put $+1$ spins on the boundary, take a point $a \in \Omega$, identify it to the closest face of Ω_δ and let $a + \delta$ be the face adjacent to that face. Then, as $\delta \to 0$

$$\text{[spin field]} \qquad \mathbb{E}_{\Omega_\delta}[\sigma_a] = 0 + C_\sigma \left|\varphi'(a)\right|^{\frac{1}{8}} \delta^{\frac{1}{8}} + o\left(\delta^{\frac{1}{8}}\right)$$

$$\text{[energy field]} \qquad \mathbb{E}_{\Omega_\delta}[\sigma_a \sigma_{a+\delta}] = \frac{\sqrt{2}}{2} + C_\epsilon \left|\varphi'(a)\right| \delta + o\,(\delta)\,,$$

where $C_\sigma, C_\epsilon > 0$ are explicit nonuniversal constants and φ is a conformal map from Ω to the unit disk $\mathbb{D}$, mapping a to the origin. The 0 and $\frac{\sqrt{2}}{2}$ on the right hand side are the *infinite-volume limits* (i.e. values on graphs approaching the full square grid $\mathbb{Z}^2$) of the quantities of the left hand side. The above results illustrate the relation between the infinite-volume limit description and the field-theoretic description: for a given local field, its correlations are described at first order by the infinite-volume limit, and the corrections are described by Conformal Field Theory quantities. The purpose of this paper is to study more general fields, to compute the infinite-volume limits, and to describe the CFT corrections.

More precisely, we look at local *pattern probabilities* (e.g. the chance that three adjacent spins are the same, the chance that a given spin is + and that its neighbor is −, etc). In the case of the dimer model, similar results, connecting pattern probabilities with conformal invariance, were obtained by Boutillier in [Bo07]. We give a way to compute the infinite-volume limit of such probabilities for the planar Ising model, and we describe the conformally covariant corrections induced by the geometry of the domain up to order δ:

$$\mathbb{P}_{\Omega_\delta}\left\{\text{pattern}\right\} = \mathbb{P}_{\mathsf{C}_\delta}\left\{\text{pattern}\right\} + \delta^{\alpha}\cdot\text{geometric correction} + o\left(\delta^{\alpha}\right),$$

where α is 1 for spin-symmetric patterns and $\frac{1}{8}$ for spin-sensitive patterns.

Another way to formulate our result is the following. The Ising CFT, conjectured to describe the scaling limit of the critical Ising model, contains three primary (conformally covariant) fields: the identity (dimension 0 – a constant field), the spin (dimension $\frac{1}{8}$) and the energy (dimension 1). We formulate pattern probabilities up to order δ in terms of these three operators (corresponding to the three terms in the above formula).

Our proof relies mainly on discrete complex analysis methods: we use lattice observables, in the form introduced in [Hon10], to connect pattern probabilities with solutions to discrete boundary value problems. We then study the scaling limits of such solutions using discrete complex analysis techniques. The new techniques introduced for this purpose, are: refined discrete analysis of multipoint observables, constructions of lattice spinor observables on the full plane, and refined analysis of convergence of observables.

Applications and Perspectives. Besides giving a general connection between pattern probabilities, Gibbs measures and conformal covariance, our results can be useful in shedding light on specific questions. One such question of Benjamini that served as one of the motivations for this work arises in the context of Ising Glauber dynamics. These dynamics are Markov chains on the space of ± 1 spin assignements, their equilibrium measures are the Ising measure. They pick a spin (uniformly at random), flip it with a certain probability, according to the state of the four neighbors, and then repeat this procedure.

A natural field, associated with these dynamics is the *flip rate* of a spin at a given location. It is tempting to relate it to the energy, as it is intuitively related to the thermal disorder of the system (the more the spin flips, the higher the disorder). Hence, at critical temperature, one would like to relate this flip rate (after correction) to the conformal invariance results of the energy density [CHI13]. This paper makes this precise: the flip rate at $x \in \Omega_\delta$ depends on the frequency of occurence of the various configurations (patterns) at the five spins $x, x \pm \delta, x \pm i\delta$ (in a spin-symmetric fashion). Hence the flip rate (at equilibrium) can be described by Theorem 1.1.

The methods introduced in this article pave the way for additional development, to appear in subsequent papers. Those will include probabilities of occurrence of multiple patterns at macroscopic distance from each other, eventually leading to a full connection between the scaling limits of the critical Ising model fields and the content of the corresponding minimal model of Conformal Field Theory [DMS97]. The methods of [ChSm11] can also be used to generalize these results to two-dimensional isoradial lattices.

Acknowledgements. Most of this research was carried out during the Research Experience for Undergraduates program at Mathematics Department of Columbia University, funded by the NSF under grant DMS-0739392. We would like to thank the T.A., Krzysztof Putyra, for his help during this program and the program coordinator Robert Lipshitz, as well as all the participants, in particular Adrien Brochard and Woo Chang Chung.

Clement Hongler would like to thank Dmitry Chelkak and Stanislav Smirnov, for sharing many ideas and insights about the Ising model and conformal invariance; Itai Benjamini and Curtis McMullen for asking questions that suggested we look at this problem; Stéphane Benoist, John Cardy, Julien Dubédat, Hugo Duminil-Copin, Konstantin Izyurov, Kalle Kytölä and Wendelin Werner for interesting discussions, the NSF under grant DMS-1106588, and the Minerva Foundation for financial support.

1.1. **Notation.** Our graph notation is largely consistent with that in [CHI13]; caution is needed since our function notation is distinct from the notation there, mixing in features from [Hon10].

1.1.1. *Graph Notation.* The Ising model is a model of spin behavior which assigns *spins* of value ± 1 to *sites* of given arrangement. In this paper, we consider spins on the faces of the discretizations Ω_δ (or $\mathbb{C}_\delta$) of a given bounded simply connected domain with smooth boundary Ω (or $\mathbb{C}$) via a rotated square graph of mesh size $\sqrt{2}\delta$. More specifically (the notations for Ω_δ are also used for $\mathbb{C}_\delta$ by putting $\mathbb{C}$ in place of Ω in the definitions):

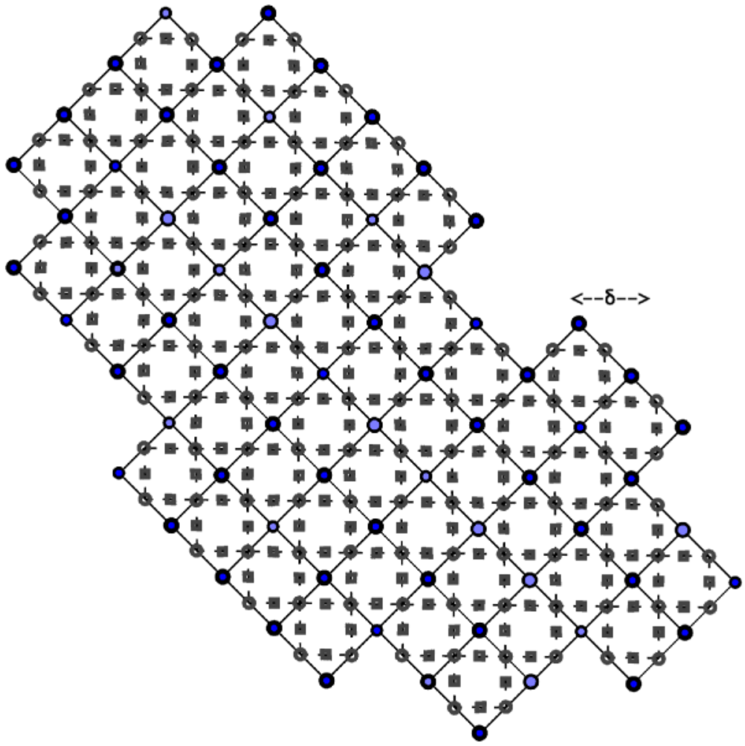

FIGURE 1.2.
$\mathcal{V}_{\Omega_\delta}$ are denoted by $\bullet$ with the solid lines connecting them being the edges in $\mathcal{E}_{\Omega_\delta}$; $\mathcal{V}^m_{\Omega_\delta}$ are denoted by $\circ$; $\mathcal{V}^c_{\Omega_\delta}$ are denoted by $\square$.

- The set of vertices $\mathcal{V}_{\Omega_\delta} := (1+i)\mathbb{Z}^2\delta \cap \Omega$; the edges $\mathcal{E}_{\Omega_\delta}$ consist of unordered pairs of vertices that are $\sqrt{2}\delta$ apart from each other; the faces $\mathcal{F}_{\Omega_\delta}$ are the square regions enclosed by four edges, the edges to which they are *incident*, the midpoints of the faces being located on $(1+i)\left[\mathbb{Z}^2 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]\delta \cap \Omega$. Two vertices or faces are *adjacent* if they have an edge between them.
- Frequently we represent edges and faces by their midpoints. In particular, the set of *medial vertices* $\mathcal{V}^m_{\Omega_\delta}$ is defined as the set of edge midpoints; given an edge $e \in \mathcal{E}_{\Omega_\delta}$, $m(e) \in \mathcal{V}^m_{\Omega_\delta}$ is its midpoint, and vice versa for $m \in \mathcal{V}^m_{\Omega_\delta}$ and $e(m) \in \mathcal{E}_{\Omega_\delta}$.
- Furthermore, we collect the *corners*, which are points $\delta/2$ off from the vertices in each of the four cardinal directions. Defining $\mathcal{V}^1_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} + \frac{\delta}{2}$, $\mathcal{V}^i_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} - \frac{\delta}{2}$, $\mathcal{V}^\lambda_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} + \frac{i\delta}{2}$, and $\mathcal{V}^{\bar\lambda}_{\Omega_\delta} := \mathcal{V}_{\Omega_\delta} - \frac{i\delta}{2}$, the set of corners is denoted as $\mathcal{V}^c_{\Omega_\delta} := \mathcal{V}^1_{\Omega_\delta} \cup \mathcal{V}^i_{\Omega_\delta} \cup \mathcal{V}^\lambda_{\Omega_\delta} \cup \mathcal{V}^{\bar\lambda}_{\Omega_\delta}$. Note that for a $\tau \in \{1, i, \lambda, \bar\lambda\}$ $\mathcal{V}^\tau_{\Omega_\delta}$ is a rotated square lattice, with nearest pairs of corners $\sqrt{2}\delta$ apart; in this lattice those corners are *adjacent*.
- Given an edge $e = \{a, b\} \in \mathcal{E}_{\Omega_\delta} \subset \mathcal{V}_{\Omega_\delta}$, an *orientation* on the edge is a choice of a unit vector o in the edge direction between $\frac{a-b}{|a-b|}$ and $\frac{b-a}{|a-b|}$. In most cases, we also arbitrarily choose one of its two square roots to get a *double orientation*, denoted $(\sqrt{o})^2$ in case of ambiguity. We denote an *oriented edge (midpoint)*, meaning a pair of an edge midpoint $m = m(e)$ and a double orientation o thereon, by $m^o = m^{(\sqrt{o})^2}$. We collect oriented edge midpoints in the set $\mathcal{V}^o_{\Omega_\delta}$. Similarly we define (double) orientations of corners, which is fixed by their type, as the unit vector in the direction to the nearest vertex and its square root.
- The domain of choice for the discrete functions in the following sections is the set of both corners and medial vertices, or $\mathcal{V}^{cm}_{\Omega_\delta} := \mathcal{V}^m_{\Omega_\delta} \cup \mathcal{V}^c_{\Omega_\delta}$. A medial vertex and a corner are *adjacent* if they are $\frac{\delta}{2}$ apart from each other.
- The *boundary faces* $\partial\mathcal{F}_{\Omega_\delta}$, *boundary medial vertices* $\partial\mathcal{V}^m_{\Omega_\delta}$, *boundary edges* in $\partial\mathcal{E}_{\Omega_\delta}$ are those faces, medial vertices, and edges that are incident but not in $\mathcal{F}_{\Omega_\delta}$, $\mathcal{V}^m_{\Omega_\delta}$, and $\mathcal{E}_{\Omega_\delta}$ respectively. Given a boundary edge, we define the *unit normal outward vector* v_{out} as the unit vector in the direction of the vertex in $\mathbb{C} \setminus \Omega$ viewed from the vertex inside Ω.

For the discrete functions with monodromy which will be introduced in Section 3, we work with graphs lifted to the double cover $[\Omega, a]$ of $\Omega \setminus \{a\}$ (or $[\mathbb{C}, a]$ of $\mathbb{C} \setminus \{a\}$) for a given fixed point $a \in \Omega$ (or $\mathbb{C}$). We assume a

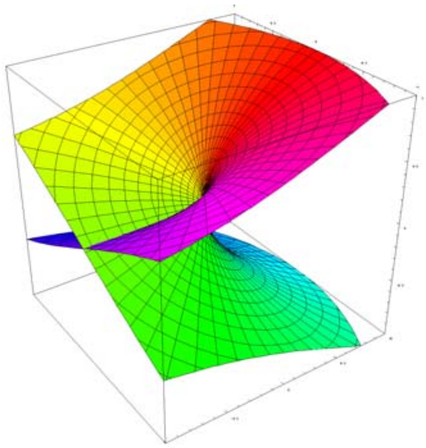

FIGURE 1.3.
The double cover of the complex plane is a Riemann surface with ramification at a point of monodromy a. Here the ramification is at the origin.

is a midpoint of a face; if not we can move the grid by less than $\sqrt{2}\delta$ so that it is the midpoint of a face. The convergence results we quote require Hausdorff convergence of the discrete domains to the continuous one–thus they hold regardless since as $\delta \to 0$ the shifted grid still converges to the continuous domain in the Hausdorff sense.

- To identify the branches of the double cover $[\mathbb{C}, a]$ we use the function $\sqrt{z-a}$ which is naturally defined on it; the left-slit plane $\mathbb{X} := \mathbb{C} \setminus (a + \mathbb{R}_-)$ lifts to two branches, $\mathbb{X}^+$ and $\mathbb{X}^-$, the former where $\operatorname{Re}\sqrt{z-a}$ is positive and the latter where it is negative. Similarly, the right-slit plane $\mathbb{Y} := \mathbb{C} \setminus (a + \mathbb{R}_+)$ lifts to $\mathbb{Y}^+$ and $\mathbb{Y}^-$, the superscripts noting the sign of $\operatorname{Im}\sqrt{z-a}$. On the discrete level, define the lift of $\mathcal{V}^1_{\Omega_\delta}$ to $\mathbb{X}^\pm$ as $\mathbb{X}^\pm_\delta$, and the lift of $\mathcal{V}^i_{\Omega_\delta}$ to $\mathbb{Y}^\pm$ as $\mathbb{Y}^\pm_\delta$.
- Our functions of interest on the double cover will be *spinors*, or functions with monodromy -1 around a; that is, we want functions that switch sign when one goes from a point on the double cover to the other point on the double cover that maps to the same point under the covering map.
- $[\Omega_\delta, a]$ is the double cover of Ω_δ with ramification at point a; in other words, the vertices, medial vertices, and corners get lifted to give the lifted vertex, edge and corner sets.
- We use similar notations for the lifted vertex, edge, and corner sets as above by replacing Ω_δ with $[\Omega_\delta, a]$.
- $[\Omega_\delta, a]$ can be naturally viewed as a subgraph of $[\mathbb{C}_\delta, a]$ in view of the natural inclusion $[\Omega, a] \subset [\mathbb{C}, a]$.

1.1.2. *Ising Model.* We consider the Ising model on the faces of Ω_δ: a *configuration* σ assigns ± 1 spins to each face in $\mathcal{F}_{\Omega_\delta}$.

- $\sigma_i \in \{\pm 1\}$ is the spin assigned to the face $i \in \mathcal{F}_{\Omega_\delta}$.
- Each configuration σ has *energy* associated with it, defined as $E(\sigma) = -\sum_{i \sim j} \sigma_i \sigma_j$, with $i \sim j$ denotes that i and j are adjacent. The Ising model declares the probability of σ, $\mathbb{P}(\sigma)$, as being proportional to $e^{-\beta E(\sigma)}$ at the *inverse temperature* $\beta > 0$.
- We only consider the critical model at inverse temperature $\beta_c = \frac{1}{2}\ln(\sqrt{2}+1)$.
- If an edge e is incident to the faces i and j, define the renormalized *energy density field* at e as $\epsilon(e) := \mu - \sigma_i \sigma_j$, where $\mu = \frac{\sqrt{2}}{2}$, the *infinite-volume limit*, is defined so that the expectation of the full-plane energy density goes to 0 as $\delta \to 0$ ([HoSm10]).
- Given a set of edges B (in particular, a spin-symmetric pattern defined in 1.13), we define the energy density of B as $\epsilon(B) := \prod_{e \in B} \epsilon(e)$.

1.1.3. *Pattern Notation.* A *base diagram* $\mathcal{B}$ is a collection of edges in the graph on the square grid $(1+i)\mathbb{Z}^2$. Denote by $F(\mathcal{B})$ the set of faces incident to edges in $\mathcal{B}$ and let a_0 be a marked midpoint of a face in $F(\mathcal{B})$. Given a graph Ω_δ and a face at $a \in \Omega_\delta$ we associate with any $b \in \mathcal{B} \cup F(\mathcal{B})$ the corresponding $a + \delta(b - a_0) \in \mathcal{E}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}$: call this embedded diagram *centered at* a, $\mathcal{B}_\delta \subset \Omega_\delta$ (omit the δ subscript when clear from context).

A *spin-sensitive pattern* is an assignment of spins ± 1 to the faces $F(\mathcal{B})$; in other words, the pattern is an element of the set $\{-1, 1\}^{F(\mathcal{B})}$. A corresponding *spin-symmetric pattern* is the union of the spin-sensitive pattern

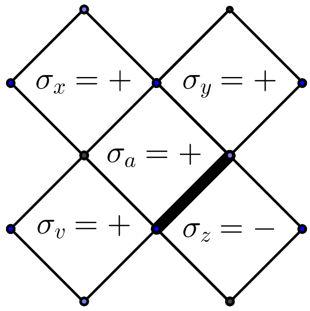

FIGURE 1.4.
Pattern Notation. The set of four edges surrounding σ_a are the elements of the base diagram $\mathcal{B}$ centered at a. The spin-symmetric pattern is defined by the presence of the edge $e_{a,z}$ so $B = \{e_{a,z}\}$ separating opposite spins. The spin-sensitive pattern associated with $\sigma_x = \sigma_y = \sigma_v = \sigma_a = +$, $\sigma_z = -$ is defined by $B|\sigma_a = +$.

and its exact negative (i.e. the spin at every face is flipped). The simplest example of a spin-sensitive pattern is the event $\{\sigma_a = +\}$ while the simplest example of a spin-symmetric pattern is the event $\{\sigma_a = \sigma_{a+\delta}\}$.

We consider the probability of such patterns occuring at a. In order to exploit the existing tools that compute the expectation of various energy densities, defined on edges, we prefer a notation based on the presence of edges. Using the low-temperature expansion, we identify each spin-symmetric pattern with an edge subset $B \subset \mathcal{B}$ by letting $e \in B$ if and only if e separates faces whose spins are different; this will be our notation for a spin-symmetric pattern from now on, with $\mathbb{P}_{\Omega_\delta}(B)$ denoting the probability that the spin-symmetric pattern B appears on $F(\mathcal{B}_\delta) \subset \mathcal{F}_{\Omega_\delta}$.

Since $F(\mathcal{B})$ is connected, taking a spin-symmetric pattern given by B and fixing the spin at a to be ± 1 specifies a spin-sensitive pattern. As such we define a spin-sensitive pattern by $[B, \sigma_a]$ with B defined as before and σ_a the spin at a.

1.1.4. *Convergence.* A family of functions $\{F_\delta : \Omega_\delta^m \to \mathbb{C}\}_{\delta>0}$ *converges on compact subsets* to the continuous function $f : \Omega \to \mathbb{C}$ if given any compact subset $\mathcal{K} \subset \Omega$ in the continuous domain for all $\epsilon > 0$, there exists a $\bar{\delta} > 0$ such that if $0 < \delta < \bar{\delta}$, $z \in \mathcal{K} \cap \mathcal{V}_{\Omega_\delta}^m \Rightarrow |F_\delta(z) - f(z)| < \epsilon$.

1.2. **Main Results.** In this section we present our main results regarding conformal invariance results of spin-symmetric and spin-sensitive patterns.

We first present the conformal invariance result on spin-symmetric spin pattern probabilities, a result that follows from the generalization of [Hon10] to edges $\mathcal{O}(\delta)$ apart from each other.

Theorem 1.1 (Conformal Invariance of Spin-Symmetric Pattern Probabilities)**.** *Given a base diagram $\mathcal{B}_\delta$ in Ω_δ centered about a, and a spin-symmetric pattern $B \subset \mathcal{B}_\delta$,*

$$\frac{1}{\delta}(\mathbb{P}_{\Omega_\delta}[B] - \mathbb{P}_{\mathsf{Z}^2}[B]) \xrightarrow[\delta \to 0]{} \langle\langle a, B\rangle\rangle_\Omega$$

where the convergence is uniform for B away from $\partial\Omega$. $\langle\langle a, B\rangle\rangle_\Omega$ is an explicitly defined function such that given a conformal map $\varphi : \Omega \to \varphi(\Omega)$,

$$\langle\langle a, B\rangle\rangle_\Omega = |\varphi'(a)| \langle\langle \varphi(a), B\rangle\rangle_{\varphi(\Omega)}.$$

Here $\mathbb{P}_{\mathsf{Z}^2}[B]$ is the infinite-volume limit *defined as*

$$\mathbb{P}_{\mathsf{Z}^2}[B] = \lim_{\Omega_\delta \to \mathsf{C}_\delta} \mathbb{P}_{\Omega_\delta}[B].$$

Remark. Since we are at critical temperature the limit exists and is unique.

We have a similar formulation of our main result for spin-sensitive pattern probabilities.

Theorem 1.2 (Conformal Invariance of Spin-Sensitive Pattern Probabilities)**.** *Given B, a spin-symmetric pattern on the base diagram $\mathcal{B}_\delta$ centered about point a in Ω, and a designated spin $\sigma_a = \pm 1$,*

$$\frac{1}{\delta^{\frac{1}{8}}}(\mathbb{P}_{\Omega_\delta}[B,\sigma_a] - \mathbb{P}_{\mathsf{Z}^2}[B,\sigma_a]) \xrightarrow[\delta\to 0]{} \langle\langle a,[B,\sigma_a]\rangle\rangle'_\Omega$$

where the convergence is uniform away from $\partial\Omega$. $\langle\langle a,[B,\sigma_a]\rangle\rangle'_\Omega$ is a function such that given a conformal map $\varphi : \Omega \to \varphi(\Omega)$ we have

$$\langle\langle a,[B,\sigma_a]\rangle\rangle'_\Omega = |\varphi'(a)|^{\frac{1}{8}} \langle\langle \varphi(a),[B,\sigma_{\varphi(a)}]\rangle\rangle'_{\varphi(\Omega)}.$$

$\mathbb{P}_{\mathsf{Z}^2}[B,\sigma_a]$ *is defined as before as the* infinite volume limit,

$$\mathbb{P}_{\mathsf{Z}^2}[B,\sigma_a] = \lim_{\Omega_\delta\to\mathsf{C}_\delta} \mathbb{P}_{\Omega_\delta}[B,\sigma_a].$$

Remark 1.3. By taking $\varphi : \mathbb{D} \to \Omega$ to be the conformal map from the unit disk to our domain with $\varphi(0) = a$, we have the following formula for the renormalized spin-sensitive pattern probability:

$$\frac{1}{\delta^{\frac{1}{8}}}(\mathbb{P}_{\Omega_\delta}[B,\sigma_a] - \mathbb{P}_{\mathsf{Z}^2}[B,\sigma_a]) \xrightarrow[\delta\to 0]{} \mathcal{C}_{[B,\sigma_a]} \cdot \mathrm{rad}^{-\frac{1}{8}}(a,\Omega),$$

where $\mathcal{C}_{[B,\sigma_a]}$ is an explicit lattice and pattern dependent constant, and $\mathrm{rad}(a,\Omega)$ is the conformal radius of Ω as seen from a. That is, $\mathrm{rad}(a,\Omega) = |\varphi'(0)|$.

1.3. **Proof Strategy.** In this section we outline the strategy for proving our main results, Theorems 1.1 and 1.2. We follow the proof structure of [Hon10]. That is, we introduce functions (discrete fermionic observables) defined on the discretized domain that take specific values related to the probability of the presence or absence of an edge in the graph. We then associate the presence or absence of edges with spin patterns on the graph and examine the continuous limits of the discrete observables. The limits, defined on the continuum obey conformal covariance properties and thus yield conformal invariance results on spin-pattern probabilities.

Specifically, we begin, in Section 2, with an overview of important definitions and results of discrete complex analysis: we construct discrete observables that satisfy certain Riemann boundary conditions with continuous counterparts. In this section, we also define the full plane discrete observables that are discrete holomorphic with a singularity and vanishing at ∞. We do so for both the complex plane, $\mathbb{C}$, and its double cover with ramification at a point $a \in \mathbb{C}$. In particular the construction of the full plane observable on the double cover with ramification at a and singularity at b is crucial to our result and relies on the explicit construction of the discrete harmonic measure on the slit plane, done in Appendix A. In constructing the full plane spinor we first argue there exists an infinite-volume limit of the fermionic spinors defined on Ω_δ, then that they converge to a continuous function in the scaling limit. The full plane fermionic observables allow us to cancel out the singularities of the functions we construct in Section 3 and prove their convergence to continuous conformally covariant functions.

In Section 3, we first define the fermionic observable on Ω_δ as first introduced in [Hon10]. Specifically we show that the results of [Hon10] extend to adjacent source points and thus the multipoint discrete fermionic observable F_{Ω_δ} can be expressed in terms of the energy density $\mathbb{E}[e_1 ... e_m]$ for adjacent edges. This allows us to relate the discrete fermionic observable to the spin-symmetric pattern probability $\mathbb{P}_{\Omega_\delta}[B]$. In order to extend this to spin-sensitive patterns we construct a multipoint version of the fermionic spinor of [CHI13], $F_{[\Omega_\delta,a]}$ living on the double cover of Ω_δ with monodromy around a. Unlike the function of [CHI13] the function we construct has points of singularity away from a point of monodromy so that we can consider a collection of edges as our pattern. We follow the same procedure as for F_{Ω_δ} to relate this discrete spinor to the spin-sensitive pattern probability $\mathbb{P}_{\Omega_\delta}[B,\sigma_a]$. This section concludes with a Pfaffian relation that defines the multipoint fermionic observables in terms of their two point counterparts.

Section 4 is the core of the paper in which we prove convergence, as the mesh size δ goes to zero, of $\delta^{-\frac{1}{2}}(F_{[\Omega_\delta,a]} - F_{[\mathsf{C}_\delta,a]})$ to a conformally covariant function. In [CHI13] such convergence was proven for a spinor with singularity located at the point of monodromy a. However, the multipoint spinors we introduce have points of singularity at all their source points, possibly all away from the point of monodromy. After expressing them using the Pfaffian relation from Section 3 in terms of two-point functions, the points of singularity approach the point of mondromy as $\delta \to 0$. We then prove convergence of these two-point functions to conformally covariant continuous functions up to higher orders of δ.

Finally in Section 5, we combine the results from Section 3 and Section 4 to prove the main results of the paper, Theorems 1.1 and 1.2. Using the Pfaffian relations from Section 3 and the relation between the discrete observables and edge pattern energy densities, we express the energy density of an arbitrary spin-symmetric pattern in terms of two-point fermionic observables. Then using the convergence results of [Hon10] and Section 4 to conformally covariant functions, we deduce that the energy densities of spin patterns converge to conformally covariant quantities. We use Appendix C to deduce conformal invariance of spin pattern probabilities in Theorems 1.1 and 1.2.

2. Discrete Complex Analysis and Full-Plane Observables

2.1. **Discrete Complex Analysis.** We introduce here basic notions of discrete complex analysis on the square lattice and introduce useful full-plane auxiliary functions $H_{\mathsf{C}_\delta}, H_{[\mathsf{C}_\delta,a]}$. Note that we denote discrete functions by upper-case alphabet letters; the continuous counterparts will be denoted by the same lower-case alphabet letters.

Definition 2.1 (s-holomorphicity,[Smi07])**.** A function $K : \mathcal{V}^{cm}_{\Omega_\delta} \to \mathbb{C}$ is said to be *s-holomorphic at* $x \in \mathcal{V}^{\tau}_{\Omega_\delta} \subset \mathcal{V}^{c}_{\Omega_\delta}$ with $\tau \in \{1, i, \lambda, \bar{\lambda}\}$ if for adjacent $z \in \mathcal{V}^{m}_{\Omega_\delta}$ we have

$$K(x) = \mathsf{P}_{\tau\mathsf{R}}\left(K(z)\right) := \frac{1}{2}\left(K(z) + \tau^2 \bar{K}(z)\right)$$

where $\mathsf{P}_{\tau\mathsf{R}}$ denotes projection onto the line $\tau\mathbb{R}$. K is s-holomorphic on Ω_δ if it is s-holomorphic at each $x \in \mathcal{V}^{c}_{\Omega_\delta}$.

For a function K defined on the double cover $[\Omega_\delta, a]$, choose an open ball $V \subset \Omega \setminus \{a\}$ around x, whose preimage by the covering map will give two disjoint copies $V_1, V_2 \subset [\Omega_\delta, a]$; we can get two bijective maps, with $i = 1,2$ in $\pi_i^{-1} : V \to V_i$. K is s-holomorphic if $K \circ \pi_i|_{\mathcal{V}^{cm}_{V_\delta}}$ is s-holomorphic for any such x, V, and i. This way of defining complex analysis notions on double covers will be frequently implied, with a point in the planar domain being identified with one of the corresponding points in the double cover.

Remark 2.2. A more obvious candidate for the discrete notion of holomorphicity, called *discrete holomorphicity*, is implied by the stronger notion of s-holomorphicity. Specifically, it is easily seen that an s-holomorphic function K satisfies $K(x + i\delta) - K(x + \delta) = i(K(x + (1 + i)\delta) - K(x))$ for all $x \in \mathcal{V}^{1,i}_{\Omega_\delta}$ where the expression makes sense. The values on $\mathcal{V}^{1,i}_{\Omega_\delta}$ in fact determine an s-holomorphic function uniquely: given a function K defined on $\mathcal{V}^{1,i}_{\Omega_\delta}$ such that the above discrete holomorphicity relations hold.

We note that a family of s-holomorphic function $\{F_\delta : \mathcal{V}^{cm}_{\Omega_\delta} \to \mathbb{C}\}|_{\mathcal{V}^{m}_{\Omega_\delta}}$ converges on compact subsets to the continuous function f if and only if the restrictions $F_\delta|_{\mathcal{V}^{1}_{\Omega_\delta}}$ and $F_\delta|_{\mathcal{V}^{i}_{\Omega_\delta}}$ converge on compact subsets respectively to $\operatorname{Re} f$ and $\operatorname{Im} f$, where the notion of convergence on (a fixed single type of) corners is defined analogously to that on medial points.

Definition 2.3 (Discrete Harmonicity)**.** For a fixed $\tau \in \{1, i\}$ a function $L : \mathcal{V}^{\tau}_{\mathsf{C}_\delta} \to \mathbb{C}$ is *discrete harmonic at* $x \in \mathcal{V}^{\tau}_{\mathsf{C}_\delta}$ if we have

$$L(x + (1 + i)\delta) + L(x + (1 - i)\delta) + L(x - (1 + i)\delta) + L(x - (1 - i)\delta) = 4L(x).$$

The function L is *discrete harmonic* if it is discrete harmonic at every point.

Remark 2.4. We only introduce the harmonicity notion on $\mathbb{C}_\delta$, but it easily generalizes to Ω_δ with some modification of the definition in case of boundary points; see [ChSm11] for details. Discrete holomorphicity implies discrete harmonicity of restrictions–the following can be checked straightforwardly:

Proposition 2.5 (Proposition 3.6, [ChSm12])**.** *The restriction of an s-holomorphic function on* $\mathcal{V}^{cm}_{\mathsf{C}_\delta}$ *to* $\mathcal{V}^{\tau}_{\mathsf{C}_\delta}$*, for some* $\tau \in \{1, i\}$*, is discrete harmonic. The restriction of an s-holomorphic function on* $\mathcal{V}^{cm}_{[\mathsf{C}_\delta,a]}$ *to* $\mathcal{V}^{\tau}_{[\mathsf{C}_\delta,a]}$ *for some* $\tau \in \{1, i\}$ *is discrete harmonic at every point except at* $a \pm \frac{\delta}{2}$*.*

Remark 2.6. As seen in Remark 2.2, an s-holomorphic function is defined by its discrete holomorphic restriction to $\mathcal{V}^{1,i}_{\Omega_\delta}$. In fact, the full restriction to $\mathcal{V}^{1,i}_{\Omega_\delta}$ can be recovered by the restriction on just one type of the corner up to a constant. If one has a harmonic function on $\mathcal{V}^{1}_{\Omega_\delta}$ and a fixed value on any one point in $\mathcal{V}^{i}_{\Omega_\delta}$ (or vice versa), there is a unique harmonic function on $\mathcal{V}^{i}_{\Omega_\delta}$ (or $\mathcal{V}^{1}_{\Omega_\delta}$), called the *harmonic conjugate*, which has the specified value at the point and forms a discrete holomorphic function together with the prescribed values on $\mathcal{V}^{1}_{\Omega_\delta}$ (or $\mathcal{V}^{i}_{\Omega_\delta}$). Harmonicity of the prescribed function ensures that the values defined by sums along different lines are well-defined (see Lemma 2.15 of [ChSm11]).

Discrete harmonic functions on a domain, like their continuous counterparts, are unique given a certain set of boundary conditions. Here, we present a discrete complex analysis version of such a result concerning s-holomorphic functions.

Proposition 2.7 (Corollary 29, [Hon10])**.** *A solution* K_δ *of* the discrete Riemann boundary value problem *on* Ω_δ *with boundary data* $L : \partial\mathcal{V}^{m}_{\Omega_\delta} \to \mathbb{C}$*, which is a discrete function defined on* $\mathcal{V}^{cm}_{\Omega_\delta}$ *satisfying the conditions*

- *K_δ is s-holomorphic in $[\Omega_\delta, a]$*
- *For each $x \in \partial\mathcal{V}^m_{[\Omega_\delta,a]}$, $[K_\delta(x) - L(x)]\sqrt{v_{out}(x)} \in \mathbb{R}$*

is unique.

Proposition 2.8. *A solution K_δ of* the discrete Riemann-Hilbert boundary value problem *on Ω_δ with boundary data $L : \partial\mathcal{V}^m_{[\Omega_\delta,a]} \to \mathbb{C}$ with monodromy -1 at $a \in \Omega$, which is a discrete function defined on $\mathcal{V}^{cm}_{[\Omega_\delta,a]}$ satisfying the conditions*

- *K_δ is s-holomorphic in $[\Omega_\delta, a]$.*
- *For each $x \in \partial\mathcal{V}^m_{[\Omega_\delta,a]}$, $[K_\delta(x) - L(x)]\sqrt{v_{out}(x)} \in \mathbb{R}$*

is unique.

Proof. In applying the proof of Proposition 28 in [Hon10] (which estimates the "area integral" of K_δ^2, which is the sum of all medial values, by the "line integral" of L^2, or the sum thereof along the boundary), the only tool needed is the superharmonic antiderivative of K_δ^2; its existence is shown in Remark 3.8 of [ChSm12]. □

2.2. **Full-plane Observables.** We now present full-plane versions of various observables constructed in Section 3. These functions encode the infinite-volume limits that the model converges to when the domain approaches $\mathbb{C}_\delta$; from a functional point of view, they have *discrete singularities*, which are used to cancel out the same singularity types in the domain-dependent observables when applying uniqueness and convergence results (examples of which we saw above in Propositions 2.7, 2.8) concerning everywhere s-holomorphic functions.

To use procedures outlined in Remarks 2.4 and 2.6, we define an important harmonic function which can be defined on most discrete domains:

Definition 2.9. Given medial vertices a_1, a_2 and an orientation $o \in \mathbb{O}^2$ on a_1, we define the *full-plane fermionic observable* $H_{\mathsf{C}_\delta}(a_1^{o_1}, a_2)$ by

$$H_{\mathsf{C}_\delta}(a_1^{o_1}, a_2) = \frac{\eta}{\sqrt{o_1}}\cos\left(\frac{\pi}{8}\right)\left(G\left(\frac{a_1}{\delta} + \frac{o_1\sqrt{2}}{2}, \frac{a_2}{\delta}\right) + G\left(\frac{a_1}{\delta} - \frac{io_2\sqrt{2}}{2}, \frac{a_2}{\delta}\right)\right)$$
$$- i\frac{\eta}{\sqrt{o_1}}\sin\left(\frac{\pi}{8}\right)\left(G\left(\frac{a_1}{\delta} - \frac{o_1\sqrt{2}}{2}, \frac{a_2}{\delta}\right) + G\left(\frac{a_1}{\delta} + \frac{io_2\sqrt{2}}{2}, \frac{a_2}{\delta}\right)\right)$$

where $G(a_{1,}a_2) = G(0, a_2 - a_1)$ and $G(0, w(1+i)) := 2C_0(0, 2w)$, and where C_0 is the coupling function defined in Section 5 of [Ken00]. $\delta^{-1}H_{\mathsf{C}_\delta}(a_1^{o_1}, a_2) \xrightarrow{\delta\to 0} h_{\mathsf{C}}(a_1^{o_1}, a_2) := \frac{\sqrt{2}}{\lambda\sqrt{o_1}}\frac{1}{a_2 - a_1}$ on compact subsets of $\Omega \times \Omega$ away from the diagonal.

Remark 2.10. By Thoerem 87 of [Hon10], the full-plane fermionic observable is the unique s-holomorphic function on $\mathcal{V}^o_{\mathsf{C}_\delta} \times \mathcal{V}^{cm}_{\mathsf{C}_\delta} \setminus \Delta$, $\Delta = \{(b^o, b) \in \mathcal{V}^o_{\mathsf{C}_\delta} \times \mathcal{V}^{cm}_{\mathsf{C}_\delta}\}$ such that for all $b^o \in \mathcal{V}^o_{[\mathsf{C}_\delta,a]}$ $H_{\mathsf{C}_\delta}(b^o, \cdot)$ has a *discrete simple pole* at b (in other words, every projection-based s-holomorphicity relation holds except for those involving $H_{\mathsf{C}_\delta}(b^o, b)$) with *residue* $\frac{i}{\sqrt{o_1}}$ at b: for $b + \frac{1\pm i}{2\sqrt{2}}o\delta$ and $b - \frac{1\pm i}{2\sqrt{2}}o\delta$ corners adjacent to b, and h^+, h^- values of $H_{\mathsf{C}_\delta}(b^o, b)$ that makes $H_{[\mathsf{C}_\delta,a]}(b^o, \cdot)$ s-holomorphic on $\{b + \frac{1\pm i}{2\sqrt{2}}o\delta\}$ and $\{b - \frac{1\pm i}{2\sqrt{2}}o\delta\}$ respectively, $h^+ - h^- = \frac{i}{\sqrt{o_1}}$. Note that an otherwise s-holomorphic function that is possibly singular at an edge is in fact s-holomorphic there if and only if the residue at that edge is zero.

On $[\Omega_\delta, a]$, [CHI13] gives a formulation of an analogous function on the double cover based on harmonic measure and its various estimates. For a discrete domain Λ and $A \subset \partial\Lambda$, the *harmonic measure of* Λ *as seen from* z $\mathrm{hm}^\Lambda_A(z)$ is the probability that a random walk starting at z hits A before hitting $\partial\Lambda\setminus A$, which, across all $z \in \Lambda$, happens to be the unique harmonic function on Λ with boundary value 1 on A and 0 elsewhere. We explicitly compute the version we will use (our needed harmonic measures are on graphs isomorphic to $\mathbb{C}_\delta \setminus \mathbb{R}_+$):

Theorem 2.11. *The discrete harmonic measure on the slit complex plane $\mathbb{C}_\delta\setminus\mathbb{R}_+$ is given by*

$$\mathrm{hm}^{\mathsf{C}_\delta\setminus\mathsf{R}_+}_{\{0\}}((s+ik)\delta) := H_0(s+ik) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C^{|k|}(\theta)}{\sqrt{1-e^{-2i\theta}}}e^{-is\theta}d\theta,$$

where $C(\theta) := \dfrac{\cos\theta}{1+|\sin\theta|}$ and the square root takes the principal value. In particular, $\mathrm{hm}^{\mathsf{C}_\delta\setminus\mathsf{R}_+}_{\{0\}}(2(s+1)\delta) = \frac{s+\frac{1}{2}}{s+1}\mathrm{hm}^{\mathsf{C}_\delta\setminus\mathsf{R}_+}_{\{0\}}(2s\delta)$*, which asymptotically gives* $\mathrm{hm}^{\mathsf{C}_\delta\setminus\mathsf{R}_+}_{\{0\}}(2s\delta) \sim \frac{1}{\sqrt{\pi s}}$.

Proof. See Appendix A.1. □

Theorem 2.12 (Theorem 2.15, [CHI13]). *There is a unique s-holomorphic spinor $H_{[\mathsf{C}_\delta,a]}$ on $[\mathbb{C}_\delta,a]\setminus\{a+\frac{\delta}{2}\}$ such that $H_{[\mathsf{C}_\delta,a]}(a+\frac{3\delta}{2})=1$ and $H_{[\mathsf{C}_\delta,a]}(z)\to 0$ uniformly as $z\to\infty$. Then $\vartheta(\delta)^{-1}H_{[\mathsf{C}_\delta,a]}$ converges on compact subsets to spinor $h_{[\mathsf{C},a]}(z):=\frac{1}{\sqrt{z-a}}$ defined on $[\mathbb{C},a]$. Here $\vartheta(\delta)$ is the normalization factor defined by $\vartheta(\delta):=H_{[\mathsf{C}_\delta,a]}\left(a+\frac{3\delta}{2}+2\delta\left\lfloor\frac{1}{2\delta}\right\rfloor\right)=\sqrt{\frac{\pi\delta}{2}}$.*

Proof. [CHI13] defines the value of $H_{[\mathsf{C}_\delta,a]}$ on $\mathcal{V}^{1,i}_{[\mathsf{C}_\delta,a]}$ by $H_{[\mathsf{C}_\delta,a]}:=\pm\operatorname{hm}^{\mathsf{X}_\delta}_{\{a+\frac{3\delta}{2}\}}$ on $\mathbb{X}^\pm_\delta$ and $H_{[\mathsf{C}_\delta,a]}:=\mp i\operatorname{hm}^{\mathsf{Y}_\delta}_{\{a+\frac{3\delta}{2}\}}$ on $\mathbb{Y}^\pm_\delta$, and zero on corners missed by either $\mathbb{X}^\pm_\delta$ or $\mathbb{Y}^\pm_\delta$. The two functions respectively converge to $\operatorname{Re}\frac{1}{\sqrt{z-a}}$ and $i\operatorname{Im}\frac{1}{\sqrt{z-a}}$ uniformly on compact sets, and extend s-holomorphically to $\mathcal{V}^{cm}_{[\mathsf{C}_\delta,a]}$ per our Remark 2.2. Theorem 2.11 provides the asymptotic estimate. Note it is impossible to extend the function s-holomorphically to $a+\frac{\delta}{2}$; the projections from the two adjacent midpoints have opposite signs. □

Now we define the two-point versions of the spinors; they are spinors with discrete singularities which can be moved to points other than $a+\frac{\delta}{2}$. Analogous to H_{C_δ}, our goal is to provide functions with singularities at edge midpoints which have the same behavior as $H_{[\Omega_\delta,a]}$, to be defined in the next section. We overload the notation $H_{[\mathsf{C}_\delta,a]}$ in order to refer to the above defined spinor and the various two-point functions below; the definitions are clearly distinguishiable once one identifies whether the argument(s) are corners or edge midpoints. Write $[\Delta,a]=\{(b^o,b)\in\mathcal{V}^o_{[\mathsf{C}_\delta,a]}\times\mathcal{V}^{cm}_{[\mathsf{C}_\delta,a]}\}$, where the function is not defined:

Theorem 2.13. *There is a unique function $H_{[\mathsf{C}_\delta,a]}:\mathcal{V}^o_{[\mathsf{C}_\delta,a]}\times\mathcal{V}^{cm}_{[\mathsf{C}_\delta,a]}\setminus[\Delta,a]\to\mathbb{C}$ such that for all $b\in\mathcal{V}^o_{[\mathsf{C}_\delta,a]}$ $H_{[\mathsf{C}_\delta,a]}(b^o,\cdot)$ satisfy properties 1,2,3 (or 1,2,4) of A.4. Then for $v_1^o\in\mathcal{V}^o_{[\mathsf{C}_\delta,a]}$ a distance $\mathcal{O}(\delta)$ from a as $\delta\to 0$, we have, uniformly over z on compact subsets away from a,*

$$\frac{1}{\vartheta(\delta)}H_{[\mathsf{C}_\delta,a]}(v_1^o,z)\xrightarrow[\delta\to 0]{}\mathcal{C}_{v_1^o}h_{[\mathsf{C},a]}(a^o,z)=\frac{\mathcal{C}_{v_1^o}}{\sqrt{z-a}}.$$

for some $C_{v_1^o}\in\mathbb{C}$.

Proof. The proof proceeds in two steps done in Appendix A. We first prove the existence and uniqueness of such a function defined on $b\in\mathcal{V}^o_{[\mathsf{C}_\delta,a]}$ with the above properties in A.4. Then we prove convergence to $C_{v_1^o}\frac{1}{\sqrt{z-a}}$ in A.8. □

Theorem 2.14. *There are s-holomorphic spinors $\{G_{[\mathsf{C}_\delta,a]}\},\{\tilde{G}_{[\mathsf{C}_\delta,a]}\}$ on $[\mathbb{C}_\delta,a]$ such that $\vartheta(\delta)^{-1}G_{[\mathsf{C}_\delta,a]},\vartheta(\delta)^{-1}\tilde{G}_{[\mathsf{C}_\delta,a]}$ converge on compact subsets respectively to spinors $g_{[\mathsf{C},a]}:=\sqrt{z-a},\tilde{g}_{[\mathsf{C},a]}:=i\sqrt{z-a}$ defined on $[\mathbb{C},a]$.*

Proof. Theorem 2.16 of [CHI13] proves this for $\{G_{[\mathsf{C}_\delta,a]}\}$ by defining $G_{[\mathsf{C}_\delta,a]}(z):=\delta\sum_{j=0}^\infty H_{[\mathsf{C}_\delta,a]}(z-2j\delta)$ on $\mathbb{X}^\pm_\delta$, zero elsewhere; [CHI13]'s objective is just to get the real corner harmonic restriction of the full s-holomorphic function. We refer to its proofs regarding convergence and harmonicity of the series; but for the limiting function we note that the sum is close to the Riemann sum of $\operatorname{Re}\frac{1}{2\sqrt{z-a}}$ that approaches $\operatorname{Re}\sqrt{z-a}$, identifying the sum.

For the imaginary part, we define $G_{[\mathsf{C}_\delta,a]}(z):=-\delta\sum_{j=1}^\infty H_{[\mathsf{C}_\delta,a]}(z+2j\delta)$ on $\mathbb{Y}^\pm_\delta$ (use $H_{[\mathsf{C}_\delta,a]}(a+\frac{\delta}{2})=\mp i$ respectively), zero elsewhere. The proofs for convergence to $\operatorname{Im}\sqrt{z-a}$ and harmonicity are exactly analogous; to get an s-holomorphic extension to $\mathcal{V}^{cm}_{[\mathsf{C}_\delta,a]}$, we need to check discrete holomorphicity. We use the explicit formula for the harmonic measure for this, and we do it in the appendix, Corollary A.3.

For $\tilde{G}$, we analogously define $\tilde{G}_{[\mathsf{C}_\delta,a]}(z):=-i\delta\sum_{j=0}^\infty H_{[\mathsf{C}_\delta,a]}(z+\delta+2j\delta)$ on $\mathbb{Y}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}$ and $\tilde{G}_{[\mathsf{C}_\delta,a]}(z):=i\delta\sum_{j=0}^\infty H_{[\mathsf{C}_\delta,a]}(z-\delta-2j\delta)$ on $\mathbb{X}^\pm\cap\mathcal{V}^i_{[\mathsf{C}_\delta,a]}$; the properties are checked exactly analogously. □

3. Discrete Holomorphic Observables

In this section, we define discrete holomorphic observables $F_{\Omega_\delta},F_{[\Omega_\delta,a]}$ on Ω_δ and $[\Omega_\delta,a]$ and prove the relations linking them to various energy densities which form the pattern expectation vector. For the rest of the paper, as hinted at in the introduction we will refer to the discrete observable F_{Ω_δ} defined on Ω_δ as the *discrete fermionic observable*. We will refer to the discrete observable $F_{[\Omega_\delta,a]}$ defined on $[\Omega_\delta,a]$, the double cover of Ω_δ with monodromy at point a, as the *discrete fermionic spinor*.

3.1. Discrete Fermionic Observable with no Monodromy.

We first reintroduce the discrete fermionic observable with no point of monodromy that was presented in [Hon10]. We generalize the observable and the most relevant results from [Hon10] to the case when source points are adjacent. This allows us to compute expectations of energy densities of edges even when the edges are $\mathcal{O}(\delta)$ apart from each other. Then the observable can be related to probabilities of arbitrary spin-symmetric patterns.

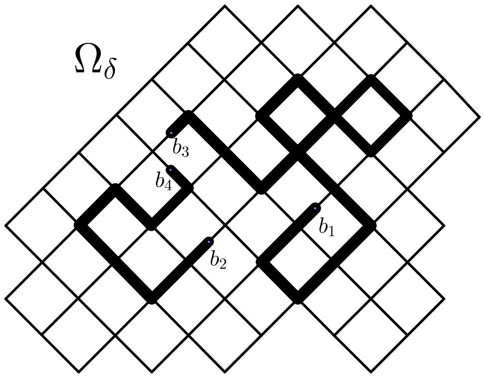

FIGURE 3.1.
An example of a two walks between b_1 and b_3 and b_2 and b_4. The winding of $\gamma : b_1 \to b_3$ is 2π while the winding of $\gamma : b_2 \to b_4$ is $-\frac{\pi}{2}$.

Remark 3.1. Here we introduce some definitions that are borrowed from [Hon10] and [CHI13] useful for defining the discrete fermionic observables on Ω_δ and $[\Omega_\delta, a]$. A point b_i on $\mathcal{V}_{\Omega_\delta}$ are medial vertices on the graph Ω_δ, each assigned a double orientation in $\mathbb{O}^2$, $o \in \{\pm 1 \pm i, \mp 1 \pm i\}$ with a fixed square root branch. A walk ω between two points b_i and b_j is a collection of edges and half edges such that there is exactly one edge at each of b_i and b_j and every other vertex in Ω_δ has an even number of incident edges. For well-definedness, we follow the conventions of [Hon10] and force the walk to go right any time it intersects itself.

The winding number $W(\omega)$ represents the number of loops around a point the path ω makes, as is typical of CFT partition functions. $c(\gamma_1, ..., \gamma_n)$ is the crossing signature induced by the walks γ_i: if we link $1, ..., 2n$ on $\mathbb{R}$ by simple paths in the upper half plane that connect the endpoints of each of the n γ_i, the number is well defined modulo 2 and is what we define to be $c(\gamma_1, ..., \gamma_n)$. For more rigorous definitions of the winding number and crossing signature see 5.2.1 of [Hon10].

Definition 3.2. Given a domain Ω_δ and $2n$ Ω_δ-distinct medial vertices $b_1, b_2, \dots, b_{2n}$ in $\mathcal{V}^m_{\Omega_\delta}$ and their double orientations $(\sqrt{o_1})^2, (\sqrt{o_2})^2, \dots, (\sqrt{o_{2n}})^2$. Define the real fermionic observable F_{Ω_δ}:

$$F_{\Omega_\delta}(b_1^{o_1}, \dots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}} \sum_{\gamma \in C(b_1^{o_1}, \dots, b_{2n}^{o_{2n}})} \alpha^{\#\gamma} (-1)^{c(\gamma_1, \dots, \gamma_n)} \prod_{i=1}^{n} \phi(\gamma_i), \text{ where}$$

$$C(b_1^{o_1}, \dots, b_{2n}^{o_{2n}}) = \{\text{edges and half-edges that form walks between } b_i^{o_i}\text{'s and loops}\}$$

$$\phi(w : \alpha^o \;\square\; \beta^{o^\square}) = i \frac{\sqrt{o'}}{\sqrt{o}} e^{-\frac{iW(w)}{2}}$$

$$\langle \gamma_1, \dots, \gamma_n \rangle\text{: admissible choice of walks from } \gamma \text{ in } C$$

$$c(\gamma_1, \dots, \gamma_n)\text{: crossing signature of } b_i\text{'s with respect to } \gamma_1, \dots, \gamma_n$$

$$\mathsf{Z}_{\Omega_\delta} = \sum_{\gamma \in C} \alpha^{\#\gamma}$$

It has been proven that $\prod_{i=1}^{n} \phi(\gamma_i)$ is well-defined for various admissible choices of walks and thus $F_{\Omega_\delta}(\cdots)$ is well defined.

Definition 3.3 ([Hon10])**.** Given a collection of doubly oriented points $(\cdots)$, a collection of signed edges $e_1, ..., e_m$ disjoint from $(\cdots)$, define the restricted real fermionic observable $F_{\Omega_\delta}^{\{e_1^{s_1}, \dots, e_m^{s_m}\}}(\cdots)$ as

$$F_{\Omega_\delta}^{\{e_1^{s_1}, \dots, e_m^{s_m}\}}(b_1^{o_1}, \dots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}} \sum_{\gamma \in C^{\{e_1^{s_1}, \dots, e_m^{s_m}\}}(b_1^{o_1}, \dots, b_{2n}^{o_{2n}})} \alpha^{\#\gamma} (-1)^{c(\gamma_1, \dots, \gamma_n)} \prod_{i=1}^{n} \phi(\gamma_i).$$

Further, given a collection of doubly oriented points (...), a collection of signed edges {...} disjoint from (...) and edges $e_1, ..., e_m$, define the fused real fermionic observable $F_{\Omega_\delta}^{[e_1, \dots, e_m]\{\dots\}}(...)$ inductively as

$$F_{\Omega_\delta}^{[e_1,...,e_m]\{...\}}(...) = F_{\Omega_\delta}^{[e_1,...,e_{m-1}]\{...,e_m^+\}}(...) - \frac{\sqrt{2}}{2} F_{\Omega_\delta}^{[e_1,...,e_{m-1}]\{...\}}(...).$$

Remark 3.4. As has been shown in [Hon10], a fused observable $F_{\Omega_\delta}^{[e_1,...,e_m]}(b_1^{o_1},...,b_{2n}^{o_{2n}})$ corresponds to a $2n+2m$ unfused observable where the $2m$ points are merged pairwise together at the edges $e_1,...,e_m$.

Definition 3.5. For a collection $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be doubly oriented medial vertices, a medial vertex b_{2n} and a configuration $\gamma \in C(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$, let us denote by $\mathsf{W}^H_{\Omega_\delta}(\gamma,o_1,...,o_{2n-1})$ the complex weight of γ, defined as

$$\mathsf{W}_{\Omega_\delta}(\gamma,o_1,...,o_{2n-1}) = \frac{i}{\sqrt{o_{2n}}}\alpha^{\#\gamma}\prod_{i=1}^{n}\phi(\gamma_i)$$

for any branch choice of o_{2n}. From here we can define the complex fermionic observable

$$H_{\Omega_\delta}^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}}\sum_{\gamma\in C^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})}\mathsf{W}^H_{\Omega_\delta}(\gamma,o_1,...,o_{2n-1}).$$

Proposition 3.6. *Let $e_1,...,e_m$ be a set of possibly adjacent interior edges. Then we have*

$$\mathbb{E}_{\Omega_\delta}[\epsilon(e_1)\cdot ...\cdot\epsilon(e_m)] = (-1)^m 2^m F_{\Omega_\delta}^{[e_1,...,e_m]}.$$

Proof. This is proven for general boundary conditions, not just plus boundary conditions in [Hon10] 5.3. □

Proposition 3.7. *Let $e_1^{s_1},...,e_m^{s_m}$ be distinct (though possibly adjacent) signed edges and $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be distinct, though possibly adjacent, doubly oriented medial vertices. Then the function*

$$b_{2n}\mapsto H_{\Omega_\delta}^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$

whose domain is $\mathcal{V}_{\Omega_\delta^m}\setminus\{m(e_1),...,m(e_m),b_1,...,b_{2n}\}$ is s-holomorphic. Moreover, it obeys boundary conditions

$$\mathrm{Im}[H_{\Omega_\delta}^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})\sqrt{v_{out}(b_{2n})}] = 0 \text{ for all } b_{2m}\in\partial\mathcal{V}^m_{\Omega_\delta}$$

where $v_{out}(b_{2n})$ is defined as the outer normal to the boundary at b_{2n}.

Proof. See Appendix B.1 for a complete proof of s-holomorphicity even in the case of adjacent edges and vertices by defining the function on corners. □

Proposition 3.8. *Let $e_1^{s_1},...,e_m^{s_m}$ be distinct signed edges and $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be distinct doubly oriented medial vertices. For each $j\in\{1,...,2n-1\}$ such that $b_j\in\mathcal{V}^m_{\Omega_\delta}\setminus\partial\mathcal{V}^m_{\Omega_\delta}$ the function*

$$b_{2n}\mapsto H_{\Omega_\delta}^{\{e_1,...,e_m\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$

has a discrete simple pole at b_j, with front and rear values given by:

$$H_j^+ = \frac{(-1)^{j+1}}{\sqrt{o_j}}F_{\Omega_\delta}^{\{e_1^{s_1},...,e_m^{s_m},e(b_j)^+\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$
$$H_j^- = \frac{(-1)^{j}}{\sqrt{o_j}}F_{\Omega_\delta}^{\{e_1^{s_1},...,e_m^{s_m},e(b_j)^-\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$

where $e(b_j)$ denotes the edge whose midpoint is b_j.

Further, the function

$$b_{2n}\mapsto H_{\Omega_\delta}^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$

can be extended to an s-holomorphic function

$$H_j^+ = \frac{1}{\sqrt{o_j}}F_{\Omega_\delta}^{\{e_1^{s_1},...,e_m^{s_m}\}}(b_1^{o_1},...b_{j-1}^{o_{j-1}},b_{j+1}^{o_{j+1}},...,b_{2n-1}^{o_{2n-1}})$$

.

Proof. This proof follows immediately from its proof where the edges and points are not allowed to be adjacent in Prop. 76 of [Hon10] and Prop 3.7 which generalizes the projection relations to the case when they are adjacent. □

Proposition 3.9. *For distinct edges $e_1, ..., e_m$ and distinct doubly oriented medial vertices $b_1^{o_1}, ..., b_{2n}^{o_{2n}}$, for each choice of orientations $q_i \in \mathbb{O}(e_i) : 1 \le i \le m$ we have that*

$$F_{\Omega_\delta}^{[e_1,...,e_m]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) = \operatorname{Pfaff}(\mathsf{A}_{\Omega_\delta}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}, b_1^{o_1}, ..., b_{2n}^{o_{2n}})$$

where we associate with e_i the medial vertex on the edge and where the $2p \times 2p$ matrix, $p = m + n$, is defined for (not necessarily distinct) doubly oriented vertices $x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}$ by

$$(\mathsf{A}_{\Omega_\delta}(x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}))_{jk} = \begin{Bmatrix} F_{\Omega_\delta}^{\mathsf{C}_\delta}(x_i^{\xi_i}, x_j^{\xi_j}) & \mathit{if}\, x_i = x_j\, \mathit{and}\, \xi_i \ne \xi_j \\ F_{\Omega_\delta}(x_i^{\xi_i}, x_j^{\xi_j}) & \mathit{if}\, x_i \ne x_j \\ 0 & \mathit{else} \end{Bmatrix}.$$

Proof. Since everything else has been generalized from [Hon10] to the case when points are adjacent and the proof of the Pfaffian relation for fused fermionic observables only relies on previously generalized Lemmas and Propositions due to the corner formulation of the projection relations, we have the same proof as in section 6.6 of [Hon10]. Thus we have reduced all problems for the real multipoint fermionic observable with points of possibly adjacent singularity to computations of two point fermionic observables. □

Remark 3.10. From the above Pfaffian relation on the fused fermionic observable we can relate the expectation of energy density of edges to $\operatorname{Pfaff}(\mathsf{A}_{\Omega_\delta})$ as

$$\mathbb{E}_{\Omega_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)] = (-1)^m 2^m \operatorname{Pfaff}(\mathbf{A}_{\Omega_\delta}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}).$$

3.2. **Discrete Fermionic Spinor with Monodromy.** We now construct a multipoint fermionic spinor with a point of monodromy around $a \in \Omega$ building off the single point spinor introduced in [CHI13] on the double cover of the complex plane. Our extension of the single-point spinor to its multipoint equivalent is based on the extension of [HoSm10] to [Hon10]. We move the source points away from the point of monodromy and allow for multiple source points and multiple end points. This allows us to compute expectations of energy densities of arbitrary edge collections while preserving information about σ_a obtained by setting the point of monodromy at a. We will eventually use convergence results on this fermionic spinor with monodromy to get expectations of spin-sensitive patterns.

Remark 3.11. Since we are now working on the double cover of the complex plane, $[\mathbb{C}_\delta, a]$ there is some additional notation to introduce, borrowed from [CHI13]. We define the double cover of a domain $\Omega_\delta \subset \mathbb{C}_\delta$ with point of monodromy around $a \in \Omega_\delta$ as in the introduction. We then define the loop number $\#L(w, a)$ as the number of loops around a (i.e. the number of loops such that a is a face contained in the interior of the loop in Ω_δ where the loops are viewed not on the double cover but on the principal domain). The sheet number is a multiplicative factor of ± 1 indicating whether a walk ends on the same sheet as it began or on a different sheet. It is defined such that one full loop around a ends on the lift of the starting point to a different sheet and is thus responsible for the spinor property of the fermionic observable with monodromy. For a more rigorous definition of the loop number and the sheet number $\mathrm{Sheet}(w)$ refer to [CHI13].

Definition 3.12. Given a domain Ω_δ and its double cover $[\Omega_\delta, a]$ ramified at $a \in \mathcal{F}_{\Omega_\delta}$, we pick $2n$ Ω_δ-distinct medial vertices $b_1, b_2, \ldots, b_{2n}$ in $[\Omega_\delta, a]$ and their double orientations $(\sqrt{o_1})^2, (\sqrt{o_2})^2, \ldots, (\sqrt{o_{2n}})^2)$. Define the multi-point real function $F_{[\Omega_\delta,a]}$:

$$F_{[\Omega_\delta,a]}(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]} \sum_{\gamma \in C(b_1^{o_1},\ldots,b_{2n}^{o_{2n}})} \alpha^{\#\gamma}(-1)^{\#L(\gamma \setminus \cup \gamma_i, a)}(-1)^{c(\gamma_1,\ldots,\gamma_n)} \prod_{i=1}^{n} \psi(\gamma_i) \text{ where}$$

$$C(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \{\text{edges and half-edges that form walks between } b_i^{o_i}\text{'s and loops}\}$$

$$\psi(w : \alpha^o \rightsquigarrow \beta^{o'}) = i\frac{\sqrt{o'}}{\sqrt{o}} e^{-\frac{iW(w)}{2}} \,\mathrm{Sheet}(w) = \phi(w)\,\mathrm{Sheet}(w)$$

$\langle\gamma_1, \ldots, \gamma_n\rangle$: admissible choice of walks from γ in C

$\#L(w, a)$: number of loops (a loop has to be connected and without crossings) in w around a

$\mathrm{Sheet}(w)$: 1 if $w : \alpha^o \rightsquigarrow \beta^{o'}$ lifted to the double cover starting at α ends at β, -1 otherwise

$c(\gamma_1, \ldots, \gamma_n)$: crossing signature of b_i's with respect to $\gamma_1, \ldots, \gamma_n$

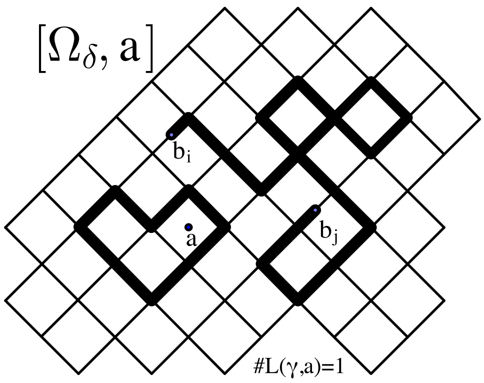

FIGURE 3.2.
An admissible collection of walks and loops, with a walk from b_i to b_j. As before the winding of the walk is 2π. The loop number of the above collection of walks and loops is 1 since there is precisely one loop with a in its interior.

Here, we define the partition functions

$$\begin{aligned}
\mathsf{Z}_{\Omega_\delta} &= \sum_{\gamma\in C} \alpha^{\#\gamma} \\
\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a] &= \sum_{\gamma\in C} \alpha^{\#\gamma}(-1)^{\#L(\gamma,a)} \\
&= \mathbb{E}_{\Omega_\delta}[\sigma_a]\mathsf{Z}_{\Omega_\delta} \\
\mathsf{Z}^{\{\dots\}}_{[\Omega_\delta,a]}[\sigma_a] &= \sum_{\gamma\in C^{\{\dots\}}} \alpha^{\gamma}(-1)^{\#L(\gamma,a)}.
\end{aligned}$$

It has been proven in [CHI13] that $(-1)^{c(\gamma_1,\dots,\gamma_n)}\prod_{i=1}^n \phi(\gamma_i)$ is well-defined for various admissible choices of walks.

Proposition 3.13. *For any two admissible choice of walks $\langle\gamma_1,\dots,\gamma_n\rangle$ and $\langle\tilde{\gamma_1},\dots,\tilde{\gamma_n}\rangle$ from γ, we have*

$$(-1)^{\#L(\gamma\setminus\bigcup_i\gamma_i,a)}\prod_{i=1}^n \mathrm{Sheet}(\gamma_i) = (-1)^{\#L(\gamma\setminus\bigcup_i\tilde{\gamma_i},a)}\prod_{i=1}^n \mathrm{Sheet}(\tilde{\gamma_i})$$

and thus $F_{[\Omega_\delta,a]}$ is well-defined.

Proof. The proof of this theorem is a lengthy set-theoretic proof that we relegate to the Appendix. See Appendix B.2 for the proof of well-definedness of the multi-point fermionic spinor. □

Definition 3.14. Given a collection of doubly oriented points $(b_1^{o_1},\dots,b_{2n}^{o_{2n}})$, a collection of signed edges $e_1,\dots,e_m$ disjoint from $(b_1^{o_1},\dots,b_{2n}^{o_{2n}})$, define the restricted real fermionic spinor as

$$F^{\{e_1^{s_1},\dots,e_m^{s_m}\}}_{[\Omega_\delta,a]}(b_1^{o_1},\dots,b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]}\sum_{\gamma\in C^{\{e_1^{s_1},\dots,e_m^{s_m}\}}(b_1^{o_1},\dots,b_{2n}^{o_{2n}})} \alpha^{\#\gamma}(-1)^{\#L(\gamma\setminus\cup\gamma_i,a)}(-1)^{c(\gamma_1,\dots,\gamma_n)}\prod_{i=1}^n\psi(\gamma_i).$$

Further, given a collection of doubly oriented points (...), a collection of signed edges {...} disjoint from (...) and edges $e_1,\dots,e_m$, define the fused real fermionic spinor $F^{[e_1,\dots,e_m]\{\dots\}}_{[\Omega_\delta,a]}(\dots)$ inductively as

$$F^{[e_1,\dots,e_m]\{\dots\}}_{[\Omega_\delta,a]}(\dots) = F^{[e_1,\dots,e_{m-1}]\{\dots,e_m^+\}}_{[\Omega_\delta,a]}(\dots) - \frac{\tilde{\mu}}{2}F^{[e_1,\dots,e_{m-1}]\{\dots\}}_{[\Omega_\delta,a]}(\dots)$$

where $\tilde{\mu} = -i(H^+_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j) + H^-_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j))$.

Remark 3.15. As has been shown in [Hon10], a fused observable $F^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}(b_1^{o_1},\dots,b_{2n}^{o_{2n}})$ corresponds to a $2n+2m$ unfused observable where the $2m$ points are merged pairwise together at the edges $e_1,\dots,e_m$.

Proposition 3.16. *Consider the critical Ising model with plus boundary conditions. Let $e_1, ..., e_m$ be a set of possibly adjacent interior edges. Then we have*

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1) \cdot ... \cdot \epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]} = (-1)^m 2^m F^{[e_1,...,e_m]}_{[\Omega_\delta,a]}.$$

Proof. We leave the proof of this to Appendix B.2 as it is simply a generalization of [Hon10] to the multipoint spinor. □

Definition 3.17. For a collection $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be doubly oriented medial vertices, a medial vertex b_{2n} and a configuration $\gamma \in C(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$, let us denote by $\mathsf{W}_{[\Omega_\delta,a]}(\gamma, o_1, ..., o_{2n-1})$ the complex weight of γ, defined as

$$\mathsf{W}_{[\Omega_\delta,a]}(\gamma, o_1, ..., o_{2n-1}) = \frac{i}{\sqrt{o_{2n}}} \alpha^{\#\gamma} (-1)^{\#L(\gamma \setminus \cup \gamma_i, a)} (-1)^{c(\gamma_1,...,\gamma_n)} \prod_{i=1}^{n} \psi(\gamma_i)$$

for any branch choice of o_{2n}. From here we can define the *complex fermionic observable*

$$H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}) = \frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]} \sum_{\gamma \in C^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}, b_{2n})} \mathsf{W}_{[\Omega_\delta,a]}(\gamma, o_1, ..., o_{2n-1}).$$

Proposition 3.18. *Let $e_1^{s_1}, ..., e_m^{s_m}$ be distinct signed edges and $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be distinct doubly oriented medial vertices, again possibly adjacent. Then the function*

$$b_{2n} \mapsto H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

whose domain is $\mathcal{V}^m_{[\Omega_\delta,a]} \setminus \{m(e_1), ..., m(e_m), b_1, ..., b_{2n}\}$ has an s-holomorphic extension to corners adjacent to two edge midpoints in the domain. It has monodromy -1 around ramification point a and obeys boundary conditions

$$\mathrm{Im}[H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}) \sqrt{v_{out}(b_{2n})}] = 0 \textit{ for all } b_{2m} \in \partial\mathcal{V}^m_{[\Omega_\delta,a]}$$

where $v_{out}(b_{2n})$ is defined as the outer normal to the boundary at b_{2n}.

Proof. We leave the proof to the Appendix B.2. □

Proposition 3.19. *Let $e_1^{s_1}, ..., e_m^{s_m}$ be distinct signed edges and $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be distinct doubly oriented medial vertices. For each $j \in \{1, ..., 2n-1\}$ such that $b_j \in \mathcal{V}_{\Omega^m_\delta} \setminus \partial\mathcal{V}_{\Omega^m_\delta}$ the function*

$$b_{2n} \mapsto H^{\{e_1,...,e_m\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

has a discrete simple pole at b_j, with front and rear values given by:

$$H_j^+ = \frac{(-1)^{j+1}}{\sqrt{o_j}} F^{\{e_1^{s_1},...,e_m^{s_m}, e(b_j)^+\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$
$$H_j^- = \frac{(-1)^{j}}{\sqrt{o_j}} F^{\{e_1^{s_1},...,e_m^{s_m}, e(b_j)^-\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}).$$

Further, for each $j \in \{1, ..., 2n-1\}$ such that $a_j \in \partial\mathcal{V}^m_{\Omega_\delta}$, the function

$$b_{2n} \mapsto H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

can be extended to an s-holomorphic function at b_j by setting the value at b_j to

$$H_j^+ = \frac{1}{\sqrt{o_j}} F^{\{e_1^{s_1},...,e_m^{s_m}\}}_{[\Omega_\delta,a]}(b_1^{o_1}, ...b_{j-1}^{o_{j-1}}, b_{j+1}^{o_{j+1}}, ..., b_{2n-1}^{o_{2n-1}}).$$

Proof. We leave the proof to Appendix B.2. □

Proposition 3.20. *For distinct edges $e_1, ..., e_m$ and distinct doubly oriented medial vertices $b_1^{o_1}, ..., b_{2n}^{o_{2n}}$, for each choice of orientations $q_i \in \mathbb{O}(e_i)$ $1 \le i \le m$ we have that*

$$F^{[e_1,...,e_m]}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) = \mathrm{Pfaff}(\mathsf{A}_{[\Omega_\delta,a]}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}, b_1^{o_1}, ..., b_{2n}^{o_{2n}})$$

where we associate with e_i the medial vertex on the edge and where the $2p \times 2p$ matrix, $p = m + n$, is defined for (not necessarily distinct) doubly oriented vertices $x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}$ by

$$(\mathsf{A}_{[\Omega_\delta,a]}(x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}))_{jk} = \begin{Bmatrix} F^{[\mathsf{C}_\delta,a]}_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) & if\, x_i = x_j\, and\, \xi_i \neq \xi_j \\ F_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) & if\, x_i \neq x_j \\ 0 & else \end{Bmatrix}.$$

Proof. We leave the detailed proof of several propositions that lead to this and this proposition to Appendix B.2. This reduces all values of the multipoint fermionic spinor to a Pfaffian of values of the two-point fermionic spinor for which we have a full plane counterpart $F_{[\mathsf{C}_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j})$ with the same singularity as $F_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j})$. □

Remark 3.21. From the above Pfaffian relation on the fused fermionic spinor we can relate the expectation of energy density of edges to $\mathrm{Pfaff}(\mathsf{A}_{[\Omega_\delta,a]})$ as

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1) \cdot ... \cdot \epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]} = (-1)^m 2^m \,\mathrm{Pfaff}(\mathbf{A}_{[\Omega_\delta,a]}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}).$$

4. Convergence of Observables

We apply discrete complex analysis results mainly cited from [HoSm10, Hon10, CHI13] in this section in order to prove convergence on the discrete fermionic spinors.

4.1. Riemann Boundary Value Problems.

Proposition 4.1 ([Hon10], Proposition 48; [CHI13], Remark 2.9)**.** *Given Ω ($[\Omega, a]$), there is at most one solution k_Ω, ($k_{[\Omega,a]}$) for each of our continuous Riemann boundary value problems, which look for functions respectively satisfying given boundary data $l : \partial\Omega \to \mathbb{C}$ ($l_{[\Omega,a]} : [\partial\Omega, a] \to \mathbb{C}$ with monodromy -1):*

- *k_Ω is holomorphic in Ω; $\forall x \in \partial\Omega, [k_\Omega(x) - l(x)]\sqrt{v_{out}(x)} \in \mathbb{R}$.*
- *$k_{[\Omega,a]}$ is a holomorphic spinor in $[\Omega, a]$; $\forall x \in \partial\Omega, [k_{[\Omega,a]}(x) - l_{[\Omega,a]}]\sqrt{v_{out}(x)} \in \mathbb{R}$; $\lim_{z\to a} k_{[\Omega,a]}(z)\sqrt{z-a} = 0$.*

Definition 4.2. The *continuous fermionic observable* and *spinor* on Ω, respectively denoted h_Ω and $h_{[\Omega,a]}$, are functions with the property that

- $h_\Omega^{\mathsf{C}} := h_\Omega - h_{\mathsf{C}}$ solves the above continuous Riemann boundary value problem on Ω, with the boundary data given as $-h_{\mathsf{C}}|_{\partial\Omega}$.
- $h_{[\Omega,a]}^{[\mathsf{C},a]} := h_{[\Omega,a]} - h_{[\mathsf{C},a]}$ solves the above continuous Riemann boundary value problem on $[\Omega, a]$ with the boundary data given as $-h_{[\mathsf{C},a]}\big|_{[\partial\Omega,a]}$.

These sets of boundary conditions, being analogous to the discrete boundary value problems, specify a unique limit for our observables, which solve the discrete problems. This is also where the conformal invariance arises; if ϕ is an injective conformal map on Ω, it is easy to see that $h_{\phi(\Omega)} = h_\Omega \circ \phi^{-1}$ and $h_{[\phi(\Omega),\phi(a)]} = h_{[\Omega,a]} \circ \phi^{-1}$.

Proposition 4.3 ([Hon10], Theorem 91)**.** $\delta^{-1} H_{\Omega_\delta}^{\mathsf{C}_\delta}(a_1^{o_1}, a_2) \xrightarrow{\delta\to 0} h_\Omega^{\mathsf{C}}(a_1^{o_1}, a_2)$ *on compact subsets of* $\Omega \times \Omega$.

4.2. Analysis near the singularity.

In this section we analyze the convergence of our s-holomorphic spinors at points order δ away from the point of monodromy a. Much of the notation used in this Section is from Section 2 where we constructed the full plane fermionic spinor $H_{[\mathsf{C}_\delta,a]}$ and introduced $G_{[\mathsf{C}_\delta,a]}(z) := \delta \sum_{j=0}^{\infty} H_{[\mathsf{C}_\delta,a]}(z - 2j\delta)$ on $\mathbb{X}_\delta^\pm$, zero everywhere else, and similarly on $\mathbb{Y}_\delta^\pm$. For details of the definitions of these full plane spinors, refer to Section 2. The s-holomorphic fermionic spinor $H_{[\Omega_\delta,a]}$ is defined as in Section 3 of the paper.

Theorem 4.4. *For $v^o \in \mathcal{V}_{\mathsf{C}_1}^o$, $H_{[\Omega_\delta,a]}((a - \delta + v\delta)^o, \cdot) \to C_{v^o} h_{[\Omega,a]}$ uniformly on compact subsets away from a.*

Proof. We note that the proof of Proposition 3.9 from [CHI13] can be applied without a problem, since $a - \delta + v\delta$ scales with δ and thus for any $\epsilon > 0$ can be included in the ϵ-ball around a if δ is small enough; a tool we possibly not have is the integral $\mathbb{I}_\delta := \mathbb{I}_\delta(H_\delta)$ of the square of $H_\delta := H_{[\Omega_\delta,a]}((a - \delta + v\delta)^o, \cdot)$ defined as in Proposition 3.6 in the same paper. $\mathbb{I}_\delta$ is well-defined locally, but global well-definedness is a priori a problem because of the simple pole at $a - \delta + v\delta$ and we have to define $\mathbb{I}_\delta$ away from it, thus creating a possible monodromy around the pole. However, we note that the monodromy of $\mathbb{I}_\delta$ along $\partial\Omega_\delta$ is zero because the increment along the boundary involves precisely adding over $\mathrm{Im}(H_\delta^2 v_{out}) = 0$. Since Ω is simply connected, the only nontrivial loops when going around the pole are those homotopic to $\partial\Omega_\delta$, and since those do not contribute any monodromy, $\mathbb{I}_\delta$ is globally well-defined on any compact subset away from a.

The other potential problem in applying the technique in the proof is the following: as noted in Remark 3.8 in the same paper, subharmonicity of the integral of the square of $H_\delta^\dagger := H_{[\Omega_\delta,a]}((a-\delta+v\delta)^o,\cdot) - H_{[\mathsf{C}_\delta,a]}((a-\delta+v\delta)^o,\cdot)$, crucial in uniformly bounding the integral near a, fails at a because the spinor branches at the point. The subharmonicity at a is used twice in the proof: in extending the uniform boundedness apart from a to a, and in the proof of Lemma 3.10 in the same paper. Lemma 3.10 is easy; they already prove the Lemma for H_δ near a branch point without singularity. We can utilize the same technique for $H_\delta^\dagger$ and deduce the result for H_δ (for small enough δ, z_{max} (for H_δ) satisfies $H_\delta^\dagger(z_{max}) > \frac{1}{2}M_\delta(\epsilon)$, since $M_\delta(\epsilon)\to\infty$ and $\vartheta(\delta)^{-1}H_{[\mathsf{C}_\delta,a]}$ converges as $\delta\to 0$ (Theorem 2.13), and the argument of the proof goes through).

To show $H_\delta^\dagger$ is bounded near a, we use a simple generalization of Theorem 2.17 in [CHI13]; the proof is exactly the same as our Theorem 4.6, but it only depends on the convergence result for the one-point $H_{[\mathsf{C}_\delta,a]}$ from the same paper. The result we need states that

$$\left[H_{[\Omega_\delta,a]} - H_{[\mathsf{C}_\delta,a]}\right](a-\delta+v\delta) = 2(\operatorname{Re}\mathcal{A}G_{[\mathsf{C}_\delta,a]} + \operatorname{Im}\mathcal{A}\tilde{G}_{[\mathsf{C}_\delta,a]})(a-\delta+v\delta) + o(\delta) = O(\delta)$$

where $H_{[\Omega_\delta,a]}(\cdot)$ is a one-point spinor, defined the same to our two-point spinor but with a different choice of normalization and only with paths originating from $a+\frac{\delta}{2}$ and culminating at another, $\mathcal{A}$ (to be defined in Remark 4.5) is a $[\Omega,a]$-dependent constant, and the second equality comes from the fact that $G_{[\mathsf{C}_\delta,a]}(a-\delta+v\delta), \tilde{G}_{[\mathsf{C}_\delta,a]}(a-\delta+v\delta) = O(\delta)$ by definition. It is obvious from the defintion of the one- and two-point spinors that

$$H_\delta^\dagger(a+\frac{\delta}{2}) = \frac{i\cos(\pi/8)}{\sqrt{o}}\overline{\mathsf{P}_{\frac{1}{i}\frac{1}{\sqrt{o}}}\left[H_{[\Omega_\delta,a]} - H_{[\mathsf{C}_\delta,a]}\right](a-\delta+v\delta)} = O(\delta)$$

then $H_\delta^{\dagger\dagger} := H_\delta^\dagger - H_\delta^\dagger(a+\frac{\delta}{2})$ vanishes at $a+\frac{\delta}{2}$ and thus $\mathbb{I}_\delta(H_\delta^{\dagger\dagger})$ is subharmonic everywhere including at a by Remark 3.8 in [CHI13]. In addition, its uniform boundedness near a (which now can be proved with the technique in [CHI13], Proposition 3.9) is equivalent to uniform boundedness of $\mathbb{I}_\delta(H_\delta^\dagger)$, since $\vartheta(\delta)^{-1}H_\delta^\dagger(a+\frac{\delta}{2}) \xrightarrow{\delta\to 0} 0$. □

Remark 4.5. As in Definition 2.11 of [CHI13], we expand $h_{[\Omega,a]} = \frac{1}{\sqrt{z-a}} + 2\mathcal{A}_{[\Omega,a]}\sqrt{z-a} + O(|z-a|^{3/2})$. As in [CHI13], $\mathcal{A}_{[\Omega,a]}$ is defined to be the coefficient of the $(z-a)^{\frac{1}{2}}$ in the expansion of $h_{[\Omega,a]}$ about $z=a$. $\mathcal{A}_{[\Omega,a]}$ depends on the domain $[\Omega,a]$. However, for ease of notation, we drop the $[\Omega,a]$ and just note that the constant depends on the domain.

Theorem 4.6. For $v_1 \in \mathcal{V}^o_{\mathsf{C}_1}, v_2 \in \mathcal{V}^m_{\mathsf{C}_1}$,

$$\begin{aligned}&\left(H_{[\Omega_\delta,a]} - H_{[\mathsf{C}_\delta,a]}\right)((a-\delta+v_1\delta)^o, a-\delta+v_2\delta)\\ &- C_{v_1^o}\left(2\operatorname{Re}\mathcal{A}\cdot G_{[\mathsf{C}_\delta,a]} + 2\operatorname{Im}\mathcal{A}\cdot\tilde{G}_{[\mathsf{C}_\delta,a]}\right)(a-\delta+v_2\delta)\\ =&o(\delta)\end{aligned}$$

where $\mathcal{A}$ is defined as in Remark 4.5 and $C_{v_1^{o_1}}$ is a nonuniversal constant.

Proof. We closely follow the strategy in the Section 3.5 of [CHI13]. Note, since the spinor $\left(H_{[\Omega_\delta,a]} - H_{[\mathsf{C}_\delta,a]}\right)((a-\delta+v_1\delta)^o,\cdot) - C_{v_1^o}\left(2\operatorname{Re}\mathcal{A}\cdot G_{[\mathsf{C}_\delta,a]} + 2\operatorname{Im}\mathcal{A}\cdot\tilde{G}_{[\mathsf{C}_\delta,a]}\right)$ is s-holomorphic, by Remark 2.2 it suffices to show the asymptotic behavior not directly at $a-\delta+v_2\delta$ on $1,i$-corners $a-\delta+v\delta$ for $v\in\mathcal{V}^c_{\mathsf{C}_1}$. Write $\mathcal{R}$ for reflection across $a+\mathbb{R}$; there is a small neighborhood Λ around a in $\Omega\cap\mathcal{R}(\Omega)$. When a is the midpoint of a face, $\mathcal{R}(\mathbb{C}_\delta)=\mathbb{C}_\delta$, and thus $\Lambda_\delta = \Lambda\cap\mathbb{C}_\delta$ is naturally a subgraph of both Ω_δ and $\mathcal{R}(\Omega_\delta)$. Since for small enough δ we can assume $a-\delta+v_i\delta, \mathcal{R}(a-\delta+v_i\delta)\in\mathcal{V}^m_{\Lambda_\delta}$, we restrict our attention to Λ_δ, where $H_\delta := H_{[\Omega_\delta,a]}((a-\delta+v_1\delta)^o,\cdot)$, $H_{[\mathsf{C}_\delta,a]} := H_{[\mathsf{C}_\delta,a]}((a-\delta+v_1\delta)^o,\cdot)$, $H^{(\mathcal{R})}_{[\mathsf{C}_\delta,a]} := H_{[\mathcal{R}(\mathsf{C}_\delta),a]}(\mathcal{R}(a-\delta+v_1\delta)^o,\cdot)$, and $H^{(\mathcal{R})}_\delta := H_{[\mathcal{R}(\Omega_\delta),a]}(\mathcal{R}(a-\delta+v_1\delta)^o,\cdot)$. Now define functions s-holomorphic everywhere on $[\Lambda_\delta,a]$:

$$S_\delta := \vartheta(\delta)^{-1}\left[\frac{1}{2}(H_\delta + H_\delta^{\mathcal{R}}) - \frac{1}{2}\left(H_{[\mathsf{C}_\delta,a]} + H^{(\mathcal{R})}_{[\mathsf{C}_\delta,a]}\right) - 2C_{v_1^o}\operatorname{Re}\mathcal{A}\cdot G_{[\mathsf{C}_\delta,a]}\right]$$

$$\tilde{S}_\delta := \vartheta(\delta)^{-1}\left[\frac{1}{2}\left(H_\delta - H_\delta^{(\mathcal{R})}\right) - \frac{1}{2}\left(H_{[\mathsf{C}_\delta,a]} - H^{(\mathcal{R})}_{[\mathsf{C}_\delta,a]}\right) - 2C_{v_1^o}\operatorname{Im}\mathcal{A}\cdot\tilde{G}_{[\mathsf{C}_\delta,a]}\right].$$

By construction, we have, on $a-\mathbb{R}_+$, $H_\delta = -\overline{H_\delta^{(\mathcal{R})}}$, and on $a+\mathbb{R}_+$, $H_\delta = \overline{H_\delta^{(\mathcal{R})}}$ (the conjugation comes from the winding angles being negated, while the additional negative sign across the monodromy comes from the sheet factor). This, as well as the construction of functions $H_{[\mathsf{C}_\delta,a]}, G_{[\mathsf{C}_\delta,a]}, \tilde{G}_{[\mathsf{C}_\delta,a]}$, shows that S_δ vanishes on $(a+\mathbb{R}_+)\cap\mathcal{V}^i_{[\Lambda_\delta,a]}, (a-\mathbb{R}_-)\cap\mathcal{V}^1_{[\Lambda_\delta,a]}$, while $\tilde{S}_\delta$ vanishes on $(a-\mathbb{R}_+)\cap\mathcal{V}^i_{[\Lambda_\delta,a]}, (a+\mathbb{R}_-)\cap\mathcal{V}^1_{[\Lambda_\delta,a]}$. Now we estimate harmonic functions $S^1_\delta := S_\delta|_{\mathsf{X}^+_\delta \cap \mathcal{V}^1_{[\Lambda_\delta,a]}}$, $S^i_\delta := S_\delta|_{\mathsf{Y}^+_\delta \cap \mathcal{V}^i_{[\Lambda_\delta,a]}}$, $\tilde{S}^1_\delta := \tilde{S}_\delta|_{\mathsf{Y}^+_\delta \cap \mathcal{V}^1_{[\Lambda_\delta,a]}}$, $\tilde{S}^i_\delta := \tilde{S}_\delta|_{\mathsf{X}^+_\delta \cap \mathcal{V}^i_{[\Lambda_\delta,a]}}$ near a (combined with spinor property, estimating those restrictions will estimate the magnitude of the function on the double cover): without loss of generality we show the $o(\delta)$ estimate for $\vartheta(\delta)S^1_\delta$, since all the restricted harmonic functions are defined on a slitted planar square lattice which are isomorphic locally around a, and all the estimates go through in the other lattices exactly the same way.

Define the discrete circle $w(r) := \{z \in \text{Domain}(S^1_\delta) : r < |z-a| < r+5\delta\}$. A similar twist of the discrete Beurling estimate (Theorem 1, [LaLi04]) as Lemma 3.3, [CHI13] gives $\text{hm}^{\mathsf{X}^+_\delta}_{\{a-\delta+v\delta\}}(z) \le C\delta^{1/2}|z-a|^{-1/2}$, reversing time on which gives $\text{hm}^{\mathsf{X}^+_\delta}_{w(r)}(a-\delta+v\delta) \le C\delta^{1/2}r^{-1/2}$, an estimate at $a-\delta+v\delta$ of a harmonic function identically 1 on $w(r)$. Comparing with S^1_δ on $w(r)$ and applying maximum principle in the interior gives

$$\left|S^1_\delta\left(a-\delta+v\delta\right)\right| \le C\delta^{\frac{1}{2}}r^{-\frac{1}{2}}\sup_{w(r)}\left|S^1_\delta\right|.$$

Now,

$$\begin{aligned} S^1_\delta &\to \text{Re}\left[\frac{1}{2}(h_{[\Omega,a]} + h^{(\mathcal{R})}_{[\Omega,a]}) - \frac{1}{2}(C_{v^o_1} + C_{\overline{v^o_1}})\frac{1}{\sqrt{z-a}}\right] - 2\,\text{Re}\,\mathcal{A}\,\text{Re}\sqrt{z-a} \\ &= O(|z-a|^{3/2}) \end{aligned}$$

where we use the fact that $h^{(\mathcal{R})}_{[\Omega,a]} = \overline{h_{[\Omega,a]} \circ \mathcal{R}}$ since the right hand side solves the boundary value problem. So we have (knowing $\vartheta(\delta) \sim \delta^{1/2}$ when δ is small)

$$\vartheta(\delta)\left|S^1_\delta(a-\delta+v\delta)\right| \le C\delta O(r)$$

and since r is arbitrary we have $\vartheta(\delta)S^1_\delta(a-\delta+v\delta) = o(\delta)$ as $\delta \to 0$. □

Corollary 4.7. For $v_1 \in \mathcal{V}^o_{\mathsf{C}_1}, v_2 \in \mathcal{V}^m_{\mathsf{C}_1}$,

$$\begin{aligned} &\left(F_{[\Omega_\delta,a]} - F_{[\mathsf{C}_\delta,a]}\right)((a-\delta+v_1\delta)^{o_1}, (a-\delta+v_2\delta)^{o_2}) \\ &- C_{v_1^{o_1}}\left(2\,\text{Re}\,\mathcal{A}\cdot G_{[\mathsf{C}_\delta,a]} + 2\,\text{Im}\,\mathcal{A}\cdot \tilde{G}_{[\mathsf{C}_\delta,a]}\right)(a-\delta+v_2\delta) \\ =& o(\delta) \end{aligned}$$

where $C_{v_1^{o_1}}$ is a nonuniversal constant that depends on $v_1^{o_1}$ and $\mathcal{A}$ is defined as before.

Proof. Note by definition of $F_{[\Omega_\delta,a]}$ for a two point function it is only a multiple of $-i\sqrt{o_2}$ of $H_{[\Omega_\delta,a]}$. We simply absorb this constant into $C_{v_1^{o_1}}$ so that Corollary 4.7 follows immediately from Theorem 4.6. This will be the convergence result we use in Section 5 to prove Theorems 1.1 and 1.2. □

5. Proof of Theorems 1.1 and 1.2

5.1. **Proof of Theorem 1.1: Spin-Symmetric Pattern Probabilities.** In this section we complete the proof of our main results beginning with Theorem 1.1 on conformal invariance of spin-symmetric pattern probabilities. We use Pfaffian relations on our discrete holomorphic obsevables from Section 3 and the convergence results from [Hon10].

In order to do this we first introduce some simple notation for the Pfaffian of a $2m \times 2m$ matrix.

Remark 5.1 (Pfaffian Notation). Call the partition of $\{1, ..., 2m\}$ into pairs $\{i_k, j_k\}_k$, $\pi \in \Pi$ where Π is the set of all $(2m-1)!!$ partitions. $\text{sgn}(\pi)$ will be the sign of the associated permutation mapping 1 to i_1, 2 to j_1, 3 to i_2, 4 to j_2 and so on. Then the definition of the Pfaffian is given as

$$\text{Pfaff}(A) = \sum_{\pi\in\Pi} \text{sgn}(\pi)\prod_{k=1}^m a_{i_k j_k}$$

Each of the $a_{i_k j_k}$ is a function of the pair $(x^{\xi_{i_k}}_{i_k}, x^{\xi_{j_k}}_{j_k})$. Call $\Pi^0 \subset \Pi$ the subset of permutations where for all k, $x_{i_k} \neq x_{j_k}$ and $\Pi^1 \subset \Pi$ the subset of permutations with exactly one k such that $x_{i_k} = x_{j_k}$.

Proposition 5.2. *Given a collection of edges $e_1, ..., e_m$ with double orientations $q_1, ..., q_m$,*

$$\mathbb{E}_{\Omega_\delta}[\epsilon(e_1)\cdot ...\cdot \epsilon(e_m)] = (-1)^m 2^m\,\text{Pfaff}(\mathsf{A}_{\Omega_\delta}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}))$$

with the matrix $\mathsf{A}_{\Omega_\delta}$ *defined as before*

$$(\mathsf{A}_{\Omega_\delta}(x_1^{\xi_1},...,x_{2p}^{\xi_{2p}}))_{i,j} = b_{ij} = \begin{cases} F_{\Omega_\delta}^{\mathsf{C}_\delta}(x_i^{\xi_i}, x_j^{\xi_j}) & if x_i = x_j \, and \xi_i \neq \xi_j \\ F_{\mathsf{C}_\delta}(x_i^{\xi_i}, x_j^{\xi_j}) + F_{\Omega_\delta}^{\mathsf{C}_\delta}(x_i^{\xi_i}, x_j^{\xi_j}) & if \, x_i \neq x_j \\ 0 & else \end{cases}$$

Proof. This follows immediately from Remark 3.10. □

Proposition 5.3. *For a given set of adjacent edges* $e_1, ..., e_m$ *centered about the face* a*, as before call*

$$(x_1^{\xi_1}, ..., x_{2m}^{\xi_{2m}}) = (e_1^{q_1}, ..., e_m^{q_m}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}).$$

Then

$$\mathbb{E}_{\Omega_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)] \xrightarrow[\Omega_\delta \to \mathsf{C}_\delta]{} (-1)^m 2^m \sum_{\pi \in \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m F_{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}),$$

$$\frac{1}{\delta}\mathbb{E}_{\Omega_\delta}^{\mathsf{C}_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)] \xrightarrow[\delta \to 0]{} (-1)^m 2^m \sum_{\pi \in \Pi^0 \cup \Pi^1} \text{sgn}(\pi) \sum_{k=1}^m f_\Omega^{\mathsf{C}}(a^o, a^{(i\sqrt{o})^2}) \prod_{k^\square \in \{1,...,n\} \setminus \{k\}} F_{\mathsf{C}_\delta}(x_{i_{k^\square}}^{\xi_{i_{k^\square}}}, x_{j_{k^\square}}^{\xi_{j_{k^\square}}}),$$

where $\mathbb{E}_{\Omega_\delta}^{\mathsf{C}_\delta}[...] = \mathbb{E}_{\Omega_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)] - \mathbb{E}_{\mathsf{C}_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)]$ *mirroring earlier notation on our fermionic observables.*

Proof. We prove these relations by expanding the Pfaffian of the matrix $\mathsf{A}_{\Omega_\delta}$.

$$\begin{aligned} \text{Pfaff}(\mathsf{A}_{\Omega_\delta}(...)) &= \sum_{\pi \in \Pi} \text{sgn}(\pi) a_{i_1 j_1} \cdot ... \cdot a_{i_m j_m} \\ &= \sum_{\pi \in \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m a_{i_k j_k} + \sum_{\pi \in \Pi \setminus \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m a_{i_k j_k} \\ &= \sum_{\pi \in \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m F_{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) + \\ &+ \sum_{\pi \in \Pi^0 \cup \Pi^1} \text{sgn}(\pi) \sum_{k=1}^m F_{\Omega_\delta}^{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) \prod_{k^\square \in \{1,...,n\} \setminus \{k\}} F_{\mathsf{C}_\delta}(x_{i_{k^\square}}^{\xi_{i_{k^\square}}}, x_{j_{k^\square}}^{\xi_{j_{k^\square}}}) \end{aligned}$$

The first convergence is immediate by taking $\Omega_\delta \to \mathbb{C}_\delta$ so terms of $F_{\Omega_\delta}^{\mathsf{C}_\delta}$ vanish. Now to get the second statement, we notice that

$$\sum_{\pi \in \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m F_{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) = \text{Pfaff}(\mathsf{A}_{\mathsf{C}_\delta}(...))$$

and thus

$$(-1)^m 2^m \sum_{\pi \in \Pi^0} \text{sgn}(\pi) \prod_{k=1}^m F_{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) = \mathbb{E}_{\mathsf{C}_\delta}[\epsilon(e_1) \cdot ... \cdot \epsilon(e_m)].$$

We then subtract this term, the infinite-volume limit, from both sides and renormalize by dividing by δ. Taking $\delta \to 0$, gives $x_{i_k}, x_{j_k} \to a$ and $\frac{1}{\delta} F_{\Omega_\delta}^{\mathsf{C}_\delta}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) \xrightarrow[\delta \to 0]{} f_\Omega^{\mathsf{C}}(a^o, a^{(i\sqrt{o})^2})$. The sum over all partitions times the sign of the partition is independent of the choice of orientation at a. □

Proposition 5.4. *Let* $\mathcal{B}$ *be a base diagram centered about point* a *in* Ω*. Here we use the notation* $\epsilon(\mathcal{B})$ *to denote the product of energy densities of edges in the base diagram* $\mathcal{B}$ *centered about* a*. Then*

$$\mathbb{E}_{\Omega_\delta}[\epsilon(\mathcal{B})] \xrightarrow[\Omega_\delta \to \mathsf{C}_\delta]{} \mathbb{E}_{\mathsf{C}_\delta}[\epsilon(\mathcal{B})]$$

$$\frac{1}{\delta}(\mathbb{E}_{\Omega_\delta}[\epsilon(\mathcal{B})] - \mathbb{E}_{\mathsf{C}_\delta}[\epsilon(\mathcal{B})]) \xrightarrow[\delta \to 0]{} \langle a^o, \mathcal{B} \rangle_\Omega,$$

where given a conformal map $\varphi : \Omega \to \varphi(\Omega)$*,* $\langle a^o, B \rangle_\Omega$ *defined as*

$$\langle a^o, \mathcal{B} \rangle_\Omega := \sum_{\pi \in \Pi^0 \cup \Pi^1} \text{sgn}(\pi) \sum_{k=1}^m f_\Omega^{\mathsf{C}}(a^o, a^{(i\sqrt{o})^2}) \prod_{k^\square \in \{1,...,n\} \setminus \{k\}} F_{\mathsf{C}_\delta}(x_{i_{k^\square}}^{\xi_{i_{k^\square}}}, x_{j_{k^\square}}^{\xi_{j_{k^\square}}}),$$

satisfies

$$\langle a^o, \mathcal{B}\rangle_\Omega = |\varphi'(a)|\langle \varphi(a)^{\tilde{o}}, \mathcal{B}\rangle_{\varphi(\Omega)},$$

where $\tilde{o}_a = (\sqrt{\varphi'(a)} \cdot \sqrt{o_a})^2$.

Proof. The first convergence result was shown in the proof of the previous proposition.

For the second convergence result, we note that $F_{\mathsf{C}_\delta}(x_{i_{k'}}, x_{j_{k'}})$ is a full plane fermionic observable and thus only depends on the relative positions of the edges, or the base diagram $\mathcal{B}$ we are examining. Hence $\langle a, \mathcal{B}\rangle$ is solely a function of the location of a and the base diagram $\mathcal{B}$ centered at a.

The conformal invariance follows directly from the fact that

$$\begin{aligned}
&\sum_{\pi\in\Pi^0\cup\Pi^1} \operatorname{sgn}(\pi)\sum_{k=1}^{m} f_\Omega^{\mathsf{C}}(a^o, a^{(i\sqrt{o})^2}) \prod_{k'\in\{1,\dots,n\}\setminus\{k\}} F_{\mathsf{C}_\delta}(x_{i_{k'}}^{\xi_{i_{k'}}}, x_{j_{k'}}^{\xi_{j_{k'}}})\\
&= f_\Omega^{\mathsf{C}}(a^o, a^{(i\sqrt{o})^2}) \sum_{\pi\in\Pi^0\cup\Pi^1} \operatorname{sgn}(\pi)\sum_{k=1}^{m} \prod_{k'\in\{1,\dots,n\}\setminus\{k\}} F_{\mathsf{C}_\delta}(x_{i_{k'}}^{\xi_{i_{k'}}}, x_{j_{k'}}^{\xi_{j_{k'}}})\\
&= |\varphi'(a)| f_{\varphi(\Omega)}^{\mathsf{C}}(\varphi(a)^{\tilde{o}}, \varphi(a)^{(i\sqrt{\tilde{o}})^2}) \sum_{\pi\in\Pi^0\cup\Pi^1} \operatorname{sgn}(\pi)\sum_{k=1}^{m} \prod_{k'\in\{1,\dots,n\}\setminus\{k\}} F_{\mathsf{C}_\delta}(x_{i_{k'}}^{\xi_{i_{k'}}}, x_{j_{k'}}^{\xi_{j_{k'}}})
\end{aligned}$$

which is a result of the conformal covariance relation on f_Ω^{C}, Proposition 92 in [Hon10]. □

Theorem (Restatement of Theorem 1.1)**.** *Given B, a connected spin-symmetric pattern on the base diagram $\mathcal{B}$ centered about point a in Ω,*

$$\mathbb{P}_{\Omega_\delta}[B] \xrightarrow[\Omega_\delta\to\mathsf{C}_\delta]{} \mathbb{P}_{\mathsf{C}_\delta}[B]$$

$$\frac{1}{\delta}(\mathbb{P}_{\Omega_\delta}[B] - \mathbb{P}_{\mathsf{C}_\delta}[B]) \xrightarrow[\delta\to 0]{} \langle\langle a, B\rangle\rangle_\Omega,$$

where $\langle\langle a, B\rangle\rangle$ is a linear combination $\sum_{i=1}^{2^{n-1}} D_i\langle a, \mathcal{B}_i\rangle_\Omega$ where $\mathcal{B}_1, ..., \mathcal{B}_{2^{n-1}}$ are 2^{n-1} subdiagrams of $\mathcal{B}$ ($n := |\mathcal{F}(\mathcal{B})|$). Given a conformal map $\varphi : \Omega \to \varphi(\Omega)$, $\langle\langle a, B\rangle\rangle$ satisfies

$$\langle\langle a, B\rangle\rangle_\Omega = |\varphi'(a)|\langle\langle \varphi(a), B\rangle\rangle_{\varphi(\Omega)}.$$

Proof. We have a matrix $\mathcal{M}$ that when multiplied by a matrix of normalized energy densities of subdiagrams of $\mathcal{B}$, $\mathcal{B}_1, ..., \mathcal{B}_{2^{n-1}}$ denoted here by $\{\mathsf{E}_{\Omega_\delta}[a, \epsilon(\mathcal{B}_i)]\}$, gives probabilites of all 2^{n-1} possible spin-symmetric diagrams B on base diagram $\mathcal{B}$. The choice of what subdiagrams, of the 2^m possible subdiagrams to use is defined along with an explicit construction of the matrix $\mathcal{M}$ in Appendix C.

For the statement, multiply the matrix $\mathcal{M}$ by both sides of the first convergence result of the previous proposition.

For the second convergence result we multiply $\mathcal{M}$ by the difference of the domain expectation matrix and the full plane expectation matrix. Denote this matrix by $\{\mathsf{E}_{\Omega_\delta}^{\mathsf{C}_\delta}[a, \epsilon(\mathcal{B}_i)]\}$. Since all the subdiagrams of $\mathcal{B}$ are also centered about a as $\delta \to 0$, we have that for some k, l

$$\begin{aligned}
\frac{1}{\delta}(\mathbb{P}_{\Omega_\delta}[B] - \mathbb{P}_{\mathsf{C}_\delta}[B]) =& \frac{1}{\delta}(\mathcal{M}\{\mathsf{E}_{\Omega_\delta}^{\mathsf{C}_\delta}[a, \epsilon(\mathcal{B}_i)]\})_{kl}\\
&\xrightarrow[\delta\to 0]{} (\sum_{i=1}^{2^{n-1}} D_i\langle a, \mathcal{B}_i\rangle_\Omega) = \langle\langle a, B\rangle\rangle_\Omega\\
&= |\varphi'(a)|(\sum_{i=1}^{2^{n-1}} D_i\langle \varphi(a), \mathcal{B}_i\rangle_{\varphi(\Omega)})\\
&= |\varphi'(a)|\langle\langle \varphi(a), B\rangle\rangle_{\varphi(\Omega).}
\end{aligned}$$

□

This completes the proof of the first part of our main result, the conformal invariance of probabilities of spin-symmetric patterns in the planar Ising model.

5.2. **Proof of Theorem 1.2: Spin-Sensitive Pattern Probabilities.** We now proceed to the proof of the general case of spin-sensitive patterns in the planar Ising model. The proof of Theorem 1.2 will be similar to the proof of Theorem 1.1 except it will use results on the discrete fermionic spinor with monodromy at a and new convergence results from Section 4.

Proposition 5.5. *Given a collection of edges $e_1, ..., e_m$ and double orientations $q_1, ..., q_m \in \mathbb{O}^2$ and a face a in Ω_δ, we have*

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_a \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]} = (-1)^m 2^m \operatorname{Pfaff}(\mathsf{A}_{[\Omega_\delta,a]}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}),$$

where the matrix $\mathsf{A}_{[\Omega_\delta,a]}$ is defined as before as

$$(\mathsf{A}_{[\Omega_\delta,a]}(x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}))_{i,j} = a_{ij} = \begin{Bmatrix} F^{[\mathsf{C}_\delta,a]}_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) & if\, x_i = x_j \, and \, \xi_i \neq \xi_j \\ F_{[\mathsf{C}_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) + F^{[\mathsf{C}_\delta,a]}_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) & if\, x_i \neq x_j \\ 0 & else \end{Bmatrix}.$$

Proof. This follows immediately from Remark 3.21. $\square$

Proposition 5.6. *For a given set of adjacent edges $e_1, ..., e_m$ centered about the face a in Ω_δ, as before call*

$$(x_1^{\xi_1}, ..., x_{2m}^{\xi_{2m}}) = (e_1^{q_1}, ..., e_m^{q_m}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}).$$

Then

$$\delta^{-\frac{1}{8}}(\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)]) \xrightarrow[\delta\to 0]{} \langle\sigma_a\rangle_\Omega \cdot (-1)^m 2^m \sum_{\pi\in\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m F_{[\mathsf{C}_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}),$$

where $\langle\sigma_a\rangle_\Omega = 2^{\frac{1}{4}} \operatorname{rad}(a,\Omega)^{-\frac{1}{8}}$. Furthermore, we have the higher order convergence result

$$\delta^{-\frac{9}{8}}(\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1) ... \cdot \epsilon(e_m)] - \langle\sigma_a\rangle_\Omega \lim_{\Omega\to\mathsf{C}} \frac{\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]}) \xrightarrow[\delta\to 0]{}$$

$$\xrightarrow[\delta\to 0]{} \langle\sigma_a\rangle_\Omega \cdot (-1)^m 2^m \sum_{\pi\in\Pi^0\cup\Pi^1} \operatorname{sgn}(\pi) \sum_{k=1}^m C_{i_k,j_k} Re\mathcal{A} + \tilde{C}_{i_k j_k} Im\mathcal{A} \prod_{k^\square\in\{1,...,n\}\setminus\{k\}} F_{[\mathsf{C}_\delta,a]}(x_{i_{k^\square}}^{\xi_{i_{k^\square}}}, x_{j_{k^\square}}^{\xi_{j_{k^\square}}}).$$

Proof. Just as in the spin-symmetric case, we expand the Pfaffian to find the delta dependence of the expectation of the energy density.

$$\begin{aligned}
\operatorname{Pfaff}(\mathsf{A}_{[\Omega_\delta,a]}(...)) &= \sum_{\pi\in\Pi} \operatorname{sgn}(\pi) a_{i_1 j_1} \cdot ... \cdot a_{i_m j_m} \\
&= \sum_{\pi\in\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m a_{i_k j_k} + \sum_{\pi\in\Pi\setminus\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m a_{i_k j_k} \\
&= \sum_{\pi\in\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m F_{[\mathsf{C}_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) + \\
&+ \sum_{\pi\in\Pi^0\cup\Pi^1} \operatorname{sgn}(\pi) \sum_{k=1}^m F^{[\mathsf{C}_\delta,a]}_{[\Omega_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}) \prod_{k^\square\in\{1,...,n\}\setminus\{k\}} F_{[\mathsf{C}_\delta,a]}(x_{i_{k^\square}}^{\xi_{i_{k^\square}}}, x_{j_{k^\square}}^{\xi_{j_{k^\square}}})
\end{aligned}$$

First we take $\Omega_\delta \to \mathbb{C}_\delta$ to get

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]} \xrightarrow[\Omega_\delta\to\mathsf{C}_\delta]{} (-1)^m 2^m \sum_{\pi\in\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m F_{[\mathsf{C}_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}).$$

From [CHI13] we have that

$$\delta^{-\frac{1}{8}} \mathbb{E}_{\Omega_\delta}[\sigma_a] \xrightarrow[\delta\to 0]{} \langle\sigma_a\rangle_\Omega = 2^{\frac{1}{4}} \operatorname{rad}(a,\Omega)^{-\frac{1}{8}}.$$

This implies

$$\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)] \xrightarrow[\Omega_\delta\to\mathsf{C}_\delta]{} 0,$$

$$\delta^{-\frac{1}{8}}(\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(e_1)\cdot ... \cdot \epsilon(e_m)]) \xrightarrow[\delta\to 0]{} \langle\sigma_a\rangle_\Omega (-1)^m 2^m \sum_{\pi\in\Pi^0} \operatorname{sgn}(\pi) \prod_{k=1}^m F_{[\mathsf{C}_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}).$$

For the higher order convergence result, first note from Corrolary 4.7 that as$\delta \to 0$, $x_i \to a$, $F^{[\mathsf{C}_\delta,a]}_{[\Omega_\delta,a]}(x_i^{\xi_i}, x_j^{\xi_j}) = C_{ij}(2\mathrm{Re}\mathcal{A}G_{[\mathsf{C}_\delta,a]} + \mathrm{Im}\mathcal{A}\tilde{G}_{[\mathsf{C}_\delta,a]})(x_i^{\xi_i})$. Then by definition of $G_{[\mathsf{C}_\delta,a]}(x_i) \xrightarrow[\delta\to 0]{} \delta C_{i,j}$ where the indices i,j indicate that the constant depends on the location of x_i and x_j. We have a similar formulation for the scaling limit of $\tilde{G}_{[\mathsf{C}_\delta,a]}$. We absorbed the full plane observables from $G_{[\mathsf{C}_\delta,a]}$, $\tilde{G}_{[\mathsf{C}_\delta,a]}$ into the constants since we focus on how they scale as $\delta \to 0$. Then subtracting the equation for the $\delta^{\frac{1}{8}}$-dependence of the expectation from both sides and dividing by δ yields the higher order convergence result. □

Proposition 5.7. *Consider the base diagram $\mathcal{B}$ centered about a face a in Ω. Given a conformal map $\varphi : \Omega \to \varphi(\Omega)$,*

$$\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(\mathcal{B})] \xrightarrow[\Omega_\delta\to\mathsf{C}_\delta]{} 0 \xleftarrow[\Omega_\delta\to\mathsf{C}_\delta]{} \mathbb{E}_{\varphi(\Omega)_\delta}[\sigma_{\varphi(a)} \cdot \epsilon(\mathcal{B})]$$
$$\delta^{-\frac{1}{8}}(\mathbb{E}_{\Omega_\delta}[\sigma_a \cdot \epsilon(\mathcal{B})]) \xrightarrow[\delta\to 0]{} C_{\frac{1}{8}}\langle a, \mathcal{B}\rangle'_\Omega$$
$$= C_{\frac{1}{8}}|\varphi'(a)|^{\frac{1}{8}}\langle \varphi(a), \mathcal{B}\rangle'_{\varphi(\Omega)}$$

where

$$\langle a, \mathcal{B}\rangle'_\Omega := \langle \sigma_a\rangle_\Omega(-1)^m 2^m \sum_{\pi\in\Pi^0} \mathrm{sgn}(\pi)\prod_{k=1}^m F_{[\mathsf{C}_\delta,a]}(x_{i_k}^{\xi_{i_k}}, x_{j_k}^{\xi_{j_k}}).$$

Proof. The first statement follows immediately from the previous proposition.

The next order conformal covariance result follows from the conformal covariance relation on $\langle\sigma_a\rangle_\Omega$ from [CHI13]:

$$\langle\sigma_a\rangle_\Omega = |\varphi'(a)|^{\frac{1}{8}}\langle\sigma_{\varphi(a)}\rangle_{\varphi(\Omega)}.$$

□

Theorem (Restatement of Theorem 1.2)**.** *Given B, a spin-symmetric pattern on the base diagram $\mathcal{B}$ centered about point a in Ω and a spin $\sigma_a \in \{\pm 1\}$ at a,*

$$\mathbb{P}_{\Omega_\delta}[B, \sigma_a] \xrightarrow[\Omega_\delta\to\mathsf{C}_\delta]{} \mathbb{P}_{\mathsf{C}_\delta}[B, \sigma_a]$$
$$\delta^{\frac{1}{8}}(\mathbb{P}_{\Omega_\delta}[B, \sigma_a] - \mathbb{P}_{\mathsf{C}_\delta}[B, \sigma_a]) \xrightarrow[\delta\to 0]{} \langle\langle a, [B, \sigma_a]\rangle\rangle'_\Omega,$$

where $\langle\langle a, [B, \sigma_a]\rangle\rangle'$ is a linear combination of spin-symmetric and spin-sensitive expectations of energy densities on subdiagrams of $\mathcal{B}$ such that given a conformal map $\varphi : \Omega \to \varphi(\Omega)$,

$$\langle\langle a, [B, \sigma_a]\rangle\rangle'_\Omega = |\varphi'(a)|^{\frac{1}{8}}\langle\langle \varphi(a), [B, \sigma_{\varphi(a)}]\rangle\rangle'_{\varphi(\Omega)}.$$

Proof. We have a matrix $\mathcal{M}$ that acts on a matrix of 2^{n-1} spin-sensitive energy densities and 2^{n-1} spin-symmetric energy densities of subdiagrams of $\mathcal{B}$, $\mathcal{B}_1, ..., \mathcal{B}_{2^{n-1}}$ denoted here by $\{\mathsf{E}_{\Omega_\delta}[a, \sigma_a \cdot \epsilon(\mathcal{B}_i), \epsilon(\mathcal{B}_i)]\}$ and gives probabilites of all 2^n possible spin-sensitive diagrams B on base diagram $\mathcal{B}$. Note again that $n := |\mathcal{F}(\mathcal{B})|$. A proof of this and an explicit construction of the matrix can be found in Appendix C.

For the first convergence result multiply the matrix by the first convergence result of the combination of the previous proposition and the first convergence result of Theorem 1.1. Since the spin-sensitive probability is a linear combination of spin-symmetric and spin-sensitive expectations both of which converge to their full plane counterparts, the spin-sensitive probability also converges to its full plane counterpart, $\mathbb{P}_{\mathsf{C}_\delta}[B, \sigma_a]$.

For the second convergence result we multiply it by the difference of the domain expectation matrix and the full plane expectation matrix (infinite-volume limit). Denote new this matrix by $\{\mathsf{E}^{\mathsf{C}_\delta}_{\Omega_\delta}[a, \sigma_a \cdot \epsilon(\mathcal{B}_i), \epsilon(\mathcal{B}_i)]\}$. Since all the subdiagrams of $\mathcal{B}$ are also centered about a as $\delta \to 0$, we have that for any spin-sensitive pattern $[\sigma_a = \pm, B]$ for some k, l

$$\delta^{-\frac{1}{8}}(\mathbb{P}^{\mathsf{C}_\delta}_{\Omega_\delta}[\sigma_a = \pm, B]) = \delta^{-\frac{1}{8}}(\mathcal{M}\{\mathsf{E}^{\mathsf{C}_\delta}_{\Omega_\delta}[a, \sigma_a \cdot \epsilon(\mathcal{B}_i), \epsilon(\mathcal{B}_i)]\})_{kl}$$
$$\xrightarrow[\delta\to 0]{} \sum_{i=1}^{2^{n-1}} D'_i\langle a, \mathcal{B}_i\rangle'_\Omega = \langle\langle a, B\rangle\rangle'_\Omega$$
$$= |\varphi'(a)|^{\frac{1}{8}}\langle\langle\varphi(a), B\rangle\rangle'_{\varphi(\Omega)}$$

for any conformal map $\varphi : \Omega \to \varphi(\Omega)$.

The convergence on the second line follows from the fact that for any spin-symmetric pattern, $\mathbb{E}_{\Omega_\delta}^{\mathsf{C}_\delta}[a,B] \xrightarrow[\delta\to 0]{} 0$ and its lowest order convergence term is δ so even after normalization by $\delta^{\frac{1}{8}}$ all higher order terms vanish. We are only left with contributions from spin-sensitive energy densities which have the same conformal covariance relation (multiplication by $|\varphi'(a)|^{\frac{1}{8}}$ which yields conformal invariance of spin-sensitive pattern probabilities. □

Appendix A. Full Plane Fermionic Spinor and Harmonic Measure

We now explicitly construct the discrete harmonic measure in the slit plane, using Fourier analysis analysis techniques (see also [CHI14]):

Proposition A.1. *The harmonic measure H_0, defined explicitly by*

$$H_0(z=s+ik) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{C^{|k|}(\theta)}{\sqrt{1-e^{-2i\theta}}} e^{-is\theta} d\theta$$

where $C(\theta) := \dfrac{\cos\theta}{1+|\sin\theta|}$ and the square root takes the principal value, is the unique harmonic function on the discrete diagonal slit plane $\mathbb{C}_{\delta=1}\setminus\mathbb{Z}_+$ with boundary values 1 at the origin and 0 elsewhere on the cut and ∞.

Proof. Since the solution to the Dirichlet problem for the discrete Laplacian is unique, it suffices to check the boundary values and check that the given function is harmonic. On the real axis, we have $k=0$, and by the generalized binomial theorem

$$\frac{1}{\sqrt{1-e^{-2i\theta}}} = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} e^{-2ni\theta} = 1 + \frac{1}{2}e^{-2i\theta} + \frac{3}{8}e^{-4i\theta} + \frac{5}{16}e^{-6i\theta} + \ldots$$

so the s-th Fourier coefficient, which is precisely $H_0(-s)$, vanishes for odd and positive even s, and $H_0(0)=1$.

For the boundary estimates at infinity, we want to show $H_0 \to 0$ uniformly in s as $|k|\to\infty$, and vice versa. For the former, we use dominated convergence:

$$|H_0(s+ik)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|C^{|k|}(\theta)\right|}{\sqrt{|1-e^{-2i\theta}|}} d\theta \xrightarrow{|k|\to\infty} 0$$

since $\frac{\left|C^{|k|}(\theta)\right|}{\sqrt{|1-e^{-2i\theta}|}} \downarrow 0$ pointwise a.e. and $\frac{\left|C^{|k|}(\theta)\right|}{\sqrt{|1-e^{-2i\theta}|}} \le \frac{1}{\sqrt{|1-e^{-2i\theta}|}} = \frac{1}{\sqrt{2|\sin\theta|}}$, which is integrable. For the $s\to\infty$ estimate, without loss of generality we show that $\tilde{H}_0(s+ik) = \int_0^{\pi} \frac{C^{|k|}(\theta)}{\sqrt{1-e^{-2i\theta}}} e^{-is\theta} d\theta \xrightarrow{s\to\infty} 0$ uniformly in k. Note, if g is any smooth function on $[0,\pi]$, we can integrate by parts and get

$$\begin{aligned}
\tilde{H}_0(s+ik) &= \int_0^{\pi}\left(\frac{1}{\sqrt{1-e^{-2i\theta}}} - g(\theta)\right) C^{|k|}(\theta)e^{is\theta}d\theta + \left[g(\theta)C^{|k|}(\theta)\frac{e^{is\theta}}{is}\right]_0^{\pi} - \int_0^{\pi}\left[g(\theta)C^{|k|}(\theta)\right]'\frac{e^{is\theta}}{is}d\theta \\
\left|\tilde{H}_0(s+ik)\right| &\le \int_0^{\pi}\left|\frac{1}{\sqrt{1-e^{-2i\theta}}} - g(\theta)\right| d\theta + \frac{2\sup g}{|s|} + \frac{1}{|s|}\int_0^{\pi}\left|g'(\theta)C^{|k|}(\theta) + |k|g(\theta)C^{|k|-1}(\theta)C'(\theta)\right| d\theta \\
&\le \int_0^{\pi}\left|\frac{1}{\sqrt{1-e^{-2i\theta}}} - g(\theta)\right| d\theta + \frac{2\sup g}{|s|} + \frac{\pi\sup g'}{|s|} + \frac{\sup g}{|s|}\int_0^{\pi}\left|kC^{|k|-1}(\theta)C'(\theta)\right| d\theta.
\end{aligned}$$

Since smooth functions are dense in L^1, we can choose a g such that the first term becomes less than $\epsilon/2$ for a given $\epsilon>0$. In controlling the remaining terms with s, the only dependence on k is in the last term; we show it is in fact uniformly bounded:

$$\begin{aligned}
\int_0^{\pi}\left|kC^{|k|-1}(\theta)C'(\theta)\right| d\theta \le |k|\int_0^{\pi}\left|\frac{\cos\theta}{1+\sin\theta}\right|^{|k|-1} d\theta &= |k|\int_0^{\pi}\left|\tan(\frac{\pi}{4}-\frac{\theta}{2})\right|^{|k|-1} d\theta \\
&= 4|k|\int_0^{\frac{\pi}{4}} (\tan\phi)^{|k|-1} d\phi \\
&\le 4|k|\int_0^{\frac{\pi}{4}} (\tan\phi)^{|k|-1}\sec^2\phi\, d\phi = 4\left[\tan^{|k|}\phi\right]_0^{\frac{\pi}{4}} = 4,
\end{aligned}$$

where we make the substitution $\phi := \frac{\pi}{4} - \frac{\theta}{2}$.

Harmonicity is easy to prove at points with $k \neq 0$ once we notice that for those points

$$\sum_{m,n\in\{1,-1\}} H_0(s+m+i(k+n)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{C^{|k|}(\theta)e^{is\theta}}{\sqrt{1-e^{-2i\theta}}}\left(C(\theta)+\frac{1}{C(\theta)}\right)\left(e^{-i\theta}+e^{i\theta}\right)d\theta$$

$$= \frac{4}{2\pi}\int_{-\pi}^{\pi}\frac{C^{|k|}(\theta)e^{is\theta}}{\sqrt{1-e^{-2i\theta}}}d\theta = 4H_0(s+ik),$$

but for $k=0$ it reduces down to

$$\sum_{m,n\in\{1,-1\}} H_0(s+m+in) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{is\theta}}{\sqrt{1-e^{2i\theta}}}\left(2C(\theta)\right)\left(e^{-i\theta}+e^{i\theta}\right)d\theta$$

$$= \frac{4}{2\pi}\int_{-\pi}^{\pi}\frac{e^{is\theta}}{\sqrt{1-e^{-2i\theta}}}(1-|\sin\theta|)d\theta = 4H_0(s) - \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{e^{is\theta}\,|\sin\theta|}{\sqrt{1-e^{-2i\theta}}}d\theta,$$

where the last term vanishes for positive s, in other words, the following function has no negative Fourier modes:

$$\frac{|\sin\theta|}{\sqrt{1-e^{-2i\theta}}} = \sqrt{\frac{\sin^2\theta}{1-e^{-2i\theta}}} == \frac{1}{2}\sqrt{1-e^{2i\theta}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}e^{2ni\theta}.$$

□

Remark A.2. The harmonic measure on the slit plane with boundary value 1 at $m \in \mathbb{Z}^+$ and 0 elsewhere, denoted H_m, can be obtained by the recursion relation $H_m(z) = H_{m-1}(z-1) - H_{m-1}(-1)H_0(z)$, since the right hand side is again harmonic and satisfies the desired boundary conditions.

Corollary A.3. $G_{[\mathsf{C}_\delta,a]}$, *as defined in Theorem 2.14, is discrete holomorphic.*

Proof. Without loss of generality, show discrete holomorphicity at $z = a + \frac{3\delta}{2} + (s+ik)\delta \in \mathbb{X}_\delta^+ \cap \mathbb{Y}^+$ $(k \geq 0)$

$$\begin{aligned}
&\frac{1}{\delta}(G_{[\mathsf{C}_\delta,a]}(z+i\delta) - G_{[\mathsf{C}_\delta,a]}(z+\delta))\\
&= -\sum_{j=1}^{\infty}\left[H_{[\mathsf{C}_\delta,a]}(z+i\delta+2j\delta) - H_{[\mathsf{C}_\delta,a]}(z+\delta+2j\delta)\right]\\
&\overset{(A)}{=} -i\sum_{j=1}^{\infty}\left[H_{[\mathsf{C}_\delta,a]}(z+(1+i)\delta+2j\delta) - H_{[\mathsf{C}_\delta,a]}(z+2j\delta)\right]\\
&= -i\sum_{j=1}^{\infty}\left[H_0((s+1)+i(k+1)+2j) - H_0(s+ik+2j)\right]\\
&= -i\sum_{j=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C^k(\theta)e^{is\theta}}{\sqrt{1-e^{-2i\theta}}}(C(\theta)e^{i\theta}-1)e^{2ji\theta}d\theta\\
&= -i\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C^k(\theta)e^{is\theta}(C(\theta)e^{i\theta}-1)}{\sqrt{1-e^{-2i\theta}}}\cdot\frac{e^{2i\theta}(1-e^{2Ni\theta})}{1-e^{2i\theta}}d\theta\\
&\overset{(B)}{=} \frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{C^k(\theta)e^{is\theta}(C(\theta)e^{i\theta}-1)}{\sqrt{1-e^{-2i\theta}}}\cdot\frac{1}{1-e^{-2i\theta}}d\theta\\
&\overset{(B)}{=} i\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{C^k(\theta)e^{is\theta}(C(\theta)e^{i\theta}-1)}{\sqrt{1-e^{-2i\theta}}}\cdot\frac{1-e^{-2Ni\theta}}{1-e^{-2i\theta}}d\theta\\
&= i\sum_{j=0}^{\infty}\left[H_{[\mathsf{C}_\delta,a]}(z+(1+i)\delta-2j\delta) - H_{[\mathsf{C}_\delta,a]}(z-2j\delta)\right]\\
&= \frac{i}{\delta}\left[G_{[\mathsf{C}_\delta,a]}(z+(1+i)\delta) - G_{[\mathsf{C}_\delta,a]}(z)\right]
\end{aligned}$$

where we use discrete holomorphicity of $H_{[\mathsf{C}_\delta,a]}$ at (A), and Riemann-Lebesgue Lemma at (B). □

We proceed to construct and prove convergence results on the full plane spinor $H_x(z) := H_{[\mathsf{C}_\delta,a]}(x^{o_x}, z)$ for x off the slit $a+\mathbb{R}_+$. Recall that we denote by $\mathcal{R}_a$ the orthogonal reflection with respect to the axis $\{a+t : t\in\mathbb{R}\}$.

First note using oriented corners $c_1^{o_1}, c_2^{o_2} \in \mathcal{V}^{\mathrm{c}}_{[\Omega_\delta,a]}$ we can define spinors $H_{[\Omega_\delta,a]}(c_1^{o_1}, c_2) = \frac{i}{\sqrt{o_2}} F_{[\Omega_\delta,a]}(c_1^{o_1}, c_2^{o_2})$ in the manner analogous to the case where the arguments are oriented edges.

Proposition A.4. *Let $a \in \mathcal{F}_{\mathsf{C}_\delta}$ be a face of $\mathbb{C}_\delta$. For $x \in \mathcal{V}^{\mathrm{cm}}_{[\mathsf{C}_\delta,a]}$, $y \in \mathcal{V}^{\mathrm{cm}}_{[\mathsf{C}_\delta,a]} \setminus \{x\}$, we have that if $\Omega_\delta^{(1)} \subset \Omega_\delta^{(2)} \subset \cdots$ is a sequence of domains with $\mathcal{R}_a\left[\Omega_\delta^{(k)}\right] = \Omega_\delta^{(k)}$ and with $\Omega_\delta^{(n)} \underset{n\to\infty}{\longrightarrow} \mathbb{C}_\delta$, the limit*

$$F_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, y^{o_y}\right) := \lim_{n\to\infty} F_{\left[\Omega_\delta^{(n)},a\right]}\left(x^{o_x}, y^{o_y}\right)$$

exists and is independent of the sequence $\left(\Omega_\delta^{(n)}\right)_n$.

The function $F_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, y^{o_y}\right)$ satisfies the following antisymmetry properties:

$$-F_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, y^{o_y}\right) = F_{[\mathsf{C}_\delta,a]}\left(x^{(i\sqrt{o_x})^2}, y^{o_y}\right) = F_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, y^{(i\sqrt{o_y})^2}\right) = F_{[\mathsf{C}_\delta,a]}\left(y^{o_y}, x^{o_x}\right).$$

For $z \in \mathcal{V}^{\mathrm{c}}_{[\mathsf{C}_\delta,a]}$, define $H_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, z\right) := \frac{i}{\sqrt{o_z}} F_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, z^{o_z}\right)$ (independent of o_z choice) and extend in the usual manner to $z \in \mathcal{V}^{m}_{[\mathsf{C}_\delta,a]}$. Then $H_x(\cdot) := H_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, \cdot\right)$ satisfies the following:

(1) $H_x(\cdot)$ has monodromy -1 around a, is s-holomorphic on $[\mathbb{C}_\delta, a]$ except at x, where it has the following discrete singularity:

$$\mathsf{P}_{\tau\mathsf{R}}\left[H_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, x \pm \frac{\delta i}{2}\right)\right] = \pm\tau\sqrt{o_x}$$

(2) $H_x(z) \to 0$ as $|z| \to \infty$.

(3) For $x \in \mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}$ and $\tilde{x} = \mathcal{R}_{a+\mathsf{R}}(x) \in \mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}$ chosen on the same sheet above $\mathbb{C} \setminus \{a - t : t > 0\}$ (or on different sheets if both are above $\{a - t : t > 0\}$), set $H_{\tilde{x}} := H_{[\mathsf{C}_\delta,a]}\left(\tilde{x}^{o_x}, \cdot\right)$. Then, we have the following cancellations on $\{a + t : t \in \mathbb{R}\}$ (except at x if $x = \tilde{x}$):
   - *$H_x + H_{\tilde{x}} = 0$ on $\mathcal{V}^{1}_{[\mathsf{C}_\delta,a]} \cap \{a + t : t > 0\}$ and $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a - t : t > 0\}$*
   - *$H_x - H_{\tilde{x}} = 0$ on $\mathcal{V}^{1}_{[\mathsf{C}_\delta,a]} \cap \{a - t : t > 0\}$ and $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a + t : t > 0\}$*

(4) For $x \in \mathcal{V}^{1}_{[\mathsf{C}_\delta,a]}$ and $\tilde{x}$ as in 3., we have the same cancellations as in 3, if we replace $t > 0$ by $t < 0$.

Furthermore, $F_{[\mathsf{C}_\delta,a]}$ is uniquely determined by the antisymmetry properties and the fact that $\left(x^{o_x}, z\right) \mapsto H_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, z\right)$ satisfies the properties 1,2,3 (or equivalently 1,2,4).

Proof. To prove this proposition, we will follow the following strategy, centered around $H_{[\mathsf{C}_\delta,a]}$ (which is equivalent to constructing and studying $F_{[\mathsf{C}_\delta,a]}$)

- We first prove the uniqueness statement (Lemma A.5).
- We then prove (Lemma A.6) that $\left(H_{\left[\Omega_\delta^{(n)},a\right]}\right)_n$ is uniformly bounded and that

$$\lim_{z\to\infty} \limsup_{n\to\infty} \left|H_{\left[\Omega_\delta^{(n)},a\right]}\left(x^{o_x}, z\right)\right| = 0$$

- By symmetry arguments we check 1,2,3 for $H_{\left[\Omega_\delta^{(n)},a\right]}$ (for x, z within $\Omega_\delta^{(n)}$). We obtain that any subsequence limit of $\left(H_{\left[\Omega_\delta^{(n)},a\right]}\right)_n$ as $n \to \infty$ satisfies 1,2,3 ; by uniqueness this concludes the convergence.
- The properties for $F_{[\mathsf{C}_\delta,a]}$ are immediate for $F_{[\Omega_\delta,a]}$ and hence are obtained by passing to the limit.

□

Lemma A.5. *With the notation of Proposition A.4, the function $F_{[\mathsf{C}_\delta,a]}$ is uniquely determined by the antisymmetry properties and the fact that $\left(x^{o_x}, y\right) \mapsto H_x(z) := H_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, z\right)$ satisfies Properties 1,2,3.*

Proof. By linearity, it is enough to show that if $F^*_{[\mathsf{C}_\delta,a]}$ is the difference of two functions $F_{[\mathsf{C}_\delta,a]}$ satisfying the above properties, then $F^*_{[\mathsf{C}_\delta,a]}$ is zero. Denote by $H^*_{[\mathsf{C}_\delta,a]}$ the difference of the two corresponding functions $H_{[\mathsf{C}_\delta,a]}$: we have that $H^*_{[\mathsf{C}_\delta,a]}$ satisfies properties 1,2,3, except that it is s-holomorphic everywhere, including if $z = x$. Set $H^*_x(z) := H^*_{[\mathsf{C}_\delta,a]}\left(x^{o_x}, z\right)$ (the choice of o_x is irrelevant: we want to show $H^*_x(z) = 0$ for all x, z).

First suppose $x \in \mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a + t : t > 0\}$. We have that $H^*_x = 0$ on $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,z]} \cap \{a + t : t > 0\}$, by Property 3 (since x is its own symmetric). Hence by harmonicity of the imaginary part and assumption about the decay at infinity, we obtain that $H^*_x = 0$ (first the imaginary part is 0, but the real part also by discrete Cauchy-Riemann equations and monodromy). Now suppose $x \in \mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a - t : t > 0\}$. By Property 3 again, we have that H^*_x vanishes on $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a - t : t > 0\}$. Notice that we cannot conclude $H^*_x = 0$ using maximum principle directly because there could be a failure of harmonicity at $a+\delta$. However, by the antisymmetry of F^* (we can swap the role of x and z) and the result when $x \in \{a+t : t>0\}$, we can deduce that $H_x^* = 0$ on $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]} \cap \{a+t : t>0\}$. Similar arguments give us that $H_x^* = 0$ for $x \in \mathcal{V}^{1}_{[\mathsf{C}_\delta,a]} \cap \{a+t : t \in \mathbb{R}\}$.

Now, for general $x \in \mathcal{V}^{1,i}_{[\mathsf{C}_\delta,a]}$ we obtain that $H_x^* = 0$ on $\mathcal{V}^{1,i}_{[\mathsf{C}_\delta,a]} \cap \{a+t : t\in\mathbb{R}\}$, by antisymmetry of F^* and the discussion above. Hence, by harmonicity (remember that H_x^* does not have singularities) and decay at infinity, $H_x^* = 0$ and we get the result. □

Lemma A.6. *With the notation of Proposition A.4, we have that* $\left(H_{\square_{\Omega_\delta^{(n)},a}\square}\right)_n$ *is uniformly bounded (in both x and z) and that*

$$\lim_{z\to\infty}\limsup_{n\to\infty}\left|H_{\square_{\Omega_\delta^{(n)},a}\square}(x^{o_x},z)\right| = 0.$$

Proof. For $x \in \mathcal{V}^{\mathrm{c}}_{[\mathsf{C}_\delta,a]}$, set $H_\delta^{n,x} := H_{\square_{\Omega_\delta^{(n)},a}\square}(x^{o_x},\cdot)$. Let $Q_\delta^{n,x}$ be the discrete analogue of the antiderivative Re $\left(H_\delta^{n,x}\right)^2$, normalized to be 0 at x as in earlier in the paper; it is single-valued. On $\mathcal{V}_{\Omega_\delta^{(n)}}$, the function $Q_\delta^{n,x}$ is subharmonic, with discrete Laplacian $\left|\sqrt{2}\partial_\delta H_\delta\right|^2$ ([HoSm10]), except at the vertex v that is adjacent to x ([CHI13]), where we can bound it from below by $1+\left|H_\delta^{n,x}(y)\right|^2$ where y is the corner opposite to x (e.g. $y = x+\delta$ with $x \in \mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}$). Moreover, its outer normal derivatives on $\partial\mathcal{V}^{\mathrm{m}}_{\Omega_\delta^{(n)}}$ are negative, proportional to $-\left|H_\delta^{n,x}\right|^2$ ([CHI13]). Summing the Laplacian of $Q_\delta^{n,x}$ over $\mathcal{V}_{[\mathsf{C}_\delta,a]}$, we obtain the inequalities

$$\sum_{w\in\mathcal{V}_{\Omega_\delta^{(n)}}} \left|\partial_\delta H_\delta^{n,x}(w)\right|^2 \le \mathrm{Cst}\cdot\left(1+\left|H_\delta^{n,x}(y)\right|^2\right).$$

$$\sum_{w\in\partial\mathcal{V}^{\mathrm{m}}_{\Omega_\delta^{(n)}}} \left|H_\delta^{n,x}(w)\right|^2 \le \mathrm{Cst}\cdot\left(1+\left|H_\delta^{n,x}(y)\right|^2\right).$$

Hence, to prove the boundedness and decay at infinity, thanks to the monodromy, it is enough to bound $\left|H_\delta^{n,x}(y)\right|^2$. For $x \in \left\{a \pm \frac{\delta}{2},\right\}$, $H_\delta^{n,x}(y)$ has a probabilistic interpretation as the ratio $\mathbb{E}_{\Omega_\delta^{(n)}}[\sigma_{a\pm2\delta}]/\mathbb{E}_{\Omega_\delta^{(n)}}[\sigma_a]$ ([CHI13]), which is uniformly bounded (it converges actually to 1) by finite-energy property of the Ising model. Hence the result holds for such x.

For $x,z \in \partial\mathcal{V}^{\mathrm{m}}_{\square_{\Omega_\delta^{(n)},a}\square}$ we have that $\left|H_\delta^{n,x}(z)\right| \le \left|\mathbb{E}^{+/-}_{\Omega_\delta^{(n)}}[\sigma_a]\right|/\mathbb{E}^{+}_{\Omega_\delta^{(n)}}[\sigma_a] \le 1$ ([ChIz13]) from high-temperature expansion, where the numerator is taken with + boundary conditions on the arc $[xz]$ and − boundary conditions on the arc $[zx]$. For $x \in \partial\mathcal{V}^{\mathrm{m}}_{\square_{\Omega_\delta^{(n)},a}\square}$ and $z \in \left\{a\pm\frac{\delta}{2}\right\}$, we have that $\left|H_\delta^{n,x}(z)\right|$ is uniformly bounded by the antisymmetry of $F_{\square_{\Omega_\delta^{(n)},a}\square}$ with respect to variable swap and the previous paragraph. By the maximum principle, $H_\delta^{n,x}(z)$ is uniformly bounded for all $x \in \partial\mathcal{V}^{\mathrm{m}}_{\square_{\Omega_\delta^{(n)},a}\square}$ and $z \in \mathcal{V}^{\mathrm{cm}}_{[\Omega_\delta,a]}$. Again, by variable swap, we have boundedness for any $x \in \mathcal{V}^{\mathrm{c}}_{[\Omega_\delta,m]}$ and any $z \in \partial\mathcal{V}^{\mathrm{m}}_{\square_{\Omega_\delta^{(n)},a}\square}$. But then if $x \in \mathcal{V}^{\mathrm{i}}_{\square_{\Omega_\delta^{(n)},a}\square}$, the $\mathrm{Im}H_\delta^{n,x}(z)$ is bounded at $z=x$ (by 1 in modulus, by construction), at $z = a\pm\frac{\delta}{2}$ and for $z \in \partial\mathcal{V}^{\mathrm{m}}_{\square_{\Omega_\delta^{(n)},a}\square}$. Hence we can apply the maximum principle to bound $H_\delta^{n,x}(z)$ with respect to all z, and the same argument applies if $x \in \mathcal{V}^{1}_{\square_{\Omega_\delta^{(n)},a}\square}$ and finally (swapping again the variables) to any $x,z \in \mathcal{V}_{\square_{\Omega_\delta^{(n)},a}\square}$.

By the first paragraph, it is enough to show boundedness to obtain decay at infinity, so this concludes the proof. □

Proposition A.7. *If $x \in \mathcal{V}^{1,i}_{[\mathsf{C}_\delta,a]} \cap \{a+t : t\in\mathbb{R}\}$, then $H_{[\mathsf{C}_\delta,a]}$ can be constructed as a linear combination of translation of harmonic measure of the tip of the slit plane $\mathcal{V}^{1}_{\mathsf{C}_\delta} \setminus \{a+t : t<0\}$.*

Proof. We use induction; note $H_{[\mathsf{C}_\delta,a]}(a+\frac{\delta}{2},z) = -H_{[\mathsf{C}_\delta,a]}(z)$. If $x \in \mathcal{V}^{i}_{\mathsf{C}_\delta} \cap \{a+t : t>0\}$, define $H_{[\mathsf{C}_\delta,a]}(x,z) = \pm i\,\mathrm{hm}^{\mathsf{Y}^{\pm}_\delta}_{\{x\}}(z)$ for $z \in \mathbb{Y}^{\pm}_\delta$. By Remark A.2,

$$i\,\mathrm{hm}^{\mathsf{Y}^{\pm}_\delta}_{\{x\}}(z) = i\,\mathrm{hm}^{\mathsf{Y}^{\pm}_\delta}_{\{x-2\delta\}}(z-2\delta) - i\,\mathrm{hm}^{\mathsf{Y}^{\pm}_\delta}_{\{x-2\delta\}}\left(a-\frac{3\delta}{2}\right)\mathrm{hm}^{\mathsf{Y}^{\pm}_\delta}_{\{a+\frac{\delta}{2}\}}(z)$$

and thus

$$H_{[\mathsf{C}_\delta,a]}(x,z) := H_{[\mathsf{C}_\delta,a]}(x-2\delta,z-2\delta) - \mathrm{hm}_{\{x-2\delta\}}^{\mathsf{Y}_\delta^\pm}(a-\frac{3\delta}{2})H_{[\mathsf{C}_\delta,a]}(a+\frac{\delta}{2},z) \tag{A.1}$$

is discrete holomorphic on $\mathbb{Y}^\pm$; extend s-holomorphically to $\mathcal{V}^{cm}_{\mathsf{Y}^\pm}$. It is easy to check that this still vanishes at ∞ and takes the appropriate values on the boundary.

There is a subtle problem because of the presence of a possible 'hidden singularity' at $a+\frac{\delta}{2}$: we must make sure that we have the right value at this point. Let us explain the case $x=a-\frac{3\delta}{2}$, the induction is then identical to the the first case. We construct the imaginary part of H_x as $\pm$ the harmonic measure of x in $\mathcal{V}^{\mathrm{i}}_{\mathsf{C}_\delta}\setminus\{a-t:t>0\}$. The difference of this harmonic measure and H_x is hence 0 on $\{a-t:t>0\}$ and harmonic on $\mathcal{V}^{\mathrm{i}}_{\mathsf{C}_\delta}\setminus\{a-t:t>0\}$, except possibly at $a+\frac{\delta}{2}$ (there could be a hidden singularity there, due to the monodromy). Hence, we must make sure the value of this difference there is 0 (we can then pretend there is no monodromy at $a+\frac{\delta}{2}$). But this follows from the antisymmetry of $F_{[\mathsf{C}_\delta,a]}$, which is precisely matched by the symmetry of this harmonic measure construction and the one of the following paragraph.

For $x\in\mathcal{V}^{1}_{\mathsf{C}_\delta}\cap\{a+t:t<0\}$, we define $H_{[\mathsf{C}_\delta,a]}(x,z)=\pm\,\mathrm{hm}_{\{x\}}^{\mathsf{Y}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}}(z)$ for $z\in\mathbb{Y}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}$ and extend analogously. Note $H_{[\mathsf{C}_\delta,a]}(a+\frac{3\delta}{2},z)=\pm i\,\mathrm{hm}_{\{a+\frac{\delta}{2}\}}^{\mathsf{X}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}}(z-2\delta)$ for $z\in\mathbb{X}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}$. In addition, we set $H_{[\mathsf{C}_\delta,a]}(a-\frac{\delta}{2},\cdot)$ to be an everywhere s-holomorphic function which extends this definition: $H_{[\mathsf{C}_\delta,a]}(a-\frac{\delta}{2},\cdot):=\pm\,\mathrm{hm}_{\{a+\frac{\delta}{2}\}}^{\mathsf{X}^\pm\cap\mathcal{V}^i_{[\mathsf{C}_\delta,a]}}$ on $\mathbb{X}^\pm\cap\mathcal{V}^i_{[\mathsf{C}_\delta,a]}$ and $H_{[\mathsf{C}_\delta,a]}(a-\frac{\delta}{2}):=\pm i\,\mathrm{hm}_{\{a-\frac{\delta}{2}\}}^{\mathsf{Y}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}}$ on $\mathbb{Y}^\pm\cap\mathcal{V}^1_{[\mathsf{C}_\delta,a]}$. This is the only spinor in the family of spinors defined in this proposition which neither vanishes nor has a singularity at $a-\frac{\delta}{2}$.

The case $x\in\mathcal{V}^1_{[\mathsf{C}_\delta,a]}\cap\{a+t:t<0\}$ is symmetric to the first one, while the case $x\in\mathcal{V}^i_{[\mathsf{C}_\delta,a]}\cap\{a+t:t>0\}$ is symmetric to the second one. □

Proposition A.8. *With the notation of Proposition A.4, assuming that $x\in\mathcal{V}^{1,\mathrm{i}}_{[\mathsf{C}_\delta,a]}$ is at distance $O(\delta)$ to a as $\delta\to 0$, we have, uniformly over z on compact subsets away from a,*

$$\frac{1}{\vartheta(\delta)}H_{[\mathsf{C}_\delta,a]}(x^{o_x},z)\xrightarrow[\delta\to 0]{}\mathcal{C}_{x^{o_x}}h_{[\mathsf{C},a]}(a^o,z)=\frac{\mathcal{C}_{x^{o_x}}}{\sqrt{z-a}}.$$

Proof. As before, set $H_x(z):=H_{[\Omega_\delta,a]}(x^{o_x},z)$. We proceed as follows:

- We first prove the result for x on $\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}\cap\{a+t:t>0\}$ and on $\mathcal{V}^1_{[\mathsf{C}_\delta,a]}\cap\{a+t:t<0\}$. We have that in this case, the function H_x can be constructed as a finite real linear combination of harmonic measures, as given by Proposition A.7. By the convergence of the harmonic measure, we obtain the desired result.
- We then extend the result for x off the real axis. Assume $x\in\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}$ and that x is below the axis $\{a+t:t\in\mathbb{R}\}$ (the other cases are similar). Now use the antisymmetry property of $F_{[\mathsf{C}_\delta,a]}$ with respect to x and z. Since $H_x(z)$ is harmonic with respect to $x\in\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}$ away from z, if $\mathrm{Im}(x)\le\mathrm{Im}(a)$ and $\mathrm{Im}(z)>\mathrm{Im}(a)$, we can represent, $H_x(z)$ as an infinite convolution of $H_w(z)$ for $w\in\mathcal{V}^{\mathrm{i}}_{[\mathsf{C}_\delta,a]}\cap\{a+t:t\in\mathbb{R}\}$ with the discrete Poisson kernel $P(w,x)$ of the lower half-plane $\mathcal{V}^{\mathrm{i}}_{\mathsf{C}_\delta}\cap\{z:\mathrm{Im}(z)<0\}$ (using in addition that $H_x(z)\to 0$ as $x\to\infty$). Since x is at distance $O(\delta)$ to $\{a+t:t\in\mathbb{R}\}$, we have that $P(w,x)\sim\frac{1}{(w-x)^2}$ (by Poisson excursion kernel estimates) as $w\to\pm\infty$ (which is summable on $\mathcal{V}^{\mathrm{i}}_{\mathsf{C}_\delta}\cap\{a+t:t\in\mathbb{R}\}$) and since $H_x(z)$ is uniformly bounded with respect to x and z (Lemma A.6) we obtain the convergence result for z above the real axis, away from a. It is easy then to extend the convergence to any z away from a: we must have boundedness, otherwise this would imply a blow-up for z above the real axis, and hence we have precompactness, and the limits must be analytic and hence are determined by their values above the axis $\{a+t:t\in\mathbb{R}\}$.

□

Appendix B. Properties of Discrete Fermionic Observables

In this section of the appendix we prove most of the results that were left unproven in Section 3. Most of the proofs are very similar to those given in [Hon10] for the discrete fermionic observable, and [CHI13] for the discrete fermionic spinor. Thus we have relegated them to the appendix and will frequently refer to both papers in the proofs.

B.1. **Discrete Fermionic Observable.**

Definition B.1 (Restatement of Defintion of Discrete Fermionic Observable)**.** Given a domain Ω_δ we pick $2n$ Ω_δ-distinct medial vertices $b_1, b_2, \ldots, b_{2n}$ in $\mathcal{V}^m_{\Omega_\delta}$ and their double orientations $(\sqrt{o_1})^2, (\sqrt{o_2})^2, \ldots, (\sqrt{o_{2n}})^2$. Define the real fermionic observable F_{Ω_δ}:

$$F_{\Omega_\delta}(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}} \sum_{\gamma \in C(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}})} \alpha^{\#\gamma}(-1)^{c(\gamma_1, \ldots, \gamma_n)} \prod_{i=1}^{n} \phi(\gamma_i), \text{ where}$$

$$C(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \{\text{edges and half-edges that form walks between } b_i^{o_i}\text{'s and loops}\}$$

$$\phi(w : \alpha^o \;\square\; \beta^{o^\square}) = i\frac{\sqrt{o'}}{\sqrt{o}} e^{-\frac{iW(w)}{2}}$$

$\langle \gamma_1, \ldots, \gamma_n \rangle$: admissible choice of walks from γ in C

$c(\gamma_1, \ldots, \gamma_n)$: crossing signature of b_i's with respect to $\gamma_1, \ldots, \gamma_n$

$$\mathsf{Z}_{\Omega_\delta} = \sum_{\gamma \in C} \alpha^{\#\gamma}$$

Given a collection of doubly oriented points (...), a collection of signed edges $e_1, ..., e_m$ disjoint from (...), define the restricted real fermionic observable $F_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}(\ldots)$ as

$$F_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}} \sum_{\gamma \in C^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}})} \alpha^{\#\gamma}(-1)^{c(\gamma_1, \ldots, \gamma_n)} \prod_{i=1}^{n} \phi(\gamma_i).$$

Further, given a collection of doubly oriented points (...), a collection of signed edges $\{...\}$ disjoint from (...) and edges $e_1, ..., e_m$, define the fused real fermionic observable $F_{\Omega_\delta}^{[e_1, \ldots, e_m]\{\ldots\}}(\ldots)$ inductively as

$$F_{\Omega_\delta}^{[e_1, \ldots, e_m]\{\ldots\}}(\ldots) = F_{\Omega_\delta}^{[e_1, \ldots, e_{m-1}]\{\ldots, e_m^+\}}(\ldots) - \frac{\sqrt{2}}{2} F_{\Omega_\delta}^{[e_1, \ldots, e_{m-1}]\{\ldots\}}(\ldots).$$

For a collection $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be doubly oriented medial vertices, a medial vertex b_{2n} and a configuration $\gamma \in C(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$, let us denote by $\mathsf{W}^H_{\Omega_\delta}(\gamma, o_1, ..., o_{2n-1})$ the complex weight of γ, defined as

$$\mathsf{W}_{\Omega_\delta}(\gamma, o_1, ..., o_{2n-1}) = \frac{i}{\sqrt{o_{2n}}} \alpha^{\#\gamma} \prod_{i=1}^{n} \phi(\gamma_i)$$

for any branch choice of o_{2n}. From here we can define the complex fermionic observable

$$H_{\Omega_\delta}^{\{\ldots\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}) = \frac{1}{\mathsf{Z}_{\Omega_\delta}} \sum_{\gamma \in C^{\{\ldots\}}(b_1^{o_1}, \ldots, b_{2n-1}^{o_{2n-1}}, b_{2n})} \mathsf{W}^H_{\Omega_\delta}(\gamma, o_1, ..., o_{2n-1})$$

Proposition B.2. *Let $e_1^{s_1}, ..., e_m^{s_m}$ be distinct (though possibly adjacent) signed edges and $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be distinct, though possibly adjacent, doubly oriented medial vertices. Then the function*

$$b_{2n} \mapsto H_{\Omega_\delta}^{\{\ldots\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

whose domain is $\mathcal{V}_{\Omega_\delta^m} \setminus \{m(e_1), ..., m(e_m), b_1, ..., b_{2n}\}$ is s-holomorphic. Moreover, it obeys boundary conditions

$$\mathrm{Im}[H_{\Omega_\delta}^{\{\ldots\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})\sqrt{v_{out}(b_{2n})}] = 0 \text{ for all } b_{2m} \in \partial\mathcal{V}^m_{\Omega_\delta}$$

where $v_{out}(b_{2n})$ is defined as the outer normal to the boundary at b_{2n}.

Lemma B.3. *Let $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be distinct (possibly adjacent) doubly oriented medial vertices and let $b_{2n}^{o_{2n}}, \tilde{b}_{2n}^{\tilde{o}_{2n}}$ be two adjacent simply oriented medial vertices distinct from $b_1, ..., b_{2n-1}$ and denote by e the medial edge $\langle b_{2n}, \tilde{b}_{2n}\rangle \in \mathcal{E}^m_{\Omega_\delta}$. Let $\gamma \in C(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$ and $\tilde{\gamma} \in C(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, \tilde{b}_{2n})$ be two configurations such that $\gamma \oplus c(e) = \tilde{\gamma}$. Then we have*

$$\mathrm{P}_{l(e)}[\mathsf{W}_{\Omega_\delta}(\gamma, o_1, ..., o_{2n-1}) = \mathrm{P}_{l(e)}[\mathsf{W}_{\Omega_\delta}(\tilde{\gamma}, o_1, ..., o_{2n-1})].$$

Proof. This has already been proven in the case that $b_1, ..., b_n$ are at least 2δ apart from each other in [Hon10]. Thus we only need to show for the case where some of the b_i are adjacent medial vertices. We define the complex weight of γ for paths ending at corners (for instance $b_{2n}+i\delta$) instead of at medial edges. Call e' the half medial edge $\langle b_{2n}, b_{2n}+\frac{i}{2}\delta\rangle$ and $c(e')=\langle b_{2n}, b_{2n}+\frac{(1+i)\delta}{2}\rangle \cup \langle b_{2n}+\frac{(1+i)\delta}{2}, b_{2n}+\frac{i}{2}\delta\rangle$. The complex weight at the corner is

$$\cos(\frac{\pi}{8})\mathsf{W}_{\Omega_\delta}(\gamma \oplus c(e'), o_1, ..., o_{2n-1})$$

where the partial edge from the vertex to the corner counts as a half edge towards the length of α and the winding is calculated with a contribution half of a right or left turn coming at the turn into the corner.

Suppose without loss of generality, $\tilde{b}_{2n}=b_{2n}+i\delta$ such that $b_{2n}+\frac{i}{2}\delta$ lies on the line $l(e)$. We claim that

$$\mathrm{P}_{l(e)}[\mathsf{W}_{\Omega_\delta}(\gamma, o_1, ..., o_{2n-1}) = \cos(\frac{\pi}{8})\mathsf{W}_{\Omega_\delta}(\gamma \oplus c(e'), o_1, ..., o_{2n-1}) = \mathrm{P}_{l(e)}[\mathsf{W}_{\Omega_\delta}(\tilde{\gamma}, o_1, ..., o_{2n-1})].$$

Note that proving the first part of the identity proves that the corner complex weight takes the value of the projection of the medial edge complex weight onto the line connecting the two points in all orientations and is thus enough to prove the second part and complete the proof of the lemma. There are two cases to consder when proving the claim: the case when $\langle b_{2n}, b_{2n}+\frac{(1+i)\delta}{2}\rangle \in \gamma \oplus c(e')$ and the case when $\langle b_{2n}, b_{2n}+\frac{(1+i)\delta}{2}\rangle \notin \gamma \oplus c(e')$. For ease of notation, set $\mathsf{W}^{H}_{\Omega_\delta}(\cdot)=\mathsf{W}^{H}_{\Omega_\delta}(\cdot, o_1, ..., o_{2n-1})$. □

- When $\langle b_{2n}, b_{2n}+\frac{(1+i)\delta}{2}\rangle \in \gamma\oplus c(e')$ the winding of the path $W(\gamma\oplus c(e'))=W(\gamma)+\frac{3\pi}{4}$ so $\mathsf{W}_{\Omega_\delta}(\gamma\oplus c(e'))=\alpha e^{-\frac{3\pi i}{8}}\mathsf{W}_{\Omega_\delta}(\gamma)$. Thus

  $$\mathsf{W}_{\Omega_\delta}(\gamma \oplus c(e')) = \alpha e^{-\frac{3\pi i}{8}}\mathsf{W}_{\Omega_\delta}(\gamma),$$

  which given the required orientation implies

  $$\mathrm{P}_{l(e)}(\mathsf{W}_{\Omega_\delta}(\gamma)) = \cos(\frac{\pi}{8})\mathsf{W}_{\Omega_\delta}(\gamma \oplus c(e')).$$

- Likewise when $\langle b_{2n}, b_{2n}+\frac{\delta}{2}\rangle \notin \gamma \oplus c(e')$, $o_{2n}=1$ we have that

  $$\mathrm{P}_{l(e)}\mathsf{W}_{\Omega_\delta}(\gamma) = \cos(\frac{\pi}{8})\mathsf{W}_{\Omega_\delta}(\gamma \oplus c(e')).$$

Proof of Proposition B.7. The proof of s-holomorphicity follows immediately now since Lemma 7 has been proven for the case where the points are not necessarily 2δ apart from each other. The boundary conditions have already been proven in chapter 6 of [Hon10]. □

Remark B.4. This completes the proof of the s-holomorphicity relation even in the case of adjacent edges and source points. The rest of the propositions in Section 3 follow immediately.

B.2. **Discrete Fermionic Spinor.**

Definition B.5 (Restatement of Definition.)**.** Given a domain Ω_δ and its double cover $[\Omega_\delta, a]$ ramified at $a \in \mathcal{F}_{\Omega_\delta}$, we pick $2n$ Ω_δ-distinct medial vertices $b_1, b_2, \ldots, b_{2n}$ in $[\Omega_\delta, a]$ and their double orientations $(\sqrt{o_1})^2, (\sqrt{o_2})^2, \ldots, (\sqrt{o_{2n}})^2)$. Define the multi-point real function $F_{[\Omega_\delta,a]}$:

$$F_{[\Omega_\delta,a]}(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]} \sum_{\gamma \in C(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}})} \alpha^{\#\gamma}(-1)^{\#L(\gamma \setminus \cup \gamma_i, a)}(-1)^{c(\gamma_1, \ldots, \gamma_n)} \prod_{i=1}^{n} \psi(\gamma_i) \text{ where}$$

$$C(b_1^{o_1}, \ldots, b_{2n}^{o_{2n}}) = \{\text{edges and half-edges that form walks between } b_i^{o_i}\text{'s and loops}\}$$

$$\psi(w : \alpha^{o} \,\square\, \beta^{o^{\square}}) = i\frac{\sqrt{o'}}{\sqrt{o}}e^{-\frac{iW(w)}{2}}\,\mathrm{Sheet}(w) = \phi(w)\,\mathrm{Sheet}(w)$$

$\langle \gamma_1, \ldots, \gamma_n\rangle$: admissible choice of walks from γ in C

$\#L(w, a)$: number of loops (a loop has to be connected and without crossings) in w around a

$\mathrm{Sheet}(w)$: 1 if $w : \alpha^{o} \,\square\, \beta^{o^{\square}}$ lifted to the double cover starting at α ends at β, -1 otherwise

$c(\gamma_1, \ldots, \gamma_n)$: crossing signature of b_i's with respect to $\gamma_1, \ldots, \gamma_n$

Here, we define the partition functions

$$
\begin{aligned}
\mathsf{Z}_{\Omega_\delta} &= \sum_{\gamma \in C} \alpha^{\#\gamma} \\
\mathsf{Z}_{[\Omega_\delta, a]}[\sigma_a] &= \sum_{\gamma \in C} \alpha^{\#\gamma} (-1)^{\#L(\gamma, a)} \\
&= \mathbb{E}_{\Omega_\delta}[\sigma_a] \mathsf{Z}_{\Omega_\delta} \\
\mathsf{Z}^{\{\dots\}}_{[\Omega_\delta, a]}[\sigma_a] &= \sum_{\gamma \in C^{\{\dots\}}} \alpha^{\gamma} (-1)^{\#L(\gamma, a)}.
\end{aligned}
$$

It has been proven in [CHI13] that $(-1)^{c(\gamma_1, \dots, \gamma_n)} \prod_{i=1}^n \phi(\gamma_i)$ is well-defined for various admissible choices of walks.

Proposition B.6 (Well-definedness of Discrete Fermionic Spinor.)**.** *For two admissible choice of walks $\langle \gamma_1, \dots, \gamma_n \rangle$ and $\langle \tilde{\gamma_1}, \dots, \tilde{\gamma_n} \rangle$ from γ, we have*

$$(-1)^{\#L(\gamma \setminus \square_i \gamma_i, a)} \prod_{i=1}^{n} \mathrm{Sheet}(\gamma_i) = (-1)^{\#L(\gamma \setminus \square_i \tilde{\gamma_i}, a)} \prod_{i=1}^{n} \mathrm{Sheet}(\tilde{\gamma_i})$$

and thus $F_{[\Omega_\delta, a]}$ is well-defined.

Lemma B.7. *If A, B are unions of disjoint loops, $\#L(A \oplus B, a) \equiv \#L(A, a) + \#L(B, a) \mod 2$.*

Proof. Interpret A and B as two collections of loops on Ω_δ. Then fill in the faces of the lattice with spins, beginning with plus boundary conditions, such that there is an edge between two faces if and only if they differ in sign. That is, A and B define two different spin configurations on the domain. Then $\#L(A, a) \mod 2$ is 0 if $\sigma_a = +$ and 1 if $\sigma_b = -$ and likewise for B. So $\sigma_a^A = (-1)^{\#L(A,a)}$.

Moreover, $A \oplus B$ is precisely the spin configuration constructed by multiplying the spins of configuration A and B pointwise. Thus

$$
\begin{aligned}
(-1)^{\#L(A \oplus B, a)} &= \sigma_a^{A \oplus B} = \sigma_a^A \sigma_a^B \\
&= (-1)^{\#L(A,a)} (-1)^{\#L(B,a)} \\
&= (-1)^{\#L(A,a) + \#L(B,a)}
\end{aligned}
$$

which proves the lemma. □

Lemma B.8. *For two admissible choice of walks $\langle \gamma_1, ..., \gamma_n \rangle$ and $\langle \tilde{\gamma_1}, ..., \tilde{\gamma_n} \rangle$ from γ, with $\gamma_i : \alpha_i^o \to \beta_i^o$ there exists permutations $s, \tilde{s} : \{1, ...n\} \to \{1, ..., n\}$ such that for $i = 1, ..., n$ $\gamma_{s(i)}$ and $\tilde{\gamma}_{\tilde{s}(i)}$ share one of their endpoints.*

Proof. Since any admissible choices of walks have to either start or end at each of the b_i's, $i \in \{1, ..., 2n\}$ we can begin by choosing γ_1 and $\tilde{\gamma_1}$ to have an endpoint at b_1. Call b_{k_1} and $b_{\tilde{k}_1}$ the other endpoints of the two walks respectively. Then we are left with a collection of walks $\langle \gamma_2, ..., \gamma_n \rangle$ and likewise for the other choice of walks such that the lemma does not necessarily hold.

Now there are two cases to consider. If $k_1 = \tilde{k}_1$ then naturally we do the same with γ_2 and $\tilde{\gamma_2}$ so they both share endpoint b_m where $m = \min(\{2, ..., 2n\} \setminus \{k_1\}$. Clearly we can continue in this manner until we reach two walks for which $k_i \neq \tilde{k}_i$. So suppose $k_i \neq \tilde{k}_i$. Then choose b_m to be the other endpoint of the walk γ_i with $b_{\tilde{k}_i}$ as one of its endpoints. Choose the permutation such that both γ_2 and $\tilde{\gamma_2}$ share endpoint b_m.

It is obvious that we are again in a situation where the other endpoints of γ_2 and $\tilde{\gamma_2}$ are different. Call the indices of the new endpoints $k_{i+1} = \tilde{k}_i$ and $\tilde{k}_{i+1}$ and continue in this manner. If at any point, $\tilde{k}_i = k_j$ for some $j \leq i$ then we restart the process with the remaining walks and choose $m = \min(\{i+1, ..., 2n\} \setminus \bigcup_{j \leq i} k_j)$. Then it is easy to continue as before and we will never run into the possibility of not having a possible choice of shared endpoint for γ or $\tilde{\gamma}$. □

Proof of Proposition B.6. We begin by noting that the value of our function $F_{[\Omega_\delta, a]}(...)$ is completely independent of the order of the walks so we can choose any permutation of the choices of walks γ_i and $\tilde{\gamma_i}$ that we desire. Hence we choose the one we proved to exist in Lemma B.8 where γ_i shares an endpoint with $\tilde{\gamma_i}$ for all i.

It follows that $\gamma_i \oplus \tilde{\gamma_i}$ is a collection of loops and possibly a path from an endpoint of γ_i to an endpoint of $\tilde{\gamma_i}$. Call the path γ_i' and call the collection of loops $\gamma_{i_0} \oplus \tilde{\gamma}_{i_0} \setminus \gamma_{i_0}'$. Note that if both endpoints are shared then $\gamma_i' = \emptyset$.

We have that $\mathrm{Sheet}(\gamma_i) \neq \mathrm{Sheet}(\tilde{\gamma}_i) \iff (-1)^{\#L(\gamma_i \oplus \tilde{\gamma}_i \setminus \gamma_i', a)} \mathrm{Sheet}(\gamma_i') = -1$. Suppose that there is an odd number of i's such that $(-1)^{\#L(\gamma_i \oplus \tilde{\gamma}_i, a)} \mathrm{Sheet}(\gamma_i') = -1$, so that $\prod_i \mathrm{Sheet}(\gamma_i) = -\prod_i \mathrm{Sheet}(\tilde{\gamma}_i)$. Then if $(-1)^{\#L(\gamma \setminus \bigoplus_i \gamma_i, a)} = -(-1)^{\#L(\gamma \setminus \bigoplus_i \tilde{\gamma}_i, a)}$ the identity is satisfied and the multi-point real fermionic spinor is well-defined:

$$
\begin{aligned}
(-1)^{\#L(\gamma \setminus \bigoplus_i \gamma_i, a)} (-1)^{\#L(\gamma \setminus \bigoplus_i \tilde{\gamma}_i, a)} &= (-1)^{\#L((\gamma \setminus \bigoplus_i \gamma_i) \oplus (\gamma \setminus \bigoplus_i \tilde{\gamma}_i), a)} \\
&= (-1)^{\#L(\bigoplus \gamma_i \oplus \tilde{\gamma}_i, a)} \\
&= (-1)^{\#L(\bigoplus \gamma_i \oplus \tilde{\gamma}_i \setminus \gamma_i', a) + \#L(\bigoplus \gamma_i')} \\
&= (-1)^{\sum_i \#L(\gamma_i \oplus \tilde{\gamma}_i \setminus \gamma_i', a)} \prod_i \mathrm{Sheet}(\gamma_i') \\
&= \prod_i (-1)^{\#L(\gamma_i \oplus \tilde{\gamma}_i, a)} \mathrm{Sheet}(\gamma_i') = -1,
\end{aligned}
$$

where we exploit Lemma B.7 twice and make use of the following elementary identity:

$$
\begin{aligned}
(\gamma \setminus \bigoplus \gamma_i) \oplus (\gamma \setminus \bigoplus \tilde{\gamma}_i) &= (\gamma \cap (\bigoplus \gamma_i)^c) \oplus (\gamma \cap (\bigoplus \tilde{\gamma}_i)^c) \\
&= \gamma \cap \left[(\bigoplus \gamma_i)^c \oplus (\bigoplus \tilde{\gamma}_i)^c \right] \\
&= \gamma \cap \left[(\bigoplus \gamma_i) \oplus (\bigoplus \tilde{\gamma}_i) \right] \\
&= \bigoplus \gamma_i \oplus \tilde{\gamma}_i
\end{aligned}
$$

and the following lemma. $\square$

Lemma B.9. *Given two admissible choice of walks $\langle \gamma_i \rangle$ and $\langle \tilde{\gamma}_i \rangle$ such that γ_i and $\tilde{\gamma}_i$ share an endpoint for each i,*

$$
(-1)^{\#L(\bigoplus \gamma_i')} = \prod_i \mathrm{Sheet}(\gamma_i'),
$$

where γ_i' is defined as before.

Proof. We first note that by construction, each $b_1, \ldots, b_{2n}$ is an endpoint of either zero or two γ_i''s. Thus $\bigoplus \gamma_i'$ is a collection of disjoint loops. Call the loops $\{\lambda_j\}$. We can break each λ_j up into its constituent components $\lambda_{j_i} = \lambda_j \cap \gamma_i'$. For any two paths γ_1 and γ_2 s.t. $\gamma_1 \oplus \gamma_2$ is connected, we have that

$$
\mathrm{Sheet}(\gamma_1 \oplus \gamma_2) = \mathrm{Sheet}(\gamma_1) \, \mathrm{Sheet}(\gamma_2).
$$

where the sheet number is defined by choosing one endpoint and choosing one of the sheets to fix it on.

Thus $\mathrm{Sheet}(\lambda_j)$, which is by definition 1 if λ_j doesn't contain a and -1 if it does, is the product of the sheet number of the λ_{j_i}'s:

$$
(-1)^{\#L(\lambda_j)} = \prod_i \mathrm{Sheet}(\lambda_{j_i}),
$$

and

$$
\mathrm{Sheet}(\gamma_i') = \prod_j \mathrm{Sheet}(\lambda_{j_i}) \prod_k \mathrm{Sheet}(\gamma_i' \cap \gamma_k').
$$

Now taking the product over all γ_i' we have

$$
\begin{aligned}
\prod_i \mathrm{Sheet}(\gamma_i') &= \prod_i (\prod_j \mathrm{Sheet}(\lambda_{j_i}) \prod_k \mathrm{Sheet}(\gamma_i' \cap \gamma_k')) \\
&= \prod_i \prod_j \mathrm{Sheet}(\lambda_{j_i}) = (-1)^{\#L(\bigoplus_j \lambda_j)} \\
&= (-1)^{\#L(\bigoplus \gamma_i')},
\end{aligned}
$$

because sheet numbers of the intersections all occur even number of times. The choice of sheet that we fix the chosen endpoint on does not matter because every endpoint is shared by an even number of constituent loop components. $\square$

This completes the proof that the multipoint fermionic spinor is well-defined.

Definition B.10. Given a collection of doubly oriented points $(b_1^{o_1},\dots,b_{2n}^{o_{2n}})$, a collection of signed edges $e_1,...,e_m$ disjoint from $(b_1^{o_1},\dots,b_{2n}^{o_{2n}})$, define the restricted real fermionic spinor as

$$F^{\{e_1^{s_1},\dots,e_m^{s_m}\}}_{[\Omega_\delta,a]}(b_1^{o_1},\dots,b_{2n}^{o_{2n}})=\frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]}\sum_{\gamma\in C^{\{e_1^{s_1},\dots,e_m^{s_m}\}}(b_1^{o_1},\dots,b_{2n}^{o_{2n}})}\alpha^{\#\gamma}(-1)^{\#L(\gamma\setminus\cup\gamma_i,a)}(-1)^{c(\gamma_1,\dots,\gamma_n)}\prod_{i=1}^n\psi(\gamma_i).$$

Further, given a collection of doubly oriented points (...), a collection of signed edges {...} disjoint from (...) and edges $e_1,...,e_m$, define the fused real fermionic spinor $F^{[e_1,\dots,e_m]\{\dots\}}_{[\Omega_\delta,a]}(\dots)$ inductively as

$$F^{[e_1,\dots,e_m]\{\dots\}}_{[\Omega_\delta,a]}(\dots)=F^{[e_1,\dots,e_{m-1}]\{\dots,e_m^+\}}_{[\Omega_\delta,a]}(\dots)-\frac{\tilde\mu}{2}F^{[e_1,\dots,e_{m-1}]\{\dots\}}_{[\Omega_\delta,a]}(\dots)$$

where $\tilde\mu=-i(F^+_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j)+F^-_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j))$.

Proposition B.11. *Consider the critical Ising model with plus boundary conditions. Let $e_1,...,e_m$ be a set of possibly adjacent interior edges and $a\in\mathcal{F}_{[\Omega_\delta,a]}$. Then we have*

$$\frac{\mathbb{E}_{\Omega_\delta}[\sigma_a\cdot\epsilon(e_1)\cdot\ldots\cdot\epsilon(e_m)]}{\mathbb{E}_{\Omega_\delta}[\sigma_a]}=(-1)^m2^mF^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}.$$

Proof. We begin by defining the fused partition function by the recursion relation

$$\mathsf{Z}^{[e_1,\dots,e_m]\{\dots\}}_{[\Omega_\delta,a]}[\sigma_a]=\mathsf{Z}^{[e_1,\dots,e_{m-1}]\{\dots,e_m^+\}}_{[\Omega_\delta,a]}[\sigma_a]-\frac{1+\tilde\mu}{2}\mathsf{Z}^{[e_1,\dots,e_{m=1}]\{\dots\}}_{[\Omega_\delta,a]}[\sigma_a]$$

with $\tilde\mu=-i(F^+_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j)+F^-_{[\mathsf{C}_\delta,a]}(b_j^{o_j},b_j))$ defined as before in terms of the full plane fermionic observable. Then it is easy to note that

$$F^{\{e_1,\dots,e_m\}}_{[\Omega_\delta,a]}=\frac{\mathsf{Z}^{\{e_1,\dots,e_m\}}_{[\Omega_\delta,a]}[\sigma_a]}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]}=\frac{\mathsf{Z}^{\{e_1,\dots,e_m\}}_{[\Omega_\delta,a]}[\sigma_a]}{\mathbb{E}^+_{[\Omega_\delta,a]}[\sigma_a]\cdot\mathsf{Z}_{\Omega_\delta}}\implies$$
$$F^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}=\frac{\mathsf{Z}^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}[\sigma_a]}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]}=\frac{\mathsf{Z}^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}[\sigma_a]}{\mathbb{E}^+_{[\Omega_\delta,a]}[\sigma_a]\cdot\mathsf{Z}_{\Omega_\delta}}.$$

Then all that remains is for us to show that we have

$$\mathbb{E}_{\Omega_\delta}[\sigma_a\cdot\epsilon(e_1)\cdot\ldots\cdot\epsilon(e_m)]=(-1)^m2^m\frac{\mathsf{Z}^{[e_1,\dots,e_m]}_{[\Omega_\delta,a]}[\sigma_a]}{\mathsf{Z}_{\Omega_\delta}}.$$

By the definition of energy density ϵ_δ we have that

$$\mathbb{E}_{\Omega_\delta}[\sigma_a\cdot\epsilon(e_1)\cdot\ldots\cdot\epsilon(e_m)]=(-1)^m\frac{\mathsf{Z}^{\langle e_1,\dots,e_m\rangle}_{[\Omega_\delta,a]}[\sigma_a]}{\mathsf{Z}_{\Omega_\delta}}$$

where $\mathsf{Z}^{\langle e_1,\dots,e_m\rangle}_{[\Omega_\delta,a]}$ is defined inductively as

$$\mathsf{Z}^{\langle e_1,\dots,e_m\rangle\{\dots\}}_{[\Omega_\delta,a]}[\sigma_a]=\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle\{\dots,e_m^+\}}[\sigma_a]-\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle\{\dots,e_m^-\}}[\sigma_a]-\tilde\mu\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle}[\sigma_a].$$

We now proceed with the proof by induction on m where when $m=0$, it is trivial that $\mathsf{Z}^{\langle\rangle}_{[\Omega_\delta,a]}=\mathsf{Z}^{[]}_{[\Omega,\delta,a]}$. Assume for the case $m-1$ that

$$\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle}_{[\Omega_\delta,a]}[\sigma_a]=2^{m-1}\mathsf{Z}^{[e_1,\dots,e_{m-1}]}_{[\Omega,\delta,a]}[\sigma_a]$$

Then contracting notation by not writing [...], we have that

$$\begin{aligned}\mathsf{Z}^{\langle e_1,\dots,e_m\rangle}_{[\Omega,\delta,a]}&=\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle\{e_m^+\}}_{[\Omega,\delta,a]}-\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle\{e_m^-\}}_{[\Omega_\delta,a]}-\tilde\mu\mathsf{Z}^{\langle e_1,\dots,e_{m-1}\rangle}_{[\Omega,\delta,a]}\\&=2^{m-1}(\mathsf{Z}^{[e_1,\dots,e_{m-1}]\{e_m^+\}}_{[\Omega,\delta,a]}-\mathsf{Z}^{[e_1,\dots,e_{m-1}]\{e_m^-\}}_{[\Omega_\delta,a]}-\tilde\mu\mathsf{Z}^{[e_1,\dots,e_{m-1}]}_{[\Omega,\delta,a]})\\&=2^{m-1}(2\mathsf{Z}^{[e_1,\dots,e_{m-1}]\{e_m^+\}}_{[\Omega,\delta,a]}-\mathsf{Z}^{[e_1,\dots,e_{m-1}]}_{[\Omega_\delta,a]}-\tilde\mu\mathsf{Z}^{[e_1,\dots,e_{m-1}]}_{[\Omega,\delta,a]})\\&=2^m(\mathsf{Z}^{[e_1,\dots,e_m]}_{[\Omega,\delta,a]})\end{aligned}$$

which completes the proof. ∎

Definition B.12. For a collection $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be doubly oriented medial vertices, a medial vertex b_{2n} and a configuration $\gamma \in C(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$, let us denote by $\mathsf{W}_{[\Omega_\delta,a]}(\gamma,o_1,...,o_{2n-1})$ the complex weight of γ, defined as

$$\mathsf{W}_{[\Omega_\delta,a]}(\gamma,o_1,...,o_{2n-1}) = \frac{i}{\sqrt{o_{2n}}}\alpha^{\#\gamma}(-1)^{\#L(\gamma\setminus\cup\gamma_i,a)}(-1)^{c(\gamma_1,...,\gamma_n)}\prod_{i=1}^{n}\psi(\gamma_i)$$

for any branch choice of o_{2n}. From here we can define the complex fermionic observable

$$H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n}) = \frac{1}{\mathsf{Z}_{[\Omega_\delta,a]}[\sigma_a]}\sum_{\gamma\in C^{\{...\}}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})}\mathsf{W}_{[\Omega_\delta,a]}(\gamma,o_1,...,o_{2n-1}).$$

Lemma B.13. *Let $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be doubly oriented medial vertices and let $b_{2n}^{o_{2n}} \in \mathcal{V}^m_{\Omega_\delta}, \tilde{b}_{2n}^{\tilde{o}_{2n}} \in \mathcal{V}^c_{\Omega_\delta}$ be a simply oriented medial vertex and a simply oriented corner distinct from $b_1,...,b_{2n-1}$ and denote by e the half-medial edge $\langle b_{2n},\tilde{b}_{2n}\rangle$. Note it suffices to check the case that $\tilde{b}_{2n} = b_{2n}+\frac{i}{2}\delta$. Let $\gamma \in C(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$ and $\tilde{\gamma} \in C(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},\tilde{b}_{2n})$ be two configurations such that $\gamma\oplus c(e) = \tilde{\gamma}$ where $c(e) = \langle b_{2n},b_{2n}+\frac{1+i}{2}\delta\rangle\cup\langle b_{2n}+\frac{1+i}{2}\delta,b_{2n}+\frac{i}{2}\delta\rangle$. Then we have*

$$\mathrm{P}_{l(e)}[\mathsf{W}_{[\Omega_\delta,a]}(\gamma,o_1,...,o_{2n-1}) = \cos(\frac{\pi}{8})\mathsf{W}_{[\Omega_\delta,a]}(\gamma\oplus c(e),o_1,...,o_{2n-1}).$$

Proof. For ease of notation, set $\mathsf{W}_{[\Omega_\delta,a]}(\cdot) = \mathsf{W}_{[\Omega_\delta,a]}(\cdot,o_1,...,o_{2n-1})$. The lemma has already been shown for the case in which $n=1$ with source point taken at the monodromy. We show this holds in general. As before, there are two cases to consider: either $\langle b_{2n},b_{2n}+\frac{1+i}{2}\delta\rangle \in \gamma$ or it isn't.

First we show that in both cases,

$$(-1)^{\#L(\gamma\setminus\cup\gamma_i,a)}(-1)^{c(\gamma_1,...,\gamma_n)}\prod_i \mathrm{Sheet}(\gamma_i) = (-1)^{\#L(\tilde{\gamma}\setminus\cup\tilde{\gamma}_i,a)}(-1)^{c(\tilde{\gamma}_1,...,\tilde{\gamma}_n)}\prod_i \mathrm{Sheet}(\tilde{\gamma}_i)$$

but only the γ_n is affected by the XOR operation so the crossing signature is the same on both sides. If $c(e)$ destroys a loop in γ that changed the sheet of $[\Omega_\delta,a]$ then the loop becomes a part of $\tilde{\gamma}_n$ so the loop number changes on the right hand side but the sheet factor also changes on the right hand side. Thus we only need to be concerned about changes to the winding factor and the number of edges in the configuration. As shown before,

- In the former case, we have

$$\mathsf{W}_{[\Omega_\delta,a]}(\gamma\oplus c(e)) = \cos(\frac{\pi}{8})\alpha e^{-\frac{3\pi i}{8}}\mathsf{W}_{[\Omega_\delta,a]}(\gamma)$$

- In the latter case, we have

$$\mathsf{W}_{[\Omega_\delta,a]}(\gamma\oplus c(e)) = \cos(\frac{\pi}{8})e^{\frac{i\pi}{8}}\mathsf{W}_{[\Omega_\delta,a]}(\gamma)$$

This completes the proof of the proposition.

□

Proposition B.14. *Let $e_1^{s_1},...,e_m^{s_m}$ be distinct signed edges and $b_1^{o_1},...,b_{2n-1}^{o_{2n-1}}$ be distinct doubly oriented medial vertices, again possibly adjacent. Then the function*

$$b_{2n} \mapsto H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})$$

whose domain is $\mathcal{V}^m_{[\Omega_\delta,a]}\setminus\{m(e_1),...,m(e_m),b_1,...,b_{2n}\}$ has an s-holomorphic extension to corners adjacent to two edge midpoints in the domain.. It has monodromy -1 around ramification point a and obeys boundary conditions

$$\mathrm{Im}[H^{\{...\}}_{[\Omega_\delta,a]}(b_1^{o_1},...,b_{2n-1}^{o_{2n-1}},b_{2n})\sqrt{v_{out}(b_{2n})}] = 0 \textit{ for all } b_{2m}\in\partial\mathcal{V}^m_{[\Omega_\delta,a]}$$

where $v_{out}(b_{2n})$ is defined as the outer normal to the boundary at b_{2n}.

Proof. The s-holomorphicity follows immediately from Lemma 13 and the definition of $H^{\{...\}}_{[\Omega_\delta,a]}(...)$ in terms of the complex weights. The monodromy -1 around ramification point a is a direct result of the multiplicative factor $\prod_i \mathrm{Sheet}(\gamma_i)$.

To show that the boundary condition holds, it suffices to recognize that any point on the boundary of the domain can only be reached from one direction and thus $\frac{i}{\sqrt{o_{2n}}}e^{-\frac{iW(\gamma_n)}{2}} = v_{out}(b_{2n}) \mod 2\pi$. □

Proposition B.15. *Let $e_1^{s_1}, ..., e_m^{s_m}$ be distinct signed edges and $b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}$ be distinct doubly oriented medial vertices. For each $j \in \{1, ..., 2n-1\}$ such that $b_j \in \mathcal{V}_{\Omega_\delta^m} \setminus \partial\mathcal{V}_{\Omega_\delta^m}$ the function*

$$b_{2n} \mapsto H_{[\Omega_\delta,a]}^{\{e_1,...,e_m\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

has a discrete simple pole at b_j, with front and rear values given by:

$$H_j^+ = \frac{(-1)^{j+1}}{\sqrt{o_j}} F_{[\Omega_\delta,a]}^{\{e_1^{s_1},...,e_m^{s_m},e(b_j)^+\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

$$H_j^- = \frac{(-1)^{j}}{\sqrt{o_j}} F_{[\Omega_\delta,a]}^{\{e_1^{s_1},...,e_m^{s_m},e(b_j)^-\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

where $e(b_j)$ denotes the edge whose midpoint is b_j.

Further, for each $j \in \{1, ..., 2n-1\}$ such that $a_j \in \partial\mathcal{V}_{\Omega_\delta}^m$, the function

$$b_{2n} \mapsto H_{[\Omega_\delta,a]}^{\{...\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n})$$

can be extended to an s-holomorphic function at b_j by setting the value at b_j to

$$H_j^+ = \frac{1}{\sqrt{o_j}} F_{[\Omega_\delta,a]}^{\{e_1^{s_1},...,e_m^{s_m}\}}(b_1^{o_1}, ...b_{j-1}^{o_{j-1}}, b_{j+1}^{o_{j+1}}, ..., b_{2n-1}^{o_{2n-1}}).$$

Proof. The proof follows exactly the methods of the analogous proposition in [Hon10] except with the multipoint fermionic spinor. □

Proposition B.16. *For each medial vertex b_j not on the boundary of the domain, the function, call it $u(b_{2n})$*

$$\begin{aligned} b_{2n} \mapsto & H_{[\Omega_\delta,a]}^{\{...\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}) + \\ & + (-1)^j F_{[\Omega_\delta,a]}^{\{...\}}(b_1^{o_1}, ...b_{j-1}^{o_{j-1}}, b_{j+1}^{o_{j+1}}, ..., b_{2n-1}^{o_{2n-1}}) H_{[\mathsf{C}_\delta,a]}(b_j^{o_j}, b_{2n}) \end{aligned}$$

extends s-holomorphically to b_j where it takes the value

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} F_{[\Omega_\delta,a]}^{\{...\}[e(b_j)]}(b_1^{o_1}, ...b_{j-1}^{o_{j-1}}, b_{j+1}^{o_{j+1}}, ..., b_{2n-1}^{o_{2n-1}}).$$

Proof. This follows immediately from the previous proposition and the analogous proof in [Hon10]. □

Proposition B.17. *Let $b_1^{o_1}, ..., b_{2n}^{o_{2n}}$ be distinct doubly oriented medial vertices. Define a $2n \times 2n$ matrix $\mathsf{A}_{[\Omega_\delta,a]}$*

$$(\mathsf{A}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}))_{jk} = \begin{Bmatrix} F_{[\Omega_\delta,a]}(b_j^{o_j}, b_k^{o_k}) & if\ j \neq k \\ 0 & if\ j = k \end{Bmatrix}$$

Then we have

$$F_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) = \mathrm{Pfaff}(\mathsf{A}_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}})).$$

Lemma B.18. *The function*

$$\begin{aligned} b_{2n} \mapsto & H_{[\Omega_\delta,a]}^{\{...\}}(b_1^{o_1}, ..., b_{2n-1}^{o_{2n-1}}, b_{2n}) + \\ & - \sum_{j=1}^{2n} (-1)^j F_{[\Omega_\delta,a]}^{\{...\}}(b_1^{o_1}, ...b_{j-1}^{o_{j-1}}, b_{j+1}^{o_{j+1}}, ..., b_{2n-1}^{o_{2n-1}}) H_{[\mathsf{C}_\delta,a]}(b_j^{o_j}, b_{2n}) \end{aligned}$$

extends to an s-holomorphic function $\mathcal{V}_{[\Omega_\delta,a]}^m \longrightarrow \mathbb{C}$, and is in fact identically zero. Further, we can immediately see that we have the following recursion relation on the real fermionic observable:

$$F_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) = \sum_{j=1}^{2n} F_{[\Omega_\delta,a]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) F_{[\Omega_\delta,a]}(b_j^{o_j}, b_{2n}^{o_{2n}}).$$

Proof. Following the same approach used in [Hon10] we see that this relation holds for the multipoint real fermionic observable with monodromy just as it does without monodromy. At this point, the proof of Proposition B.17 is immediate from the Lemma and from the recursion formula for the Pfaffian. □

We follow this proposition with an important generalization providing an equivalent Pfaffian for the fused fermionic observable.

Proposition B.19. *For distinct edges $e_1, ..., e_m$ and distinct doubly oriented medial vertices $b_1^{o_1}, ..., b_{2n}^{o_{2n}}$, for each choice of orientations $q_i \in \mathbb{O}(e_i)$ $1 \leq i \leq m$ we have that*

$$F_{[\Omega_\delta, a]}^{[e_1,...,e_m]}(b_1^{o_1}, ..., b_{2n}^{o_{2n}}) = \mathrm{Sheet}(\mathsf{A}_{[\Omega_\delta, a]}(e_1^{(\sqrt{q_1})^2}, ..., e_m^{(\sqrt{q_m})^2}, e_m^{(i\sqrt{q_m})^2}, ..., e_1^{(i\sqrt{q_1})^2}, b_1^{o_1}, ..., b_{2n}^{o_{2n}})$$

where we associate with e_i the medial vertex on the edge and where the $2p \times 2p$ matrix, $p = m + n$, is defined for (not necessarily distinct) doubly oriented vertices $x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}$ by

$$(\mathsf{A}_{[\Omega_\delta, a]}(x_1^{\xi_1}, ..., x_{2p}^{\xi_{2p}}))_{jk} = \begin{Bmatrix} F_{[\Omega_\delta, a]}^{[\mathsf{C}_\delta, a]}(x_i^{\xi_i}, x_j^{\xi_j}) & \mathit{if}\, x_i = x_j \,\mathit{and}\, \xi_i \neq \xi_j \\ F_{[\Omega_\delta, a]}(x_i^{\xi_i}, x_j^{\xi_j}) & \mathit{if}\, x_i \neq x_j \\ 0 & \mathit{else} \end{Bmatrix}.$$

Proof. The proof of this is exactly as in the proof of the Pfaffian relation in [Hon10] except with the fermionic observable with a monodromy point and on the double cover. This difference, however affects none of the steps since projection relations and Lemma B.18 are the same in this case as in the real fermionic observable with no monodromy. □

Appendix C. Mapping between Energy Densities and Probabilities

A spin-symmetric pattern on a base diagram $\mathcal{B}$ is a member of $Sp(\mathcal{B}) := \{-1, 1\}^{\mathcal{F}(\mathcal{B})}/\{-1, 1\}$. Given $B \in Sp(\mathcal{B})$, and an edge $i \in \mathcal{B}$, whether i separates two faces with different spins is a well-defined property, invariant after flipping all spins in the pattern: let $B_i = -1$ if i separate spins, $B_i = 1$ if not.

Given a subset B of edges in $\mathcal{B}'$, $\mathbb{E}(B) := \mathbb{E}(\prod_{i \in B} \epsilon_i) = \sum_{B' \in Sp(\mathcal{B})} \prod_{i \in B}(\mu - B_i')\mathbb{P}(B')$. So the idea is to invert the matrix $EP_{BB'} = \prod_{i \in B}(\mu - B_i')$. There is a potential complication: when we write the expectation and probability vectors of a base diagram $\mathcal{B}$, whose components have the edge expectation and probability of a given spin-symmetric pattern, as $E_{\mathcal{B}} := (\mathbb{E}(B))_{B \subset \mathcal{B}}, P_{\mathcal{B}} := (\mathbb{P}(B))_{B \in Sp(\mathcal{B})}$, the dimension of the former, $2^{\mathcal{B}}$, in general is different from that of the latter, $2^{\mathcal{F}(\mathcal{B})-1}$. However, we note that there exists a set $\mathcal{B}'$ of interior edges, which are any edges separating faces in $\mathcal{F}(\mathcal{B})$, of size $\mathcal{F}(\mathcal{B}) - 1$ such that $\mathcal{F}(\mathcal{B}') = \mathcal{F}(\mathcal{B})$ (or, $\mathcal{B}'$ *spans* $\mathcal{F}(\mathcal{B})$) and there exists a path between any two faces in $\mathcal{F}(\mathcal{B})$ given by crossing only the edges in $\mathcal{B}'$: indeed, start from a face $F_0 \in \mathcal{F}(\mathcal{B})$ and for each of the neighboring faces, if there is not an acceptable path yet thereto, add the edge crossed between F_0 and that face. Since $\mathcal{F}(\mathcal{B})$ is connected, inductively we can get all faces in at most $\mathcal{F}(\mathcal{B}) - 1$ steps. So we restrict the collection of edges as members of $\mathcal{P} := \{-1, 1\}^{\mathcal{B}'} \subset \{-1, 1\}^{\mathcal{B}}$ (the inclusion comes by fixing the non-$\mathcal{B}'$ components as 1), which has the same cardinality as $Sp(\mathcal{B})$, and define $E_{\mathcal{B}} := (\mathbb{E}(B))_{B \in \mathcal{P}} = \mathbb{E}(\prod_{i: B_i = -1} \epsilon_i)$ and we invert the matrix $EP_{BB'} = \prod_{i: B_i = -1}(\mu - B_i')$.

Define for $D \in \mathcal{P}$ the matrix

$$PE_{B'D} = (-1)^{\#\{i \in \mathcal{B}': D_i B_i' = -1\}} \prod_{i: D_i = 1} (\mu + B_i').$$

We claim $EP \times PE = (\mu + 1 - (\mu - 1))^n Id_{\mathsf{R} \otimes \mathcal{P}} = 2^n Id_{\mathsf{R} \otimes \mathcal{P}}$, where $n = \#\mathcal{B}'$. When $D = B$

$$\begin{aligned} \sum_{B'} EP_{BB'} PE_{B'B} &= \sum_{B'} (-1)^{\#\{i: B_i B_i' = -1\}} \prod_{i \in \mathcal{B}} (\mu + B_i B_i') \\ &= \sum_{B'} \left[-(\mu - 1)\right]^{\#\{i: B_i B_i' = -1\}} (\mu + 1)^{\#\{i: B_i B_i' = 1\}} \\ &= \sum_{m=0}^{n} \binom{n}{m} (\mu + 1)^m \left[-(\mu - 1)\right]^{n-m} = (\mu + 1 - (\mu - 1))^n \end{aligned}$$

where we convert the sum over $B' \in Sp(\mathcal{B})$ to a sum over natural numbers by weighting the respective summands. The weight turns out to be the binomial coefficient, because of the following fact: given a subset of $\mathcal{B}'$, there is precisely one pattern $B' \in Sp(\mathcal{B})$ such that $B_i' = 1$ on that subset and $B_i' = -1$ on its complement in $\mathcal{B}'$. Indeed, one can start off from a face, declare a spin there, and go to any other face along the path through $\mathcal{B}'$ discussed above to define a spin on that face, which will define uniquely the wanted spin pattern if there is any. Existence comes from the fact that there are 2^n spin patterns and 2^n choice of subsets. So given $0 \leq m \leq n$ there are exactly $\binom{n}{m}$ spin patterns with $\#\{i : B_i B_i' = -1\} = m$, since there are $\binom{n}{m}$ choices of subsets in $\mathcal{B}'$ of size m.

Now for $D \neq B$

$$\begin{aligned}\sum_{B'} EP_{BB'}PE_{B'D} &= \sum_{B'}(-1)^{\#\{i:D_iB_i'=-1\}} \prod_{i:B_i=D_i}(\mu+D_iB_i') \prod_{i:B_i=-D_i=-1}(\mu-B_i')(\mu+B_i') \\ &= \sum_{B'}(-1)^{\#\{i:D_iB_i'=-1\}} \prod_{i:B_i=D_i}(\mu+D_iB_i') \prod_{i:B_i=-D_i=-1}(\mu^2-1)\end{aligned}$$

note that, if $i_0 \in \mathcal{B}'$ is such that $B_{i_0} \neq D_{i_0}$, we can pair any B' with $\bar{B}'$ obtained by flipping the sign of B_{i_0} (which uniquely exists by above argument). In that case only the power of (-1) in front of the product would switch sign; in other words,

$$(-1)^{\#\{i:D_iB_i'=-1\}} \prod_{i:B_i=D_i}(\mu+D_iB_i') + (-1)^{\#\{i:D_i\bar{B}'_i=-1\}} \prod_{i:B_i=D_i}(\mu+D_i\bar{B}'_i) = 0$$

or

$$\prod_{i:B_i=-D_i=1}(\mu^2-1)\sum_{B'}(-1)^{\#\{i:D_iB_i'=-1\}} \prod_{i:B_i=D_i}(\mu+D_iB_i') = 0.$$

So, in conclusion we have

$$\begin{aligned}EP^{-1} &= \frac{PE}{2^n} \\ \mathbb{P}(B') &= \frac{1}{2^n}\sum_{D}[(-1)^{\#\{i:D_iB_i=-1\}} \prod_{i:D_i=1}(\mu+B_i')]\mathbb{E}(D).\end{aligned}$$

In addition, if we define $\mathbb{P}^{a\pm}(X) := \mathbb{P}(X|\sigma_a = \pm 1)$, $\mathbb{P}^{a+}$ and $\mathbb{P}^{a-}$ are new probability measures, with expectation operations $\mathbb{E}^{a\pm}(X) := \mathbb{E}(X|\sigma_a = \pm 1)$. We have, meaning that we have the same EP and PE matrices. We get $\mathbb{E}^{a\pm}$ by combining the following two equations:

$$\begin{aligned}\mathbb{E}(\sigma_a X) &= \mathbb{E}^{a+}(X)\mathbb{P}(\sigma_a = 1) - \mathbb{E}^{a-}(X)\mathbb{P}(\sigma_a = -1) \\ \mathbb{E}(X) &= \mathbb{E}^{a+}(X)\mathbb{P}(\sigma_a = 1) + \mathbb{E}^{a-}(X)\mathbb{P}(\sigma_a = -1)\end{aligned}$$

thus

$$\mathbb{E}^{a\pm}(X) = \frac{\mathbb{E}(X) \pm \mathbb{E}(\sigma_a X)}{2\mathbb{P}(\sigma_a = \pm 1)} = \frac{\mathbb{E}(X) \pm \mathbb{E}(\sigma_a X)}{1 \pm \mathbb{E}(\sigma_a)}.$$

It gives a way to calculate the probability vector with the spin at a fixed:

$$P_{(\mathcal{B},a\pm)} = \mathbb{P}(\sigma_a = \pm 1)P^{a\pm}_{\mathcal{B}} = \mathbb{P}(\sigma_a = \pm 1)PE(E^{a\pm}_{\mathcal{B}}) = \frac{1}{2}PE(E_{\mathcal{B}} \pm E_{a\mathcal{B}}).$$

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