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The energy density in the planar Ising model

by

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1. Introduction

1.1. The model

The Lenz–Ising model in two dimensions is probably one of the most studied models for an order-disorder phase transition, exhibiting very rich and interesting behavior, yet well understood both from the mathematical and physical viewpoints [B], [MW2], [P].

After Kramers and Wannier [KW] derived the value of the critical temperature and Onsager [O] analyzed the behavior of the partition function for the Ising model on the two-dimensional square lattice, a number of exact derivations were obtained by a variety of methods. Thus it is often said that the 2D Ising model is *exactly solvable* or *integrable*. Moreover, it has a conformally invariant scaling limit at criticality, which allows one to use *conformal field theory* (*CFT*) or *Schramm's SLE* techniques. CFT provides predictions for quantities like the correlation functions of the spin or the energy fields, which in principle can then be related to SLE.

In this paper, we obtain a rigorous exact derivation of the one-point function of the energy density, matching the CFT predictions [DMS], [C], [BG]. We exploit the integrable structure of the 2D Ising model, but in a different way from the one employed in the classical literature. Our approach is rather similar to Kenyon's approach to the dimer model [Ken].

We write the energy density in terms of discrete fermionic correlators, of the form introduced in [S1]. These correlators solve a discrete version of a Riemann boundary value problem, which identifies them uniquely. In principle, this could be used to give an exact, albeit very complicated, formula, that one could try to simplify—a strategy similar to most of the earlier approaches. Instead we pass to the scaling limit, showing that the solution to the discrete boundary value problem approximates well its continuous counterpart, which can be easily written using conformal maps. Thus we obtain a short expression, approximating the energy density to the first order. Moreover, our method works in any simply connected planar domain, and the answer is, as expected, conformally covariant.

The fermionic approach to the Ising model was introduced by Kaufman [Ka]. The fermionic correlators were in particular studied by Kadanoff and Ceva [KC] and later in Mercat [M], but their scaling limits with boundary conditions were not discussed before [S1]. Our results have been generalized in [Ho], where the limits of n-point correlation functions of the energy density are obtained, via the introduction of multipoint fermionic correlators. Scaling limits of the n-point *spin* correlations are obtained in [CHI], via the introduction of spinors and the analysis of their scaling limits.

Recall that the Ising model on a graph $\mathcal{G}$ is defined by a Gibbs probability measure on configurations of ± 1 (or *up*/*down*) spins located at the vertices: it is a random assignment $(\sigma_x)_{x\in\mathcal{V}}$ of ± 1 spins to the vertices $\mathcal{V}$ of $\mathcal{G}$ and the probability of a state is proportional to its Boltzmann weight $e^{-\beta H}$, where $\beta>0$ is the inverse temperature of the model and H is the Hamiltonian, or energy, of the state σ. In the Ising model with no external magnetic field, we have $H:=-\sum_{i\sim j}\sigma_i\sigma_j$, where the sum is over all the pairs of adjacent vertices of $\mathcal{G}$.

1.2. The energy density

Let Ω be a Jordan domain and let Ω_δ be a discretization of it by a subgraph of the square grid of mesh size $\delta>0$. We consider the Ising model on the graph Ω_δ at the critical inverse temperature $\beta_c=\frac{1}{2}\log(\sqrt{2}+1)$; on the boundary of Ω_δ, we may impose the value $+1$ to the spins or let them free (we call these $+$ and *free* boundary conditions respectively). Our main result about the energy density is the following.

THEOREM. *Let $a\in\Omega$ and for each $\delta>0$, let $\langle x_\delta, y_\delta\rangle$ be the closest edge to a in Ω_δ. Then, as $\delta\to 0$, we have*

$$\mathsf{E}_+\left[\sigma_{x_\delta}\sigma_{y_\delta}-\frac{\sqrt{2}}{2}\right]=\frac{l_\Omega(a)}{2\pi}\delta+o(\delta)\quad \textit{and}\quad \mathsf{E}_{\mathsf{free}}\left[\sigma_{x_\delta}\sigma_{y_\delta}-\frac{\sqrt{2}}{2}\right]=-\frac{l_\Omega(a)}{2\pi}\delta+o(\delta),$$

where the subscripts $+$ and free *denote the boundary conditions and l_Ω is the element of the hyperbolic metric of Ω.*

A precise version of this theorem in terms of the energy density field is given in §1.4. This result has been predicted for a long time by CFT methods (see [DMS] and [BG] for instance), notably using Cardy's celebrated mirror image technique [C]. However, the CFT approach does not allow one to determine the lattice-specific constant $1/2\pi$ appearing in front of the hyperbolic metric element.

This is one of the first results where full conformal invariance (i.e. not only Möbius invariance) of a correlation function for the Ising model is actually shown. The proof does not appeal to the SLE machinery, although the fermionic correlator that we use is very similar to the one employed to prove convergence of Ising interfaces to SLE(3) [CS2]. A generalization of our result with mixed boundary conditions could also be used to deduce convergence to SLE.

In the case of the full plane, the energy density correlations have been first computed by Hecht [He] using transfer matrix techniques. These results were later generalized by Boutillier and De Tiliére, using dimer model techniques [BT1], [BT2]. However, their approach works only in the infinite-volume limit or in periodic domains and does not directly apply to arbitrary bounded domains.

In the case of the half-plane, the energy density one-point function has been recently obtained by Assis and McCoy [AM] (passing to the limit the finite-scale results of [MW1]), using transfer matrix techniques.

The strategy for the proof of our theorem relies mainly on:

- The introduction of a discrete fermionic correlator, which is a complex deformation of a certain partition function, and of an infinite-volume version of this correlator.
- The expression of the energy density in terms of discrete fermionic correlators.

• The proof of the convergence of the discrete fermionic correlators to continuous ones, which are holomorphic functions.

1.3. Graph notation

Let us first give some general graph notation. Let $\mathcal{G}$ be a graph embedded in the complex plane C.

• We denote by $\mathcal{V}_{\mathcal{G}}$ the set of the vertices of $\mathcal{G}$, by $\mathcal{E}_{\mathcal{G}}$ the set of its (unoriented) edges, and by $\vec{\mathcal{E}}_{\mathcal{G}}$ the set of its oriented edges.

• We identify the vertices $\mathcal{V}_{\mathcal{G}}$ with the corresponding points in the complex plane (since $\mathcal{G}$ is embedded). An oriented edge is identified with the difference of the final vertex minus the initial one.

• Two vertices $v_1, v_2 \in \mathcal{V}_{\mathcal{G}}$ are said to be *adjacent* if they are the endpoints of an edge, denoted $\langle v_1, v_2\rangle$, and two distinct edges $e_1, e_2 \in \mathcal{E}_{\mathcal{G}}$ are said to be *incident* if they share an endvertex.

1.3.1. Discrete domains

• We denote by C_δ the square grid of mesh size $\delta>0$. Its vertices and edges are defined by

$$\mathcal{V}_{\mathsf{C}_\delta} := \{\delta(j+ik) : j,k \in \mathsf{Z}\} \quad \text{and} \quad \mathcal{E}_{\mathsf{C}_\delta} := \{\langle v_1, v_2\rangle : v_1, v_2 \in \mathcal{V}_{\mathsf{C}_\delta} \text{ and } |v_1 - v_2| = \delta\}.$$

• In order to keep the notation as simple as possible, we will only look at finite *induced* subgraphs Ω_δ of C_δ (two vertices of Ω_δ are linked by an edge in Ω_δ whenever they are linked in C_δ), that we will also call *discrete domains*.

• For a discrete domain Ω_δ, we denote by Ω_δ^* the *dual graph* of Ω_δ: its vertices $\mathcal{V}_{\Omega_\delta^*}$ are the centers of the bounded faces of Ω_δ and two vertices of $\mathcal{V}_{\Omega_\delta^*}$ are linked by an edge of $\mathcal{E}_{\Omega_\delta^*}$ if the corresponding faces of Ω_δ share an edge.

• We denote by $\partial\mathcal{V}_{\Omega_\delta}$ the set of vertices of $\mathcal{V}_{\mathsf{C}_\delta} \setminus \mathcal{V}_{\Omega_\delta}$ that are at distance δ from a vertex of $\mathcal{V}_{\Omega_\delta}$ (i.e. that are adjacent in C_δ to a vertex of $\mathcal{V}_{\Omega_\delta}$) and by $\partial\mathcal{E}_{\Omega_\delta} \subset \mathcal{E}_{\mathsf{C}_\delta}$ the set of edges between a vertex of $\mathcal{V}_{\Omega_\delta}$ and a vertex of $\partial\mathcal{V}_{\Omega_\delta}$. The vertices in $\partial\mathcal{V}_{\Omega_\delta}$ *appear with multiplicity:* if a vertex of $\mathcal{V}_{\mathsf{C}_\delta} \setminus \mathcal{V}_{\Omega_\delta}$ is at distance δ to several vertices of $\mathcal{V}_{\Omega_\delta}$, then it appears as so many distinct elements of $\partial\mathcal{V}_{\Omega_\delta}$. In other words, there is a one-to-one correspondence between $\partial\mathcal{V}_{\Omega_\delta}$ and $\partial\mathcal{E}_{\Omega_\delta}$.

• We denote by $\partial\mathcal{V}_{\Omega_\delta^*}$ the centers of the faces of C_δ that are adjacent to a face of Ω_δ. We denote by $\partial\mathcal{E}_{\Omega_\delta^*} \subset \mathcal{E}_{\mathsf{C}_\delta^*}$ the set of dual edges between a vertex of $\mathcal{V}_{\Omega_\delta^*}$ and a vertex of $\partial\mathcal{V}_{\Omega_\delta^*}$. For an edge $e \in \mathcal{E}_{\Omega_\delta}$ we denote by $e^* \in \mathcal{E}_{\Omega_\delta^*}$ its dual (e and e^* intersect at their midpoint).

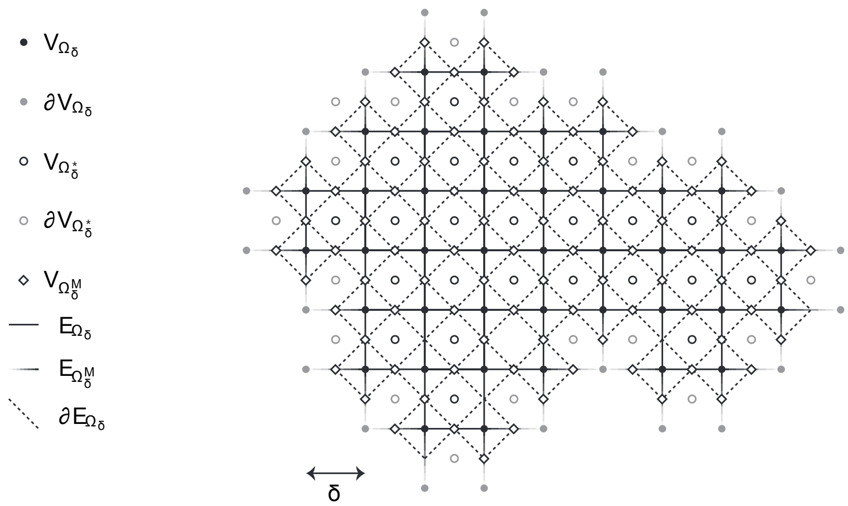

Figure 1.1. Notation for discrete domains

- We write $\bar{\mathcal{V}}_{\Omega_\delta}$ for $\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{V}_{\Omega_\delta}$ and $\bar{\mathcal{V}}_{\Omega_\delta^*}$ for $\mathcal{V}_{\Omega_\delta^*} \cup \partial\mathcal{V}_{\Omega_\delta^*}$.
- We denote by $\mathcal{E}_{\Omega_\delta}^h \subset \mathcal{E}_{\Omega_\delta}$ the set of the horizontal (i.e. parallel to the real axis) edges of Ω_δ and by $\mathcal{E}_{\Omega_\delta}^v := \mathcal{E}_{\Omega_\delta} \setminus \mathcal{E}_{\Omega_\delta}^h$ the set of the vertical ones.
- We denote by Ω_δ^M the *medial graph of* Ω_δ: its vertices $\mathcal{V}_{\Omega_\delta^M}$ are the midpoints of the edges of $\mathcal{E}_{\Omega_\delta} \cup \partial\mathcal{E}_{\Omega_\delta}$ and the medial edges $\mathcal{E}_{\Omega_\delta^M}$ link midpoints of incident edges of $\mathcal{E}_{\Omega_\delta} \cup \partial\mathcal{E}_{\Omega_\delta}$.
- We say that a family $(\Omega_\delta)_{\delta>0}$ of discrete domains (with $\Omega_\delta \subset \mathsf{C}_\delta$ for each $\delta>0$) *approximates* or *discretizes* a continuous domain Ω if for each $\delta>0$, Ω_δ is the largest connected induced subgraph of C_δ contained in Ω.

1.3.2. The Ising model with boundary conditions

The Ising model (with *free boundary conditions*) on a finite graph $\mathcal{G}$ (in this paper, $\mathcal{G}$ will be a discrete domain Ω_δ or its dual Ω_δ^*) at *inverse temperature* $\beta>0$ is a model whose state space $\Xi_\mathcal{G}$ is given by $\Xi_\mathcal{G} := \{(\sigma_x)_{x\in\mathcal{V}_\mathsf{G}} : \sigma_x \in \{\pm 1\}\}$: a state assigns to every vertex x of $\mathcal{G}$ a *spin* $\sigma_x \in \{\pm 1\}$. The probability of a configuration $\sigma \in \Xi_\mathcal{G}$ is

$$\mathsf{P}_\mathcal{G}^{\beta,\mathsf{free}}\{\sigma\} := \frac{1}{\mathcal{Z}_\mathcal{G}^{\beta,\mathsf{free}}} e^{-\beta H_\mathsf{G}^{\beta,\mathsf{free}}(\sigma)},$$

with the *energy* (or Hamiltonian) $H_\mathcal{G}^{\beta,\mathsf{free}}$ of a configuration σ given by

$$H_\mathcal{G}^{\beta,\mathsf{free}}(\sigma) := -\sum_{\langle x,y\rangle \in \mathcal{E}_\mathsf{G}} \sigma_x \sigma_y,$$

and the *partition function* $\mathcal{Z}_{\mathcal{G}}^{\beta,\mathsf{free}}$ by

$$\mathcal{Z}_{\mathcal{G}}^{\beta,\mathsf{free}} := \sum_{\sigma \in \Xi_{\mathsf{G}}} e^{-\beta H(\sigma)}.$$

Given a graph $\mathcal{G}$ with boundary vertices $\partial \mathcal{V}_{\mathcal{G}}$ (like Ω_δ^* with $\partial \mathcal{V}_{\Omega_\delta^*}$) the Ising model on $\mathcal{G}$ with + *boundary* condition is defined as the Ising model on $\mathcal{G}$, with extra spins located at the vertices of $\partial \mathcal{V}_{\mathcal{G}}$ that are set to +1 and with energy

$$H_{\mathcal{G}}^{\beta,+}(\sigma) := -\sum_{\langle x,y\rangle \in \bar{\mathcal{E}}_{\mathsf{G}}} \sigma_x \sigma_y,$$

where $\bar{\mathcal{E}}_{\mathcal{G}}$ is the set of edges linking vertices of $\mathcal{V}_{\mathcal{G}} \cup \partial \mathcal{V}_{\mathcal{G}}$.

In this paper, we will be interested in the Ising model with free and + boundary conditions on discrete square grid domains Ω_δ at the critical inverse temperature

$$\beta_c := \frac{1}{2} \log(\sqrt{2}+1),$$

when the mesh size δ is small.

We will from now on omit the inverse temperature parameter β in the notation and will denote by $\mathsf{P}_{\mathcal{G}}^{\mathsf{free}}$ and $\mathsf{P}_{\mathcal{G}}^{+}$ the probability measures of the Ising model on $\mathcal{G}$ at $\beta = \beta_c$ with free and + boundary conditions and by $\mathsf{E}_{\mathcal{G}}^{\mathsf{free}}$ and $\mathsf{E}_{\mathcal{G}}^{+}$ the corresponding expectations.

1.4. The energy density

Let Ω_δ be a discrete domain and let $a_\delta \in \mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ be the midpoint of a horizontal edge of Ω_δ. We introduce the two quantities $\langle \varepsilon_\delta(a_\delta)\rangle_{\Omega_\delta}^{\mathsf{free}}$ and $\langle \varepsilon_\delta(a_\delta)\rangle_{\Omega_\delta^*}^{+}$, called *average energy density* (with free and + boundary conditions), defined by

$$\langle \varepsilon_\delta(a_\delta)\rangle_{\Omega_\delta}^{\mathsf{free}} := \mathsf{E}_{\Omega_\delta}^{\mathsf{free}}\left[\sigma_{e_\delta}\sigma_{w_\delta} - \frac{\sqrt{2}}{2}\right] \quad \text{and} \quad \langle \varepsilon_\delta(a_\delta)\rangle_{\Omega_\delta^*}^{+} := \mathsf{E}_{\Omega_\delta^*}^{+}\left[\sigma_{n_\delta}\sigma_{s_\delta} - \frac{\sqrt{2}}{2}\right],$$

where $\langle e_\delta, w_\delta\rangle \in \mathcal{E}_{\Omega_\delta}$ and $\langle n_\delta, s_\delta\rangle \in \mathcal{E}_{\Omega_\delta^*}$ are respectively the (horizontal) edge and the dual (vertical) edge, the midpoint of both of which is a_δ (see Figures 1.2 and 1.3). The quantity $\frac{1}{2}\sqrt{2}$ is the infinite-volume limit of the product of two adjacent spins (it can be found in [MW2, Chapter VIII, Formula 4.12], for instance). The energy density field is the fluctuation of the product of adjacent spins around this limit: it measures the distribution of the energy H among the edges, as a function of their locations. We are considering horizontal edges on Ω_δ and vertical edges on Ω_δ^* for concreteness and for making the notation simpler, but our results are rotationally invariant.

We can now state the main result of this paper, which is the conformal covariance of the average energy density.

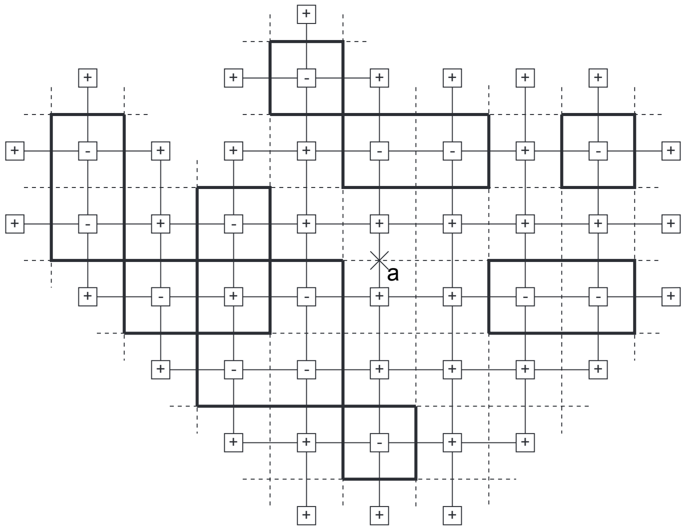

Figure 1.2. The Ising model on $\mathcal{V}_{\Omega_\delta^\star}$ with + boundary condition, with the contours corresponding to its low-temperature expansion.

THEOREM 1.1. *Let* Ω *be a* $\mathcal{C}^1$ *simply connected domain and let* $a\in\Omega$. *Consider a family* $(\Omega_\delta)_{\delta>0}$ *of discrete domains approximating* Ω *and for each* $\delta>0$, *let* $a_\delta\in\mathcal{V}_{\Omega_\delta^M}$ *be the midpoint of the horizontal edge that is the closest to* a. *Then, as* δ ! 0, *uniformly on the compact subsets of* Ω, *we have*

$$\frac{1}{\delta}\langle\varepsilon_\delta(a_\delta)\rangle^{+}_{\Omega_\delta^\star} \;!\; \frac{1}{2\pi}\ell_\Omega(a) \quad \textit{and} \quad \frac{1}{\delta}\langle\varepsilon_\delta(a_\delta)\rangle^{\mathrm{free}}_{\Omega_\delta^\star} \;!\; -\frac{1}{2\pi}\ell_\Omega(a),$$

$\ell_\Omega(a)$ *being the hyperbolic metric element of* Ω *at* a. *Namely,* $\ell_\Omega(a):=2\psi_a'(a)$, *where* ψ_a *is the conformal mapping from* Ω *to the unit disk* $\mathsf{D}:=\{z\in\mathsf{C}:|z|<1\}$ *such that* $\psi_a(a)=0$ *and* $\psi_a'(a)>0$.

The proof will be given in §1.6.

COROLLARY 1.2. *The conclusions of Theorem* 1.1 *hold under the assumption that* Ω *is a Jordan domain.*

Proof. We have that $\langle\varepsilon_\delta(a_\delta)\rangle^{+}_{\Omega_\delta}$ and $\langle\varepsilon_\delta(a_\delta)\rangle^{\mathrm{free}}_{\Omega_\delta}$ are, respectively, non-increasing and non-decreasing with respect to the discrete domain Ω_δ, as follows easily from the Fortuin–Kasteleyn–Ginibre inequality applied to the Fortuin–Kasteleyn representation of the model (see [G, Chapters 1 and 2], for instance): if $\Omega_\delta\subset\widetilde{\Omega}_\delta$, then

$$\langle\varepsilon_\delta(a_\delta)\rangle^{+}_{\Omega_\delta^\star} \;\square\; \langle\varepsilon_\delta(a_\delta)\rangle^{+}_{\widetilde{\Omega}_\delta^\star} \quad \text{and} \quad \langle\varepsilon_\delta(a_\delta)\rangle^{\mathrm{free}}_{\Omega_\delta} \;\square\; \langle\varepsilon_\delta(a_\delta)\rangle^{\mathrm{free}}_{\widetilde{\Omega}_\delta}.$$

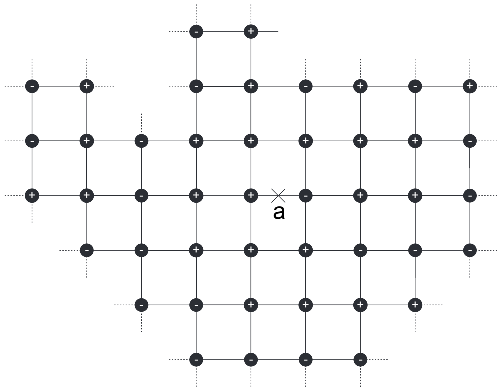

Figure 1.3. The Ising model on $\mathcal{V}_{\Omega_\delta}$ with free boundary condition.

If Ω is a Jordan domain, we can approximate Ω by increasing and decreasing sequences of smooth domains, for which Theorem 1.1 applies, and deduce the result for Ω. $\square$

The central idea for proving Theorem 1.1 is to introduce a discrete fermionic correlator which is a two-point function $f_{\Omega_\delta}(a,z)$; it is defined in the next subsection. We then relate f_{Ω_δ} to the average energy density and prove its convergence to a holomorphic function f_Ω.

1.5. Contour statistics and discrete fermionic correlators

1.5.1. Contour statistics

Let Ω_δ be a discrete domain. We denote by $\mathcal{C}_{\Omega_\delta}$ the set of edge collections $\omega \subset \mathcal{E}_{\Omega_\delta}$ such that every vertex $v \in \mathcal{V}_{\Omega_\delta}$ belongs to an even number of edges of ω: in other words, by Euler's theorem for walks, the edge collections $\omega \in \mathcal{C}_{\Omega_\delta}$ are the ones that consist of edges forming (not necessarily simple) closed contours. For an edge $e \in \mathcal{E}_{\Omega_\delta}$, we denote by $\mathcal{C}_{\Omega_\delta}^{\{e+\}}$ the set of configurations $\omega \in \mathcal{C}_{\Omega_\delta}$ that *do not contain* e and by $\mathcal{C}_{\Omega_\delta}^{\{e-\}}$ the set of configurations that *do contain* e.

Set $\alpha := \sqrt{2}-1$. For a collection of edges $\omega \subset \mathcal{E}_{\Omega_\delta}$, we denote by $|\omega|$ its cardinality. For $e \in \mathcal{E}_{\Omega_\delta}$ we define

$$\mathbf{Z}_{\Omega_\delta} := \sum_{\omega \in \mathcal{C}_{\Omega_\delta}} \alpha^{|\omega|}, \quad \mathbf{Z}_{\Omega_\delta}^{\{e+\}} := \sum_{\omega \in \mathcal{C}_{\Omega_\delta}^{\{e+\}}} \alpha^{|\omega|} \quad \text{and} \quad \mathbf{Z}_{\Omega_\delta}^{\{e-\}} := \sum_{\omega \in \mathcal{C}_{\Omega_\delta}^{\{e-\}}} \alpha^{|\omega|}.$$

We now have the following representation of the energy density in terms of contour statistics.

PROPOSITION 1.3. *Let* $e\in\mathcal{E}^h_{\Omega_\delta}$ *be a horizontal edge and let its midpoint be* $a\in\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$. *Then we have*

$$\langle\varepsilon_\delta(a)\rangle^+_{\Omega_\delta^*}=\frac{\mathbf{Z}^{\{e+\}}_{\Omega_\delta}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\mathbf{Z}^{\{e-\}}_{\Omega_\delta}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\sqrt{2}}{2}=2\frac{\mathbf{Z}^{\{e+\}}_{\Omega_\delta}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\sqrt{2}+2}{2}\quad \textit{and}\quad \langle\varepsilon_\delta(a)\rangle^{\mathsf{free}}_{\Omega_\delta}=-\langle\varepsilon_\delta(a)\rangle^+_{\Omega_\delta^*}.$$

Proof. From the low-temperature expansion of the Ising model (see [P, Chapter 1], for instance), there is a natural bijection between the configurations of spins σ on $\mathcal{V}_{\Omega_\delta^*}$ with + boundary condition on $\partial\mathcal{V}_{\Omega_\delta^*}$, and the edge collections $\omega\in\mathcal{C}_{\Omega_\delta}$: one puts an edge $e\in\mathcal{E}_{\Omega_\delta}$ in the edge collection ω if the spins of σ at the endpoints of the dual edge $e^*\in\mathcal{E}_{\Omega_\delta^*}$ are different. It is easy to see that the probability measure on $\mathcal{C}_{\Omega_\delta}$ induced by this bijection gives to each edge collection $\omega\in\mathcal{C}_{\Omega_\delta}$ a weight proportional to $(e^{-2\beta})^{|\omega|}$, and hence to $\alpha^{|\omega|}$, where $\alpha=\sqrt{2}-1$ as above (since $\beta=\beta_c=\frac{1}{2}\log(\sqrt{2}+1)$). The event that the spins at two adjacent dual vertices $x,y\in\mathcal{V}_{\Omega_\delta^*}$ are the same (respectively are different) corresponds through the natural bijection to $\mathcal{C}^{\{e+\}}_{\Omega_\delta}$ (respectively $\mathcal{C}^{\{e-\}}_{\Omega_\delta}$), where $e\in\mathcal{E}_{\Omega_\delta}$ is such that $e^*=\langle x,y\rangle$. Using that $\mathbf{Z}^{\{e+\}}_{\Omega_\delta}+\mathbf{Z}^{\{e-\}}_{\Omega_\delta}=\mathbf{Z}_{\Omega_\delta}$, we deduce the first identity.

From the so-called high-temperature expansion (see [P, Chapter 1]) we have that for the Ising model on Ω_δ with free boundary condition, the correlation of two spins $z_1,z_2\in\mathcal{V}_{\Omega_\delta}$ is equal to

$$\frac{\sum_{\widetilde{\omega}\in\mathcal{C}_{\Omega_\delta}(z_1,z_2)}(\tanh\beta)^{|\widetilde{\omega}|}}{\sum_{\omega\in\mathcal{C}_{\Omega_\delta}}(\tanh\beta)^{|\omega|}},$$

where $\mathcal{C}_{\Omega_\delta}(z_1,z_2)$ is the set of edge collections $\widetilde{\omega}$ such that every vertex in $\mathcal{V}_{\Omega_\delta}\setminus\{z_1,z_2\}$ belongs to an even number of edges of $\widetilde{\omega}$ and such that z_1 and z_2 both belong to an odd number of edges of $\widetilde{\omega}$. At $\beta=\beta_c$, we have $\tanh\beta=\alpha$ (the fact that $\tanh\beta_c=e^{-2\beta_c}$ actually characterizes β_c).

Let us now take z_1 and z_2 adjacent, set $e:=\langle z_1,z_2\rangle\in\mathcal{E}_{\Omega_\delta}$, and denote by $\mathcal{C}^+_{\Omega_\delta}(z_1,z_2)$ and $\mathcal{C}^-_{\Omega_\delta}(z_1,z_2)$ the sets of $\widetilde{\omega}\in\mathcal{C}_{\Omega_\delta}(z_1,z_2)$ such that $e\in\widetilde{\omega}$ and $e\notin\widetilde{\omega}$ respectively. From each $\widetilde{\omega}\in\mathcal{C}^+_{\Omega_\delta}(z_1,z_2)$, we can remove e and obtain an edge collection in $\mathcal{C}^{\{e+\}}_{\Omega_\delta}$ (this map $\mathcal{C}^+_{\Omega_\delta}(z_1,z_2)\,!\,\mathcal{C}^{\{e+\}}_{\Omega_\delta}$ is bijective) and to each $\widetilde{\omega}\in\mathcal{C}^-_{\Omega_\delta}(z_1,z_2)$, we can add e and obtain an edge collection in $\mathcal{C}^{\{e-\}}_{\Omega_\delta}$. Hence we have

$$\begin{aligned}\langle\varepsilon_\delta(a)\rangle^{\mathsf{free}}_{\Omega_\delta}&=\frac{\sum_{\widetilde{\omega}\in\mathcal{C}^+_{\Omega_\delta}(z_1,z_2)}\alpha^{|\widetilde{\omega}|}}{\mathbf{Z}_{\Omega_\delta}}+\frac{\sum_{\widetilde{\omega}\in\mathcal{C}^-_{\Omega_\delta}(z_1,z_2)}\alpha^{|\widetilde{\omega}|}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\sqrt{2}}{2}\\&=\frac{\alpha\sum_{\omega\in\mathcal{C}^{\{e+\}}_{\Omega_\delta}}\alpha^{|\omega|}}{\mathbf{Z}_{\Omega_\delta}}+\frac{\alpha^{-1}\sum_{\omega\in\mathcal{C}^{\{e-\}}_{\Omega_\delta}}\alpha^{|\omega|}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\sqrt{2}}{2}=\frac{\alpha\mathbf{Z}^{\{e+\}}_{\Omega_\delta}+\alpha^{-1}\mathbf{Z}^{\{e-\}}_{\Omega_\delta}}{\mathbf{Z}_{\Omega_\delta}}-\frac{\sqrt{2}}{2}.\end{aligned}$$

Using the relation $\mathbf{Z}^{\{e+\}}_{\Omega_\delta}+\mathbf{Z}^{\{e-\}}_{\Omega_\delta}=\mathbf{Z}_{\Omega_\delta}$, we obtain the second identity. □

1.5.2. The discrete fermionic correlator in bounded domains

If $a \in \mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ is the midpoint of a horizontal edge $e_1 \in \mathcal{E}_{\Omega_\delta}^h$ and $z \in \mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ is the midpoint of an arbitrary edge $e_2 \in \mathcal{E}_{\Omega_\delta} \cup \partial \mathcal{E}_{\Omega_\delta}$, we denote by $\mathcal{C}_{\Omega_\delta}(a, z)$ the set of γ consisting of edges of $\mathcal{E}_{\Omega_\delta} \setminus \{e_1, e_2\}$ and of two *half-edges* (half of an edge between its midpoint and one of its ends) such that

- one of the half-edges has endpoints a and $a + \frac{1}{2}\delta$;
- the other half-edge is incident to z;
- every vertex $v \in \mathcal{V}_{\Omega_\delta}$ belongs to an even number of edges or half-edges of γ.

For a configuration $\gamma \in \mathcal{C}_{\Omega_\delta}(a, z)$, we say that a sequence $e_0, e_1, ..., e_n$ is an *admissible walk along* γ if

- e_0 is the half-edge incident to a;
- e_n is the half-edge incident to z;
- $e_1, ..., e_{n-1} \in \mathcal{E}_{\Omega_\delta}$ are edges;
- e_j and e_{j+1} are incident for each $j \in \{0, ..., n-1\}$;
- each edge appears at most once in the walk;
- when one follows the walk and arrives at a vertex that belongs to four edges or half-edges of γ (we call this an *ambiguity*), one either turns left or right (going straight in that case is forbidden).

It is easy to see that, for any $\gamma \in \mathcal{C}_{\Omega_\delta}(a, z)$, such a walk always exists, though in general it is not unique (see Figure 1.4).

Given a configuration $\gamma \in \mathcal{C}_{\Omega_\delta}(a, z)$ and an admissible walk along γ, we define the *winding number* $\mathbf{W}$ of γ, denoted $\mathbf{W}(\gamma) \in \mathsf{R}/4\pi\mathsf{Z}$, by

$$\mathbf{W}(\gamma) := \tfrac{1}{2}\pi(n_l - n_r),$$

where n_l and n_r are the number of left turns and right turns, respectively, that the admissible walk makes from a to z: it is the total rotation of the walk between a and z, measured in radians. More generally, we define the winding number of a rectifiable curve as its total rotation from its initial point to its final point, measured in radians. The following lemma shows that the winding number (modulo 4π) of a configuration $\gamma \in \mathcal{C}_{\Omega_\delta}(a, z)$ is actually independent of the choice of the walk on γ.

LEMMA 1.4. *For any* $\gamma \in \mathcal{C}_{\Omega_\delta}(a, z)$, *the winding number* $\mathbf{W}(\gamma) \in \mathsf{R}/4\pi\mathsf{Z}$ *is independent of the choice of admissible walk along* γ.

The proof is given in Appendix A.

Due to Lemma 1.4, we can now define the discrete fermionic correlator f_{Ω_δ} that will be instrumental in our studies of the energy density.

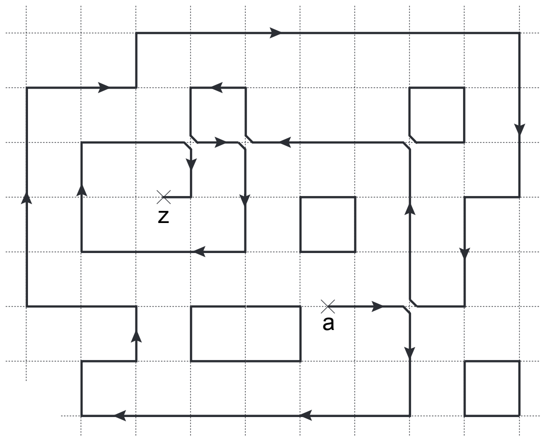

Figure 1.4. An admissible walk in a configuration in $\mathcal{C}_{\Omega_\delta}(\mathsf{a}, \mathsf{z})$. There are sixteen different choices of such walks in this configuration.

Definition 1.5. For any midpoint of a horizontal edge $a \in \mathcal{V}_{\Omega_\delta^{M}}$, we define the discrete fermionic correlator $f_{\Omega_\delta}(a, \cdot) : \mathcal{V}_{\Omega_\delta^{M}} \to \mathbb{C}$ by

$$f_{\Omega_\delta}(a, z) := \frac{1}{\mathbf{Z}_{\Omega_\delta}} \sum_{\gamma \in \mathcal{C}_{\Omega_\delta}(a,z)} \alpha^{|\gamma|} e^{-(i/2)W(\gamma)} \quad \text{and} \quad f_{\Omega_\delta}(a, a) := \frac{\mathbf{Z}_{\Omega_\delta}^{\{e_+\}}}{\mathbf{Z}_{\Omega_\delta}},$$

where $|\gamma|$ denotes the number of edges and half-edges of γ, with the half-edges contributing $\frac{1}{2}$ each.

In this way, $z \mapsto f_{\Omega_\delta}(a, z)$ is a function whose value at $z = a$ gives (up to an additive constant) the average energy density at a. As we will see, moving the point z across the domain will allow us to gain information about the effect of the geometry of the domain on the energy density.

1.5.3. The discrete fermionic correlator in the full plane

As mentioned above, the $\frac{1}{2}\sqrt{2}$ appearing in the definition of the average energy density (§1.4) is the infinite volume limit (or full-plane) average product of two adjacent spins, which one has to subtract in order for the effect of the shape of the domain to be studied. We now introduce a full-plane version of the discrete fermionic correlator, whose definition a priori seems quite different from the bounded domain version. It will allow us to represent the energy density (with the correct additive constant) in terms of the difference of two discrete fermionic correlators.

Definition 1.6. For $a,z\in\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ with $a\neq z$ and a being the midpoint of a horizontal edge, define f_{C_δ} by

$$f_{\mathsf{C}_\delta}(a,z):=\cos\left(\frac{\pi}{8}\right)e^{\pi i/8}\left(C_0\left(\frac{2\left(a+\frac{1}{2}\delta\right)}{\delta},\frac{2z}{\delta}\right)+C_0\left(\frac{2\left(a-\frac{1}{2}i\delta\right)}{\delta},\frac{2z}{\delta}\right)\right)$$
$$+\sin\left(\frac{\pi}{8}\right)e^{-3\pi i/8}\left(C_0\left(\frac{2\left(a-\frac{1}{2}\delta\right)}{\delta},\frac{2z}{\delta}\right)+C_0\left(\frac{2\left(a+\frac{1}{2}i\delta\right)}{\delta},\frac{2z}{\delta}\right)\right),$$

where $C_0(z_1,z_2):=C_0(0,z_2-z_1)$ is the dimer coupling function of Kenyon (see [Ken]), defined (on $\{(z_1,z_2)\in\mathsf{Z}^2:z_1+z_2 \text{ is odd}\}$) by

$$C(0,x+iy):=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{i(x\theta-y\phi)}}{2i\sin\theta+2\sin\phi}\,d\theta\,d\phi.$$

We set $f_{\mathsf{C}_\delta}(a,a):=\frac{1}{4}(2+\sqrt{2})$. This value corresponds to the limit of $f_{\Omega_\delta}(a,a)=\mathbf{Z}_{\Omega_\delta}^{\{e+\}}/\mathbf{Z}_{\Omega_\delta}$ when Ω_δ ! C_δ. We will not use this fact, though, but rather that $f_{\mathsf{C}_\delta}(a,\cdot)$ and $f_{\Omega_\delta}(a,\cdot)$ have the same discrete singularity at a (see Propositions 2.6 and 2.9).

From our definitions up to now and from Proposition 1.3, we deduce the following result.

Lemma 1.7. *Let Ω_δ be a discrete domain and $a\in\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ be the midpoint of a horizontal edge of Ω_δ. Then the average energy density can be represented as*

$$\langle\varepsilon_\delta(a)\rangle^{+}_{\Omega_\delta^\star}=2(f_{\Omega_\delta}-f_{\mathsf{C}_\delta})(a,a)\quad\textit{and}\quad\langle\varepsilon_\delta(a)\rangle^{\mathsf{free}}_{\Omega_\delta}=-2(f_{\Omega_\delta}-f_{\mathsf{C}_\delta})(a,a).$$

1.6. Convergence results and proof of Theorem 1.1

The core of this paper is the convergence of the discrete fermionic correlators to continuous ones, which are holomorphic functions. Let us define these functions first: for $a,z\in\Omega$ with $a\neq z$, we define

$$f_\Omega(a,z):=\frac{1}{2\pi}\sqrt{\psi_a'(a)}\sqrt{\psi_a'(z)}\,\frac{\psi_a(z)+1}{\psi_a(z)}\quad\text{and}\quad f_{\mathsf{C}}(a,z):=\frac{1}{2\pi(z-a)},$$

where ψ_a is the unique conformal mapping from Ω to the unit disk D with $\psi_a(a)=0$ and $\psi_a'(a)>0$ (this mapping exists by the Riemann mapping theorem). Note that z! $f_\Omega(a,z)$ and z! $f_{\mathsf{C}}(a,z)$ both have a simple pole of residue $1/2\pi$ at $z=a$ and that $(f_\Omega-f_{\mathsf{C}})(a,z)$ hence extends holomorphically to $z=a$.

We can now state the key theorem of this paper.

THEOREM 1.8. *For each* $\delta>0$, *identify* $a\in\mathbb{C}$ *with the closest midpoint of a horizontal edge of* $\mathbb{C}_\delta$ *and* $z\in\mathbb{C}$ *with the closest midpoint of an edge of* $\mathbb{C}_\delta$. *Then, as* $\delta\to 0$, *we have the following convergence results:*

$$\begin{aligned} \frac{f_{\Omega_\delta}(a,z)}{\delta} &\to f_\Omega(a,z) && \text{for all } a,z\in\Omega \text{ with } a\neq z,\\ \frac{f_{\mathbb{C}_\delta}(a,z)}{\delta} &\to f_{\mathbb{C}}(a,z) && \text{for all } a,z\in\mathbb{C} \text{ with } a\neq z,\\ \frac{(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,z)}{\delta} &\to (f_\Omega-f_{\mathbb{C}})(a,z) && \text{for all } a,z\in\Omega, \end{aligned}$$

where the convergence of f_{Ω_δ}/δ *is uniform on the compact subsets of* $\Omega\times\Omega$ *away from the diagonal* $\{(w,w):w\in\Omega\}$, *the convergence of* $f_{\mathbb{C}_\delta}/\delta$ *is uniform on* $\mathbb{C}\times\mathbb{C}$ *away from the diagonal* $\{(w,w):w\in\mathbb{C}\}$ *and the convergence of* $(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta$ *is uniform on the compact subsets of* $\Omega\times\Omega$.

From this result, the proof of the main theorem follows readily: since we have (Lemma 1.7)

$$\langle\varepsilon_\delta(a)\rangle^{+}_{\Omega^*_\delta}=2(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,a) \quad\text{and}\quad \langle\varepsilon_\delta(a)\rangle^{\mathrm{free}}_{\Omega_\delta}=-2(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,a),$$

and since $(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta$ converges to $(f_\Omega-f_{\mathbb{C}})(a,a)$, it suffices to check that

$$\left(\sqrt{\psi_a'(a)}\sqrt{\psi_a'(z)}\,\frac{\psi_a(z)+1}{\psi_a(z)}-\frac{1}{z-a}\right)\to \psi_a'(a) \quad \text{as } z\to a,$$

which follows readily by verifying that

$$\frac{\sqrt{\psi_a'(a)}\sqrt{\psi_a'(z)}}{\psi_a(z)}-\frac{1}{z-a}\to 0 \quad\text{as } z\to a.$$

To prove this, notice that since

$$\frac{\sqrt{\psi_a'(a)}\sqrt{\psi_a'(z)}}{\psi_a(z)}=\frac{1}{z-a}+A+O(z-a) \quad\text{as } z\to a,$$

by squaring this expression, it suffices to check that the residue of $\psi_a'(z)/\psi_a^2(z)$ at $z=a$ vanishes. By contour integrating on a small circle C around a and using the change of variable formula (since ψ_a is conformal), we have

$$\frac{4\pi iA}{\psi_a'(a)}=\oint_C\frac{\psi_a'(z)\,dz}{\psi_a(z)^2}=\oint_{\psi_a^{-1}(C)}\frac{dw}{w^2}=0.$$

1.7. Proof of convergence

The rest of this paper is devoted to the proof of the key theorem (Theorem 1.8). This proof consists mainly of two parts:

- The analysis of the discrete fermionic correlators f_{Ω_δ} and $f_{\mathbb{C}_\delta}$ as functions of their second variable (§2):
  - We prove that $f_{\Omega_\delta}(a,\cdot)$ and $f_{\mathbb{C}_\delta}(a,\cdot)$ are discrete holomorphic (in a specific sense) on $\mathcal{V}_{\Omega_\delta^{\mathsf{M}}} \setminus \{a\}$ and $\mathcal{V}_{\mathbb{C}_\delta^{\mathsf{M}}} \setminus \{a\}$, respectively (Propositions 2.5 and 2.6).
  - We show that $f_{\Omega_\delta}(a,\cdot)$ and $f_{\mathbb{C}_\delta}(a,\cdot)$ have the same discrete singularity at a: their difference $(f_{\Omega_\delta} - f_{\mathbb{C}_\delta})(a,\cdot)$ is hence discrete holomorphic on $\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ (Propositions 2.7–2.9).
  - We observe that $f_{\Omega_\delta}(a,\cdot)$ has some specific boundary values on $\partial_0\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ (Proposition 2.10).
- The proof of convergence of the functions f_{Ω_δ}/δ, $f_{\mathbb{C}_\delta}/\delta$ and $(f_{\Omega_\delta} - f_{\mathbb{C}_\delta})/\delta$ (§3):
  - The convergence of $f_{\mathbb{C}_\delta}/\delta$ follows directly from the convergence result of Kenyon for the dimer coupling function (Theorem 3.1).
  - We show that the family of functions $((f_{\Omega_\delta} - f_{\mathbb{C}_\delta})/\delta)_{\delta>0}$ is precompact on the compact subsets of $\Omega\times\Omega$: it admits subsequences which converge as $\delta \to 0$ (Proposition 3.2). Hence $(f_{\Omega_\delta}/\delta)_{\delta>0}$ is also precompact on the compact subsets of $\Omega\times\Omega$ away from the diagonal.
  - We identify the $\delta \to 0$ limits of subsequences of f_{Ω_δ}/δ with the function f_Ω (Proposition 3.6). This allows us to conclude the proof of Theorem 1.8.

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2. Analysis of the discrete fermionic correlators

In this section, we study the properties of the discrete fermionic correlators f_{Ω_δ} and $f_{\mathbb{C}_\delta}$ that follow from their constructions. In the next section, we will use these properties to prove Theorem 1.8.

We study both correlators as functions $f_{\Omega_\delta}(a,\cdot)$ and $f_{\mathbb{C}_\delta}(a,\cdot)$ of their second variable, keeping fixed the medial vertex $a\in\mathcal{V}_{\Omega^M_\delta}$ (which is the midpoint of a horizontal edge of Ω_δ).

Let us first introduce the discrete versions of the differential operators $\bar\partial$ and Δ that will be useful in this paper: for a $\mathbb{C}$-valued function f, we define, wherever it makes sense (i.e. for vertices of $\mathcal{V}_{\Omega_\delta}\cup\mathcal{V}_{\Omega^*_\delta}$),

$$\bar\partial_\delta f(x) := f\big(x+\tfrac{1}{2}\delta\big)-f\big(x-\tfrac{1}{2}\delta\big)+i\big(f\big(x+\tfrac{1}{2}i\delta\big)-f\big(x-\tfrac{1}{2}i\delta\big)\big),$$
$$\Delta_\delta f(x) := f(x+\delta)+f(x+i\delta)+f(x-\delta)+f(x-i\delta)-4f(x).$$

In the case where one has that a vertex $y\in\{x\pm\delta, x\pm i\delta\}$ belongs to $\partial\mathcal{V}_{\Omega_\delta}$ in the definition of Δ_δ, the boundary vertex is the one identified with the edge $\langle x,y\rangle\in\partial\mathcal{E}_{\Omega_\delta}$.

If $e=\overrightarrow{xy}\in\vec{\mathcal{E}}_{\Omega_\delta}$ is an oriented edge with $x,y\in\bar{\mathcal{V}}_{\Omega_\delta}$ and f is a function $\bar{\mathcal{V}}_{\Omega_\delta}\to\mathbb{C}$ we denote by $\partial_e f$ the discrete partial derivative defined by $\partial_e f:=f(y)-f(x)$.

2.1. Discrete holomorphicity

It turns out that the functions $f_{\Omega_\delta}(a,\cdot)$ and $f_{\mathbb{C}_\delta}(a,\cdot)$ are discrete holomorphic in a specific sense, which we call *s-holomorphicity* or *spin-holomorphicity*.

Let us first define this notion. With any medial edge $e\in\mathcal{E}_{\mathbb{C}^M_\delta}$, we associate a line $\ell(e)\subset\mathbb{C}$ of the complex plane defined by

$$\ell(e) := (d-v)^{-1/2}\mathbb{R} = \{(d-v)^{-1/2}t : t\in\mathbb{R}\},$$

where $v\in\mathcal{V}_{\mathbb{C}_\delta}$ is the closest vertex to e and $d\in\mathcal{V}_{\mathbb{C}^*_\delta}$ is the closest dual vertex to e. On the square lattice, the four possible lines that we obtain are $e^{\pm\pi i/8}\mathbb{R}$ and $e^{\pm 3\pi i/8}\mathbb{R}$. When $\ell:=e^{i\theta}\mathbb{R}$ is a line in the complex plane passing through the origin, let us denote by $\mathtt{P}_\ell$ the orthogonal projection on ℓ, defined by

$$\mathtt{P}_\ell[z] := \tfrac{1}{2}(z+e^{2i\theta}\bar z)\quad\text{for all } z\in\mathbb{C}.$$

Definition 2.1. Let $\mathcal{M}_\delta\subset\mathcal{V}_{\mathbb{C}^M_\delta}$ be a collection of medial vertices. We say that $f:\mathcal{M}_\delta\to\mathbb{C}$ is *s-holomorphic on* $\mathcal{M}_\delta$ if for any two medial vertices $x,y\in\mathcal{M}_\delta$ that are adjacent in $\mathbb{C}^M_\delta$,

$$\mathtt{P}_{\ell(e)}[f(x)] = \mathtt{P}_{\ell(e)}[f(y)],$$

where $e=\langle x,y\rangle\in\mathcal{E}_{\mathbb{C}^M_\delta}$.

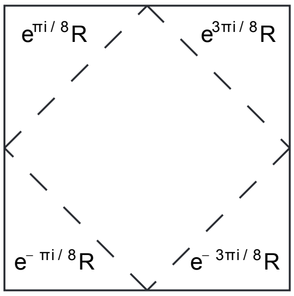

Figure 2.1. The lines associated with the medial edges of Ω_δ.

Remark 2.2. Our definition is the same as the one introduced in [S2], except that the lines that we consider are rotated by a phase of $e^{\pi i/8}$, and that our lattice is rotated by an angle of $\frac{1}{4}\pi$, compared to the definitions there. In [CS2] this definition is also used, in the more general context of isoradial graphs.

Remark 2.3. The definition of s-holomorphicity implies that a discrete version of the Cauchy–Riemann equations is satisfied: if $f:\mathcal{M}_\delta\subset\mathcal{V}_{\mathbb{C}_\delta^M}\to\mathbb{C}$ is s-holomorphic and $v\in\mathcal{V}_{\mathbb{C}_\delta}\cup\mathcal{V}_{\mathbb{C}_\delta^*}$ is such that the four medial vertices $v\pm\frac{1}{2}\delta$ and $v\pm\frac{1}{2}i\delta$ are in $\mathcal{M}_\delta$, then we have

$$\bar{\partial}_\delta f(v)=0.$$

This can be found in [S2] (it follows by taking a linear combination of the four s-holomorphicity relations between the values $f\big(v\pm\frac{1}{2}\delta\big)$ and $f\big(v\pm\frac{1}{2}i\delta\big)\big)$, as well as the fact that satisfying this difference equation is strictly weaker than being s-holomorphic.

Remark 2.4. If $\lambda_\delta=\{\overrightarrow{v_iv_{i+1}}\in\vec{\mathcal{E}}_{\Omega_\delta^M}:i\in\mathbb{Z}/n\mathbb{Z}\}$ is a simple counterclockwise-oriented closed discrete contour of medial edges and Λ_δ is the collection of points in $\mathcal{V}_{\Omega_\delta}\cup\mathcal{V}_{\Omega_\delta^*}$ surrounded by λ_δ, then it is easy to check that for any function $f:\mathcal{V}_{\Omega_\delta^M}\to\mathbb{C}$ we have

$$\sum_{\overrightarrow{v_iv_{i+1}}\in\lambda_\delta}\frac{f(v_i)+f(v_{i+1})}{2}(v_{i+1}-v_i)=i\delta\sum_{z\in\Lambda_\delta}\bar{\partial}_\delta f(v).$$

In particular this sum vanishes if f is discrete holomorphic.

PROPOSITION 2.5. *The function* $f_{\Omega_\delta}(a,\cdot)$ *is s-holomorphic on* $\mathcal{V}_{\Omega_\delta^M}\setminus\{a\}$.

Proof. Let $z,w\in\mathcal{V}_{\Omega_\delta^M}\setminus\{a\}$ be two adjacent medial vertices and let $e\in\mathcal{E}_{\Omega_\delta^M}$ be the medial edge linking them. Suppose that z is the midpoint of a horizontal edge and that w is the midpoint of a vertical edge. We prove the result in the case where $w=z+\frac{1}{2}(1+i)\delta$ (the other ones are symmetric). Denote by h the half-edge between z and $z+\frac{1}{2}\delta\in\mathcal{V}_{\Omega_\delta}$ and by $\tilde{h}$ the half-edge between $z+\frac{1}{2}\delta$ and w. For any $\gamma\in\mathcal{C}_{\Omega_\delta}(a,z)$, define $\varphi(\gamma):=\gamma\oplus h\oplus\tilde{h}$

as the symmetric difference of γ with $\{h,\tilde{h}\}$: if h is not in γ, add it, otherwise remove it, and similarly for $\tilde{h}$. Clearly, φ is an involution mapping $\mathcal{C}_{\Omega_\delta}(a,z)$ to $\mathcal{C}_{\Omega_\delta}(a,w)$ and vice versa. Moreover, for $\gamma\in\mathcal{C}_{\Omega_\delta}(a,z)$, we have

$$\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[\alpha^{|\gamma|}e^{-(i/2)\mathrm{W}(\gamma)}]=\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[\alpha^{|\varphi(\gamma)|}e^{-(i/2)\mathrm{W}(\varphi(\gamma))}]. \tag{2.1}$$

This identity follows from considering the four possible cases, as shown in Figure 2.2:

(1) If $h\notin\gamma$ and $\tilde{h}\notin\gamma$, then we have $e^{-(i/2)\mathrm{W}(\gamma)}\in\mathbb{R}$, $|\varphi(\gamma)|=|\gamma|+1$ and

$$e^{-(i/2)\mathrm{W}(\varphi(\gamma))}=e^{-\pi i/4}e^{-(i/2)\mathrm{W}(\gamma)}.$$

(2) If $h\notin\gamma$ and $\tilde{h}\in\gamma$, we have $e^{-(i/2)\mathrm{W}(\gamma)}\in\mathbb{R}$, $|\varphi(\gamma)|=|\gamma|$ and

$$e^{-(i/2)\mathrm{W}(\varphi(\gamma))}=e^{-3\pi i/4}e^{-(i/2)\mathrm{W}(\gamma)};$$

there are a number of subcases, as shown in Figure 2.2, for which these relations are satisfied.

(3) If $h\in\gamma$ and $\tilde{h}\in\gamma$, we have $e^{-(i/2)\mathrm{W}(\gamma)}\in i\mathbb{R}$, $|\varphi(\gamma)|=|\gamma|-1$ and

$$e^{-(i/2)\mathrm{W}(\varphi(\gamma))}=e^{-\pi i/4}e^{-(i/2)\mathrm{W}(\gamma)}:$$

in this case, we can always choose an admissible walk on γ that is like in Figure 2.2.

(4) If $h\in\gamma$ and $\tilde{h}\notin\gamma$, we have $e^{-(i/2)\mathrm{W}(\gamma)}\in i\mathbb{R}$, $|\varphi(\gamma)|=|\gamma|$ and

$$e^{-(i/2)\mathrm{W}(\varphi(\gamma))}=e^{\pi i/4}e^{-(i/2)\mathrm{W}(\gamma)}.$$

In all the four cases, it is then straightforward to check that equation (2.1) is satisfied. By the definition of f_{Ω_δ} (§1.5), we finally deduce that

$$\begin{aligned}\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[f_{\Omega_\delta}(a,z)]&=\frac{1}{\mathbf{Z}_{\Omega_\delta}}\sum_{\gamma\in\mathcal{C}_{\Omega_\delta}(a,z)}\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[\alpha^{|\gamma|}e^{-(i/2)\mathrm{W}(\gamma)}]\\&=\frac{1}{\mathbf{Z}_{\Omega_\delta}}\sum_{\tilde{\gamma}\in\mathcal{C}_{\Omega_\delta}(a,w)}\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[\alpha^{|\tilde{\gamma}|}e^{-(i/2)\mathrm{W}(\tilde{\gamma})}]=\mathrm{P}_{e^{-3\pi i/8}\mathbb{R}}[f_{\Omega_\delta}(a,w)],\end{aligned}$$

which is the s-holomorphicity equation. □

The full-plane discrete correlator is also s-holomorphic.

PROPOSITION 2.6. *The function* $f_{\mathbb{C}_\delta}(a,\cdot)$ *is s-holomorphic on* $\mathcal{V}_{\mathbb{C}_\delta^{\mathrm{M}}}\setminus\{a\}$.

Proof. This follows directly from the definition of $f_{\mathbb{C}_\delta}$ (Definition 1.6) and from Lemma B.1. □

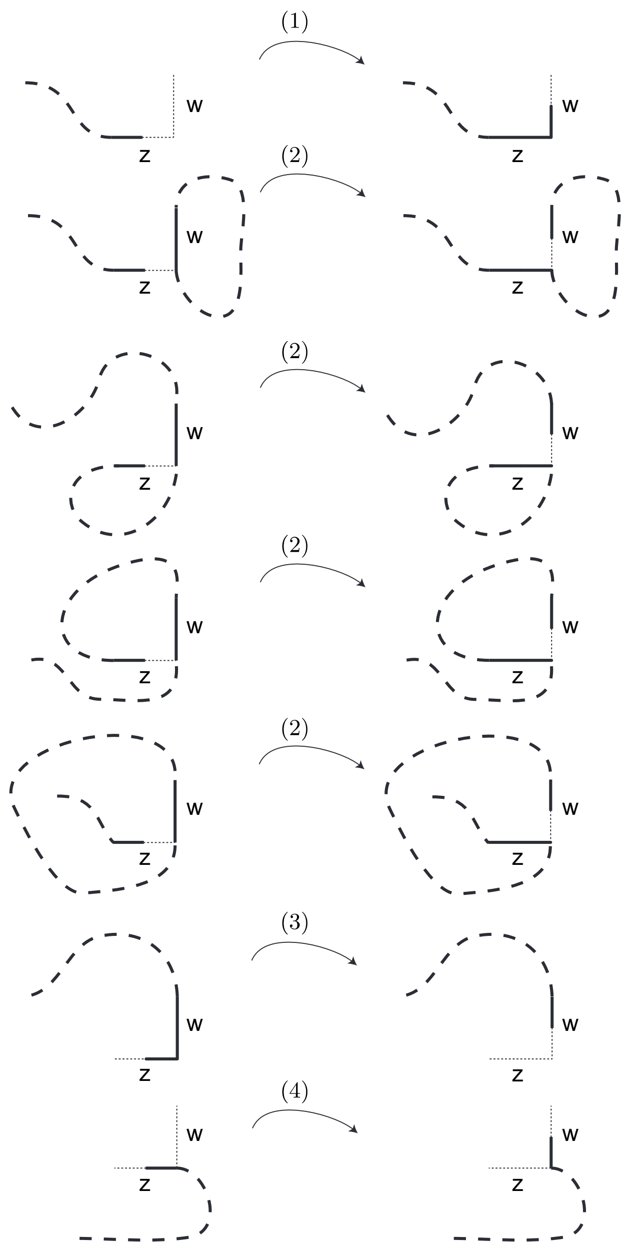

Figure 2.2. The four possible cases in the proof of Proposition 2.5: how a configuration is changed after two half-edges between z and w are removed or added.

2.2. Singularity

Near the medial vertex a, the functions $f_{\Omega_\delta}(a,\cdot)$ and $f_{\mathbb{C}_\delta}(a,\cdot)$ are not s-holomorphic: they both have a discrete singularity, but of the same nature, and consequently the difference $(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,\cdot)$ is s-holomorphic on $\mathcal{V}_{\Omega_\delta^{\mathrm{m}}}$, including the point a. Rather than defining the notion of discrete singularity, let us simply describe the relations that these functions satisfy near a. For $x\in\{\pm1\pm i\}$, let us denote by $a_x:=a+\frac{1}{2}x\delta\in\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$ the medial vertex adjacent to a, by $e_x\in\mathcal{E}_{\Omega_\delta^{\mathrm{M}}}$ the medial edge between a and a_x, and by ℓ_x the line $\ell(e_x)$ (as in Definition 2.1). Let $e\in\mathcal{E}^h_{\Omega_\delta}$ be the horizontal edge with midpoint a. Recall that $f_{\Omega_\delta}(a,a)$ is defined as $\mathbf{Z}^{\{e+\}}_{\Omega_\delta}/\mathbf{Z}_{\Omega_\delta}$ (see Definition 1.5).

Proposition 2.7. *Near a, the function $f_{\Omega_\delta}(a,\cdot)$ satisfies the relations*

$$
\begin{aligned}
\mathrm{P}_{\ell_{1+i}}[f_{\Omega_\delta}(a,a)] &= \mathrm{P}_{\ell_{1+i}}[f_{\Omega_\delta}(a,a_{1+i})],\\
\mathrm{P}_{\ell_{1-i}}[f_{\Omega_\delta}(a,a)] &= \mathrm{P}_{\ell_{1-i}}[f_{\Omega_\delta}(a,a_{1-i})],\\
\mathrm{P}_{\ell_{-1+i}}[f_{\Omega_\delta}(a,a)-1] &= \mathrm{P}_{\ell_{-1+i}}[f_{\Omega_\delta}(a,a_{-1+i})],\\
\mathrm{P}_{\ell_{-1-i}}[f_{\Omega_\delta}(a,a)-1] &= \mathrm{P}_{\ell_{-1-i}}[f_{\Omega_\delta}(a,a_{-1-i})].
\end{aligned}
$$

Proof. The first two relations are the s-holomorphicity relations and they are obtained in exactly the same way as the s-holomorphicity relations away from a. Indeed, let us use the same notation as in the proof of Proposition 2.5. First recall that for $\gamma\in\mathcal{C}_{\Omega_\delta}(a,a_{1\pm i})$ the half-edge of γ starting at a goes to a_1. Now consider the involutions $\varphi_{1+i}:\mathcal{C}^{\{e+\}}_{\Omega_\delta}\to\mathcal{C}_{\Omega_\delta}(a,a_{1+i})$ and $\varphi_{1-i}:\mathcal{C}^{\{e+\}}_{\Omega_\delta}\to\mathcal{C}_{\Omega_\delta}(a,a_{1-i})$ defined by

$$\varphi_{1+i}(\gamma):=\gamma\oplus\langle a,a_1\rangle\oplus\langle a_1,a_{1+i}\rangle \quad\text{and}\quad \varphi_{1-i}(\gamma):=\gamma\oplus\langle a,a_1\rangle\oplus\langle a_1,a_{1-i}\rangle,$$

respectively: as in the proof of Proposition 2.7, we have that these involutions preserve the projections on ℓ_{1+i} and ℓ_{1-i} respectively (a configuration $\gamma\in\mathcal{C}^{\{e+\}}_{\Omega_\delta}$ is interpreted as a configuration with winding number 0).

For the last two relations, we have that the involutions $\varphi_{-1+i}:\mathcal{C}^{\{e-\}}_{\Omega_\delta}\to\mathcal{C}_{\Omega_\delta}(a,a_{-1+i})$ and $\varphi_{-1-i}:\mathcal{C}^{\{e-\}}_{\Omega_\delta}\to\mathcal{C}_{\Omega_\delta}(a,a_{-1-i})$, respectively defined by

$$\varphi_{-1+i}(\gamma):=\gamma\oplus\langle a,a_{-1}\rangle\oplus\langle a_{-1},a_{-1+i}\rangle \quad\text{and}\quad \varphi_{-1-i}(\gamma):=\gamma\oplus\langle a,a_{-1}\rangle\oplus\langle a_{-1},a_{-1-i}\rangle,$$

are such that for any $\gamma\in\mathcal{C}^{\{e-\}}_{\Omega_\delta}$ we have

$$
\begin{aligned}
-\mathrm{P}_{\ell_{-1+i}}[\alpha^{|\gamma|}] &= \mathrm{P}_{\ell_{-1+i}}[\alpha^{|\gamma|}e^{-(i/2)\mathrm{W}(\gamma)}] = \mathrm{P}_{\ell_{-1+i}}[\alpha^{|\varphi_{-1+i}(\gamma)|}e^{-(i/2)\mathrm{W}(\varphi_{-1+i}(\gamma))}],\\
-\mathrm{P}_{\ell_{-1-i}}[\alpha^{|\gamma|}] &= \mathrm{P}_{\ell_{-1-i}}[\alpha^{|\gamma|}e^{-(i/2)\mathrm{W}(\gamma)}] = \mathrm{P}_{\ell_{-1-i}}[\alpha^{|\varphi_{-1-i}(\gamma)|}e^{-(i/2)\mathrm{W}(\varphi_{-1-i}(\gamma))}],
\end{aligned}
$$

where γ is interpreted as a configuration with a path from a to a that makes a loop, with winding number $\pm2\pi$. This follows from the same considerations as in the proof of Proposition 2.5. Hence, since $\mathbf{Z}^{\{e-\}}_{\Omega_\delta}/\mathbf{Z}_{\Omega_\delta}=1-f_{\Omega_\delta}(a,a)$, we obtain, by summing the above equations over all $\gamma\in\mathcal{C}^{\{e-\}}_{\Omega_\delta}$, the last two identities of Proposition 2.7. $\square$

The function $f_{\mathbb{C}_\delta}(a,\cdot)$ has the same type of discrete singularity as $f_{\Omega_\delta}(a,\cdot)$.

PROPOSITION 2.8. *Near a, the function $f_{\mathbb{C}_\delta}(a,\cdot)$ satisfies exactly the same projection relations as the ones satisfied by the function $f_{\Omega_\delta}(a,\cdot)$, given by Proposition 2.7.*

Proof. See Proposition B.2 in Appendix B. □

From Propositions 2.5–2.8, we readily deduce the following result.

PROPOSITION 2.9. *The function $(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,\cdot)\colon \mathcal{V}_{\Omega_\delta^{\mathrm{M}}} \to \mathbb{C}$ is s-holomorphic on $\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$.*

2.3. Boundary values

A crucial piece of information to understand the effect of the geometry of the discrete domain Ω_δ on the average energy density at $a\in\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$ is the boundary behavior of $f_{\Omega_\delta}(a,\cdot)$. On the set of boundary medial vertices $\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$, which link a vertex of Ω_δ and a vertex of $\partial\Omega_\delta$, the argument of $f_{\Omega_\delta}(a,\cdot)$ is determined modulo π. For each $z\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$, with z being the midpoint of an edge $e\in\partial\mathcal{E}_{\Omega_\delta}$ between a vertex $x\in\mathcal{V}_{\Omega_\delta}$ and a vertex $y\in\partial\mathcal{V}_{\Omega_\delta}$, denote by $\nu_{\mathrm{out}}(z)\in\partial\vec{\mathcal{E}}_{\Omega_\delta}$ the oriented outward-pointing edge at z, identified with the number $y-x$: it is a discrete analogue of the outward-pointing normal to the domain.

PROPOSITION 2.10. *On $\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$, the argument of the value of $f_{\Omega_\delta}(a,\cdot)$ is determined (modulo π): for each $z\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$, we have*

$$\operatorname{Im}(f_{\Omega_\delta}(a,z)\nu_{\mathrm{out}}^{1/2}(z))=0.$$

Proof. From topological considerations, we have that if $z\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$ and $\gamma\in\mathcal{C}_{\Omega_\delta}(a,z)$, then $\operatorname{Im}(e^{-(i/2)\mathrm{W}(\gamma)}\nu_{\mathrm{out}}^{1/2}(z))=0$: the winding number of any admissible walk from a to z is determined modulo 2π (see Figure 2.3) and it is easy to check that $e^{-(i/2)\mathrm{W}(\gamma)}$ is a real multiple of $\nu_{\mathrm{out}}^{-1/2}(z)$. Hence, the result follows from the definition of f_{Ω_δ}. □

Remark 2.11. This is the same kind of Riemann-type boundary conditions as in [S2] and [CS2]. Notice that in these papers, the argument of the function on the boundary is fully determined (not only modulo π).

2.4. Discrete integration

An essential tool that we will use for deriving the convergence of $f_{\Omega_\delta}(a,\cdot)$ is the possibility to define a discrete version of the antiderivative of the square of an s-holomorphic function, cf. [S2].

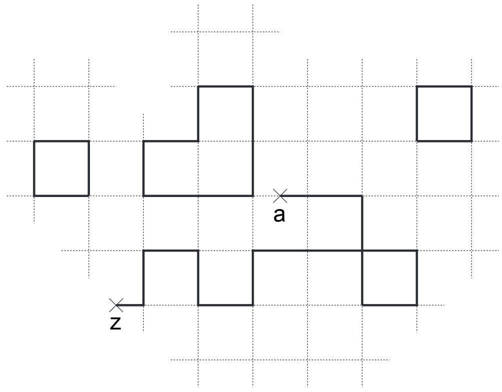

Figure 2.3. When $z\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$, the winding number of the walks on $\mathcal{C}_{\Omega_\delta}(a,z)$ is determined modulo 2π.

PROPOSITION 2.12. *Let $f\colon\mathcal{V}_{\mathcal{D}_\delta^{\mathrm{M}}}\to\mathbb{C}$ be an s-holomorphic function on a discrete domain $\mathcal{D}_\delta$ and let $x\in\bar{\mathcal{V}}_{\mathcal{D}_\delta}\cup\bar{\mathcal{V}}_{\mathcal{D}_\delta^*}$ (where $\bar{\mathcal{V}}_{\mathcal{D}_\delta}=\mathcal{V}_{\mathcal{D}_\delta}\cup\partial\mathcal{V}_{\mathcal{D}_\delta}$ and $\bar{\mathcal{V}}_{\mathcal{D}_\delta^*}=\mathcal{V}_{\mathcal{D}_\delta^*}\cup\partial\mathcal{V}_{\mathcal{D}_\delta^*}$). Then there exists a (possibly multivalued) discrete analogue $\mathsf{I}_{x,\delta}[f]\colon\bar{\mathcal{V}}_{\mathcal{D}_\delta}\cup\bar{\mathcal{V}}_{\mathcal{D}_\delta^*}\to\mathbb{R}$ of the antiderivative*

$$z\longmapsto -\mathrm{Re}\biggl(\int_x^z f^2\biggr),$$

uniquely defined by the equation

$$\mathsf{I}_{x,\delta}[f](b)-\mathsf{I}_{x,\delta}[f](w)=\sqrt{2}\delta|\mathrm{P}_{\ell(e^*)}[f(y)]|^2=\sqrt{2}\delta|\mathrm{P}_{\ell(e^*)}[f(z)]|^2$$

for all $b\in\mathcal{V}_{\Omega_\delta}$ and $w\in\mathcal{V}_{\Omega_\delta^*}$ such that $|b-w|=\delta/\sqrt{2}$, where $e^*=\langle y,z\rangle\in\mathcal{E}_{\mathcal{D}_\delta^{\mathrm{M}}}$ is the medial edge which is between b and w, and by the condition $\mathsf{I}_{x,\delta}[f](x)=0$. If $\mathcal{D}_\delta$ is simply connected, then the function $\mathsf{I}_{x,\delta}[f]$ is globally well defined (single-valued). When the choice of the point x is irrelevant, we will omit it and simply write $\mathsf{I}_\delta[f]$.

Remark 2.13. It follows from the definition of $\mathsf{I}_\delta[f]$ that for any pair of adjacent vertices $x,y\in\bar{\mathcal{V}}_{\mathcal{D}_\delta}$ we have

$$\mathsf{I}_\delta[f](x)-\mathsf{I}_\delta[f](y)=-\mathrm{Re}\Bigl(f\Bigl(\frac{x+y}{2}\Bigr)^2(y-x)\Bigr),$$

and similarly if $x,y\in\bar{\mathcal{V}}_{\mathcal{D}_\delta^*}$ are adjacent dual vertices. From there it is easy to see that if the mesh size is small, $\mathsf{I}_\delta[f]$ is a good approximation of $-\mathrm{Re}(\int_x f^2)$.

We denote by $\mathsf{I}^\bullet_{x,\delta}[f]$ and $\mathsf{I}^\circ_{x,\delta}[f]$ the restrictions of $\mathsf{I}_{x,\delta}[f]$ to $\bar{\mathcal{V}}_{\mathcal{D}_\delta}$ and $\bar{\mathcal{V}}_{\mathcal{D}_\delta^*}$ respectively. We have the following result.

Proposition 2.14. *The function* $I_\delta^\bullet[f]: \bar{\mathcal{V}}_{\mathcal{D}_\delta} \to \mathbb{R}$ *is discrete subharmonic and the function* $I_\delta^\circ[f]: \bar{\mathcal{V}}_{\mathcal{D}_\delta^*} \to \mathbb{R}$ *is discrete superharmonic*: *we have*

$$\Delta_\delta I_\delta^\bullet[f](v) \geq 0 \quad \textit{for all } v \in \mathcal{V}_{\mathcal{D}_\delta},$$
$$\Delta_\delta I_\delta^\circ[f](v) \leq 0 \quad \textit{for all } v \in \mathcal{V}_{\mathcal{D}_\delta^*}.$$

If $m \in \partial \mathcal{V}_{\mathcal{D}_\delta^{\mathrm{M}}}$, *then we have*

$$\partial_{\nu_{\mathrm{out}}(m)} I_\delta^\bullet[f] = \mathrm{Im}(f(m)\nu_{\mathrm{out}}^{1/2}(m))^2 - \mathrm{Re}(f(m)\nu_{\mathrm{out}}^{1/2}(m))^2,$$

where $\nu_{\mathrm{out}}(m) \in \partial \vec{\mathcal{E}}_{\mathcal{D}_\delta}$ *is the oriented edge from* $a \in \mathcal{D}_\delta$ *to* $b \in \partial \mathcal{D}_\delta$, *the midpoint of which is* m.

Proof. For the subharmonicity/superharmonicity deduced from the s-holomorphicity of f, see [S2, Lemma 3.8] (the fact that the phases are different does not affect the result). The normal derivative statement follows directly from the definition of $I_\delta[f]$. □

In the case of the discrete fermionic correlator $f_{\Omega_\delta}(\cdot,\cdot)$, the boundary condition for $I_\delta[f_{\Omega_\delta}(a,\cdot)](\cdot)$ becomes particularly simple.

Proposition 2.15. *The function* $I_\delta^\circ[f_{\Omega_\delta}(a,\cdot)]: \bar{\mathcal{V}}_{\Omega_\delta^*} \to \mathbb{R}$ *is constant on* $\partial \mathcal{V}_{\Omega_\delta^*}$ *and for each* $m \in \partial_0 \mathcal{V}_{\Omega_\delta^{\mathrm{M}}}$,

$$\partial_{\nu_{\mathrm{out}}(m)} I_\delta^\bullet[f_{\Omega_\delta}(a,\cdot)] = -|f_{\Omega_\delta}(a,m)|^2.$$

Proof. The first statement follows from the construction of $I_\delta^\circ[f_{\Omega_\delta}(a,\cdot)]$ and from the boundary condition for f_{Ω_δ} (Proposition 2.10).

The statement for $I_\delta^\bullet[f_{\Omega_\delta}(a,\cdot)]$ follows directly from Proposition 2.14 and the boundary condition for f_{Ω_δ} (Proposition 2.10 again). □

Remark 2.16. Note that $I_\delta[f_{\Omega_\delta}(a,\cdot)]$ is single-valued (as a consequence of Proposition 2.15) and well defined on $\bar{\mathcal{V}}_{\Omega_\delta} \cup \bar{\mathcal{V}}_{\Omega_\delta^*}$ but that the presence of a singularity near a implies that and $I_\delta^\bullet[f_{\Omega_\delta}(a,\cdot)]$ and $I_\delta^\circ[f_{\Omega_\delta}(a,\cdot)]$ are (at least a priori) not subharmonic or superharmonic near a (more precisely at $a \pm \frac{1}{2}\delta$ and $a \pm \frac{1}{2} i\delta$).

3. Convergence of the discrete fermionic correlators

We now turn to the convergence of the three functions f_{Ω_δ}/δ, $f_{\mathbb{C}_\delta}/\delta$ and $(f_{\Omega_\delta} - f_{\mathbb{C}_\delta})/\delta$ as $\delta \to 0$ (Theorem 1.8). For this, we use the discrete results derived in the previous section: the s-holomorphicity, the discrete singularity and the boundary values. As we will discuss convergence questions, we will always, when necessary, identify the points of the complex plane with the closest vertices on the graphs considered. In this way, we will extend functions defined on the vertices of the graphs Ω_δ, Ω_δ^* and Ω_δ^m to functions defined on Ω. In particular, for the discrete holomorphic fermionic correlators, when we write $f_{\Omega_\delta}(a,z)$ or $f_{\mathbb{C}_\delta}(a,z)$ for $a,z\in\Omega$, we identify a with the closest midpoint of a horizontal edge of $\mathcal{E}^h_{\Omega_\delta}$ and z with the closest midpoint of an arbitrary edge of $\mathcal{E}_{\Omega_\delta}$.

The convergence of $f_{\mathbb{C}_\delta}$ almost immediately follows from the work of Kenyon [Ken].

THEOREM 3.1. *For any $\varepsilon>0$, we have*

$$\frac{f_{\mathbb{C}_\delta}(a,z)}{\delta} \to f_{\mathbb{C}}(a,z) \quad \textit{as } \delta\to 0$$

uniformly on $\{(a,z)\in\mathbb{C}^2:|a-z|\geq\varepsilon\}$, *where*

$$f_{\mathbb{C}}(a,z)=\frac{1}{2\pi(z-a)}.$$

Proof. See the last paragraph of Appendix B. □

For the convergence of f_{Ω_δ}/δ and $(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta$, we proceed in two steps: we first show that the family of functions $((f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta)_{\delta>0}$ is precompact. Precompactness for $(f_{\Omega_\delta}/\delta)_{\delta>0}$ will then readily follow from Theorem 3.1. We then identify uniquely the limits of subsequences of $(f_{\Omega_\delta}/\delta)_{\delta>0}$; this also identifies the ones of $((f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta)_{\delta>0}$.

3.1. Precompactness

We now state our main precompactness result.

PROPOSITION 3.2. *The family of functions*

$$\left((a,z)\mapsto\frac{(f_{\Omega_\delta}-f_{\mathbb{C}_\delta})(a,z)}{\delta}\right)_{\delta>0}$$

is precompact in the topology of uniform convergence on the compact subsets of $\Omega\times\Omega$, *and hence the family of functions*

$$\left((a,z)\mapsto\frac{f_{\Omega_\delta}(a,z)}{\delta}\right)_{\delta>0}$$

is precompact in the topology of uniform convergence on the compact subsets of $\Omega\times\Omega$ *that are away from the diagonal.*

Proof. Set $f^{\mathbb{C}_\delta}_{\Omega_\delta}:=f_{\Omega_\delta}-f_{\mathbb{C}_\delta}$. By Proposition 2.10, we have that for any $x\in\partial_0\mathcal{V}_{\Omega^{\mathrm{M}}_\delta}$,

$$\mathrm{Im}(f^{\mathbb{C}_\delta}_{\Omega_\delta}(a,x)\nu^{1/2}_{\mathrm{out}}(x))=-\mathrm{Im}(f_{\mathbb{C}_\delta}(a,x)\nu^{1/2}_{\mathrm{out}}(x)).$$

By Theorem 3.1 and the fact that $\partial\Omega$ is smooth (and hence the number of medial vertices in $\partial_0\mathcal{V}_{\Omega_\delta^M}$ is $O(\delta^{-1})$) we deduce that the family of functions

$$\left(a\mapsto \sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}} \operatorname{Im}\left(\frac{f_{\Omega_\delta}^{\mathbb{C}_\delta}(a,x)}{\delta}\nu_{\mathrm{out}}^{1/2}(x)\right)^2\delta\right)_{\delta>0}$$

is uniformly bounded and equicontinuous on the compact subsets of Ω (since $f_{\mathbb{C}_\delta}/\delta$ is uniformly convergent by Theorem 3.1). By Proposition 3.3 below, we obtain that the family of functions

$$\left(\frac{f_{\Omega_\delta}^{\mathbb{C}_\delta}}{\delta}:\mathcal{V}_{\Omega_\delta^M}\times\mathcal{V}_{\Omega_\delta^M}\to\mathbb{C}\right)_{\delta>0}$$

is uniformly bounded and equicontinuous and hence we get the desired result by extending the functions $f_{\Omega_\delta}^{\mathbb{C}_\delta}$ in a uniformly continuous way to $\Omega\times\Omega$ (for instance by piecewise-linear interpolation) and by using then Arzelà–Ascoli theorem. □

PROPOSITION 3.3. *There exists a universal constant $C>0$ such that for each $\delta>0$ and any s-holomorphic function $u_\delta:\mathcal{V}_{\Omega_\delta^M}\to\mathbb{C}$, we have, for any $v\in\mathcal{V}_{\Omega_\delta^M}\setminus\partial_0\mathcal{V}_{\Omega_\delta^M}$,*

$$|u_\delta(v)|\leq C\sqrt{\frac{\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}\operatorname{Im}(u_\delta(x)\nu_{\mathrm{out}}^{1/2}(x))^2\delta}{\operatorname{dist}(v,\partial_0\mathcal{V}_{\Omega_\delta^M})}},$$

$$\frac{\|\nabla_\delta u_\delta(v)\|^2}{\delta}\leq C\sqrt{\frac{\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}\operatorname{Im}(u_\delta(x)\nu_{\mathrm{out}}^{1/2}(x))^2\delta}{\operatorname{dist}(v,\partial_0\mathcal{V}_{\Omega_\delta^M})^3}},$$

where $\nabla_\delta u_\delta(v)=(u_\delta(v+\delta)-u_\delta(v),u_\delta(v+i\delta)-u_\delta(v))$.

Proof. Consider the function $H_\delta[u_\delta]$, normalized to be 0 at an arbitrary point. By subharmonicity (Proposition 2.14), a discrete integration by parts, and again by Proposition 2.14, we obtain

$$\begin{aligned}0&\leq\sum_{b\in\mathcal{V}_{\Omega_\delta}}\Delta_\delta H_\delta^\bullet[u_\delta]=\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}\partial_{\nu_{\mathrm{out}}(x)}H_\delta^\bullet[u_\delta]\\&=\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}((\operatorname{Im}(u_\delta(x)\nu_{\mathrm{out}}^{1/2}(x)))^2-(\operatorname{Re}(u_\delta(x)\nu_{\mathrm{out}}^{1/2}(x)))^2)\end{aligned}$$

and from the last identity we also deduce that

$$\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}|u_\delta(x)|^2\leq 2\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^M}}(\operatorname{Im}(u_\delta(x)\nu_{\mathrm{out}}^{1/2}(x)))^2.$$

On the other hand, from the construction of $\mathsf{I}_\delta[u]$, it is easy to see that

$$\max_{z\in\partial\mathcal{V}_{\Omega_\delta}\cup\partial\mathcal{V}_{\Omega_\delta^\star}} |\mathsf{I}_\delta[u_\delta](z)| \leqslant \sqrt{2}\bigg(\sum_{x\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}} |u_\delta(x)|^2\delta\bigg).$$

By subharmonicity of $\mathsf{I}_\delta^\bullet[u_\delta]$, superharmonicity of $\mathsf{I}_\delta^\circ[u_\delta]$ and the construction of $\mathsf{I}_\delta[u_\delta]$ (Proposition 2.12), we have

$$\max_{w\in\bar{\mathcal{V}}_{\Omega_\delta}\cup\bar{\mathcal{V}}_{\Omega_\delta^\star}} |\mathsf{I}_\delta[u_\delta](w)| = \max_{z\in\partial\mathcal{V}_{\Omega_\delta}\cup\partial\mathcal{V}_{\Omega_\delta^\star}} |\mathsf{I}_\delta[u_\delta](z)|.$$

By [CS2, Theorem 3.12] (the construction there is the same as the one of our paper, up to a multiplication by an overall complex factor, which does not affect the result), there exists then a universal constant $\widetilde{C}>0$ such that for any $v\in\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}\setminus\partial_0\mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$,

$$|u_\delta(v)|^2 \leqslant \widetilde{C}\frac{\max_{w\in\bar{\mathcal{V}}_{\Omega_\delta}\cup\bar{\mathcal{V}}_{\Omega_\delta^\star}} |\mathsf{I}_\delta[u_\delta](w)|}{\operatorname{dist}(v,\partial_0\mathcal{V}_{\Omega_\delta^{\mathsf{M}}})},$$

$$\|\nabla_\delta u_\delta(v)\|^2 \leqslant \widetilde{C}\frac{\max_{w\in\bar{\mathcal{V}}_{\Omega_\delta}\cup\bar{\mathcal{V}}_{\Omega_\delta^\star}} |\mathsf{I}_\delta[u_\delta](w)|}{\operatorname{dist}(v,\partial_0\mathcal{V}_{\Omega_\delta^{\mathsf{M}}})^3}.$$

We therefore deduce the desired inequalities. □

3.2. Identification of the limit

We can now uniquely identify the limits of subsequences of $(f_{\Omega_\delta}/\delta)_{\delta>0}$ as $\delta\to 0$ (we will often make a slight of abuse of notation and simply denote the family of functions by f_{Ω_δ}/δ). Let us start with a characterization of the continuous fermionic correlator $f_\Omega(a,\cdot)$ (defined in §1.6).

Lemma 3.4. *The function $f_\Omega(a,\cdot)$ is the unique holomorphic function such that*

$$z\longmapsto f_\Omega(a,z)-\frac{1}{2\pi(z-a)}$$

is bounded near $z=a$ and such that

$$\operatorname{Im}(f_\Omega(a,z)\nu_{\mathsf{out}}^{1/2}(z))=0 \quad \textit{for all } z\in\partial\Omega, \tag{3.1}$$

where ν_{out} denotes the outward-pointing normal to $\partial\Omega$.

The boundary condition (3.1) is equivalent to the condition that the antiderivative

$$F(z)=-\operatorname{Re}\bigg(\int^z f_\Omega^2(a,w)\,dw\bigg)$$

is single-valued on $\Omega\setminus\{a\}$, constant on $\partial\Omega$ and satisfies

$$\partial_{\nu_{\mathsf{out}}(z)}F\geqslant 0 \quad \textit{for all } z\in\partial\Omega,$$

where $\partial_{\nu_{\mathsf{out}}(z)}F$ denotes the normal derivative of F in the outward-pointing direction.

Proof. It is straightforward to check from the definition (§1.6) that $f_\Omega(a,\cdot)$ has a simple pole of order 1 and residue $1/2\pi$ at $z=a$, and satisfies the boundary condition (3.1).

Let $\tilde{f}$ be another function with the same pole and boundary condition. Then the function g defined by $g(z):=f_\Omega(a,z)-\tilde{f}(z)$ extends holomorphically to Ω and satisfies

$$\operatorname{Im}(g(z)\nu_{\mathrm{out}}^{1/2}(z))=0 \quad \text{for all } z\in\partial\Omega.$$

The function $G:\Omega\to\mathbb{C}$ defined by $G(w):=\int^w g^2(z)\,dz$ has constant real part on $\partial\Omega$ and hence is constant on Ω, by the maximum principle and the Cauchy–Riemann equations. Hence $f_\Omega(a,\cdot)=\tilde{f}(\cdot)$.

For the second part of the statement, notice that the boundary condition (3.1) implies that $f_\Omega^2(a,\cdot)\nu_{\mathrm{out}}(\cdot)$ is purely real on $\partial\Omega$, and hence F must be constant on $\partial\Omega$ (when going along the boundary, one integrates $f_\Omega^2(a,\cdot)\,d\tau$, where τ is the tangent to the boundary, which is orthogonal to the normal ν_{out}); this implies that F is single-valued on $\Omega\setminus\{a\}$ (since Ω is simply connected) and that

$$\partial_{\nu_{\mathrm{out}}(z)}F=-|f_\Omega(a,z)|^2 \quad \text{for all } z\in\partial\Omega.$$

Conversely, it is easy to check that if F is constant on $\partial\Omega$, then for any $z\in\partial\Omega$, we have $f_\Omega^2(a,z)\nu_{\mathrm{out}}(z)\in\mathbb{R}$. Moreover, for any $z\in\partial\Omega$, we have that $\partial_{\nu_{\mathrm{out}}(z)}F\,\square\,0$ implies $f_\Omega^2(a,z)\nu_{\mathrm{out}}(z)\,\square\,0$, which is equivalent to $\operatorname{Im}(f_\Omega(a,z)\nu_{\mathrm{out}}^{1/2}(z))=0$. $\square$

Let us also give a lemma which will be useful to connect the discrete correlators to the continuous ones.

LEMMA 3.5. *We have the uniform bound*

$$\sup_{\delta>0}\sum_{z\in\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{m}}}}|f_{\Omega_\delta}(z)|^2\delta<\infty.$$

Proof. This follows directly from the the proof of precompactness of $((f_{\Omega_\delta}-f_{\mathbb{C}_\delta})/\delta)_\delta$ (Proposition 3.2), the convergence of $f_{\mathbb{C}_\delta}/\delta$ (Theorem 3.1) and the fact that $\partial\Omega$ is smooth (the number of medial vertices in $\partial_0\mathcal{V}_{\Omega_\delta^{\mathrm{m}}}$ is $O(\delta^{-1})$). $\square$

We now identify the limits of subsequences of $f_{\Omega_\delta}(a,\cdot)/\delta$ as $\delta\to 0$.

PROPOSITION 3.6. *Let δ_n be a sequence with $\delta_n\to 0$ as $n\to\infty$ such that*

$$\frac{f_{\Omega_{\delta_n}}(a,\cdot)}{\delta_n}\to f(\cdot) \quad \text{as } n\to\infty$$

uniformly on the compact subsets of $\Omega\setminus\{a\}$. Then $f(\cdot)=f_\Omega(a,\cdot)$, where $f_\Omega(a,\cdot)$ is defined in §1.6.

Proof. For each $\delta>0$, set $f_\delta(\cdot):=f_{\Omega_\delta}(a,\cdot)/\delta$.

Let us first remark that $f(\cdot)$ is holomorphic on $\Omega\setminus\{a\}$, as it satisfies Morera's condition: the integral of $f(\cdot)$ on any contractible contour vanishes, since it can be approximated by a Riemann sum involving $f_{\delta_n}(\cdot)$ (for δ_n small), which vanishes identically as explained in Remark 2.4. Fix a point $p\in\Omega\setminus\{a\}$. By Lemma 3.4, to identify f with $f_\Omega(a,\cdot)$, it suffices to check that F, defined by

$$F(z):=-\operatorname{Re}\left(\int_p^z f^2(w)\,dw\right),$$

satisfies the conditions of the second part of that lemma.

Let $F_\delta\colon \overline{\mathcal{V}}_{\Omega_\delta}\cup\overline{\mathcal{V}}_{\Omega_\delta^*}\to\mathbb{R}$ be the discrete antiderivative $\mathrm{I}_{p,\delta}[f_\delta]$, as defined in Proposition 2.12, and let $F_\delta^\bullet$ and F_δ° denote the restrictions of F_δ to $\overline{\mathcal{V}}_{\Omega_\delta}$ and $\overline{\mathcal{V}}_{\Omega_\delta^*}$, which are discrete subharmonic and superharmonic respectively (away from a), by Proposition 2.14. By Proposition 2.15, the function F_δ° is constant on $\partial\mathcal{V}_{\Omega_\delta^*}$; denote by $F_\delta(\partial\Omega)$ this value. Fix a smooth doubly connected domain $\Upsilon\subset\Omega\setminus\{a\}$ such that $\partial\Omega\subset\partial\Upsilon$ and $\operatorname{dist}(a,\partial\Upsilon)>0$ (one of the components of $\partial\Upsilon$ is $\partial\Omega$ and the other is a simple loop surrounding a). Let us write $F_\delta^\bullet=:H_\delta^\bullet+S_\delta^\bullet$ and $F_\delta^\circ=:H_\delta^\circ+S_\delta^\circ$, where

- $H_\delta^\bullet\colon\overline{\mathcal{V}}_{\Upsilon_\delta}\to\mathbb{R}$ is discrete harmonic, with $H_\delta^\bullet:=F_\delta^\bullet$ on $\partial\mathcal{V}_{\Upsilon_\delta}$,
- $S_\delta^\bullet\colon\overline{\mathcal{V}}_{\Upsilon_\delta}\to\mathbb{R}$ is discrete subharmonic, with $S_\delta^\bullet:=0$ on $\partial\mathcal{V}_{\Upsilon_\delta}$,
- $H_\delta^\circ\colon\overline{\mathcal{V}}_{\Upsilon_\delta^*}\to\mathbb{R}$ is discrete harmonic, with $H_\delta^\circ:=F_\delta^\circ$ on $\partial\mathcal{V}_{\Upsilon_\delta^*}$,
- $S_\delta^\circ\colon\overline{\mathcal{V}}_{\Upsilon_\delta^*}\to\mathbb{R}$ is discrete superharmonic, with $S_\delta^\circ:=0$ on $\partial\mathcal{V}_{\Upsilon_\delta^*}$.

Let us further decompose $H_\delta^\bullet$ as $A_\delta^\bullet+B_\delta^\bullet$, where

- $A_\delta^\bullet$ is discrete harmonic with

$$A_\delta^\bullet:=\begin{cases} F_\delta(\partial\Omega) & \text{on } \partial\mathcal{V}_{\Omega_\delta},\\ H_\delta^\bullet & \text{on } \partial\mathcal{V}_{\Upsilon_\delta}\setminus\partial\mathcal{V}_{\Omega_\delta}.\end{cases}$$

- $B_\delta^\bullet$ is discrete harmonic with

$$B_\delta^\bullet:=\begin{cases} H_\delta^\bullet-F_\delta(\partial\Omega) & \text{on } \partial\mathcal{V}_{\Omega_\delta},\\ 0 & \text{on } \partial\mathcal{V}_{\Upsilon_\delta}\setminus\partial\mathcal{V}_{\Omega_\delta}.\end{cases}$$

The situation is hence the following: for any $z\in\mathcal{V}_{\Omega_\delta}$ and $w\in\mathcal{V}_{\Omega_\delta^*}$ such that $|z-w|=\delta/\sqrt{2}$, from the construction of F_δ, the superharmonicity of F_δ° and the subharmonicity of $F_\delta^\bullet$, we have

$$H_\delta^\circ(w)\leq F_\delta^\circ(w)\leq F_\delta^\bullet(z)\leq H_\delta^\bullet(z)=A_\delta^\bullet(z)+B_\delta^\bullet(z). \tag{3.2}$$

It follows easily from Remark 2.13 that, as $n\to\infty$, we have that $F_{\delta_n}\to F$, uniformly on the compact subsets of $\Omega\setminus\{a\}$ (since $f_{\delta_n}\to f$).

Let us now check that F satisfies the conditions of Lemma 3.4. $H^{\circ}_{\delta_n}$ and $H^{\bullet}_{\delta_n}$ are uniformly close to each other on $\partial\Upsilon\setminus\partial\Omega$ (they are equal to $F^{\circ}_{\delta_n}$ and $F^{\bullet}_{\delta_n}$ there, and these functions are uniformly close to each other near $\partial\Upsilon\setminus\partial\Omega$, as follows easily from the convergence of f_{δ_n}). To control B_{δ_n}, we use the following lemma, which is proven at the end of the section.

Lemma 3.7. *As $n\to\infty$, $B^{\bullet}_{\delta_n}\to 0$ uniformly on the compact subsets of Υ.*

Observe that $F_{\delta_n}(\partial\Omega)$ is uniformly bounded. Suppose indeed that it would not be the case and (by extracting a subsequence) that $F_{\delta_n}(\partial\Omega)\to\infty$ (say). We would have $H^{\circ}_{\delta_n}\to\infty$, since $H^{\circ}_{\delta_n}$ is harmonic and bounded on $\partial\Upsilon\setminus\partial\Omega$ (as it is equal to $F^{\circ}_{\delta_n}$ there). We also would have $A^{\bullet}_{\delta_n}\to\infty$, for the same reasons. By equation (3.2) and Lemma 3.7, it would imply that F_{δ_n} would blow up on Υ, which would contradict the fact that it converges uniformly to F on the compact subsets of Υ.

We deduce that $H^{\circ}_{\delta_n}$ and $A^{\bullet}_{\delta_n}$ are uniformly bounded on $\overline{\Upsilon}$.

We have that $H^{\circ}_{\delta_n}\to F$ and $A^{\bullet}_{\delta_n}\to F$ as $n\to\infty$, uniformly on the compact subsets of Υ. From the discrete Beurling estimate (see [Kes]) and the uniform boundedness of $H^{\circ}_{\delta_n}$ and $A^{\bullet}_{\delta_n}$ near $\partial\Omega$, we readily obtain

$$\limsup_{n\to\infty}|A^{\bullet}_{\delta_n}(z)-F_{\delta_n}(\partial\Omega)|\to 0 \quad \text{as } z\to\partial\Omega,$$
$$\limsup_{n\to\infty}|H^{\circ}_{\delta_n}(z)-F_{\delta_n}(\partial\Omega)|\to 0 \quad \text{as } z\to\partial\Omega,$$

and we deduce that F continuously extends to $\partial\Omega$ and is constant there.

To show that $\partial_{\nu_{\mathrm{out}}(z)}F\leq 0$ for all $z\in\partial\Omega$, we consider the harmonic conjugate C: it is the unique function C (defined on the universal cover of $\Omega\setminus\{a\}$ and normalized to be 0 at an arbitrary interior point x) such that $F+iC$ is holormophic. By the Cauchy–Riemann equations, we have

$$\partial_{\nu_{\mathrm{out}}(z)}F=\partial_{\tau_{\mathrm{ccw}}(z)}C \quad \text{for all } z\in\partial\Omega,$$

where $\partial_{\tau_{\mathrm{ccw}}(z)}$ is the tangential derivative on $\partial\Omega$ in the counterclockwise direction and the condition $\partial_{\nu_{\mathrm{out}}(z)}F\leq 0$ becomes $\partial_{\tau_{\mathrm{ccw}}(z)}C\leq 0$. This latter condition is equivalent to the one that C is non-increasing when going counterclockwise along (the universal cover of) $\partial\Omega$.

Let us now check that this condition is satisfied. Take Υ as before and denote its universal cover by $\widetilde{\Upsilon}$.

For each $\delta>0$, let $C^{\circ}_{\delta}:\overline{\mathcal{V}}_{\tilde{\Upsilon}^{\circ}_{\delta}}\to\mathbb{R}$ be the discrete harmonic conjugate of $H^{\bullet}_{\delta}$ (lifted to $\mathcal{V}_{\tilde{\Upsilon}_{\delta}}$), defined by integrating the discrete Cauchy–Riemann equations

$$\bar{\partial}_{\delta}(H^{\bullet}_{\delta}+iC^{\circ}_{\delta})(z)=0 \quad \text{for all } z\in\mathcal{V}_{\tilde{\Upsilon}^{\diamond}_{\delta}},$$

and with the normalization $C_\delta^\circ(x)=0$. By subharmonicity of $F_\delta^\bullet$, we have $F_\delta^\bullet \leq H_\delta^\bullet$ on $\mathcal{V}_{\Upsilon_\delta^{\mathsf{M}}}$ and hence, since $F_\delta^\bullet = H_\delta^\bullet$ on $\partial \mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$,

$$\partial_{\nu_{\mathrm{out}}(z)} H_\delta^\bullet \leq \partial_{\nu_{\mathrm{out}}(z)} F_\delta^\bullet \leq 0 \quad \text{for all } z \in \partial \mathcal{V}_{\Omega_\delta^{\mathsf{M}}},$$

and we deduce by the discrete Cauchy–Riemann equations that C_δ° is non-increasing when going along the universal cover of $\partial \mathcal{V}_{\Omega_\delta^\bullet}$ in counterclockwise direction.

On the compact subsets of $\widetilde{\Upsilon}$, since the (normalized) discrete derivatives of $H_{\delta_n}^\bullet$ converge uniformly (see [CS1, Remark 3.2]) to the derivatives of F, it is easy to check that $C_{\delta_n}^\circ$ also converges uniformly to C. Since C_δ° is non-increasing (when going along the universal cover of $\partial\Omega$), we have that $C_{\delta_n}^\circ$ is locally uniformly bounded (uniformly with respect to n) on the universal cover of $\partial\Omega$ (if it would blow up there as $n \to \infty$, it would also blow up on $\widetilde{\Upsilon}$), and hence it is bounded everywhere on the closure of $\widetilde{\Upsilon}$.

From this, we deduce that C is non-increasing on the (counterclockwise-oriented) universal cover of $\partial\Omega$: if it would not be the case, using again the discrete Beurling estimate [Kes], we would obtain a contradiction (in the $n \to \infty$ limit) to the fact that $C_{\delta_n}^\circ$ is non-decreasing. □

Proof of Lemma 3.7. For $z \in \partial \mathcal{V}_{\Upsilon_\delta}$, let us write $P_\delta(z, \cdot) : \overline{\mathcal{V}}_{\Upsilon_\delta} \to \mathsf{R}$ for the discrete harmonic function such that $P_\delta(z, \cdot) = \mathbf{1}_{\{z\}}(\cdot)$ on $\partial \mathcal{V}_{\Upsilon_\delta}$ (this is the discrete harmonic measure of $\{z\}$). By uniqueness of the solution to the discrete Dirichlet problem, we can write

$$B_\delta(y) = \sum_{z \in \partial \mathcal{V}_{\Omega_\delta}} B_\delta(z) P(z, y) \quad \text{for all } y \in \mathcal{V}_{\Upsilon_\delta}.$$

As $\delta \to 0$, we have that $P_\delta(x, \cdot) \to 0$ on the compact subsets of Υ, uniformly with respect to x (this follows directly from [CS1, Proposition 2.11]). By the construction of F_δ and the boundary conditions (Propositions 2.10 and 2.15), we have

$$B_\delta(z) = F_\delta(z) - F_\delta(\partial\Omega) = \sqrt{2}\cos\left(\tfrac{3}{8}\pi\right) |f_\delta(m)|^2 \delta$$

for any $z \in \partial \mathcal{V}_{\Omega_\delta}$, where $m \in \partial_0 \mathcal{V}_{\Omega_\delta^{\mathsf{M}}}$ is the midpoint of the edge between z and its neighbor in $\mathcal{V}_{\Omega_\delta}$. Since

$$\sum_{m \in \partial_0 \mathcal{V}_{\Omega_{\delta_n}^{\mathsf{M}}}} |f_{\delta_n}(m)|^2 \delta_n$$

is uniformly bounded by Lemma 3.5, we readily deduce that $B_{\delta_n} \to 0$ uniformly on the compact subsets of Υ. □

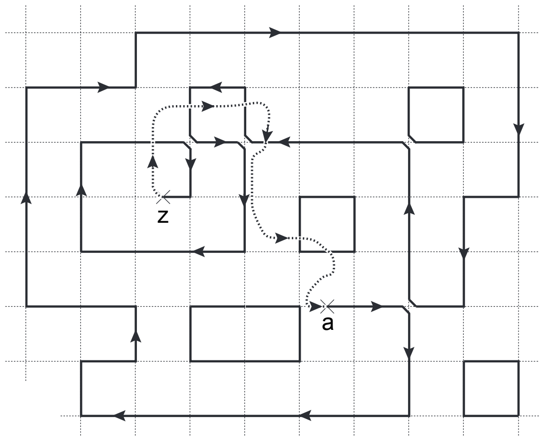

Figure A.1. The oriented loop $\mathcal{L}$ formed by adding the curve μ (dotted) to λ. In this case, $N_1=6$ and $N_2=4$.

Appendix A. Proof of Lemma 1.4

We here give the proof of Lemma 1.4: for a configuration $\gamma \in \mathcal{C}_{\Omega_\delta}(a,z)$, the winding number (modulo 4π) of an admissible walk on γ (see Figure 1.4) is independent of the choice of that walk.

Proof of Lemma 1.4. Without loss of generality, half-edges of γ emanate from z and a in the same direction, so the winding is a multiple of 2π.

Add a curve μ from z to a, which emanates in opposite direction from γ and run slightly off the lattice, so that μ is transversal to γ when an intersection occurs (see Figure A.1). Let N_1 be the number of intersections of μ with γ.

Take any admissible walk λ along γ. The rest of γ can be split into disjoint cycles. So, if N_2 is the number of intersections of μ with λ, then $N_2 \equiv N_1 \pmod 2$. Indeed, their difference comes from cycles, which are disjoint from λ and so intersect μ an even number of times (see Figure A.1).

The concatenation of λ and μ (when oriented) forms a loop $\mathcal{L}$, which has several intersections (when λ and μ run transversally). At each of those, change the connection so that there is no intersection, but instead two turns—one left and one right. Each of N_2 rearrangements either adds or removes one loop, so after the procedure $\mathcal{L}$ splits into N_3 simple loops with $N_3 \equiv N_2 \pmod 2$ (see Figure A.2).

Each of the N_3 simple loops has winding number 2π or -2π, so $\mathbf{W}(\mathcal{L}) \equiv 2\pi N_3 \equiv 2\pi N_1 \pmod{4\pi}$. We conclude that, modulo 4π, $\mathbf{W}(\lambda) = \mathbf{W}(\mathcal{L}) - \mathbf{W}(\mu) = N_1 - \mathbf{W}(\mu)$ and so $\mathbf{W}(\lambda)$ (modulo 4π) is independent of its particular choice. $\square$

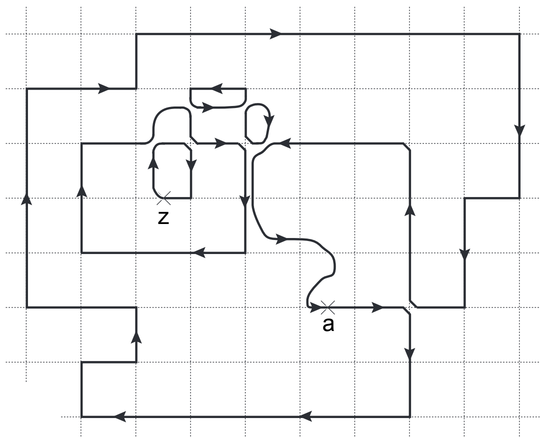

Figure A.2. The five simple loops obtained from $\mathcal{L}$ after four rearrangments (and discarding the loops that were not part of λ).

Appendix B. The full-plane fermionic correlator

We here prove some technical results concerning the discrete full-plane correlator, introduced in §1.5.3. Let us denote by $C_\delta(\cdot,\cdot):=C_0(2\cdot/\delta, 2\cdot/\delta)$ the rescaled version of Kenyon's coupling function (defined in [Ken]).

LEMMA B.1. *With the notation and assumptions of Proposition 2.6, the functions*

$$\begin{aligned} G_1\colon \mathcal{V}_{\mathbb{C}^M_\delta}\setminus\{a\} &\longrightarrow \mathbb{C},\\ z &\longmapsto e^{\pi i/8}\big(C_\delta(a+\tfrac{1}{2}\delta, z)+C_\delta\big(a-\tfrac{1}{2}i\delta, z\big)\big),\\ G_2\colon \mathcal{V}_{\mathbb{C}^M_\delta}\setminus\{a\} &\longrightarrow \mathbb{C},\\ z &\longmapsto e^{5\pi i/8}\big(C_\delta\big(a-\tfrac{1}{2}\delta, z\big)+C_\delta\big(a+\tfrac{1}{2}i\delta, z\big)\big) \end{aligned}$$

are s-holomorphic.

Proof. Set $\eta:=e^{\pi i/8}$ and for any vertex z and any $\mu\in\{\pm1,\pm i\}$, set $z_\mu:=z+\frac{1}{2}\mu\delta$. By translation invariance, we have

$$G_1(z)=\eta(C_\delta(a,z_{-1})+C_\delta(a,z_i)) \quad\text{and}\quad G_2(z)=i\eta(C_\delta(a,z_1)+C_\delta(a,z_{-i})),$$

where, on the right-hand sides, the two values of $C_\delta(a,\cdot)$ are orthogonal: one is purely real and the other purely imaginary. Let $x,y\in\mathcal{V}_{\mathbb{C}^M_\delta\setminus\{a\}}$ be two adjacent medial vertices, with x being the midpoint of a horizontal edge of $\mathcal{E}_{\Omega_\delta}$ and y the midpoint of a vertical one, and let $e:=\langle x,y\rangle\in\mathcal{E}_{\mathbb{C}_\delta^{\mathsf{M}}}$. Then, there are four possibilities for the line $\ell:=\ell(e)$:

- If $\ell=\eta\mathbb{R}$, we have $x=y+\frac{1}{2}(1+i)\delta$ and

$$\mathtt{P}_\ell[G_1(x)]=\eta C_\delta(a,x_{-1})=\eta C_\delta(a,y_i)=\mathtt{P}_\ell[G_1(y)],$$
$$\mathtt{P}_\ell[G_2(x)]=i\eta C_\delta(a,x_{-i})=i\eta C_\delta(a,y_1)=\mathtt{P}_\ell[G_2(y)].$$

- If $\ell=\bar\eta^3\mathbb{R}$, we have $x=y-\frac{1}{2}(1+i)\delta$ and

$$\mathtt{P}_\ell[G_1(x)]=\eta C_\delta(a,x_i)=\eta C_\delta(a,y_{-1})=\mathtt{P}_\ell[G_1(y)],$$
$$\mathtt{P}_\ell[G_2(x)]=i\eta C_\delta(a,x_1)=i\eta C_\delta(a,y_{-i})=\mathtt{P}_\ell[G_2(y)].$$

- If $\ell=\bar\eta\mathbb{R}$, we have $x=y+\frac{1}{2}(1-i)\delta$ and

$$\begin{aligned}\mathtt{P}_\ell[G_1(x)-G_1(y)]&=\frac{\bar\eta}{\sqrt2}(C_\delta(a_1,x_{-1})+iC_\delta(a_1,x_i)-iC_\delta(a,y_{-1})-C_\delta(a,y_i))\\&=\frac{i\bar\eta}{\sqrt2}(\bar\partial_\delta C_\delta(a,\cdot))(y)=0,\end{aligned}$$

and similarly

$$\mathtt{P}_\ell[G_2(x)-G_2(y)]=\frac{\bar\eta}{\sqrt2}(-C_\delta(a,x_1)+iC_\delta(a,x_{-i})+C_\delta(a,y_{-i})-iC_\delta(a,y_1)).$$

- If $\ell=\eta^3\mathbb{R}$, we have $x=y+\frac{1}{2}(i-1)\delta$ and

$$\begin{aligned}\mathtt{P}_\ell[G_1(x)-G_1(y)]&=\frac{1}{\sqrt2}(C_\delta(a,x_{-1})-iC_\delta(a,x_i)+iC_\delta(a,y_{-1})-C_\delta(a,y_i))\\&=-\frac{\eta^3}{\sqrt2}(\bar\partial_\delta C_\delta(a,\cdot))(x)=0,\end{aligned}$$

and similarly

$$\begin{aligned}\mathtt{P}_\ell[G_2(x)-G_2(y)]&=\frac{\eta^3}{\sqrt2}(-C_\delta(a,x_1)-iC_\delta(a,x_{-i})+C_\delta(a,y_{-i})+iC_\delta(a,y_1))\\&=\frac{i\eta^3}{\sqrt2}(\bar\partial_\delta C_\delta(a,\cdot))(y)=0.\end{aligned}$$

This concludes the proof of the lemma. □

We now turn to the singularity of $f_{\mathbb{C}_\delta}$ (Proposition 2.8).

PROPOSITION B.2. *Near the midpoint of a horizontal edge* $a\in\mathcal{V}_{\mathbb{C}_\delta}$, *for* $x\in\{\pm1,\pm i\}$, *set* $a_x:=a+\frac{1}{2}x\delta\in\mathcal{V}_{\mathbb{C}_\delta^{\mathsf{M}}}$ *and by* $e_x:=\langle a,a_x\rangle\in\mathcal{E}_{\mathbb{C}_\delta^{\mathsf{M}}}$. *Then the function* $f_{\mathbb{C}_\delta}(a,\cdot)$ *satisfies the relations*

$$\mathtt{P}_{\ell(e_{1+i})}[f_{\mathbb{C}_\delta}(a,a)]=\mathtt{P}_{\ell(e_{1+i})}[f_{\mathbb{C}_\delta}(a,a_{1+i})],$$
$$\mathtt{P}_{\ell(e_{1-i})}[f_{\mathbb{C}_\delta}(a,a)]=\mathtt{P}_{\ell(e_{1-i})}[f_{\mathbb{C}_\delta}(a,a_{1-i})],$$
$$\mathtt{P}_{\ell(e_{-1+i})}[f_{\mathbb{C}_\delta}(a,a)-1]=\mathtt{P}_{\ell(e_{-1+i})}[f_{\mathbb{C}_\delta}(a,a_{-1+i})],$$
$$\mathtt{P}_{\ell(e_{-1-i})}[f_{\mathbb{C}_\delta}(a,a)-1]=\mathtt{P}_{\ell(e_{-1-i})}[f_{\mathbb{C}_\delta}(a,a_{-1-i})].$$

Proof. Set $c:=\cos\left(\frac{1}{8}\pi\right)$ and $s:=\sin\left(\frac{1}{8}\pi\right)$ and $\eta:=e^{i\pi/8}$. The following exact values of the coupling function C_0 can be found in [Ken] (see Figure 6 there):

$$\begin{aligned}
C_0(0,1) &= -C_0(0,-1) = \frac{1}{4},\\
C_0(0,i) &= -C_0(0,-i) = -\frac{i}{4},\\
C_0(0,2+i) &= C_0(0,-2+i) = -i\left(\frac{1}{\pi}-\frac{1}{4}\right),\\
C_0(0,1+2i) &= C_0(1-2i) = \frac{1}{\pi}-\frac{1}{4},\\
C_0(0,2-i) &= C_0(0,-2-2i) = i\left(\frac{1}{\pi}-\frac{1}{4}\right),\\
C_0(0,-1-2i) &= C_0(0,-1+2i) = \frac{1}{4}-\frac{1}{\pi}.
\end{aligned}$$

Using these values and the definition of $f_{\mathbb{C}_\delta}$, a straightforward computation gives

$$\begin{aligned}
f_{\mathbb{C}_\delta}(a,a_{1+i}) &= \frac{\eta}{2}\left(c\left(\frac{2}{\pi}-\frac{1+i}{2}\right)-is\left(-\frac{2i}{\pi}+\frac{1+i}{2}\right)\right),\\
f_{\mathbb{C}_\delta}(a,a_{1-i}) &= \frac{\eta}{2}\left(c\left(\frac{1+i}{2}\right)-is\left(\frac{2i+2}{\pi}-\frac{1+i}{2}\right)\right),\\
f_{\mathbb{C}_\delta}(a,a_{-1+i}) &= \frac{\eta}{2}\left(c\left(-\frac{2+2i}{\pi}+\frac{1+i}{2}\right)+is\left(\frac{1+i}{2}\right)\right),\\
f_{\mathbb{C}_\delta}(a,a_{-1-i}) &= \frac{\eta}{2}\left(c\left(\frac{2i}{\pi}-\frac{1+i}{2}\right)+is\left(\frac{2}{\pi}-\frac{1+i}{2}\right)\right).
\end{aligned}$$

If we compute the projections of these values on the lines associated with the medial edges e_x, a straightforward computation gives

$$\begin{aligned}
\mathtt{P}_{\bar{\eta}^3\mathbb{R}}[f_{\mathbb{C}_\delta}(a,a_{1+i})] &= \frac{\bar{\eta}^3c}{2\sqrt{2}} = \mathtt{P}_{\bar{\eta}^3\mathbb{R}}\left[\frac{2+\sqrt{2}}{4}\right],\\
\mathtt{P}_{\eta^3\mathbb{R}}[f_{\mathbb{C}_\delta}(a,a_{1-i})] &= \frac{\eta^3c}{2\sqrt{2}} = \mathtt{P}_{\eta^3\mathbb{R}}\left[\frac{2+\sqrt{2}}{4}\right],\\
\mathtt{P}_{\bar{\eta}\mathbb{R}}[f_{\mathbb{C}_\delta}(a,a_{-1+i})] &= -\frac{\bar{\eta}s}{2\sqrt{2}} = \mathtt{P}_{\bar{\eta}\mathbb{R}}\left[\frac{2+\sqrt{2}}{4}-1\right],\\
\mathtt{P}_{\eta\mathbb{R}}[f_{\mathbb{C}_\delta}(a,a_{-1-i})] &= -\frac{\eta s}{2\sqrt{2}} = \mathtt{P}_{\eta\mathbb{R}}\left[\frac{2+\sqrt{2}}{4}-1\right],
\end{aligned}$$

which is the desired result. □

We now recall the result of Kenyon concerning the convergence of the function C_0.

THEOREM B.3. ([Ken, Theorem 11]) *As* $|z| \to \infty$, *we have*

$$C_0(0,z) = \begin{cases} \operatorname{Re}\left(\dfrac{1}{\pi z}\right) + O\left(\dfrac{1}{|z|^2}\right), & \textit{if } z = 2m+(2n+1)i, \textit{ with } m,n \in \mathbb{Z}, \\ i \operatorname{Im}\left(\dfrac{1}{\pi z}\right) + O\left(\dfrac{1}{|z|^2}\right), & \textit{if } z = (2m+1)+2ni, \textit{ with } m,n \in \mathbb{Z}. \end{cases}$$

From this, we can prove Theorem 3.1.

Proof of Theorem 3.1. By rescaling the lattice of the theorem above, one readily deduces that

$$C_0\left(\frac{2}{\delta}\left(a+\frac{\delta}{2}\right), \frac{2}{\delta}z\right) + C_0\left(\frac{2}{\delta}\left(a-\frac{i\delta}{2}\right), \frac{2}{\delta}z\right) \to \frac{1}{2\pi(z-a)} \quad \text{as } \delta \to 0,$$

uniformly on the sets $\{(a,z) : |a-z| \geq \varepsilon\}$. The proof of the theorem then follows from the definition of $f_{\mathbb{C}_\delta}$. □

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