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Hardy spaces and boundary conditions from the Ising model

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Abstract Functions in Hardy spaces on multiply-connected domains in the plane are given an explicit characterization in terms of a boundary condition inspired by the two-dimensional Ising model. The key underlying property is the positivity of a certain operator constructed inductively on the number of components of the boundary.

1 Introduction

A remarkable property of critical phenomena in two dimensions is their local conformal invariance. This has resulted in a rich interaction between statistical physics and many branches of mathematics, including probability, complex analysis, Riemann surfaces, and infinite-dimensional Lie algebras (see, e.g. [1–5, 14, 15, 17] and references therein).

The goal of the present paper is to show how ideas from two-dimensional statistical physics can help answer an important question in complex analysis, namely how to characterize explicitly the boundary values of holomorphic functions on a smooth multiply-connected domain Ω. See [8] for an elementary introduction to analytic function spaces on planar domains.

The simply-connected case is well known. In this case, Ω can be assumed to be the unit disk $\mathbb{D}$ by the Riemann mapping theorem. The space of holomorphic functions admitting L^2 boundary values is the Hardy space $H^2(\mathbb{D})$, and their boundary values can be characterized by the condition that all their Fourier coefficients of negative index vanish (see, e.g. [18]). The projection operator from $L^2(\partial\mathbb{D})$ to the space of L^2 boundary values of holomorphic functions is given in terms of the Hilbert transform, which is the primary example of a singular

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integral operator. Underlying this is the basic fact that, for any real-valued L^2 function f on $\partial\mathbb{D}$, there is a single-valued holomorphic function F on $\mathbb{D}$ with f as its real part on $\partial\mathbb{D}$.

In general, there is no such function F if Ω is multiply-connected (see [6], for instance). However, we shall show that the following boundary value problem always admits a unique solution, and provides a natural remedy to this difficulty:

$$F \quad \text{holomorphic} \quad \text{on} \quad \Omega, \qquad \Im\mathfrak{m}((F-f)\nu^{\frac{1}{2}}) = 0 \quad \text{on} \quad \partial\Omega \tag{1.1}$$

Here ν is the outer normal to the boundary, viewed as a complex number. This boundary value problem is suggested by recent studies of the Ising model [9,11,16]. An n-connected domain Ω in the plane admits 2^{n-1} different spin structures. Since Ω admits a global frame, for one of the spin structures, which we refer to as the trivial one, the spinors can be identified with scalar single-valued functions. We shall show that the boundary value problem (1.1) can always be solved for the trivial spin structure. This will give a simple explicit characterization of L^2 boundary values of holomorphic functions. In the process, we shall also find an analogue of the Hilbert transform for multiply-connected domains.

In this paper, we provide a functional-analytic proof of the existence and uniqueness of the solution to the problem 1.1.

Our result can in principle be applied to the study of the Schramm's-SLE curves SLE(3) and SLE(16/3) infinitely-connected geometries, for which solutions of the boundary value problem (1.1) are crucial [12].

2 Statement of the main results

Let Ω be a bounded domain in $\mathbb{R}$ with smooth boundary $\partial\Omega$. Let N_{in} and N_{out} be the unit inward and outward pointing normals to the boundary $\partial\Omega$. They can be identified with complex numbers ν_{in} and ν_{out} by writing,

$$N_{\text{in}} = 2\Re\mathfrak{e}\left(\nu_{\text{in}}\frac{\partial}{\partial z}\right), \qquad N_{\text{out}} = 2\Re\mathfrak{e}\left(\nu_{\text{out}}\frac{\partial}{\partial z}\right), \tag{2.1}$$

with $\nu_{\text{in}} = -\nu_{\text{out}}, |\nu_{\text{in}}| = |\nu_{\text{out}}| = 1$.

Let $L^2(\partial\Omega)$ denote the space of L^2 complex-valued functions defined on $\partial\Omega$. Even though this is clearly a (complex) Hilbert space, we will most of the time view it as a **real** Hilbert space. Define the (real-linear) projection operators $\mathsf{P}_{\text{in}} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ and $\mathsf{P}_{\text{out}} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$\mathsf{P}_{\text{in}}[f](z) := \frac{1}{2}(f(z) + \overline{\nu_{\text{in}}}(z)\overline{f}(z)),$$
$$\mathsf{P}_{\text{out}}[f](z) := \frac{1}{2}(f(z) + \overline{\nu_{\text{out}}}(z)\overline{f}(z)),$$

Observe that the orthogonal projection $\text{Proj}_{e^{i\theta}\mathbb{R}}$ of $\mathbb{C}$ on the real line $e^{i\theta}\mathbb{R}$ in $\mathbb{C}$ can be expressed as $\text{Proj}_{e^{i\theta}\mathbb{R}}(\zeta) = \frac{1}{2}(\zeta + e^{2i\theta}\overline{\zeta}), \quad \forall\zeta \in \mathbb{C}, \forall\theta \in \mathbb{R}$. Thus $\mathsf{P}_{\text{in}}[f](z)$ and $\mathsf{P}_{\text{out}}[f](z)$ are just the projections of the complex number $f(z)$ on the two perpendicular lines in $\mathbf{C}$ defined by $\nu_{\text{in}}^{-\frac{1}{2}}(z)\mathbb{R}$ and $\nu_{\text{out}}^{-\frac{1}{2}}(z)\mathbb{R}$. As such, they are just twisted versions of the projections of complex numbers onto their real and imaginary parts. They provide a simple way of formulating boundary conditions of the form (1.1), e.g.,

$$\Im\mathfrak{m}(f(z)\nu_{\text{in}}^{\frac{1}{2}}(z)) = 0 \quad \Leftrightarrow \quad Pin[f](z) = 0, \qquad z \in \partial\Omega. \tag{2.2}$$

Clearly $\mathsf{P}_{\text{in}}^2 = \mathsf{P}_{\text{in}}$, $\mathsf{P}_{\text{out}}^2 = \mathsf{P}_{\text{out}}$, and $\mathsf{P}_{\text{in}} + \mathsf{P}_{\text{out}} = \mathbf{Id}$. We can now define the real Hilbert subspaces $L^2_{\text{in}}(\partial\Omega)$ and $L^2_{\text{out}}(\partial\Omega)$ by

$$L^2_{\text{in}}(\partial\Omega) = \text{Ker}(\mathsf{P}_{\text{out}}) = \text{Range}(\mathsf{P}_{\text{in}}), \tag{2.3}$$

$$L^2_{\text{out}}(\partial\Omega) = \text{Ker}(\mathsf{P}_{\text{in}}) = \text{Range}(\mathsf{P}_{\text{out}}), \tag{2.4}$$

and we have the direct-sum decomposition,

$$L^2(\partial\Omega) = L^2_{\text{in}}(\partial\Omega) \oplus L^2_{\text{out}}(\partial\Omega). \tag{2.5}$$

Let $H^2(\Omega)$ be the Hardy space of holomorphic functions on Ω. It can be defined in several ways, and one way is as the Banach space of holomorphic functions $F(z)$ on Ω satisfying

$$\sup_{\delta>0} \int\limits_{\partial\Omega_\delta} |F(z)|^2 d\sigma(z) < \infty, \tag{2.6}$$

where Ω_δ is the subset of Ω consisting of points at a distance $> \delta$ from $\partial\Omega$. For δ sufficiently small, the orthogonal projection of $\partial\Omega_\delta$ on $\partial\Omega$ is a diffeomorphism, and functions on $\partial\Omega_\delta$ can be identified with functions on $\partial\Omega$. What is important for our purposes is the fact that for each function $F \in H^2(\Omega)$, the restrictions of F to $\partial\Omega_\delta$, viewed in this way as functions on $\partial\Omega$, converge in $L^2(\partial\Omega)$ and pointwise a.e. to a function $R_{\partial\Omega}F$. The "restriction operator" R_Ω is a bounded, injective operator from $H^2(\Omega)$ to $L^2(\Omega)$.

Then our main result can be formulated as follows:

Theorem 1 *Let Ω be a bounded domain with smooth boundary. Then for any function $f \in L^2(\partial\Omega)$, there exists a unique function $F \in H^2(\Omega)$ satisfying the boundary condition*

$$\mathsf{P}_{\text{in}}(R_\Omega(F) - f) = 0. \tag{2.7}$$

This theorem is essentially equivalent to another theorem, which is actually the one that we shall prove first. Let the operators $T_\Omega : H^2(\Omega) \to L^2_{\text{in}}(\partial\Omega)$ and $U_\Omega : H^2(\Omega) \to L^2_{\text{out}}(\partial\Omega)$ be defined by

$$T_\Omega := \mathsf{P}_{\text{in}} \circ R_{\partial\Omega}, \quad U_\Omega := \mathsf{P}_{\text{out}} \circ R_{\partial\Omega}. \tag{2.8}$$

It is useful to depict this graphically as

$$\begin{array}{ccccc} & & H^2\,(\Omega) & & \\ & \swarrow_{T_\Omega} & \downarrow R_{\partial\Omega} & \searrow_{U_\Omega} & \\ L^2_{\text{in}}\,(\partial\Omega) & \leftarrow & L^2\,(\partial\Omega) & \rightarrow & L^2_{\text{out}}\,(\partial\Omega) \end{array}$$

Theorem 2 *Let Ω be a bounded domain with smooth boundary. Then*

(a) *the operators $T_\Omega : H^2(\Omega) \to L^2_{\text{in}}(\partial\Omega)$ and $U_\Omega : H^2(\Omega) \to L^2_{\text{out}}(\partial\Omega)$ are real-linear isomorphisms.*

(b) *Define the operator $W_\Omega : L^2_{\text{in}}(\partial\Omega) \to L^2_{\text{out}}(\partial\Omega)$ by*

$$W_\Omega = U_\Omega \circ T_\Omega^{-1}. \tag{2.9}$$

Then W is a one-to-one and onto operator satisfying

$$W_\Omega \circ (\mathbf{j}) \circ W_\Omega = -\text{Id}$$

$$(-\mathbf{j}) \circ W_\Omega \circ (\mathbf{j}) = T_\Omega \circ U_\Omega^{-1},$$

where $\mathbf{j}$ *denotes* $i \cdot \text{Id}$ (*multiplication by* i *operator*).

To see how Theorem 1 follows from Theorem 2, it suffices to observe that, if T_Ω^{-1} exists, then for any function $f \in L^2(\partial\Omega)$, the function $F = T_\Omega^{-1}(\mathsf{P}_{\rm in}(f))$ satisfies the desired property.

Part (b) of Theorem 2 can be easily seen by tracing back the definitions of the operators T_Ω and U_Ω, once Part (a) has been proved. Thus we shall henceforth concentrate on the proof of Part (a) of Theorem 2.

The spaces $L^2_{\rm in}(\partial\Omega)$ and $L^2_{\rm out}(\partial\Omega)$ can be viewed as the analogues, for multiply-connected domains, of the spaces of L^2 functions with only non-vanishing Fourier coefficients of respectively positive and negative indices in the case of the unit disk. The operator W_Ω is a variant of the Hilbert transform, for the twisted line bundle $\nu_{\rm out}^{-\frac{1}{2}}(z)\mathbb{R}$ on the boundary of Ω.

As stressed in the introduction, the analogue of the preceding theorem would fail if the projection operators $\mathsf{P}_{\rm in}$ and $\mathsf{P}_{\rm out}$ were replaced by the projections on the trivial line bundles $\mathbb{R}\times\partial\Omega$ and $i\mathbb{R}\times\partial\Omega$ on the boundary of Ω. In this case, there would have been an obstruction of a non-trivial finite-dimensional subspace. That this difficulty can be eliminated by considering boundary conditions of the form (1.1) is a key insight provided by recent advances in the study of the critical Ising model. There is a discrete variant of the above theorems which can be established explicitly for discrete fermions (see [10], Section 13).

3 Proof of Theorem 2

We begin by proving the injectivity of the operators T_Ω and U_Ω.

Let $F \in H^2(\Omega)$, and assume that $T_\Omega(F) = 0$. If we set $f = R_\Omega(F)$, this means that $f = \rho(z)\nu_{\rm in}^{-\frac{1}{2}}(z)$ for some real-valued scalar function $\rho(z)$ on the boundary. We claim that $\rho(z) = 0$ identically, and hence $F = 0$ identically in Ω. Indeed, the holomorphicity of F implies

$$\oint_{\partial\Omega_\delta} F^2(z)dz = 0 \tag{3.1}$$

for all $\delta > 0$ sufficiently small. But the convergence of the restrictions of $F(z)$ to $\partial\Omega_\delta$ to f, viewed as L^2 functions on $\partial\Omega$ as explained in the previous section, implies in turn that

$$\oint_{\partial\Omega} f^2(z)dz = 0. \tag{3.2}$$

On the other hand, a key motivation for the boundary condition (1.1) is the following identity,

$$\mathfrak{Re}(f^2(z)dz) = \rho^2(z)\mathfrak{Re}\left(\frac{dz}{\nu_{\rm in}(z)}\right) = \rho^2(z)\,ds \tag{3.3}$$

where ds is the element of arc-length along $\partial\Omega$. This can be seen by picking a local defining function $r(z)$ for the boundary $\partial\Omega$. Then $\nu(z) = \frac{1}{|\nabla r|}(\partial_x r + i\partial_y r)$, and

$$\frac{dz}{\nu} = ds + i(\partial_x r\,dy - \partial_y r\,dx), \tag{3.4}$$

which implies (3.3). It follows that $\rho(z) = 0$ identically, as was to be shown. The argument for the injectivity of U_Ω is exactly the same.

The proof of Theorem 2 reduces then to the proof of the surjectivity of the operators T_Ω and U_Ω. This will be done by induction on the number n of components of the boundary $\partial\Omega$ of the domain Ω. The precise statements that we shall prove are the following. Let

$$\partial\Omega = \partial_1\Omega \cup \cdots \cup \partial_n\Omega, \tag{3.5}$$

where the $\partial_j\Omega$'s are the connected components of $\partial\Omega$.

For each $j \in \{1, \ldots, n\}$, we denote by $H^2(\Omega, \partial_j\Omega)$ the subspace of $H^2(\Omega)$ defined by

$$H^2(\Omega, \partial_j\Omega) := \{f \in H^2(\Omega) : T_\Omega(f)|_{\partial\Omega\setminus\partial_j\Omega} = 0\}.$$

Let $T_\Omega^{\partial_j\Omega} : H^2(\Omega) \to L^2_{\text{in}}(\partial\Omega_j)$ be the projection onto $L^2_{\text{in}}(\partial\Omega_j)$ of T_Ω,

$$f \mapsto T_\Omega^{\partial_j\Omega}(f) = (T_\Omega(f))|_{\partial\Omega_j}. \tag{3.6}$$

The restriction of $T_\Omega^{\partial_j\Omega}$ to $H^2\left(\Omega, \partial_j\Omega\right)$ will be denoted by T_Ω^j.

Then Part (a) of Theorem 2 is an immediate consequence of the following two lemmas, the first being the case $n = 1$, and the second the induction step from n to $n + 1$:

Lemma 3 (Simply-connected case) *Let Ω be a simply-connected domain. Then the mappings $T_\Omega : H^2(\Omega) \to L^2_{\text{in}}(\partial\Omega)$ and $U_\Omega : H^2(\Omega) \to L^2_{\text{out}}(\partial\Omega)$ are isomorphisms.*

Lemma 4 (Induction step) *Let $n \geq 1$, and assume that for any n-connected smooth domain Ω, the mapping $T_\Omega : H^2(\Omega) \to L^2_{\text{in}}(\partial\Omega)$ is an isomorphism. Let Λ be any $(n + 1)$-connected smooth domain. Then the operator $T_\Lambda : H^2(\Lambda) \to L^2_{\text{in}}(\partial\Lambda)$ is an isomorphism.*

We give now the proofs of Lemmas 3 and 4. An essential ingredient is the following conformal invariance property. Its proof is straightforward, since a conformal equivalence between two smooth domains extends to a diffeomorphism of the boundaries:

Lemma 5 (Conformal equivalence) *Let Λ and Ξ be two conformally equivalent smooth domains and let $\psi : \Xi \to \Lambda$ be a conformal map. Let $\partial_j\Xi$ be a connected component of $\partial\Xi$ and $\partial_j\Lambda := \psi(\partial_j\Xi)$. Then the following diagram commutes*

$$\begin{array}{ccc} H^2(\Lambda, \partial_j\Lambda) & \xrightarrow{T_\Lambda^{\partial_j\Lambda}} & L^2_{\text{in}}(\partial\Lambda_j) \\ \downarrow \Psi & & \downarrow \Psi|_{L^2_{\text{in}}(\partial\Xi j)} \\ H^2(\Xi, \partial_j\Xi) & \xrightarrow{T_\Xi^{\partial_j\Xi}} & L^2_{\text{in}}(\partial_j\Xi), \end{array}$$

where the isomorphism $\Psi : H^2(\Lambda) \to H^2(\Xi)$ is defined by

$$(z \mapsto f(z)) \mapsto (w \mapsto f(\psi_j(w))\sqrt{\psi_j'(w)}).$$

We observe that the square root $\sqrt{\psi_j'(w)}$ is well-defined (up to a global harmless sign) even when Ω is multiply-connected (see [9], Chapter 4). Thus the trivial spin structure is mapped into the trivial spin structure under global conformal transformations.

Proof of Lemma 3 If Ω is simply-connected, then there exists a conformal equivalence between Ω and the unit disk $\mathbb{D}$ in $\mathbb{C}$, which extends to a diffeomorphism between the boundary $\partial\Omega$ and the unit circle $\mathbb{S}$. By Lemma 5, it suffices to prove the desired statement when $\Omega = \mathbb{D}$ and $\partial\Omega = \mathbb{S}$.

Let $\psi : \mathbb{D} \to \Omega$ be a conformal mapping. By Lemma 5 (defining the operator Ψ as in that lemma), we have the following commuting diagram

$$\begin{array}{ccc} H^2(\Omega, \partial\Omega) & \xrightarrow{T_\Omega^{\partial\Omega}} & L^2_{\mathrm{in}}(\partial\Omega) \\ \downarrow \Psi & & \downarrow \Psi|_{L^2_{\mathrm{in}}(\partial\Omega)} \\ H^2(\mathbb{D}, \mathbb{S}) & \xrightarrow{T_\mathbb{D}^{\mathbb{S}}} & L^2_{\mathrm{in}}(\partial\mathbb{S}), \end{array}$$

We should now solve the problem on the unit disk: we should show that $T_\mathbb{D}^\mathbb{S} = T_\mathbb{D}$ is invertible.

So, let us construct the inverse $S_\mathbb{D}$. Let f be a function in $L^2_{\mathrm{in}}(\mathbb{S})$. Let $(c_k)_{k\in\mathbb{Z}}$ be the Fourier coefficients of f, so that the Fourier series of f reads $\sum_{k\in\mathbb{Z}} c_k e_k$, where $e_k(\theta) := e^{ik\theta}$. By definition of $L^2_{\mathrm{in}}(\mathbb{S})$, we have that $\mathsf{P}_{\mathrm{out}}[f] = \frac{1}{2}(f + e_{-1}\bar{f}) = 0$. It is hence easy to see that we have $c_k + \overline{c_{-k-1}} = 0$ for all $k \in \mathbb{Z}$. We define $S_\mathbb{D}(f) := g$, where $g \in H^2(\mathbb{D})$ is defined by

$$g(z) := 2\sum_{k=0}^{\infty} c_k z^k.$$

This clearly defines a bounded operator. Let us check that $T_\mathbb{D}(g) = f$. The Fourier series of $T_\mathbb{D}(g)$ reads

$$\sum_{k=0}^{\infty} c_k e_k - \sum_{k=-\infty}^{-1} \overline{c_{-k-1}}\, e_k.$$

The nonnegative Fourier coefficients $T_\mathbb{D}(g)$ are clearly the same as the ones of f, and using $c_k + \overline{c_{-k-1}} = 0$ for all $k \in \mathbb{Z}$, we get that $T_\mathbb{D}(g) = f$. Using exactly the same arguments, it is easy to check that $S_\mathbb{D} \circ T_\mathbb{D}$ is the identity.

Remark It is also possible to construct $S_\mathbb{D}$ by writing it explicitly as a convolution kernel. This is in spirit closer to Ising model techniques: the convolution kernel corresponds then to a fermionic correlator.

We turn next to the proof of Lemma 4. We need two simple observations. The first is a superposition principle, which allows one to reduce the inversion of the operator T_Ω to the inversion of operators $T_\Omega^{\partial_j\Omega}$ associated to the components $\partial_1\Omega, \ldots, \partial_n\Omega$ of $\partial\Omega$.

Lemma 6 (Superposition) *Let Ξ be an $(n+1)$-connected domain with $\partial\Xi = \partial_1\Xi \cup \cdots \cup \partial_{n+1}\Xi$. Suppose that for each $j \in \{1, \ldots, n+1\}$, the restriction $T_\Xi^j : H^2(\Xi, \partial_j\Xi) \to L^2_{\mathrm{in}}(\partial_j\Xi)$ of the operator $T_\Xi^{\partial_j\Xi}$ (originally defined on $H^2(\Xi)$) is an isomorphism. For each $j \in \{1, \ldots, n+1\}$, denote by $S_\Xi^{\partial\Xi_j} : L^2_{\mathrm{in}}(\partial_j\Xi) \to H^2(\Xi)$ the inverse of $T_\Xi^{\partial_j}$, injected into $H^2(\Xi)$ (the range of $(T_\Xi^{\partial_j})^{-1}$ is contained in $H^2(\Xi, \partial_j\Xi)$). Then we have*

$$\begin{aligned} T_\Xi^{\partial_j\Xi} \circ S_\Xi^{\partial_j\Xi} &= \mathrm{Id} \quad \forall j \\ T_\Xi^{\partial_j\Xi} \circ S_\Xi^{\partial_k\Xi} &= 0 \quad \forall j \neq k, \end{aligned}$$

and T_Ξ is invertible, with inverse $S_\Xi := S_\Xi^{\partial_1\Xi} \oplus \cdots \oplus S_\Xi^{\partial_{j+1}\Xi}$ in the decomposition $L^2(\partial\Xi) = L^2(\partial_1\Xi) \oplus \cdots \oplus L^2(\partial_{n+1}\Xi)$.

The proof of this lemma is again straightforward.

The second observation is the following version of the Riemann mapping theorem for multi-connected domains:

Lemma 7 (Riemann mapping theorem) *Let Ξ be an $n+1$-connected domain, with $n \geq 1$. Then for any component $\partial_j \Xi$ of the boundary $\partial \Xi$, there exists a conformal equivalence between Ξ and $\Omega \backslash \overline{\mathbb{D}}$, where Ω is an n-connected domain containing the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{C}$, and $\partial_j \Xi$ is mapped onto $\partial \mathbb{D}$.*

Proof of Lemma 7 Assume first that $\partial_j \Xi$ is an inner component of the boundary of Σ. Let D_1 be the connected component of $\mathbb{C} \backslash \Sigma$ enclosed by $\partial_j \Xi$. Let I_p be the inversion map $z \to (z-p)^{-1}$, for any point $p \in \mathbb{C}$. Choose a point $p_1 \in D_1$ and a point $p_0 \in \mathbb{C} \backslash \overline{\Omega}$. If we apply I_{p_1}, then the domain Σ will be mapped to a domain $I_{p_1}(\Sigma)$ lying within $\mathbb{C} \backslash \overline{I_{p_1}(D_1)}$. By the Riemann mapping theorem, there exists a conformal equivalence Ψ between the simply-connected domain $\mathbb{C} \backslash \overline{I_{p_1}(D_1)}$ and the unit disk $\mathbb{D}$, which extends to a diffeomorphism between $I_{p_1}(\partial_1 \Sigma)$ and the unit circle $\mathbb{S}$. Then Σ is conformally equivalent to the domain $I_{I_{p_0}} \Psi I_{p_1}(\Sigma)$, one of whose boundary components is the circle $I_{I_{p_0}} \Psi I_{p_1}(\partial_1 \Sigma) = I_{I_{p_0}}(\mathbb{S})$. Translating and dilating so that this last circle is the unit circle, we obtain the desired domain Ω. When $\partial_j \Xi$ is the outer component of $\partial \Xi$, we can apply an inversion I_p with respect to a point outside $\overline{\Xi}$ to transform Ξ into another domain with $\partial_j \Xi$ transformed into an inner component of the boundary. This reduces the problem to the case already treated, and the proof of Lemma 7 is complete.

The point of the two observations Lemmas 6 and 7 is that, in conjunction with the conformal equivalence property, it suffices to prove that each individual operator $T_\Omega^{\partial_j \Omega}$ for each fixed j, $1 \leq j \leq n$, is invertible, when $\partial_j \Omega$ is an inner boundary and a unit circle. This is the content of the next lemma, which is the hardest part of our argument, and which will be proved in the next section:

Lemma 8 (Key Lemma) *Let $n \geq 2$. Let Ω be an n-connected domain containing the unit disk $\mathbb{D}$. Let Ξ be the $(n+1)$-connected domain defined by $\Omega \backslash \overline{\mathbb{D}}$. Assume that $T_\Omega : H^2(\Omega) \to L^2_{\text{in}}(\partial \Omega)$ is an isomorphism. Then $T_\Xi^{\mathbb{S}} : H^2(\Xi, \mathbb{S}) \to L^2_{\text{in}}(\mathbb{S})$ is an isomorphism.*

4 Proof of Lemma 8

Recall that $\Xi = \Omega \backslash \overline{\mathbb{D}}$. Our goal is to construct an inverse to the operator $T_\Xi^{\mathbb{S}}$, using the operator $(T_\Omega)^{-1}$ and function theory on $\mathbb{S}$.

4.1 Function theory on $\mathbb{S}$

It is convenient to identify $L^2(\mathbb{S})$ with $\ell^2(\mathbb{Z})$ by the Fourier transform, $\mathcal{F} : L^2(\mathbb{S}) \to \ell^2(\mathbb{Z})$, defined by

$$(f \in L^2(\mathbb{S})) \mapsto \left(c_k(f) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x) \, \mathrm{d}x \right)$$

We denote by $\mathcal{F}^{-1}$ the inverse of $\mathcal{F}$.

Within $\ell^2(\mathbb{Z})$, let us introduce the following real-linear subspaces,

$$\ell^2_-(\mathbb{Z}) := \{(c_k)_{k\in\mathbb{Z}} : c_k = 0 \ \forall k \geq 0\},$$
$$\ell^2_+(\mathbb{Z}) := \{(c_k)_{k\in\mathbb{Z}} : c_k = 0 \ \forall k < 0\},$$
$$\ell^2_{\text{in}}(\mathbb{Z}) := \{(c_k)_{k\in\mathbb{Z}} : c_k - \overline{c_{-1-k}} = 0 \ \forall k\},$$
$$\ell^2_{\text{out}}(\mathbb{Z}) := \{(c_k)_{k\in\mathbb{Z}} : c_k + \overline{c_{-1-k}} = 0 \ \forall k\}.$$

We denote by $\mathcal{P}_\pm$ the orthogonal projection on $\ell^2_\pm(\mathbb{Z})$ and by $\mathcal{P}_{\text{in}} : \ell^2(\mathbb{Z}) \to \ell^2_{\text{in}}(\mathbb{Z})$ the orthogonal projection on ℓ^2_{in}. In coordinates:

$$\mathcal{P}_+ : (c_k)_{k\in\mathbb{Z}} \mapsto (\mathbf{1}_{\{k\geq 0\}} c_k)_{k\in\mathbb{Z}}$$
$$\mathcal{P}_- : (c_k)_{k\in\mathbb{Z}} \mapsto (\mathbf{1}_{\{k<0\}} c_k)_{k\in\mathbb{Z}}$$
$$\mathcal{P}_{\text{in}} : (c_k)_{k\in\mathbb{Z}} \mapsto \frac{1}{2}(c_k + \overline{c_{-k-1}})$$
$$\mathcal{P}_{\text{out}} : (c_k)_{k\in\mathbb{Z}} \mapsto \frac{1}{2}(c_k - \overline{c_{-k-1}})$$

Clearly, we have

$$\mathcal{P}_- \circ 2\mathcal{P}_{\text{in}} = \text{Id}_{\ell^2_-(\mathbb{Z})}, \qquad 2\mathcal{P}_{\text{in}} \circ \mathcal{P}_- = \text{Id}_{\ell^2_{\text{in}}(\mathbb{Z})} \tag{4.1}$$

as well as the commuting diagram,

$$\begin{array}{ccc} L^2(\mathbb{S}) & \xrightarrow[\mathsf{P}_{\text{in}}]{} & L^2_{\text{in}}(\mathbb{S}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \ell^2(\mathbb{Z}) & \xrightarrow[\mathcal{P}_{\text{in}}]{} & \ell^2_{\text{in}}(\mathbb{Z}) \end{array}$$

where

$$L^2_{\text{in}}(\mathbb{S}) : = \{f \in L^2(\mathbb{S}) : \Im\mathfrak{m}(f(e^{i\theta})e^{i\theta/2}) = 0 \quad \text{for almost every } \theta\}.$$

(this choice of notation is made as in our case the inner normal on $\mathbb{S}$ is actually pointing *towards the exterior of the unit disk* $\mathbb{D}$, as $\mathbb{D}$ is in the *complement* of our domain Ξ).

We also need the operator $\mathcal{J} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$\mathcal{J} : (c_k)_{k\in\mathbb{Z}} \mapsto (\overline{c_{-1-k}})_{k\in\mathbb{Z}}. \tag{4.2}$$

which exchanges $\ell^2_+(\mathbb{Z})$ and $\ell^2_-(\mathbb{Z})$, i.e., $\mathcal{J}(\ell^2_\pm(\mathbb{Z})) = \ell^2_\mp(\mathbb{Z})$. Finally, we set

$$\mathcal{F}_+ := \mathcal{P}_+ \circ \mathcal{F} \qquad \mathcal{F}_- := \mathcal{P}_- \circ \mathcal{F}.$$

4.2 The operator $\Phi : \ell^2_-(\mathbb{Z}) \to H^2(\mathbb{Z}, \mathbb{S})$

We come now to the main building block of the proof, which is the operator $\Phi : \ell^2_-(\mathbb{Z}) \to H^2(\mathbb{Z}, \mathbb{S})$ constructed from the operator S_Ω (which exists by induction hypothesis) and Fourier series on $\ell^2(\mathbb{Z})$ as follows.

For each negative integer $k \in \mathbb{Z}_-$, set

$$\varphi_k^{\Re\mathfrak{e}}(z) := z^k - S_\Omega(\mathsf{P}_{\text{in}}[\zeta^k]), \qquad \varphi_k^{\Im}(z) := iz^k - S_\Omega(\mathsf{P}_{\text{in}}[i\zeta^k])(z), \tag{4.3}$$

where $S_\Omega : L^2_{\text{in}}(\partial\Omega) \to H^2(\Omega)$ is the inverse of T_Ω (which exists by assumption). Note that, while ζ^k has a pole at 0, $\mathsf{P}_{\text{in}}[\zeta^k]$ is a well-defined L^2 function on $\partial\Omega$, and hence $S_\Omega(\mathsf{P}_{\text{in}}[\zeta^k])$

is well-defined as a holomorphic function on Ω. Thus the functions $\varphi_k^{\Re}$ and $\varphi_k^{\Im}$ are the unique holomorphic functions on $\Omega\setminus\{0\}$ such that

$$\begin{aligned} &\varphi_k^{\Re}(z) - z^k \text{ and } \varphi_k^{\Im}(z) - iz^k \quad \text{are holomorphic in } \Omega \\ &\mathsf{P}_{\mathrm{in}}[\varphi_k^{\Re}(z)] = \mathsf{P}_{\mathrm{in}}[\varphi_k^{\Im}(z)] = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Lemma 9 *Define the real-linear operator* $\Phi : \ell^2_-(\mathbb{Z}) \to H^2(\Xi, \mathbb{S})$ *by*

$$\Phi : (c_k)_{k\in\mathbb{Z}} \mapsto \left(\sum_{k=-\infty}^{-1} \mathfrak{Re}(c_k)\varphi_k^{\Re} + \mathfrak{Im}(c_k)\varphi_k^{\Im} \right). \tag{4.4}$$

(a) *Then the operator* Φ *is well-defined and bounded as a bounded operator from* $\ell^2_-(\mathbb{Z})$ *to* $H^2(\Xi, \mathbb{S})$.
(b) *The operator* Φ *is invertible, and* $\Phi^{-1} : H^2(\Xi, \mathbb{S}) \to \ell^2_-(\mathbb{S})$ *is equal to* $\mathcal{F}R^{\mathbb{S}}_{\partial\Xi}$.

Proof of Lemma 9 To prove Part (a), we have to show that the series defining Φ converges and is bounded in $H^2(\Xi, \mathbb{S})$ for $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Since the boundary $\partial\Omega$ of Ω lies entirely within the region $\{|\zeta| > \rho\}$ for some fixed $\rho > 1$, the functions ζ^k decay exponentially fast for k negative,

$$\|\zeta^k\|_{C^0(\partial\Omega)} \leq \rho^{-k}. \tag{4.5}$$

By the assumption of Lemma 8, the operator S_Ω is bounded from $L^2(\partial\Omega)$ to $H^2(\Omega)$. Thus we have

$$\begin{aligned} \|(\varphi_k^{\Re} - z^k)_{k<0}\|_{H^2(\Omega)} &\leq C\rho^{-k} \\ \|(\varphi_k^{\Im} - iz^k)_{k<0}\|_{H^2(\Omega)} &\leq C\rho^{-k} \end{aligned}$$

for a constant C independent of k. It follows that the series

$$\sum_{k=-\infty}^{-1} \mathfrak{Re}(c_k)(\varphi_k^{\Re} - z^k) + \mathfrak{Im}(c_k)(\varphi_k^{\Im} - iz^k) \tag{4.6}$$

converges in $H^2(\Omega)$ and defines a function in $H^2(\Omega) \subset H^2(\Xi)$. Furthermore, by the Cauchy–Schwarz inequality, its $H^2(\Omega)$ norm is bounded by

$$C\sum_{k=-\infty}^{-1} |c_k|\rho^{-k} \leq C\left(\sum_{k=-\infty}^{-1} |c_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=-\infty}^{-1} \rho^{-2k}\right)^{\frac{1}{2}} \leq C'\left(\sum_{k=-\infty}^{-1} |c_k|^2\right)^{\frac{1}{2}}. \tag{4.7}$$

Next, we consider the series $\mathfrak{Re}\,(c_k)\,z^k + i\mathfrak{Im}\,(c_k)\,z^k$. It converges uniformly on compact subsets and defines a holomorphic function on Ξ. As we have seen, on the components of $\partial\Xi$ which are also in $\partial\Omega$, it converges exponentially fast. On the component $\mathbb{S}$ of $\partial\Xi$, it is an L^2 function, since the functions z^k form an orthonormal system in $L^2(\mathbb{S})$,

$$\left\| \sum_{k=-\infty}^{-1} \mathfrak{Re}\,(c_k)\,z^k + i\mathfrak{Im}\,(c_k)\,z^k \right\|_{L^2(\mathbb{S})} = \sum_{k=-\infty}^{-1} |c_k|^2 < \infty.$$

It follows that the $H^2(\Xi)$ norm of $\mathfrak{Re}(c_k)z^k + i\mathfrak{Im}(c_k)z^k$ is also bounded by $C(\sum_{k=-\infty}^{-1} |c_k|^2)^{\frac{1}{2}}$. Altogether, the expression

$$\Phi((c_k)_{k\in\mathbb{Z}}) = \sum_{k=-\infty}^{-1} \mathfrak{Re}(c_k)(\varphi_k^{\Re} - z^k) + \mathfrak{Im}(c_k)(\varphi_k^{\Im} - iz^k) + \mathfrak{Re}(c_k)z^k + i\mathfrak{Im}(c_k)z^k,$$

defines a function in $H^2(\Xi)$, whose $H^2(\Xi)$ norm is bounded by $C(\sum_{k=-\infty}^{-1}|c_k|^2)^{\frac{1}{2}}$. This shows that Φ is a well-defined and bounded operator, proving (a).

Next, we prove (b). Checking that $\mathcal{F}_- \circ \Phi = \mathrm{Id}_{\ell^2_-(\mathbb{Z})}$ is easy. To get that $\Phi \circ \mathcal{F}_- = \mathrm{Id}_{H^2(\Omega,\mathbb{S})}$, it just suffices to check that $\mathcal{F}_-$ is injective. Suppose that $f \in H^2(\Xi, \mathbb{S})$ is such that $\mathcal{F}_-(f)$ is zero. Since it does not have any negative Fourier coefficients, f can be extended to a holomorphic function on Ω. But $T_\Omega(f) = 0$, and hence by the injectivity of the operator T_Ω established earlier in Sect. 3, it follows that $f = 0$. □

4.3 The operators $\mathcal{O}, \mathcal{Q} : \ell^2_-(\mathbb{Z}) \to \ell^2_-(\mathbb{Z})$

Let $\mathcal{O} : \ell^2_-(\mathbb{Z}) \to \ell^2_-(\mathbb{Z})$ be the bounded operator defined by the composition

$$\mathcal{O} : \ell^2_-(\mathbb{Z}) \underset{\Phi}{\longrightarrow} H^2(\Xi, \mathbb{S}) \underset{T_\Xi^{\mathbb{S}}}{\longrightarrow} L^2_{\mathrm{in}}(\mathbb{S}) \underset{\mathcal{F}_-}{\longrightarrow} \ell^2_-(\mathbb{Z}),$$

Then we have the following formula:

Lemma 10 (a) *The operator $T_\Xi^{\mathbb{S}}$ is invertible with bounded inverse if and only if the operator $\mathcal{O}$ is invertible with bounded inverse.*
(b) *The operator $\mathcal{O}$ can be expressed as*

$$\mathcal{O} = \mathrm{Id}_{\ell^2_-(\mathbb{Z})} + \mathcal{Q} \tag{4.8}$$

where $\mathcal{Q}$ is the operator on $\ell^2_-(\mathbb{Z})$ defined as the composition

$$\mathcal{Q} : \ell^2_-(\mathbb{Z}) \underset{\Phi}{\longrightarrow} H^2(\Xi, \mathbb{S}) \underset{\mathcal{F}_+}{\longrightarrow} \ell^2_+(\mathbb{Z}) \underset{\mathcal{J}}{\longrightarrow} \ell^2_-(\mathbb{Z}),$$

where $\mathcal{J}$ is the operator of exchanging Fourier coefficients of positive and negative indices defined in Sect. 4.1

Proof of Lemma 10 Part (a) follows immediately from the fact that the other operators besides $T_\Xi^{\mathbb{S}}$ in the composition defining the operator $\mathcal{O}$ are invertible with bounded inverses.

To prove Part (b), let $(c_k)_{k\in\mathbb{Z}} \in \ell^2_-(\mathbb{Z})$. Set $f := \Phi((c_k)_{k\in\mathbb{Z}}) \in H^2(\Omega, \mathbb{S})$, i.e.,

$$f(z) = \sum_{k=-\infty}^{-1} \mathfrak{Re}(c_k)\varphi_k^{\mathfrak{R}}(z) + \mathfrak{Im}(c_k)\varphi_k^{\mathfrak{I}}(z).$$

Set $g := T_\Xi^{\mathbb{S}}(f) \in L^2_{\mathrm{in}}(\mathbb{S})$. Thus

$$g(e^{i\theta}) = \frac{1}{2}(f(e^{i\theta}) + e^{-i\theta}\overline{f(e^{i\theta})})$$

We compute the Fourier coefficients of g with negative indices. Set $e_k(z) := z^k$ (defined on $\mathbb{S}$) and $\langle f, g\rangle := \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta}) \mathrm{d}\theta$. We have

$$\begin{aligned}\langle e_k, g\rangle &= \langle e_k, f\rangle + \langle e_k, e_{-1}\overline{f}\rangle \\ &= \langle e_k, f\rangle + \langle e_{k+1}, \overline{f}\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle e_k, f\rangle + \overline{\langle e_{-k-1}, f\rangle} \\
&= \left\langle e_k, \sum_{j=-\infty}^{-1} \mathfrak{Re}\left(c_j\right) \varphi_j^{\mathfrak{R}} + \mathfrak{Im}\left(c_k\right) \varphi_j^{\mathfrak{I}} \right\rangle + \overline{\langle e_{-k-1}, f\rangle} \\
&= \sum_{j=-\infty}^{-1} (\mathfrak{Re}(c_j)\langle e_k, \varphi_j^{\mathfrak{R}}\rangle + \mathfrak{Im}(c_k)\langle e_k, \varphi_k^{\mathfrak{I}}\rangle) + \overline{\langle e_{-k-1}, f\rangle}, \\
&= c_k + \overline{\langle e_{-k-1}, f\rangle}
\end{aligned}
$$

where in the before last equation, we use that $\varphi_j^{\mathfrak{R}} - z^j$ and $\varphi_j^{\mathfrak{I}} - iz^j$ are regular in the unit disk (and hence have no negative Fourier coefficients). From there we get $\mathcal{F} \circ T_{\Xi}^{\mathbb{S}} \circ \Phi(c_k) = c_k + \mathcal{Q}c_k$, which is the desired result. □

We arrive now at the key property of the operator $\mathcal{Q}$, which is perhaps surprising in itself:

Lemma 11 (Positive definiteness) *The operator $\mathcal{Q} : \ell^2_-(\mathbb{Z}) \to \ell^2_-(\mathbb{Z})$ is positive semi-definite with respect to the following real inner-product $\bullet_{\mathbb{R}}$ on $\ell^2_-(\mathbb{Z})$,*

$$
(a_k)_k \bullet_{\mathbb{R}} (b_k)_k := \sum_{k\in\mathbb{Z}} \mathfrak{Re}(a_k)\mathfrak{Re}(b_k) + \mathfrak{Im}(a_k)\mathfrak{Im}(b_k).
$$

Proof of Lemma 11 Let $(c_k)_{k\in\mathbb{Z}} \in \ell^2_-(\mathbb{Z})$ and set $f := \Phi((c_k)_k) \in H^2(\Xi, \mathbb{S})$. Then, from the proof of Lemma 10, we get:

$$
\begin{aligned}
(c_k)_k \bullet_{\mathbb{R}} (\mathcal{Q}(c_k)_k) &= \sum_{k=-\infty}^{-1} \mathfrak{Re}(c_k)\mathfrak{Re}(\overline{\langle e_{-k-1}, f\rangle}) + \mathfrak{Im}(c_k)\mathfrak{Im}(\overline{\langle e_{-k-1}, f\rangle}) \\
&= \sum_{k=-\infty}^{-1} \mathfrak{Re}(\langle e_k, f\rangle)\mathfrak{Re}(\langle e_{-k-1}, f\rangle) - \mathfrak{Im}(\langle e_k, f\rangle)\mathfrak{Im}(\langle e_{-k-1}, f\rangle)
\end{aligned}
$$

On the other hand, by Fourier analysis, the counterclockwise-oriented integral of f^2 on $\mathbb{S}$ gives

$$
\begin{aligned}
\mathfrak{Re}\left(\frac{1}{2\pi i}\oint_{\mathbb{S}} f^2(z\right)\mathrm{d}z) &= \left\langle e_{-1}, \left(\sum_{k=-\infty}^{-1} \langle e_k, f\rangle e_k\right)^2 \right\rangle \\
&= \sum_{k=-\infty}^{-1} \mathfrak{Re}(\langle e_k, f\rangle\langle e_{-k-1}, f\rangle) \\
&= \sum_{k=-\infty}^{-1} \mathfrak{Re}(\langle e_k, f\rangle)\mathfrak{Re}(\langle e_{-k-1}, f\rangle) - \mathfrak{Im}(\langle e_k, f\rangle)\mathfrak{Im}(\langle e_{-k-1}, f\rangle) \\
&= (c_k)_k \bullet_{\mathbb{R}} (\mathcal{Q}(c_k)_k).
\end{aligned}
$$

Because $f \in H^2(\Xi)$, we can deform the integration contour of $f^2(z)dz$ as in Sect. 3 to get

$$
\oint_{\mathbb{S}} f^2(z)\,\mathrm{d}z = \oint_{\partial\Omega} f^2(z)\,\mathrm{d}z,
$$

where the orientation of the inner components of $\partial\Omega$ is clockwise, and the orientation of the outer component of $\partial\Omega$ is counterclockwise. But, as shown in Sect. 3, the fact that $f(z)$

satisfies the boundary condition on $\partial\Omega$ implies that $\Re(\frac{1}{i} f^2(z)\, \nu_{\mathrm{out}})(z) \geq 0$ on $\partial\Omega$. Hence

$$\frac{1}{2\pi i} \oint_{\partial\Omega} f^2(z)\, \mathrm{d}z \geq 0.$$

We deduce that $(c_k)_k \bullet_{\mathbb{R}} \left(\mathcal{Q}\,(c_k)_k\right) \geq 0$. This proves the lemma. □

4.4 End of proof of Lemma 8

Finally, we can complete the proof of Lemma 8: it suffices to consider the operator

$$\begin{aligned}\mathcal{O}\mathcal{O}^T &= (\mathrm{Id}_{\ell^2_-(\mathbb{Z})} + \mathcal{Q})(\mathrm{Id}_{\ell^2_-(\mathbb{Z})} + \mathcal{Q}^T)\\ &= \mathrm{Id}_{\ell^2_-(\mathbb{Z})} + (\mathcal{Q} + \mathcal{Q}^T) + \mathcal{Q}\mathcal{Q}^T\end{aligned}$$

which is symmetric (for the scalar product $\bullet_{\mathbb{R}}$). Its spectrum is bounded from below by 1. Thus $\mathcal{O}\mathcal{O}^T$ is invertible, and we can write

$$\mathcal{O}^{-1} = \mathcal{O}^T(\mathcal{O}\mathcal{O}^T)^{-1},$$

which is clearly a bounded operator. As noted in Lemma 10, Part (a), the invertibility of $\mathcal{O}$ is equivalent to the invertibility of the operator $T^{\mathbb{S}}_{\Xi}$. The proof of Lemma 8, and hence of Theorem 2 is complete.

5 Intrinsic formulation of the Ising boundary condition

The Ising boundary condition 1.1 can be formulated intrinsically for spinors on an arbitrary Riemann surface Ω with smooth boundary $\partial\Omega$. Recall that a spin structure δ on Ω is a holomorphic line bundle L_δ on Ω with $L_\delta^2 = K_\Omega$, where K_Ω is the canonical bundle of Ω, that is, the bundle of $(1,0)$-forms over Ω. Spinors on Ω with respect to the spin structure are then sections of L_δ. By abuse of notation, we shall often denote spinors by $f(z)(dz)^{\frac{1}{2}}$.

Fix a spin structure L_δ. Choose any metric $ds^2 = g_{\bar{z}z} dz d\bar{z}$ on Ω with $z, \bar{z}$ as isothermal coordinates, and let $N(z) = 2\Re(\nu(z)\frac{\partial}{\partial z})$ be the inward-pointing unit normal. Then we say that a section $f(z)(dz)^{\frac{1}{2}}$ of L_δ satisfies the Ising boundary condition if

$$f(z)\nu(z) g_{\bar{z}z}^{\frac{1}{2}} = \overline{f(z)}\,(\nu(z)\bar{\nu}(z) g_{\bar{z}z})^{\frac{1}{2}}. \tag{5.1}$$

It is easily seen that this condition is equivalent to the condition $\Im(f(z)\nu(z)^{\frac{1}{2}}) = 0$. However, it is clearly intrinsic: a square root $(g_{\bar{z}z})^{\frac{1}{2}}$ of a metric $g_{\bar{z}z}$ on the surface X is a well-defined metric on the spin bundle L_δ^{-1}. As such, it is a section of the bundle $L_\delta \otimes \bar{L}_\delta$. The left hand side is thus a section of $L_\delta \otimes L_\delta^{-2} \otimes (L_\delta \otimes \bar{L}_\delta) = \bar{L}_\delta$. Since $(\nu\bar{\nu} g_{\bar{z}z})$ is a scalar, the right hand side is also a section of $\bar{L}_\delta$, and the equation is intrinsic. Note that it is invariant under a Weyl scaling $g_{\bar{z}z} \to e^{2\sigma(z)} g_{\bar{z}z}$ of the metric, so it is an equation that depends only on the complex structure of X.

6 Ellipticity of the Ising boundary condition

The main goal of this section is to show that, for generic even spin structures, the Ising boundary condition defines an elliptic boundary value problem for the Cauchy–Riemann operator $\bar{\partial}$. This is essentially a consequence of classic arguments for pseudo-differential operators, and we shall be brief. For simplicity, we assume that Ω is an open subset of a compact Riemann surface X, whose boundary $\partial\Omega$ is a smooth, simple closed curve in a domain holomorphically equivalent with the coordinate chart $D = \{z \in \mathbb{C}; |z| < 2\}$. We assume also that X is equipped with a spin structure δ, and we equip Ω with the induced spin structure. For generic even spin structures, the dimension of the space of holomorphic spinors on X is 0.

We adapt the method of multiple-layer potentials. Let $S_\delta(z, w)(dz)^{\frac{1}{2}} \otimes (dw)^{\frac{1}{2}} \in L_\delta(z) \otimes L_\delta(w)$ be the Szegö kernel for L_δ, where L_δ is the spin bundle defined by δ, $L_\delta^2 = K_X$, and K_X is the line bundle of (1, 0)-forms on X. Then $S_\delta(z, w)$ has exactly one simple pole in z at w when δ is a generic even spin structure (see, e.g. [3,7]). Using the coordinate system on the chart D, we can write any section of L_δ over $\partial\Omega$ as $f(w)(dw)^{\frac{1}{2}}$. Thus we can define the operator

$$F(z) = \left(\frac{1}{\pi i}\int_{\partial\Omega} S_\delta(w, z) f(w)dw\,(dz)^{\frac{1}{2}}\right) \tag{6.1}$$

which maps sections of L_δ over $\partial\Omega$ to sections of L_δ over Ω. Clearly $F(z)$ is holomorphic on Ω.

(a) The boundary values of the spinor $F(z)$ exist and are given by

$$\lim_{z\to w_0} F(z) = \left(\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\partial\Omega_\varepsilon(w_0)} S_\delta(w, w_0) f(w)dw\,(dw_0)^{\frac{1}{2}} - f(w_0)(dw_0)^{\frac{1}{2}}\right)$$

for any $w_0 \in \partial\Omega$. Here $\partial\Omega_\varepsilon(w_0)$ is the complement in $\partial\Omega$ of the disk centered at w_0 and of radius ε with respect to the Euclidian metric. This follows by the well-known Plemelj arguments (see, e.g. [13]): the function $f(w)$ in (6.1) can be replaced by $f(w) - f(w_0)$, since the integral of $S_\delta(w, z)dw$ over the contour $\partial\Omega$ for fixed $z \in \Omega$ can be deformed to the origin of the disk D. The resulting integral over $\partial\Omega$ converges when $z \to w_0$ and can be replaced by the limit of integrals over $\partial\Omega_\varepsilon(w_0)$. The contribution of the term $f(w_0)$ can now be evaluated separately: the contour $\partial\Omega_\varepsilon(w_0)$ can be viewed as the difference between a closed contour $\partial\tilde{\Omega}_\varepsilon(w_0)$ consisting of $\partial\Omega_\varepsilon(w_0)$ completed by a small half-loop $C_\varepsilon(w_0)$ of radius ε around w_0, in the exterior of Ω, and the half-loop $C_\varepsilon(w_0)$ itself. The contribution over $\partial\tilde{\Omega}_\varepsilon(w_0)$ is again 0 by the holomorphicity of $S_\delta(w, w_0)$, while the one over $C_\varepsilon(w_0)$ can be calculated exactly in the limit $\varepsilon \to 0$, using the short-distance asymptotics of $S_\delta(z, w) = (z - w)^{-1} + O((z - w)^3)$.

(b) Let H be the operator on $\partial\Omega$ defined by

$$Hf(w_0) = \lim_{\epsilon\to 0}\frac{1}{\pi i}\int_{\partial\Omega_\varepsilon(w_0)} S_\delta(w, w_0) f(w)dw\,(dw_0)^{\frac{1}{2}}.$$

Then H is a pseudo-differential operator of order 0. Its leading term maps real functions to real functions, and its symbol $\sigma(z, \xi)$, if we parametrize $\partial\Omega$ by the arc-length with respect to the Euclidian metric, is given by

$$\sigma(z, \xi) = i(\operatorname{sgn}\xi). \tag{6.2}$$

This statement is local. Since we can work near the diagonal $z = w$ and drop smoothing errors, we can replace $S_\delta(z, w)$ by $(z - w)^{-1}$. If we parametrize $\partial\Omega$ by the arc-length from a fixed point $P \in \partial\Omega$, $s \to w(s) \in \partial\Omega$, we can express H as a one-dimensional singular integral operator with kernel

$$K(s, s_0) = \frac{t(s)}{w(s) - w(s_0)} = \frac{t(s)}{(s - s_0)(t(s_0) + (s - s_0)E(s, s_0))} \tag{6.3}$$

where $t(s) = dw/ds$, and $E(s, s_0)$ is a smooth function. The last expression can be recognized as $(s - s_0)^{-1}$ up to a smooth kernel. Thus H is, up to smoothing errors, just the classic Hilbert transform, and it is well-known that (6.2) is its symbol.

(c) The Ising boundary condition (5.1) can be interpreted as the problem of finding F with boundary values admitting a given projection along each direction $\nu^{\frac{1}{2}}$. To check the ellipticity of this boundary value problem, we can again work locally and restrict ourselves to the terms of leading order. Then the complex number $\nu(z)g_{\bar{z}z}^{\frac{1}{2}}$ has modulus 1, and can be expressed locally as $\nu(z)g_{\bar{z}z}^{\frac{1}{2}} = e^{2i\theta}$ for some real-function θ on $\partial\Omega$. The (signed) length of the projection of a complex number ζ on the line $e^{i\theta}\mathbb{R}$ is given by

$$\frac{1}{2}e^{-i\theta}(\zeta + e^{2i\theta}\bar{\zeta}) = \frac{1}{2}(e^{-i\theta}\zeta + e^{i\theta}\bar{\zeta}). \tag{6.4}$$

Thus the ellipticity of the Ising boundary condition is just the ellipticity of the operator on real functions

$$f \to Mf := \frac{1}{2}(e^{-i\theta}(iH - I) + e^{i\theta}(-iH - I))f = (\sin\theta\, H - \cos\theta\, I)f. \tag{6.5}$$

Since the principal symbol of the operator M is $i(\operatorname{sgn}\xi)\sin\theta - \cos\theta$, which has norm 1, the ellipticity of M follows at once.

(d) As a consequence, the operator M admits a parametrix which is a pseudodifferential operator of order 0. In particular, it is bounded on Schauder spaces and on Sobolev spaces. Thus, when the operator T_Ω^{-1} exists, it is bounded on Schauder and on Sobolev spaces.
(e)In general, the dimensions of the kernel of M and of its co-range are finite-dimensional.

7 Canonical metrics

The solvability of the Ising boundary condition yields a new canonical metric for smooth multi-connected domains in **C**. In the case of the trivial spin structure, this metric corresponds to the energy density one-point function of the model, with locally constant +/− boundary conditions (see [9], Chapter 7).

Let $\Omega \subset \mathbb{C}$ be a multi-connected domain with smooth boundary $\partial\Omega$. Theorem 1 implies the existence and uniqueness of the solution to the boundary value problem

$$\partial_{\bar{z}}G(z, w) = \delta(z, w) \text{ in } \Omega, \quad \Im(G(\cdot, w)\sqrt{\nu}) = 0 \tag{7.1}$$

for any given $w \in \Omega$. We set

$$\ell(w) = \lim_{z \to w}(G(z, w) - \frac{1}{z - w}). \tag{7.2}$$

It is then easy to show, by a similar argument as in the proof of the uniqueness part of Theorem 1, that $\ell(w)$ is always a strictly positive number. Thus

$$ds_\Omega^2 := \ell(w)^2 dw d\bar{w} \tag{7.3}$$

defines a metric on Ω. Lemma 5 implies that, for any conformal equivalence Φ,

$$ds^2_{\Phi(\Omega)} = \Phi_*(ds^2_\Omega). \tag{7.4}$$

In this sense, the "Ising energy metric" ds^2 is a canonical metric, which is actually different from the many other canonical metrics known in the literature. This can be verified explicitly in the case of an annulus $\Omega = \{z \in \mathbb{C}; 1 < |z| < R\}$ for some fixed $R > 1$. Then it is not difficult to verify that the Ising model metric is given by

$$\ell(w) = \sum_{n\in\mathbb{Z}} \frac{1}{1+R^{2n+1}} |w|^{2n} \tag{7.5}$$

for the even spin structure (the spinors are anti-periodic when one goes around the circles centered at 0), and

$$\ell(w) = \sum_{n\in\mathbb{Z}} \frac{1}{1+R^{2n}} |w|^{2n+1} \tag{7.6}$$

for the odd spin structure (the spinors are now periodic). On the other hand, the Bergman metric $K(z)$ is given by

$$K(z) = \frac{1}{\pi \log R^2}|z|^{-2} + \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{1}{(R^p|z|^2 - R^{-p})^2} \tag{7.7}$$

while the Robin metric is given in terms of the θ-function

$$\theta\left(\frac{1}{2\pi i}\log\frac{R}{|w|^2} + \frac{1}{2}\Big|\frac{i}{\pi}\log R\right), \tag{7.8}$$

up to a factor independent of w (in fact, a Dedekind function in R).

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