

Conformal Invariance of Ising Model Correlations

C. Hongler

Mathematics Department, Columbia University,
New York, NY 10027, New York

□ E-mail: hongler@math.columbia.edu
www.math.columbia.edu/ hongler

We review recent results with D. Chelkak and K. Izuyurov,¹⁰ where we rigorously prove existence and conformal invariance of scaling limits of magnetization and multi-point spin correlations in the critical Ising model on an arbitrary simply connected planar domain. This solves a number of conjectures coming from physical and mathematical literatures. The proof is based on convergence results for discrete holomorphic spinor observables.

Keywords: Ising Model, Conformal Field Theory, Conformal Invariance, Exact Solvability, Scaling Limits, Critical Phenomena, Spinors, Discrete Complex Analysis

1. Introduction

The Ising model plays a central role in equilibrium statistical mechanics. In dimension two and above it exhibits an order-disorder phase transition. Besides its mathematical interest, the phase transition has found successful applications in the study of ferromagnetism, lattice gases, chemical adsorbates, ecology and image processing.

The thermodynamics of the 2D Ising model is understood thanks to the work of Onsager,²⁷ who exactly computed the free energy, using the transfer matrix technique. The fine description of the phase transition remained mysterious for a long time, limited to the computation of the critical exponents of the model.

Renormalization group arguments¹⁶ suggested that at critical temperature, the 2D Ising model has a universal scaling limit. The introduction of an operator algebra for the model²² and the insight that 2D critical models should have conformal symmetry in the scaling limit led to the introduction of Conformal Field Theory (CFT).^{3,4}

The Ising model corresponds to a minimal model of CFT, with central charge $\frac{1}{2}$,¹⁴ with two nontrivial primary fields: the spin and the energy. Boundary CFT allows one to deal with various geometries and boundary conditions: +, □ and free.⁸

The connection between the scaling limit of the Ising model and CFT however has remained conjectural. In particular, one of the most emblematic achievements of

CFT techniques has been out of mathematical reach: the formulae for the scaling limit the Ising magnetization and spin-spin correlations,^{7,8} which are for instance used in condensed matter physics. Recently, most of the CFT predictions for the Ising model have been proven. In this report, we focus on the spin correlations.¹⁰ See Section 1.2 for a short review of related results.

1.1. Ising model and conformal invariance

The Ising model on a graph G is a random assignment of ± 1 spins to the vertices of G . The probability of a spin configuration $\sigma \in \{\pm 1\}^G$ is proportional to the Boltzmann weight $e^{-\beta H(\sigma)}$, where $\beta > 0$ is the inverse temperature and $H(\sigma) := \sum_{x \sim y} \sigma_x \sigma_y$ is the energy of the configuration σ (the sum is over all pairs of adjacent vertices).

In this paper, we will consider the Ising model on rescaled subgraphs of the square grid Z^2 . We are interested in the model at its phase transition, which occurs at the parameter value $\beta_c = \frac{1}{2} \ln \sqrt{2} + 1$.

The spin correlations of the Ising model have conformally covariant scaling limits. In,¹⁰ the following result about all n -point correlation functions is proven:

Theorem 1.1. Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain. Consider the critical Ising model with + boundary conditions, on discretizations $(\Omega_\delta)_\delta$ of Ω by the square grid of mesh size δ (the distance between adjacent spins is $\sqrt{2}\delta$). Suppose that $\Omega_\delta \rightarrow \Omega$ in the Hausdorff metric as $\delta \rightarrow 0$. Then we have

$$\frac{1}{\delta^{\frac{n-1}{8}}} E_{\Omega_\delta}[\sigma_{a_1} \dots \sigma_{a_n}] \underset{\delta \rightarrow 0}{\sim} C^n h_{a_1, \dots, a_n}^{\pm};$$

where C is an explicit (lattice-dependent) constant and $h_{a_1, \dots, a_n}^{\pm}$ is an explicit conformally covariant tensor of degree $\frac{1}{8}$ in each variables.

The formulae for the one-point and two-point functions are given by:

$$\begin{aligned} h_a^{\pm} &= (\phi^0(a))^{\pm \frac{1}{8}}; \\ h_{a,b}^{\pm} &= h_a^{\pm} h_b^{\pm} \left(1 \pm e^{\pm 2d_H^{\pm}(a;b)} \right)^{\pm \frac{1}{4}}; \end{aligned}$$

where ϕ is the unique conformal mapping from Ω to the unit disk D with $\phi(a) = 0$ and $\phi^0(a) > 0$ (given by Riemann's mapping theorem) and where d_H^{\pm} is the hyperbolic metric on Ω .

1.2. Related results about Ising model correlations

1.2.1. Spin correlations in the the plane

In the plane (without boundary), a number of massive limits (perturbations around criticality) were computed using transfer matrix. The two-point function (a Painlevé III function) was computed by Wu, McCoy, Tracy and Barouch.³⁶ Sato, Miwa and Jimbo (SMJ) represented spin correlations in terms of tau functions for twisted Dirac operators.^{31,34} SMJ's approach was justified rigorously by Palmer

and Tracy,²⁹ and recently Palmer derived the CFT predictions by taking the zero-mass limit of SMJ's tau functions.²⁸

In the critical regime, the diagonal and horizontal spin-spin correlations were computed by Wu.²⁶ Pinson obtained the spin two-point function using nonlinear analysis.³⁰ Dubédat derived the $2n$ -point functions using exact bosonization techniques and recent results on monomer correlations.¹⁵

1.2.2. Energy field

In the full plane, energy two-point correlations were computed by Hecht.¹⁸ Using Temperley-Fisher representation and dimer techniques, Boutillier and de Tilière^{5,6} computed the n -point energy correlations, on periodic isoradial graphs.

In bounded domains, the energy correlations functions have been computed recently.^{19,20} The scaling limit of the one-point function in the half-plane was obtained by Assis and McCoy,¹ using transfer matrix, in the near-critical window.

2. Sketch of the Proof of Theorem 1.1

The overall strategy to prove Theorem 1.1 is the following:

- (1) We compute the logarithmic derivatives of the spin correlations (with + boundary conditions) in any domain Ω :
 - (a) We represent the discrete logarithmic derivatives of the spin correlations in terms of lattice spinors.
 - (b) We formulate the lattice spinors as solutions to discrete Riemann boundary value problems with monodromies.
 - (c) We prove the convergence of the lattice spinors to continuous spinors.
 - (d) We perform a local analysis of the lattice spinors near to extract the asymptotic behavior of the discrete logarithmic derivatives.
- (2) We obtain the limit of the ratio of the two-spin correlations with + and free boundary conditions at the same location in the same domain Ω .
 - (a) We represent this ratio in terms of the same lattice spinors as in Part 1(a).
 - (b) We get that the spinors converge by Part 1(c).
 - (c) We perform a local analysis of the lattice spinors to extract the asymptotic behavior of this ratio.
- (3) We integrate the logarithmic derivatives and calibrate the integrals to obtain domain-independent results:
 - (a) We integrate the discrete logarithmic derivatives and obtain the ratios of spin correlations computed at different locations in a fixed domain.
 - (b) We prove the rotational invariance of the two-spin correlation in the full plane.

- (c) We calibrate the two-point function: we obtain that $E_{\square_\delta}[\square_a \square_b] \sim E_{C_\delta}[\square_a \square_b]$ taking the limit $\delta \rightarrow 0$ and then $|\square_a - \square_b| \rightarrow 0$, where C_δ is the discretization of the full plane.
- (d) We obtain that $\frac{1}{\delta^{1-4}} E_{C_\delta}[\square_a \square_b] \sim C^2 \frac{1}{|x - y|^{1-4}}$ as $\delta \rightarrow 0$.
- (e) We calibrate the one-point function: we get that $E_{\square_\delta}[\square_a \square_b] \sim E_{\square_\delta}[\square_a] E_{\square_\delta}[\square_b]$ taking the limit $\delta \rightarrow 0$ and then $a \rightarrow \infty$.
- (f) We calibrate the n-point functions: we have that $E_{\square_\delta}[\square_{a_0} \dots \square_{a_n}] \sim E_{\square_\delta}[\square_{a_0}] \dots E_{\square_\delta}[\square_{a_n}]$ taking the limit $\delta \rightarrow 0$ and then $a_0 \rightarrow \infty$.
- (g) We calibrate the continuous formulae, given by the integrals of the logarithmic derivatives computed in Part 1(c).

The technical details of Part 1 are outlined (in the case of the one-point function) in Section 3 below. The techniques of Part 2 are fairly similar to those of Part 1. Part 3 uses correlation inequalities (FGK and GHS inequalities), results of Wu about spin correlations in the full plane (see also⁹ for a derivation using lattice spinors), and explicit complex analysis computations.

3. Logarithmic Derivatives Convergence

In this section, we state and sketch the proof of one of the key results to obtain Theorem 1.1 (Part 1 of the proof in Section 1.1). For simplicity and definiteness, we just consider the case of the one-point function. The proof of the analogous result for the n-point functions contains additional subtleties, but follows roughly the same strategy.

Theorem 3.1. With the notation and assumptions of Theorem 1.1, we have

$$\frac{1}{2\delta} \frac{E_{\square_\delta}[\square_{a+2\delta}]}{E_{\square_\delta}[\square_a]} \sim 1 + \sum_{\substack{x+iy=a \\ x,y \in \mathbb{Z}}} \delta \log h_{x+iy}$$

The proof of Theorem 3.1 is given in Section 3.5 below. The strategy is to represent the discrete logarithmic derivative in terms of a lattice spinor (analogous to the spinors considered by the Kyoto school^{31{34}}), to prove that the lattice spinor converges to a continuous one and finally to analyze the values that give the logarithmic derivatives.

3.1. Lattice spinor

Definition 3.1. Let \square_δ be a discrete domain and a be the center of a face of \square_δ , as in Theorem 1.1. We denote by $[\square_\delta; a]$ the double cover of \square_δ ramified at a . We will identify $\square_\delta \cap \{x : x \in \mathbb{R}\}$ with one of the sheets $\square_\delta^+ \subset [\square_\delta; a]$ that lives above it. We will mostly consider the vertices, (centers of) faces, medial vertices (midpoints of edges) and corners of $[\square_\delta; a]$, which we will denote by $V_{[\square_\delta; a]}$, $V_{[\square_\delta; a]}^\square$, $V_{[\square_\delta; a]}^m$ and $V_{[\square_\delta; a]}^c$ respectively (see Figure 1).

We now introduce the lattice spinor, which is used to connect the Ising model with discrete complex analysis.

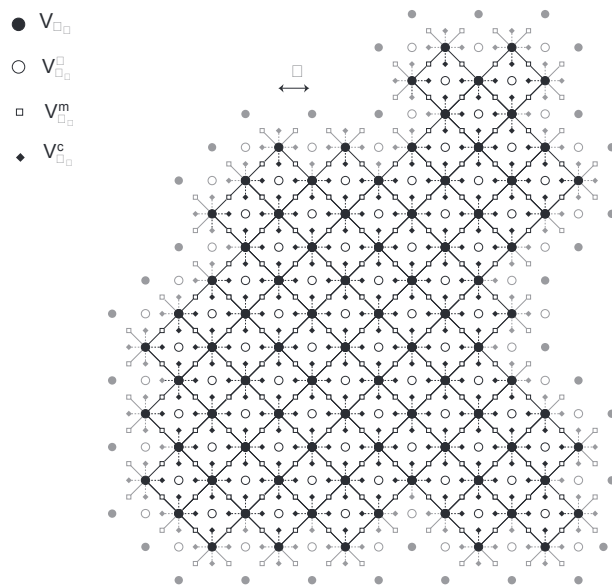


Fig. 1. The sets V_{\square} , V_{\square}^{\square} , V_{\square}^m and V_{\square}^c .

Definition 3.2. We define the lattice spinor $F_{[\square; a]}$ at $z \in V_{\square}^c$ by

$$F_{[\square; a]}(z) = \frac{\sum_{\square \in C_{\square}(a + \frac{\square}{2}; z)} \# \text{edges}(\square) e^{\frac{i}{2} W(\square)} (\square 1)^{L(\square; a)} \square(\square; z)}{\sum_{! \in C_{\square}(a + \frac{\square}{2}; z)} \# \text{edges}(!) (\square 1)^{L(!; a)}}; \quad (1)$$

where (see Figure 2):

- $\square C_{\square}$ is the set of collections of closed loops made of edges of \square (a collection of edges $!$ is in C_{\square} if every vertex $x \in \square$ is incident to an even number of edges of $!$),
- $\square C_{\square}(a + \frac{\square}{2}; z)$ is the set of collection of edges consisting in closed loops plus a path from (the projection to the single cover of) $a + \frac{\square}{2}$ to (the projection of) z .
- $\square \# \text{edges}(\square)$ is the number of (full) edges of \square .
- $\square W(\square)$ is the winding number of \square from $a + \frac{\square}{2}$ to z , i.e. $\frac{\square}{2} (n_l - n_r)$, where n_l and n_r are the numbers of left and right turns when going from $a + \frac{\square}{2}$ to z .
- $\square L(\square; a)$ is the number of closed loops of \square surrounding a .

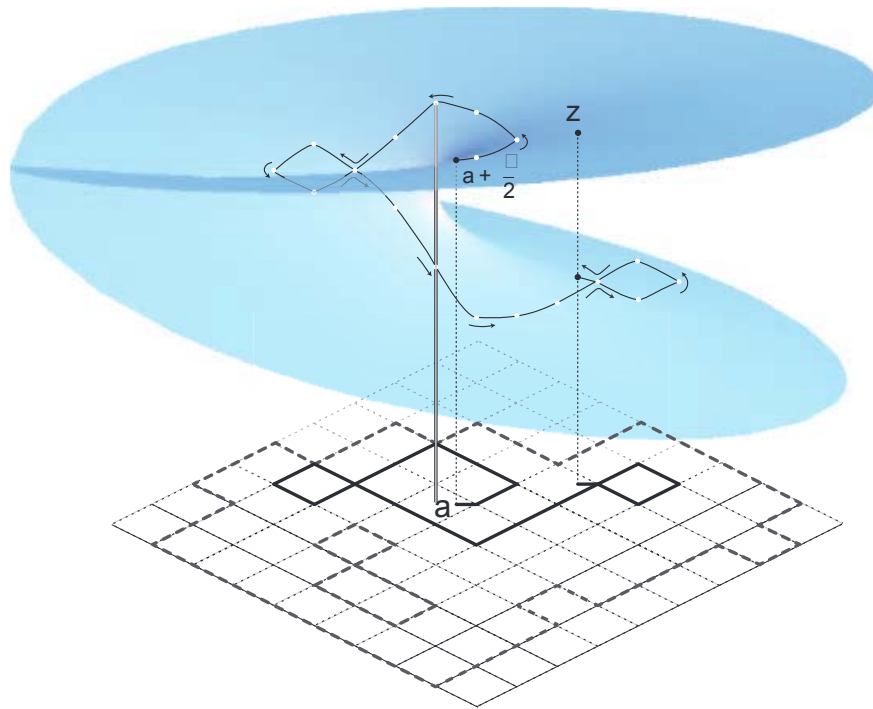


Fig. 2. The lattice spinor.

$\square(\square; z) = 1$ if the lift of \square to $[\square_\square; a]$ starting from \square_\square^+ (the sheet of $a + \frac{3}{2}$) ends on the same sheet as z and $\square(\square; z) = -1$ otherwise.

We extend the definition of $F_{[\square_\square; a]}$ to medial vertices of $V_{[\square_\square; a]}^m$ by taking the right-hand side of (1), multiplied by a factor $\frac{1}{\cos(\frac{\pi}{8})}$.

Remark 3.1. This definition (on the corners) corresponds essentially to the lattice operator $\frac{h(z)\square(a)i}{h(a)i}$ in the formalism of Kadano and Ceva.²²

Proposition 3.1. We have that

$$\frac{E_{\square_\square}[\square_{a+2\square}]}{E_{\square_\square}[\square_a]} = F_{[\square_\square; a]} \left(a + \frac{3\square}{2} \right);$$

where $a + \frac{3\square}{2} \in V_{[\square_\square; a]}^c$ is taken on the sheet \square_\square^+ (as in Definition 3.1).

Proof. The proof of this fact follows from elementary considerations: the configurations on the numerator of $F_{[\square_\square; a]} \left(a + \frac{3\square}{2} \right)$ can be interpreted as the low-temperature expansion of $E_{\square_\square}[\square_{a+2\square}]$, and similarly the denominator as the low-temperature expansion of $E_{\square_\square}[\square_a]$. See¹⁰ for more details. \square

3.2. Discrete analysis of the lattice spinor

Definition 3.3. We say that a function $f : V_{[\square;a]} \rightarrow \mathbb{C}$ is s -holomorphic if for any corner $c \in V_{[\square;a]}^c$ adjacent to a medial vertex $m \in V_{[\square;a]}^m$, we have $f(m) = \frac{1}{2} f(c) + \overline{(c-x)}f(m)$, where $x \in V_{[\square;a]}$ is the vertex adjacent to c and m . In other words $f(m) = P_{(c)}[f(c)]$, where $P_{(c)}(A) = \frac{1}{2} A + \overline{(c-x)}A$.

Proposition 3.2. The lattice spinor $F_{[\square;a]}$ is the unique solution to the following discrete Riemann Boundary value problem:

- $F_{[\square;a]}$ has monodromy $\square 1$ around a .
- $F_{[\square;a]}$ is s -holomorphic on $[\square;a] \setminus a + \frac{\square}{2}$.
- $F_{[\square;a]}$ has a discrete singularity next to $a + \frac{\square}{2}$: on the sheet \square^+ , if we set $A^\square = F_{[\square;a]}(a + \frac{1-\square i}{2})$, we have $P_{iR}[A^\square] = \square P_{iR}[A^\square] = \square i$.
- for any midpoint $z \in V_{[\square;a]}^m$ of an edge hw , where $v \in V_{[\square;a]}$ and $w \notin V_{[\square;a]}$, we have $\operatorname{Re} F_{[\square;a]}(z) \frac{p}{w-z} = 0$.

Proof. The first and last properties follow from elementary topological considerations. The second and third follow from XOR bijections between configurations contributing to a medial vertex and a corner. See¹⁰ for details and for the proof that the solution to the above problem is unique. □

3.3. Scaling limit of the lattice spinor

Definition 3.4. We define $\#(\square)$ as the probability that a simple random walk on C_\square (the square grid rotated by 45 degrees with sidelength of a square equal to $\sqrt{2}\square$) starting from (an approximation of) 1 hits 0 before $\square x < 0$.

Remark 3.2. In other words $\# \frac{\square}{n}$ is the probability, on the standard square grid Z^2 , that a simple random walk starting from $(n;n)$ hits $(0;0)$ before $\square(k;k); k < 0$. A variant of Kesten's Beurling estimate²³ shows that $\# \frac{1}{n}$ is of order $\frac{1}{n}$.

Definition 3.5. Let $[\square;a]$ denote the double cover of $\square n$ g ramified around a and let \square^+ be the sheet of $[\square;a]$ defined as the limit of \square_{\square} (see Definition 3.2).

We define the continuous spinor $f_{[\square;a]} : [\square;a] \rightarrow \mathbb{C}$ as $\frac{1}{\square}$; where \square is the unique conformal mapping from \square to the unit disk D with $\square(a) = 0$ and $\square'(a) > 0$. We choose the square root branches such that $\operatorname{Re} f_{[\square;a]}(z) > 0$ as $z \rightarrow a$ on \square^+ .

Proposition 3.3. We have that

$$\frac{1}{\#(\square)} F_{[\square;a]}(z) \xrightarrow[\square \rightarrow 0]{} f_{[\square;a]}(z);$$

uniformly over all $a; z$ away from each other and from ∂ .

Proof. We first obtain a precompactness result, and then identify the subsequential scaling limits. The proof of both steps in particular on the possibility to define a lattice analogue of $\langle e^{-f_{[\square;a]}^2} \rangle$. See¹⁰ for more details, and also^{11,13,20,35} for proofs of convergence based on similar techniques. \square

3.4. Local spinor analysis

We need to extract the behavior of $F_{[\square;a]}$ right next to the singularity at a . There is a nontrivial interchange of limit:

Proposition 3.4. We have

$$\lim_{\square \rightarrow 0} \frac{1}{\#(\square)} F_{[\square;a]} \left(a + \frac{3}{2}\square \right) = \langle e^{-\lim_{z \rightarrow a} \frac{1}{p} \frac{1}{z-a} f_{[\square;a]}(z)} \rangle : \quad (1)$$

Proof. This part requires delicate analysis, in particular discrete versions of $\frac{1}{p} \frac{1}{z-a}$ and of $\frac{1}{p} \frac{1}{z-a}$, constructed on the full-plane lattice C_\square , in terms of random walks. Call them F_\square and R_\square respectively. It follows from their constructions that $F_\square \left(a + \frac{3}{2}\square \right) = 1$ and $R_\square \left(a + \frac{3}{2}\square \right) = \square$ (on C_\square^+) and that F_\square is the lattice derivative of R_\square . We have $\frac{1}{\#(\square)} F_\square \sim \frac{1}{p} \frac{1}{z-a}$ and $\frac{1}{\#(\square)} R_\square \sim \frac{1}{p} \frac{1}{z-a}$ as $\square \rightarrow 0$. Proving the statement of the proposition is equivalent to showing that

$$F_{[\square;a]} \left(a + \frac{3}{2}\square \right) - a_{[\square;a]} = o(\square) \text{ as } \square \rightarrow 0; \quad (2)$$

where $a_{[\square;a]} > 0$ is the right-hand side in the statement of the proposition.

Let us define $G_{[\square;a]} : C_\square \rightarrow \mathbb{C}$ as $\frac{1}{\#(\square)} F_\square - a_{[\square;a]} R_\square$. By Proposition 3.3 and the convergence of $\frac{1}{\#(\square)} F_\square$ and $\frac{1}{\#(\square)} R_\square$, we can prove that $G_\square \rightarrow g$ as $\square \rightarrow 0$, where $g : \mathbb{R} \rightarrow \mathbb{R}$, where $g(z) = o\left(\frac{1}{p} \frac{1}{z-a}\right)$ as $z \rightarrow a$.

Let $R(\square)$ denote the domain \square reflected with respect to the axis $f + x; x \in \mathbb{R}$, let G_\square^+ be defined as $G_{[\square;a]} - G_{[R(\square);a]}$. We have that $G_\square^+ \sim g^+$ as $\square \rightarrow 0$, where $g^+(z) = o\left(\frac{1}{p} \frac{1}{z-a}\right)$ as $z \rightarrow a$. By symmetry considerations, one can show that G_\square^+ vanishes on the half-line $a + \frac{1+4k}{2}\square; k \in \mathbb{N}$. Similarly, one can show that G_\square^- vanishes on the half-line $a - \frac{1+4k}{2}\square; k \in \mathbb{N}$.

A variant of Kesten's Beurling estimate allows us to deduce that in any $O(\square)$ -neighborhood of a , G_\square^+ and G_\square^- are of order $o\left(\frac{1}{p} \frac{1}{z-a}\right)$. As $2G_{[\square;a]} = G_\square^+ - G_\square^-$, we deduce that $G_{[\square;a]} \left(a + \frac{3}{2}\square \right) = o\left(\frac{1}{p} \frac{1}{z-a}\right)$. So, we get the left-hand side of (2) is of order $o\left(\frac{1}{p} \frac{1}{z-a}\right)$. As we know that $\#(\square)$ is of order \square^{-2} (see Remark 3.2), we get (2). \square

3.5. Proof of Theorem 3.1

Proof. By Propositions 3.1 and 3.4, we obtain that

$$\lim_{\square \rightarrow 0} \frac{1}{\#(\square)} \frac{E_{\square_\square}[\square_{a+2\square}]}{E_{\square_\square}[\square_a]} - 1 = \langle e^{-\lim_{z \rightarrow a} \frac{1}{p} \frac{1}{z-a} f_{[\square;a]}(z)} \rangle : \quad (3)$$

It therefore remains to check that the right-hand side indeed equals $2\pi \log h_{x+iy}$. This follows from an explicit computation. \square

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