LATTICE REPRESENTATIONS OF THE VIRASORO ALGEBRA I: DISCRETE GAUSSIAN FREE FIELD

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Abstract. Most two-dimensional massless field theories carry representations of the Virasoro algebra as consequences of their conformal symmetry. Recently, conformal symmetry has been rigorously established for scaling limits of lattice models by means of discrete complex analysis, which efficiently expresses the integrability of these models.

In this paper we study the discrete Gaussian free field on the square grid. We show that the lattice integrability of this model gives explicit representations of the Virasoro algebra acting on the Gibbs measures of the model. Thus, somewhat surprisingly, the algebraic structure of Conformal Field Theory describing the scaling limit of the model is already present on lattice level.

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1. Introduction and preliminaries

1.1. **Statistical mechanics and Conformal Field Theory.** Physical arguments suggest that 2D lattice models at continuous phase transitions have conformally invariant scaling limits that can be described by Conformal Field Theories (CFTs). The 2D CFTs are exactly solvable: they can be studied in terms of representations of the Virasoro algebra. One can in particular obtain exact formulae for the (conjectural) scaling limits of correlations, partition functions, and critical exponents of a large family of 2D lattice models. See, e.g., [BPZ84a, BPZ84b, DMS97] and the references in the latter.

While CFT has been tremendously successful and allowed for the derivation of deep and spectacular conjectures about lattice models, it usually lends itself only to a non-rigorous approach to statistical mechanics. Indeed, one needs to assume that the fields of the models have conformally invariant scaling limits and that they can be described within the framework of certain quantum field theories. In the special case of the Gaussian free field, the field can be studied rigorously and representations can be found in the continuum at the level of insertions [Gaw99, KaMa11].

An alternative approach to conformal invariance was introduced by Schramm, with his discovery of the SLE$_\kappa$ processes [Sch00]. The idea here is to consider random curves that arise in the lattice models and to describe their scaling limits using Loewner's differential equation. This approach has been carried out rigorously for random curves in the following models: critical percolation [Smi01], loop-erased random walk and uniform spanning trees [LSW04], the FK-Ising model [Smi10], and the discrete Gaussian free field [ScSh09]. Besides describing scaling limits of discrete interfaces (and so scaling limits of many natural observables), the SLE process is amenable to calculations and SLE techniques have been successfully applied to produce rigorous proofs of a number of the conjectures of CFT. More systematic connections between SLE and CFT have been studied by a number of authors, with various degrees of rigor, see, e.g., [BaBe03, FrWe03, BaBe04, DRC06, Kyt07, KaMa11].

Still, we are far from having a complete correspondence and, overall, it is fair to say that the application of CFT to statistical mechanics remains somewhat mysterious from a mathematical perspective. One of the difficulties is to pass to the scaling limits: one has to identify fields on lattice level, and to assume (or prove) that they have conformally invariant scaling limits described by a local action. Moreover, the fields do not always admit a probabilistic interpretation. A natural approach to improve the understanding is to try to find precursors of various CFT objects on the lattice level, and to see what relations they satisfy.

1.2. **Exactly solvable lattice models.** A number of two-dimensional lattice models are considered exactly solvable. Typically, this means that relations such as the Yang-Baxter equations or discrete holomorphicity are present on the discrete level. This implies in particular that the partition function or the correlation functions of these models admit exact formulae. In fact, this is how most of the rigorous results about lattice models were derived in the 20th century. See, e.g., [Bax89, JM94] and references therein.

Recent mathematical progress has allowed for rigorous proofs that a number of exactly solvable lattice models (in the discrete sense) have conformally invariant scaling limits, and to rigorously derive exact formulae for their correlation functions (predicted by exactly solved CFTs). The central tool to establish conformal invariance of these lattice models is their discrete holomorphicity relations, allowing for the use of discrete complex analysis techniques, see, e.g., [Ken00, Smi01, Smi10, CHI12].

Once the (lattice) exact solvability has been used to prove conformal invariance in the limit, one can try to use conformal invariance to reveal the algebraic structure of the Virasoro algebra, and in this way connect the lattice solvability with the CFT solvability. For this, one can either connect the model with SLE or the (continuum) Gaussian free field and use the techniques described above, or try to find lattice precursors of CFT objects [BeHo13, Dub13]. A natural question thus arises:

- Is there a direct connection between the exact solvability on the discrete level and the one on the continuum level?

We will show that, surprisingly, the answer is positive in two natural cases. On the square lattice for the discrete Gaussian free field (discrete GFF) and the critical Ising model, the same algebraic structures that arise in the continuum as consequences of the conformal symmetry are already present on discrete level as consequences of discrete holomorphicity. In this paper we will give the proofs for the discrete GFF and in a companion paper those for the Ising model.

This question has been investigated in the physics literature in the case of the 8-vertex model (using the Corner transfer matrix) and the Ising model, see [ItTh87, KoSa94] and the references in the latter. However, our results are the first in which discrete holomorphicity is connected with the Virasoro algebra and our approach gives the exact Virasoro algebras (and not deformations thereof) with the right physical content, that is, the correct central charge. Moreover, the action of the operators takes place on the level of Gibbs measures, and this gives concrete insight into the role of CFT for the description of probabilistic lattice models.

Our constructions can be modified to cover the case of the discrete GFF in a half plane with Dirichlet boundary condition. This corresponds to boundary CFT (cf., e.g., [Car84]): besides the two commuting bulk representations, we recover in the vicinity of the boundary a representation of the Virasoro algebra with correct central charge, "mixing" the analytic and antianalytic sectors.

1.3. **Measure-theoretic framework.** An advantage of considering lattice models before the scaling limit is that measure-theoretic questions become much easier to deal with.

- We consider *complex Gibbs measures* on an infinite graph $\mathbb{L}$ (typically $\mathbb{L}$ is a square grid), that is, assignments of a (finite) complex Borel measure μ_G on $\mathbb{R}^G$ to the finite subgraphs $G \subset \mathbb{L}$ satisfying the following compatibility condition: for $G \subset G'$, integrating $\mu_{G'}$ over $\mathbb{R}^{G' \setminus G}$ gives μ_G.
- If μ and ν are complex Gibbs measures on $\mathbb{L}$, we say that ν is *a change of measure* of μ if ν_G is absolutely continuous with respect to μ_G for all finite $G \subset \mathbb{L}$.
- For a linear space of complex Gibbs measures $\mathcal{M}$, we call a linear operator $T : \mathcal{M} \to \mathcal{M}$ such $T\mu$ is a change of measure of μ a *change of measure operator*.

Our main results are constructions of explicit families of change of measure operators acting on the Gibbs measures of the model while forming exact representations of the Virasoro algebra that arises in CFT. In the statements of the results, we furthermore use the following terminology:

- We call a complex Gibbs measure μ *symmetric* if for any G and any Borel set $A \subset \mathbb{R}^G$, we have $\mu_G(-A) = \mu_G(A)$. We call μ *antisymmetric* if $\mu_G(-A) = -\mu_G(A)$.
- A change of measure operator T is said to be *parity preserving* (resp. *parity reversing*) if it maps symmetric Gibbs measures to symmetric (resp. antisymmetric) Gibbs measures and antisymmetric Gibbs measures to antisymmetric (resp. symmetric) Gibbs measures.
- The *complex conjugate* of a complex Gibbs measure μ is the Gibbs measure $\bar{\mu}$ determined by $\bar{\mu}_G(A) = \overline{\mu_G(A)}$ for all Borel sets $A \subset \mathbb{R}^G$.
- The *complex conjugate* of a change of measure operator T is the change of measure operator $\bar{T}$ determined by $\bar{T}\mu = \overline{T\bar{\mu}}$. Note that $\bar{T}$ is still a $\mathbb{C}$-linear operator.

1.4. **Discrete GFF.** The central object of this paper is the discrete Gaussian free field (GFF) which we will consider on the square grid $\mathbb{Z}^2$. The discrete GFF on the infinite square grid $\mathbb{Z}^2$ is defined only up to an additive constant, see, e.g., [Fun05]. To resolve this ambiguity we choose to consider the (whole-plane) discrete Gaussian free field *pinned at 0*, that is, conditioned to vanish at the origin. An alternative essentially equivalent approach would be to define a discrete gradient Gaussian field on the edges.

The discrete GFF on a finite connected graph G with Dirichlet boundary condition on $\partial G \neq \emptyset$ is a collection of centered real Gaussian random variables $(\Phi(x))_{x \in G}$ with covariances given by the Green's kernel of G, that is, $\mathbb{E}[\Phi(x)\Phi(y)]$ is the expected number of visits to y of a simple random walk $(X_t)_{t=0}^{\tau}$ on G started from x and stopped at the time τ of hitting ∂G. See, for instance, [She05] or [Jan97, Chapter 9]. This definition also works for some infinite graphs, including a case that we will also treat explicitly: the upper half plane square lattice $\mathbb{H} = \mathbb{Z} \times \mathbb{N}$ with boundary $\partial\mathbb{H} = \mathbb{Z} \times \{0\}$.

When G is a finite subgraph of $\mathbb{Z}^2$, the discrete GFF Φ can equivalently be defined as a Gaussian vector of $\mathbb{R}^G$ with density proportional to $\exp\left(-\frac{1}{2}E(\Phi)\right)$, where $E(\Phi) := \frac{1}{4}\sum_{x \sim y}(\Phi(x) - \Phi(y))^2$ is the Dirichlet energy, the sum being over all pairs of adjacent vertices $x, y \in G \cup \partial G$, and we set $\Phi \equiv 0$ on ∂G.

Simple random walk is recurrent on the infinite square grid $\mathbb{Z}^2$, and so the Green's function as defined above diverges; an "infra-red" regularization is needed. We will use the massive regularization: the discrete GFF on $\mathbb{Z}^2$ with mass $m > 0$ is a collections of centered real Gaussian random variables $(\Phi_m(x))_{x\in\mathbb{Z}^2}$ with covariances given by the following massive regularization of the Green's kernel $G_m(x,y) = \sum_{t=0}^{\infty}(1+m^2)^{-1-t}\,\mathsf{P}_{X_0=x}[X_t = y]$, where $(X_t)_{t\in\mathbb{N}}$ is simple random walk on $\mathbb{Z}^2$.

- The *pinned discrete GFF on the infinite square grid* is a collection of centered real Gaussian random variables $(\Phi(x))_{x\in\mathbb{Z}^2}$ obtained as the massless limit of the pinned massive discrete GFFs as

$$\Phi = \lim_{m\searrow 0}\left(\Phi_m - \Phi_m(0)\right).$$

The pinned discrete GFF could also be defined as the infinite-volume limit of the discrete GFFs conditioned to vanish at the origin, that is, the limit as $N\to\infty$ of discrete GFF on finite subgraphs $\Lambda_N \subset \mathbb{Z}^2$ with boundary conditions 0 on $\partial\Lambda_N \cup \{0\}$, such that the increasing sequence of finite subgraphs exhausts the whole lattice $\mathbb{Z}^2$.

- We denote by μ_{GFF} the Gibbs measure of the discrete GFF on $\mathbb{Z}^2$, pinned at 0. For a finite $G = \{x_1,\ldots,x_k\}\subset\mathbb{Z}^2$ and $f:\mathbb{R}^G\to\mathbb{C}$, we denote by $\left\langle f(\Phi|_G)\right\rangle$ the expectation $\mathsf{E}[f(\Phi(x_1),\ldots,\Phi(x_k))]$ under the measure μ_{GFF}. (Typically we consider a *correlation function* and take $f = \Phi(x_{j_1})\cdots\Phi(x_{j_m})$ for $x_{j_1},\ldots,x_{j_m}\in G$ not necessarily distinct.) Similarly, if ν is a change of measure of μ_{GFF}, we write $\left\langle f(\Phi|_G)\right\rangle_\nu$ for the integral of f with respect to $\nu|_G$ (or with respect to $\nu|_{G'}$ for any $G'\supset G$).

1.5. Main result.

Our main result about the discrete Gaussian free field on the full plane is the following, the more precise version of which will be stated as Theorem 4.2 in Section 4.

Theorem. *Let μ_{GFF} be the Gibbs measure of the discrete Gaussian free field on $\mathbb{Z}^2$, pinned at 0. There exists a space $\mathcal{M}$ of changes of measure of μ_{GFF} and explicit parity preserving change of measure operators $L_n:\mathcal{M}\to\mathcal{M}$ for $n\in\mathbb{Z}$ such that $(L_n)_{n\in\mathbb{Z}}$ yields a representation of the Virasoro algebra of central charge $c=1$:*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}\,\mathrm{Id}.$$

Each operator L_n has a complex conjugate $\bar{L}_n$ and $(L_n)_{n\in\mathbb{Z}}$ and $\left(\bar{L}_n\right)_{n\in\mathbb{Z}}$ yield two commuting representations of the Virasoro algebra with central charge $c=1$.

This result is an exact discrete analogue of the classical result that the Gaussian free field is described by a CFT of central charge $c = 1$, see for instance [Gaw99, KaMa11]. Through the CFT–SLE correspondence [BaBe03, KaMa11], the parameter for the corresponding SLE$_\kappa$ is $\kappa = 4$; and indeed, the level lines of a discrete GFF with appropriate boundary conditions have been shown to converge to chordal SLE$_4$ in the lattice size scaling limit [ScSh09].

It is interesting and quite surprising that the central charge can already be seen on lattice level.

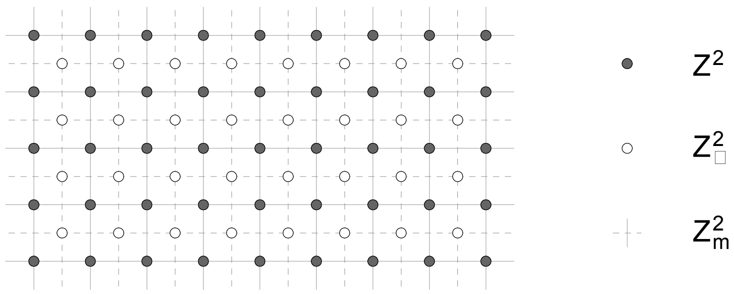

FIGURE 1.1. The Figure illustrates the square lattice $\mathbb{Z}^2$ and its dual $\mathbb{Z}^2_\star$. The medial lattice $\mathbb{Z}^2_m$ consists of midpoints of edges, and the diamond graph $\mathbb{Z}^2_\diamond$ consists of points of the square lattice and its dual.

1.6. **Further results.** Similar ideas can be carried out in other settings, too.

In particular, we shall use a Coulomb gas construction to build representations of the Virasoro algebra with other central charges, see Section 5.1 for precise statements. (We remark, however, that only the representations with $c = 1$ preserve the symmetry of the GFF with respect to multiplication by -1.)

We also construct representations for the discrete GFF in the upper half-plane with Dirichlet boundary condition, see Theorem 5.2.

1.7. **Overview of construction.** Let us give an overview of the proof of Theorem 4.2 and reduce it to a number of statements to be proven in the rest of the paper:

- We construct discrete current modes (Section 2), which are lattice analogs of $\oint \partial\varphi(z)z^n dz$, $n \in \mathbb{Z}$.
- The discrete current modes act on field insertions by discrete contour integrals (Lemma 3.2).
- We define a space of complex Gibbs measures $\mathcal{M}$ containing $\mathbb{P}_{\mathrm{GFF}}$ (Definition 1.1).
- The discrete current modes lift to change of measure operators $(\mathbf{a}_n : \mathcal{M} \to \mathcal{M})_{n \in \mathbb{Z}}$ (Proposition 3.4).
- We show that the commutation relations of the $(\mathbf{a}_n)_{n \in \mathbb{Z}}$ are those of the Heisenberg algebra (Theorem 3.5).
- We define the Virasoro generators $(\mathbf{L}_n : \mathcal{M} \to \mathcal{M})_{n \in \mathbb{Z}}$ from the discrete current mode operators using the Sugawara construction (Proposition 4.1).
- The construction extends to the antianalytic sector, yielding two commuting representations of the Virasoro algebra (Proposition 3.6, Theorem 4.2).

Let us denote by $\mathbb{Z}^2_\star$ the dual of $\mathbb{Z}^2$, by $\mathbb{Z}^2_\diamond$ the diamond graph $\mathbb{Z}^2 \cup \mathbb{Z}^2_\star$, and by $\mathbb{Z}^2_m$ the medial graph of $\mathbb{Z}^2$ (the set of midedges of $\mathbb{Z}^2$), see Figure 1.1. The lattice representation of Theorem 4.2 relies on the construction of discrete currents and monomials that live on these graphs.

Proposition (Sections 2 and 3)**.** For $k \in \mathbb{Z}$, there exist natural discrete analogues $[\partial\Phi] : \mathbb{Z}^2_m \to \mathbb{C}$, $z^{[k]} : \mathbb{Z}^2_\diamond \cup \mathbb{Z}^2_m \to \mathbb{C}$, and $\oint_{[\gamma]} [\partial\Phi](z) z^{[k]} dz$ of the current $\partial\Phi$, the monomial functions $z \mapsto z^k$, and the contour integrals $\oint_\gamma \partial\Phi(z) z^k dz$, respectively, such that the following holds: If

$$J(z) := [\partial\Phi](z)$$

is the discrete current, then

$$z^{[0]} \equiv 1 \qquad \text{and} \qquad \langle J(z)\Phi(x)\rangle = \frac{1}{2\pi}\left(z^{[-1]} - (z-x)^{[-1]}\right)$$

for $z \in \mathbb{Z}^2_m; x \in \mathbb{Z}^2$. Moreover, we have

$$2\mathrm{i}\pi \oint_{[\gamma]} \langle J(w)J(z)\rangle z^{[k]} dz = k\, w^{[k-1]};$$

for all $k \in \mathbb{Z}$, $w \in \mathbb{Z}^2_m$, whenever $[\gamma]$ is a sufficiently large closed simple positively oriented discrete contour.

Let us now define the space of Gibbs measures relevant to our framework.

Definition 1.1. We define $\mathcal{M}$ to be the vector space of complex Gibbs measures μ such that:

- The Gibbs measure $\mu \in \mathcal{M}$ is a change of measure of $\mathsf{P}_{\mathrm{GFF}}$;
- For every finite $G \subset \mathbb{Z}^2$, the Radon-Nikodym derivative of $\mu|_G$ with respect to $\mathsf{P}_{\mathrm{GFF}}|_G$, denoted by g^μ_G, lies in $L^p(\mathsf{P}_{\mathrm{GFF}}|_G)$ for every $p < \infty$.

We also define the subspaces $\mathcal{S} \subset \mathcal{M}$ and $\mathcal{A} \subset \mathcal{M}$ of symmetric and antisymmetric measures in $\mathcal{M}$, respectively. In other words, $\mathcal{S}$ and $\mathcal{A}$ consist of the measures in $\mathcal{M}$ with even and odd Radon-Nikodym derivatives g^μ_G, respectively.

Remark 1.2. For each G, the linear span of polynomials in the Gaussian variables $\{\Phi(z) : z \in G\}$ is dense in $L^p\left(\mathsf{P}_{\mathrm{GFF}}|_G\right)$ for every $0 \leq p < \infty$, see, e.g., [Jan97, Theorem 2.11].

Lemma (Lemma 3.2)**.** For any (not necessarily distinct) $x_1, \ldots, x_m \in \mathbb{Z}^2$ and every $n \in \mathbb{Z}$ the discrete current mode a_n acting on field insertions by the contour integral

$$\langle a_n \Phi(x_1)\cdots\Phi(x_m)\rangle = \frac{1}{\sqrt{\pi}} \oint_{[\gamma]} \langle J(z)\Phi(x_1)\cdots\Phi(x_m)\rangle z^{[n]} dz$$

is independent of the choice of the contour $[\gamma]$ provided that $[\gamma]$ encircles a large enough neighborhood of origin in the positive direction.

More generally, the product of discrete current modes acting on field insertions by contour integrals

$$\begin{aligned}\left\langle a_{n_j}\cdots a_{n_1}\Phi(x_1)\cdots\Phi(x_m)\right\rangle \\ := \frac{1}{\pi^{j/2}} \oint_{[\gamma_j]}\cdots\oint_{[\gamma_1]} \langle J(w_j)\cdots J(w_1)\Phi(x_1)\cdots\Phi(x_m)\rangle w_1^{[n_1]}\cdots w_j^{[n_j]} dw_1\cdots dw_j\end{aligned}$$

is independent on the choice of sequence of discrete contours $[\gamma_1]; [\gamma_2]; \ldots; [\gamma_j]$ provided the contours encircle sufficiently large neighborhoods of the origin and are radially ordered: for $l < k$ the contour $[\gamma_l]$ (corresponding to a_{n_l}) is contained in the set encircled by the contour $[\gamma_k]$ (corresponding to a_{n_k}).

Remark 1.3. For any given realization, the contour integral $\oint_{[\gamma]} J(z) z^{[n]} dz$ does depend on the contour $[\gamma]$, but the expected value against any insertion $\phi(x_1)\cdots\phi(x_n)$ is the same for all positively oriented discrete contours $[\gamma]$ enclosing a large enough neighborhood of the origin.

By Lemma 3.2, the insertions of $\oint_{[\gamma]} J(z) z^{[n]} dz$ (and the corresponding products) into correlation functions of the discrete GFF (and its changes of measure) are well-defined provided we take a large enough contour $[\gamma]$ (or radially ordered sequence of contours). We show that the current modes $(a_n)_{n\in\mathbb{Z}}$ (and their products) lift to unique change of measure operators $(a_n)_{n\in\mathbb{Z}}$ which act on $\mathcal{M}$ and which can be though of as formal adjoint operators to the a_n. (This point of view is behind the use of a_{-n} in the definition of a_n below.) These change of measure operators are the building blocks used in the construction of the operators $(L_n)_{n\in\mathbb{Z}}$ of Theorem 4.2.

Proposition (Proposition 3.4)**.** *For each $n\in\mathbb{Z}$, there exists a unique operator $a_{-n}:\mathcal{M}\to\mathcal{M}$ such that for any $\mu\in\mathcal{M}$ and any (not necessarily distinct) $x_1,\dots,x_k\in\mathbb{Z}^2$ and any finite $G\subset\mathbb{Z}^2$ containing the points, we have*

$$\begin{aligned}\langle\phi(x_1)\cdots\phi(x_k)\rangle_{a_{-n}\mu} &= \langle a_n\phi(x_1)\cdots\phi(x_k)\rangle_{\mu}\\ &:= \frac{1}{\sqrt{\pi}}\oint_{[\gamma]}\langle J(z)\phi(x_1)\cdots\phi(x_m)\,g(\phi)\rangle z^{[n]}dz;\end{aligned}$$

where g is the Radon-Nikodym derivative of $\mu|_G$ with respect to $\mathbb{P}_{GFF}|_G$, and $[\gamma]$ is any discrete contour that encircles a large enough neighborhood of the origin. The operator a_{-n} is parity reversing.

We then show that the operators $(a_n)_{n\in\mathbb{Z}}$ satisfy the commutation relations of the Heisenberg algebra:

Theorem (Theorem 3.5)**.** *The operators $(a_n)_{n\in\mathbb{Z}}$ satisfy the commutation relations*

$$[a_m, a_n] = m\,\delta_{n+m,0}\,\mathrm{id}_{\mathcal{M}}$$

Finally, we can rely on the classical Sugawara construction [Sug68, Som68] to define Virasoro operators L_n. See [Mic89] for a nice textbook about the algebra of the current modes. In the following statement and in the rest of the paper, a function $f:\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{C}$ is said to be a *cylinder function* if it depends only on finitely many coordinates, called its *base*.

Theorem (Theorem 4.2)**.** *For each $n\in\mathbb{Z}$, in the formally infinite sum*

$$L_n := \frac{1}{2}\sum_{k\geq 0} a_{n-k}a_k + \frac{1}{2}\sum_{k<0} a_k a_{n-k};$$

only finitely many terms are non-zero when acting on an insertion of a cylinder function. The action of L_{-n} on insertions lifts to a well-defined parity preserving change of measure operator $L_n:\mathcal{M}\to\mathcal{M}$.

The $(L_n)_{n\in\mathbb{Z}}$ yield a representation of the Virasoro algebra with central charge $c=1$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\,\delta_{m+n,0}\,\mathrm{id}_{\mathcal{M}}:$$

Remark 1.4. The above expression for Virasoro generators as a formally infinite sum of quadratic polynomials in the currents is standard in Sugawara constructions. In usual algebraic Sugawara constructions, the expression has well defined action in representations of an appropriate type, owing to its truncation to only finitely many terms when applied to any particular vector — it is sufficient that the representation has a grading by L_0-eigenvalues that are bounded from below. Our action on insertions by L_n truncates in a similar manner, now because of the finite base of the inserted cylinder function. We remark, however, that the action on interesting Gibbs measures by L_n can not be recovered as a finite linear combination of quadratic expressions in $(a_m)_{m\in\mathbb{Z}}$.

The operators above build a discrete version of the analytic sector of the CFT of the free field. (One could perhaps call it a *discrete analytic sector*.) As in the continuum case, we can also build the antianalytic sector in the same manner, and we thus obtain two commuting representations of the Virasoro algebra.

Proposition (Proposition 3.6, Theorem 4.2). *For $n \in \mathbb{Z}$, let $\bar{a}_n : \mathcal{M} \to \mathcal{M}$ be the complex conjugate of the change of measure operator a_n. For all $n, m \in \mathbb{Z}$, we have $[\bar{a}_m, a_n] = 0$ and $[\bar{a}_m, \bar{a}_n] = m\,\delta_{m+n,0}\,\mathrm{id}_{\mathcal{M}}$. Defining L_n as in Proposition 4.2 and $\bar{L}_n$ as its complex conjugate, we have that $(L_n)_{n\in\mathbb{Z}}$ and $(\bar{L}_n)_{n\in\mathbb{Z}}$ yield two commuting representations of the Virasoro algebra with central charge $c = 1$:*

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n+m}\,\mathrm{id}_{\mathcal{M}} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n+m}\,\mathrm{id}_{\mathcal{M}} \\ [L_n, \bar{L}_m] &= 0. \end{aligned}$$

Both representations consist of parity preserving change of measure operators.

Finally, we can use the background charge idea in the Coulomb gas construction to yield representations of the Virasoro algebra at arbitrary central charges. When $c \neq 1$, however, the operators $L^b_n, \bar{L}^b_n$ are not parity preserving, and thus one loses a symmetry of the original problem.

Proposition (Proposition 5.1). *Let $c \leq 1$ and let $b = \pm\sqrt{\frac{1-c}{12}}$. If we define the change of measure operators $L^b_n, \bar{L}^b_n : \mathcal{M} \to \mathcal{M}$ by $L^b_n := L_n + b(n+1)a_n$, with L_n and a_n as above, we have that $(L^b_n)_{n\in\mathbb{Z}}$ and $(\bar{L}^b_n)_{n\in\mathbb{Z}}$ yield two commuting representations of the Virasoro algebra with central charge c.*

1.8. **General remarks.** In conformal field theory, the basis $(L_n)_{n\in\mathbb{Z}}$ that spans the Virasoro algebra is given by modes of the (analytic component of the) stress-energy tensor $T(z)$. By comparing with the continuum GFF one might therefore attempt to define discrete L_n modes as discrete modes of (a constant times) $[T(z)] = [J(z)]^2$. However, there seems to be no natural way to do this so that one has independence on the choice of integration contour. Instead, the Sugawara construction of the L_n modes from the discrete current modes can be viewed as taking place directly in "Fourier space" on the level of operator valued Laurent modes. Our constructions give concrete meaning to these objects.

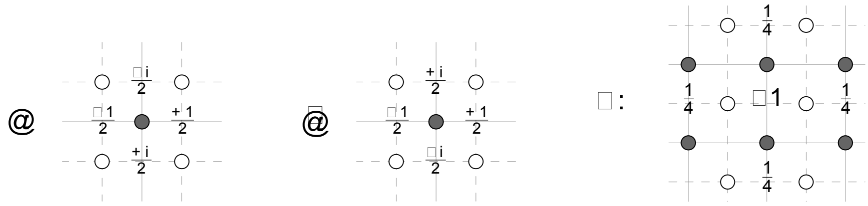

FIGURE 2.1. Illustrations of the weights in the finite difference operators ∂ and $\bar\partial$ and $\Delta = \partial\bar\partial$ on particular sublattices. The weights of other sublattices are translations of the ones shown.

Acknowledgments. CH is supported by the Minerva Foundation and by the National Science Foundation grant number DMS-1106588. FJV is supported by the Simons foundation. KK is supported by the Academy of Finland. We would like to thank Stéphane Benoist, Dmitry Chelkak, Julien Dubédat, Krzysztof Gawędzki, Igor Krichever, Antti Kupiainen, Jouko Mickelsson, Andrei Okounkov, Eveliina Peltola, Stanislav Smirnov and Valerio Toledano Laredo for interesting discussions.

2. Discrete currents, polynomials, and contour integrals

We now describe in detail the basic building blocks in our construction.

2.1. **Finite difference operators.** Recall the definitions of the square grid $\mathbb{Z}^2$ with vertices $\{m + in : m, n \in \mathbb{Z}\}$, its dual $\mathbb{Z}^2_*$ with vertices $\{m + \frac{1}{2} + i(n + \frac{1}{2}) : m, n \in \mathbb{Z}\}$, the diamond graph with vertices $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$, and the medial graph $\mathbb{Z}^2_m$ with a vertex at the midpoint of every edge of $\mathbb{Z}^2$. We will sometimes identify an edge of a graph with its midpoint.

For $f : \mathbb{Z}^2_\diamond \to \mathbb{C}$ (and $f : \mathbb{Z}^2_m \to \mathbb{C}$, respectively), we define $[\partial f] : \mathbb{Z}^2_m \to \mathbb{C}$ (and $[\partial f] : \mathbb{Z}^2_\diamond \to \mathbb{C}$, respectively) by

$$[\partial f](z) = \sum_{a \in \{\pm\frac{1}{2}, \pm\frac{i}{2}\}} \bar{a} f(z + a).$$

Similarly, we define $[\bar\partial f]$ by

$$[\bar\partial f](z) = \sum_{a \in \{\pm\frac{1}{2}, \pm\frac{i}{2}\}} a f(z + a).$$

We say that $f : \mathbb{Z}^2_\diamond \to \mathbb{C}$ (or $f : \mathbb{Z}^2_m \to \mathbb{C}$) is *discrete holomorphic* at z if $[\bar\partial f](z) = 0$.

For a function f defined on the square grid $\mathbb{Z}^2$ (more generally, on a shifted square grid, e.g., the dual $\mathbb{Z}^2_*$), the discrete Laplacian $[\Delta f] : \mathbb{Z}^2 \to \mathbb{C}$ is defined by

$$[\Delta f](z) = \frac{1}{4} \sum_{\xi \in \{\pm 1, \pm i\}} f(z + \xi) - f(z);$$

where the latter sum is over the four nearest neighbors w of z. Note that for f defined on $\mathbb{Z}^2_\diamond$ (or on $\mathbb{Z}^2_m$), the composition $\partial\bar\partial = \bar\partial\partial = \Delta$ is the above Laplacian, which in particular splits to two blocks corresponding to $\mathbb{Z}^2$ and $\mathbb{Z}^2_*$ (on $\mathbb{Z}^2_m$ the two blocks correspond to midpoints of horizontal and vertical edges).

We record a property that is used in some calculations below.

Lemma 2.1. *The transposes of the finite difference operators ∂ are $-\partial$ (the operator $\mathbb{C}^{\mathbb{Z}^2_m} \to \mathbb{C}^{\mathbb{Z}^2_\diamond}$ has transpose $\mathbb{C}^{\mathbb{Z}^2_\diamond} \to \mathbb{C}^{\mathbb{Z}^2_m}$ and vice versa). In particular, if $f : \mathbb{Z}^2_m \to \mathbb{C}$ and $g : \mathbb{Z}^2_\diamond \to \mathbb{C}$ are functions, at least one of which has finite support, then*

$$\sum_{z \in \mathbb{Z}^2_m} f(z)\, [\partial g](z) = -\sum_{z \in \mathbb{Z}^2_\diamond} [\partial f](z)\, g(z).$$

Similarly, the transposes of $\bar\partial$ are $-\bar\partial$.

2.2. **Potential kernel and discrete Cauchy kernel.** In the full plane, the expected number of visits to a point by simple random walk $(X_t)_{t \in \mathbb{N}}$ is infinite, and for this reason a naive definition of the (massless) discrete Gaussian free field will not work and one needs to consider a regularized version instead.

We shall use the massive regularization based on the massive Green's function $G_m : \mathbb{Z}^2 \to \mathbb{R}$, $m > 0$, which is the function decaying at infinity that satisfies $m^2 G_m - [\Delta G_m] = \delta_0$. If we subtract a suitable diverging constant, for example the value at the origin, the massless limit $m \searrow 0$ can be taken. We define the *potential kernel* $a : \mathbb{Z}^2 \to \mathbb{R}$ by

$$a(z) = \lim_{m \searrow 0} \big(G_m(0) - G_m(z) \big).$$

It is a function satisfying $[\Delta a](z) = \delta_0(z)$ and having logarithmic growth at infinity, and it is determined up to an additive constant by these conditions. In fact we have

$$a(z) = \frac{2}{\pi} \log|z| + \frac{2\gamma + \log 8}{\pi} + O(|z|^{-2})$$

as $z \to \infty$, see, e.g., [LaLi10, Sections 4 and 6]. For convenience, we extend the potential kernel to the diamond lattice $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$ by setting the values on $\mathbb{Z}^2_*$ to zero. With this extension, we can consider the function

$$[\partial a] : \mathbb{Z}^2_m \to \mathbb{C},$$

which is purely real on the horizontal edges and purely imaginary on the vertical edges of $\mathbb{Z}^2$.

See [Hon10, Ken00] for discussion and references to the following result.

Lemma 2.2. *The function $K = [\partial a]$ is the unique function $K : \mathbb{Z}^2_m \to \mathbb{C}$ satisfying $[\bar\partial K](z) = \delta_0(z)$ and $K(z) \to 0$ as $|z| \to \infty$. We call K the* discrete Cauchy kernel *with pole at 0.*

Proof. Clearly $K = [\partial a]$ satisfies $[\bar\partial K] = [\bar\partial \partial a] = [\Delta a] = \delta_0$, and K decays at infinity.

To show uniqueness, consider the difference of two functions satisfying the properties. The difference is a function on $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$, which is annihilated by $\bar\partial$ and in particular its restrictions to $\mathbb{Z}^2$ and $\mathbb{Z}^2_*$ are harmonic and decaying at infinity, thus zero. ∎

2.3. **Discrete Laurent monomials.** We next define discrete Laurent monomial functions on both $\mathbb{Z}^2_\diamond$ and $\mathbb{Z}^2_m$. The appropriate neighborhoods of origin are measured in the norm $\|x + iy\|_1 = |x| + |y|$, and we denote these neighborhoods on the two lattices by

$$\overline{B}_{\diamond/m}(r) = \left\{ z \in \mathbb{Z}^2_{\diamond/m} : \|z\|_1 \leq r \right\}.$$

Lemma 2.3. There exist unique functions $z \mapsto z^{[k]}$, $k \in \mathbb{Z}$, defined for $z \in \mathbb{Z}^2_\diamond \cup \mathbb{Z}^2_m$ such that:

- The functions $z \mapsto z^{[k]}$ for $k \in \{-1, 0, 1\}$ are given by
$$z^{[0]} \equiv 1; \qquad z^{[1]} = z$$
and
$$z^{[-1]} = \begin{cases} 2 \cdot K(z) & \text{for } z \in \mathbb{Z}^2_m \\ \frac{\pi}{2} \sum_{a \in \{\pm\frac{1}{2}, \pm\frac{i}{2}\}} K(z-a) & \text{for } z \in \mathbb{Z}^2_\diamond \end{cases}.$$
- For all $k \in \mathbb{Z}$ we have

(2.1) $$[\partial z^{[k]}] = k z^{[k-1]}.$$

- For $k > 0$ the function $z \mapsto z^{[k]}$ is discrete holomorphic in the sense that
$$[\bar\partial z^{[k]}] \equiv 0.$$
- For all $k > 0$ we have the vanishing at origin conditions
$$0^{[k]} = 0; \qquad \left(\frac{1}{2}\right)^{[k]} + \left(\frac{-1}{2}\right)^{[k]} = 0; \qquad \left(\frac{i}{2}\right)^{[k]} + \left(\frac{-i}{2}\right)^{[k]} = 0;$$
$$\text{and} \quad \left(\frac{1+i}{2}\right)^{[k]} + \left(\frac{1-i}{2}\right)^{[k]} + \left(\frac{-1+i}{2}\right)^{[k]} + \left(\frac{-1-i}{2}\right)^{[k]} = 0.$$

The functions $z \mapsto z^{[k]}$ are called the Laurent monomials, and they have the following further properties:

- For all $k \in \mathbb{Z}$ and $z \in \mathbb{Z}^2_\diamond \cup \mathbb{Z}^2_m$ we have $\bar{z}^{[k]} = \overline{z^{[k]}}$.
- For all $k \in \mathbb{Z}$ and $z \in \mathbb{Z}^2_\diamond \cup \mathbb{Z}^2_m$ we have $(iz)^{[k]} = i^k z^{[k]}$.
- For $k > 0$ the function $z \mapsto z^{[k]}$ vanishes for $z \in \overline{B}_{\diamond/m}(\frac{k-1}{2})$.
- For $k \leq -1$ the function $z \mapsto z^{[k]}$ is discrete holomorphic outside a neighborhood of the origin in the sense that $[\bar\partial z^{[k]}] = 0$ for $z \notin \overline{B}_{\diamond/m}(\frac{|k|}{2})$.

In the proof below we use discrete integrations that differ slightly from the important integrals later (for the attentive reader there should be no confusion, however, as later we will use integration contours on a different lattice, and we will integrate products of two functions). Below, for a function f defined at the midpoints of edges of $\mathbb{Z}^2$ (i.e., on $\mathbb{Z}^2_m$), we define the integral along the oriented edge $\overrightarrow{zw}$ ($z, w \in \mathbb{Z}^2$, $|z-w| = 1$) by the evident expression $(w-z) \cdot f(\frac{z+w}{2})$. The integral along an oriented path is the sum of the integrals along the oriented edges of the path. Note that the positively oriented integral around a plaquette surrounding $z \in \mathbb{Z}^2_\diamond$ of $f : \mathbb{Z}^2_m \to \mathbb{C}$ is just $2i[\bar\partial f](z)$. The set of horizontal edges $(\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z} \subset \mathbb{Z}^2_m$, the set of vertical edges $\mathbb{Z} \times (\mathbb{Z} + \frac{1}{2}) \subset \mathbb{Z}^2_m$, and the dual $\mathbb{Z}^2_\diamond = (\mathbb{Z} + \frac{1}{2})^2$ are all translates of the standard square grid $\mathbb{Z}^2$, and we define integrals on these with obvious modifications.

Proof of Lemma 2.3. The case of negative k is obvious once $z^{[-1]}$ has been defined by the formula given above. Indeed, to have the correct derivatives of Equation 2.1, we need to set
$$z^{[-1-m]} = \frac{(-1)^m}{m!} [\partial^m z^{[-1]}].$$

It follows from the property $[\bar\partial z^{[-1]}] = 2\pi\delta_0(z) + \frac{\pi}{2}\sum_a \delta_a(z)$ and the definition of the difference operator $\bar\partial$ that $[\bar\partial z^{[-1-m]}]$ vanishes outside the neighborhood $\overline{B}_{\diamond/m}(\frac{1+m}{2})$ of the origin. The symmetry $(iz)^{[-1-m]} = i^{-1-m}z^{[-1-m]}$ follows from the symmetry $a(iz) = a(z)$ of the potential kernel, and the expression for ∂. The symmetry $\bar{z}^{[-1-m]} = \overline{z^{[-1-m]}}$ similarly follows from $a(\bar{z}) = a(z)$ and the expression for ∂.

For positive k, we proceed recursively. We first define $z^{[0]}$ and $z^{[1]}$ by the given formulas (they obviously satisfy the stated properties). Then for $k \geq 2$ we define $z^{[k]}$ by integrating $kz^{[k-1]}$. For $z \in \mathbb{Z}^2$ the path of integration is any path from $0 \in \mathbb{Z}^2$ to z. The result is independent of the choice of such path because $[\bar\partial z^{[k-1]}] \equiv 0$. For z (a midpoint of) a horizontal edge the path of integration is any path from $\frac{1}{2} \in (\mathbb{Z}+\frac{1}{2}) \times \mathbb{Z}$ to z (we could alternatively take the starting point $-\frac{1}{2}$ without any change, since $0^{[k-1]} = 0$), and for z (a midpoint of) a vertical edge the path of integration is any path from $\frac{i}{2} \in \mathbb{Z} \times (\mathbb{Z}+\frac{1}{2})$ to z (we could alternatively take $-\frac{i}{2}$ as the starting point). For $z \in \mathbb{Z}^2_\diamond$ the path of integration is any path from $\frac{1+i}{2} \in \mathbb{Z}^2_\diamond$ to z, and we include an additive constant so as to guarantee $\sum_{a \in \{\pm\frac{1+i}{2}, \pm\frac{1-i}{2}\}} a^{[k]} = 0$ (in fact, this additive constant is zero except when $k = 2$, in which case we have to set $(\frac{1+i}{2})^{[2]} = \frac{i}{2}$).

To calculate $[\partial z^{[k]}]$, write it as

$$[\partial z^{[k]}] = \frac{1}{2}\left[\left(z+\frac{1}{2}\right)^{[k]} - \left(z-\frac{1}{2}\right)^{[k]}\right] - \frac{i}{2}\left[\left(z+\frac{i}{2}\right)^{[k]} - \left(z-\frac{i}{2}\right)^{[k]}\right]$$

and note that the two terms are both equal to $\frac{1}{2}kz^{[k-1]}$ by the construction of $z^{[k]}$ as an integral. Similarly, in the calculation of $[\bar\partial z^{[k]}]$ the two contributions cancel. We have thus constructed Laurent monomials with the desired defining properties, and uniqueness is clear since only constant functions are annihilated by both ∂ and $\bar\partial$.

The fact that $z^{[k]}$ is 0 in $\overline{B}_{\diamond/m}(\frac{k-1}{2})$ follows inductively: the vanishing of $z^{[k-1]}$ in a neighborhood of origin makes its integral $z^{[k]}$ constant in a slightly bigger neighborhood, and this constant is fixed by the vanishing at origin conditions. The symmetries $(iz)^{[k]} = i^k z^{[k]}$ and $\bar{z}^{[k]} = \overline{z^{[k]}}$ are clear by the construction. □

2.4. **Discrete contour integrals of products of two functions.** We now define the important notion of discrete contour integral of a product of two functions. The contour of integration will be a path on the shifted half-mesh square lattice $(\frac{1}{2}\mathbb{Z}+\frac{1}{4})^2$. Note that any edge of this lattice is between a point of $\mathbb{Z}^2_m$ and a point of $\mathbb{Z}^2_\diamond$, both at distance $\frac{1}{4}$ of the edge. Let $f : \mathbb{Z}^2_m \to \mathbb{C}$ and $g : \mathbb{Z}^2_\diamond \to \mathbb{C}$ be two functions. We define the integral of the product of f and g along an oriented edge $[\vec{uv}]$ ($u, v \in (\frac{1}{2}\mathbb{Z}+\frac{1}{4})^2$, $|u-v| = \frac{1}{2}$) as

$$\int_{[\vec{uv}]} f(z)g(z)dz := (v-u) \cdot f(z_m)g(z_\diamond);$$

where $z_m \in \mathbb{Z}^2_m$ and $z_\diamond \in \mathbb{Z}^2_\diamond$ are the points at distance $\frac{1}{4}$ of $[\vec{uv}]$. The integral along an oriented path $[\gamma]$ on $(\frac{1}{2}\mathbb{Z}+\frac{1}{4})^2$ is the sum of the integrals along the oriented edges of the path, and is denoted by $\int_{[\gamma]} f(z)g(z)dz$ (by convention, we write the function defined on $\mathbb{Z}^2_m$ first, and the function defined on $\mathbb{Z}^2_\diamond$ second).

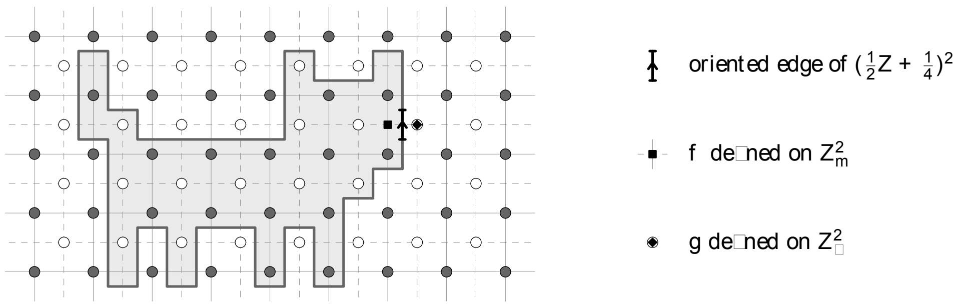

FIGURE 2.2. A discrete contour is a closed counterclockwise oriented path on the shifted half-mesh square lattice $(\frac{1}{2}\mathbb{Z}^2+\frac{1}{4})^2$.

By discrete contour we mean a positively oriented (i.e. counterclockwise) closed simple path $[\gamma]$ on $(\frac{1}{2}\mathbb{Z}+\frac{1}{4})^2$. We denote by

$$\mathrm{int}_\diamond[\gamma] \subset \mathbb{Z}^2_\diamond \qquad \text{(respectively } \mathrm{int}_m[\gamma] \subset \mathbb{Z}^2_m\text{)}$$

the set of points of the diamond lattice (respectively the medial lattice) encircled by $[\gamma]$. The discrete closure $\overline{\mathrm{int}}_\diamond[\gamma] \subset \mathbb{Z}^2_\diamond$ (respectively $\overline{\mathrm{int}}_m[\gamma] \subset \mathbb{Z}^2_m$) is the union of $\mathrm{int}_\diamond[\gamma] \subset \mathbb{Z}^2_\diamond$ (respectively $\mathrm{int}_\diamond[\gamma] \subset \mathbb{Z}^2_\diamond$) and the set of points of $\mathbb{Z}^2_\diamond$ (respectively $\mathbb{Z}^2_m$) at distance $\frac{1}{4}$ from $[\gamma]$. The integrals over such closed discrete contours are denoted by

$$\oint_{[\gamma]} f(z)g(z)dz,$$

and they can be evaluated with the help of the following combination of discrete Leibnitz rule and Stokes' formula.

Lemma 2.4. Let $f : \mathbb{Z}^2_m \to \mathbb{C}$ and $g : \mathbb{Z}^2_\diamond \to \mathbb{C}$ and $[\gamma]$ a discrete contour. Then we have

$$\oint_{[\gamma]} f(z)g(z)dz = i \sum_{z\in \mathrm{int}_m[\gamma]} f(z)\,[\bar\partial g](z) + i \sum_{z\in \mathrm{int}_\diamond[\gamma]} [\bar\partial f](z)\,g(z).$$

Proof. Decompose the integral to a sum of integrals around elementary plaquettes of the shifted half-mesh square lattice $(\frac{1}{2}\mathbb{Z}+\frac{1}{4})^2$, and notice that the integral around a plaquette that surrounds $z \in \mathbb{Z}^2_m$ is $if(z)[\bar\partial g](z)$ and the integral around a plaquette that surrounds $z \in \mathbb{Z}^2_\diamond$ is $i[\bar\partial f](z)\,g(z)$. □

2.5. **Discrete residue calculus.** As a consequence of the above simple lemma, we have the following discrete analog of a standard residue calculation.

Lemma 2.5. Let $m, n \in \mathbb{Z}$, let $r = \max\{0, -\frac{m}{2}, -\frac{n}{2}\}$, and let $[\gamma]$ be a discrete contour that separates the neighborhood $\overline{B}_{\diamond/m}(r)$ of the origin from infinity. Then we have

$$\frac{1}{2\pi i}\oint_{[\gamma]} z^{[m]} z^{[n]} dz = \delta_{m+n,-1}, \tag{2.2}$$

where $z^{[m]}$ and $z^{[n]}$ are discrete Laurent monomial functions restricted to $\mathbb{Z}^2_m$ and $\mathbb{Z}^2_\diamond$, respectively.

Proof. By Lemma 2.4 we have

$$\frac{1}{i}\oint_{[\gamma]} z^{[m]} z^{[n]}\, dz = \sum_{z\in \mathrm{int}_m[\gamma]} z^{[m]} [\bar\partial z^{[n]}] + \sum_{z\in \mathrm{int}_\diamond[\gamma]} [\bar\partial z^{[m]}] z^{[n]}; \tag{2.3}$$

and we note that by the assumption on $[\gamma]$, the sets $\mathrm{int}_m[\gamma]$ and $\mathrm{int}_\diamond[\gamma]$ contain the supports of $[\bar\partial z^{[n]}]$ and $[\bar\partial z^{[m]}]$, respectively. Also note that if $m, n \geq 0$, then (2.2) follows immediately from (2.3), since $z^{[n]}$ and $z^{[m]}$ are discrete holomorphic.

Suppose $n \geq 0, m \leq -1$. Then only the second sum on the right hand side of (2.3) remains. Using $\partial^* = -\partial$ repeatedly we get that

$$\begin{aligned} \sum_{z\in \mathrm{int}_\diamond[\gamma]} [\bar\partial z^{[m]}] z^{[n]} &= \frac{(-1)^{-m-1}}{(-m-1)!} \sum_{z\in \mathrm{int}_\diamond[\gamma]} [\bar\partial \partial^{-m-1} z^{[-1]}] z^{[n]} \\ &= \frac{1}{(-m-1)!} \sum_{z} [\bar\partial z^{[-1]}] [\partial^{-m-1} z^{[n]}] \\ &= \frac{n!}{(-m-1)!(n+m+1)!} \sum_{z} [\bar\partial z^{[-1]}] z^{[n+m+1]}, \end{aligned} \tag{2.4}$$

where the sums in the last two expressions are over $z \in \mathbb{Z}^2_\diamond$ or $z \in \mathbb{Z}^2_m$, according to whether $-m-1$ is even or odd, respectively. In the two cases, we have that $[\bar\partial z^{[-1]}]$ is $2\pi\delta_0(z)$ or $\frac{\pi}{2}\sum_{a\in\{\pm\frac12,\pm\frac{i}{2}\}} \delta_a(z)$, respectively, and in either case the result of the whole expression evaluates to δ_{n+m+1}. The case $n \leq -1; m \geq 0$ is similar.

Finally, in the case $m, n \leq -1$ rewrite both sums on the right hand side of (2.3) as above. If n and m have the same parity, then each of the rewritten sums vanishes separately because the summands are odd: $[\bar\partial K](-z)\cdot(-z)^{[n+m+1]} = -[\bar\partial K](z)\cdot z^{[n+m+1]}$. If n and m are of different parities, then both rewritten sums are sums over the same sublattice — they are otherwise identical, but they come with opposite signs and cancel each other. □

Remark 2.6. Discrete holomorphic monomials and related constructions have been considered by several authors, apparently beginning with the work of Duffin [Duf56]. We mention also the more recent work of Mercat [Mer01] which was also in part motivated by applications to statistical mechanics models.

3. Discrete current modes

3.1. **The discrete Gaussian free field pinned at the origin.** While the (massless) discrete Gaussian free field can not directly be defined in the full plane, the massive Green's function G_m introduced in Section 2 is the covariance of the well-defined massive discrete Gaussian free field $(\Phi_m(z))_{z\in\mathbb{Z}^2}$,

$$\langle \Phi_m(z)\, \Phi_m(w)\rangle = G_m(z-w).$$

In the massless limit $m \searrow 0$, differences of values of the field remain well-behaved. The pinned discrete Gaussian free field Φ is the limit as $m \searrow 0$ of the field $\Phi_m - \Phi_m(0)$. We can write its correlation functions in terms of the potential kernel a introduced in

Section 2,

$$\begin{aligned}\langle \varphi(z)\,\varphi(w)\rangle &= \lim_{m\searrow 0} \langle (\varphi_m(z)-\varphi_m(0))\,(\varphi_m(w)-\varphi_m(0))\rangle \\ &= a(z)+a(w)-a(z-w).\end{aligned}$$

3.2. **The discrete current.** Let φ be the discrete GFF on $\mathbb{Z}^2$, pinned at 0. To define the *current*

$$J(z) = [\partial\varphi](z) \qquad (z \in \mathbb{Z}^2_m);$$

we first extend the definition of φ to $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$ by setting the value of φ to zero on $\mathbb{Z}^2_*$. By differentiating the two point correlation function

$$\langle \varphi(z)\varphi(w)\rangle = a(z)+a(w)-a(z-w)$$

we can now conveniently write the correlations of the current in terms of $K=[\partial a]$. For example:

$$\begin{aligned}\langle J(z)\varphi(x)\rangle &= \begin{cases} K(z)-K(z-x) & \text{if } x\in\mathbb{Z}^2,\ z\in\mathbb{Z}^2_m, \\ 0 & \text{if } x\in\mathbb{Z}^2_*,\ z\in\mathbb{Z}^2_m\end{cases}, \\ \langle J(z)J(w)\rangle &= [\partial K](w-z).\end{aligned}$$

3.3. **Wick's formula.** Let φ be the discrete GFF on $\mathbb{Z}^2_\diamond$, pinned at 0. We have given the two-point correlations for its current $J=[\partial\varphi]$ in terms of $[\partial K]$. Wick's formula lets us compute higher order correlations of a Gaussian field from its two-point correlations; see [Jan97, Chapter 1] for a proof.

Lemma 3.1 (Bosonic Wick's formula). *Suppose that $(X_i)_{i\in I}$ is a finite collection of centered Gaussian random variables. Then we have*

$$\left\langle \prod_{i\in I} X_i \right\rangle = \sum_P \prod_{\{a,b\}\in P} \langle X_a X_b\rangle, \tag{3.1}$$

where the sum is over all pair partitions P of I, i.e. over collections P of $\frac{|I|}{2}$ mutually disjoint two element subsets of I (the sum is empty if $|I|$ is odd).

We will make calculations involving correlations of the discrete current $J(z)$ and the field $\varphi(x)$. In particular, the Wick expansion of

$$\Big\langle J(z)J(w)\varphi(x_1)\cdots\varphi(x_k)\Big\rangle$$

contains terms of two types: those in which the two currents are paired together

$$\big\langle J(z)J(w)\big\rangle \prod_{\{a,b\}\in P} \big\langle \varphi(x_a)\varphi(x_b)\big\rangle \tag{3.2}$$

$$(P \text{ a pair partition of } \{1,2,\ldots,k\})$$

and those in which both are paired with something else

$$\big\langle J(z)\varphi(x_i)\big\rangle\big\langle J(w)\varphi(x_j)\big\rangle \prod_{\{a,b\}\in P'} \big\langle \varphi(x_a)\varphi(x_b)\big\rangle \tag{3.3}$$

$$(i\neq j \text{ and } P' \text{ a pair partition of } \{1,2,\ldots,k\}\setminus\{i,j\}).$$

3.4. **Discrete current modes acting on insertions.** Suppose that $G \subset \mathbb{Z}^2$ is finite. For any $\mu \in \mathcal{M}$ and any $n \in \mathbb{Z}$ we define discrete current modes a_n acting by discrete contour integrals on insertions of functions $f : \mathbb{R}^G \to \mathbb{C}$ in $L^p(\mathbb{P}_{\mathrm{GFF}}|_G)$ for some $p > 1$:

$$\left\langle a_n f(\phi|_G)\right\rangle_\mu := \frac{1}{\sqrt{\pi}} \oint_{[\gamma]} \left\langle J(z) f(\phi|_G) g(\phi|_G)\right\rangle z^{[n]}\, dz \tag{3.4}$$

where $[\gamma]$ is any discrete contour separating $G \cup \overline{B}_{\delta/m}(\max\{0, -\frac{n}{2}\})$ from ∞, and $g = g_G^\mu$ is the Radon-Nikodym derivative of $\mu|_G$ with respect to $\mathbb{P}_{\mathrm{GFF}}|_G$.

More generally, we define the iterated action (or "product") of multiple discrete current modes on field insertions by contour integrals

$$\begin{aligned}\left\langle a_{n_j} \cdots a_{n_1} f(\phi|_G)\right\rangle_\mu \\ := \frac{1}{\pi^{j/2}} \oint_{[\gamma_j]} \cdots \oint_{[\gamma_1]} \left\langle J(w_j) \cdots J(w_1) f(\phi|_G) g(\phi|_G)\right\rangle w_1^{[n_1]} \cdots w_j^{[n_j]}\, dw_1 \cdots dw_j,\end{aligned}$$

where the contours are radially ordered: the contour $[\gamma_1]$ of integration of w_1 separates $G \cup \overline{B}_{\delta/m}(\max\{0, -\frac{n_1}{2}\})$ from ∞, and for $k = 2, 3, \ldots, j$, the contour $[\gamma_k]$ of integration of w_k separates $\overline{\mathrm{int}}_m[\gamma_{k-1}] \cup \overline{\mathrm{int}}_\diamond[\gamma_{k-1}] \cup \overline{B}_{\delta/m}(\max\{0, -\frac{n_k}{2}\})$ from ∞.

It follows from Hölder's inequality that the integrals $\langle a_n f(\phi|_G)\rangle_\mu$ and $\langle a_{n_j} \cdots a_{n_1} f(\phi|_G)\rangle_\mu$ make sense for any given choice of contours, but we need to verify that the definition is independent of the choices of contours.

Lemma 3.2. *Suppose that $G \subset \mathbb{Z}^2$ is finite and that $f : \mathbb{R}^G \to \mathbb{C}$ is in $L^p(\mathbb{P}_{\mathrm{GFF}}|_G)$ for some $p > 1$. Then,*

$$\oint_{[\gamma_j]} \cdots \oint_{[\gamma_1]} \left\langle J(w_j) \cdots J(w_1) f(\phi|_G)\right\rangle w_1^{[n_1]} \cdots w_j^{[n_j]}\, dw_1 \cdots dw_j,$$

is independent of the sequence of contours $[\gamma_1], [\gamma_2], \ldots, [\gamma_j]$, provided the contours are radially ordered as above.

Proof. We consider the case of one contour first, and we assume $G \neq \emptyset$. By the domain Markov property of ϕ under $\mathbb{P}_{\mathrm{GFF}}$, if we condition on the values of ϕ on G, the conditional distribution of ϕ outside of G is that of a Gaussian free field with Dirichlet zero boundary conditions plus a discrete harmonic function $h : \mathbb{Z}^2 \setminus G \to \mathbb{C}$ with boundary values on $\partial(\mathbb{Z}^2 \setminus G)$ given by the values of $\phi|_G$. Consequently, by conditioning inside the expected value, since $f(\phi|_G)$ is measurable with respect to the values of ϕ on G, we have

$$\left\langle J(z) f(\phi|_G)\right\rangle = \left\langle [\partial h](z) f(\phi|_G)\right\rangle, \qquad z \in \mathbb{Z}_m^2. \tag{3.5}$$

Now suppose $[\gamma']$ is another positively oriented closed discrete contour that also separates $G \cup \overline{B}_{\delta/m}(\max\{0, -\frac{n}{2}\})$ from infinity, and which we may for simplicity assume to encircle at least the same set as $[\gamma]$. The identity (3.5) now shows that

$$\left(\oint_{[\gamma']} - \oint_{[\gamma]}\right)\left\langle J(z) f(\Phi|_G)\right\rangle z^{[n]}dz = \left(\oint_{[\gamma']} - \oint_{[\gamma]}\right)\left\langle [\partial h](z) f(\Phi|_G)\right\rangle z^{[n]}dz$$
$$= \left\langle f(\Phi|_G) \left(\oint_{[\gamma']} - \oint_{[\gamma]}\right)[\partial h](z) z^{[n]}dz\right\rangle = 0.$$

Indeed, the last equality follows since $z^{[n]}$ is discrete holomorphic outside of $\overline{B}_{\ell/m}(-\frac{n}{2})$, and so for any discrete harmonic function h as above, the formula of Lemma 2.4 gives

$$\frac{1}{i}\left(\oint_{[\gamma']} - \oint_{[\gamma]}\right)[\partial h](z) z^{[n]}dz$$
$$= \sum_{z\in \mathrm{int}_m[\gamma']\setminus \mathrm{int}_m[\gamma]} [\partial h](z)\underbrace{[\bar\partial z^{[n]}]}_{=0} + \sum_{z\in \mathrm{int}_\diamond[\gamma']\setminus \mathrm{int}_\diamond[\gamma]} \underbrace{[\Delta h](z)}_{=0} z^{[n]}.$$

It remains to check the case with several radially ordered contours. By conditioning on the values of Φ on G and on the contours $[\gamma_{k-1}]$ and $[\gamma_{k+1}]$ (more precisely, on the vertices of $\mathbb{Z}^2$ at distance $\frac{1}{4}$ from these contours), a similar argument as the one just given implies that we can move $[\gamma_k]$ as long as it is between $[\gamma_{k-1}]$ and $[\gamma_{k+1}]$. $\square$

3.5. **Current modes as change of measure operators.** Each discrete current mode a_n induces a change of measure operator acting on $\mathcal{M}$, denoted by a_{-n}.

We first state an auxiliary result on the reconstruction of the change of measure operator from a consistent family of operators acting on insertions. Below we denote briefly $L^p_G = L^p(\mathsf{P}_{\mathrm{GFF}}|_G)$. Note that a Gibbs measure $\mu \in \mathcal{M}$ is determined by the collection of Radon-Nikodym derivatives $(g_G)_{G\subset\mathbb{Z}^2 \text{ finite}}$,

$$g_G = \frac{d\mu|_G}{d\mathsf{P}_{\mathrm{GFF}}|_G} \in \bigcap_{p<\infty} L^p_G,$$

which satisfy the consistency conditions that for any $G\subset G'$ we have

$$g_G(\Phi|_G) = \mathsf{E}\left[g_{G'}(\Phi|_{G'})\,\middle|\,\Phi|_G\right],$$

where the right hand side is the conditional expected value over the discrete GFF Φ given its values on G. Conversely, any collection $(g_G)_{G\subset\mathbb{Z}^2 \text{ finite}}$ of functions $g_G \in \bigcap_{p<\infty} L^p_G$ satisfying the above consistency condition determines a Gibbs measure $\mu\in\mathcal{M}$.

Lemma 3.3. *Suppose that $(A_G)_{G\subset\mathbb{Z}^2 \text{ finite}}$ is a family of linear functionals*

$$A_G : \bigcup_{p>1} L^p_G \to \mathbb{C}$$

such that

- *for any $p > 1$, the functional A_G restricted to L^p_G is a bounded operator*
- *for all $G\subset G'$ and $f\in L^p_G$, the consistency $A_{G'}(f) = A_G(f)$ holds.*

Then there exists a unique change of measure operator $A : \mathcal{M} \to \mathcal{M}$ such that for any $\mu \in \mathcal{M}$ and G and any $f \in L^p_G$, $p > 1$, we have

$$\langle f \rangle_{A\mu} = A_G(f\, g^\mu_G).$$

Proof. Fix a Gibbs measure $\mu \in \mathcal{M}$. Uniqueness of the Gibbs measure $A\mu$ with the above correlation functions is clear, so it suffices to construct one and prove that $A\mu \in \mathcal{M}$.

Fixing first also G, we note that the map

$$f \mapsto A_G(f\, g^\mu_G)$$

is a bounded linear functional on any L^p_G, $p > 1$. It follows from L^p-space duality, that there is a unique function $\rho^\mu_G \in \bigcap_{p<\infty} L^p_G$ such that we can write

$$A_G(f\, g^\mu_G) = \left\langle f\, \rho^\mu_G \right\rangle.$$

Consistency of the operators implies that for all $f \in L^p_G$ and for any $G' \supset G$ we have

$$\left\langle f\, \rho^\mu_{G'} \right\rangle = \left\langle f\, \rho^\mu_G \right\rangle.$$

Thus the collection $(\rho_G)_{G \subset \mathbb{Z}^2 \text{ finite}}$ is consistent and can be used to define the Gibbs measure $A\mu$ such that the Radon-Nikodym derivative of $(A\mu)\big|_G$ is ρ_G. By construction $A\mu$ has the desired correlation functions, the Radon-Nikodym derivative ρ^μ_G is in L^p_G for all $p < \infty$, and $(A\mu)\big|_G$ is absolutely continuous with respect to $\mu\big|_G$ — so we have that $A\mu \in \mathcal{M}$ is a change of measure of μ. ∎

Proposition 3.4. *For each $n \in \mathbb{Z}$, there exists a unique operator $a_n : \mathcal{M} \to \mathcal{M}$ such that for any $\mu \in \mathcal{M}$ and any $x_1, \ldots, x_k \in \mathbb{Z}^2$,*

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle_{a_n \mu} = \langle a_{-n}\, \phi(x_1) \cdots \phi(x_k) \rangle_\mu.$$

The operator a_n is parity reversing. Moreover, for any $n_1, \ldots, n_j \in \mathbb{Z}$, the composition of operators $a_{n_1} \cdots a_{n_j} : \mathcal{M} \to \mathcal{M}$ is characterized by the formulas

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle_{a_{n_1} \cdots a_{n_j} \mu} = \left\langle a_{-n_j} \cdots a_{-n_1}\, \phi(x_1) \cdots \phi(x_k) \right\rangle_\mu;$$

for all $\mu \in \mathcal{M}$ and $x_1, \ldots, x_k \in \mathbb{Z}^2$, and we recall that the product of current modes $a_{-n_j} \cdots a_{-n_1}$ acting on insertions is taken radially ordered.

Proof. Since polynomials in $(\phi(x))_{x \in G}$ are dense in any L^p_G, and the association $f \mapsto \langle a_{-n} f \rangle$ is bounded in any L^p_G, the existence and uniqueness of the change of measure operator a_n follows from Lemma 3.3. The formula for the correlation functions of $a_{n_1} \cdots a_{n_j} \mu$ is obtained iteratively, and the uniqueness of the measure with these correlation functions is again clear. To show that a_n takes symmetric measures to antisymmetric measures and vice versa, just note that the Radon-Nikodym derivatives ρ^μ_G for $a_n \mu$, as constructed in Lemma 3.3, have the opposite parity compared to the Radon-Nikodym derivatives g^μ_G for μ. ∎

Theorem 3.5. *The change of measure operators (a_n), $n \in \mathbb{Z}$, form a representation of the Heisenberg algebra:*

$$[a_m, a_n] = m\, \delta_{m+n,0}\, \mathrm{id}_{\mathcal{M}}.$$

Proof. Let $x_1,\dots,x_k$ be points in $\mathbb{Z}^2$ and let $[\gamma_0],[\gamma_1],[\gamma_2]$ be discrete contours such that $[\gamma_0]$ encircles $\{x_1,\dots,x_k\}\cup\overline{B}_{\cdot/m}(\max\{0,-\frac{m}{2},-\frac{n}{2}\})$, and $[\gamma_1]$ encircles $[\gamma_0]$, and $[\gamma_2]$ encircles $[\gamma_1]$. We shall start by computing

$$
\begin{aligned}
(3.6)\quad &\langle a_m a_n \phi(x_1)\cdots\phi(x_k)\rangle - \langle a_n a_m \phi(x_1)\cdots\phi(x_k)\rangle \\
&= \frac{1}{\pi}\oint_{[\gamma_2]}\oint_{[\gamma_1]}\langle J(z)J(w)\phi(x_1)\cdots\phi(x_k)\rangle\, w^{[n]}z^{[m]}\,dw\,dz \\
&\quad - \frac{1}{\pi}\oint_{[\gamma_1]}\oint_{[\gamma_0]}\langle J(w)J(z)\phi(x_1)\cdots\phi(x_k)\rangle\, z^{[m]}w^{[n]}\,dz\,dw.
\end{aligned}
$$

A Wick expansion of the correlation function produces terms of the form (3.3) that factor and thus cancel in (3.6), and one non-cancelling term of the form (3.2). By fixing $w\in\mathbb{Z}^2_m$ at distance $\frac{1}{4}$ from $[\gamma_1]$ in each of the two integrals contributing to (3.6), and rearranging the integrations with the help of Lemma 2.4, we see that the contribution of the non-cancelling terms to (3.6) can be written as

$$
\langle\phi(x_1)\cdots\phi(x_k)\rangle\cdot\frac{1}{\pi}\oint_{[\gamma_1]}\left(\oint_{[\sigma_w]}\langle J(z)J(w)\rangle\, z^{[m]}dz\right)w^{[n]}dw,
$$

where $[\sigma_w]$ is a positively oriented discrete closed contour encircling w. For each w, the inner integral therefore becomes

$$
\begin{aligned}
\oint_{[\sigma_w]}\langle J(z)J(w)\rangle\, z^{[m]}dz &= \oint_{[\sigma_w]}[\partial K](z-w)\, z^{[m]}dz \\
&= i\sum_{z\in\mathrm{int}_\diamond[\sigma_w]}[\bar\partial\partial K](z-w)\, z^{[m]} \\
&= -i\sum_{z\in\overline{\mathrm{int}}_m[\sigma_w]}[\bar\partial K](z-w)\,[\partial z^{[m]}] \\
&= -i\frac{m}{2}w^{[m-1]},
\end{aligned}
$$

where we used Lemma 2.4, the fact that $[\bar\partial z^{[m]}]=0$ inside $[\sigma_w]$ and the antisymmetry of ∂ stated in Lemma 2.1. We then integrate around $[\gamma_1]$ and apply Lemma 2.5:

$$
\oint_{[\gamma_1]}\left(\oint_{[\sigma_w]}\langle J(z)J(w)\rangle\, z^{[m]}dz\right)w^{[n]}dw = \frac{-im}{2}\oint_{[\gamma_1]}w^{[m-1]}w^{[n]}dw = \pi m\,\delta_{m+n,0}.
$$

Now, let $\Phi\in\mathcal{M}$ and a finite $G\subset\mathbb{Z}^2$ be given. The computations above describe the action of $(a_ma_n - a_na_m)$ on insertions of polynomials P in the Gaussian variables $(\phi(x))_{x\in G}$:

$$
(3.7)\qquad \left\langle (a_ma_n - a_na_m)P(\phi|_G)\right\rangle = m\,\delta_{0,m+n}\left\langle P(\phi|_G)\right\rangle.
$$

Since this holds for all polynomials P, we obtain

$$
[a_{-n},a_{-m}]\Phi = m\,\delta_{0,m+n}\,\Phi,
$$

and so the proof is complete. □

By analogous arguments we may construct antianalytic current modes $\bar{a}_n$ corresponding to $\oint \bar{J}(z)\bar{z}^{[n]}d\bar{z}$, where $\bar{J}(z) := [\bar{\partial}\Phi](z)$, and show that these operators also yield a representation of the Heisenberg algebra.

Proposition 3.6. *For each $k \in \mathbb{Z}$, there exists a unique operator $\bar{a}_k : \mathcal{M} \to \mathcal{M}$, with $\bar{a}_k\mathcal{S} \subset \mathcal{A}$ and $\bar{a}_k\mathcal{A} \subset \mathcal{S}$ such that for any $\mu \in \mathcal{M}$ and any (not necessarily distinct) $x_1, \ldots, x_m \in \mathbb{Z}^2_\delta$ we have*

$$\langle \Phi(x_1)\cdots\Phi(x_m)\rangle_{\bar{a}_k\mu} = \overline{\langle \Phi(x_1)\cdots\Phi(x_m)\rangle_{a_k\bar{\mu}}}.$$

Moreover, the change of measure operators $(\bar{a}_n)_{n\in\mathbb{Z}}$ form a representation of the Heisenberg algebra:

$$[\bar{a}_m, \bar{a}_n] = m\,\delta_{m+n,0}\,\mathrm{id}_{\mathcal{M}}$$

and they commute with the discrete analytic current modes

$$[a_m, \bar{a}_n] = 0.$$

Proof. The construction of $\bar{a}_n$ and the calculation of their commutation relations is identical to those of a_n. To see that the two representations of Heisenberg algebra commute, note that in a calculation like in the proof of Proposition 3.5, a Wick expansion of $\langle J(z)\bar{J}(w)\Phi(x_1)\ldots\Phi(x_k)\rangle$ yields cancelling terms plus a term containing the factor $\langle J(z)\bar{J}(w)\rangle = [\partial\bar{\partial}K](w-z)$. But this function of z is identically 0 on the contour γ_w. □

4. Sugawara construction of Virasoro modes

We are now ready to construct the operators that span the commuting Virasoro algebras.

4.1. **Virasoro modes acting on insertions.** For any $n \in \mathbb{Z}$, the action on insertions of the formally infinite linear combinations

$$L_n := \frac{1}{2}\sum_{j\geq 0} a_{n-j}a_j + \frac{1}{2}\sum_{j\leq -1} a_j a_{n-j} \tag{4.1}$$

$$\bar{L}_n := \frac{1}{2}\sum_{j\geq 0} \bar{a}_{n-j}\bar{a}_j + \frac{1}{2}\sum_{j\leq -1} \bar{a}_j\bar{a}_{n-j} \tag{4.2}$$

of current modes are well defined, since only finitely many terms ever contribute when we act on insertions of cylinder functions. Indeed, for fixed finite subgraph $G \subset \mathbb{Z}^2$, denote

$$M_G = 1 + 2\cdot\max\{\|z\|_\infty : z \in G\}.$$

Then if $x_1, \ldots, x_k \in G$, we have that $\langle a_j\Phi(x_1)\cdots\Phi(x_k)\rangle$ vanishes for all $j > M_G$, because the integration contour in the definition of a_j can then be taken so that in the integration the monomial $z^{[j]}$ only ever gets evaluated in the set $\overline{B}_{\delta/m}(\frac{j-1}{2})$, where it is zero. Thus for $f \in L^p_G$ the action of L_n is actually defined as the finite sum

$$\Big\langle L_n f(\Phi_G)\Big\rangle = \frac{1}{2}\sum_{j=0}^{M_G}\Big\langle a_{n-j}a_j f(\Phi_G)\Big\rangle + \frac{1}{2}\sum_{j=n-M_G}^{-1}\Big\langle a_j a_{n-j} f(\Phi_G)\Big\rangle;$$

and similarly for $\bar{L}_n$.

4.2. **Virasoro modes as change of measure operators.** Like in Section 3 with current modes, we can lift the action of L_n on insertions to an action on Gibbs measures.

Proposition 4.1. *For each $n \in \mathbb{Z}$, there exists unique operators L_n and $\bar{L}_n$ on $\mathcal{M}$ such that for any $\mu \in \mathcal{M}$ and any $x_1, \ldots, x_k \in \mathbb{Z}^2$*

$$\begin{aligned}
\langle \varphi(x_1) \cdots \varphi(x_k) \rangle_{L_n \mu} &= \langle L_{-n} \varphi(x_1) \cdots \varphi(x_k) \rangle_{\mu} \\
\langle \varphi(x_1) \cdots \varphi(x_k) \rangle_{\bar{L}_n \mu} &= \langle \bar{L}_{-n} \varphi(x_1) \cdots \varphi(x_k) \rangle_{\mu} .
\end{aligned}$$

The operators $L_n, \bar{L}_n$ are parity preserving.

Proof. For any $n \in \mathbb{Z}$, the operators L_{-n} and $\bar{L}_{-n}$ form consistent families of operators that are bounded on each L^p_G, $p \geq 1$, as follows immediately from the fact that for a fixed G and $n \in \mathbb{Z}$ they coincide with finite linear combinations of products of current modes. Thus by an earlier Lemma, they lift to change of measure operators $L_n \colon \mathcal{M} \to \mathcal{M}$ and $\bar{L}_n \colon \mathcal{M} \to \mathcal{M}$, respectively, uniquely determined by the above formulas on the dense subspace of polynomials. $\square$

Keeping in mind that a_n is the change of measure operator obtained by lifting the action on insertions of a_{-n}, and that the lift of a composition is a composition of lifts in the reverse order (the change of measure operator is a formal adjoint of the action on insertions), we have a formal expression for the L_n as

$$L_n = \frac{1}{2} \sum_{j \geq 0} a_{-j} a_{j+n} + \frac{1}{2} \sum_{j \leq -1} a_{j+n} a_{-j} = \frac{1}{2} \sum_{j \leq -1} a_j a_{n-j} + \frac{1}{2} \sum_{j \geq 0} a_{n-j} a_j ;$$

and similarly for $\bar{L}_n$. However, one would have to make sense of these infinite linear combinations of (compositions of) change of measure operators. Our approach of lifting the action could be thought of as considering the pointwise convergence of the above series of change of measure operators when the underlying space of Gibbs measures is equipped with an appropriate weak topology.

4.3. **Virasoro commutation relations.** To be self-contained, and to be explicit about the well-definedness of the needed compositions of operators, we include here the well-known calculation of the commutation relations of the operators L_n obtained by a Sugawara-type construction.

Theorem 4.2. *The two families of change of measure operators $(\bar{L}_n), (L_n)$, $n \in \mathbb{Z}$, yield commuting representations of the Virasoro algebra of central charge $c = 1$:*

$$\begin{aligned}
[L_m, L_n] &= (m-n) L_{m+n} + \frac{1}{12}(m^3 - m) \delta_{m+n,0} \mathrm{Id}; \\
[\bar{L}_m, \bar{L}_n] &= (m-n) \bar{L}_{m+n} + \frac{1}{12}(m^3 - m) \delta_{m+n,0} \mathrm{Id}; \\
[L_m, \bar{L}_n] &= 0.
\end{aligned}$$

We have $[L_n, a_m] = -m a_{n+m}$, $[\bar{L}_n, \bar{a}_m] = -m \bar{a}_{n+m}$, and $[L_n, \bar{a}_m] = 0 = [\bar{L}_n, a_m]$.

Proof. By the remark at the end of Section 3 it is clear that $[L_m, \bar{L}_n] = 0$ for all $m, n \in \mathbb{Z}$. We will check the Virasoro relations for (L_n) by first treating the actions (L_n) on insertions; the argument for $(\bar{L}_n)$ is identical. Using the commutation relations of the a_n

(see the proof of Theorem 3.5 and Proposition 3.4) and the simple commutator identity $[A, BC] = [A, B]C + B[A, C]$, we have

$$\begin{aligned} 2[L_m, a_n] &= \sum_{j\geq 0}([a_{m-j}, a_n]a_j + a_{m-j}[a_j, a_n]) + \sum_{j\leq -1}([a_j, a_n]a_{m-j} + a_j[a_{m-j}, a_n]) \\ &= -2na_{m+n}, \end{aligned}$$

where at each step of the calculation the expression has a well-defined action on insertions of polynomials P of $(\phi(x))_{x\in G}$ due to a truncation to finitely many terms. Consequently, again by Theorem 3.5, Proposition 3.4, and the same commutator identity, we have

$$\begin{aligned} 2[L_m, L_n] &= \sum_{j\geq 0}[L_m, a_{n-j}a_j] + \sum_{j\leq -1}[L_m, a_j a_{n-j}] \\ &= \sum_{j\geq 0}(-(n-j)a_{m+n-j}a_j - j\,a_{n-j}a_{m+j}) \\ &\qquad + \sum_{j\leq -1}(-(n-j)a_j a_{m+n-j} - j\,a_{m+j}a_{n-j}) \\ &= \sum_{j\geq 0}(m-n+j-m)a_{m+n-j}a_j - \sum_{j\geq m}(j-m)a_{m+n-j}a_j \\ &\qquad + \sum_{j\leq -1}(m-n+j-m)a_j a_{m+n-j} - \sum_{j\leq m-1}(j-m)a_j a_{m+n-j}. \end{aligned}$$

We can rewrite this last expression as

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2}\sum_{j\geq 0}(m-n)a_{m+n-j}a_j + \frac{1}{2}\sum_{j\leq -1}(m-n)a_j a_{m+n-j} + R_{m,n} \\ &= (m-n)L_{m+n} + R_{m,n}, \end{aligned}$$

where

$$R_{m,n} := \begin{cases} \frac{1}{2}\sum_{j=0}^{m-1}(m-j)(a_j a_{m+n-j} - a_{m+n-j}a_j) & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ \frac{1}{2}\sum_{j=m}^{-1}(j-m)(a_j a_{m+n-j} - a_{m+n-j}a_j) & \text{if } m < 0. \end{cases}$$

We then show that $R_{m,n} = \frac{1}{12}(m^3 - m)\delta_{m+n,0}$. If $m \geq 1$ this is seen by the calculation

$$\begin{aligned} R_{m,n} &= \frac{1}{2}\sum_{j=0}^{m-1}(m-j)(a_j a_{m+n-j} - a_{m+n-j}a_j) \\ &= \frac{1}{2}\sum_{j=0}^{m-1}(m-j)j\,\delta_{m+n,0} && \text{(by Proposition 1.7)} \\ &= \frac{1}{12}(m^3 - m)\delta_{m+n,0}, \end{aligned}$$

and if $m \leq -1$, the calculation is similar.

As in the proof of Theorem 3.5, the commutation relations for the actions on insertions give the commutation relations for the change of measure operators, because $L_n a_m - a_m L_n$ is the operator obtained by lifting the action of $a_{-m}L_{-n} - L_{-n}a_{-m}$ in the sense of Lemma 3.3, and $L_nL_m - L_mL_n$ is the operator obtained by lifting the action of $\mathcal{L}_{-m}\mathcal{L}_{-n} - \mathcal{L}_{-n}\mathcal{L}_{-m}$ etc. $\square$

5. Further results

5.1. **Coulomb gas.** By adding a multiple of the current mode to each L_n one obtains representations of the Virasoro algebra with other central charges. For $b \in \mathbb{R}$ we define change of measure operators

$$L_n^b := L_n + b(n+1)a_n; \qquad \bar{L}_n^b := \bar{L}_n + b(n+1)\bar{a}_n; \tag{5.1}$$

which will both satisfy the Virasoro commutation relations with central charge $c = 1 - 12b^2$. For $b \neq 0$, these operators are not parity preserving (nor are they parity reversing), as opposed to the L_n corresponding to $c = 1$, and they can therefore be considered less natural.

Proposition 5.1. For $c \leq 1$ and $n \in \mathbb{Z}$, if we define the change of measure operators $L_n^b, \bar{L}_n^b : M \to M$ by (5.1) with $b = \pm\sqrt{\frac{1-c}{12}}$, we have that $\left(L_n^b\right)_{n\in\mathbb{Z}}$ and $\left(\bar{L}_n^b\right)_{n\in\mathbb{Z}}$ yield two commuting representations of the Virasoro algebra with central charge c, that is $[L_n^b, \bar{L}_m^b] = 0$,

$$[L_m^b, L_n^b] = (m-n)L_{m+n}^b + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,\mathrm{id}_M;$$

and similarly for $(\bar{L}_n^b)$.

Proof. Starting from the observation $[L_m^b, a_n] = -na_{m+n} + bm\delta_{m+n,0}\mathrm{id}_M$, the proof is a straightforward modification of that of Theorem 4.2. $\square$

5.2. **Discrete GFF in a half plane.** A similar construction based on current modes can be carried out for the discrete GFF in the upper half-plane $\mathbb{H} = \mathbb{Z} \times \mathbb{N}$ with Dirichlet boundary condition on $\partial\mathbb{H} = \mathbb{Z} \times \{0\}$. We sketch the construction.

In the half-plane $\mathbb{H}$, the Green's function $G^{\mathbb{H}}(z, w)$ is well defined even without an infrared regularization: for a fixed $w \in \mathbb{H}$ it is determined by $-[\Delta G^{\mathbb{H}}](\cdot, w) = \delta_w$ and $G^{\mathbb{H}}(z, w) = 0$ for $z \in \partial\mathbb{H}$. The discrete GFF in the upper half-plane is a collection $(\Phi^{\mathbb{H}}(z))_{z\in\mathbb{H}}$ of centered real Gaussians with covariance given by the Green's function

$$\mathbb{E}\left[\Phi^{\mathbb{H}}(z)\Phi^{\mathbb{H}}(w)\right] = G^{\mathbb{H}}(z, w).$$

We note that since $G^{\mathbb{H}}(z, w) = a(z - \bar{w}) - a(z - w)$, the field $\Phi^{\mathbb{H}}$ can be constructed by a reflection: if $\Phi = (\Phi(z))_{z\in\mathbb{Z}^2}$ is the pinned discrete GFF on the infinite square lattice $\mathbb{Z}^2$, then the field

$$\Phi^{\mathbb{H}}(z) = \frac{1}{\sqrt{2}}\left(\Phi(z) - \Phi(\bar{z})\right)$$

is a discrete GFF in the upper half-plane. We define the current by

$$J^{\mathbb{H}}(z) = [\partial\Phi^{\mathbb{H}}](z)$$

with $\Phi^{\mathbb{H}}$ extended to the dual as zero. With the reflection trick, the current makes sense for all $z \in \mathbb{Z}_m^2$, its correlation functions are recovered by Wick's formula from the two-point functions

$$\left\langle J^{\mathbb{H}}(z)\Phi^{\mathbb{H}}(x)\right\rangle = K(z - x) - K(z - \bar{x})$$

for $x \in \mathbb{Z}^2$ (discrete holomorphic for $z \notin \{x, \bar{x}\}$), and we can define the action on insertions by integration

$$\left\langle a_n^{\mathbb{H}} f\left(\Phi^{\mathbb{H}}_{G}\right)\right\rangle_\mu := \frac{1}{\sqrt{\pi}} \oint_{[\gamma]} \left\langle J^{\mathbb{H}}(z) f\left(\Phi^{\mathbb{H}}_{G}\right) g\left(\Phi^{\mathbb{H}}_{G}\right)\right\rangle z^{[n]}\, dz$$

where $[\gamma]$ is any sufficiently large simple closed discrete contour and $g = g^{\mu}_{G}$ is a Radon-Nikodym derivative. If one prefers not to resort to the reflection trick, the above expression can be written entirely in terms of $(\Phi^{\mathbb{H}}(z))_{z\in\mathbb{H}}$ by noting that the contour may be assumed to be symmetric with respect to a reflection in the real axis, and then in the integrand we may use $\bar{z}^{[n]} = \overline{z^{[n]}}$ and $J^{\mathbb{H}}(\bar{z}) = -\overline{J^{\mathbb{H}}(z)}$ to rewrite everything in terms of only the part of the contour that is in the upper half plane.

From the two-point function of the current

$$\left\langle J^{\mathbb{H}}(z) J^{\mathbb{H}}(w)\right\rangle = [\partial K](w-z) - \delta_{z,\bar{w}}$$

one recovers the Heisenberg commutation relations

$$a^{\mathbb{H}}_m a^{\mathbb{H}}_n - a^{\mathbb{H}}_n a^{\mathbb{H}}_m = m\,\delta_{n+m,0}\,\mathrm{id}$$

as before. The construction of the change of measure operators $(a^{\mathbb{H}}_n)_{n\in\mathbb{Z}}$ and the Virasoro operators $(L^{\mathbb{H}}_n)_{n\in\mathbb{Z}}$ is entirely parallel to the case treated earlier, and one obtains the following result:

Theorem 5.2. *Let $\mathcal{M}^{\mathbb{H}}$ be the vector space of complex Gibbs measures that are changes of measure of law of the discrete GFF in $\mathbb{H}$, and whose Radon-Nikodym derivatives with respect to it are in L^p for all $p < \infty$.*

There exists a family of change of measure operators $(a^{\mathbb{H}}_n)_{n\in\mathbb{Z}}$ on $\mathcal{M}^{\mathbb{H}}$ that are parity reversing and that satisfy the Heisenberg algebra commutation relations and a family of change of measure operators $(L^{\mathbb{H}}_n)_{n\in\mathbb{Z}}$ on $\mathcal{M}^{\mathbb{H}}$ that are parity preserving and that satisfy the Virasoro algebra commutation relations with central charge $c = 1$ and such that $[L_n, a_m] = -m a_{n+m}$.

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