Conformal Invariance of Ising Model Correlations

Clément Hongler

Université de Genève

Thesis Advisor: Stanislav Smirnov

Contents

Remerciements/Acknowledgements

Cette thèse doit beaucoup à un grand nombre de personnes, que je souhaiterais toutes remercier ici ; que celles que j'aurai oubliées me pardonnent – et me demandent de leur offrir un verre en compensation.

Ma gratitude va tout d'abord et en particulier à Stas Smirnov, pour avoir été un directeur de thèse fantastique, par son enthousiasme, ses idées brillantes et son soutien constant. Il m'a initié aux magnifiques sujets de l'invariance conforme et de l'analyse complexe discrète, en me faisant découvrir des concepts nouveaux et des points de vue éclairants, a proposé le sujet de cette thèse et a introduit une grande partie des techniques qu'elle utilise ; il m'a aidé quand j'en ai eu besoin, m'a laissé une grande liberté le reste du temps et m'a toujours porté une confiance dont je suis très honoré ; sa manière de voir les mathématiques m'a toujours enchanté et il m'a transmis un optimisme qui m'accompagnera, je l'espère, longtemps. Il ne faut pas oublier sa bonne humeur et ses anecdotes hilarantes, qui auront fait de cette thèse une expérience encore plus mémorable !

I would like to warmly thank Dima Chelkak for his creativity, his kindness and his help during of these three years. All the discussions we had were enlightening and many of his advice could be transformed into decisive ideas. Especially at the beginning, his help was crucial. For this and all the rest, I am greatly indebted to him.

J'aimerais exprimer ma reconnaissance à Kalle Kytölä, pour sa gentillesse, toutes nos discussions passionnantes et son aide en toutes circonstances. Il m'a beaucoup appris, que ce soit en mathématiques ou en physique, a été un interlocuteur patient, critique et avisé. J'ai grand plaisir à travailler avec lui et je n'oublierai pas les nuits qu'il a passées pour boucler nos projets quand j'en ai eu besoin. Finalement, cela a été un collègue de bureau des plus agréables, et nos innombrables pauses café et dîners chinois me manqueront.

Un grand merci à mes deux chers camarades de thèse à Genève, Hugo Duminil-Copin et Kostya Izyurov, qui m'ont toujours impressionné par leur sagacité et leur vivacité d'esprit. C'est toujours un grand plaisir de travailler avec Hugo, qui est d'une efficacité redoutable, ainsi que d'une grande créativité ; c'est aussi un ami très sympathique, toujours prêt à aider, et que je dois particulièrement remercier pour ses commentaires très utiles sur les versions préliminaires de cette thèse. Kostya est un compagnon de route très apprécié et admiré depuis bientôt trois ans ; sa rapidité incroyable, son imagination et ses vastes connaissances m'auront été très précieuses.

Mes remerciements vont également à Pierre Nolin pour nos discussions intéressantes et notre collaboration fructueuse, ainsi que pour son accueil à New York.

Je tiens aussi à témoigner ma gratitude à Vincent Beffara pour tout ce qu'il m'a appris, pour ses conseils avisés, sa patience, son immense culture et tous les bons moments passés à tant de conférences.

Je suis très reconnaissant à Wendelin Werner pour nos discussions dont j'ai beaucoup bénéficié, pour ses suggestions, pour sa très gentille invitation à Paris et son aide mathématique aussi bien que pratique en de nombreuses circonstances. Enfin, pour m'avoir fait l'honneur de le compter dans mon jury de thèse.

J'ai aussi la chance de compter Yvan Velenik dans mon jury, qui m'a toujours accueilli gentiment dans son bureau pour répondre à mes questions et qui a toujours partagé avec enthousiasme sa culture.

Je tiens à remercier Philippe Di Francesco pour avoir également accepté d'être juré, ainsi que pour son fameux “livre jaune”, qui m'a beaucoup éclairé.

I am very honored to count Nicolai Reshetikhin as a member of my thesis committee.

I also would like to thank Scott Sheffield for his hospitality in New York and Boston and all the discussions we had together. I also thank Lincoln Chayes, Geoffrey Grimmett and John Palmer for interesting conversations and suggestions.

Merci à tous les membres de notre groupe à Genève, en particulier Dima Beliaev, Jürgen Angst, Thierry Lévy et Yacine Ikhlef, pour m'avoir beaucoup appris sur de nombreuses questions intéressantes, pour tous les bons moments passés ensemble et pour leur humour inimitable.

A Lausanne, j'ai eu la chance de découvrir la mécanique statistique avec Reda Messikh et Jiří Černý, ainsi qu'avec Gérard Ben Arous et Tom Mountford, qui ont largement contribué à mes premiers pas dans ce sujet. Il ne faut pas oublier Antoine Derighetti, avec qui j'ai eu le plaisir depuis de longues années de parler d'analyse et d'autres choses, et qui m'a toujours soutenu et encouragé.

Outre les sus-mentionnés, j'aimerais remercier tous les gens qui m'ont invité à donner des séminaires ou pour de courts séjours : Béatrice de Tilière, à Neuchâtel, pour son accueil et les discussions éclairantes que j'ai pu avoir avec elle et Cédric Boutillier ; Christophe Garban, pour son invitation à Lyon, ainsi que pour nos nombreuses conversations au fil des conférences ; Antonio Costa pour sa très gentille invitation à Madrid; Michel Bauer et Denis Bernard pour un séjour très instructif et sympathique à Saclay; Julien Dubédat pour son accueil à New York.

Besides the above-mentioned people, I would like to thank all those who have invited and warmly welcomed me for short stays or seminars: Igor Krichever and Duong Hong Phong at Columbia, where I will be next year; Chuck Newman and Ron Peled at Courant; Curt McMullen, Horng-Tzer Yau and Shing-Tung Yau at Harvard.

I would like also to thank many nice young people I met at conferences and with whom I had the pleasure to do some math or other things, and from whom I learned a lot: Alan, Anne, Antti, Erik, Fredrik, Grégory, Hubert, Istvan, Ivan, Laure, Laurent, Leo, Marie, Marco, Mohammad, Oscar, Romain, Sébastien, Stéphane, Tom and Yen.

Je tiens aussi à mentionner des amis avec lesquels j'ai eu le plaisir discuter ou de travailler sur de questions mathématiques ou scientifiques et qui m'ont apporté beaucoup ; à Lausanne, en particulier Olivier, mais aussi Anna, Julian, Mathieu et Peter, ainsi qu'Adrien, Bastien, Burak, David, Diego, Julien, Justine, Maroussia, Philippe, Samuel, Shahin, Sylvain et Yannick ; à Paris, Cyrille, Gilles, Ivan, Nicolas et Olivier.

A la Section de Mathématiques de Genève, je souhaite remercier tout le monde, et en particulier Cyrille, David, Fabrice, Félix, Loren, Luc, Martin, Michel, Mucio, Nicola, Sylvain et Yves, pour avoir été des collègues très sympathiques et toujours prêts à donner un coup de main; également, Annick et Najat, pour leur aide. Sans oublier mes chers étudiants ! Je suis aussi très reconnaissant à la Section de m'avoir laissé une grande liberté et permis d'assister à de nombreuses conférences.

Il faut aussi mentionner tous les amis qui ne figurent pas ci-dessus et qui d'une certaine manière ont aussi contribué indirectement à cette thèse, en particulier Christelle, Gaëlle, Jérémie, Jeanne, Joana, Kathinka et Quynh Thu.

Finalement, je remercie ma famille, en particulier ma soeur Isadora et mes parents Vân et Max.

CHAPTER 1

Introduction

1.1. History

The Ising model was introduced by Wilhelm Lenz in 1920 as a model for ferromagnetism. The Ising model is a random assignment of ± 1 spins to the vertices V_G of a graph G. The probability of a spin configuration $(\sigma_x)_{x \in V_G}$ is proportional to $e^{-\beta H(\sigma)}$, where $\beta > 0$ is called the inverse temperature and the energy H is given by $H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$, the sum being over the pairs of adjacent vertices of G. The model favors local alignment of spins, and the strength of this effect is modulated by the parameter β, which in statistical mechanics is set to $\frac{1}{k_B T}$, where k_B is Boltzmann's constant and T is the temperature. Hence, the higher the temperature, the smaller the ordering effect and the higher the disorder.

In his doctoral thesis, his student Ernst Ising determined the absence of phase transition in the model in one dimension (i.e. when the graph is $\mathbb{Z}$), and showed that the model is disordered at any temperature. He generalized incorrectly this result to higher dimensions, thus concluding that the model was not suited to model ferromagnetism. This conclusion, which was believed to be correct by the physical community for many years, led physicists to introduce and consider alternative models, for instance the Heisenberg model. In 1936, Peierls showed however that the model undergoes a phase transition in any dimension greated than one (i.e. on $\mathbb{Z}^d$ for $d \geq 2$), using estimates on the length of interfaces between the spins, thus showing, said informally, that at low enough temperature, Ising ferromagnets can exhibit spontaneous magnetization.

In 1941, Kramers and Wannier identified the critical inverse temperature of the two-dimensional Ising model as $\beta_c = \frac{1}{2} \ln \left(\sqrt{2} + 1 \right)$, thanks to the remarkable observation of a duality between subcritical and supercritical temperatures that is named after them. In 1944, Onsager computed explicitly the free energy of the model, using the celebrated transfer matrix technique, thus allowing for a rigorous and precise analysis of all the thermodynamical properties of the model. After this, the Ising model became one of the fundamental examples for an order-disorder transition.

Onsager's analysis was later simplified and made more conceptual by Kaufman, who exhibited deep relations between the understanding of the correlation functions of the model and the spin representations. A number of alternative methods were developed to study the model, notably combinatorial approaches in what is referred to as the "Pfaffian approach", allowing for many fine results, notably certain scaling correlation functions to be computed.

Despite the immense progress induced by Onsager's calculation, the critical regime at the inverse temperature β_c and its neighborhood, of special physical interest, since they are the key to understand the apparition of magnetization, remained somewhat mysterious. The introduction of the renormalization group at the end of the 1960's allowed for a unified and more conceptual understanding of the critical and near-critical regimes of the statistical physics model, in particular the ones of the Ising model, from a physical point of view. The idea is to look at the *scaling limit* of the model, which consists informally in looking at the model "from very far away", or equivalently in considering the model on a large graph of very small mesh size. Although non-rigorous, block-spin renormalization ideas gave convincing evidence of why such at scaling limit should exist at criticality.

Renormalization group suggests the idea of universality: the scaling limit of a model at criticality should in some sense be independent of many of its local details, such as the lattice on which it is defined. A large number of similar models should moreover belong to the same universality class. This led in particular to the hypothesis of scaling, translation and rotational invariance of the model, and gave additional evidence for the relevance of the study of the Ising model to understand more complicated models conjectured to belong to its universality class. By adding the assumption of invariance of the model under inversions, Polyakov was able to predict more accurate information, the plausibility thereof led to suspect Möbius invariance of the model.

In the same time, an operator algebra formalism was introduced for the two-dimensional model, notably by Kadanoff and Ceva, that suggested the existence of quantum field theory underlying the model and describing its scaling limit. Notably, the idea that a finite number of generating operators could represent all the correlation functions of the Ising model and other models was developed, which would be later generalized in the theory of so-called minimal models.

In the 1980's, the existence of a much stronger symmetry for the scaling limit of two-dimensional critical systems was suggested by Belavin-Polyakov-Zamolodchikov, which led to the development of Conformal Field Theory: these scaling limits should be *conformally invariant*, that is, invariant under conformal mapping between arbitrary domains. Informally, if $\Sigma, \tilde{\Sigma}$ are Riemann surfaces and $\phi : \Sigma \to \tilde{\Sigma}$ is a conformal mapping, then the Ising model on a very fine discretization of $\tilde{\Sigma}$ should be the image by ϕ of the Ising model on a very fine discretization of Σ. Combined with the operator formalism ideas, this lead to the postulate that the scaling limits could be described by quantum field theories with an infinitely-dimensional Lie algebra of symmetry, called Conformal Field Theories (CFT). These theories form a one-parameter family, indexed by a real parameter c, called the central charge. The universality classes of the conformally invariant scaling limits of many statistical models correspond to CFT with specific rational central charges, called the minimal models.

Using the techniques of CFT, the nature of the critical regime of many models, for instance the Ising model, could be understood with an unprecedently level of resolution. In particular exact formulae for the scaling limits of correlations of the local fields of the models could be derived, revealing deep and spectacular connections between the models and the conformal geometry of the Riemann surface on which they live. Concerning the Ising model, the correlation functions of the two local fields of the model, the spin σ, which measures the repartition of the magnetization on the surface, and the energy density ε, which measures the repartition of the energy H across the surface, could be computed on various geometries with various boundary conditions. In particular, on simply connected bounded domains, the celebrated mirror-image technique of Cardy [Car84] allowed for the computations of the spin and the energy in a conceptual and elegant way. However almost all these predictions are very far from being mathematical results, the very existence of a scaling limit of the model remaining unproven, the conformal symmetry of it being even more conjectural and the proof of existence of a CFT describing the model staying an open problem.

About the same years, another two-dimensional theory, called quantum holonomic fields theory, was developed by Sato, Miwa and Jimbo, that encompasses the scaling limit of the Ising model and allowed to represent its massive or near-critical correlation functions in terms of solutions to so-called holonomic differential equations, enabling notably the exact computation of a number of these functions. This remarkable theory has the advantage of being more rigorous, many of its aspects having been developed in a completely rigorous way, in particular by Palmer and Tracy. Unfortunately, it seems that the extent to which a mathematical use of this theory can be applied to the Ising model is limited to specific geometries, like the full plane, the cylinder or the torus, since the key tool to pass to the scaling limit, the transfer matrix method, does not behave well on general geometries.

Conformal invariance of the Ising model or of other models remained out of mathematical reach until a revival of the subject with the introduction of Schramm-Loewner Evolution (SLE), in the late 1990's. The point of view is different, since it focuses on the random curves appearing in two-dimensional critical systems, and mathematically precise; the SLE processes form a one-parameter family, indexed by a positive parameter κ, which can be put in correspondence with the central charge c of the CFT. Several links between SLE and discrete systems were recently shown, for percolation, the loop-erased random walk, the uniform spanning tree, the discrete Gaussian free field, and the Ising model, as well as connections with continuous process, like the planar Brownian motion and the continuous Gaussian free field. In the Ising model, the natural candidates for curves are the interfaces between $+$ and $-$ spin clusters, and they were recently shown to converge to SLE with $\kappa = 3$, by Smirnov [Smi06] (on the square lattice) and Chelkak-Smirnov (on more general lattices) in a so-called Dobrushin setup, by Kytölä and the author in more general setups [HoKy10] (on the square lattice).

While SLE shed a new light on conformal invariance as well as mathematical rigor and allowed for many exact computations, a number of predictions of CFT seem beyond the reach of SLE techniques, or that at least one has to use additional techniques in conjuction with them. This seems to be the case for percolation, and also for the Ising model: there does not seem be a straighforward way to compute the spin and energy correlation functions directly with SLE techniques. Moreover, SLE theory is almost only developed for simply connected geometries and specific boundary conditions.

In this text, we prove several predictions of Conformal Field Theory for correlation functions of the critical Ising model in bounded geometries (which can be found in [BuGu93], generalizing the results of [Car84]), generalizing the results of [HoSm10]. These predictions concern the following two local fields:

- The bulk energy density field □, which is basically the product of two adjacent spins in the bulk, and hence measures the repartition of the energy of the model across the surface.

- The boundary spin field σ, which gives the value of a spin on the boundary of the surface.

For the energy density, the boundary conditions that are treated are:

- Free: we do not impose anything on the boundary
- Locally monochromatic with boundary changing operator: we condition the boundary spin to be locally always $+$ or always $-$ and to change at given locations.
- In the simply connected case, mixed $+$ and $-$ boundary conditions.

These are the boundary conditions for which we have found predictions in the CFT literature. For the spin, the only boundary condition treated in this text is free. However, with the additional help of SLE techniques, one can compute boundary spin correlations with more general boundary conditions [HoKy10]. The results of this text are the following

- On multiply connected domains, we show the existence of the scaling limits of the correlation functions of the fields above and show their conformal covariance.
- On simply connected domains, we obtain exact formulae for these correlation functions.

Our techniques rely on the introduction of so-called fermionic observables, which can be viewed as complexified deformations of partition functions, and generalize the fermions introduced in [Smi06], [ChSm09], [HoSm10] (see also [Smi07] for similar fermions), which are themselve complexified versions of fermions appearing in the physics literature (see for instance [KaCe71]). These are n-point functions, which are harmonic in each of their variables (in the scaling limit) and which can be identified by certain boundary conditions. Some details of our construction are reminiscent to the one of the holomorphic spinors introduced by Mercat.

The key for proving our results is the analysis of these fermions, first on the discrete level, and then on the continuous one, once we pass to the scaling limit. Notably, we show that they solve certain discrete versions of Riemann-Hilbert boundary value problems, allowing us to use and develop discrete complex analysis tools to obtain relations between them and to show their convergence.

The introduction consists of the following:

- In Section 1.2, we introduce some notation, mostly related to graphs.
- In Section 1.4, we define the Ising model and the observables that we are interested it.
- In Section 1.5, we state our main results about the energy density and the spin.
- In Section 1.6, we give an overview of the stategy used to show the main results and give a detailed summary of the text.
- In Section 1.7, some additional notation, which is needed in the proofs, is given.

1.2. Notation

In this section, we define most of the notation and conventions that will be used in this text. The few remaining notation will be given at the end of this section.

1.3. Graph notation

Let us first give some general graph notation. Let G be a planar graph.

- We denote by V_G the set of the vertices of G, by E_G the set of its (unoriented) edges, by $\vec{E}_G$ the set of its oriented edges, by F_G the set of its faces.
- For two vertices $x, y \in V_G$, we write $x \sim y$ if they are adjacent and in that case, we denote by $\langle x, y\rangle \in E_G$ the unoriented edge between them, by $\vec{xy} \in \vec{E}_G$ the oriented edge from x to y (x and y are called the **initial** and **final vertex** of $\vec{xy}$ respectively) by $-\vec{xy} \in \vec{E}_G$ the oriented edge from y to x.
- If the graph is embedded in the complex plane, we identify the vertices with the corresponding points in the complex plane. An oriented edge is identified with the difference of the final vertex minus the initial one.
- For two edges $e_1, e_2 \in E_G$, we write $e_1 \sim e_2$ if they share an endvertex and in that case we denote by $e_1 \cap e_2 \in V_G$ that endvertex.
- For two faces $f_1, f_2 \in F_G$, we write $f_1 \sim f_2$ if they share an edge and in that case we denote by $f_1 \cap f_2 \in E_G$ that edge.

1.3.1. Discrete domains.

- We denote by C_δ the square grid of mesh size $\delta > 0$, viewed as a subset of the complex plane.

$$\begin{aligned} V_{C_\delta} &= \{j + ik : j, k \in \mathbb{Z}\}, \\ E_{C_\delta} &= \{\langle v_1, v_2\rangle : v_1, v_2 \in V_{C_\delta}, |v_1 - v_2| = \delta\}. \end{aligned}$$

- We will mostly be interested in finite **induced** subgraphs Ω_δ of C_δ (two vertices of Ω_δ are linked by an edge if they are linked in C_δ), that we will also call **discrete domains**.
- The faces $F_{\Omega_\delta} \subset F_{C_\delta}$ are the faces whose four edges are in E_{Ω_δ}.
- We call **inner boundary vertices** of Ω_δ and denote by $\partial_0 V_{\Omega_\delta} \subset V_{\Omega_\delta}$ the set of vertices that are at distance δ or $\sqrt{2} \cdot \delta$ from a vertex in $V_{C_\delta \setminus \Omega_\delta}$. We call **outer boundary vertices** of Ω_δ and denote by $\partial V_{\Omega_\delta} \subset V_{C_\delta \setminus \Omega_\delta}$ or $\partial_1 V_{\Omega_\delta}$ the set of vertices at distance δ or $\sqrt{2} \cdot \delta$ from a vertex of V_{Ω_δ}.
- We call **boundary edges** and denote by $\partial E_{\Omega_\delta} \subset E_{C_\delta}$ the set of edges between a vertex of $\partial_0 V_{\Omega_\delta}$ and a vertex of $\partial V_{\Omega_\delta}$, by $\partial_0 E_{\Omega_\delta}$ the set of edges $e \in E_{\Omega_\delta}$ between vertices of $\partial_0 V_{\Omega_\delta}$ such that e is adjacent to a face of $F_{C_\delta \setminus \Omega_\delta}$, and by $\partial_0 \Omega_\delta$ the graph with vertex set $\partial_0 V_{\Omega_\delta}$ and edge set $\partial_0 E_{\Omega_\delta}$. We denote by $\partial_1 E_{\Omega_\delta}$ the set of edges $e \in E_{C_\delta}$ between vertices of $\partial_1 V_{\Omega_\delta}$ such that e is adjacent to a face of F_{Ω_δ}, and by $\partial_1 \Omega_\delta$ the graph with vertex set $\partial_1 V_{\Omega_\delta}$ and edge set $\partial_1 E_{\Omega_\delta}$.
- We call a discrete domain **simply connected** if $C_\delta \setminus \Omega_\delta$ has only one connected component.
- When needed, we identify discrete domains with the union of their faces.

1.3.2. Dual graph.

We define the following corresponding notions for the dual graph:

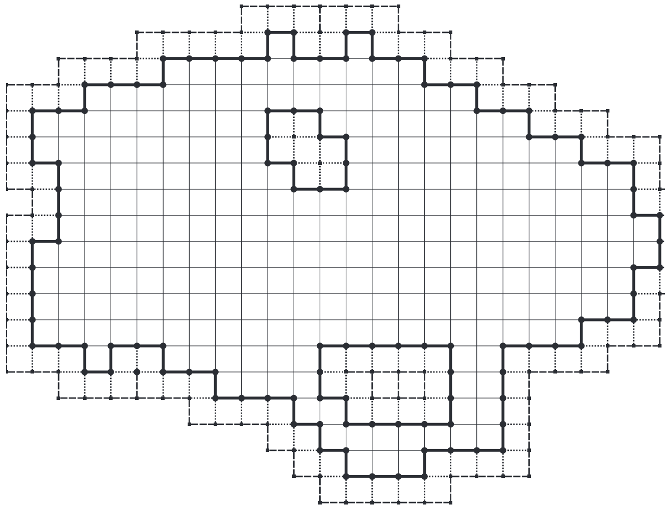

Figure 1.3.1. A discrete domain Ω_δ, with the vertices of $\partial_0 V_{\Omega_\delta}$ marked by bold points and the ones of $\partial_1 V_{\Omega_\delta}$ by small squares and with the edges of $\partial_0 E_{\Omega_\delta}$ drawn in bold, the ones of $\partial E_{\Omega_\delta}$ dotted and the ones of $\partial_1 E_{\Omega_\delta}$ dashed.

- For a discrete domain Ω_δ, we denote by $m(f)$ the center of a face $f \in F_{\Omega_\delta}$ and by Ω_δ^* the **dual graph** of Ω_δ, defined by

$$V_{\Omega_\delta^*} = \{m(f) : f \in F_{\Omega_\delta}\}$$
$$E_{\Omega_\delta^*} = \{\langle m(f_1), m(f_2)\rangle : f_1, f_2 \in F_{\Omega_\delta}, f_1 \sim f_2\},$$

  the vertices of Ω_δ^* being identified with the corresponding points in the complex plane.
- We call **inner boundary medial vertices** and denote by $\partial_0 V_{\Omega_\delta^*} \subset V_{\Omega_\delta^*}$ the set of the centers of the faces of F_{Ω_δ} that are adjacent to a face of $F_{C_\delta \setminus \Omega_\delta}$ or touch a face of $F_{C_\delta \setminus \Omega_\delta}$ at a corner, and call **outer boundary medial vertices** and denote by $\partial V_{\Omega_\delta^*} \subset V_{C_\delta^* \setminus \Omega_\delta^*}$ or $\partial_1 V_{\Omega_\delta^*}$ the set of the centers of faces of $F_{C_\delta \setminus \Omega_\delta}$ that are adjacent to a face of F_{Ω_δ} or touch a face of F_{Ω_δ} at a corner.
- We denote by $\partial E_{\Omega_\delta^*}$ the set of edges of $E_{\Omega_\delta^*}$ between a vertex of $\partial_0 V_{\Omega_\delta^*}$ and a vertex of $\partial V_{\Omega_\delta^*}$. We denote by $\partial_0 E_{\Omega_\delta^*}$ the set of edges $e \in E_{\Omega_\delta^*}$ between vertices of $\partial_0 V_{\Omega_\delta^*}$ such that e is adjacent to a face in $F_{C_\delta^* \setminus \Omega_\delta^*}$, and by $\partial_0 \Omega_\delta^*$ the graph with vertex set $\partial_0 V_{\Omega_\delta^*}$ and edge set $\partial_0 E_{\Omega_\delta^*}$. We denote by $\partial_1 E_{\Omega_\delta^*}$ the set of edges $e \in E_{C_\delta^*}$ between vertices of $\partial_1 V_{\Omega_\delta^*}$, such that e is adjacent to a face in $F_{\Omega_\delta^*}$.
- For an edge $e \in E_{\Omega_\delta}$ we denote by $e^* \in E_{\Omega_\delta^*}$ the edge of Ω_δ^* that intersects e. Conversely, for an edge $a \in E_{\Omega_\delta^*}$, we denote by $a^* \in E_{\Omega_\delta}$ the edge of Ω_δ that crosses it.

1.3.3. Approximation of continuous domains.

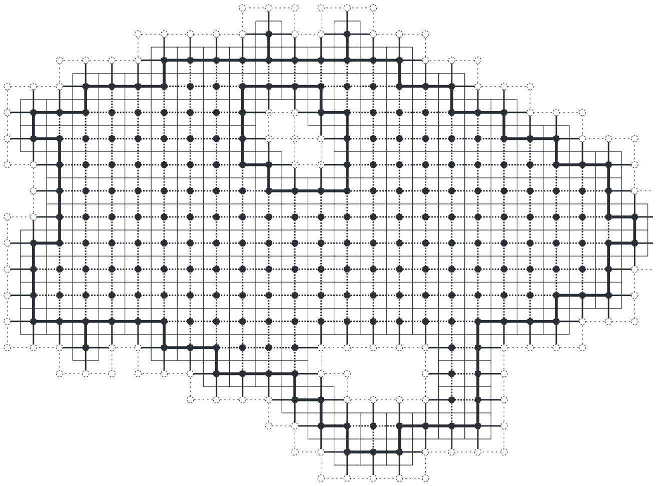

Figure 1.3.2. The dual Ω^*_δ of the discrete domain Ω_δ of Figure 1.3.1 (drawn with light stroke), with the vertices of $V_{\Omega^*_\delta}$ depicted by black points, the vertices of $\partial_1 V_{\Omega^*_\delta}$ by black points. The edges of $E_{\Omega^*_\delta} \setminus \partial_0 E_{\Omega^*_\delta}$ are depicted by densely dotted strokes, the edges of $\partial_0 E_{\Omega^*_\delta}$ by bold strokes, the edges of $\partial E_{\Omega^*_\delta}$ by normal strokes and the ones of $\partial_1 E_{\Omega^*_\delta}$ by sparsely dotted strokes.

- We say that a family $(\Omega_\delta)_{\delta>0}$ of discrete domains (with $\Omega_\delta \subset C_\delta$ for each $\delta > 0$) approximates or discretizes a continuous domain Ω if for each $\delta > 0$, Ω_δ is the largest connected induced subgraph of C_δ contained in Ω.
- Given a domain $\Omega \subset \mathbb{C}$ we call a straight part of the boundary $\partial^s\Omega \subset \partial\Omega$, a piece of the boundary made of finite number of pieces parallel to either the real or the imaginary line.
- We say that a domain is smooth if its boundary is piecewise C^1.
- For a set of vertices $V \subset V_{G_\delta}$ with G_δ equal to Ω_δ, Ω^*_δ, Ω^m_δ or Ω^{m*}_δ and a subset $K \subset \mathbb{C}$, we denote by $V \cap K$ the set of vertices of V that are at distance at most δ from K.

1.3.4. Linear algebra and Pfaffians. Let us finish this subsection by giving some general linear algebra notation.

- We denote by $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ the algebras of $n \times n$ real and complex matrices.
- For an antisymmetric $2n \times 2n$ matrix A, we denote by $\mathrm{Pfaff}(A)$ the Pfaffian of A, defined by

$$\mathrm{Pfaff}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} A_{\sigma(2j-1),\sigma(2j)},$$

where S_{2n} is the set of the permutations of $\{1, \ldots, 2n\}$ and $\mathrm{sgn}(\cdot)$ denotes the signature of a permutation.

- In particular, we have

$$\mathrm{Pfaff}\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} = x.$$

- If for a matrix $A \in M_{2n}(\mathbb{C})$, and $j, k \in \{1, \ldots, 2n\}$ we denote by $A \setminus_{jk} \in M_{2n-2}(\mathbb{C})$ obtained by removing the j-th line and column and the k-th line and column, we have,

$$\mathrm{Pfaff}(A) = \sum_{j=1} (-1)^j A_{j1} \mathrm{Pfaff}(A \setminus_{j1}).$$

- For any antisymmetric matrix $A \in M_{2n}(\mathbb{C})$, we have

$$\mathrm{Pfaff}(A)^2 = \det(A).$$

- If $A \in M_{2n}(\mathbb{C})$ is an antisymmetric matrix and $B \in M_{2n}(\mathbb{C})$ is any matrix, we have

$$\mathrm{Pfaff}\left(B^t A B\right) = \det(B)\,\mathrm{Pfaff}(A).$$

1.4. Ising model

Recall that the Ising model is a random assignment of ± 1 spins to the vertices of a graph with a parameter β called the inverse temperature controlling the strength of the interactions between the spins. In our setup, the notion of boundary conditions, i.e. the values that we assign to the spins at the boundary vertices of the graph, will be crucial: one of the aim of this text is to examine the effect of these boundary conditions on the behavior of the model in the bulk.

1.4.1. With free boundary condition.

More formally, the Ising model **with free boundary condition** on a graph G (in this text, G will be a discrete domain Ω_δ or its dual Ω_δ^*) at inverse temperature $\beta > 0$ is a model whose state space is

$$\Xi_G^{\mathrm{free}} = \left\{ (\sigma_x)_{x \in V_G} : \sigma_x \in \{-1, 1\} \ \forall x \in V_G \right\},$$

where the probability of **spin configuration** $\sigma \in \Xi_G^{\mathrm{free}}$ is equal to

$$\mathbb{P}^{\mathrm{free}}\{\sigma\} = \frac{1}{Z_{G,\beta}^{\mathrm{free}}} \exp\left(-\beta H_G^{\mathrm{free}}(\sigma)\right),$$

with the **Hamiltonian** or **energy** H_G^{free} of a configuration σ given by

$$H_G^{\mathrm{free}}(\sigma) = -\sum_{\langle x,y \rangle \in E_G} \sigma_x \sigma_y$$

and the **partition function** $Z_{G,\beta}^{\mathrm{free}}$ given by

$$Z_{G,\beta}^{\mathrm{free}} = \sum_{\sigma \in \Xi_G} e^{-\beta H_G^{\mathrm{free}}(\sigma)}.$$

For $x \in V_G$, the random number $\sigma_x \in \{\pm 1\}$ is called the **spin** at x.

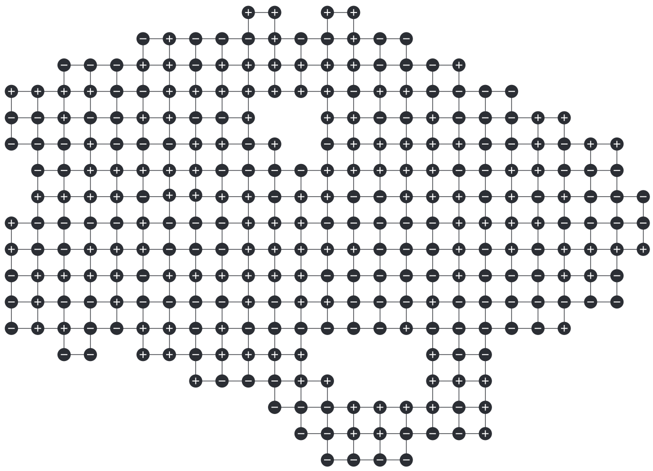

Figure 1.4.1. The Ising model with free boundary condition on the discrete domain Ω_δ of Figure 1.3.1.

1.4.2. With mixed boundary conditions. We define the notion of boundary conditions only in the setup that we will study in this text. Given a (possibly empty) collection $b = \{b_1, \ldots, b_{2n}\}$ of vertices on $\partial_0 V_{\Omega^m_\delta}$, we define the Ising model on Ω^*_δ **with the (locally monochromatic) boundary condition** b as the Ising model with state space $\Xi^b_{\Omega^*_\delta}$ defined as

$$\left\{ (\sigma_x)_{x \in V_{\Omega^*_\delta}} \in \{\pm 1\}^{V_{\Omega^*_\delta} \cup \partial V_{\Omega^*_\delta}} : \sigma_{v_1} = \sigma_{v_2} \iff \langle v_1, v_2 \rangle^* \notin b \ \forall v_1, v_2 \in \partial V_{\Omega^*_\delta} \right\},$$

and with the energy $H^\partial_{\Omega^*_\delta}$ given by

$$H^\partial_{\Omega^*_\delta} = - \sum_{\langle x,y \rangle \in E_{\Omega^*_\delta} \cup \partial E_{\Omega^*_\delta}} \sigma_x \sigma_y .$$

In other words, the Ising model with boundary condition b is the Ising model with spins on $V_{\Omega^*_\delta} \cup \partial V_{\Omega^*_\delta}$ with the boundary spins on $\partial V_{\Omega^*_\delta}$ conditioned to be locally constant, and to switch at the locations of the vertices of b. The model is well-defined only if for each of the connected components of $\partial_0 \Omega_\delta$ is adjacent to an even (possibly zero) number of boundary changing operators. The edges $b_1, \ldots, b_{2n} \in \partial E_{\Omega_\delta}$ are called **boundary changing operators**. If there are no boundary changing operators, we write $b = \emptyset$

The probability of a configuration is as before given by

$$P\{\sigma\} = \frac{1}{Z^b_{\Omega^*_\delta, \beta}} e^{-\beta H^b_{\Omega^*_\delta}(\sigma)}$$

with the partition function $Z^b_{\Omega^*_\delta, \beta}$ defined as

$$\sum_{\sigma \in \Xi^b_{\Omega^*_\delta}} e^{-\beta H^b_{\Omega^*_\delta}(\sigma)} .$$

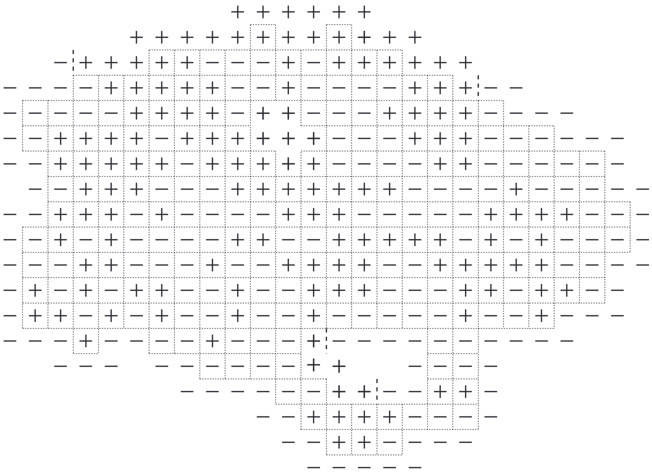

Figure 1.4.2. A realization of the Ising model with mixed boundary conditions on the graph Ω^*_δ of Figure 1.3.2 with locally monochromatic boundary condition and boundary changing operators at the dashed edges.

We also define the Ising model on Ω^*_δ with $+$ **boundary condition** as the Ising model with state space

$$\Xi^+_{\Omega^*_\delta} = \left\{ (\sigma_x)_{x\in \overline{V}_{\Omega^*_\delta}} \in \{\pm 1\} : \sigma_v = 1 \ \forall v \in \partial V_{\Omega^*_\delta} \right\},$$

with the energy $H^{\partial}_{\Omega^*_\delta}$ and the corresponding probability measure and partition function. The Ising model with $-$ boundary condition is defined exactly in the same way.

Let us finally define the Ising model with **alternating $+/-$ boundary conditions** as follows, on a simply connected domain Ω_δ (the only case where we will consider these boundary conditions). Given edges $b_1, \ldots, b_{2m} \in \partial E_{\Omega_\delta}$ enumerated in counterclockwise order and alternating signs $s_1, \ldots, s_{2m} \in \{\pm 1\}$, the Ising model with the boundary condition $b_1^{s_1} b_2^{s_2} \ldots b_{2m-1}^{s_{2m-1}} b_{2m}^{s_{2m}}$ is the Ising model with state space

$$\Xi^+_{\Omega^*_\delta} = \left\{ (\sigma_x)_{x\in \overline{V}_{\Omega^*_\delta}} \in \{\pm 1\} : \sigma_y = s_j \ \forall y \in \partial V_{\Omega^*_\delta} \cap \overline{b_j b_{j+1}} \right\},$$

where $\overline{b_j b_{j+1}}$ denotes the counterclockwise arc between b_j and b_{j+1} (with the indices taken modulo $2m$).

1.4.3. Critical Ising model. In this text, we will be interested in the **critical Ising model**, that is, the Ising model at **critical inverse temperature** $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

1.4.4. Observables. Let Ω_δ be a discrete domain. The two main observables of Conformal Field Theory for the Ising model are.

- The spin: for each $x \in V_{\Omega_\delta}$, we denote by $\sigma_\delta(x)$ the (random) value of the **spin** at x.

- The energy density: for each edge $e \in E_{\Omega_\delta}$, with $e = \langle x.y \rangle$, we denote by $\varepsilon_\delta(e)$ the **energy density** at e, which is defined by

$$\varepsilon_\delta(e) = \mu - \sigma_x \sigma_y,$$

where $\mu = \sqrt{2}/2$.

Remark. The value of μ set this way corresponds to the **infinite-volume limit** of the product of adjacent spins σ_x and σ_y: as will be shown later, when $\delta \to 0$, if x and y stay away from the boundary, we have $\sigma_x \sigma_y \to \mu$.

1.5. Main results

We can now state our main results, whose proofs are given in Section 7.4. Recall that we consider the critical Ising model on (finite) subgraphs of the square grid with boundary conditions. Our results concern mostly the scaling limit of the model, that is, when the mesh size of the square grid goes to zero (and we rescale properly the quantities that we look at). Let us just make the following conventions:

- For a domain Ω, let us denote by $(\Omega_\delta)_{\delta > 0}$ the family of discrete domains approximating it.
- For a family $b = \{b_1, \ldots, b_{2k}\}$ of points on $\partial\Omega$ and a family $a_1, \ldots, a_n$ of points in Ω, let us denote for each $\delta > 0$ by $b_\delta = \{b_{1,\delta}, \ldots, b_{2k,\delta}\}$ the family of edges in $\partial E_{\Omega_\delta}$ that are the closest to $\{b_1, \ldots, b_{2k}\}$ and by $a_{1,\delta}, \ldots, a_{n,\delta}$ the family of edges in E_{Ω_δ} that are the closest to $a_1, \ldots, a_n$.
- For points distinct $x_1, \ldots, x_{2p} \in \mathbb{C}$, denote by $K(x_1, \ldots, x_p) \in M_p(\mathbb{R})$ the antisymmetric matrix defined by

$$K(x_1, \ldots, x_p)_{jk} = \begin{cases} \frac{1}{x_j - x_k} & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

The main theorem of this paper is the following:

Theorem 1. *Let Ω be a finitely-connected domain with a collection $b = \{b_1, \ldots, b_{2k}\}$ of boundary points, such that each connected component of $\partial\Omega$ contains an even number of b_j's, and let $a_1, \ldots, a_n \in \Omega$ be a collection of distinct interior points.*

There exists a correlation function $\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b \in \mathbb{R}$, such that for any conformal mapping $\phi : \Omega \to \Omega'$, we have

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b = \left(\prod_{j=1}^n |\phi'(a_j)| \right) \cdot \langle \varepsilon(\phi(a_1)) \cdot \ldots \cdot \varepsilon(\phi(a_n)) \rangle_{\Omega'}^{\phi(b)},$$

where $\phi(b) = \{\phi(b_1), \ldots, \phi(b_{2n})\}$ and such that the following convergence result holds:

If Ω is smooth and all the boundary points of b are located on a straight part $\partial^s\Omega \subset \partial\Omega$ of the boundary, and if we consider the critical Ising model on Ω_δ with the above notation, we have

$$\frac{1}{\delta^n} \mathbb{E}_{\Omega_\delta^*}^{b_\delta} \left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta}) \right] \underset{\delta \to 0}{\longrightarrow} \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b,$$

uniformly on the compact subsets of

$$\{(b_1, \ldots, b_{2k}) \in \partial^s\Omega \times \ldots \times \partial^s\Omega : b_j \neq b_l \; \forall j \neq l\}$$
$$\times \{(a_1, \ldots, a_n) \in \Omega \times \ldots \times \Omega : a_j \neq a_l \; \forall j \neq l\}.$$

In particular, the limit is independent of the (horizontal or vertical) orientation of the edges $a_{1,\delta}, \ldots, a_{n,\delta}$.

On the upper half-plane, the correlation function is given by

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\mathbb{H}}^{b} = \frac{1}{(\pi i)^n} \cdot \frac{\mathrm{Pfaff}(K(a_1 \ldots, a_n, \overline{a_n}, \ldots, \overline{a_1}, b_1, \ldots, b_{2m}))}{\mathrm{Pfaff}(K(b_1, \ldots, b_{2m}))},$$

where the matrix K is as defined above.

Corollary 2. With the above notation, in the case of free boundary condition, we have

$$\frac{1}{\delta^n} \mathbb{E}_{\Omega_\delta}^{\mathrm{free}} [\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})] \underset{\delta \to 0}{\longrightarrow} \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{\mathrm{free}},$$

where

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{\mathrm{free}} = (-1)^n \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{\varnothing},$$

where $\varnothing$ denotes the locally monochromatic boundary condition with no boundary changing operators. The convergence is uniform on the compact subsets of

$$\{(a_1, \ldots, a_n) \in \Omega \times \ldots \times \Omega : a_j \neq a_k \ \forall j \neq k\}.$$

Corollary 3. With the above notation, if Ω is simply connected and we consider the Ising model with $+$ boundary condition on its discretizations, we have

$$\frac{1}{\delta^n} \mathbb{E}_{\Omega_\delta^*}^{+} [\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})] \underset{\delta \to 0}{\longrightarrow} \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{+},$$

where

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{+} = \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{\varnothing}.$$

The convergence is uniform on the compact subsets of

$$\{(a_1, \ldots, a_n) \in \Omega \times \ldots \times \Omega : a_j \neq a_k \ \forall j \neq k\}.$$

Let $\partial^s \Omega$ be a straight part of $\partial \Omega$ and $b_1, \ldots, b_{2k} \in \partial^s \Omega$ be distinct boundary points appearing in counterclockwise order. With the above notation, consider the critical Ising model with alternating $+/-$ boundary condition $a_\delta = b_{1,\delta}^{+} b_{2,\delta}^{-} \ldots b_{2k-1,\delta}^{-} b_{2k,\delta}^{+}$. Then, we have

$$\frac{1}{\delta^n} \mathbb{E}_{\Omega_\delta^*}^{a_\delta} [\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})] \underset{\delta \to 0}{\longrightarrow} \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{(b_1^{+} b_2^{-} \ldots b_{2k-1}^{+} b_{2k}^{-})},$$

with

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{(b_1^{+} b_2^{-} \ldots b_{2k-1}^{+} b_{2k}^{-})} = \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_{\Omega}^{\{b_1, b_2, \ldots b_{2k-1}, b_{2k}\}},$$

where the right term is as in Theorem 1. The convergence is uniform on the compact subsets of

$$\{(b_1, \ldots, b_{2k}) \in (\partial^s \Omega) : b_j \neq b_l \ \forall j \neq l\}$$
$$\times \{(a_1, \ldots, a_n) \in \Omega \times \ldots \times \Omega : a_j \neq a_l \ \forall j \neq l\}.$$

Remark 4. Notice that if Ω is simply connected, our result says that the one-point function $\langle \varepsilon(a_1) \rangle_{\Omega}^{\varnothing}$ is proportional to the hyperbolic metric element of Ω at a – this is the result obtained in [HoSm10].

Our other result concerns the boundary spin correlation. For a domain Ω, let us denote as before $(\Omega_\delta)_{\delta>0}$ the discretization of it. For boundary points $x_1, \ldots, x_{2n} \in \partial \Omega$, let us denote this time for each $\delta > 0$ by $x_{1,\delta}, \ldots, x_{2n,\delta}$ the closest vertices of $\partial_0 V_{\Omega_\delta}$ to $x_1, \ldots, x_{2n}$.

Theorem 5. Let Ω be a finitely-connected domain and let $x_1, \ldots, x_{2n} \in \partial\Omega$ be distinct boundary points on smooth parts of $\partial\Omega$, such that each connected component of $\partial\Omega$ contains an even number of x_j's. Then there exists a correlation function

$$\langle \sigma(x_1) \cdot \ldots \cdot \sigma(x_{2n}) \rangle_{\Omega}^{\text{free}} \in \mathbb{R}$$

such that for any conformal mapping $\phi : \Omega \to \Omega'$ (with $\phi(x_1), \ldots, \phi(x_{2n})$ on smooth parts of $\partial\Omega'$), we have

$$\langle \sigma(x_1) \cdot \ldots \cdot \sigma(x_{2n}) \rangle_{\Omega}^{\text{free}} = \left(\prod_{j=1}^{2n} |\phi'(x_j)|^{\frac{1}{2}} \right) \langle \sigma(\phi(x_1)) \cdot \ldots \cdot \sigma(\phi(x_{2n})) \rangle_{\Omega'}^{\text{free}}$$

and such that the following convergence result holds:

If Ω is a smooth bounded domain and the points $x_1, \ldots, x_{2n}$ are all on a straight boundary part $\partial^s\Omega$, then with the above notation we have

$$\frac{1}{\delta^n} \mathbb{E}_{\Omega_\delta}^{\text{free}} [\sigma_\delta(x_{1,\delta}) \cdot \ldots \cdot \sigma_\delta(x_{2n,\delta})] \underset{\delta \to 0}{\longrightarrow} \langle \sigma(x_1) \cdot \ldots \cdot \sigma(x_{2n}) \rangle_{\Omega}^{\text{free}},$$

uniformly on the compact subsets of

$$\{(x_1, \ldots, x_{2n}) \in \partial^s\Omega \times \ldots \times \partial^s\Omega : x_j \neq x_k \, \forall j \neq k\}$$

On the upper half-plane, the correlation functions is given by:

$$\langle \sigma(x_1) \cdot \ldots \cdot \sigma(x_{2n}) \rangle_{\mathbb{H}}^{\text{free}} = \left(\frac{2}{\pi\alpha} \right)^n |\mathrm{Pfaff}(K(x_1, \ldots, x_{2n}))|,$$

with the matrix K as defined above and $\alpha = \sqrt{2} - 1$.

1.6. Overall strategy and summary

1.6.1. Strategy. Informally speaking (and if we concentrate on the arguments directly used to prove the main theorems given in the previous section) the strategy that we use is the following:

- We introduce appropriate representations of the Ising model, to represent all the discrete quantities of interest in terms of statistics over suitable families of contours (Section 5.1).
- We introduce so-called fermionic observables which are functions defined on the midpoints of the edges on which the Ising model is defined (Section 5.2).
- We obtain the quantities of interest as special values of the fermionic observables (Section 5.3). Very informally, as will be justified later (Section 6.6), the idea to compute the n-point energy correlation with $2k$ boundary changing operator is to introduce a $2n + 2k$-point fermionic observable, and to move $2k$ of those points to the locations of boundary changing operators and to merge pairwise (with some renormalization) the remaining $2n$ points at the locations where we want to compute the energy (we call this *fusion of observables*).
- We obtain integrability properties for the fermionic observables that we translate in terms of discrete complex analyticity (Sections 6.1 and 6.2)
- We perform an analysis of the boundary values and singularities of the different observables, summarizing them as solutions of discrete Riemann-Hilbert boundary value problems (Sections 6.3, 6.4, 6.5).

- We translate the relations in terms of Pfaffians, reducing the computation of all the fermionic observables to the computation of two of them (6.6), which are both two-point observables.
- We apply discrete complex analysis techniques (developed in Chapters 2, 3, 4) to obtain convergence of the two-point observables (Sections 7.1 and 7.2).
- We pass the Pfaffians of discrete functions to the limit and show conformal covariance properties for them, and obtain exact formulae (Sections 7.3 and 7.5).

1.6.2. Summary. Let us now give a more detailed and systematic summary of this text. Roughly speaking, this text is divided in two parts:

- A first part, consisting of Chapters 2, 3 and 4, where discrete complex analysis techniques are developed, including convergence questions. This part is independent of the Ising model, although the questions are motivated by applications to the Ising model.
- A second part, consisting of Chapters 5, 6 and 7, that are concerned with the Ising model, and that are notably devoted to an analysis of so-called fermionic observables, in terms of which the correlation functions of the main theorems above can be represented.

More precisely:

(1) In Chapter 2, we introduce and adapt results (mostly existing ones) from discrete complex analysis that will be useful for the study of the observables:
  (a) In Section 2.1, we introduce the discretizations ∂_δ, $\overline{\partial}_\delta$, Δ_δ and $\overline{\nabla}_\delta$ of the classical differential operators ∂, $\overline{\partial}$, Δ and $\overline{\nabla}$ that will be useful for us, and obtain discrete analogues of the classical integral formulae for them.
  (b) In Section 2.2, we introduce the classical notions of discrete harmonicity and discrete holomorphicity, as well as a stronger notion of discrete holomorphicity, which we call s-holomorphicity, and define discrete singularities.
  (c) In Section 2.3, we introduce discrete versions of the Green's functions for the $\overline{\partial}_\delta$ and Δ_δ operators that will play a very important role throughout the text, providing us with a discrete version of Cauchy's formula and a representation of solutions to discrete Poisson equation.
  (d) In Section 2.4, we introduce the s-holomorphic version of the Green's function for $\overline{\partial}_\delta$, that are the full-plane analogues of the fermionic observables introduced in Section 5.2.
  (e) In Section 2.5, we discuss the integration of s-holomorphic function, in particular the remarkable feature that the square of s-holomorphic functions can be integrated, and study certain properties of the discrete integral.

(f) In Section 2.6, we formulate the boundary value problems that are relevant in this text, which are discrete inhomogenenous Riemann-Hilbert boundary value problems, discuss some of their basic properties and study relations with the discrete integrals defined in the previous subsection.

(2) In Chapter 3, we review and adapt to our setup existing notions and results about regularity and convergence of discrete harmonic, holomorphic or s-holomorphic functions to continuous ones. Those will be in particular used in the next section to study the convergence of solutions of discrete Riemann-Hilbert boundary value problems.
   (a) In Section 3.1, we discuss the notion of convergence of discrete harmonic functions.
   (b) In Section 3.2, we discuss the convergence the Green's functions introduced in Section 2.3 and of harmonic measure.
   (c) In Section 3.3, we discuss regularity results for discrete harmonic and holomorphic functions, in particular about how to transform integral control into uniform control.

(3) In Chapter 4, we study the convergence of solutions of discrete Riemann-Hilbert boundary value problems to continuous ones.
   (a) In Section 4.1, we define continuous Riemann-Hilbert boundary value problems, which are the natural candidates for the limit of discrete ones.
   (b) In Section 4.2, we obtain regularity and precompactness estimates for solutions to discrete Riemann-Hilbert problems for the topology of convergence on compact subsets.
   (c) In Section 4.3, we identify the subsequential limits of solutions of Riemann-Hilbert problems for the topology of convergence on the compact subsets.
   (d) In Section 4.4, we extend the convergence results of the previous subsection up to the boundary, where it is nice enough.
   (e) In Section 4.5, we summarize the results of the previous subsections, formulating a convergence result that will be used in the next section to show the convergence of observables.

(4) In Chapter 5, we introduce contour statistics that are relevant for computing the quantities of interest, representing them as special values of so-called fermionic observables, that are the central tools in this text.
   (a) In Section 5.1, we introduce two classical contour representations of the Ising model, which are dual to each other and allow to treat in a unified way the various discrete correlation functions of interest.
   (b) In Section 5.2, we introduce the so-called fused fermionic observables, that are constructed from more basic ones, called unfused fermionic observables. These observables come in two variants, a real and a complex one, which can be viewed as signed or complexified versions of classical contour representations.
   (c) In Section 5.3, we connect the classical contour representations with the fermionic observables, obtaining the correlation functions of the

main theorems in terms special values of certain real fermionic observables. Hence, establishing the convergence of the fermionic observables will give convergence of the correlation functions.

(5) In Chapter 6, using integrability properties of the critical Ising model, we derive discrete complex analysis properties of the model, formulating the observables as solutions to the discrete Riemann-Hilbert boundary value problems introduced in the previous section and use this to obtain representations of the observables in terms of each other.
   (a) In Section 6.1, we discuss the integrability of the model, that appears in this case as a collection of conservation laws for the fermionic weights introduced in Section 5.2 under a family of combinatorial involutions.
   (b) In Section 6.2, we translate the integrability properties of the previous subsection into s-holomorphicity properties for the complex fermionic observables.
   (c) In Section 6.3, we study the discrete singularities of the complex fermionic observables.
   (d) In Section 6.4, we study the boundary behavior of the complex fermionic observables.
   (e) In Section 6.5, we summarize the results of the previous subsections, formulating the complex fermionic observables as solutions to discrete Riemann-Hilbert boundary value problems, notably introducing a so-called boundary effect observable.
   (f) In Section 6.6, we use the results of the previous subsections and the discrete complex analysis results of the previous section to deduce recursion relations between the observables, yielding Pfaffian formulae for them, that allow for the representation of all the fermionic observables in terms of the two-point fermionic observable and of the boundary effect observable.

(6) In Chapter 7, we apply the results of the previous section to show the convergence of the discrete observables to continuous ones, to obtain scaling formulae and eventually to prove the main theorems.
   (a) In Section 7.1, we define full-plane two-point continuous fermionic observables and obtain the convergence of the discrete full-plane observables to them, using the convergence of the $\overline{\partial}_\delta$-Green's function.
   (b) In Section 7.2, we define continuous two-point fermionic observables and continuous boundary effect observable and show, using the results of Section 4, the convergence of the discrete observables to them.
   (c) In Section 7.3, we define general continuous fused fermionic observables and use the results of Section 6.6 and of the previous subsection to obtain the convergence of the general discrete fermionic observables to them.
   (d) In Section 7.5, we summarize the convergence results of the previous subsections and use the representations of Section 5.3 to prove the main theorems.

- In the Appendices A and B, we give the proofs of certain propositions of Chapter 5.

- In Appendix C, we give a small by-product of the discrete analysis in Chapter 5 and 6, that seems to be of independent interest and to be hard to obtain without discrete complex analysis, although completely elementary in its statement.

1.7. More notation

Let us conclude this chapter by giving some additional notation that will be used in this text.

1.7.1. Constants. Throughout this text, we use the same letters for certain constants, some of which have already been defined above, that we will use frequently. Their definitions will often be recalled throughout the text, but for facilitating the reading, we give them here:

- $\alpha = \sqrt{2} - 1$,
- $\beta_c = \frac{1}{\sqrt{2}} \ln\left(\sqrt{2} + 1\right)$,
- $\mu = \frac{\sqrt{2}}{2}$,
- $\lambda = e^{\pi i/4}$,
- $\eta = e^{\pi i/8}$.

1.7.2. Orientations and double-orientations. We will often use the following notation

- We denote by $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle of the complex plane. We will often refer to the elements of $\mathbb{S}$ as **(simple) orientations**.
- We denote by $(\mathbb{S})^2 = \left\{(z)^2 : z \in \mathbb{S}\right\}$ the double covering of the unit circle by itself. We will represent the elements of $(\mathbb{S})^2$ as elements of $\mathbb{S}$ with a specified square root (in $\mathbb{S}$). We will denote by $\sqrt{x}$ this square root for $x \in (\mathbb{S})^2$. Conversely, we will denote by $(z)^2$ the element of $(\mathbb{S})^2$ such that $\sqrt{(z)^2} = z \in \mathbb{S}$. We will often refer to the elements of $\mathbb{S}$ as **double orientations**.
- We denote by $\mathbb{S}_\square \subset \mathbb{S}$ the set of fourth roots of unity $\{\pm 1, \pm i\}$ and by $(\mathbb{S})^2_\square$ the set of squares of eight roots of unity
$$\left\{(\pm 1)^2, (\pm i)^2, (\pm \lambda)^2, \left(\pm \bar{\lambda}\right)^2\right\}.$$
We call $\mathbb{S}_\square$ and $(\mathbb{S})^2_\square$ the sets of **lattice simple** and **double orientations**.

1.7.3. More graphs. We give some more definitions concerning graphs, in particular, we define two types of graphs that will be central in this text: the medial graph and its dual.

Medial graph.

- For a discrete domain Ω_δ, we denote by $m(e)$ the midpoint of an edge $e \in E_{\Omega_\delta}$ and by Ω^m_δ the **medial graph** of Ω_δ, defined by
$$\begin{aligned} V_{\Omega^m_\delta} &= \{m(e) : e \in E_{\Omega_\delta} \cup \partial E_{\Omega_\delta}\} \\ E_{\Omega^m_\delta} &= \{\langle m(e_1), m(e_2)\rangle : e_1, e_2 \in E_{\Omega_\delta} \cup \partial E_{\Omega_\delta}, e_1 \sim e_2\}, \end{aligned}$$
where we identify as usual the vertices with the corresponding points in the complex plane.

- We denote by $\partial_0 V_{\Omega_\delta^m} \subset V_{\Omega_\delta^m}$ the set of the centers of the edges in $\partial E_{\Omega_\delta}$ and by $\partial V_{\Omega_\delta^m} \subset V_{C_\delta^m \setminus \Omega_\delta^m}$ the set of medial vertices at distance $\frac{\sqrt{2}}{2}\delta$ from a vertex of $V_{\Omega_\delta^m}$.
- We denote by $V^h_{\Omega_\delta^m}$ and $V^v_{\Omega_\delta^m}$ the set of midpoints of horizontal and vertical edges of Ω_δ respectively.
- For a medial vertex $x \in V_{\Omega_\delta^m}$, we denote by $O(x) \subset \mathbb{S}$ the set of admissible (simple) orientations of x, defined as $\{\pm 1\}$ if $x \in V^h_{\Omega_\delta^m}$ and as $\{\pm i\}$ if $x \in V^v_{\Omega_\delta^m}$ and by $(O)^2(x) \subset (\mathbb{S})^2$ the set of admissible double orientations of x, defined as $\{(\pm 1)^2, (\pm i)^2\}$ if $x \in V^h_{\Omega_\delta^m}$ and as $\{(\pm\lambda)^2, (\pm\bar{\lambda})^2\}$ if $x \in V^v_{\Omega_\delta^m}$.
- We denote by $S_{\Omega_\delta^m}$ the set of simply-oriented medial vertices defined by
$$S_{\Omega_\delta^m} = \left\{ x^o : x \in V_{\Omega_\delta^m}, o \in O(x) \right\}$$
and by $D_{\Omega_\delta^m}$ the set of doubly-oriented medial vertices defined as
$$D_{\Omega_\delta^m} = \left\{ x^o : x \in V_{\Omega_\delta^m}, o \in (O)^2(x) \right\}.$$
We will often identify an oriented medial vertex $x^o \in S_{\Omega_\delta^m}$ or in $D_{\Omega_\delta^m}$ to the medial vertex x. In particular, when we speak about "distinct oriented medial vertices", we mean that the corresponding medial vertices are distinct.
- We denote by $E^\Sigma_{\Omega_\delta}$ the set of signed edges of Ω_δ, defined by
$$E^\Sigma_{\Omega_\delta} = \{ e^s : e \in E_\Omega, s \in \{\pm 1\} \}.$$
As for oriented medial vertices, we will often identify a signed edge $e^s \in E^\Sigma_{\Omega_\delta}$ with the edge e, and when we speak about "distinct signed edges", we mean that the corresponding edges are distinct.
- For a boundary middlepoint $x \in \partial_0 V_{\Omega_\delta^m}$, we call inward-pointing orientation at x the orientation $o \in \mathbb{S}$ such that $x + \frac{1}{2} o\delta \in \partial_0 V_{\Omega_\delta}$, and inward-pointing double orientation a double orientation $o \in (\mathbb{S})^2$ that gets identified with a simple inward-pointing orientation.
- We denote by $H_{\Omega_\delta} = \left\{ \langle v, x \rangle : v \in V_{\Omega_\delta}, x \in V_{\Omega_\delta^m} \right\}$ the set of half-edges of Ω_δ and by $\vec{H}_{\Omega_\delta}$ the set of oriented half-edges of Ω_δ. Each edge $e = \langle v_1, v_2 \rangle \in E_{\Omega_\delta}$ is identified with the union of its two half-edges $\langle v_1, m(e) \rangle, \langle m(e), v_2 \rangle \in H_{\Omega_\delta}$.
- We denote by
$$\begin{aligned} \partial_0 H_{\Omega_\delta} &= \left\{ \langle v, x \rangle : v \in \partial_0 V_{\Omega_\delta}, x \in \partial_0 V_{\Omega_\delta^m} \right\}, \\ \partial H_{\Omega_\delta} &= \left\{ \langle v, x \rangle : v \in \partial V_{\Omega_\delta}, x \in \partial_0 V_{\Omega_\delta^m} \right\} \end{aligned}$$
the sets of boundary half-edges. We denote by
$$\begin{aligned} \partial_{ext} \vec{H}_{\Omega_\delta} &= \left\{ \vec{xv} : x \in \partial_0 V_{\Omega_\delta^m}, v \in \partial V_{\Omega_\delta} \right\} \\ \partial_{int} \vec{H}_{\Omega_\delta} &= \left\{ \vec{xv} : x \in \partial_0 V_{\Omega_\delta^m}, v \in \partial_0 V_{\Omega_\delta} \right\} \end{aligned}$$
the set of outward-pointing boundary half-edges and for a medial boundary vertex $x \in \partial_0 V_{\Omega_\delta^m}$, we denote by $\vec{v}_{ext}(x) \in \partial_{ext} \vec{H}_{\Omega_\delta}$ the outward-pointing boundary half-edge with intial vertex x and by $\vec{v}_{int}(x) \in \partial_{int} \vec{H}_{\Omega_\delta}$ the inward-pointing boundary half-edge with initial vertex x.

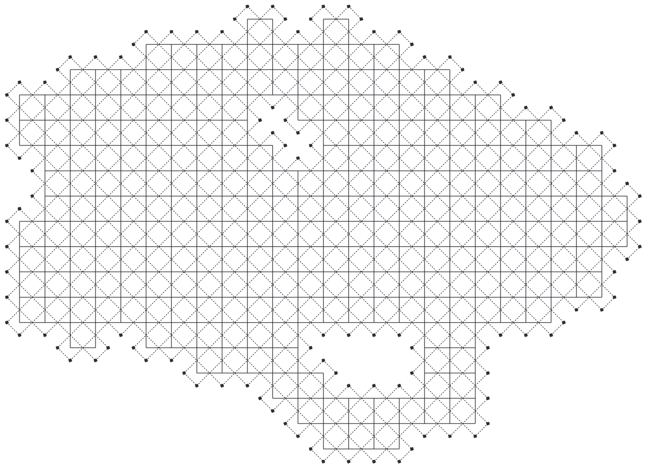

Figure 1.7.1. The medial graph Ω_δ^m of the graph Ω_δ of Figure 1.3.1, with the edges of $E_{\Omega_\delta^m}$ depicted by dotted strokes and with the vertices of $\partial_0 V_{\Omega_\delta^m}$ marked by black dots.

- We call **(half-edge) configuration** a subset of H_{Ω_δ}. For two half-edges configurations ω_1, ω_2, we denote by $\omega_1 \oplus \omega_2$ the symmetric difference of ω_1 and ω_2, defined as $(\omega_1 \cup \omega_2) \setminus (\omega_1 \cap \omega_2)$.

Dual of the medial graph.

- For a discrete domain Ω_δ, we denote by $\Omega_\delta^{m\,*}$ the **dual of the medial** of Ω_δ the graph defined by

$$
\begin{aligned}
V_{\Omega_\delta^{m\,*}} &= V_{\Omega_\delta} \cup V_{\Omega_\delta^*} \\
E_{\Omega_\delta^{m\,*}} &= \left\{ \langle x_1, x_2 \rangle : x_1, x_2 \in V_{\Omega_\delta^{m\,*}} : |x_1 - x_2| = \frac{\sqrt{2}}{2}\delta \right\}.
\end{aligned}
$$

Closure of graphs.

- For a graph G_δ equal to Ω_δ, Ω_δ^*, Ω_δ^m or $\Omega_\delta^{m\,*}$, we denote by $\overline{G}_\delta$ the **closure** of G_δ, defined as the union $G_\delta \cup \partial G_\delta$, where ∂G_δ is defined above for each of those graphs.

CHAPTER 2

Discrete Complex Analysis

In this chapter, we shortly review a number of concepts in discrete complex analysis and adapt them to the framework that will be useful for us. We do not address convergence results here. In this chapter, Ω_δ denotes a finite connected subgraph of the square grid $\mathbb{C}_\delta = \delta\mathbb{Z}^2$ of mesh size δ.

2.1. Discrete differential operators

2.1.1. Definitions. In this text, we will consider discrete differentiations of complex-valued functions defined on V_{G_δ} with G_δ usually equal to Ω_δ, Ω_δ^*, Ω_δ^m or $\Omega_\delta^{m\,*}$. For an oriented edge $\vec{e} = \vec{xy} \in \vec{E}_{G_\delta}$ and a function $f : V_{G_\delta} \to \mathbb{C}$, we define $\partial_{\vec{e}} f$ as $f(y) - f(x)$.

When G_δ is Ω_δ^m or $\Omega_\delta^{m\,*}$, for a function $f : \overline{V}_{G_\delta} \to \mathbb{C}$, we define $\partial_\delta f : V_{G_\delta^*} \to \mathbb{C}$ and $\overline{\partial}_\delta f : V_{G_\delta^*} \to \mathbb{C}$ by

$$\begin{aligned} \partial_\delta f(x) &= \frac{1}{2}\left[\left(f\left(x+\frac{\delta}{2}\right) - f\left(x-\frac{\delta}{2}\right)\right) - i\left(f\left(x+i\frac{\delta}{2}\right) - f\left(x-i\frac{\delta}{2}\right)\right)\right], \\ \overline{\partial}_\delta f(x) &= \frac{1}{2}\left[\left(f\left(x+\frac{\delta}{2}\right) - f\left(x-\frac{\delta}{2}\right)\right) + i\left(f\left(x+i\frac{\delta}{2}\right) - f\left(x-i\frac{\delta}{2}\right)\right)\right]. \end{aligned}$$

For $f : \overline{V}_{G_\delta} \to \mathbb{C}$, with G_δ equal to Ω_δ or Ω_δ^*, we define $\nabla_\delta f : V_{G_\delta} \to \mathbb{C}^2$ by

$$\nabla_\delta f(v) = (f(v+\delta) - f(v), f(v+i\delta) - f(v)).$$

When G_δ is Ω_δ or Ω_δ^*, for a function $f : \overline{V}_{G_\delta} \to \mathbb{C}$, we define $\Delta_\delta f : V_{G_\delta} \to \mathbb{C}$ by

$$\Delta_\delta f(v) = \sum_{o \in \{\pm 1, \pm i\}} (f(v+o\delta) - f(v)).$$

In case of vertex ambiguities near the boundary, the vertex $v + o\delta$ designates the one which is adjacent to v. We will sometimes write $\Delta_\delta^{\bullet}$ for the Laplacian acting on functions $\overline{V}_{\Omega_\delta} \to \mathbb{C}$ and Δ_δ° for the one acting on functions $\overline{V}_{\Omega_\delta^*} \to \mathbb{C}$.

As $\delta \to 0$, with the following renormalizations, the discrete operators converge (in the sense of distributions) to their continuous versions:

$$\begin{aligned} \frac{1}{\delta}\partial_\delta &\underset{\delta\to 0}{\longrightarrow} \partial = \frac{1}{2}(\partial_x - i\partial_y), \\ \frac{1}{\delta}\overline{\partial}_\delta &\underset{\delta\to 0}{\longrightarrow} \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y), \\ \frac{1}{\delta^2}\Delta_\delta &\underset{\delta\to 0}{\longrightarrow} \Delta = (\partial_{xx} + \partial_{yy}), \\ \frac{1}{\delta}\nabla_\delta &\underset{\delta\to 0}{\longrightarrow} \nabla = (\partial_x, \partial_y). \end{aligned}$$

As for the continuous operators, we have

$$\Delta_\delta = 4\partial_\delta\bar\partial_\delta = 4\bar\partial_\delta\partial_\delta.$$

2.1.2. Discrete formulae. As in the continuum, the above-defined differential operators satisfy discrete integral formulae, which are very reminiscent of the classical integral formulae of vector calculus. We only state them in the contexts where we will use them. The following lemma is a discrete version of the integral formula

$$\iint_\Omega \Delta f = \int_{\partial\Omega} \partial_n f,$$

where n denotes the outward-pointing normal vector, for differentiable domains and functions.

Lemma 6. *For any function* $f : \overline{V}_{\Omega_\delta} \to \mathbb{C}$, *we have*

$$\sum_{v\in V_{\Omega_\delta}} \Delta^\bullet_\delta f(v) = \sum_{\vec{n}\in\partial_{ext}\vec{E}_{\Omega_\delta}} \partial_{\vec{n}} f.$$

Proof. If Ω_δ consists of a single vertex, this is by definition. It suffices to notice that the right hand-side is additive, in the sense that for $f : \overline{V}_{\Omega_\delta\cup\tilde\Omega_\delta} \to \mathbb{C}$, with $\Omega_\delta \cap \tilde\Omega_\delta = \varnothing$ we have

$$\sum_{\vec{n}\in\partial_{ext}\vec{E}_{\Omega_\delta}} \partial_{\vec{n}} f + \sum_{\vec{n}\in\partial_{ext}\vec{E}_{\tilde\Omega_\delta}} \partial_{\vec{n}} f = \sum_{\vec{n}\in\partial_{ext}\vec{E}_{\Omega_\delta\cup\tilde\Omega_\delta}} \partial_{\vec{n}} f,$$

which follows from the fact that the inner contributions come with opposite signs. □

The following lemma is a discrete version of the integral variant of Green-Riemann's formula

$$\frac{1}{2i}\cdot\iint_\Omega \bar\partial f = \oint_{\partial\Omega} f.$$

for differentiable domains and functions.

Lemma 7. *For any function* $f : V_{\Omega^m_\delta} \to \mathbb{C}$, *we have*

$$\frac{1}{2i}\cdot\left(\sum_{v\in V_{\Omega^{m*}_\delta}} \bar\partial_\delta f(v)\right) = \sum_{\vec{xy}\in\partial_0\vec{E}_{\Omega^m_\delta}} \frac{f(x)+f(y)}{2}(y-x),$$

where $\partial_0\vec{E}_{\Omega^m_\delta}$ *is the set of edges of* $E_{\Omega^m_\delta}$ *between the midpoints of the edges in* $\partial_0 E_{\Omega_\delta}$ *and those of* $\partial E_{\Omega_\delta}$, *oriented counterclockwise on the outer component of* $\partial_0\Omega^m_\delta$ *and clockwise on the inner components.*

Proof. For Ω_δ consisting of a single square face, this is a straightforward. We have that the right hand-side is additive, since inner contributions cancel, and the result follows. □

Let us now give the following useful discrete integration by parts lemma:

Lemma 8. For any two functions $f : V_{\Omega_\delta^m} \to \mathbb{C}$ and $g : \overline{V}_{\Omega_\delta} \cup \overline{V}_{\Omega_\delta^*} \to \mathbb{C}$, we have

$$\sum_{x \in V_{\Omega_\delta^m}} f(x) \cdot \left(\overline{\partial}_\delta g\right)(x) = -\sum_{y \in V_{\Omega_\delta} \cup V_{\Omega_\delta^*}} \left(\partial_\delta f\right)(y) \cdot g(y) + \frac{1}{2i} \sum_{\vec{e} \in \partial_0 \vec{E}_{\Omega_\delta}} f(m(\vec{e})) \cdot g\left(m(\vec{e}) + \frac{i\vec{e}}{2}\right) \cdot \vec{e} + \frac{1}{2i} \sum_{\vec{e} \in \partial \vec{E}_{\Omega_\delta^*}} f(m(\vec{e})) \cdot g\left(m(\vec{e}) + \frac{i\vec{e}}{2}\right) \cdot \vec{e},$$

where $\partial_0 \vec{E}_{\Omega_\delta}$ denotes the set of edges of $\partial_0 \Omega_\delta$ and $\partial \vec{E}_{\Omega_\delta^*}$ the set of edges of $\partial \Omega_\delta^*$, both oriented counterclockwise on the outer component of $\partial_0 \Omega_\delta$ and $\partial \Omega_\delta^*$ and oriented clockwise on the inner ones.

Proof. This follows from a straightforward computation. □

2.2. Discrete harmonicity and holomorphicity

In this section, we introduce the types of functions defined on graphs that will play a role in this text:

- The discrete harmonic, subharmonic and superharmonic functions.
- The discrete holomorphic functions, which are a particular type of harmonic functions.
- The s-holomorphic functions, which are a particular type of discrete holomorphic functions.

We then introduce the notion of discrete singularities, which as in the continuum, can basically be represented as defects of discrete harmonicity, discrete holomorphicity or s-holomorphicity.

2.2.1. Discrete harmonicity.

Discrete harmonicity can be defined in terms of the discrete operator Δ_δ:

Definition 9. Let G_δ be Ω_δ, Ω_δ^*, Ω_δ^m or Ω_δ^{m*}. We call a function $f : \overline{V}_{G_\delta} \to \mathbb{C}$

- *discrete harmonic* on G_δ if $\Delta_\delta f(v) = 0$ for each $v \in V_{G_\delta}$,
- *discrete subharmonic* on G_δ if $\Delta_\delta f(v) \geq 0$ for each $v \in V_{G_\delta}$,
- *discrete superharmonic* on G_δ if $\Delta_\delta f(v) \leq 0$ for each $v \in V_{G_\delta}$.

As in the continuum, it is easy to see that discrete subharmonic and superharmonic functions satisfy the maximum and minumum principle respectively: a subharmonic (respectively superharmonic) function reaches its maximum (respectively minimum) on the boundary of the discrete domain.

2.2.2. Discrete holomorphicity.

Discrete holomorphicity can in turn be defined in terms of $\overline{\partial}_\delta$:

Definition 10. Let G_δ be either Ω_δ^m or Ω_δ^{m*}. We call a function $f : \overline{V}_{G_\delta} \to \mathbb{C}$ *discrete holomorphic* of *discrete analytic* if $\overline{\partial}_\delta f(x) = 0$ for each $x \in V_{G_\delta^*}$. We call the equation $\overline{\partial}_\delta f(x) = 0$ the *discrete Cauchy-Riemann equation*.

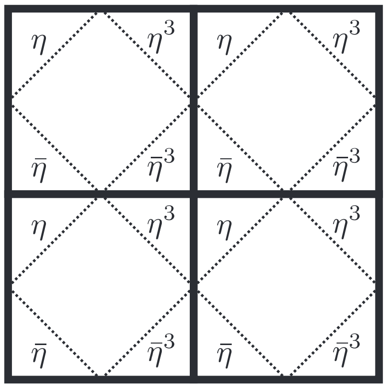

Figure 2.2.1. Two faces of $\mathbb{C}_\delta$, with the medial edges drawn with dotted strokes, and the lines associated with them.

In particular, with these definitions, a discrete holomorphic function is discrete harmonic. Conversely if G_δ is either Ω_δ or Ω_δ^*, and if $f : \overline{V}_{G_\delta} \to \mathbb{C}$ is a discrete harmonic function, then f can be locally extended to a discrete holomorphic function $f : \overline{V}_{G_\delta^{m*}} \to \mathbb{C}$ by solving the equation

$$\overline{\partial}_\delta f(x) = 0 \ \forall x \in V_{G_\delta^m}.$$

If G_δ is simply connected, a global extension always exists and is unique up to a constant.

2.2.3. S-holomorphicity. We now turn to the notion of s-holomorphicity (for *spin* holomorphicity) which is central in this text. Unlike the ones of discrete harmonicity and holomorphicity, we do not define it in terms of differential operators: this notion is not $\mathbb{C}$-linear. Also, we will only consider s-holomorphic functions defined on the vertices of the medial graph Ω_δ^m. Recall that with each medial edge $e \in E_{\Omega_\delta^m}$, we associate a complex line $\ell(e) \subset \mathbb{C}$ defined by

$$\ell(e) = (m(e) - c(e))^{-\frac{1}{2}} \mathbb{R},$$

where $m(e)$ is the middlepoint of e and $c(e) \in V_{\Omega_\delta}$ is the opposite vertex of the corner at e, that is, the vertex of Ω_δ that is the closest to e. Hence the different lines associated with medial edges on the square lattice are $\eta, \overline{\eta}, \eta^3, \overline{\eta}^3$.

Definition 11. We call a function $f : \overline{V}_{\Omega_\delta^m} \to \mathbb{C}$ *s-holomorphic* if for each medial edge $e \in E_{\Omega_\delta^m}$ with endpoints $x, y \in V_{\Omega_\delta^m}$, we have

$$\mathrm{P}_{\ell(e)}[f(x)] = \mathrm{P}_{\ell(e)}[f(y)],$$

where P_ℓ denotes the orthogonal projection on ℓ in $\mathbb{C}$.

We have that s-holomorphicity implies discrete holomorphicity (it is easy to check that the converse is not true):

Lemma 12. *If $f : \overline{V}_{\Omega_\delta^m} \to \mathbb{C}$ is s-holomorphic, then it is also discrete holomorphic.*

Proof. Let $v \in V_{\Omega_\delta^{m*}}$ be a vertex. We want to show that $\bar\partial_\delta f(v) = 0$. Let us suppose that $v \in V_{\Omega_\delta}$, the other case $v \in V_{\Omega_\delta^*}$ is identical. Denote by $v_1, v_i, v_{-1}, v_{-i} \in V_{\Omega_\delta^m}$ the four medial vertices $v + \frac{\delta}{2}, v + i\frac{\delta}{2}, v - \frac{\delta}{2}, v - i\frac{\delta}{2}$ and by $e_\lambda, e_{-\bar\lambda}, e_{-\lambda}, e_{\bar\lambda} \in E_{\Omega_\delta^m}$ the four medial edges $\langle v_1, v_i\rangle, \langle v_i, v_{-1}\rangle, \langle v_{-1}, v_{-i}\rangle, \langle v_{-i}, v_1\rangle$. By definition, we have

$$\begin{aligned} P_{\bar\eta\mathbb{R}}[f(v_1)] &= P_{\bar\eta\mathbb{R}}[f(v_i)], \\ P_{\bar\eta^3\mathbb{R}}[f(v_i)] &= P_{\bar\eta^3\mathbb{R}}[f(v_{-1})], \\ P_{\eta^3\mathbb{R}}[f(v_{-1})] &= P_{\eta^3\mathbb{R}}[f(v_{-i})], \\ P_{\eta\mathbb{R}}[f(v_{-i})] &= P_{\eta\mathbb{R}}[f(v_1)]. \end{aligned}$$

Rewriting this, we have

$$\begin{aligned} f(v_1) + \overline{\lambda f(v_1)} &= f(v_i) + \overline{\lambda f(v_i)}, \\ f(v_i) - \lambda\overline{f(v_i)} &= f(v_{-1}) - \lambda\overline{f(v_{-1})}, \\ f(v_{-1}) - \overline{\lambda f(v_{-1})} &= f(v_{-i}) - \overline{\lambda f(v_{-i})}, \\ f(v_{-i}) + \lambda\overline{f(v_{-i})} &= f(v_1) + \lambda\overline{f(v_1)}. \end{aligned}$$

Multiplying the first equation by $\frac{\lambda}{\sqrt{2}}$, the second by $\frac{\bar\lambda}{\sqrt{2}}$, the third by $-\frac{\lambda}{\sqrt{2}}$ and the fourth by $-\frac{\bar\lambda}{\sqrt{2}}$, and summing, we obtain

$$0 = i\left(f(v_1) - f(v_{-1})\right) - \left(f(v_i) - f(v_{-i})\right) = i \cdot \bar\partial_\delta f(v),$$

which is the desired result. □

2.2.4. Discrete singularities. An important aspect of discrete complex analysis is the one of singularities, which arise as defects of discrete harmonicity, analyticity or s-holomorphicity:

Definition 13. Let G_δ be Ω_δ, Ω_δ^*, Ω_δ^m or Ω_δ^{m*}, $f : V_{G_\delta} \to \mathbb{C}$ be a function and $a \in V_{G_\delta}$. We say that f has a discrete Δ_δ-singularity at a if $\Delta_\delta f(a) \neq 0$. Similarly, we say that f has a discrete $\bar\partial_\delta$-singularity at a if $\bar\partial_\delta f(a) \neq 0$.

For s-holomorphic functions, we define the notion of simple pole as follows:

Definition 14. Let $a^o \in V_{\Omega_\delta^m}$ be an oriented medial vertex and $f : V_{\Omega_\delta^m} \setminus \{a\} \to \mathbb{C}$ be an s-holomorphic function. We say that f has a simple pole at a^o, if the two complex numbers $f\left(a^o_+\right)$ and $f\left(a^o_-\right)$ differ (which we respectively call **front** and **rear values**), where $f\left(a^o_+\right)$ and $f\left(a^o_-\right)$ are defined by

$$\begin{aligned} P_{\langle a, a_{o\lambda}\rangle} f\left(a^o_+\right) &= P_{\langle a, a_{o\lambda}\rangle} f\left(a_{o\lambda}\right) \\ P_{\langle a, a_{o\bar\lambda}\rangle} f\left(a^o_+\right) &= P_{\langle a, a_{o\bar\lambda}\rangle} f\left(a_{o\bar\lambda}\right) \\ P_{\langle a, a_{-o\lambda}\rangle} f\left(a^o_-\right) &= P_{\langle a, a_{-o\lambda}\rangle} f\left(a_{-o\lambda}\right) \\ P_{\langle a, a_{-o\bar\lambda}\rangle} f\left(a^o_-\right) &= P_{\langle a, a_{-o\bar\lambda}\rangle} f\left(a_{-o\bar\lambda}\right) \end{aligned}$$

where a_x denotes $a + \frac{\sqrt{2}}{2}x\delta$. We call **discrete residue** at a^o the quantity $o \cdot \left(f\left(a^o_+\right) - f\left(a^o_-\right)\right) \in \mathbb{C}$.

The following lemma justifies the definitions of discrete simple pole and residue:

Lemma 15. Let $a^o \in D_{\Omega_\delta^m}$ be a doubly-oriented medial vertex. Let $f : V_{\Omega_\delta^m \setminus \{a\}} \to \mathbb{C}$ be an s-holomorphic function with a simple pole with residue ρ at a^o. If we extend f to $V_{\Omega_\delta^m}$ by setting $f(a) = \tau$, for any $\tau \in \mathbb{C}$, we have

$$\begin{aligned} \left[\overline{\partial}_\delta f\right]\left(a + \frac{o\delta}{2}\right) + \left[\overline{\partial}_\delta f\right]\left(a - \frac{o\delta}{2}\right) &= \rho, \\ \left[\overline{\partial}_\delta f\right]\left(a + i\frac{o\delta}{2}\right) + \left[\overline{\partial}_\delta f\right]\left(a - i\frac{o\delta}{2}\right) &= 0. \end{aligned}$$

For any simple contractible counterclockwise $\gamma \subset E_{\Omega_\delta^m \setminus \{a\}}$ path winding around a, we have

$$\sum_{\overline{xy} \in \gamma} \frac{f(x) + f(y)}{2} \cdot (y - x).$$

Proof. A straightforward computation, similar to the one of the proof Lemma 12, gives the first assertion. The second one follows from Lemma 7. □

2.3. Green's functions

A central tool in the discrete complex analysis theory that we will need is the one of discrete Green's functions for the $\overline{\partial}_\delta$ and Δ_δ operators. The theory of such functions is well developed and will be of great usefulness for our purpose, mostly thanks to the following two features:

- They are explicitly computable: we can explicitly obtain the value of these functions at any given point.
- Their convergence is known: as the mesh size δ goes to zero, these functions converge to their continuous counterparts.

The latter feature will be discussed in Chapter 3.

2.3.1. Dirac Green's function.

Green's functions for the discrete Dirac's operator $\overline{\partial}_\delta$ play different important roles in this text. We present in this subsection the usual version, which is discrete holomorphic except at a dual vertex, that allows to formulate a discrete analogue of Cauchy's formula. In the next section, we will present an s-holomorphic version of this Green's function, which is crucial for the analysis of the fermionic observables.

Theorem 16. Let $a \in V_{\mathbb{C}_\delta}$ be a vertex. There exists a unique function $V_{\mathbb{C}_\delta^m} \to \mathbb{C}$, which we denote by $G^{\overline{\partial}_\delta}(a,\cdot)$ and call the discrete Green's function for the $\overline{\partial}_\delta$ operator such that

$$\begin{aligned} \left[\overline{\partial}_\delta G^{\overline{\partial}_\delta}(a,\cdot)\right](v) &= 0 \ \forall v \in V_{\mathbb{C}_\delta^{m*}} \setminus \{a_1\}. \\ \left[\overline{\partial}_\delta G^{\overline{\partial}_\delta}(a,\cdot)\right](a) &= 1 \\ G^{\overline{\partial}_\delta}(a,b) &\underset{b\to\infty}{\longrightarrow} 0 \\ G^{\overline{\partial}_\delta}(a,\cdot)|_{V^h_{\mathbb{C}_\delta^m}} &\in \mathbb{R} \\ G^{\overline{\partial}_\delta}(a,\cdot)|_{V^v_{\mathbb{C}_\delta^m}} &\in i\mathbb{R} \end{aligned}$$

Proof. This follows directly from [Ken00], where the $\overline{\partial}_\delta$ Green's function is denoted by C_0 (and whose normalization differ by a factor of 2). □

For $v \in V_{C_\delta^*}$ we define $G^{\overline{\partial}_\delta}(v,\cdot) : V_{C_\delta^m} \to \mathbb{C}$ by translation, by setting $G^{\overline{\partial}_\delta}(v,w) = G^{\overline{\partial}_\delta}\left(v - \frac{1+i}{2}\delta, w - \frac{1+i}{2}\delta\right)$ for each $w \in V_{C_\delta^m}$. We define also $G^{\overline{\partial}_\delta}(x,\cdot) : V_{\Omega_\delta^{m*}} \to \mathbb{C}$ for $x \in V_{C_\delta^m}$ by translation, by setting $G^{\overline{\partial}_\delta}(x,\cdot) = G^{\overline{\partial}_\delta}\left(x - \frac{\delta}{2}, \cdot - \frac{\delta}{2}\right)$ for $x \in V^h_{\Omega_\delta^m}$ and $G^{\overline{\partial}_\delta}(y,\cdot) = G^{\overline{\partial}_\delta}\left(y - \frac{i\delta}{2}, \cdot - \frac{i\delta}{2}\right)$ for $y \in V^v_{\Omega_\delta^m}$.

The explicit values of the Green's function can be explicitly computed. Let us give the values that we will use:

Proposition 17. *Near the point a, the function $G^{\overline{\partial}_\delta}$ takes the following values:*

$$\begin{aligned}
G^{\overline{\partial}_\delta}\left(a, a + \frac{\delta}{2}\right) &= -G^{\overline{\partial}_\delta}\left(a, a - \frac{\delta}{2}\right) = \frac{1}{2} \\
G^{\overline{\partial}_\delta}\left(a, a + i\frac{\delta}{2}\right) &= -G^{\overline{\partial}_\delta}\left(a, a - i\frac{\delta}{2}\right) = -\frac{i}{2} \\
G^{\overline{\partial}_\delta}\left(a, a + \left(1 + \frac{i}{2}\right)\delta\right) &= G^{\overline{\partial}_\delta}\left(a, a - \left(1 - \frac{i}{2}\right)\delta\right) = \frac{i}{2} - \frac{2i}{\pi} \\
G^{\overline{\partial}_\delta}\left(a, a + \left(1 - \frac{i}{2}\right)\delta\right) &= G^{\overline{\partial}_\delta}\left(a, a - \left(1 + \frac{i}{2}\right)\delta\right) = \frac{2i}{\pi} - \frac{i}{2} \\
G^{\overline{\partial}_\delta}\left(a, a + \left(\frac{1}{2} + i\right)\delta\right) &= G^{\overline{\partial}_\delta}\left(a, a + \left(\frac{1}{2} - i\right)\delta\right) = \frac{2}{\pi} - \frac{1}{2} \\
G^{\overline{\partial}_\delta}\left(a, a - \left(\frac{1}{2} + i\right)\delta\right) &= G^{\overline{\partial}_\delta}\left(a, a - \left(\frac{1}{2} - i\right)\delta\right) = \frac{1}{2} - \frac{2}{\pi}.
\end{aligned}$$

Proof. This follows from [Ken00]. The normalization of the function C_0 defined there differs from the one of $G^{\overline{\partial}_\delta}$ by a factor of 2, namely we have $G^{\overline{\partial}_\delta}(\cdot,\cdot) = 2C_0$. The computation of the values is made in Figure 6 there (the other values that we need can be computed by symmetry). ∎

This $\overline{\partial}_\delta$-Green's function allows us to formulate a discrete version of Cauchy's formula.

Proposition 18. *Let Ω_δ be a simply connected discrete domain and $f : V_{\Omega_\delta^m} \to \mathbb{C}$ a discrete holomorphic function. Then for each $v \in V_{\Omega_\delta^m}$, we have*

$$\begin{aligned}
f(v) = \frac{1}{2i}\Bigg[&\sum_{\vec{e} \in \partial_0 \vec{E}_{\Omega_\delta}} f(m(\vec{e})) \cdot G^{\overline{\partial}_\delta}\left(v, m(\vec{e}) + \frac{i\vec{e}}{2}\right) \cdot \vec{e} \\
+ &\sum_{\vec{e} \in \partial \vec{E}_{\Omega_\delta^*}} f(m(\vec{e})) \cdot G^{\overline{\partial}_\delta}\left(v, m(\vec{e}) + \frac{i\vec{e}}{2}\right) \cdot \vec{e}\Bigg],
\end{aligned}$$

where $\partial_0 \vec{E}_{\Omega_\delta}$ and $\partial \vec{E}_{\Omega_\delta^*}$ denote the sets of edges of $\partial_0\Omega_\delta$ and $\partial\Omega_\delta^*$, oriented in counterclockwise direction on the outer component of $\partial_0\Omega_\delta$ and $\partial\Omega_\delta^*$ and in clockwise direction on the inner ones.

Proof. For each $v \in V_{\Omega_\delta^m}$, by definition of $G^{\overline{\partial}_\delta}(\cdot,\cdot)$, by discrete integration by parts (Lemma 8) and by the discrete Cauchy-Riemann equation, we have

$$\begin{aligned} f(v) &= \sum_{x \in V_{\Omega_\delta^m}} f(x) \cdot \left[\overline{\partial}_\delta G^{\overline{\partial}_\delta}(v,\cdot) \right](x) \\ &= \frac{1}{2i} \left[\sum_{e \in \partial_0 E_{\Omega_\delta}} f(m(e)) \cdot G^{\overline{\partial}_\delta}\left(v, m(e) + \frac{ie}{2} \right) \cdot e \right. \\ &\quad \left. + \sum_{e \in \partial E_{\Omega_\delta^*}} f(m(e)) \cdot G^{\overline{\partial}_\delta}\left(v, m(e) + \frac{ie}{2} \right) \cdot e \right], \end{aligned}$$

which shows the result. ∎

2.3.2. Laplace Green's function. Another important Green's function for us is the one of the Δ_δ operator. We only consider here the Dirichlet Green's function in bounded domains:

Theorem 19. *Let G_δ be Ω_δ or Ω_δ^*. Then there exists a unique function $G_{G_\delta}^{\Delta_\delta} : \overline{V}_{G_\delta} \times \overline{V}_{G_\delta} \to \mathbb{R}$ such that for each $a_1 \in V_{G_\delta}$ the following properties are satisfied:*

- *The function $a_2 \mapsto G_{G_\delta}^{\Delta_\delta}(a_1, a_2)$ is harmonic on $G_\delta \setminus \{a_1\}$.*
- *For each $a_2 \in \partial G_\delta$, we have $G_{G_\delta}^{\Delta_\delta}(a_1, a_2) = 0$.*

Using the Laplacian Green's function, we can readily solve the discrete Poisson equation $\Delta_\delta u = f$ with Dirichlet boundary values:

Proposition 20. *Let G_δ be Ω_δ or Ω_δ^*. The solution to the discrete boundary value problem*

$$\begin{aligned} \Delta_\delta u &= f \\ u|_{\partial V_{G_\delta}} &= 0, \end{aligned}$$

where $f : V_{G_\delta} \to \mathbb{R}$ is the data and $u : \overline{V}_{G_\delta} \to \mathbb{R}$ is the unknown is given by

$$u(v) = \sum_{w \in V_{G_\delta}} G_{G_\delta}^{\Delta_\delta}(w, v) \cdot f(w).$$

2.4. Full-plane fermionic observables

In this section, we introduce an s-holomorphic version of the Green's function for the $\overline{\partial}_\delta$. The role of this function is extremely important in this text, since it possesses a direct physical interpretation, beyond being a fundamental tool to establish later the convergence of the observables and hence of the scaling limit of the correlation functions: it corresponds informally to the infinite-volume limit of the two-point fermionic observable. We do not make rigorous or precise sense of this fact, since we never use it directly, but this consideration plays an important motivation role for our strategy and justifies the notation.

Definition 21. Let $a_1^{o_1} \in D_{\mathbb{C}_\delta^m}$ be a doubly-oriented medial vertex. We call *full-plane discrete complex fermionic observable* the function $h_{\mathbb{C}_\delta}(a_1^{o_1}, \cdot) : V_{\mathbb{C}_\delta^m \setminus \{a_1\}} \to$

$\mathbb{C}$ defined by

$$\begin{aligned} h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2) &= \eta \cdot \sqrt{\overline{o_1}} \cdot \cos\left(\frac{\pi}{8}\right)\left(G^{\overline{\partial}_\delta}\left(a_1 + \frac{o_1\delta}{2}, a_2\right) + G^{\overline{\partial}_\delta}\left(a_1 - \frac{io_1\delta}{2}, a_2\right)\right) \\ &\quad - i \cdot \eta \cdot \sqrt{\overline{o_1}} \cdot \sin\left(\frac{\pi}{8}\right)\left(G^{\overline{\partial}_\delta}\left(a_1 - \frac{o_1\delta}{2}, a_2\right) + G^{\overline{\partial}_\delta}\left(a_1 + \frac{io_1\delta}{2}, a_2\right)\right), \end{aligned}$$

where $G^{\overline{\partial}_\delta}$ is the discrete $\overline{\partial}_\delta$-Green's function defined in Theorem 16.

The following important properties of $h_{\mathbb{C}_\delta}$ (that characterize it) are the following:

Proposition 22. Let $a_1^{o_1} \in \mathcal{D}_{\mathbb{C}_\delta^m}$ be a doubly-oriented medial vertex. Then $h_{\mathbb{C}_\delta}(a_1^{o_1}, \cdot) : \mathcal{V}_{\Omega_\delta^m \setminus \{a_1\}} \to \mathbb{C}$ is the unique s-holomorphic function such that

- As $a_2 \to \infty$, we have $h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2) \to 0$.
- The function $h_{\mathbb{C}_\delta}(a_1^{o_1}, \cdot)$ has a discrete simple pole at $a_1^{o_1}$, of residue $\frac{1}{\sqrt{o_1}}$.

The two front anb rear values h_+ and h_- of $h_{\mathbb{C}_\delta}(a_1^{o_1}, \cdot)$ at a_1 are given by

$$\begin{aligned} h_+(a_1^{o_1}) &= \frac{1}{\sqrt{o_1}}\frac{1+\mu}{2}, \\ h_-(a_1^{o_1}) &= \frac{1}{\sqrt{o_1}}\frac{\mu-1}{2}, \end{aligned}$$

where $\mu = \frac{\sqrt{2}}{2}$.

Proof. Clearly, we have that $h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2)$ tends to 0 as $a_2 \to \infty$ and we readily obtain that it is uniquely determined by the two conditions above: the difference of two s-holomorphic functions $\mathcal{V}_{\mathbb{C}_\delta^m \setminus \{a_1\}} \to \mathbb{C}$ with these conditions extends to an s-holomorphic function $\mathcal{V}_{\mathbb{C}_\delta^m} \to \mathbb{C}$, which is harmonic (since it is in particular discrete holomorphic) and tends to 0 at infinity, and hence is identically equal to 0 (by maximum's principle). By the following lemma, $a_2 \mapsto h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2)$ is s-holomorphic:

Lemma 23. The functions

$$G_1 : a_2 \mapsto \eta \cdot \sqrt{\overline{o_1}}\left(G^{\overline{\partial}_\delta}\left(a_1 + \frac{o_1\delta}{2}, a_2\right) + G^{\overline{\partial}_\delta}\left(a_1 - \frac{io_1\delta}{2}, a_2\right)\right)$$

and

$$G_2 : a_2 \mapsto i \cdot \eta \cdot \sqrt{\overline{o_1}}\left(G^{\overline{\partial}_\delta}\left(a_1 - \frac{o_1\delta}{2}, a_2\right) + G^{\overline{\partial}_\delta}\left(a_1 + \frac{io_1\delta}{2}, a_2\right)\right)$$

are both s-holomorphic on $\mathcal{V}_{\mathbb{C}_\delta \setminus \{a_1\}}$.

Set $c = \cos(\pi/8)$ and $s = \sin(\pi/8)$. Once we have the lemma, it it suffices to check that the front and rear values of $h_{\mathbb{C}_\delta}$ are the ones claimed. Using translation invariance of $G^{\overline{\partial}_\delta}(\cdot,\cdot)$ and the exact values supplied by Proposition 17, a straightforward computation gives:

$$
\begin{aligned}
h_{\Omega_\delta}\left(a_1^{o_1}, a_1 + o_1\frac{1+i}{2}\delta\right) &= \sqrt{\frac{\eta}{o_1}}\left(c\left(\frac{2}{\pi} - \frac{1+i}{2}\right) - i\cdot s\cdot\left(-\frac{2i}{\pi} + \frac{1+i}{2}\right)\right)\\
h_{\Omega_\delta}\left(a_1^{o_1}, a_1 + o_1\frac{1-i}{2}\delta\right) &= \sqrt{\frac{\eta}{o_1}}\left(c\left(\frac{1+i}{2}\right) - i\cdot s\cdot\left(\frac{2i+2}{\pi} - \frac{1+i}{2}\right)\right)\\
h_{\Omega_\delta}\left(a_1^{o_1}, a_1 - o_1\frac{1-i}{2}\delta\right) &= \sqrt{\frac{\eta}{o_1}}\left(c\left(-\frac{2+2i}{\pi} + \frac{1+i}{2}\right) + i\cdot s\cdot\left(\frac{1+i}{2}\right)\right)\\
h_{\Omega_\delta}\left(a_1^{o_1}, a_1 - o_1\frac{1+i}{2}\delta\right) &= \sqrt{\frac{\eta}{o_1}}\left(c\left(\frac{2i}{\pi} - \frac{1+i}{2}\right) + i\cdot s\cdot\left(\frac{2}{\pi} - \frac{1+i}{2}\right)\right)
\end{aligned}
$$

Let us compute the projections on lines corresponding to the edges $\left[a_1, a_1 + o_1\frac{1+i}{2}\delta\right]$

$$
\begin{aligned}
P_{\sqrt{\frac{\bar\eta^3}{o_1}}\mathbb{R}}\left(h_{\Omega_\delta}\left(a_1^{o_1}, a_1 + o_1\frac{1+i}{2}\delta\right)\right) &= \sqrt{\frac{1}{o_1}}\cdot\frac{\bar\eta^3}{2}(c+s)\\
&= \sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\bar\eta^3}{2}}\cdot c\\
P_{\sqrt{\frac{\eta^3}{o_1}}\mathbb{R}}\left(h_{\Omega_\delta}\left(a_1^{o_1}, a_1 + o_1\frac{1-i}{2}\delta\right)\right) &= \sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\eta^3}{2}}\cdot c\\
P_{\sqrt{\frac{\bar\eta}{o_1}}\mathbb{R}}\left(h_{\Omega_\delta}\left(a_1^{o_1}, a_1 - o_1\frac{1-i}{2}\delta\right)\right) &= -\sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\bar\eta}{2}}\cdot s\\
P_{\sqrt{\frac{\eta}{o_1}}\mathbb{R}}\left(h_{\Omega_\delta}\left(a_1^{o_1}, a_1 - o_1\frac{1+i}{2}\delta\right)\right) &= -\sqrt{\frac{1}{o_1}}\cdot\frac{\eta}{2}(c-s)\\
&= -\sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\eta}{2}}\cdot s
\end{aligned}
$$

Another straightforward computation gives

$$
\begin{aligned}
P_{\sqrt{\frac{\bar\eta^3}{o_1}}\mathbb{R}}\left(\sqrt{\frac{1}{o_1}}\frac{1+\mu}{2}\right) &= \sqrt{\frac{1}{o_1}}\cdot\frac{\bar\eta^3}{2}\cdot c\\
P_{\sqrt{\frac{\eta^3}{o_1}}\mathbb{R}}\left(\sqrt{\frac{1}{o_1}}\frac{1+\mu}{2}\right) &= \sqrt{\frac{1}{o_1}}\cdot\frac{\eta^3}{2}\cdot c\\
P_{\sqrt{\frac{\bar\eta}{o_1}}\mathbb{R}}\left(\sqrt{\frac{1}{o_1}}\frac{\mu-1}{2}\right) &= -\sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\bar\eta}{2}}\cdot s\\
P_{\sqrt{\frac{\eta}{o_1}}\mathbb{R}}\left(\sqrt{\frac{1}{o_1}}\frac{\mu-1}{2}\right) &= -\sqrt{\frac{1}{o_1}}\cdot\sqrt{\frac{\eta}{2}}\cdot c
\end{aligned}
$$

and hence we obtain the desired result. □

Proof of Lemma 23. Fix a doubly-oriented medial vertex $a_1^{o_1}$. Let us use the translation invariance to rewrite these two functions

$$
\begin{aligned}
G_1(a_2) &= \eta\cdot\sqrt{o_1}\left(G^{\bar\partial_\delta}\left(a_1, a_2 - \frac{o_1\delta}{2}\right) + G^{\bar\partial_\delta}\left(a_1, a_2 + \frac{io_1\delta}{2}\right)\right)\\
G_2(a_2) &= i\cdot\eta\cdot\sqrt{o_1}\left(G^{\bar\partial_\delta}\left(a_1, a_2 + \frac{o_1\delta}{2}\right) + G^{\bar\partial_\delta}\left(a_1, a_2 - \frac{io_1\delta}{2}\right)\right),
\end{aligned}
$$

where on the right hand sides, the two $G^{\bar\partial_\delta}(a_1,\cdot)$ terms are orthogonal: one of the $G^{\bar\partial_\delta}(a_1,\cdot)$ terms is purely real and the other is purely imaginary.

Let $e \in E_{\Omega_\delta^m}$ be a medial edge and let $x \in V^h_{C_\delta^m \setminus \{a_1\}}, y \in V^v_{C_\delta^m \setminus \{a_1\}}$ be its endpoints with x being the horizontal and y the vertical one. Then there are four possibilities for the edge $\langle x, y\rangle \in E_{\Omega_\delta^m}$:

- If $\ell(e) = \eta \cdot \sqrt{o_1}\mathbb{R}$: we have that $x = y + \frac{1+i}{2} o_1 \delta$ and

$$
\begin{aligned}
\mathrm{P}_{\ell(e)}[G_1(x)] &= \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\partial_\delta}\left(a_1, x - \frac{o_1\delta}{2}\right) \\
&= \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\partial_\delta}\left(a_1, y + \frac{i o_1\delta}{2}\right) \\
&= \mathrm{P}_{\ell(e)}[G_1(y)]
\end{aligned}
$$

  and similarly

$$
\begin{aligned}
\mathrm{P}_{\ell(e)}[G_2(x)] &= i \cdot \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, x - \frac{i o_1\delta}{2}\right) \\
&= i \cdot \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, y + \frac{o_1\delta}{2}\right) \\
&= \mathrm{P}_{\ell(e)}[G_2(y)].
\end{aligned}
$$

- If $\ell(e) = \overline{\eta}^3 \cdot \sqrt{o_1}\mathbb{R}$: we have that $x = y - \frac{1+i}{2} o_1 \delta$ and

$$
\begin{aligned}
\mathrm{P}_{\ell(e)}[G_1(x)] &= \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, x + \frac{i o_1\delta}{2}\right) \\
&= \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, y - \frac{o_1\delta}{2}\right) \\
&= \mathrm{P}_{\ell(e)}[G_1(y)]
\end{aligned}
$$

  and similarly

$$
\begin{aligned}
\mathrm{P}_{\ell(e)}[G_2(x)] &= i \cdot \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, x + \frac{o_1\delta}{2}\right) \\
&= i \cdot \eta \cdot \sqrt{\overline{o_1}} \cdot G^{\overline{\partial}_\delta}\left(a_1, y - \frac{i o_1\delta}{2}\right).
\end{aligned}
$$

- If $\ell(e) = \overline{\eta} \cdot \sqrt{o_1}\mathbb{R}$: we have that $x = y + \frac{1-i}{2} o_1 \delta$ and

$$
\begin{aligned}
&\mathrm{P}_{\ell(e)}[G_1(x) - G_1(y)] \\
&= \frac{\overline{\eta}}{\sqrt{2}} \cdot \sqrt{\overline{o_1}} \left(G^{\overline{\partial}_\delta}\left(a_1, x - \frac{o_1\delta}{2}\right) + i G^{\overline{\partial}_\delta}\left(a_1, x + \frac{i o_1\delta}{2}\right) \right. \\
&\quad \left. - i G^{\overline{\partial}_\delta}\left(a_1, y - \frac{o_1\delta}{2}\right) - G^{\overline{\partial}_\delta}\left(a_1, y + \frac{i o_1\delta}{2}\right)\right) \\
&= \frac{\overline{\eta}}{\sqrt{2}} \cdot \sqrt{\overline{o_1}} \cdot i \left(\partial_\delta G^{\overline{\partial}_\delta}(a_1, \cdot)\right)(y) \\
&= 0.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned} & \mathbb{P}_{\tau(e)}\left[G_2(x)-G_2(y)\right] \\ = \; & \frac{\eta}{\sqrt{2}}\cdot\sqrt{o_1}\Big(-G^{\overline{\partial}_\delta}\big(a_1,x+\frac{o_1\delta}{2}\big)+iG^{\overline{\partial}_\delta}\big(a_1,x-\frac{io_1\delta}{2}\big) \\ & +G^{\overline{\partial}_\delta}\big(a_1,y-\frac{io_1\delta}{2}\big)-iG^{\overline{\partial}_\delta}\big(a_1,y+\frac{o_1\delta}{2}\big)\Big) \\ = \; & -\frac{\eta}{\sqrt{2}}\cdot\sqrt{o_1}\big(\partial_\delta G^{\overline{\partial}_\delta}(a_1,\cdot)\big)(x) \\ = \; & 0. \end{aligned}$$

- If $\tau(e)=\eta^3\cdot\sqrt{o_1}\mathbb{R}$: we have that $x=y-\frac{1-i}{2}o_1\delta$ and

$$\begin{aligned} & \mathbb{P}_{\tau(e)}\left[G_1(x)-G_1(y)\right] \\ = \; & \frac{\eta^3}{\sqrt{2}}\cdot\sqrt{o_1}\Big(G^{\overline{\partial}_\delta}\big(a_1,x-\frac{o_1\delta}{2}\big)-iG^{\overline{\partial}_\delta}\big(a_1,x+\frac{io_1\delta}{2}\big) \\ & +iG^{\overline{\partial}_\delta}\big(a_1,y-\frac{o_1\delta}{2}\big)-G^{\overline{\partial}_\delta}\big(a_1,y+\frac{io_1\delta}{2}\big)\Big) \\ = \; & -\frac{\eta^3}{\sqrt{2}}\cdot\sqrt{o_1}\big(\partial_\delta G^{\overline{\partial}_\delta}(a_1,\cdot)\big)(x) \end{aligned}$$

and similarly

$$\begin{aligned} & \mathbb{P}_{\tau(e)}\left[G_2(x)-G_2(y)\right] \\ = \; & \frac{\eta^3}{\sqrt{2}}\cdot\sqrt{o_1}\Big(-G^{\overline{\partial}_\delta}\big(a_1,x+\frac{o_1\delta}{2}\big)-iG^{\overline{\partial}_\delta}\big(a_1,x-\frac{io_1\delta}{2}\big) \\ & +G^{\overline{\partial}_\delta}\big(a_1,y-\frac{io_1\delta}{2}\big)+iG^{\overline{\partial}_\delta}\big(a_1,y+\frac{o_1\delta}{2}\big)\Big) \\ = \; & \frac{\eta^3}{\sqrt{2}}\cdot\sqrt{o_1}\cdot i\cdot\big(\partial_\delta G^{\overline{\partial}_\delta}(a_1,\cdot)\big)(y). \end{aligned}$$

This concludes the proof of the lemma. □

2.5. Discrete integration

Except in special cases, the product (or even the square) of discrete holomorphic (or s-holomorphic) functions is no longer discrete holomorphic. However, a specificity of s-holomorphic functions is that the (real part of the) antiderivative of the square of an s-holomorphic function can be defined, in the following way:

Proposition 24. Let $f: V_{\Omega_\delta^m}\to\mathbb{C}$ be an s-holomorphic function and $x\in V_{\Omega_\delta^{m*}}$. Then there exists a locally well-defined discrete analogue $I_{x,\delta}[f]:\overline{V}_{\Omega_\delta^{m*}}\to\mathbb{R}$ of the antiderivative

$$-\Re e\left(\int_x f^2\right),$$

obtained by integrating:

$$\begin{aligned} I_{x,\delta}[f](x) &= 0 \\ I_{x,\delta}[f](b)-I_{x,\delta}[f](w) &= \sqrt{2}\cdot\delta\cdot\left|\mathbb{P}_{\tau(e^*)}[f(x)]\right|^2 \\ &= \sqrt{2}\cdot\delta\cdot\left|\mathbb{P}_{\tau(e^*)}[f(y)]\right|^2, \end{aligned}$$

for each edge $e = \langle b, w\rangle \in \overline{E}_{\Omega_\delta^{m*}}$, with $b \in \overline{V}^{\bullet}_{\Omega_\delta^{m*}}$ and $w \in \overline{V}^{\circ}_{\Omega_\delta^{m*}}$, with $e^* = \langle x, y\rangle \in E_{\Omega_\delta^m}$ and with $\ell(e^*)$ denoting the complex line associated with e^*. The function $I_{x,\delta}[f]$ is globally well-defined if Ω_δ is simply connected.

When the choice of the vertex x is not relevant, we will merely write $I_\delta[f]$ for $I_{x,\delta}[f]$.

Proof. It is sufficient to check that for each $v \in V_{\Omega_\delta^m}$, if for $\rho \in \{\pm\lambda, \pm\bar{\lambda}\}$ if we set $v_\rho = v + \rho\frac{\delta}{\sqrt{2}} \in V_{\Omega_\delta^m}$ and $e_\rho = \langle v, v_\rho\rangle \in E_{\Omega_\delta^m}$, we have

$$\left|P_{\ell(e_\lambda)}[f(v)]\right|^2 + \left|P_{\ell(e_{-\lambda})}[f(v)]\right|^2 = \left|P_{\ell(e_{\bar{\lambda}})}[f(v)]\right|^2 + \left|P_{\ell(e_{-\bar{\lambda}})}[f(v)]\right|^2,$$

since it gives that the increment when going around v is zero, and hence shows local well-definedness. But this follows from the fact that for each $\rho \in \{\pm\lambda, \pm\bar{\lambda}\}$ the lines $\ell(e_\rho)$ and $\ell(e_{-\rho})$ are orthogonal and that hence $\left|P_{\ell(e_\rho)}[f(v)]\right|^2 + \left|P_{\ell(-e_\rho)}[f(v)]\right|^2 = |f(v)|^2$. □

The following elementary lemma that immediately follows from the definition of $I_\delta[\cdot]$ justifies the analogy with $-\mathrm{Re}\left[\int (\cdot)^2\right]$:

Lemma 25. Let $f : V_{\Omega_\delta} \to \mathbb{C}$ be an s-holomorphic function. Then for any edge $\langle b_1, b_2\rangle \in \overline{E}_{\Omega_\delta}$ with middlepoint $x \in V_{\Omega_\delta^m}$, we have

$$I_\delta[f](b_2) - I_\delta[f](b_1) = -\mathrm{Re}\left[f(x)^2 \cdot (b_2 - b_1)\right].$$

and any dual edge $\langle w_1, w_2\rangle \in \overline{E}_{\Omega_\delta^*}$ with middlepoint $y \in V_{\Omega_\delta^m}$, we have

$$I_\delta[f](w_2) - I_\delta[f](w_1) = -\mathrm{Re}\left[f(y)^2 \cdot (w_2 - w_1)\right].$$

The function $I_x[\cdot]$ is in general not harmonic:

Proposition 26. Let $f : V_{\Omega_\delta^m} \to \mathbb{C}$ be an s-holomorphic function. Then the restrictions $I^\bullet_\delta[f] : \overline{V}_{\Omega_\delta} \to \mathbb{R}$ and $I^\circ_\delta[f] : \overline{V}_{\Omega_\delta^*} \to \mathbb{R}$ (identifying V_{Ω_δ} with $V^\bullet_{\Omega_\delta^m}$ and $V_{\Omega_\delta^*}$ with $V^\circ_{\Omega_\delta^m}$) are subharmonic and superharmonic respectively: we have

$$\begin{aligned}
\Delta^\bullet_\delta I^\bullet_\delta[f](b) &= \sqrt{2}\cdot\delta\cdot|\partial_\delta f(b)|^2 \quad \forall b \in V_{\Omega_\delta},\\
\Delta^\circ_\delta I^\circ_\delta[f](w) &= -\sqrt{2}\cdot\delta\cdot|\partial_\delta f(w)|^2 \quad \forall w \in V_{\Omega_\delta^*}.
\end{aligned}$$

Proof. Let us first remark that for any complex number $\tau \in \mathbb{C}$, $f - \tau$ is still s-holomorphic and that $\partial_\delta(f - \tau)(x) = \partial_\delta f(x)$ for each $x \in V_{\Omega_\delta} \cup V_{\Omega_\delta^*}$. Let us show that $\Delta_\delta I_\delta[f - \tau](x) = \Delta_\delta I_\delta[f](x)$ for each $x \in V_{\Omega_\delta} \cup V_{\Omega_\delta^*}$. From Lemma 25, for each $x \in V_{\Omega_\delta} \cup V_{\Omega^*_\delta}$, we have

$$
\begin{aligned}
& \Delta_\delta I_\delta[f-\tau](x) \\
= \;& \mathrm{Re}\left[\left(f\left(x-\frac{\delta}{2}\right)-\tau\right)^2 - \left(f\left(x+\frac{\delta}{2}\right)-\tau\right)^2\right] \\
& + \mathrm{Re}\left[i\left(\left(f\left(x-i\frac{\delta}{2}\right)-\tau\right)^2 - \left(f\left(x+i\frac{\delta}{2}\right)-\tau\right)^2\right)\right] \\
= \;& \mathrm{Re}\left[f\left(x-\frac{\delta}{2}\right)^2 - f\left(x+\frac{\delta}{2}\right)^2 + if\left(x-i\frac{\delta}{2}\right)^2 - if\left(x+i\frac{\delta}{2}\right)^2\right] \\
& + \mathrm{Re}\left[4\tau\overline{\partial}f(x)\right] \\
= \;& \Delta_\delta I_\delta[f](x),
\end{aligned}
$$

where the last identity follows from the discrete Cauchy-Riemann equation. Let now $x \in V_{\Omega_\delta}$; to show that

$$
\Delta^\bullet_\delta I^\bullet_\delta[f](b) = \sqrt{2}\cdot\delta\cdot|\partial_\delta f(b)|^2,
$$

we may suppose that $f\left(b-\frac{\delta}{2}\right) = 0$, by substracting a constant to f if necessary. In that case, by the Cauchy-Riemann equation, we have $|\partial_\delta f(b)|^2 = \left|f\left(b+\frac{\delta}{2}\right)\right|^2$. By definition, we now have

$$
\begin{aligned}
\Delta^\bullet_\delta I^\bullet_\delta[f](b) \;=\;& -\frac{1}{2}\left(\mathbb{P}_{\eta R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 + \frac{1}{2}\left(\mathbb{P}_{\eta^3 R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 \\
& -\frac{1}{2}\left(\mathbb{P}_{\overline{\eta} R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 + \frac{1}{2}\left(\mathbb{P}_{\overline{\eta}^3 R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 \\
& -\left(\mathbb{P}_{\overline{\eta}^3 R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right)^2 + \left(\mathbb{P}_{\eta^3 R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right)^2 \\
& -\left(\mathbb{P}_{\eta^3 R}\left[f\left(b-i\frac{\delta}{2}\right)\right]\right)^2 + \left(\mathbb{P}_{\overline{\eta}^3}\left[f\left(b-i\frac{\delta}{2}\right)\right]\right)^2 \\
=\;& -\frac{1}{2}\left(\mathbb{P}_{\overline{\eta} R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right)^2 + \frac{1}{2}\left(\mathbb{P}_{\overline{\eta}^3 R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 \\
& -\frac{1}{2}\left(\mathbb{P}_{\eta R}\left[f\left(b-i\frac{\delta}{2}\right)\right]\right)^2 + \frac{1}{2}\left(\mathbb{P}_{\eta^3 R}\left[f\left(b+\frac{\delta}{2}\right)\right]\right)^2 \\
& +\left(\mathbb{P}_{\eta^3 R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right)^2 + \left(\mathbb{P}_{\overline{\eta}^3 R}\left[f\left(b-i\frac{\delta}{2}\right)\right]\right)^2,
\end{aligned}
$$

where we have used the s-holomorphicity and the relations $\mathbb{P}_{\overline{\eta}^3 R}\left[f\left(b+i\frac{\delta}{2}\right)\right] = 0$ and $\mathbb{P}_{\eta^3 R}\left[f\left(b-i\frac{\delta}{2}\right)\right] = 0$ that come from the s-holomorphicity and the fact that $f\left(b-\frac{\delta}{2}\right) = 0$. These relations give also $\left(\mathbb{P}_{\eta^3 R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right) = \left(\mathbb{P}_{\overline{\eta} R}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right)$

and $\mathbb{P}_{\bar\eta^3\mathbb{R}}\left[f\left(b-i\frac{\delta}{2}\right)\right] = \mathbb{P}_{\eta\mathbb{R}}\left[f\left(b-i\frac{\delta}{2}\right)\right]$. Hence we obtain

$$\begin{aligned}\Delta^{\bullet}_{\delta} I^{\bullet}_{\delta}[f](b) &= \frac{1}{2}\left|\mathbb{P}_{\bar\eta\mathbb{R}}\left[f\left(b+i\frac{\delta}{2}\right)\right]\right|^2 + \frac{1}{2}\left|\mathbb{P}_{\bar\eta^3\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2 \\ &\quad + \frac{1}{2}\left|\mathbb{P}_{\eta\mathbb{R}}\left[f\left(b-i\frac{\delta}{2}\right)\right]\right|^2 + \frac{1}{2}\left|\mathbb{P}_{\eta^3\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2 \\ &= \frac{1}{2}\left|\mathbb{P}_{\bar\eta\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2 + \frac{1}{2}\left|\mathbb{P}_{\bar\eta^3\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2 \\ &\quad + \frac{1}{2}\left|\mathbb{P}_{\eta\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2 + \frac{1}{2}\left|\mathbb{P}_{\eta^3\mathbb{R}}\left[f\left(b+\frac{\delta}{2}\right)\right]\right|^2,\end{aligned}$$

where we used once more s-holomorphicity. Using that for any $z \in \mathbb{C}$,

$$|z|^2 = \; = |\mathbb{P}_{\eta\mathbb{R}}[z]|^2 + \left|\mathbb{P}_{\eta^3\mathbb{R}}[z]\right|^2 = |\mathbb{P}_{\bar\eta\mathbb{R}}[z]|^2 + \left|\mathbb{P}_{\eta^3\mathbb{R}}[z]\right|^2,$$

we obtain

$$\begin{aligned}\Delta^{\bullet}_{\delta} I^{\bullet}_{\delta}[f](b) &= \left|f\left(b+\frac{\delta}{2}\right)\right|^2 \\ &= |(\partial_\delta f)(b)|^2,\end{aligned}$$

which is the desired result. Using exactly the same reasoning, we obtain $\Delta^{\circ}_{\delta} I^{\circ}_{\delta}[f](w) = -|(\partial_\delta f)(w)|^2$ for each $w \in V_{\Omega^{\circ}_{\delta}}$. ∎

2.6. Discrete Riemann-Hilbert boundary value problems

The purpose of this section is not to give general definitions or a general theory of discrete versions of Riemann-Hilbert boundary value problems, but rather to focus on the specific type of boundary value problems we will study. They are the tool that we use to represent the observables, to obtain relations between them and finally to pass to the scaling limit.

We will only give here the definition of these problems and a useful uniqueness result.

Definition 27. Let $u : V_{\Omega^m_\delta} \to \mathbb{C}$ be an s-holomorphic function. We say that u solves the inhomogeneous Riemann-Hilbert boundary value problem $(\spadesuit_{\Omega_\delta}, f)$ for a function $f : \partial_0 V_{\Omega^m_\delta} \to \mathbb{C}$ if for each $x \in \partial_0 V_{\Omega^m_\delta}$, we have

$$(u(x) - f(x)) \parallel \frac{1}{\sqrt{v_{ext}(x)}},$$

or equivalently

$$\mathbb{P}_{\frac{i}{\sqrt{v_{ext}(x)}}\mathbb{R}}[u(x) - f(x)] = 0.$$

Let us give the following proposition, that gives an a priori control on solutions to the problems $(\spadesuit_{\Omega_\delta}, f)$ will be very useful in this text:

Proposition 28. *If $u : V_{\Omega^m_\delta} \to \mathbb{C}$ solves the Riemann-Hilbert boundary value problem $(\spadesuit_{\Omega_\delta}, f)$, then we have*

$$\sum_{x \in \partial_0 V_{\Omega^m_\delta}} |u(x)|^2 \leq 2\left(\sum_{x \in \partial_0 V_{\Omega^m_\delta}} |f(x)|^2\right).$$

Proof. We can suppose, by replacing f by $f - \mathbb{P}_{\frac{1}{\sqrt{\nu_{ext}}}\mathbb{R}}[f]$ if necessary, that for each $x \in \partial_0 V_{\Omega_\delta^m}$, we have

$$f(v) = \mathbb{P}_{\frac{1}{\sqrt{-\nu_{ext}(x)}}\mathbb{R}}[u(x)],$$

so that we can write $u(x) = f(x) + g(x)$, where

$$g(x) = \mathbb{P}_{\frac{1}{\sqrt{\nu_{ext}(x)}}\mathbb{R}}[u(x)].$$

Consider a branch of the discrete antiderivative $I_\delta[u] : \overline{V}_{\Omega_\delta^{m*}} \to \mathbb{R}$ and its restriction $I_\delta^\bullet[u] : \overline{V}_{\Omega_\delta} \to \mathbb{R}$ to $\overline{V}_{\Omega_\delta}$. For each $x \in \partial_0 V_{\Omega_\delta^m}$, we have

$$\partial_{\nu_{ext}(x)} I_\delta^\bullet[u] = \left(\mathbb{P}_{\sqrt{\frac{\eta^3}{\nu_{ext}(x)}}\mathbb{R}}[u(x)]\right)^2 - \left(\mathbb{P}_{\sqrt{\frac{\eta}{\nu_{ext}(x)}}\mathbb{R}}[u(x)]\right)^2.$$

So, we obtain, using that $\Re\left(\overline{f(m)}g(m)\right) = 0$,

$$\begin{aligned}
\frac{1}{\delta}\partial_{\nu_{ext}(x)} I_\delta^\bullet[u] &= \left(\sqrt{2}\left(\cos\left(\frac{\pi}{8}\right) f(m) + \cos\left(\frac{3\pi}{8}\right) g(m)\right)\right)^2 \\
&\quad - \left(\sqrt{2}\left(\cos\left(\frac{3\pi}{8}\right) f(m) + \cos\left(\frac{\pi}{8}\right) g(m)\right)\right)^2 \\
&= |f(m)|^2 - |g(m)|^2.
\end{aligned}$$

Using that $I_x^\bullet[u]$ is subharmonic and Lemma 6 (which can be applied without problem even if $I_x[u]$ is not single-valued, since only the differences of values at adjacent vertices are used), we obtain

$$\begin{aligned}
0 &\leq \sum_{v \in V_{\Omega_\delta}} \Delta_\delta^\bullet I_\delta^\bullet[u] \\
&= \sum_{x \in \partial_0 V_{\Omega_\delta^m}} \partial_{\nu_{ext}(x)} I_\delta^\bullet[u] \\
&= \sum_{x \in \partial_0 V_{\Omega_\delta^m}} \left(|f(x)|^2 - |g(x)|^2\right),
\end{aligned}$$

and hence

$$\sum_{x \in \partial_0 V_{\Omega_\delta^m}} |g(x)|^2 \leq \sum_{x \in \partial_0 V_{\Omega_\delta^m}} |f(x)|^2,$$

Finally, using that for each $x \in \partial_0 V_{\Omega_\delta^m}$, $|u(x)|^2 = |f(x)|^2 + |g(x)|^2$, we obtain the result. □

This proposition gives in particular the following uniqueness result:

Corollary 29. *Any discrete Riemann-Hilbert boundary value problem $(\spadesuit_{\Omega_\delta}, f)$ has at most one solution.*

Proof. By linearity, the difference of two solutions solves the problem $(\spadesuit_{\Omega_\delta}, 0)$. From Proposition 28, the solution of $(\spadesuit_{\Omega_\delta}, 0)$ must be zero. □

Let us now give the following integral reformulation of the boundary condition $\square_{\frac{1}{\sqrt{\nu_{ext}}}}$ that is very useful, since it allows to translate a boundary condition which seems a priori badly suited for passing to the continuum limit (since the discrete normal vector only takes value in a discrete space) into a much more robust one:

Lemma 30. *Let $u : V_{\Omega_\delta^m} \to \mathbb{C}$ be an s-holomorphic function and let $I \subset \partial_0\Omega_\delta$ be a connected part of $\partial_0\Omega_\delta$ such that*

$$u(x) \parallel \frac{1}{\sqrt{\nu_{\mathrm{ext}}(x)}} \quad \forall x \in \partial_0 V_{\Omega_\delta^m} \cap I.$$

Then $I_\delta[u]$ is locally constant on $\partial_0 V_{\Omega_\delta^*} \cap I$. In particular, if $I = \partial_0\Omega_\delta^m$, then $I_\delta[u]$ is globally well-defined on $V_{\Omega_\delta^{m*}}$ and locally constant on $\partial_0 V_{\Omega_\delta^*}$.

For each $x \in \partial_0 V_{\Omega_\delta^m} \cap I$, we have

$$\partial_{\nu_{\mathrm{ext}}(x)} \left(I_\delta^\bullet[u]\right) = -|u(x)|^2.$$

Proof. It is straightforward to check from Lemma 25 and the boundary condition that for adjacent vertices $x, y \in \partial_0 V_{\Omega_\delta^*} \cap I$, $I_\delta^\circ[u](x) = I_\delta^\circ[u](y)$. If $I = \partial_0\Omega_\delta$, the global well-definedness follows from the fact that there is no monodromy of $I_\delta[u]$ around the components of $\partial\Omega_\delta$ and hence no monodromy at all. For the second property, we use that, as in the proof of Proposition 28, for each $x \in \partial_0 V_{\Omega_\delta^m}$, we have

$$\begin{aligned}\frac{1}{\delta}\partial_{\nu_{\mathrm{ext}}(x)} I_\delta^\bullet[u] &= \sqrt{2}\left(\mathbb{P}_{\sqrt{\frac{\eta^3}{\nu_{\mathrm{ext}}(x)}}\mathbb{R}}[u(x)]\right)^2 - \sqrt{2}\left(\mathbb{P}_{\sqrt{\frac{\eta}{\nu_{\mathrm{ext}}(x)}}\mathbb{R}}[u(x)]\right)^2 \\ &= -|u(x)|^2.\end{aligned}$$

∎

2.6.1. Boundary modification trick. If $u : V_{\Omega_\delta^m} \to \mathbb{C}$ is an s-holomorphic function with the boundary condition $\parallel \frac{1}{\sqrt{\nu_{\mathrm{ext}}}}$ on a part I of $\partial V_{\Omega_\delta^m}$, then we have seen in Lemma 30 that $I_\delta[u]$ is locally constant on $\partial_0 V_{\Omega_\delta^*} \cap I$, but not in principle on $\partial V_{\Omega_\delta} \cap I$. We can artificially set the value of $I_\delta^\bullet[u]$ on $\partial V_{\Omega_\delta} \cap I$ to be equal to the one on the dual vertices of $\partial_0 V_{\Omega_\delta^*} \cap I$. We denote by $\tilde{I}_\delta^\bullet[u]$ the function $\overline{V}_{\Omega_\delta} \to \mathbb{R}$ equal to $I_\delta^\bullet[u]$ on $\overline{V}_{\Omega_\delta} \setminus (\partial_1 V_{\Omega_\delta} \cap I)$ and where, for each $x \in \partial_1 V_{\Omega_\delta} \cap I$ adjacent to a dual vertex $y \in \partial_0 V_{\Omega_\delta^*} \cap I$, we set $\tilde{I}_\delta^\bullet[u](x) = I_\delta^\circ[u](y)$.

However this spoils the subharmonicity of $I_\delta^\bullet[u]$ established in Proposition 26. The **boundary modification trick**, introduced in [ChSm09] and used in the same way as in [DHN09] that we use to overcome this problem consists in defining a modified version $\tilde{\Delta}_\delta^\bullet$ of the Laplacian $\Delta_\delta^\bullet$ acting on functions $\overline{V}_{\Omega_\delta} \to \mathbb{R}$ by

$$\left(\tilde{\Delta}_\delta^\bullet f\right)(v) = \begin{cases} \sum_{y \in V_{\Omega_\delta} : y \sim x} \partial_{\vec{vy}} f + 2\alpha \sum_{x \in \partial V_{\Omega_\delta} : x \sim v} \partial_{\vec{vx}} f & \text{if } x \in \partial_0 V_{\Omega_\delta} \cap I, \\ \Delta_\delta f(v) & \text{else} \end{cases}$$

where $\alpha = \sqrt{2} - 1$ as usual. The following lemma justifies the introduction of $\tilde{\Delta}_\delta^\bullet$ and $\tilde{I}_\delta^\bullet[u]$:

Lemma 31. *For each $v \in V_{\Omega_\delta}$, we have that $\left(\tilde{\Delta}_\delta^\bullet \tilde{I}_\delta^\bullet[u]\right)(v) = (\Delta_\delta^\bullet I_\delta^\bullet[u])(v)$.*

Proof. Let $v \in \partial_0 V_{\Omega_\delta} \cap I$ (since otherwise there is nothing to check). For each $x \in \partial V_{\Omega_\delta}$ with $x \sim v$, we have

$$\begin{aligned}
\partial_{\vec{vx}} I^\bullet_\delta[u] &= \delta \cdot |u(m\langle vx\rangle)|^2 \\
\partial_{\vec{vx}} \tilde{I}^\bullet_\delta[u] &= \delta \cdot \left(\sqrt{2} \cdot \cos\frac{\pi}{8}\right)^2 |u(m\langle vx\rangle)|^2 \\
&= \frac{1}{2\alpha} \cdot \partial_{\vec{vx}} I^\bullet_\delta[u]
\end{aligned}$$

hence

$$\begin{aligned}
(\Delta^\bullet_\delta I^\bullet_\delta[u])(v) &= \sum_{y \in V_{\Omega_\delta} : y \sim v} \partial_{\vec{vy}} I^\bullet_\delta[u] + \sum_{x \in \partial V_{\Omega_\delta} : x \sim v} \partial_{\vec{vx}} I^\bullet_\delta[u] \\
&= \sum_{y \in V_{\Omega_\delta} : y \sim v} \partial_{\vec{vy}} \tilde{I}^\bullet_\delta[u] + 2\alpha \sum_{x \in \partial V_{\Omega_\delta} : x \sim v} \partial_{\vec{vx}} \tilde{I}^\bullet_\delta[u] \\
&= \left(\tilde{\Delta}^\bullet_\delta I^\bullet_\delta[u]\right)(v).
\end{aligned}$$

□

CHAPTER 3

Convergence Results for Discrete Functions

The main purpose of this chapter is to review and adapt to our case existing tools for showing the convergence of discrete functions to continuous ones that will be used in the next chapter to study the convergence of solutions of discrete Riemann-Hilbert boundary value problems. Most of these results come from [ChSm08, ChSm09, Ken00, Kes87].

3.1. Convergence of discrete harmonic and holomorphic functions

Let us first define what we mean by convergence of discrete domains and of functions defined on them to continuous ones.

Recall that we say that a family $(\Omega_\delta)_{\delta>0}$ is a discretization or an approximation of a domain Ω if for each $\delta > 0$, Ω_δ is the largest induced subgraph of C_δ contained in Ω.

For notation convenience reasons, we will extend naturally the functions defined on G_δ (with G_δ equal to Ω_δ, Ω_δ^*, Ω_δ^m, Ω_δ^{m*}) to functions defined on Ω, by defining the value at $x \in \Omega$ of a function $f_\delta : V_{G_\delta} \to C$ as $f_\delta(x_\delta)$, where x_δ is the closest vertex (or one of them if there are several) in V_{G_δ} to x. We will never use the precise definition of the extension, and it will never be used in any other way than to simplify notation.

For a function $h_\delta : S_{\Omega_\delta^m} \to C$ (respectively $D_{\Omega_\delta^m} \to C$), a lattice orientation $o \in S_\square$ (respectively in $(S)_\square^2$) and a point $x \in \Omega$, we define $h_\delta(x^o)$ as $h_\delta(x_\delta^{o_\delta})$, where x_δ is the closest medial vertex (or one of them if there are several) for which o is an admissible orientation.

Let us start by stating the classical result that ensures that the limits of discrete holomorphic functions are continuous holomorphic – the same is true for discrete harmonic functions.

Lemma 32. *Let $(\Omega_\delta)_{\delta>0}$ be a discretization of a domain Ω, let $(f_\delta : V_{\Omega_\delta} \to C)_{\delta>0}$ be a family of discrete holomorphic functions and let $f : \Omega \to C$ be a continuous function such that $f_\delta \to f$ uniformly on the compact subsets of Ω as $\delta \to 0$. Then f is holomorphic.*

Proof. We can use Morera's condition. Let $\gamma \subset \Omega$ be a simple contractible contour oriented counterclockwise and denote by $Y \subset \Omega$ the subdomain such that $\partial Y = \gamma$. Then, by approximating by Riemann sums, we have

$$\oint_\gamma f(z)\,dz = \lim_{\delta\to 0} \sum_{xy \in \partial_0 \vec{E}_{Y_\delta^m}} \frac{f(x) + f(y)}{2}(y - x),$$

where for each $\delta > 0$, $\partial_0 \vec{E}_{Y_\delta^m}$ denotes the set of edges of $E_{\Omega_\delta^m}$ between midpoints of $\partial_0 E_{Y_\delta^m}$ and midpoints of $\partial E_{Y_\delta^m}$, oriented counterclockwise. By the convergence assumption, we have

$$\lim_{\delta\to 0} \sum_{xy \in \partial_0 E_{\Upsilon^m_\delta}} \frac{f(x)+f(y)}{2}(y-x) = \lim_{\delta\to 0} \sum_{xy \in \partial_0 E_{\Upsilon^m_\delta}} \frac{f_\delta(x)+f_\delta(y)}{2}(y-x)$$

and by Lemma 7 and discrete holomorphicity, we have

$$\sum_{xy \in \partial_0 E_{\Upsilon^m_\delta}} \frac{f_\delta(x)+f_\delta(y)}{2}(y-x) = 0$$

for each $\delta > 0$, which shows the result. □

3.2. Green's functions and harmonic measure convergence

The convergence of discrete Green's functions and of harmonic measure is a classical subject, and a lot of results, especially on the square lattice, have been known since many years. Let us start with the convergence of the full-plane $\overline{\partial}_\delta$-Green's function.

Theorem 33. *Consider the full-plane Green's function $G^{\overline{\partial}_\delta}_{C^m_\delta}$. Then, there exists a universal constant $C > 0$ such that for each $\delta > 0$ and each $x \in V_{C_\delta}$, we have*

$$\left| \frac{1}{\delta} G^{\overline{\partial}_\delta}_{C^m_\delta}(x,y) - \mathrm{Re}\left(G^{\overline{\partial}}_{\mathbb{C}}(x,y) \right) \right| \leq C \cdot \frac{\delta}{|x-y|^2} \quad \forall y \in V^h_{C^m_\delta},$$

$$\left| \frac{1}{\delta} G^{\overline{\partial}_\delta}_{C^m_\delta}(x,y) - \mathrm{Im}\left(G^{\overline{\partial}}_{\mathbb{C}}(x,y) \right) \right| \leq C \cdot \frac{\delta}{|x-y|^2} \quad \forall y \in V^v_{C^m_\delta},$$

where $V^h_{C^m_\delta}$ and $V^v_{C^m_\delta}$ are the sets of horizontal and vertical medial vertices respectively, and where

$$G^{\overline{\partial}}_{\mathbb{C}}(x,y) = \frac{1}{\pi(y-x)}.$$

Proof. This follows from [Ken00], Theorem 11, by rescaling the lattice (which is there of fixed mesh size 1). □

Let us now give convergence results concerning Δ_δ-Green's function G^{Δ_δ} with Dirichlet boundary conditions. The main one, which is very classical, is the convergence of G^{Δ_δ} to the continuous Δ-Green's function G^{Δ}:

Proposition 34. *Let $(\Omega_\delta)_{\delta>0}$ be a family of discrete domains approximating a bounded domain Ω as $\delta \to 0$ and let $(G_\delta)_{\delta>0}$ denote either $(\Omega_\delta)_{\delta>0}$ or $(\Omega^*_\delta)_{\delta>0}$. Consider the Green's function $G^{\Delta_\delta}_{G_\delta} : V_{G_\delta} \times V_{G_\delta} \to \mathbb{R}$ with Dirichlet boundary conditions. Then there exists a universal constant $C > 0$ such that for any compact subset $K \subset \Omega$, there exists $\varepsilon_K(\delta) > 0$ with $\varepsilon_K(\delta) \underset{\delta\to 0}{\longrightarrow} 0$ such that*

$$\left| G^{\Delta_\delta}_{G_\delta}(x,y) - G^{\Delta}_{\Omega}(x,y) \right| \leq C \cdot \frac{\delta^2}{|x-y|^2} + \varepsilon_K(\delta) \quad \forall x, y \in V_{G_\delta} \cap K$$

$$\left| \left(\frac{1}{\delta} \nabla_\delta G^{\Delta_\delta}_{G_\delta}(x,\cdot) \right)(y) - \left(\nabla G^{\Delta}_{\Omega}(x,\cdot) \right)(y) \right| \leq C \cdot \frac{\delta}{|x-y|^2} + \varepsilon_K(\delta) \quad \forall x, y \in V_{G_\delta} \cap K.$$

where G^{Δ}_{Ω} is the continuous Dirichlet Δ-Green's function of Ω.

Proof. Suppose to simplify the notation that $(G_\delta)_{\delta>0} = (\Omega_\delta)_{\delta>0}$ (the other case is similar). Then we have that $G^{\Delta^\bullet_\delta}_{\Omega_\delta}(x,y) = G^{\Delta^\bullet_\delta}_{C_\delta}(x,y) - \tilde{G}_{\Omega_\delta}$, where $G^{\Delta^\bullet_\delta}_{C_\delta} : V_{C_\delta} \times V_{C_\delta} \to \mathbb{R}$ is the Green's function of the full-plane (see [ChSm08]), defined by

$$\left(\Delta^\bullet_\delta G^{\Delta^\bullet_\delta}_{C_\delta}(x,\cdot)\right)(y) = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

$$G^{\Delta^\bullet_\delta}_{C_\delta}(x,y) - \frac{1}{2\pi}\log|y-x| \underset{y\to\infty}{\longrightarrow} 0.$$

and where $\tilde{G}_{\Omega_\delta} : \overline{V}_{\Omega_\delta} \times \overline{V}_{\Omega_\delta} \to \mathbb{R}$ is such that

$$\begin{aligned}\left(\Delta^\bullet_\delta \tilde{G}_{\Omega_\delta}(x,\cdot)\right)(y) &= 0 \;\forall x \in \overline{V}_{\Omega_\delta}, \forall y \in V_{\Omega_\delta} \\ \tilde{G}_{\Omega_\delta}(x,y) &= G^{\Delta^\bullet_\delta}_{C_\delta}(x,y) \;\forall x \in \overline{V}_{\Omega_\delta}, \forall y \in \partial V_{\Omega_\delta}.\end{aligned}$$

Similarly, we have that

$$G^{\Delta}_{\Omega}(x,y) = G^{\Delta}_{C}(x,y) - G^{*}_{\Omega}(x,y),$$

where

$$G^{\Delta}_{C}(x,y) = \frac{1}{2\pi}\log|y-x|.$$

and $\tilde{G}_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ is defined by

$$\begin{aligned}\left(\Delta \tilde{G}_{\Omega}(x,\cdot)\right)(y) &= 0 \;\forall x \in \overline{\Omega}, \forall y \in \Omega. \\ \tilde{G}_{\Omega}(x,y) &= G^{\Delta}_{C}(x,y) \;\forall x \in \overline{\Omega}, \forall y \in \partial\Omega\end{aligned}$$

By Lemma 12 in [Ken00], we have that there exists a universal constant C_0 such that

$$\left|G^{\Delta^\bullet_\delta}_{C_\delta}(x,y) - G^{\Delta}_{C}(x,y)\right| \leq C_0 \cdot \frac{\delta^2}{|x-y|^2} \;\forall x,y \in V_{C_\delta}.$$

By Theorem 3.9 in [ChSm08], we have that for each K, there exists $\tilde{\varepsilon}_K(\delta)$ with $\tilde{\varepsilon}_K(\delta) \underset{\delta\to 0}{\longrightarrow} 0$ such that for each $\delta > 0$, we have

$$\left|\tilde{G}_{\Omega_\delta}(x,y) - \tilde{G}_{\Omega}(x,y)\right| \leq \tilde{\varepsilon}_K(\delta) \;\forall x,y \in \overline{V}_{\Omega_\delta},$$

which gives the first estimate. For the second one, we have that there exists C_1 such that

$$\left|\frac{1}{\delta}\left(\nabla_\delta G^{\Delta^\bullet_\delta}_{C_\delta}(x,\cdot)\right)(y) - \left(\nabla G^{\Delta}_{C}(x,\cdot)\right)(y)\right| \leq 2C_1 \cdot \frac{\delta}{|x-y|^2} \;\forall x,y \in V_{C_\delta}$$

and by Corollary 2.8 in that paper there exists C_2 such that

$$\left|\frac{1}{\delta}\left(\nabla \tilde{G}^{\Delta^\bullet_\delta}_{\Omega_\delta}(x,\cdot)\right)(y) - \left(\nabla \tilde{G}^{\Delta}_{\Omega}(x,\cdot)\right)(y)\right| \leq \frac{C_2 \cdot \tilde{\varepsilon}_K(\delta)}{\operatorname{dist}(K,\partial\Omega)}.$$

From there, we readily obtain the desired result. □

A useful corollary is a lower bound on the integral of G^{Δ_δ} (written in [ChSm09]).

Corollary 35. With the notation and assumptions of Proposition 34, there exists a universal $C > 0$ such that for each $d > 0$, each $\delta > 0$ sufficiently small and each $v \in V_{G_\delta}$ at distance at least d from $\partial\Omega$, we have

$$\left|\sum_{x \in V_{G_\delta} \cap D\left(v, \frac{1}{2}d\right)} G^{\Delta_\delta}_{G_\delta}(v, x) \cdot \delta^2\right| \geq C \cdot d^2.$$

Proof. Fix d and $v \in V_{G_\delta}$. From Proposition 34, we have that for each $\delta > 0$ sufficiently small

$$G^{\Delta_\delta}_{G_\delta}(v, x) \leq G^{\Delta}_{\Omega}(v, x) + \frac{1}{4\pi}\log(2) \ \ \forall x \in V_{G_\delta} \cap \left(D\left(v, \frac{2}{3}d\right) \setminus D\left(v, \frac{1}{2}d\right)\right).$$

Since

$$G^{\Delta}_{\Omega}(v, x) \leq G_{D(v,d)}(v, x) = \frac{1}{2\pi}\log(|v| / d),$$

we obtain, using that $G^{\Delta_\delta}_{G_\delta}(v, \cdot)$ reaches its minimum at v,

$$G^{\Delta_\delta}_{G_\delta}(v, x) \leq -\frac{1}{4\pi}\log(2) \ \ \forall v \in V_{G_\delta} \cap D\left(v, \frac{1}{2}d\right).$$

Hence, there exists $C > 0$ such that

$$\sum_{x \in V_{G_\delta} \cap D\left(v, \frac{1}{2}d\right)} G^{\Delta_\delta}_{G_\delta}(v, x) \cdot \delta^2 \leq -C \cdot d^2,$$

which shows the result. $\square$

Another useful corollary is an L^1 upper bound on the derivatives of G^{Δ_δ}, which will later allow us to obtain such a control on the derivatives of subharmonic functions.

Corollary 36. With the notation and assumptions of Proposition 34, there exists $C > 0$ such that for each $v \in V_{G_\delta}$ and each $0 < d \leq \frac{\text{dist}(v,\partial\Omega)}{2}$, we have

$$\sum_{x \in V_{G_\delta} \cap D(v,d)} \left|\frac{1}{\delta} \cdot \left(\nabla_\delta G^{\Delta_\delta}_{G_\delta}(v, \cdot)\right)(x)\right| \cdot \delta^2 \leq C \cdot d,$$

uniformly with respect to δ.

Proof. By Proposition 34, for each $x \in V_{G_\delta} \cap D(v, d)$, we have

$$\frac{1}{\delta} \cdot \left(\nabla_\delta G^{\Delta_\delta}_{G_\delta}(v, \cdot)\right)(x) \leq \left|\left(\nabla G^{\Delta}_{\Omega}(v, \cdot)\right)(x)\right| + C \cdot \left(\frac{\delta}{|v - x|^2} + \varepsilon_d(\delta)\right)$$

There exists a constant $C_1 > 0$ such that

$$\sum_{x \in V_{G_\delta} \cap D(v,d)} \left|\left(\nabla G^{\Delta}_{\Omega}(v, \cdot)\right)(x)\right| \cdot \delta^2 \leq C_1 \int_{D(v,d)} \left|\left(\nabla G^{\Delta}_{\Omega}(v, \cdot)\right)(x)\right| dx,$$

Since $\left(\nabla G^{\Delta}_{\Omega}(v, \cdot)\right)(x) \sim \frac{1}{v-x}$ as $x \to v$, we have that

$$\int_{D(v,d)} \left|\left(\nabla G^{\Delta}_{\Omega}(v, \cdot)\right)(x)\right| dx \leq M_2 \cdot d$$

for a finite constant $M_2 > 0$, as can easily be checked. For the remainder, we again use an integral comparison, noticing that

$$\begin{aligned}\sum_{x\in V_{G_\delta}\cap D(v,d)} \frac{\delta}{|v-x|^2}\cdot\delta^2 &\leq C_3\cdot\int_{D(v,d)}\frac{1}{|v-x|}dx\\ &\leq M_4\cdot d,\end{aligned}$$

for universal constants $C_3, C_4 > 0$. Since $\varepsilon_d(\delta)$ is uniformly bounded with respect to δ, the result follows readily. □

Near the boundary, we have also the following estimate, which follows from a weak discrete version of Beurling's estimate.

Proposition 37. With the same notation as above, for each compact subset $K \subset \Omega$ and each $\varepsilon > 0$ there exists $d(K,\varepsilon) > 0$ such that

$$\left|G^{\Delta_\delta}_{G_\delta}(v_1,v_2)\right| \leq \varepsilon$$

for each $v_1 \in K$ and v_2 such that $\mathrm{dist}(v_2, \partial G_\delta) \leq d(K,\varepsilon)$, uniformly over all $\delta > 0$.

Proof. See [Kes87]. □

The discrete harmonic measure is well-known to converge to its continuous part. Let us mention without proof the statement that we will use:

Proposition 38. Let Ω be a domain, let $I \subset \partial\Omega$ be a boundary arc and let $(\Omega_\delta)_{\delta>0}$ be a discretization of Ω. For each $\delta > 0$, let $\tilde{\Delta}^\bullet_\delta$ with boundary modification on $\partial V_{\Omega_\delta} \cap I$, as explained in Section 2.6.1. For each $\delta > 0$, let $H^\circ_\delta : \overline{V}_{\Omega^*_\delta} \to \mathbb{R}$ and $\tilde{H}^\bullet_\delta : \overline{V}_{\Omega_\delta} \to \mathbb{R}$ be the harmonic measures of $\partial V_{\Omega^*_\delta} \cap I$ and $\partial V_{\Omega_\delta} \cap I$ for the Laplacians Δ°_δ and $\tilde{\Delta}^\bullet_\delta$ respectively, defined by

$$\begin{aligned}H^\circ_\delta(x) &= 1_{\partial V_{\Omega^*_\delta}\cap I}(x)\ \ \forall x \in \partial V_{\Omega^*_\delta}\\ \tilde{H}^\bullet_\delta(y) &= 1_{\partial V_{\Omega_\delta}\cap I}(x)\ \ \forall x \in \partial V_{\Omega_\delta}\cap I\\ \Delta^\circ_\delta H^\circ_\delta(x) &= 0\ \forall x \in V_{\Omega^*_\delta}\\ \Delta^\bullet_\delta \tilde{H}^\bullet_\delta(x) &= 0\ \forall x \in V_{\Omega_\delta}\end{aligned}$$

Then we have that

$$\begin{aligned}H^\circ_\delta &\underset{\delta\to 0}{\longrightarrow} H\\ \tilde{H}^\bullet_\delta &\underset{\delta\to 0}{\longrightarrow} H\end{aligned}$$

uniformly on the compact subsets of Ω, where H is the continuous harmonic measure, defined by $H_{|_I} = 1$, $H_{|\partial\Omega\setminus I} = 0$, $\Delta H = 0$.

Let us finish this section by giving estimates for the discrete harmonic measure on rectangles (similar to the ones used in [DHN09]):

Lemma 39. Let R be a rectangle with horizontal and vertical sides, and denote by $I \subset \partial R$ its lower side. For each $\delta > 0$, let $\tilde{\Delta}^\bullet_\delta$ be the Laplacian modified on $\partial V_{R_\delta} \cap I$, as defined in Section 2.6.1. Let $H_\delta : \overline{V}_{R_\delta} \to \mathbb{R}$ and $\tilde{H}_\delta : \overline{V}_{R_\delta} \to \mathbb{R}$ be the harmonic measures of $\partial R_\delta \setminus I$ with respect to the Laplacians $\Delta^\bullet_\delta$ and $\tilde{\Delta}^\bullet_\delta$ defined as in Proposition 38. Then we have

$$H_\delta(x) \leq \tilde{H}_\delta(x)\ \ \forall x \in V_{R_\delta},$$

and there exists $C_1 > 0$ such that

$$C_2 \cdot \delta \leq H_\delta(z + i\delta) \quad \forall z \in \partial V_{R_\delta} \cap l.$$

and for each $\varepsilon > 0$, if we denote by $l_\varepsilon \subset l$ the part of the segment l defined as $\{z \in l : \mathrm{dist}(\partial R \setminus l) \geq \varepsilon\}$, there exists $C_3 > 0$ such that

$$\tilde{H}_\delta(z + i\delta) \leq C_3 \cdot \delta \ \forall z \in \partial V_{R_\delta} \cap l_\varepsilon.$$

Proof. Let $u \subset \partial R$ be the upper side of R. Then by standard random walk coupling arguments (by representing H_δ and $\tilde{H}_\delta$ in terms of random walks with generators $\Delta^\bullet_\delta$ and $\tilde{\Delta}^\bullet_\delta$), we have

$$H_\delta(x) \leq \tilde{H}_\delta(x) \quad \forall x \in \overline{V}_{R_\delta}.$$

Let S be the horizontal infinite strip such that $l \cup u \subset \partial S$ and denote by $\underline{l}$ and $\underline{u}$ its lower and upper sides. Let $H^S_\delta : \overline{V}_{S_\delta} \to \mathbb{R}$ and $\tilde{H}^S_\delta : \overline{V}_{S_\delta} \to \mathbb{R}$ be the harmonic measures of the upper side of S with respect to $\Delta^\bullet_\delta$ and $\tilde{\Delta}^\bullet_\delta$. Then by standard random walk arguments, it is easy to see that we have

$$H^S_\delta(x) \leq H_\delta(x) \quad \forall x \in \overline{V}_{R_\delta}.$$

By symmetry of S, the computation of $H^S_\delta(x)$ reduces to a one-dimensional problem and in particular we have

$$H^S_\delta(z + i\delta) = \frac{\delta}{\mathrm{height}(R_\delta)} \quad \forall z \in \partial V_{S_\delta} \cap \underline{l},$$

which gives us the second estimate.

Let $H^*_\delta, \tilde{H}^*_\delta : \overline{V}_{R_\delta} \to \mathbb{R}$ be the harmonic measures of $\partial R_\delta \cap u$ with respect to the Laplacians Δ_δ and $\tilde{\Delta}_\delta$ (which is modified on $\partial V_{R_\delta} \cap l$ only). Then, again by standard random walk coupling arguments, we have

$$\frac{H^*_\delta(x)}{H_\delta(x)} = \frac{\tilde{H}^*_\delta(x)}{\tilde{H}_\delta(x)} \leq 1.$$

Still by random walks arguments, it is easy to see that for each $\varepsilon > 0$, there exists a constant $C_3 > 0$ such that for all $\delta > 0$

$$\frac{H^*_\delta(z + i\delta)}{H_\delta(z + i\delta)} \geq \frac{1}{C_3} \quad \forall z \in \partial V_{R_\delta} \cap l_\varepsilon.$$

By symmetry arguments, the computation of $\tilde{H}^S_\delta$ reduces to a one-dimensional problem and we have that there exists $C_4 > 0$ such that for all $\delta > 0$, we have

$$\tilde{H}^S_\delta(z + i\delta) \leq C_4 \cdot \delta \ \forall z \in \partial V_{S_\delta} \cap \underline{l},$$

which yields the desired result:

$$H_\delta(z + i\delta) \leq C_3 \cdot H^*_\delta(z + i\delta) \leq C_3 \cdot C_4 \cdot \tilde{H}^S_\delta(z + i\delta) \quad \forall z \in \partial V_{R_\delta} \cap l_\varepsilon.$$

□

3.3. Regularity estimates for discrete harmonic and holomorphic functions

The following result gives uniform control on the derivatives of discrete harmonic functions, thus allowing to obtain that they are uniformly Lipschitz-continuous on the compact subsets:

Proposition 40. *Let $f_\delta : V_{G_\delta} \to \mathbb{C}_{\delta>0}$ be a discrete harmonic function (with G_δ equal to Ω_δ, Ω^*_δ, Ω^m_δ or Ω^{m*}_δ). Then there exists a universal constant $C > 0$ such that*

$$|f_\delta(v_1) - f_\delta(v_2)| \leq C \cdot \frac{|v_1 - v_2| \cdot \max_{v\in\partial_0 V_{G_\delta}} |f_\delta(v)|}{\min\left(\operatorname{dist}(v_1, \partial V_{G_\delta}), \operatorname{dist}(v_2, \partial V_{G_\delta})\right)},$$

for all $v_1, v_2 \in V_{G_\delta}$.

Proof. This follows directly from Corollary 2.8 in [ChSm08]. □

In the case of s-holomorphic functions, we have the following useful result, which allows to transform L^1 boundary control into L^∞ bulk control and hence to deduce precompactness in some cases:

Proposition 41. *Let $f_\delta : V_{\Omega^m_\delta} \to \mathbb{C}$ be a discrete s-holomorphic function. Then there exists a universal constant $C > 0$ such that*

$$|f_\delta(v)| \leq C \cdot \frac{\sum_{w\in\partial_0 V_{\Omega^m_\delta}\cup\partial_{-1/2}V_{\Omega^m_\delta}} |f_\delta(w)|\cdot\delta}{\operatorname{dist}(v, \partial\Omega_\delta)},$$

$$\left|\frac{1}{\delta}\nabla_\delta f_\delta(x)\right| \leq C \cdot \frac{\sum_{w\in\partial_0 V_{\Omega^m_\delta}\cup\partial_{-1/2}V_{\Omega^m_\delta}} |f_\delta(w)|\cdot\delta}{\left(\operatorname{dist}(v, \partial\Omega_\delta)\right)^2},$$

where $\partial_{-1/2}V_{\Omega^m_\delta} = \left\{x \in V_{\Omega^m_\delta} : \operatorname{dist}\left(x, \partial_0 V_{\Omega^m_\delta}\right) = \frac{\sqrt{2}}{2}\delta\right\}$.

Proof. From the discrete Cauchy's formula (Proposition 18), we have

$$f_\delta(v) = \frac{1}{2i}\Bigg(\sum_{\vec{e}\in\partial_0\vec{E}_{\Omega_\delta}} f(m(\vec{e}))\cdot G^{\bar\partial_\delta}\left(v, m(\vec{e}) + \frac{i\vec{e}}{2}\right)\cdot\vec{e}$$
$$+ \sum_{\vec{e}\in\partial\vec{E}_{\Omega^*_\delta}} f(m(\vec{e}))\cdot G^{\bar\partial_\delta}\left(v, m(\vec{e}) + \frac{i\vec{e}}{2}\right)\cdot\vec{e}\Bigg),$$

where $\partial_0\vec{E}_{\Omega_\delta}$ and $\partial\vec{E}_{\Omega^*_\delta}$ are the sets of edges of $\partial_0\Omega_\delta$ and $\partial\Omega^*_\delta$, oriented in counterclockwise direction on the outer component of the boundary and in clockwise direction on the inner ones. Hence we have

$$\nabla_\delta f_\delta(x) = \frac{1}{2i}\Bigg(\sum_{\vec{e}\in\partial_0\vec{E}_{\Omega_\delta}} f(m(\vec{e}))\left(\nabla_\delta G^{\bar\partial_\delta}\left(\cdot, m(\vec{e}) + \frac{i\vec{e}}{2}\right)\right)(x)\cdot\vec{e}$$
$$+ \sum_{\vec{e}\in\partial\vec{E}_{\Omega^*_\delta}} f(m(\vec{e}))\left(\nabla_\delta G^{\bar\partial_\delta}\left(\cdot, m(\vec{e}) + \frac{i\vec{e}}{2}\right)\right)(x)\cdot\vec{e}\Bigg).$$

Since by Proposition 33, there exists a constant C such that $\frac{1}{\delta}\left|G^{\overline{\partial}_\delta}(x,y)\right| \leq C\left|G^{\overline{\partial}}(x,y)\right|$ and such that

$$\frac{1}{\delta^2}\left|\left(\nabla_\delta G^{\overline{\partial}_\delta}(\cdot,y)\right)(x)\right| \leq \left|\left(\nabla G^{\overline{\partial}}(\cdot,y)\right)(x)\right|,$$

uniformly over all $\{(x,y) \in C \times C\}$, we obtain the desired result. □

From there we deduce another useful result, which translates L^1 bulk control into L^∞ control.

Proposition 42. Let $f_\delta : V_{\Omega_\delta^m} \to C_\delta$ be a discrete holomorphic function. Then there exists a universal constant $C > 0$ such that

$$\begin{aligned} |f_\delta(v)| &\leq C \cdot \frac{\sum_{w\in V_{\Omega_\delta}} |f_\delta(w)| \cdot \delta^2}{(\mathrm{dist}(v,\partial V_{\Omega_\delta}))^2}, \\ \frac{1}{\delta}\left|\nabla_\delta u_\delta(v)\right| &\leq C \cdot \frac{\sum_{w\in V_{\Omega_\delta}} |f_\delta(w)| \cdot \delta^2}{(\mathrm{dist}(v,\partial V_{\Omega_\delta}))^3}, \end{aligned}$$

for each $v \in V_{\Omega_\delta}$.

Proof. Let $d = \mathrm{dist}(v,\partial V_{\Omega_\delta})$. By integrating the estimates of Proposition 41 on the following family of squares

$$\left\{R_{k,\delta} = \left(\left[v - \frac{k\delta}{2}, v + \frac{k\delta}{2}\right] \times \left[v - i\frac{k\delta}{2}, v + i\frac{\delta}{2}\right]\right) \cap C_\delta : k \in \left[\frac{d}{4\delta}, \frac{d}{2\delta}\right] \cap N\right\},$$

we obtain the desired result. □

An important tool for us is the control of values of s-holomorphic functions by the values of the discrete antiderivatives of their squares, developed in [ChSm09] (whose proof we follow closely):

Proposition 43. Let $f_\delta : V_{\Omega_\delta^m} \to C$ be an s-holomorphic function. Then there exists a constant $C > 0$ such that for $\delta > 0$ and any $x \in V_{\Omega_\delta^{m*}}$, we have

$$\begin{aligned} |f_\delta(v)| &\leq C \cdot \sqrt{\frac{\max_{w\in V_{\Omega_\delta^{m*}}} |I_{\delta,x}[f_\delta](w)|}{\mathrm{dist}\left(v, \partial_0 V_{\Omega_\delta^m}\right)}} \quad \forall v \in V_{\Omega_\delta^m}. \\ \frac{1}{\delta}\left|\nabla_\delta u_\delta(v)\right| &\leq C \cdot \frac{\sqrt{\max_{w\in V_{\Omega_\delta^{m*}}} |I_{\delta,x}[f_\delta](w)|}}{\left(\mathrm{dist}\left(v, \partial_0 V_{\Omega_\delta^m}\right)\right)^{\frac{3}{2}}} \quad \forall v \in V_{\Omega_\delta^m}. \end{aligned}$$

Proof. Let M_δ be $\max_{w\in V_{\Omega_\delta^{m*}}} |I_{\delta,x}[f_\delta](w)|$. Let us give the two following lemmas:

Lemma 44. There exists a universal constant $C_0 > 0$ such that for any $v \in V_{\Omega_\delta^m}$, we have

$$\begin{aligned} \sum_{w\in V_{\Omega_\delta}\cap D\left(v,\frac{3}{4}\cdot\mathrm{dist}\left(v,\partial V_{\Omega_\delta}\right)\right)} \Delta_\delta^\bullet I_\delta^\bullet[f_\delta](w) &\leq C_0 \cdot M_\delta, \\ \sum_{w\in V_{\Omega_\delta^*}\cap D\left(v,\frac{3}{4}\cdot\mathrm{dist}\left(v,\partial V_{\Omega_\delta}\right)\right)} \Delta_\delta^\circ I_\delta^\circ[f_\delta](w) &\leq C_0 \cdot M_\delta. \end{aligned}$$

Lemma 45. There exists a universal constant $C_1 > 0$ such that for any $v \in V_{\Omega_\delta^m}$, we have

$$\sum_{x \in V_{\Omega_\delta^m} \cap D\left(v, \frac{2}{3}\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)\right)} |f_\delta(x)|^2 \cdot \delta^2 \leq C_1 \cdot M_\delta \cdot \mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right).$$

Once these two lemmas are proven, by the Cauchy-Schwarz inequality, we obtain that there exists a universal constant $C_2 > 0$ such that

$$\begin{aligned}
& \sum_{x \in V_{\Omega_\delta^m} \cap D\left(v, \frac{2}{3}\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)\right)} |f_\delta(x)| \cdot \delta^2 \\
\leq\ & C_2 \cdot \mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right) \cdot \sqrt{\sum_{x \in V_{\Omega_\delta^m} \cap D\left(v, \frac{2}{3}\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)\right)} |f_\delta(x)|^2 \cdot \delta^2} \\
\leq\ & C_2 \cdot C_1 \cdot \sqrt{M_\delta} \cdot \mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)^{\frac{3}{2}}.
\end{aligned}$$

From Proposition 42, we deduce that there exists a universal constant $C_3 > 0$ such that

$$\begin{aligned}
\max_{x \in V_{\Omega_\delta^m} \cap D\left(v, \frac{1}{2}\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)\right)} |f_\delta(x)| &\leq C_3 \cdot \sqrt{\frac{M_\delta}{\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)}} \\
\max_{x \in V_{\Omega_\delta^m} \cap D\left(v, \frac{1}{2}\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)\right)} |(\nabla_\delta f_\delta)(x)| &\leq C_4 \cdot \frac{\sqrt{M_\delta}}{\mathrm{dist}\left(v, \partial V_{\Omega_\delta}\right)^{\frac{3}{2}}},
\end{aligned}$$

which is the desired result. ∎

Let us now give the proofs of the two lemmas:

Proof of Lemma 44. Set $d = \mathrm{dist}(v, \partial V_{\Omega_\delta})$. Let us denote by $H_\delta^\bullet, S_\delta^\bullet : \overline{V}_{\Omega_\delta} \to \mathbb{R}$ the harmonic and subharmonic parts of $I_\delta^\bullet[f_\delta]$ and by $H_\delta^\circ, S_\delta^\circ : \overline{V}_{\Omega_\delta^*} \to \mathbb{R}$ the harmonic superharmonic parts of $I_\delta^\circ[f_\delta]$ defined by

$$\begin{aligned}
H_\delta^\bullet(v) &= I_\delta^\bullet[f_\delta](v) \ \ \forall v \in \partial V_{\Omega_\delta}, \\
H_\delta^\circ(w) &= I_\delta^\circ[f_\delta](w) \ \ \forall w \in \partial V_{\Omega_\delta^*} \\
\Delta_\delta^\bullet H_\delta^\bullet(v) &= 0 \ \forall v \in V_{\Omega_\delta}, \\
\Delta_\delta^\circ H_\delta^\circ(w) &= 0 \ \forall v \in V_{\Omega_\delta^*}, \\
H_\delta^\bullet(v) + S_\delta^\bullet(v) &= I_\delta^\bullet[f_\delta](v) \ \ \forall v \in \overline{V}_{\Omega_\delta}, \\
H_\delta^\circ(w) + S_\delta^\circ(w) &= I_\delta^\bullet[f_\delta](w) \ \ \forall w \in \overline{V}_{\Omega_\delta^*}.
\end{aligned}$$

By the maximum principle, we have that $H_\delta^\bullet$ and H_δ° are bounded by M_δ and hence that $S_\delta^\bullet$ and S_δ° are bounded by $2M\delta$. Since $S_\delta^\bullet$ and S_δ° have zero boundary values, by Proposition 20, we have

$$\begin{aligned}
S_\delta^\bullet(v) &= \sum_{x \in V_{\Omega_\delta}} \Delta_\delta S_\delta^\bullet(x)\, G_{\Omega_\delta}^{\Delta_\delta}(x, v) \ \ \forall v \in V_{\Omega_\delta}, \\
S_\delta^\circ(w) &= \sum_{y \in V_{\Omega_\delta^*}} \Delta_\delta S_\delta^\circ(y)\, G_{\Omega_\delta^*}^{\Delta_\delta}(y, w) \ \ \forall w \in V_{\Omega_\delta^*}.
\end{aligned}$$

Since $G^{\Delta^\bullet_\delta}_{\Omega_\delta}(\cdot,\cdot) \le 0$, we obtain that

$$\left(\sum_{w\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} S^\bullet_\delta(w)\right) \cdot \min_{x\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} G^{\Delta^\bullet_\delta}_{\Omega_\delta}(x,w) \le 2M_\delta.$$

and hence, summing over $w \in V_{\Omega_\delta}$, we obtain

$$\left(\sum_{w\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} S^\bullet_\delta(w)\right) \cdot \min_{x\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} \sum_{w\in V_{\Omega_\delta}} G^{\Delta^\bullet_\delta}_{\Omega_\delta}(x,w)$$
$$\le \quad C_0 \cdot M_\delta \cdot d^2,$$

for a universal constant C_0. By Corollary 35, there exists a universal constant C_1 such that

$$\min_{x\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} \sum_{w\in V_{\Omega_\delta}} G^{\Delta^\bullet_\delta}_{\Omega_\delta}(x,w) \ge C_1 \cdot d^2$$

Hence we obtain

$$\left(\sum_{w\in V_{\Omega_\delta}\cap D(v,\frac{3}{4}d)} S^\bullet_\delta(w)\right) \le \frac{C_0}{C_1} M_\delta$$

which finishes the proof of Lemma 44. □

Proof of Lemma 45. Set $d = \mathrm{dist}(v, \partial V_{\Omega_\delta})$. We have

$$|f_\delta(v)| = \left|\frac{1}{\delta}(\partial_\delta I_\delta[f_\delta])(v)\right|$$

as can easily be checked from the definition of $I_\delta[f_\delta]$ and hence we have to show that there exists $C_0 > 0$ such that

$$\sum_{v\in V_{\Omega^m_\delta}\cap D(v,\frac{2}{3}d)} \left|\frac{1}{\delta}(\partial_\delta I_\delta[f_\delta])(v)\right| \cdot \delta^2 \le C_0 \cdot M_\delta \cdot d$$

Let $\underline{\Omega}_\delta$ be $\Omega_\delta \cap D\left(v, \frac{3}{4}d\right)$ and $\tilde{\Omega}_\delta$ be $\Omega_\delta \cap D\left(d, \frac{2}{3}d\right)$. Let $\underline{H}^\bullet_\delta, \underline{S}^\bullet_\delta : \overline{V}_{\underline{\Omega}_\delta} \to \mathbb{R}$ be the harmonic and subharmonic parts of $I^\bullet_\delta[f_\delta]$ and let $\underline{H}^\circ_\delta, \underline{S}^\circ_\delta : \overline{V}_{\underline{\Omega}^*_\delta} \to \mathbb{R}$ be the harmonic and superharmonic parts of $I^\circ_\delta[f_\delta]$, defined by $\underline{H}^\bullet_\delta + \underline{S}^\bullet_\delta = I^\bullet_\delta[f_\delta]$, $\Delta^\bullet_\delta \underline{H}^\bullet_\delta = 0$, $S^\bullet_\delta|_{\partial V_{\underline{\Omega}_\delta}} = 0$, $\underline{H}^\circ_\delta + \underline{S}^\circ_\delta = I^\circ_\delta[f_\delta]$ $\Delta^\circ_\delta \underline{H}^\circ_\delta = 0$ and $S^\circ_\delta|_{\partial V_{\underline{\Omega}^*_\delta}} = 0$. By Proposition 40, there exists $C_1 > 0$ such that

$$\left|\frac{1}{\delta}\partial_\delta \underline{H}^\bullet_\delta(x)\right| \le \frac{C_1 \cdot M_\delta}{d} \quad \forall x \in V_{\tilde{\Omega}_\delta},$$
$$\left|\frac{1}{\delta}\partial_\delta \underline{H}^\circ_\delta(y)\right| \le \frac{C_1 \cdot M_\delta}{d} \quad \forall y \in V_{\tilde{\Omega}^*_\delta}.$$

Hence

$$\sum_{x\in V_{\tilde{\Omega}_\delta}} \left|\frac{1}{\delta}\partial_\delta \underline{H}^\bullet_\delta(x)\right| \le C_1 \cdot M_\delta \cdot d,$$
$$\sum_{y\in V_{\tilde{\Omega}^*_\delta}} \left|\frac{1}{\delta}\partial_\delta \underline{H}^\circ_\delta(y)\right| \le C_1 \cdot M_\delta \cdot d.$$

On the other hand, by Proposition 20, we have

$$
\begin{aligned}
\sum_{v\in V_{\tilde{\Omega}_\delta}} \left|\frac{1}{\delta}\cdot \partial_\delta \underline{S}^\bullet_\delta(x)\right|\cdot \delta^2 \;&\le\; \sum_{v\in V_{\underline{\Omega}_\delta}} \left|\Delta^\bullet_\delta \underline{S}^\bullet_\delta(v)\right| \\
&\quad \cdot \max_{z\in V_{\tilde{\Omega}_\delta}} \left(\sum_{x\in V_{\tilde{\Omega}_\delta}} \left|\frac{1}{\delta}\cdot \left(\partial_\delta G^{\Delta_\delta}_{\Omega_\delta}(x,\cdot)\right)(z)\right|\cdot \delta^2\right), \\
\sum_{w\in V_{\tilde{\Omega}^*_\delta}} \left|\frac{1}{\delta}\cdot \partial_\delta \underline{S}^\circ_\delta(w)\right|\cdot \delta^2 \;&\le\; \sum_{w\in V_{\underline{\Omega}^*_\delta}} \left|\Delta^\circ_\delta \underline{S}^\circ_\delta(w)\right| \\
&\quad \cdot \max_{z\in V_{\tilde{\Omega}^*_\delta}} \left(\sum_{y\in V_{\tilde{\Omega}^*_\delta}} \left|\frac{1}{\delta}\cdot \left(\partial_\delta G^{\Delta_\delta}_{\Omega_\delta}(y,\cdot)\right)(z)\right|\cdot \delta^2\right).
\end{aligned}
$$

By Lemma 44, the first terms in the right hand sides are uniformly bounded by $C_2 \cdot M_\delta$, for some universal constant C_2. By Corollary 36, the second terms in the right hand sides are uniformly bounded by $C_3 \cdot d$. The lemma follows. □

CHAPTER 4

Convergence of Solutions of Discrete Riemann-Hilbert Boundary Value Problems

In this chapter, we study convergence of solutions to discrete Riemann-Hilbert boundary value problems to continuous ones.

(1) We first define continuous Riemann-Hilbert boundary value problems.
(2) We then obtain precompactness results for solutions of discrete Riemann-Hilbert boundary value problems with bounded boundary data.
(3) We identify the subsequential limits of discrete Riemann-Hilbert boundary value problems with convergent boundary data.
(4) We finally improve the convergence of the previous point up to the pieces of the boundary where it is nice enough.

In the next chapters, these results will be applied to show the convergence of the discrete fermionic observables.

4.1. Continuous Riemann-Hilbert boundary value problems

In this section, we define the continuous Riemann-Hilbert boundary value problems which are the natural candidates for the limits of these converging subsequences, as will be proved in the next section.

Definition 46. Let $\Omega \subset \mathbb{C}$ be a finitely connected domain with $\Omega \neq \mathbb{C}$. Let $f : Y \to \mathbb{C}$ be a holomorphic function defined on some neighborhood Y of $\partial\Omega$ in $\overline{\Omega}$. We say that a holomorphic function $u : \Omega \to \mathbb{C}$ solves the *continuous Riemann-Hilbert boundary value problem* $(\spadesuit_{\Omega}, f)$ if we have

$$(u - f)(z) \parallel \frac{1}{\sqrt{\nu_{ext}(z)}} \quad \forall z \in \partial\Omega$$

in the following *integral sense*: for any $a \in \Omega$, the real part $\zeta \mapsto \Re\mathfrak{e}\left(\int_a^{\zeta} (u(z) - f(z))^2 \, dz\right)$ of the antiderivative of $(u - f)^2$ is well-defined on Y, extends continuously to $Y \cup \partial\Omega$, is locally constant on $\partial\Omega$ and is non-increasing as $z \to \partial\Omega$.

It is easy to check that when the boundary has enough regularity to define the normal vector at a point, the above definition coincides with the one that $(u - f)$ extends to the boundary and satisfies the condition $\parallel \frac{1}{\sqrt{\nu_{ext}}}$ in the usual sense. Let us also remark that this boundary condition is conformally covariant:

Proposition 47. *Let $\varphi : \Omega \to \tilde{\Omega}$ be a conformal mapping. Then there exists a globally well-defined holomorphic function such that $\psi : \Omega \to \mathbb{C}$ such that $\psi^2 = \varphi'$, which we denote by $\sqrt{\varphi'}$ when its branch choice is clear or irrelevant. If $u : \Omega \to \mathbb{C}$*

is a holomorphic function such that on a connected component $b \subset \partial\Omega$ of the boundary, we have

$$u(z) \parallel \frac{1}{\sqrt{\nu_{ext}(z)}} \quad \forall z \in I,$$

in the sense that $\mathrm{Re}\left(\int u^2\right)$ is globally well-defined on a neighborhood of b and extends continously to a constant on b. Then for any branch choice of $\sqrt{\varphi'}$, we have that

$$u\left(\varphi^{-1}(\zeta)\right) \cdot \sqrt{\varphi'(\varphi^{-1}(\zeta))} \parallel \frac{1}{\sqrt{\nu_{ext}(\zeta)}} \quad \forall \zeta \in \varphi(I),$$

in the same sense as above.

Proof. Let b be a connected component of $\partial\Omega$. Then, by argument's principle, we have that

$$\begin{aligned}\oint_b d\log(\varphi'(z)) &= \oint_b d\log(\varphi' \cdot \tau(z)) - d\log(\tau(z)) \\ &= \pm 2\pi i - 2\pi i,\end{aligned}$$

where τ denotes the counterclockwise-oriented tangent direction of b, since $\varphi : b \to \varphi(b)$ is a homeomorphism (in the sense of prime ends), either orientation-preserving (in which case the above quantity is 0) or orientation-reversing (in which case we obtain -4π). Since this is true, we have that

$$\psi(z) = \exp\left(\frac{1}{2}\log(\varphi'(z))\right)$$

is globally well-defined and satisfies $\psi^2 = \varphi'$.

The boundary condition follows from the change of variable formula. ∎

As in the discrete world, we have that the boundary data of such problems identifies uniquely their solutions when they exist:

Proposition 48. *A Riemann-Hilbert boundary value problem $(\spadesuit_\Omega, f)$ for given Ω and f has at most one solution.*

Proof. By linearity, it suffices to show that the unique solution of the problem $(\spadesuit_\Omega, 0)$ is 0. Let us assume that Ω is smooth. If $h : \Omega \to \mathbb{C}$ is a solution to $(\spadesuit_\Omega, 0)$, then we have that $H(\zeta) = \mathrm{Re}\left(\int_a^\zeta h(z)^2 dz\right)$ is globally well-defined, harmonic on Ω and constant on $\partial\Omega$. By Gauss' formula and the boundary condition, we have

$$\begin{aligned}0 &= \iint_\Omega \Delta H(x+iy)\,dxdy \\ &= \oint_{\partial\Omega} \partial_{\nu_{ext}(z)} H(z)\,dz \\ &= \oint_{\partial\Omega} |h(z)|^2\,dz,\end{aligned}$$

which shows that h is equal to 0 on $\partial\Omega$.

If Ω is not smooth, by Proposition 47, we can map Ω conformally to smooth domain $\tilde{\Omega}$ by a conformal mapping φ and we have that $h \circ \varphi^{-1} \cdot \sqrt{\varphi' \circ \varphi^{-1}}$ is a solution to the problem $(\spadesuit_{\tilde{\Omega}}, 0)$, hence reducing to the previous case. ∎

4.2. Discrete Riemann-Hilbert boundary value problems: regularity and precompactness

In this section, we establish precompactness and regularity result for solutions to discrete Riemann-Hilbert boundary value problems, under the assumption that the boundary of the domain considered essentially has rectifiable boundary. The central result of this section is the following, which gives control on the solutions and their derivatives:

Proposition 49. There exists a universal constant $C > 0$ such that for each $\delta > 0$ and any s-holomorphic function $u_\delta : V_{\Omega_\delta^m} \to \mathbb{C}$ solving the discrete Riemann-Hilbert boundary value problem $(\spadesuit_{\Omega_\delta}, f_\delta)$ for $f_\delta : \partial_0 V_{\Omega_\delta^m} \to \mathbb{C}$, we have

$$|u_\delta(v)|^2 \leq C \cdot \left(\frac{\sqrt{\sum_{x\in\partial_0 V_{\Omega_\delta^m}} |f_\delta(x)|\cdot\delta}}{\mathrm{dist}\left(v, \partial_0 V_{\Omega_\delta^m}\right)} \right) \quad \forall v \in V_{\Omega_\delta^m}, \tag{4.2.1}$$

$$\frac{1}{\delta}\cdot \left|\nabla_\delta u_\delta(v)\right| \leq C \cdot \frac{\sqrt{\sum_{x\in\partial_0 V_{\Omega_\delta^m}} |f_\delta(x)|\cdot\delta}}{\mathrm{dist}\left(v, \partial_0 V_{\Omega_\delta^m}\right)^{\frac{3}{2}}} \quad \forall v \in V_{\Omega_\delta^m}. \tag{4.2.2}$$

Proof. From Proposition 28, we have that

$$\sum_{x\in\partial_0 V_{\Omega_\delta^m}} |u_\delta(x)|^2 \leq \sum_{v\in\partial_0 V_{\Omega_\delta^m}} |f_\delta(x)|^2.$$

Fix $y \in \partial_0 V_{\Omega_\delta}$ and consider the antiderivative $I_{y,\delta}[u_\delta] : V_{\Omega_\delta^{m*}} \to \mathbb{R}$ as defined in Section 2.5, normalized to be equal to 0 at y. By definition of $I_{y,\delta}[u_\delta]$, integrating along the boundary, we obtain

$$|I_{y,\delta}[u_\delta](z)| \leq 2 \sum_{x\in\partial_0 V_{\Omega_\delta^m}} |f_\delta(x)|^2 \cdot \delta \ \forall z \in \partial V_{\Omega_\delta}.$$

From the subharmonicity of $I^\bullet_{y,\delta}[u_\delta]$ and the superharmonicity of $I^\circ_{y,\delta}[u_\delta]$, by the maximum/minimum principle, we easily deduce

$$\sup_x |I_{y,\delta}[u_\delta](x)| \leq 2 \sum_{x\in\partial_0 V_{\Omega_\delta^m}} |f_\delta(x)|^2 \cdot \delta.$$

The estimates 4.2.1 and 4.2.2 then follow from Proposition 43. □

From this, we obtain the following precompactness resul:

Theorem 50. Let $(\Omega_\delta)_{\delta>0}$ be a family of discrete domains approximating a smooth domain Ω. Let $P \subset \mathbb{R}^n$ be a parameter space and let $\left(u_\delta^p\right)_{\delta>0}^{p\in P}$ be a family of functions such that for each $\delta > 0$ and each $p \in P$, $u_\delta^p : V_{\Omega_\delta^m} \to \mathbb{C}$ solves the discrete problem $\left(\spadesuit_{\Omega_\delta}, f_\delta^p\right)$ with $f_\delta^p : \partial_0 V_{\Omega_\delta^m} \to \mathbb{C}$. Then if the the family of functions

$$\left(\partial_0 V_{\Omega_\delta^m} \times P \ni (x,p) \mapsto (f_\delta(x))_\delta^p\right)_{\delta>0}$$

is uniformly equicontinuous and bounded, the family of functions

$$\left(V_{\Omega_\delta^m} \times P \ni (v,p) \mapsto \left(u_\delta^p(v)\right)\right)_{\delta>0}$$

is uniformly equicontinuous and bounded for the topology of the convergence on the compact subsets of $\Omega \times P$.

Proof. For each $d > 0$, denote by $K_d = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) \geq d\}$. Let $\varepsilon > 0$ and a compact subset $P \subset \mathcal{P}$ be given. From Proposition 49 and the assumption on $\partial\Omega$, there exists $M > 0$ such that we have, uniformly in $\delta > 0$,

$$\left|u_\delta^p(v)\right| \leq M \quad \forall v \in V_{\Omega_\delta^m} \cap K_{\frac{d}{2}}, \forall p \in P.$$

From Proposition 40, we obtain that there exists $C > 0$ such that we have, uniformly in $\delta > 0$

$$\left|u_\delta^p(v) - u_\delta^p(\tilde{v})\right| \leq C \cdot M \cdot |v - \tilde{v}| \quad \forall v, \tilde{v} \in V_{\Omega_\delta^m} \cap K_d, \forall p \in P.$$

From Proposition 49, we also obtain that for each $\varepsilon > 0$, there exists $\tau > 0$ such that

$$\left|u_\delta^p(v) - u_\delta^{\tilde{p}}(v)\right| \leq \varepsilon \quad \forall v \in V_{\Omega_\delta^m} \cap K_d \ \forall p, \tilde{p} \in P : |p - \tilde{p}| \leq \tau.$$

By a classical application of the triangular inequality, we obtain the desired result, once we extend the functions in a suitable way, for instance by. piecewise affine interpolation. □

4.3. Riemann-Hilbert boundary value problems: identification of subsequential limits

In this section, we study subsequential limits of solutions to discrete Riemann-Hilbert boundary value problems and identify them as (unique) solutions to the continuous problems defined in Section 4.1.

Proposition 51. *Let Ω be a domain and $(\Omega_\delta)_{\delta>0}$ be a discretization of Ω and let $Y \subset \Omega$ be a neighborhood of $\partial\Omega$ such that $\partial\Omega \subset \overline{Y}$. Let $\left(g_{\delta_n} : V_{Y_{\delta_n}^m} \to \mathbb{C}\right)_{n \geq 0}$ be a sequence of s-holomorphic functions such that*

$$g_{\delta_n}(x) \parallel \frac{1}{\sqrt{\nu_{\mathrm{ext}}(x)}} \quad \forall x \in \partial_0 V_{\Omega_\delta^m}.$$

Suppose that, as $n \to \infty$, $\delta_n \to 0$ and, uniformly on the compact subsets of Y, we have $g_{\delta_n} \to g$ for some continuous function $g : Y \to \mathbb{C}$. Then we have that g is holomorphic and that $\Im\mathrm{m} \int g^2$ is globally well-defined on Y and extends continuously to a locally constant function on $\partial\Omega$, and has nonpositive outer normal derivative there.

Corollary 52. *Let $(\Omega_\delta)_{\delta>0}$ be a family of discrete domains approximating a domain Ω and let $(u_\delta)_{\delta>0}$ be a family of s-holomorphic functions such that for each $\delta > 0$, $u_\delta : V_{\Omega_\delta^m} \to \mathbb{C}$ solves the discrete problem $(\spadesuit_{\Omega_\delta}, f_\delta)$, with $f_\delta \to f$ as $\delta \to 0$. Suppose that there exists a neighborhood $Y \subset \Omega$ with $\partial\Omega \subset \overline{Y}$ such that for each $\delta > 0$, $f_\delta : \partial_0 V_{\Omega_\delta^m} \to \mathbb{C}$ extends to s-holomorphic function $f_\delta : V_{Y_\delta^m} \to \mathbb{C}$, and such that $f_\delta \to f$ uniformly on $\overline{Y}$ for some holomorphic function $f : \overline{Y} \to \mathbb{C}$ as $\delta \to 0$.*

Then any convergent subsequence of $(u_\delta)_{\delta>0}$ (for the topology of uniform convergence on the compact subsets) converges to the solution to the continuous problem $(\spadesuit_\Omega, f)$ as $\delta \to 0$, which is in particular guaranteed to exist.

Proof of Corollary 52. From Proposition 51, we have that any subsequential limit $u = \lim_{n\to\infty} u_{\delta_n}$ is the unique (by Proposition 48) solution to the Riemann-Hilbert boundary value problem $(\spadesuit_\Omega, f)$, the existence thereof is in particular guaranteed. By uniqueness, we conclude that $u_\delta \to u$ as $\delta \to 0$. □

Proof of Proposition 51. Let us first remark that by Lemma 32, the function g is holomorphic on Y and that hence $\Re\mathrm{e}\left(\int g^2\right)$ is at least locally well-defined.

We treat the boundary values separately on each of the connected components of $\partial\Omega$. Let $b \subset \partial\Omega$ be such a connected component and let $\Lambda \subset Y$ be a neighborhood of b in $\overline{\Omega}$, such that $\partial\Lambda \setminus \partial\Omega$ is smooth and contained in Y. Consider the antiderivative $I_\delta[g_\delta] : \overline{V}_{\Lambda^{m}_{\delta}{}^{*}}$, which, thanks to its boundary values, is constant on $V_{b^{m}_{\delta}{}^{*}}$ and globally well-defined on $\Lambda^{m}_{\delta}{}^{*}$ by Lemma 30. We know by assumption that $g_{\delta_n} \to g$ and that $I_{\delta_n}[g_{\delta_n}] \to -\Re\mathrm{e}\left(\int g^2\right)$ on the compact subsets of $\Omega \cap \Lambda$ as $n \to \infty$. So, to prove that $\Re\mathrm{e}\left(\int g^2\right)$ extends continuously to a locally constant function on $\partial\Omega$, it is sufficient to show that we have

$$\limsup_{n\to\infty}\left|I_{\delta_n}[g_{\delta_n}](v) - I^{\circ}_{\delta_n}[g_{\delta_n}](b_\delta)\right| \underset{v\to b}{\longrightarrow} 0.$$

where $I^{\circ}_{\delta_n}[g_\delta](b_\delta)$ denotes the (constant) value of $I_\delta[g_\delta]$ on $\partial_0 V_{\Lambda^{*}_{\delta}} \cap \partial\Omega$. This is given by Lemma 53 below.

We should finally prove that the normal derivative of $-\Re\mathrm{e}\left(\int g^2\right)$ is nonpositive on $\partial\Omega$. This follows from the discrete the fact that this property is true on discrete level. We introduce the boundary modified antiderivative $\tilde{I}^{\bullet}_{\delta}[g_\delta]$ defined in Section 2.6.1 and use the fact that the normal derivative of $\partial_{\overline{V}_{ext}}\tilde{I}^{\bullet}_{\delta}[g_\delta] \leq 0$ on $\partial_0 V_{\Omega^{m}_{\delta}}$: since $\tilde{I}^{\bullet}_{\delta}[g_\delta]$ is constant on $\partial\Omega$, if the normal derivative of $-\Re\mathrm{e}\left(\int g^2\right)$ would be positive at a point of $\partial\Omega$, it would yield a contradiction as $n \to \infty$. □

Lemma 53. *Let Λ be a doubly-connected domain and denote by b one of the two connected components of $\partial\Lambda$. Let $(g_{\delta_n})_{n>0}$ be a family of s-holomorphic functions with $g_{\delta_n} : V_{\Lambda_{\delta_n}} \to \mathbb{C}$ such that we have*

$$g_{\delta_n}(v) \parallel \frac{1}{\sqrt{\overline{\nu}_{ext}(x)}} \quad \forall x \in \partial_0 V_{\Lambda_{\delta_n}} \cap b.$$

Then if g_{δ_n} is uniformly bounded on the compact subsets of Λ, we have

$$\limsup_{n\to\infty}\left|I_{\delta_n}[g_{\delta_n}](v) - I^{\circ}_{\delta_n}[g_{\delta_n}](b_\delta)\right| \underset{v\to b}{\longrightarrow} 0.$$

Proof. We use the boundary modification trick introduced in Section 2.6.1: for each $\delta \in \{\delta_n : n \geq 0\}$ let $\tilde{I}_\delta : \overline{V}_{\Lambda^{m}_{\delta}{}^{*}} \to \mathbb{R}$ denote the antiderivative of g_δ, boundary modified on $\partial V_{\Lambda_\delta} \cap b$, such that $\tilde{I}^{\bullet}_{\delta}(x) = I^{\circ}_{\delta}[g_\delta](b_\delta)$ for each $x \in \partial V_{\Lambda_\delta} \cap b$ and that $\tilde{\Delta}^{\bullet}_{\delta}\tilde{I}^{\bullet}_{\delta}[g_\delta] \geq 0$, where $\tilde{\Delta}^{\bullet}_{\delta}$ is the boundary-modified Laplacian also defined in that section. Let Y be a doubly-connected subdomain of Λ with $\partial Y \cap \partial\Lambda = b$ and denote by $d \subset \partial Y$ the other connected component of ∂Y. Let $\tilde{H}^{\bullet}_{\delta} : \overline{V}_{Y_\delta} \to \mathbb{R}$ and $H^{\circ}_{\delta} : \overline{V}_{Y^{*}_{\delta}} \to \mathbb{R}$ be the harmonic parts of $\tilde{I}^{\bullet}_{\delta}$ and I°_{δ}, defined by

$$\begin{aligned}
\tilde{H}^{\bullet}_{\delta}(x) &= \tilde{I}^{\bullet}_{\delta}[u](x) \;\; \forall x \in \partial V_{Y_\delta}, \\
H^{\circ}_{\delta}(y) &= \tilde{I}^{\circ}_{\delta}[u](y) \;\; \forall y \in \partial V_{Y^{*}_{\delta}}, \\
\tilde{\Delta}^{\bullet}_{\delta}\tilde{H}^{\bullet}_{\delta}(x) &= 0 \;\; \forall x \in V_{Y_\delta}, \\
\Delta^{\circ}_{\delta}H^{\circ}_{\delta}(y) &= 0 \;\; \forall y \in V_{Y^{*}_{\delta}}.
\end{aligned}$$

From super and subharmonicity, we have, if we extend the functions defined on $\overline{V}_{Y^{*m}_{\delta}}$ to functions on Y in the usual, piecewise constant, way,

$$\tilde{H}^{\circ}_{\delta}(z) \leq I^{\circ}_{\delta}[u](z) \leq \tilde{I}^{\bullet}_{\delta}[u](z) \leq \tilde{H}^{\bullet}_{\delta}(z) \;\; \forall z \in Y.$$

By assumption, we have that $(g_{\delta_n})_{n\geq 0}$ is uniformly bounded on the compact subsets of Λ and hence that $I^{\bullet}_{\delta}$ and I°_{δ} are uniformly close near d and it follows that H°_{δ} and $\tilde{H}^{\bullet}_{\delta}$ are uniformly close near d; they are moreover equal on b. We deduce that since $\tilde{I}^{\bullet}_{\delta_n}$ and $I^{\circ}_{\delta_n}$ are uniformly bounded on the compact subsets of Λ, $I^{\circ}_{\delta_n}[g_{\delta_n}](b_{\delta_n})$ is uniformly bounded: if this were not the case, then $\tilde{H}^{\bullet}_{\delta_n}$ and $H^{\circ}_{\delta_n}$ would blow up in the same direction, forcing $I_{\delta_n}[u]$ to blow up, a contradiction. From there, we easily deduce that

$$\limsup_{n\to\infty} \left| \tilde{H}^{\bullet}_{\delta_n}(z) - I^{\circ}_{\delta_n}[g_{\delta_n}](b_{\delta_n}) \right| \underset{z\to b}{\longrightarrow} 0,$$
$$\limsup_{n\to\infty} \left| H^{\circ}_{\delta_n}(z) - I^{\circ}_{\delta_n}[g_{\delta_n}](b_{\delta_n}) \right| \underset{z\to b}{\longrightarrow} 0,$$

and the desired result follows. □

4.4. Extension of convergence to the boundary

The convergence results obtainable by the precompactness and identification results of the previous sections only apply in the bulk of the domain. However, the computation of the energy density with mixed boundary condition involve the values of s-holomorphic observables on the boundary on the domain and to treat many interesting cases we hence need to establish boundary convergence results for solutions to Riemann-Hilbert boundary value problems. We do this in the simplest case: when the boundary is *straight*, by which we mean parallel to the lattice: either horizontal or vertical in our case.

In such a case, we can obtain convergence result (using mostly the same techniques as [ChSm09]).

Proposition 54. *Let $(\Omega_\delta)_{\delta>0}$ be a discretization of a domain Ω with a straight boundary part $\partial^s\Omega \subset \partial\Omega$. Let $Y \subset \Omega$ be a neighborhood of $\partial\Omega$ with $\partial\Omega \subset \overline{Y}$. Let $\left(g_\delta : V_{Y^m_\delta} \to \mathbb{C}\right)$ be a family of s-holomorphic function such that for each $\delta > 0$, we have*

$$g_\delta(x) \parallel \frac{1}{\sqrt{\nu_{ext}(x)}} \quad \forall x \in \partial_0 V_{\Omega^m_\delta}$$

and such that as $\delta \to 0$, we have $g_\delta \to g$ uniformly on the compact subsets of Y for some holomorphic function g. Then the convergence extends uniformly to the compact subsets of $Y \cup \partial^s\Omega$.

Corollary 55. *Let $(\Omega_\delta)_{\delta>0}$ be a family of discrete domains approximating a domain Ω with a straight boundary part $\partial^s\Omega \subset \partial\Omega$. Let $\left(u_\delta : V_{\Omega^m_\delta} \to \mathbb{C}\right)_{\delta>0}$ be a family of s-holomorphic functions that are the solutions to discrete problems $(\spadesuit_{\Omega_\delta}, f_\delta)$. Suppose that there exists a neighborhood $Y \subset \Omega$ of $\partial\Omega$ with $\partial\Omega \subset \overline{Y}$ such that for each $\delta > 0$, f_δ extend to an s-holomorphic function $V_{Y^m_\delta} \to \mathbb{C}$ and such that $f_\delta \to f$ uniformly on $\overline{Y}$ for some f as $\delta \to 0$. Then if $u_\delta \to u$ uniformly on the compact subsets of Ω, where u is the solution to $(\spadesuit_\Omega, u)$, we have that u extends to $\Omega \cup \partial^s\Omega$ and that the convergence $u_\delta \to u$ extends uniformly to the compact subsets of $\Omega \cup \partial^s\Omega$.*

Proof of Corollary 55. Set $g_\delta = u_\delta - f_\delta$. Then it follows from Proposition 54 that g_δ converges uniformly on the compact subsets of $Y \cup \partial^s\Omega$. Since f_δ also converges on the compact subsets of $\overline{Y}$, we obtain the convergence of u_δ. □

Proof of Proposition 54. By the following two lemmas, the family of functions in uniformly bounded near $\partial^s\Omega$ and converges uniformly on the segments compactly contained in $\partial^s\Omega$ to a continuous function. Using harmonic measure estimates (since everything is bounded), is easy to conclude that the convergence extends to the compact subsets of $Y \cup \partial^s\Omega$.

Lemma 56. *With the notation and under the assumptions of Proposition 55, we have that for each straight segment s compactly contained in $\partial^s\Omega$, there exists a rectangle $R \subset \Omega$ with $\overline{R} \cap \partial\Omega = s$ such that the family of functions $(u_\delta)_{\delta>0}$ is uniformly bounded on $\overline{R}$.*

Lemma 57. *Let $(R_\delta)_{\delta>0}$ be a family of discrete domains approximating a rectangle R. Let s be a side of R and let $\left(g_\delta : V_{R^m_\delta} \to \mathbb{C}\right)_{\delta>0}$ be a family of s-holomorphic functions such that for each $\delta > 0$, we have the boundary condition*

$$g_\delta(x) \,\|\, \frac{1}{\sqrt{\nu_{ext}(x)}} \quad \forall x \in \partial_0 V_{R^\delta_m} \cap I_\delta.$$

Then if there exists a holomorphic function $g : \overline{R} \to \mathbb{C}$ such that

$$g_\delta \underset{\delta\to 0}{\longrightarrow} g$$

uniformly on the compact subsets of $\overline{R} \setminus s$ and if $(g_\delta)_\delta$ is uniformly bounded on $\overline{R}$, then the convergence $g_\delta \to g$ is uniform on the compact subsets of s.

$\square$

Let us now give the proof of these two lemmas.

Proof of Lemma 56. . We adapt the ideas of [ChSm09]. Let I be a segment compactly containing s and compactly contained in $\partial^s\Omega$. Let Λ be the connected component of $Y \cap \Omega$ near I and let R be a rectangle contained in Λ such that $R \cap \partial\Omega = I$. Let $\left(g_\delta : V_{\Lambda^m_\delta} \to \mathbb{C}\right)_\delta$ be the family of s-holomorphic functions defined, for each $\delta > 0$ by $g_\delta = u_\delta - f_\delta$. Then we have

$$g_\delta(x) \,\|\, \frac{1}{\sqrt{\nu_{ext}(x)}} \quad \forall x \in \partial_0 V_{\Lambda^m_\delta} \cap \partial\Omega$$

and hence we can use the boundary modification trick introduced in Section 2.6.1, defining the modified version $\tilde{I}_\delta[g_\delta] : \overline{V}_{\Lambda_\delta} \to \mathbb{R}$ of the antiderivative $I_\delta[u]$, which is locally constant on $\left(\partial_1 V_{\Lambda^*_\delta} \cup \partial_1 V_{\Lambda_\delta}\right) \cap \partial\Omega$, superharmonic on $V_{\Lambda^*_\delta}$ for the usual Laplacian and subharmonic on V_{Λ_δ} for the modified Laplacian $\tilde{\Delta}_\delta$ introduced there. Let us first show the following lemma:

Lemma 58. *For each $\varepsilon > 0$, there exists $C > 0$ such that for each $\delta > 0$ and each $v \in V_{R^m_\delta}$ such that $\mathrm{dist}(v, \partial R \setminus I) \geq \varepsilon$, we have*

$$\left|\tilde{I}_\delta[g_\delta](v) - \tilde{I}[g_\delta](I_\delta)\right| \leq C \cdot \mathrm{dist}(v, I).$$

Proof of Lemma 58. First of all, let us remark that the above quantity is uniformly bounded. Indeed, let $\gamma \subset \Lambda$ be a curve disconnecting $\partial\Lambda \cap \partial\Omega$ from $\partial\Lambda \setminus \partial\Omega$ and let Γ be the connected component of $\Lambda \setminus \gamma$ near $\partial\Lambda \cap \partial\Omega$. By superharmonicity, subharmonicity and the maximum's principle, we have, if we extend $\tilde{I}^{\bullet}_{\delta}$ and $\tilde{I}^{\circ}_{\delta}$ to functions $\Lambda \to \mathbb{C}$, for each $x \in \Gamma$,

$$\begin{aligned} \min\left(\tilde{I}_{\delta}[g_{\delta}](l_{\delta}), m_{\gamma}\right) &\leq \tilde{I}^{\circ}_{\delta}[g_{\delta}](x) \\ &\leq \tilde{I}^{\bullet}_{\delta}[g_{\delta}](x) \\ &\leq \max\left(\tilde{I}[g_{\delta}](l_{\delta}), M_{\gamma}\right), \end{aligned}$$

where $m_{\gamma} = \inf_{\delta>0} \inf_{z\in\gamma} \tilde{I}[g_{\delta}](z)$ and $M_{\gamma} = \sup_{\delta>0} \sup_{z\in\gamma} \tilde{I}[g_{\delta}](z)$. By Lemma, we have that 53 $\tilde{I}[g_{\delta}](l_{\delta}) - m_{\gamma}$ and $\tilde{I}[g_{\delta}](l_{\delta}) - M_{\gamma}$ are uniformly bounded and hence the uniform boundedness of $\left|\tilde{I}_{\delta}[g_{\delta}](v) - \tilde{I}[g_{\delta}](l_{\delta})\right|$ follows easily.

To simplify the notation, let us fix the additive constant of $\tilde{I}_{\delta}[g_{\delta}]$ so that $\tilde{I}[g_{\delta}](l_{\delta}) = 0$. Set $M_R = \sup_{\delta>0} \sup_{z\in R} \left|\tilde{I}[g_{\delta}]\right|$ and denote by $H^{\bullet}_{\delta} : V_{R_{\delta}} \to \mathbb{R}$ the harmonic measure of $\partial_1 V_{R_{\delta}\setminus l_{\delta}}$ in R_{δ} with respect to $\tilde{\Delta}_{\delta}$ and by $H^{\circ}_{\delta} : V_{R^*_{\delta}} \to \mathbb{R}$ the harmonic measure of $\partial_1 V_{R^*_{\delta}\setminus l_{\delta}}$ in R^*_{δ}, as defined in Proposition 38. Then, from superharmonicity and subharmonicity, we obtain that for each $z \in R$, we have (if we extend the discrete functions to functions $R \to \mathbb{R}$)

$$-M_R H^{\circ}_{\delta}(z) \leq \tilde{I}^{\circ}_{\delta}[g_{\delta}](z) \leq \tilde{I}^{\bullet}_{\delta}[g_{\delta}](z) \leq M_R H^{\bullet}_{\delta}(z).$$

Now, from the estimates of Lemma 39 on discrete harmonic measure, it is easy to obtain that for each $\varepsilon > 0$, there exists a constant $C > 0$ such that, uniformly for all $\delta > 0$ and $z \in R$ with $\mathrm{dist}(z, R \setminus l) \geq \varepsilon$, we have

$$\begin{aligned} H^{\circ}_{\delta}(z) &\leq C\,\mathrm{dist}(z, l), \\ H^{\bullet}_{\delta}(z) &\leq C\,\mathrm{dist}(z, l), \end{aligned}$$

which finishes the proof of Lemma 58. □

Let us now finish the proof of Lemma 56. By Lemma 58, we obtain that taking $R \subset \mathbb{R}$ such that $R \cap \partial\Omega = s$, we have, if we normalize $\tilde{I}_{\delta}[g_{\delta}]$ to be equal to 0 on s, that there exists C such that for each $\delta > 0$ and each $v \in R$

$$\left|\tilde{I}_{\delta}[g_{\delta}](v)\right| \leq C \cdot \mathrm{dist}(v, \partial\Omega).$$

By Proposition, 43, we readily obtain that g_{δ} is uniformly bounded. □

Proof of Lemma 57. Let us suppose without loss of generality that s is the right side of R (the three other cases are symmetric, modulo an adaptation of the phases). Then the boundary condition becomes

$$g_{\delta}(v) \in \mathbb{R} \;\forall v \in \partial_0 V_{R^m_{\delta}} \cap s_{\delta}$$

and we moreover have

$$g(v) \in \mathbb{R} \;\forall v \in s.$$

Let us remark that applying Schwarz reflection principle, we readility see that $g(v)$ is uniformly Lipschitz continuous near the compact subsets of s. What we have to show is that for each $z \in s$, we have

$$\limsup_{\delta\to 0} \sup_{x\in[z-i\varepsilon, z+i\varepsilon]} |g_{\delta}(x) - g(z)| \underset{\delta\to 0}{\longrightarrow} 0.$$

Let us denote by $h_{\delta} : V_{R^m_{\delta}} \to \mathbb{C}$ the function $x \mapsto g_{\delta}(x) - g(z)$, and by $h : R \to \mathbb{C}$ the function $x \mapsto g(x) - g(z)$, both of which still satisfy the same boundary condition as g_δ and g respectively. Let $\tilde{I}_\delta[h_\delta]$ denote the discrete antiderivative of h_δ^2 with the boundary modification trick applied on s, which is constant on $\left(\partial_0 V_{R_\delta^*} \cup \partial_0 V_{R_\delta}\right) \cap s$ and let $\tilde{I}_\delta^\bullet[h_\delta]$ denote its restriction to $V_{R_\delta^m}$, which is subharmonic with respect to the modified Laplacian. From Proposition 52, we have that

$$\tilde{I}_\delta[h_\delta] \xrightarrow[\delta\to 0]{} -\Re\left(\int h^2\right),$$

where the convergence is uniform on $\overline{R}$. Let us normalize $\tilde{I}_\delta[h_\delta]$ and $-\Re\left(\int h^2\right)$ to be equal to 0 on s. From the boundary condition, we deduce that

$$-\partial_x \Re\left(\int h^2\right) \leq 0$$

and hence we obtain that $-\Re\left(\int g^2\right)$ is nonnegative on a neighorhood of s. From there and the convergence $\tilde{I}_\delta[g_\delta] \to -\Re\left(\int g^2\right)$, we easily deduce that for each $\varepsilon > 0$ sufficiently small, the rectangle $R^\varepsilon = [z-\varepsilon] \times [z - i\varepsilon, z + i\varepsilon]$ is contained in R and is such that for each $\delta > 0$ sufficiently small, we have

$$\tilde{I}_\delta[h_\delta](x) \geq 0 \ \forall x \in V_{R_\delta^m} \cap R_\delta^\varepsilon.$$

Notice that we have

$$\partial_{\overline{v}_{ext}(x)} \tilde{I}_\delta^\bullet[h_\delta] = -\delta \cdot h_\delta^2 \ \forall x \in \partial_0 V_{R_\delta^m} \cap s_\delta$$

and that on the other hand, by subharmonicity of $\tilde{I}_\delta^\bullet$, we have, for each $\varepsilon > 0$ sufficiently small

$$\partial_{\overline{v}_{ext}(x)} \tilde{I}_\delta^\bullet[h_\delta] \leq \left(\max_{y \in R^\varepsilon} \tilde{I}_\delta^\bullet[h_\delta](y)\right) \cdot H_\delta^{\bullet \varepsilon}\left(x - \frac{\delta}{2}\right) \ \forall x \in \partial_0 V_{R_\delta^m} \cap [z - i\varepsilon, z + i\varepsilon],$$

where $H_\delta^{\bullet\varepsilon} : V_{R_\delta^m} \cap R^\varepsilon \to R$ denotes the discrete harmonic measure of $\partial_1 V_{R_\delta} \partial R^\varepsilon \setminus [z - i\varepsilon, z + i\varepsilon]$ in $\overline{V}_{R_\delta} \cap R^\varepsilon$. From the estimates of Lemma 39, we have that there exists a constant $C > 0$ such that for each $\varepsilon > 0$ and any $\delta > 0$, we have

$$H_\delta^{\bullet\varepsilon}\left(x - \frac{\delta}{2}\right) \leq C \cdot \delta \ \forall x \in \partial_0 V_{R_\delta^m} \cap \left[z - i\frac{\varepsilon}{2}, z + i\frac{\varepsilon}{2}\right].$$

Hence, for each $\varepsilon > 0$ and each $\delta > 0$, we have

$$h_\delta^2 \leq C \cdot \left(\max_{y \in R^\varepsilon} \tilde{I}_\delta^\bullet[h_\delta](y)\right),$$

and hence

$$\limsup_{\delta \to 0} \left(\max_{x \in \partial_0 V_{R_\delta^m} \cap [z - i\frac{\varepsilon}{2}, z + i\frac{\varepsilon}{2}]}\right) \leq C_0 \limsup_{\delta\to 0}\left(\max_{y \in R^\varepsilon} \tilde{I}_\delta^\bullet[h_\delta](y)\right),$$

so it remains to show that the right hand side converges to 0 as $\varepsilon \to 0$.

Since h_δ converges on the compact subsets of $\overline{R} \setminus \overline{s}$ to h, which is equal to 0 at z and which is Lipschitz near z, we have that there exists $C_1 > 0$ such for each $0 < \varepsilon_1 < \varepsilon$, there exists $\delta_0(\varepsilon_1)$ such that for $\delta \leq \delta_0(\varepsilon_1)$,

$$|h_\delta(x)| \leq C_1 |x - z| \ \forall x \in V_{R_\delta} \cap [z - \varepsilon, z - \varepsilon_1].$$

Since h_δ is uniformly bounded on $\overline{R}$ by a constant $M > 0$, by integrating h_δ^2 along straight segments, we deduce that for $\delta \leq \delta_0(\varepsilon_1)$

$$\tilde{I}_\delta^\bullet[h_\delta](x) \leq M^2 \varepsilon_1 + \frac{1}{3} C_1 |x - z|^3 \ \forall x \in V_{R_\delta} \cap R^\varepsilon.$$

And hence

$$\limsup_{\delta\to 0} \left(\max_{y\in R^{\varepsilon}} \tilde{\mathrm{I}}^{\bullet}_{\delta}[h_\delta](y) \right) \le M^2\varepsilon_1 + C_1\varepsilon^2.$$

Letting $\varepsilon_1 \to 0$, we obtain

$$\limsup_{\delta\to 0} \left(\max_{y\in R^{\varepsilon}} \tilde{\mathrm{I}}^{\bullet}_{\delta}[h_\delta](y) \right) \le C_1\varepsilon^2,$$

which converges to 0 as $\varepsilon \to 0$ and hence proves the desired result. □

4.5. Convergence of solutions to Riemann-Hilbert boundary value problems

Let us finish this chapter and summarize its results by the following convergence theorem.

Theorem 59. *Let Ω be a smooth domain with (possibly empty) straight boundary parts $\partial^s\Omega \subset \Omega$, $(\Omega_\delta)_{\delta>0}$ a discretization of Ω. Let $Y \subset \Omega$ be a neighborhood of $\partial\Omega$ such that $\partial\Omega \subset \overline{Y}$ and let P be a parameter space. Let $\left(u^p_\delta\right)^{p\in P}_{\delta>0}$ be a family of s-holomorphic functions such that for each $\delta > 0$ and each $p \in P$, $u^p_\delta : V_{\Omega^m_\delta} \to \mathbb{C}$ is the solution to the boundary value problem $\left(\spadesuit_{\Omega_\delta}, f^p_\delta\right)$ such that f^p_δ extends to an s-holomorphic function $V_{Y^m_\delta} \to \mathbb{C}$. Then if $f^p_\delta(z) \to f^p(z)$ in a uniformly continuous way on the compact subsets of $\left\{(z,p) \in \overline{Y} \times P\right\}$, we have that*

$$u^p_\delta(z) \underset{\delta\to 0}{\longrightarrow} u^p(z)$$

uniformly on the compact subsets of $\{(z,p) \in (\Omega \cup \partial^s\Omega) \times P\}$, where u^p is the solution to $(\spadesuit_\Omega, f^p)$, the existence thereof in particular exists, which depends continuously on p.

Proof. By Theorem 50, we have that the family $\left(u^p_\delta(z) : V_{\Omega^m_\delta} \times P \to \mathbb{C}\right)_\delta$ is uniformly equicontinuous and bounded on the compact subsets of $\{(z,p) \in \Omega \times P\}$. By extending the functions u^p_δ in a piecewise-linear way and by Arzelà-Ascoli's theorem, we have that this family admits convergent subsequences. By Corollary 52, we deduce

$$u^p_\delta(z) \underset{\delta\to 0}{\longrightarrow} u^p(z),$$

uniformly on the compact subsets of $\{(z,p) \in \Omega \times P\}$, continuously with respect to z and p, and in particular that the limit exists. By Corollary 55, we have that this convergence extends to the compact subsets of $\{(z,p) \in (\Omega \cup \partial^s\Omega) \times P\}$, and it is easy to see that it is continuous with respect to z and p. □

CHAPTER 5

Contour Statistics and Fermionic observables

In this chapter, we define the fermionic observables, that will allow us to prove the main theorems of this paper. More precisely, we will:

- Introduce classical contour representations of the Ising model:
  - The low-temperature representation, which is a way of mapping spin configurations to contour configurations, and hence to translate Ising partition functions as weighted sums over families of contours.
  - The high-temperature expansion, which is a way of representing partition functions and correlation functions as statistics over contours.
  - We explain the Kramers-Wannier duality which relates them.
  - We derive consequences for the energy density and spin fields.
- Introduce the discrete fermionic observables, which are complexified versions of these contour representations.
  - We introduce a complex phase on certain families contours, which is a compactification of its winding number and show its well-definedness.
  - We introduce the real discrete fermionic observables, which are antisymmetric functions defined on collection double-oriented medial vertices.
  - We express the discrete correlation functions of interest to us in terms of these observables
  - We introduce the discrete complex fermionic observables, which are modified versions of the real observables and will fit in our discrete complex analysis framework.

All the above quantities are basically signed weighted sums over certain families of contours. Let Ω_δ be, as in the rest of this chapter, a discrete domain, that is, an induced connected subgraph of the square grid $C_\delta = \delta Z^2$. We call contour or (edge) configuration a subcollection of E_{Ω_δ} or of its half-edges set H_{Ω_δ}. Let us define the contours that we will use:

- We denote by C_{Ω_δ} the set of subcollections $\omega \subset E_{\Omega_\delta}$ such that each vertex $v \in V_{\Omega_\delta}$ belongs to an even number of edges of ω. In other words, the configurations of C_{Ω_δ} are the set of contours that consist of (non necessarily simple) loops.
- For vertices $v_1, \ldots, v_n \in V_{\Omega_\delta}$, we denote by $C_{\Omega_\delta}(v_1, \ldots, v_n)$ the set of subcollections $\omega \subset E_{\Omega_\delta}$ such that each vertex $v \in V_{\Omega_\delta} \setminus \{v_1, \ldots, v_n\}$ belongs to an even number of edges of ω and such that each vertex $v \in \{v_1, \ldots, v_n\}$ belongs to an odd number of edges of ω. Informally, a configuration in $C_{\Omega_\delta}(v_1, \ldots, v_n)$ consists of a set of (non necessarily simple) loops, plus $\frac{n}{2}$ paths linking pairwise the points $v_1, \ldots, v_n$ (it is easy to see that this set is empty if n is odd).

- For half-edges $h_1, \ldots, h_n \in H_{\Omega_\delta}$ (or equivalently oriented medial vertices $a_1^{o_1}, \ldots, a_n^{o_n}$), we denote by $C_{\Omega_\delta}(h_1, \ldots, h_n)$ the set of subcollections of edges and half-edges $\omega \subset H_{\Omega_\delta}$ consisting of full edges plus the n half-edges $h_1, \ldots, h_n$, such that each vertex $v \in V_{\Omega_\delta}$ belongs to an even number of edges and half-edges of ω.
- For a contour set C like the above ones and ± 1-signed edges $e_1^{s_1}, \ldots, e_m^{s_m} \in E_{\Omega_\delta}$ we denote by $C^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}$ the subfamily of C defined by

$$C^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}} = \{\gamma \in C : e_k \in \gamma \iff s_k = -1 \ \forall k = 1, \ldots, m\}.$$

5.1. Classical Contour Representations

5.1.1. Low-temperature expansion. The low-temperature expansion of the Ising model is a natural graphical representation of the model which was introduced by Peierls in 1933 to show the existence of a phase transition in the Ising model. Informally, the idea is to represent a configuration by tracing the contours of its spin clusters, or equivalently its interfaces, and to express partition functions as weighted sums over families of contours.

Consider the Ising model on the dual graph Ω_δ^*. With each spin configuration $\sigma \in \{\pm 1\}^{V_{\Omega_\delta^*}}$ we associate and edge configuration $\omega(\sigma) \subset E_{\Omega_\delta^*}$ defined by

$$\langle v, w \rangle^* \in \omega(\sigma) \iff \sigma_v \neq \sigma_w \ \forall \langle v, w \rangle \in E_{\Omega_\delta^*}.$$

The proof of the following is elementary:

Proposition 60. *The map $\sigma \mapsto \omega(\sigma)$ is a surjective two-to-one mapping between the set of spin configurations with locally monochromatic boundary condition on $\partial\Omega^*$ and the set of contours C_{Ω_δ}, and more generally between the spin configurations with locally monochromatic boundary conditions b alternating at $b_1, \ldots, b_{2n} \in \partial E_{\Omega_\delta}$ (with an even number of b_j's adjacent to each connected component of $\partial_0\Omega_\delta$) and the set of contours $C_{\Omega_\delta}(x_1, \ldots, x_{2n})$, where $x_1, \ldots, x_{2n}$ are the endpoints in V_{Ω_δ} of $b_1, \ldots, b_{2n}$ respectively.*

Remark 61. If we fix the sign of a given spin, then $\sigma \mapsto \omega(\sigma)$ becomes a bijection. In particular, if Ω_δ^* is simply connected, it realizes a bijection between the spin configurations of the Ising model $+$ boundary condition and C_{Ω_δ}.

It is also easy to check the following:

Proposition 62. *The probability on $C_{\Omega_\delta}(x_1, \ldots, x_{2n})$ induced by the mapping $\sigma \mapsto \omega(\sigma)$ is such that for each $\omega \in C_{\Omega_\delta}(x_1, \ldots, x_{2n})$,*

$$P\{\omega\} \propto \alpha_\square^{|\omega|},$$

where $|\omega|$ denotes the number of edges of ω and $\alpha_\square = e^{-2\beta}$, where β is the inverse temperature of the Ising model. In particular at the critical inverse temperature $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$, we have $\alpha_\square = \alpha = \sqrt{2}-1$.

Given a boundary condition $b_1, \ldots, b_{2n}$, we denote by $Z_{\Omega_\delta}(b_1, \ldots, b_{2n})$ the *low-temperature partition function*, defined by

$$Z_{\Omega_\delta}(b_1, \ldots, b_{2n}) = \sum_{\omega \in C_{\Omega_\delta}(b_1^\times, \ldots, b_{2n}^\times)} \alpha^{|\omega|}.$$

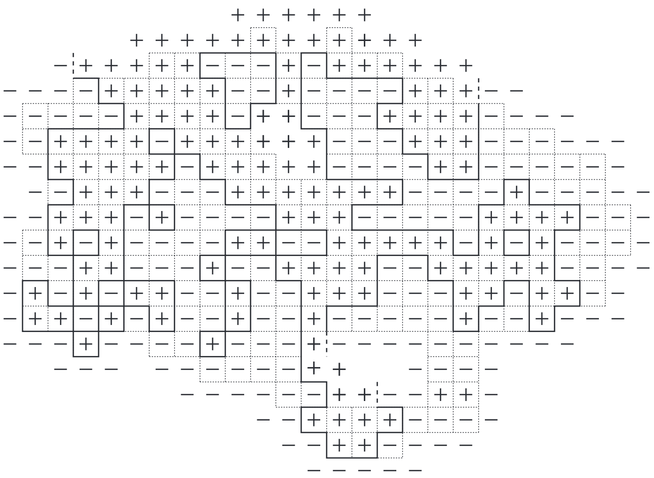

Figure 5.1.1. The edge configuration associated with the Ising configuration of Figure 1.4.2, with the edge of the configuration drawn with bold strokes, the other edges of E_{Ω_δ} drawn with dotted strokes and the boundary changing operators with dashed strokes.

Given signed edges $e_1^{s_1}, \ldots, e_m^{s_m}$, we define the **restricted low-temperature partition function** as

$$Z_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(b_1, \ldots, b_{2n}) = \sum_{\omega \in \mathcal{C}_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(b_1^\times,\ldots,b_{2n}^\times)} \alpha^{|\omega|}.$$

5.1.2. **High-temperature expansion.** The high-temperature expansion of the Ising model is a slightly more involved representation of the Ising model, which was introduced by Kramers and Wanniers in 1941 and allowed for the first derivation of the critical temperature of the Ising model, through the Kramers-Wannier duality (see below). Unlike the low-temperature expansion, it does not consists of a mapping between spin and edge configurations, but rather in a way of computing in a graphical way partition functions, allowing notably for a powerful representation of spin correlations. A notable difference is also that in this case, the contours involved live on the same graph as the Ising model – and not on the dual graph. We state here the version that we will need in this text. The Kramers-Wannier duality in general will be shorty discussed in the next section.

Proposition 63. *Consider the Ising model on Ω_δ with free boundary condition at inverse temperature β and denote by $Z_{\Omega_\delta}^{\text{free}}$ its partition function. Then we have*

$$Z_{\Omega_\delta}^{\text{free}} = 2^{|V_{\Omega_\delta}|} (\cosh \beta)^{|E_{\Omega_\delta}|} \sum_{\omega \in \mathcal{C}_{\Omega_\delta}} \alpha_h^{|\omega|},$$

where $\alpha_h = \tanh \beta$ and $|\omega|$ is the number of edges of ω. In particular, at the critical value $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$ of β, we have $\alpha_h = \alpha = \sqrt{2}-1$. More generally, for distinct vertices $v_1, \ldots, v_{2n} \in V_{\Omega_\delta}$, if we denote by $Z_{\Omega_\delta}^{\text{free}}(v_1, \ldots, v_{2n})$ the partition function

$$Z^{\mathrm{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n}) = \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \sigma_{v_1} \cdot \ldots \cdot \sigma_{v_{2n}} \cdot e^{-\beta H(\sigma)},$$

we have

$$Z^{\mathrm{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n}) = 2^{|V_{\Omega_\delta}|} (\cosh \beta)^{|E_{\Omega_\delta}|} \sum_{\omega \in C_{\Omega_\delta}(v_1,\ldots,v_{2n})} \alpha_h^{|\omega|}.$$

In particular, we have

$$\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_\delta(v_1) \cdot \ldots \cdot \sigma_\delta(v_{2n})] = \left(\sum_{\omega \in C_{\Omega_\delta}(v_1,\ldots,v_{2n})} \alpha_h^{|\omega|} \right) / \left(\sum_{\omega \in C_{\Omega_\delta}} \alpha_h^{|\omega|} \right).$$

In the critical case $\beta = \beta_c$, we can rewrite this latter ratio as

$$Z_{\Omega_\delta}(v_1, \ldots, v_{2n}) / Z_{\Omega_\delta},$$

where Z_{Ω_δ} is as defined in the previous paragraph.

Proof. See Appendix A. □

5.1.3. Kramers-Wannier duality. It is remarkable that contours appearing in the low-temperature representation of Ising model on Ω_δ^* and in the high-temperature expansion of correlations on Ω_δ are the same (for the appropriate boundary conditions). This is a particular case of the general Kramers-Wannier duality, which informally "exchanges" data about the model, in this sense: low-temperature expansions of certain quantities with certain data in one model are equal (up to multiplicative constant) to the high-temperature expansions of dual quantities with dual data. The exchanged quantities and data are the following data:

- The graph Ω_δ and its dual Ω_δ^*.
- The temperature β and a dual temperature β^* – the critical temperature is self-dual.
- Locally monochromatic boundary conditions and free boundary conditions.
- Boundary condition changing operators and boundary spin operators.

There is a way to make precise a general version of this duality, with an involutive operation that involves complex terms in the Hamiltonian. We will not describe it here, but rather focus on the implications

Proposition 64. Consider both the critical Ising model on Ω_δ^* with locally monochromatic boundary condition and the critical Ising model on Ω_δ with free boundary condition. Let $e_1, \ldots, e_m \in E(\Omega_\delta)$ be distinct edges. Then we have

$$\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\epsilon_\delta(e_1) \cdot \ldots \cdot \epsilon_\delta(e_m)] = (-1)^m \, \mathbb{E}^{\varnothing}_{\Omega_\delta^*}[\epsilon_\delta(e_1^*) \cdot \ldots \cdot \epsilon_\delta(e_m^*)],$$

where $\varnothing$ denotes the locally monochromatic boundary condition with no boundary changing operators.

In other words, the discrete energy field with free boundary condition is equal to minus the discrete energy field with locally monochromatic boundary condition.

Proof. See Appendix A. □

5.2. Discrete fermionic observables

We now introduce the discrete fermionic observables, that are the central tool to compute the correlation functions. They are related to both low- and high-temperature expansions, in the following sense:

- They can be viewed as discrete deformations of the low-temperature expansions of energy correlations.
- They are very similar to high-temperature expansion of correlation functions, with the difference that complex phases are added to the weights of the contours.

We introduce first the real-valued versions of these observables, for which convenient and clean formulae will be obtained in further chapters and which give nice representations of the discrete correlation functions of interest. Further in this section, we will introduce the complex version, which are just slightly rephased and which fit in the discrete complex analysis setting that we will develop later, a fact that will permit to derive the above mentioned formulae and eventually to obtain convergence to continuous observables.

We then define the real and complex versions of the full-plane observable, introduced in Chapter 2.

As mentioned above, the central point in the observables is the presence of a complex phase in the weights of the contours, this complex phase being a compactification of a topological notion: the winding number. The first part of this section is devoted to elementary properties of this phase, in particular, its well-definedness.

The next ones are devoted to the definition of the observables themselves.

5.2.1. Winding numbers and complex phases.

Let $a, b \in V_{\Omega_\delta^m}$ be two medial vertices and let $\gamma = \langle a, v_1, \ldots, v_n, b\rangle$ be a **walk** from a to b, consisting of (non-necessarily distinct) vertices $v_1, \ldots, v_n \in V_{\Omega_\delta}$ with $v_j \sim v_{j+1}$ for each $j \in \{1, \ldots, n-1\}$ and with $a \sim v_1$ and $v_{2n} \sim b$.

We define the winding number $w(\gamma)$ of γ as the total rotation of γ from a_1 to a_2, when going along the sequence of edges $\langle a_1, v_1\rangle, \langle v_1, v_2\rangle, \ldots, \langle v_{n-1}, v_n\rangle, \langle v_n, a_2\rangle$: it is defined as $\frac{\pi}{2}(n_\ell - n_r)$, where n_ℓ and n_r denote the number of left and right turns made by γ respectively.

Let us recall that, as defined in Section 1.7.3, we denote by $D_{\Omega_\delta^m}$ the set of doubly-oriented medial vertices of Ω_δ, that is, the set of medial vertices

$$\left\{x^o : x \in V_{\Omega_\delta^m}, o \in (O)^2(x)\right\}$$

equipped with a double orientation (i.e. an orientation with a specified square root, as defined in Section 1.7.2).

We now introduce the notion of winding phase. Let $a_1^{o_1}, a_2^{o_2} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices and γ a walk from a_1 to a_2 using the two half-edges specified by the orientations o_1 and o_2. We call **oriented winding phase** of γ and denote by $\varphi(\gamma, o_1, o_2) \in S$ the phase defined by

$$\varphi(\gamma, o_1, o_2) = i \cdot \sqrt{\frac{o_2}{o_1}} \cdot \exp\left(-\frac{i}{2} w(\gamma)\right).$$

Let us now define the **winding phase** of a configuration. Let $a_1^{o_1}, \ldots, a_{2n}^{o_{2n}} \in D_{\Omega_\delta^m}$ be distinct doubly-oriented medial vertices and let $C_{\Omega_\delta}(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}})$ be the set of configurations defined at the beginning of this chapter. For a configuration

$\omega \in C_{\Omega_\delta}(a_1^{o_1}, \dots, a_{2n}^{o_{2n}})$, we call admissible choice of walks on ω a collection of n walks $\gamma_1, \dots, \gamma_n \subset \omega$ linking pairwise the medial vertices $a_1, \dots, a_{2n}$, each of them being oriented from the medial vertex of lower index to the one of higher index. It is easy to see that such an admissible choice of walks always exists and is in general not unique – for non-trivalent graphs like the square grid.

Definition 65. We define the winding phase $\varphi(\gamma_1, \dots, \gamma_n)$ by

$$\varphi(\gamma_1, \dots, \gamma_n, o_1, \dots, o_{2n}) = (-1)^{c(\gamma_1,\dots,\gamma_n)} \prod_{\gamma_j : a_{\iota_j} \to a_{\tau_j}} \varphi\left(\gamma_j, o_{\iota_j}, o_{\tau_j}\right),$$

where $(-1)^{c(\gamma_1,\dots,\gamma_n)}$ is the crossing signature in the upper half-plane of the pair partition $\{\{\iota_j, \tau_j\} : j \in \{1, \dots, n\}\}$ of $\{1, \dots, 2n\}$ induced by the paths γ_j: if we link the numbers $\{1, \dots, 2n\} \subset \mathbb{R}$ by simple paths in the upper half-plane in general position, it is easy to see that the number of crossings points of these paths is well-defined modulo 2, and we define $(-1)^{c(\gamma_1,\dots,\gamma_n)}$ to be 1 if this number is even and -1 if this number is odd.

Remark 66. Another way of defining $(-1)^{c(\gamma_1,\dots,\gamma_n)}$ is as follows: if we reorder the indices $j \in \{1, \dots, n\}$ in such a way that $(\iota_j)_j$ is increasing, it is easy to see that we have $(-1)^{c(\gamma_1,\dots,\gamma_n)} = \mathrm{sgn}\left(\sigma_{(\gamma_1,\dots,\gamma_n)}\right)$, where $\sigma_{(\gamma_1,\dots,\gamma_n)} \in S_{2n}$ is the permutation defined by $\sigma_{(\gamma_1,\dots,\gamma_n)}(2k-1) = \iota_k$ and $\sigma_{(\gamma_1,\dots,\gamma_n)}(2k) = \tau_k$ for $k = 1, \dots, n$.

The following proposition allows us to define the oriented phase of a configuration:

Proposition 67. *Let $a_1^{o_1}, \dots, a_{2n}^{o_{2n}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices. Then for each configuration $\omega \in C_{\Omega_\delta}(a_1^{o_1}, \dots, a_{2n}^{o_{2n}})$, the winding phases $\varphi(\gamma_1, \dots, \gamma_n, o_1, \dots, o_{2n})$ and $\varphi(\tilde\gamma_1, \dots, \tilde\gamma_n, o_1, \dots, o_{2n})$ of any two admissible choices of walks on ω are the same.*

We denote by $\varphi(\omega, o_1, \dots, o_{2n})$ the winding phase of ω defined as the winding phase of its admissible choices of walks. The winding phase is antisymmetric with respect to the permutations of the indices $\{1, \dots, 2n\}$.

Proof. See Appendix B. ∎

Another important feature of the winding phase is that it is fixed when the medial vertices are all on the boundary. Recall that for a boundary middlepoint $x \in \partial_0 V_{\Omega_\delta^m}$, we call inward-pointing (simple) orientation at x the orientation

$$o = \frac{\overline{v}_{\mathrm{int}}(x)}{|\overline{v}_{\mathrm{int}}(x)|} \in O(x),$$

and that we call a double-orientation $\tilde o \in (O)^2(x)$ inward-pointing a double orientation if it gets identified with the inward-pointing simple orientation at x.

Proposition 68. *Let $a_1, \dots, a_{2n} \in \partial_0 V_{\Omega_\delta^m}$ be boundary medial vertices such that each connected component of $\partial\Omega_\delta$ contains an even number of a_j's let and $o_1, \dots, o_{2n} \in (S)^2_{\square}$ be inward-pointing double orientations at $a_1, \dots, a_{2n}$. Then for each $\omega \in C_{\Omega_\delta}(a_1^{o_1}, \dots, a_{2n}^{o_{2n}})$, the winding phase $\varphi(\omega)$ is the same.*

Proof. See Appendix B. ∎

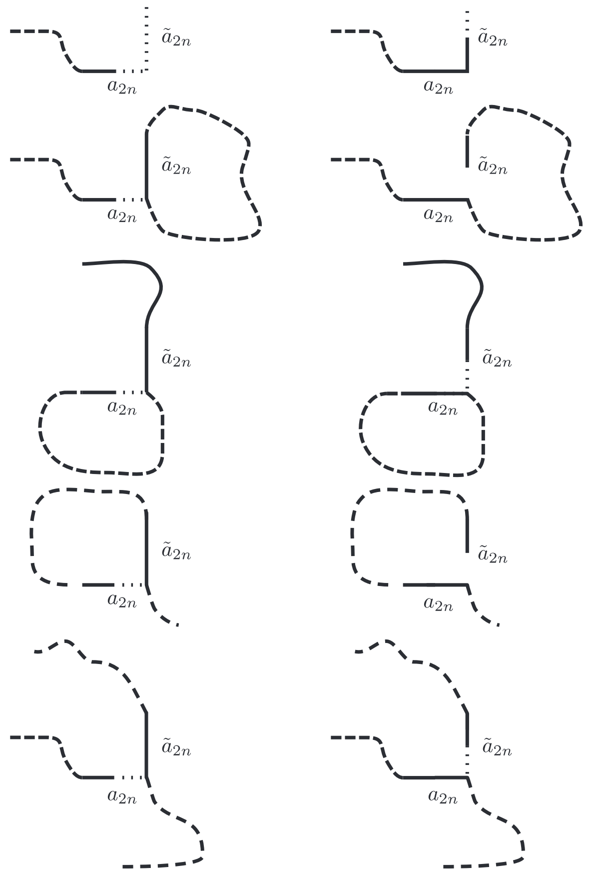

Figure 5.2.1. A configuration in $C_{\Omega_\delta}^{\{e_1^+, e_2^-, e_3^-\}}(a_1, \ldots, a_{10})$ with an admissible choice of walks (drawn with white on black paths) $\gamma_1 : a_1 \rightsquigarrow a_7$, $\gamma_2 : a_2 \rightsquigarrow a_8$, $\gamma_3 : a_3 \rightsquigarrow a_4$, $\gamma_4 : a_5 \rightsquigarrow a_6$, $\gamma_5 : a_9 \rightsquigarrow a_{10}$. If we put for instance the orientations $o_1 = (1)^2$, $o_2 = (\bar{\lambda})^2$, $o_3 = (-1)^2$, $o_4 = (1)^2$, $o_5 = (-\bar{\lambda})^2$, $o_6 = (1)^2$, $o_7 = (i)^2$, $o_8 = (\lambda)^2$, $o_9 = (\lambda)^2$, $o_{10} = (-i)^2$ on $a_1, \ldots, a_{10}$, we have $\varphi(\gamma_1, o_1, o_7) = 1$, $\varphi(\gamma_2, o_2, o_8) = i$, $\varphi(\gamma_3, o_3, o_4) = -1$, $\varphi(\gamma_4, o_5, o_6) = 1$, $\varphi(\gamma_5, a_9, a_{10}) = 1$ and $(-1)^{c(\gamma_1, \ldots, \gamma_n)} = -1$, which gives $\varphi(\gamma_1, \ldots, \gamma_5, o_1, \ldots, o_{10}) = -1$.

5.2.2. The real fermionic observables. The previous proposition allows us, for doubly oriented medial vertices $a_1^{o_1}, \ldots, a_{2n}^{o_{2n}} \in D_{\Omega_\delta^m}$ to define the **real fermionic**

observable $f_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$ by

$$f_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = \frac{1}{Z_{\Omega_\delta}}\left(\sum_{\omega\in C_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})} \alpha^{|\omega|}\varphi(\omega,o_1,\ldots,o_{2n})\right),$$

where Z_{Ω_δ} is as defined at the end of the previous section and $|\omega|$ denotes the number of edges of ω, with each half-edge counting $\frac{1}{2}$. As for the partition function, given signed edges $e_1^{s_1},\ldots,e_m^{s_m}$ distinct from $a_1,\ldots,a_{2n}$, we define the **restricted real fermionic observable** by

$$f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = \frac{1}{Z_{\Omega_\delta}}\left(\sum_{\omega\in C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(a_1^{o_1})} \alpha^{|\omega|}\varphi(\omega,o_1,\ldots,o_{2n})\right).$$

Given a collection of doubly-oriented edges $(\ldots)$, a collection of signed edges $\{\ldots\}$ disjoint from $(\ldots)$ and edges $e_1,\ldots,e_m$, we define the **fused real fermionic observable** $f_{\Omega_\delta}^{[e_1,\ldots,e_m]\{\ldots\}}(\ldots)$ inductively by

$$f_{\Omega_\delta}^{[e_1,\ldots,e_m]\{\ldots\}}(\ldots) = f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]\{\ldots,e_m^+\}}(\ldots) - \frac{1+\mu}{2}\cdot f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]\{\ldots\}}(\ldots)$$

where $\mu = \sqrt{2}-1$. It is easy to check that this definition does not depend on the order of $e_1,\ldots,e_m$.

Remark 69. As will be shown later (in Chapter 6), a fused observable $f_{\Omega_\delta}^{[e_1,\ldots,e_m]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$ corresponds informally to a $2n+2m$-point unfused observable, where $2m$ points are merged pairwise together at the edges $e_1,\ldots,e_m$, and the other points are $a_1^{o_1},\ldots,a_{2n}^{o_{2n}}$, which justifies the denomination.

5.2.3. **Complex fermionic observables and weights.** We now define a slightly modified complex variant of the real fermionic observable that will fit the discrete complex analysis framework detailed in the next chapter, hence enabling to derive formulae for the real observables and later to pass to the scaling limit. Let $\{\ldots\}$ be a collection of signed edges, $[\ldots]$ a collection of edges and $a_1^{o_1},\ldots,a_{2n}^{o_{2n}} \in D_{\Omega_\delta^m}$ be doubly oriented medial vertices. We define the **complex fermionic observable** $h_{\Omega_\delta}^{[\ldots]\{\ldots\}}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$ by

$$h_{\Omega_\delta}^{[\ldots]\{\ldots\}}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = -\sqrt{\frac{i}{o_{2n}}}\cdot f_{\Omega_\delta}^{[\ldots]\{\ldots\}}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}).$$

The complex fermionic observable hence does not depend on the branch choice of o_{2n} (i.e. we can take $o_{2n}\in S$ rather than in $(S)^2$). When we do not specify any orientation for the last medial vertex $a_{2n}\in V_{\Omega_\delta^m}$ (and keep $a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}}\in D_{\Omega_\delta^m}$), we define $h_{\Omega_\delta}^{[\ldots]\{\ldots\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$ by

$$h_{\Omega_\delta}^{[\ldots]\{\ldots\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) = \sum_{o_{2n}\in O(a_{2n})} h_{\Omega_\delta}^{[\ldots]\{\ldots\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}^{o_{2n}}\right),$$

where $O(a_{2n})$ is the set of the two admissible simple orientations of a_{2n} (± 1 if it is a horizontal medial vertex, $\pm i$ if it is a vertical one). We call the observables **fused** if they contain edges in the brackets $[\ldots]$ and **unfused** otherwise.

Finally, let us define a quantity which will be very useful in the proofs of Chapter 6. For a collection $a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ of doubly-oriented medial vertices, a medial vertex $a_{2n} \in V_{\Omega_\delta^m}$, and a configuration $\omega \in C_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$, let us denote by $W_h(\omega, o_1, \ldots, o_{2n-1})$ the **complex weight** of ω, defined by

$$W_h(\omega, o_1, \ldots, o_{2n-1}) = \sqrt{\frac{i}{o_{2n}}} \cdot \alpha^{|\omega|} \varphi(\omega, o_1, \ldots, o_{2n})$$

for any choice branch choice of o_{2n} such that $\omega \in W_h(\omega, o_1, \ldots, o_{2n-1})$ (it is easy to check that this is independent of the branch choice). With this notation, we have in particular

$$h_{\Omega_\delta}^{\{\ldots\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) = \frac{1}{Z_{\Omega_\delta}} \left(\sum_{\omega \in C_{\Omega_\delta}^{\{\ldots\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)} W_h(\omega, o_1, \ldots, o_{2n-1}) \right).$$

Remark 70. The use of double-orientations to define complex discrete observables is reminiscent of the spinors used in the treatment of the Ising model (see [KaCe71, McWu73], for instance).

5.2.4. Full-plane: real and complex versions. In Section 2.4, we defined a two-point function $h_{\mathbb{C}_\delta}(a_1^{o_1}, \cdot) : V_{\mathbb{C}_\delta^m \setminus \{a_1\}} \to \mathbb{C}$ for $a_1^{o_1} \in D_{\mathbb{C}_\delta^m}$, called full-plane complex fermionic observable, using the Green's function for the $\overline{\partial}_\delta$ operator. Somewhat in reverse order of construction, compared to the fermionic observables for domains, we define the following functions:

Definition 71. We $h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2})$ for doubly-oriented medial vertices $a_1^{o_1}, a_2^{o_2} \in D_{\mathbb{C}_\delta^m}$ with $a_1 \neq a_2$ by

$$h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2}) = \mathsf{P}_{\sqrt{\frac{i}{o_2}} \cdot \mathbb{R}}\left[h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2)\right]$$

and we define the **real full-plane fermionic observable** $f_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2})$ by

$$f_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2}) = i\sqrt{\overline{o_2}} \cdot h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2}).$$

As before, notice that $h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2})$ does not depend on the branch choice of o_2.

5.3. Representations of discrete correlation functions

We finish this chapter by connecting the fermionic observables with the discrete correlation functions of the main theorems, allowing for a unified representation of the latter by the former.

Proposition 72. *Consider the critical Ising model on Ω_δ^* with locally monochromatic boundary condition with boundary changing operators at the edges $b_1, \ldots, b_{2p} \in \partial E_{\Omega_\delta}$ such that an even number of b_j's are incident to each component of $\partial_0 \Omega_\delta$, and let $a_1, \ldots, a_n \in E_{\Omega_\delta}$ be interior edges. Then, if we denote by $v_1, \ldots, v_{2p} \in \partial_0 V_{\Omega_\delta^m}$ the medial vertices $m(b_1), \ldots, m(b_{2p})$, for any choice of inward-pointing normal double orientations $o_1, \ldots, o_p \in (\mathbb{S})^2$ at $v_1, \ldots, v_{2p}$, we have*

$$\mathbb{E}_{\Omega_\delta^*}^{(b_1, \ldots, b_{2p})}\left[\epsilon_\delta(a_1) \cdot \ldots \cdot \epsilon_\delta(a_n)\right] = (-1)^n \cdot 2^n \cdot \frac{f_{\Omega_\delta}^{[a_1, \ldots, a_n]}\left(v_1^{o_1}, \ldots, v_{2p}^{o_{2p}}\right)}{f_{\Omega_\delta}\left(v_1^{o_1}, \ldots, v_{2p}^{o_{2p}}\right)}. \tag{5.3.1}$$

Proof. The winding phase $\varphi(\omega, o_1, \dots, o_{2p})$ is the same for all $\omega \in \mathcal{C}_{\Omega_\delta}\left(v_1^{o_1}, \dots, v_{2p}^{o_{2p}}\right)$, by Proposition 68. Hence the complex phase φ factors out of the numerator and the denominator of the right hand side of Equation 5.3.1. From each of the configurations in $\mathcal{C}_{\Omega_\delta}\left(v_1^{o_1}, \dots, v_{2p}^{o_{2p}}\right)$, we remove the half-edges $\langle v_1, x_1\rangle, \dots, \langle v_{2p}, x_{2p}\rangle \in \partial_0 H_{\Omega_\delta^m}$ with $v_j = m(b_j)$ and $x_j \in \partial_0 V_{\Omega_\delta}$ incident to b_j for each $j \in \{1, \dots, 2p\}$, hence dividing the weights $\alpha^{|\cdot|}$ of each of these configurations by α^p. We obtain that

$$\frac{f_{\Omega_\delta}^{[a_1,\dots,a_n]}\left(v_1^{o_1}, \dots, v_{2p}^{o_{2p}}\right)}{f_{\Omega_\delta}\left(v_1^{o_1}, \dots, v_{2p}^{o_{2p}}\right)} = \frac{Z_{\Omega_\delta}^{[a_1,\dots,a_n]}(x_1, \dots, x_{2p})}{Z_{\Omega_\delta}(x_1, \dots, x_{2p})},$$

where $Z_{\Omega_\delta}(x_1, \dots, x_{2p})$ is as defined above and $Z_{\Omega_\delta}^{[a_1,\dots,a_n]}(x_1, \dots, x_{2p})$ is inductively defined, similarly to the fused observable $f^{[\dots]}(\dots)$ above, by

$$\begin{aligned} Z_{\Omega_\delta}^{[a_1,\dots,a_n]\{\dots\}}(x_1, \dots, x_{2p}) &= Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]\{\dots,a_n^+\}}(x_1, \dots, x_{2p}) \\ &\quad - \frac{1+\mu}{2} \cdot Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]\{\dots\}}(x_1, \dots, x_{2p}), \end{aligned}$$

where $\mu = \frac{\sqrt{2}}{2}$.

What remains to show is that we have:

$$\mathbb{E}_{\Omega_\delta^*}^{(b_1,\dots,b_{2p})}\left[\epsilon_\delta(a_1) \cdot \ldots \cdot \epsilon_\delta(a_n)\right] = (-1)^n \cdot 2^n \cdot \frac{Z_{\Omega_\delta}^{[a_1,\dots,a_n]}(x_1, \dots, x_{2p})}{Z_{\Omega_\delta}(x_1, \dots, x_{2p})}.$$

From Proposition 60, it is easy to show that we have

$$\mathbb{E}_{\Omega_\delta^*}^{(b_1,\dots,b_{2p})}\left[\epsilon_\delta(a_1) \cdot \ldots \cdot \epsilon_\delta(a_n)\right] = (-1)^n \cdot \frac{Z^{\langle a_1,\dots,a_n\rangle}(x_1, \dots, x_{2p})}{Z_{\Omega_\delta}(x_1, \dots, x_{2p})},$$

where $Z_{\Omega_\delta}^{\langle a_1,\dots,a_n\rangle}(x_1, \dots, x_{2p})$ is inductively defined by

$$\begin{aligned} Z_{\Omega_\delta}^{\langle a_1,\dots,a_n\rangle\{\dots\}}(x_1, \dots, x_{2p}) &= Z_{\Omega_\delta}^{\langle a_1,\dots,a_{n-1}\rangle\{\dots,a_n^+\}}(x_1, \dots, x_{2p}) \\ &\quad - Z_{\Omega_\delta}^{\langle a_1,\dots,a_{n-1}\rangle\{\dots,a_n^-\}}(x_1, \dots, x_{2p}) \\ &\quad - \mu Z_{\Omega_\delta}^{\langle a_1,\dots,a_{n-1}\rangle}(x_1, \dots, x_{2p}). \end{aligned}$$

Assuming by induction (the case $n = 1$ being trivially true) that

$$Z^{\langle a_1,\dots,a_{n-1}\rangle} = 2^{n-1} \cdot Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]}(x_1, \dots, x_{2p}),$$

let us show that

$$Z^{\langle a_1,\dots,a_n\rangle} = 2^n \cdot Z_{\Omega_\delta}^{[a_1,\dots,a_n]}(x_1, \dots, x_{2p}).$$

If we expand (omitting the boundary points $(x_1, \dots, x_{2p})$), using the induction assumption and that $Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]\{a_n^+\}} + Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]\{a_n^-\}} = Z_{\Omega_\delta}^{[a_1,\dots,a_{n-1}]}$ and putting together, we obtain

$$
\begin{aligned}
& Z^{\langle a_1,\ldots,a_n\rangle}(\ldots) \\
= \; & Z_{\Omega_\delta}^{\langle a_1,\ldots,a_{n-1}\rangle\{a_n^+\}}(\ldots) - Z_{\Omega_\delta}^{\langle a_1,\ldots,a_{n-1}\rangle\{a_n^-\}}(\ldots) - \mu Z_{\Omega_\delta}^{\langle a_1,\ldots,a_{n-1}\rangle}(\ldots) \\
= \; & 2^{n-1}\left(Z_{\Omega_\delta}^{[a_1,\ldots,a_{n-1}]\{a_n^+\}}(\ldots) - Z_{\Omega_\delta}^{[a_1,\ldots,a_{n-1}]\{a_n^-\}}(\ldots) - \mu Z_{\Omega_\delta}^{[a_1,\ldots,a_{n-1}]}(\ldots)\right) \\
= \; & 2^{n-1}\left(2\cdot Z_{\Omega_\delta}^{[a_1,\ldots,a_{n-1}]\{a_n^+\}}(\ldots) - (1+\mu) Z_{\Omega_\delta}^{[a_1,\ldots,a_{n-1}]}(\ldots)\right) \\
= \; & 2^{n}\left(Z_{\Omega_\delta}^{[a_1,\ldots,a_n]}(\ldots)\right),
\end{aligned}
$$

which is the desired result. ∎

Proposition 73. Consider the critical Ising model on Ω_δ with free boundary condition and let $v_1,\ldots,v_{2n} \in \partial_0 V_{\Omega_\delta}$ be boundary vertices such that there are an even number of v_j's on each connected component of $\partial_0\Omega_\delta$. Let $w_1,\ldots,w_{2n} \in \partial_0 V_{\Omega_\delta^m}$ be the closest boundary medial vertices to $v_1,\ldots,v_{2n}$. Then, for any choice of inward-pointing normal double orientations $o_1,\ldots,o_{2n} \in (\mathbb{S})^2$ at $v_1,\ldots,v_{2n}$, we have

$$\mathbb{E}_{\Omega_\delta}^{\mathrm{free}}[\sigma_\delta(v_1)\cdot\ldots\cdot\sigma_\delta(v_{2n})] = \frac{1}{\alpha^n}\,|f_{\Omega_\delta}(w_1^{o_1},\ldots,w_{2n}^{o_{2n}})|.$$

Proof. By Proposition 68, since $w_1,\ldots,w_{2n}$ are on the boundary, the winding phase of all the configurations in $C_{\Omega_\delta}\left(w_1^{o_1},\ldots,w_p^{o_p}\right)$ is the same and hence factors out from f_{Ω_δ}, giving

$$|f_{\Omega_\delta}(w_1^{o_1},\ldots,w_{2n}^{o_{2n}})| = \frac{1}{Z_{\Omega_\delta}}\left(\sum_{\omega\in C_{\Omega_\delta}(w_1,\ldots,w_{2n})}\alpha^{|\omega|}\right).$$

By removing from each of the configurations the half-edges $\langle v_1,w_1\rangle,\ldots,\langle v_{2n},w_{2n}\rangle$, we divide their weights by α^n, we obtain

$$\frac{1}{Z_{\Omega_\delta}}\left(\sum_{\omega\in C_{\Omega_\delta}(v_1,\ldots,v_{2n})}\alpha^{|\omega|}\right).$$

By Proposition 63, this equals

$$\mathbb{E}_{\Omega_\delta}^{\mathrm{free}}[\sigma_\delta(v_1)\cdot\ldots\cdot\sigma_\delta(v_{2n})].$$

∎

CHAPTER 6

Discrete Analysis of the Observables

In this chapter, we study the fermionic observables defined in Section 5.2, from a discrete complex analysis point of view. As before, $\Omega_\delta \subset \mathbb{C}_\delta$ is a discrete domain, that is, an induced connected subgraph of the square grid $\mathbb{C}_\delta = \delta\mathbb{Z}^2$ of mesh size $\delta > 0$. The strategy is the following:

- Three types of properties on the complex fermionic observables h_Ω are obtained:
  - Away from the signed edges in $\{\ldots\}$ and from the oriented medial vertices and $(\ldots)$, the observables $h_{\Omega_\delta}^{\{\ldots\}}(\ldots)$ are s-holomorphic in their last variable (Section 6.2).
  - Near the oriented medial vertices in $(\ldots)$, the observables $h_{\Omega_\delta}^{\{\ldots\}}(\ldots)$ have discrete simple poles with identifiable singularities (Section 6.3).
  - On the boundary, the complex fermionic observables satisfies discrete Riemann-Hilbert boundary conditions (Section 6.4).
- We then use these properties to express the observables $h_{\Omega_\delta}^{[\ldots]}(\ldots)$ in terms of solutions to discrete Riemann-Hilbert boundary value problems (Section 6.5).
- Using the results of Section 6.5, we obtains recursion relations between the different observables, yielding Pfaffian formulae for them (Section 6.6). The strategy is to first obtain Pfaffian formulae for the unfused version of the observables and then to merge points two by two in a suitable manner to obtain formulae for the fused observables (which justifies their name).

6.1. Integrability

Before entering the discrete complex analysis considerations, let us introduce the elementary integrability relation that is central for the analysis of the observables.

By integrability of a system, one usually means the existence of a large number of relations or of conservation laws that allow to solve the system exactly. The Ising model is known to be integrable and in our (critical) case, integrability arises in terms of relations between the values of the observables, which will get translated in terms of s-holomorphicity.

For each edge $e \in E_{\Omega_\delta^m}$, we denote by $\kappa_e \subset H_{\Omega_\delta}$ the *corner at* e, consisting of the two half-edges forming a right triangle together with e. Let us denote by $\omega \mapsto \omega \oplus \kappa_e$ the involutive operation on half-edge configurations that perform a symmetric difference between ω and the two half-edges of κ_e.

The following elementary lemma is instrumental in the next two sections and concentrates all the integrability of the model that we will use; it also justifies the introduction of the winding phase. Let us remark that this is the only place (besides

Kramers-Wannier duality) where we use the value of the inverse temperature $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

Lemma 74. Let $a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices at distance at least δ from each other, let $a_{2n}^{o_{2n}}, \tilde{a}_{2n}^{\tilde{o}_{2n}} \in S_{\Omega_\delta^m}$ be two adjacent simply-oriented medial vertices distinct from $a_1, \ldots, a_{2n-1}$ and denote by $e \in E_{\Omega_\delta^m}$ the medial edge $\langle a_{2n}, \tilde{a}_{2n}\rangle$. Let $\omega \in C_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$ and $\tilde{\omega} \in C_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, \tilde{a}_{2n}\right)$ be two configurations such that $\omega \oplus c(e) = \tilde{\omega}$. Then we have

$$\mathbb{P}_{(e)}\left[W(\omega, o_1, \ldots, o_{2n-1})\right] = \mathbb{P}_{(e)}\left[W(\tilde{\omega}, o_1, \ldots, o_{2n-1})\right].$$

Proof. Set $W(\cdot) = W(\cdot, o_1, \ldots, o_{2n-1})$ to shorten the notation. Assume that a_{2n} is a horizontal medial vertex, and that hence $\tilde{a}_{2n}$ is a vertical one. Denote by e_{2n} and $\tilde{e}_{2n}$ the edges whose midpoints are a_{2n} and $\tilde{a}_{2n}$. Assume that $\tilde{a}_{2n} = a_{2n} + (1+i)\frac{\delta}{2}$ (the other cases are symmetric).

Let us denote by $o_{2n}, \tilde{o}_{2n} \in S$ the simple orientations such that $\omega \in C_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}^{o_{2n}}\right)$ and $\tilde{\omega} \in C_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, \tilde{a}_{2n}^{\tilde{o}_{2n}}\right)$, by $c_1 \in H_{\Omega_\delta}^h$ and $c_i \in H_{\Omega_\delta}^v$ the horizontal and vertical half-edges of κ_e, and by $v_{2n} \in V_{\Omega_\delta}$ the vertex $c_1(e) \cap c_2(e)$. Then we have four possibilities:

- We have $o_{2n} = 1$ and $\tilde{o}_{2n} = i$: in that case, we have that $W(\omega) \in \mathbb{R}$ and $W(\tilde{\omega}) = \alpha\bar{\lambda}W(\omega)$. Indeed, we have $\omega \cap \kappa_e = \emptyset$ and $\tilde{\omega} \cap \kappa_e = \kappa_e$, so the modulus of the weight of ω gets multiplied by α and for any admissible choice of walks on ω, the one arriving at a_{2n} can be extended by κ_e to a walk arriving at $\tilde{a}_{2n}$, thus adding one more left turn to the winding of the path, and hence multiplying the total winding phase by $\bar{\lambda}$ (we keep the other walks unchanged).
- We have $o_{2n} = 1$ and $\tilde{o}_{2n} = -i$: in that case, we have that $W(\omega) \in \mathbb{R}$ and $W(\tilde{\omega}) = -\lambda W(\omega)$. Indeed, we have $\omega \cap \kappa_e = c_i$ and $\tilde{\omega} \cap \kappa_e = c_1$, so the modulus of the weight gets unchanged. Let $\gamma_1, \ldots, \gamma_n$ be an admissible collection of walks on ω with for each j, $\gamma_j : a_{\iota_j} \rightsquigarrow a_{\tau_j}$ and with the indices chosen such that $\tau_n = n$. Then, we have three subcases:
  - We have $\tilde{e}_{2n} \notin \gamma_1 \cup \ldots \cup \gamma_n$ and hence $\tilde{e}_{2n}$ belongs to a simple loop $\Lambda \subset \omega \setminus (\gamma_1 \cup \ldots \cup \gamma_n)$. Then $\gamma_1, \ldots, \gamma_{n-1}, \tilde{\gamma}_n$ is an admissible collection of walks on $\tilde{\omega}$, where $\tilde{\gamma}_n$ is the walk γ_n extended by c_1 and then following Λ in the counterclockwise direction up to $\tilde{a}_{2n}$, hence giving $w(\tilde{\gamma}_{2n}) = w(\gamma_{2n}) + \frac{3\pi}{2}$. Hence we have $W(\tilde{\omega}) = -\lambda W(\omega)$.
  - We have $\tilde{e}_{2n} \in \gamma_n$ and it is easy to see that $\gamma_1, \ldots, \gamma_{n-1}, \tilde{\gamma}_n$ is an admissible collection of walks on $\tilde{\omega}$, where $\tilde{\gamma}_n$ is the walk γ stopped at $\tilde{a}_{2n}$, if $\tilde{e}_{2n}$ is ran from top to bottom by γ_n, in which case we have $w(\tilde{\gamma}_{2n}) = w(\gamma_{2n}) + \frac{3\pi}{2}$ and where $\tilde{\gamma}_n$ is the walk γ_n stopped at v_{2n}, following then c_1 and finally running along the rest of γ_n backwards from a_{2n} to $\tilde{a}_{2n}$, in which case we have $w(\tilde{\gamma}_{2n}) = w(\gamma_{2n}) - \frac{5\pi}{2} + 4k\pi$. In both cases, we have $W(\tilde{\omega}) = -\lambda W(\omega)$.
  - We have $\tilde{e}_{2n} \in \gamma_j$ for some $j \in \{1, \ldots, n-1\}$. Then we have the several possibilities:
    - * If $\iota_n < \iota_j < \tau_j$ and $\tilde{e}_{2n}$ is ran from top to bottom, by γ_j then $\gamma_1, \ldots, \gamma_{j-1}, \tilde{\gamma}_j, \gamma_{j+1}, \ldots, \gamma_{n-1}, \tilde{\gamma}_{2n}$ is an admissible collection of walks, where $\tilde{\gamma}_j : a_{\iota_n} \rightsquigarrow a_{\tau_j}$ is obtained by following γ_n from a_{ι_n} to a_{2n}, then c_1 and then running along γ_j from v_{2n}

to a_{τ_j}, and where γ_n is obtained by following γ_j from a_{ι_j} to $\tilde{a}_{2n}$. In this case, the crossing signature of the pair partition of $\{1,\ldots,2n\}$ induced by $\gamma_1,\ldots,\gamma_n$ is changed and we have $w(\tilde{\gamma}_j)+w(\tilde{\gamma}_n)=w(\gamma_j)+w(\gamma_n)-\pi/2+4k\pi$ and we hence have $W(\tilde{\omega})=-\lambda W(\omega)$.

  * If $\iota_n<\iota_j<\tau_j$ and $\tilde{e}_{2n}$ is ran from bottom to top by γ_j, then $\gamma_1,\ldots,\gamma_{j-1},\tilde{\gamma}_j,\gamma_{j+1},\ldots,\gamma_{n-1},\tilde{\gamma}_{2n}$ is an admissible collection of walks, where $\tilde{\gamma}_j : a_{\iota_n} \to a_{\iota_j}$ is obtained by following γ_n from a_{ι_n} to a_{2n}, then e_1 and finally running along γ_j backwards from v_{2n} to a_{ι_j} and $\tilde{\gamma}_n : a_{\tau_n} \to a_{2n}$ is obtained by running along γ_j backwards from a_{τ_n} to $\tilde{a}_{2n}$. In this case, the crossing signature of the pair partition of $\{1,\ldots,2n\}$ induced by $\gamma_1,\ldots,\gamma_n$ is unchanged and we have $w(\tilde{\gamma}_j)+w(\tilde{\gamma}_n)=w(\gamma_j)+w(\gamma_n)+3\pi/2+4k\pi$ and hence $W(\tilde{\omega})=-\lambda W(\omega)$.
  * All the other subcases ($\iota_j<\iota_n<\tau_j$ and $\iota_j<\tau_j<\iota_n$) can be treated in a similar manner.

- We have $o_{2n}=-1$ and $\tilde{o}_{2n}=i$: in that case, we have that $W(\omega)\in i\mathbb{R}$ and $W(\tilde{\omega})=\lambda W(\omega)$: if $\gamma_1,\ldots,\gamma_n$ is an admissible choice with γ_n arriving at a_{2n}, then $\gamma_1,\ldots,\gamma_{n-1},\tilde{\gamma}_n$ is an admissible choice, where $\tilde{\gamma}_n$ is obtained by removing c_1 from γ_n and adding c_i to it, thus giving $w(\tilde{\gamma}_n)=w(\gamma_n)-\frac{\pi}{2}$ and keeping the number of edges unchanged.
- We have $o_{2n}=-1$ and $\tilde{o}_{2n}=-i$: in that case, we have that $W(\omega)\in i\mathbb{R}$ and $W(\tilde{\omega})=\alpha^{-1}\bar{\lambda}W(\omega)$. Indeed, we have $\omega\cap\kappa_e=\kappa_e$ and $\tilde{\omega}\cap\kappa_e=\emptyset$ so modulus of the weight of ω gets divided by α. For the phases, if we fix admissible choices $\gamma_1,\ldots,\gamma_n$ of walks on ω with γ_n arriving at a_{2n}, as for the case when $o_{2n}=1$ and $\tilde{o}_{2n}=i$, there are three subcases:
  - We have $\tilde{e}_{2n}\in\gamma_n$. If $\tilde{e}_{2n}$ is ran from top to bottom by γ_n, then $\gamma_1,\ldots,\gamma_{n-1},\tilde{\gamma}_n$ is an admissible choice of walks, where $\tilde{\gamma}_n$ is obtained by following γ_n and stopping it when it arrives at $\tilde{a}_{2n}$ and removing κ_e. In this case, the number of edges has decreased by 1 and we have $w(\tilde{\gamma}_n)=w(\gamma_n)+\pi/2$, which gives $W(\tilde{\omega})=\alpha^{-1}\bar{\lambda}W(\omega)$. If $\tilde{e}_{2n}$ is ran from bottom to top, then γ_n arrives from $v_{2n}+\delta$ before passing through $\tilde{e}_{2n}$ and comes back to v_{2n} from $v_{2n}-i\delta$, and hence makes a loop around a_{2n} in counterclockwise direction between the two times it hits v_{2n}. We can reverse the direction in which this loop is made without changing the admissibility of γ_n and hence we can assume that $\tilde{e}_{2n}$ is ran from top to botton.
  - We have $\tilde{e}_{2n}\in\omega\setminus(\gamma_1\cup\ldots\cup\gamma_n)$. Then γ_n arrives at v_n from $v_n-i\delta$ and turns left to arrive to a_{2n} and hence there is a simple loop in $\omega\setminus(\gamma_1\cup\ldots\cup\gamma_n)$ touching v_n and we can extend γ_n by this loop, so that we are in the previous subcase.
  - We have $\tilde{e}\in\gamma_1\cup\ldots\cup\gamma_{n-1}$. Then there is a path $\gamma_j:\iota_j\to\tau_j$ that interesects with γ_n at v_{2n} such that γ_n arrives at v_{2n} from $v_{2n}-i\delta$ and with γ_j either arriving at v_{2n} from $v_{2n}-\delta$ and leaving to $v_{2n}+i\delta$ or the converse. Then we can construct another admissible choice of walks $\gamma_1,\ldots,\gamma_{j-1},\gamma_j^*,\gamma_{j+1},\ldots,\gamma_{n-1},\gamma_n^*$ on ω, with γ_j^* and γ_n^* obtained by gluing the two components of $\gamma_j\setminus\{v_{2n}\}$ with the two components of

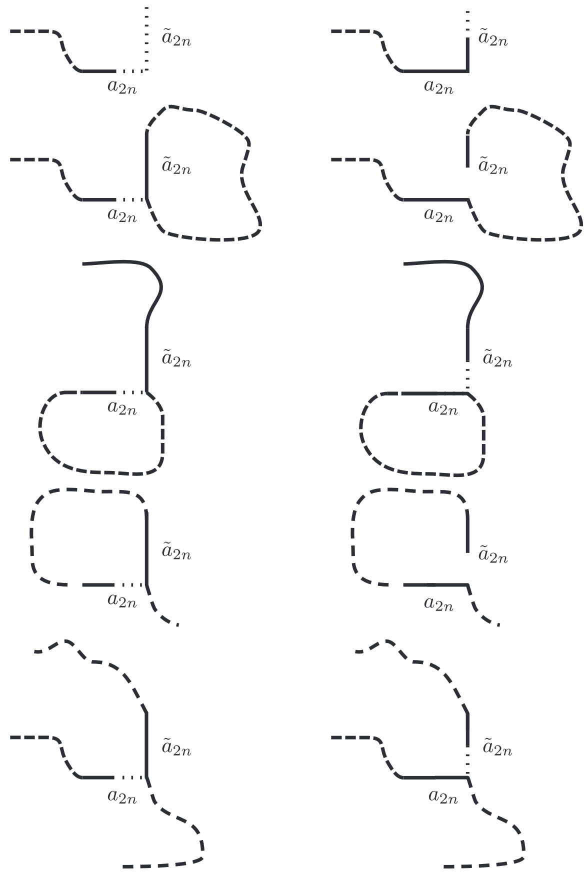

Figure 6.1.1. The possible configurations when $o_{2n} = 1$ and their images by the involution $\omega \mapsto \omega \oplus \kappa_e$.

$\gamma_n \setminus \{v_{2n}\}$ respectively (there is only one admissible way to do this), so that γ_n^* ends at a_{2n}, and we are back to the first subcase.

□

This relation will both enable for the derivation of the s-holomorphicity and the analysis of the singularities.

6.2. S-holomorphicity

We now turn to the first (and most important) ingredient of the analysis of the complex fermionic observables, which is their s-holomorphicity.

Proposition 75. *Let $e_1^{s_1}, \ldots, e_m^{s_m} \in E_{\Omega_\delta}^{\Sigma}$ be signed edges and $a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices such that $m(e_1), \ldots, m(e_m), a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}$*

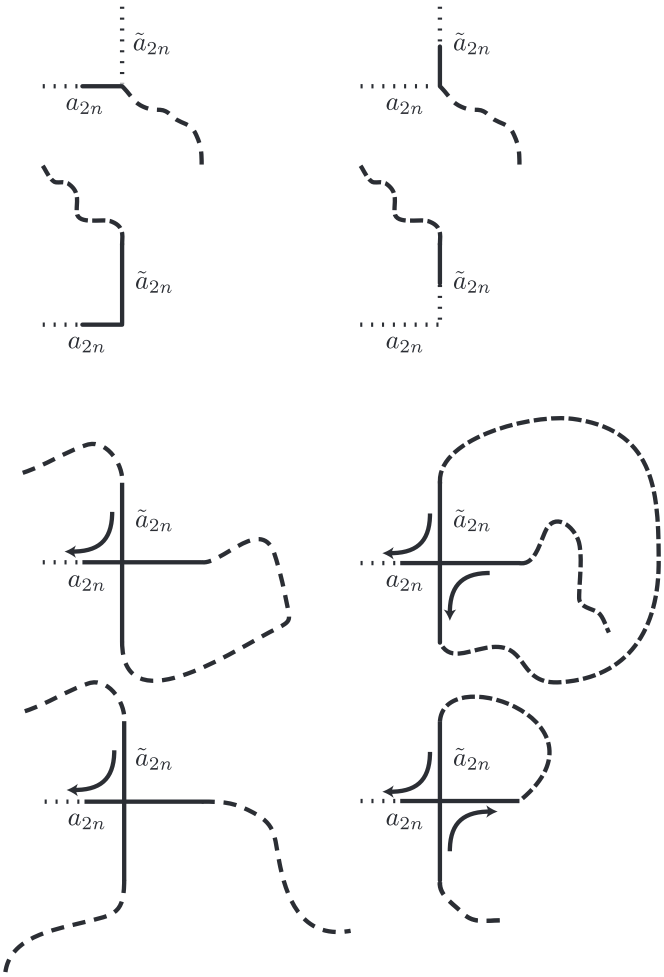

Figure 6.1.2. The possible configurations when $o_{2n} = -1$ and the way to make the appropriate choices of walks.

are at distance at least δ from each other. Then the function

$$\begin{aligned} V_{\Omega_\delta^m \setminus \{m(e_1),\dots m(e_m),a_1,\dots,a_{2n-1}\}} &\to \mathbb{C} \\ a_{2n} &\mapsto h_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \end{aligned}$$

is s-holomorphic.

Proof. We have

$$\begin{aligned} &h_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \\ &= \frac{1}{Z_{\Omega_\delta}}\left(\sum_{\omega\in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)} W(\omega, o_1,\dots,o_{2n-1})\right). \end{aligned}$$

For two adjacent medial vertices $a_{2n}, \tilde{a}_{2n} \in V_{\Omega_\delta \setminus \{m(e_1),\ldots,m(e_m),a_1,\ldots,a_{2n-1}\}}$, the involution $\omega \mapsto \omega \oplus \kappa_{\langle a_{2n},\tilde{a}_{2n}\rangle}$ is a bijection between $C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$ and $C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},\tilde{a}_{2n}\right)$. From Lemma 74, we have that

$$\mathrm{P}_{\langle a_{2n},\tilde{a}_{2n}\rangle}[W(\omega,o_1,\ldots,o_{2n-1})] = \mathrm{P}_{\langle a_{2n},\tilde{a}_{2n}\rangle}\left[W\left(\omega \oplus \kappa_{\langle a_{2n},\tilde{a}_{2n}\rangle},o_1,\ldots,o_{2n-1}\right)\right]$$

for each $\omega \in C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$. By real linearity of the projections we obtain

$$\mathrm{P}_{\langle a_{2n},\tilde{a}_{2n}\rangle}\left[\sum_{\omega\in C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n})} W(\omega,o_1,\ldots,o_{2n-1})\right]$$

$$= \mathrm{P}_{\langle a_{2n},\tilde{a}_{2n}\rangle}\left[\sum_{\omega\in C_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},\tilde{a}_{2n})} W(\omega,o_1,\ldots,o_{2n-1})\right]$$

and the result follows, since $Z_{\Omega_\delta} \in \mathbb{R}$. □

6.3. Singularities

With the notation and assumptions of Proposition 6.2, near the medial vertices $m(e_1),\ldots m(e_m)$ and $a_1,\ldots a_{2n-1}$, the function

$$a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$$

is not discrete holomorphic: it has simple poles (near the medial vertices that are in the interior of Ω_δ). We only analyze here the behavior near $a_1,\ldots,a_{2n-1}$, since it is the only case that we need to understand for what follows, although the behavior near $e_1,\ldots,e_m$ is very similar and can be studied in the same way.

Proposition 76. Let $e_1^{s_1},\ldots,e_m^{s_m} \in E_{\Omega_\delta}^{\Sigma}$ be signed edges and let $a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices, such that $m(e_1),\ldots,m(e_m),a_1,\ldots,a_{2n-1}$ are at distance at least δ from each other.

For each $j \in \{1,\ldots 2n-1\}$ such that $a_j \in V_{\Omega_\delta^m} \setminus \partial_0 V_{\Omega_\delta^m}$, the function

$$a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1,\ldots,e_m\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$$

has a discrete simple pole at a_j, with front and rear values given by:

$$h_j^+ = \frac{(-1)^{j+1}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right),$$

$$h_j^- = \frac{(-1)^{j}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m},e(a_j)^-\}}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right),$$

where $e(a_j) \in E_{\Omega_\delta}$ denotes the edge whose midpoint is a_j.

For each $j \in \{1,\ldots,2n-1\}$ such that $a_j \in \partial_0 V_{\Omega_\delta^m}$, the function

$$a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$$

can be extended to an s-holomorphic function to a_j, by setting the value at a_j to

$$h_j^+ = \sqrt{\frac{1}{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\dots,a_{2n-1}^{o_{2n-1}}\right).$$

Proof. It follows from an adaptation of Lemma 74; let us treat the case of $j \in \{1,\dots,2n-1\}$ such that $a_j \in V_{\Omega_\delta^m} \setminus \partial_0 V_{\Omega_\delta^m}$ first.

If we denote by $v_j, \tilde{v}_j \in V_{\Omega_\delta^m}$ the two medial vertices $a_j + o_j \cdot (1+i)\frac{\delta}{2}$ and $a_j + o_j \cdot (1-i)\frac{\delta}{2}$, the contour set

$$C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\dots,a_{2n-1}^{o_{2n-1}}\right)$$

is the image under the involutions $\omega \mapsto \omega \oplus \kappa_{\langle a_j,v_j\rangle}$ and $\omega \mapsto \omega \oplus \kappa_{\langle a_j,\tilde{v}_j\rangle}$ of the contour sets

$$C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},v_j\right)$$

and

$$C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},\tilde{v}_j\right)$$

and we can apply Lemma 74: we have

$$\begin{aligned}
\mathbb{P}_{\langle a_j,v_j\rangle}\left[W(\omega,o_1,\dots,o_{2n-1})\right] &= \mathbb{P}_{\langle a_j,v_j\rangle}\left[W(\omega\oplus\langle a_j,v_j\rangle,o_1,\dots,o_{2n-1},o_j)\right],\\
\mathbb{P}_{\langle a_j,\tilde{v}_j\rangle}\left[W(\tilde{\omega},o_1,\dots,o_{2n-1})\right] &= \mathbb{P}_{\langle a_j,\tilde{v}_j\rangle}\left[W(\tilde{\omega}\oplus\langle a_j,\tilde{v}_j\rangle,o_1,\dots,o_{2n-1},o_j)\right]
\end{aligned}$$

respectively for each

$$\begin{aligned}
\omega &\in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},v_j\right),\\
\tilde{\omega} &\in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},\tilde{v}_j\right),
\end{aligned}$$

where on the right hand sides, the configurations $\omega\oplus\kappa_{\langle a_j,v_j\rangle}$ and $\tilde{\omega}\oplus\kappa_{\langle a_j,\tilde{v}_j\rangle}$ should be interpreted as configurations in

$$C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},a_j^{o_j}\right),$$

where there is a "walk of length zero from a_j to a_j" (of zero winding), and in particular with no edge at a_j. By definition of the weight W, we have

$$\begin{aligned}
W(\omega\oplus\langle a_j,v_j\rangle,o_1,\dots,o_{2n-1},o_j) &= \frac{(-1)^{j+1}}{\sqrt{o_j}}\cdot\alpha^{\left|\omega\oplus\kappa_{\langle a_j,v_j\rangle}\right|}\\
&\quad\cdot\varphi(\omega\oplus\langle a_j,v_j\rangle,o_1,\dots,o_{j-1},o_{j+1},\dots,o_{2n-1}),\\
W(\tilde{\omega}\oplus\langle a_j,\tilde{v}_j\rangle,o_1,\dots,o_{2n-1},o_j) &= \frac{(-1)^{j+1}}{\sqrt{o_j}}\cdot\alpha^{\left|\tilde{\omega}\oplus\kappa_{\langle a_j,\tilde{v}_j\rangle}\right|}\\
&\quad\cdot\varphi(\tilde{\omega}\oplus\langle a_j,\tilde{v}_j\rangle,o_1,\dots,o_{j-1},a_{j+1},\dots,o_{2n-1}).
\end{aligned}$$

where on the right hand sides, $\omega\oplus\kappa_{\langle a_j,v_j\rangle}$ and $\tilde{\omega}\oplus\kappa_{\langle a_j,\tilde{v}_j\rangle}$ are interpreted as a configurations in

$$C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},a_{2n-1}^{o_{2n-1}}\right).$$

Summing over all $\omega \in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},v_j\right)$ and $\tilde{\omega} \in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}},\tilde{v}_j\right)$, we obtain

$$h_j^+ = (-1)^{j+1}\sqrt{\frac{i}{o_j}}\cdot f_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\dots,a_{2n-1}^{o_{2n-1}}\right).$$

Similary, if we denote by $w_j, \tilde{w}_j \in V_{\Omega_\delta^m}$ the two medial vertices $a_j - o_j \cdot (1-i)\frac{\delta}{2}$ and $a_j - o_j \cdot (1+i)\frac{\delta}{2}$, we can apply Lemma 74 to obtain

$$\begin{aligned}
\mathbb{P}_{\langle a_j, w_j\rangle}[W(\omega, o_1, \ldots, o_{2n-1})] &= \mathbb{P}_{\langle a_j, w_j\rangle}\left[W\left(\omega \oplus \kappa_{\langle a_j, w_j\rangle}, o_1, \ldots, o_{2n-1}, o_j\right)\right], \\
\mathbb{P}_{\langle a_j, \tilde{w}_j\rangle}[W(\tilde{\omega}, o_1, \ldots, o_{2n-1})] &= \mathbb{P}_{\langle a_j, \tilde{w}_j\rangle}\left[W\left(\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}, o_1, \ldots, o_{2n-1}, o_j\right)\right]
\end{aligned}$$

respectively for each

$$\begin{aligned}
\omega &\in C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, w_j\right), \\
\tilde{\omega} &\in C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, \tilde{w}_j\right),
\end{aligned}$$

where, this time, the configurations $\omega \oplus \kappa_{\langle a_j, w_j\rangle}$ and $\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}$ should be interpreted as configurations in

$$C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_j^{o_j}\right),$$

where there is a "walk from a_j to a_j that makes a loop" (of $\pm 2\pi$ winding), and in particular there is an edge at a_j. By definition of the weight W, we have

$$\begin{aligned}
W\left(\omega \oplus \kappa_{\langle a_j, w_j\rangle}, o_1, \ldots, o_{2n-1}\right) &= \frac{(-1)^j}{\sqrt{o_j}} \cdot \alpha^{\left|\omega \oplus \kappa_{\langle a_j, w_j\rangle}\right|} \\
&\quad \cdot \varphi\left(\omega \oplus \langle a_j, w_j\rangle, o_1, \ldots, o_{j-1}, o_{j+1}, \ldots, o_{2n-1}\right), \\
W\left(\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}, o_1, \ldots, o_{2n-1}, o_j\right) &= \frac{(-1)^j}{\sqrt{o_j}} \cdot \alpha^{\left|\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}\right|} \\
&\quad \cdot \varphi\left(\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}, o_1, \ldots, o_{j-1}, a_{j+1}, \ldots, o_{2n-1}\right).
\end{aligned}$$

where on the right hand sides, $\omega \oplus \kappa_{\langle a_j, w_j\rangle}$ and $\tilde{\omega} \oplus \kappa_{\langle a_j, \tilde{w}_j\rangle}$ are interpreted as a configurations in

$$C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}, e(a_j)^-\}}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, a_{2n-1}^{o_{2n-1}}\right).$$

.Summing over all

$$\begin{aligned}
\omega &\in C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, w_j\right), \\
\tilde{\omega} &\in C_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, \tilde{w}_j\right),
\end{aligned}$$

we obtain

$$h_j^- = (-1)^j \sqrt{\frac{i}{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}, e(a_j)^-\}}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right).$$

For $j \in \{1, \ldots, 2n-1\}$ such that $a_j \in \partial_0 V_{\Omega_\delta^m}$, it is clear that

$$a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$$

can be extended to a_j, since the only two adjacent medial vertices to a_j are v_j and $\tilde{v}_j$. Using again Lemma 74 in the same way as for interior medial vertices, it is easy to see that the value of this extension is the one claimed. □

The following proposition (and its corollary) is a key in the fusion procedure, since it allows to make the fused observables appear as special values of extensions of unfused ones.

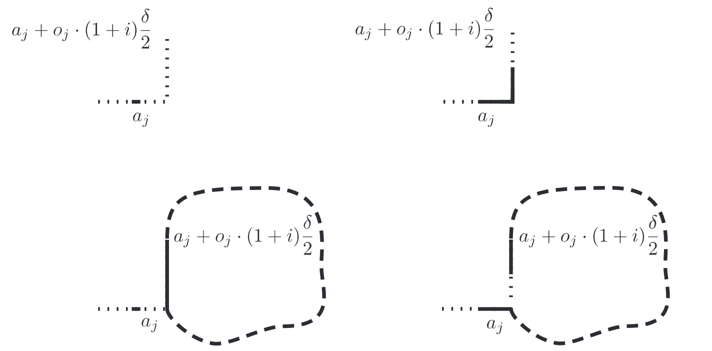

Figure 6.3.1. The configurations contributing to the front value of the observable and their images by the involution $\omega \mapsto \omega \oplus K_{\langle a_j, a_j + o_\delta \cdot (1+i)\frac{\delta}{2} \rangle}$.

Proposition 77. Let $e_1^{s_1}, \ldots, e_m^{s_m} \in E_{\Omega_\delta}^{\Sigma}$ be signed edges and $a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices such that $m(e_1), \ldots, m(e_m), a_1, \ldots, a_{2n-1}$ are at distance at least δ from each other. Then for each $a_j \in V_{\Omega_\delta^m} \setminus \partial_0 V_{\Omega_\delta^m}$, we have that

$$\begin{aligned} a_{2n} \mapsto\ & h_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) \\ & + (-1)^j \cdot f_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right) h_{C_\delta}\left(a_j^{o_j}, a_{2n}\right) \end{aligned}$$

extends in an s-holomorphic way to a_j. The value of the extension to a_j is equal to

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}[e(a_j)]}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right),$$

where $e(a_j) \in E_{\Omega_\delta}$ denotes the edge whose middlepoint is a_j.

A useful corollary, which follows from the definitions of the fused observables, it the following:

Corollary 78. With the notation above, for each $a_j \in V_{\Omega_\delta^m} \setminus \partial_0 V_{\Omega_\delta^m}$ we have that

$$\begin{aligned} a_{2n} \mapsto\ & h_{\Omega_\delta}^{[e_1, \ldots, e_m]}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) \\ & + (-1)^j \cdot f_{\Omega_\delta}^{[e_1, \ldots, e_m]}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right) h_{C_\delta}\left(a_j^{o_j}, a_{2n}\right) \end{aligned}$$

extends in an s-holomorphic way to a_j and that the value of this extension at a_j equals

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{[e_1, \ldots, e_m, e(a_j)]}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right).$$

Proof of Proposition 77. It is sufficient to check that the above function, which we will denote by $a_{2n} \mapsto u(a_{2n})$ has discrete residue zero at a_j for each $j = 1, \ldots, 2n$. By definition of $h_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}$ and Proposition 76, the difference between the front and rear values of $a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1^{s_1}, \ldots, e_m^{s_m}\}}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$ near

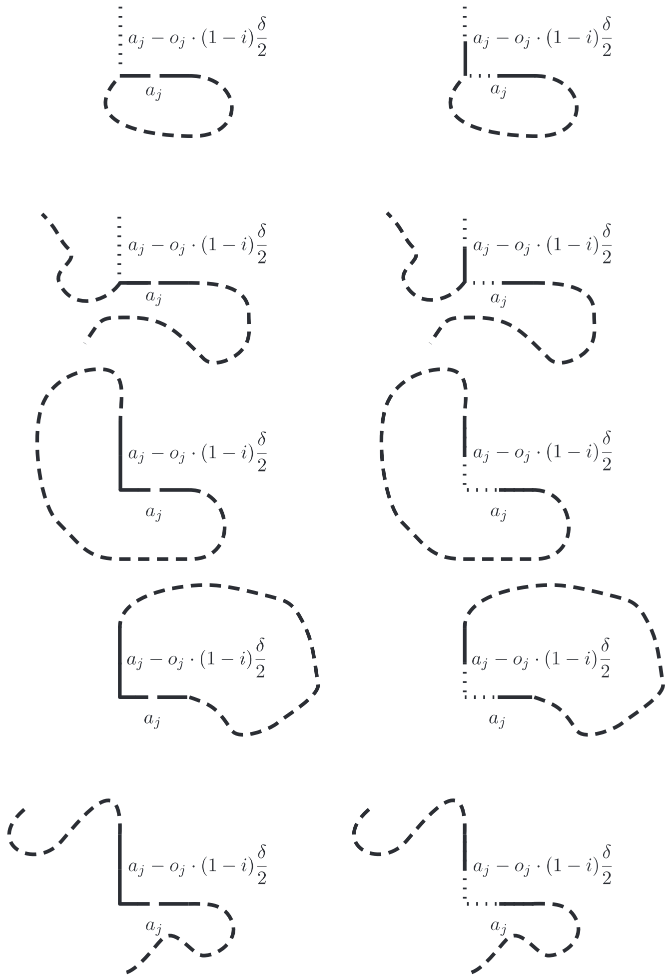

Figure 6.3.2. The configurations contributing to the rear value of the observable and their images by the involution $\omega \mapsto \omega \oplus \kappa_{\langle a_j, a_j - o_j \cdot (1-i)\frac{\delta}{2}\rangle}$.

a_j is equal to

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\dots,a_{2n-1}^{o_{2n-1}}\right).$$

By definition, the difference between the front and rear values of $a_{2n} \mapsto h_{C_\delta}\left(a_j^{o_j}, a_{2n}\right)$ is equal to $\frac{1}{\sqrt{o_j}}$ and hence the front and rear values of u are the same and we can extend it by them. The front value of $a_{2n} \mapsto h_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}\left(a_1^{o_1},\dots,a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$ is again given by Proposition 76 and is equal to

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} \cdot f_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\dots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\dots,a_{2n-1}^{o_{2n-1}}\right).$$

By Theorem 22, the front value of $a_{2n} \mapsto h_{C_\delta}\left(a_j^{o_j}, a_{2n}\right)$ is equal to $\frac{\mu+1}{2}$. Hence the front value of $u(\cdot)$ (which is equal to its rear value) is equal to

$$\frac{(-1)^{j+1}}{\sqrt{o_j}} f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m},e(a_j)^+\}}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right)$$
$$+(-1)^j\cdot\frac{1+\mu}{2\sqrt{o_j}}\cdot f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right)$$
$$=\frac{(-1)^{j+1}}{\sqrt{o_j}}\cdot f_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}[e(a_j)]}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right),$$

which is the desired result. □

6.4. Boundary conditions

The boundary values of the complex fermionic observable are very simple to study:

Proposition 79. *Let $e_1^{s_1},\ldots,e_m^{s_m} \in E_{\Omega_\delta}^{\pm}$ be signed edges and $a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices such that $m(e_1),\ldots,m(e_m),a_1,\ldots,a_{2n-1}$ at distance δ from each other. Then for each $a_{2n} \in \partial_0 V_{\Omega_\delta^m}\setminus\{a_1,\ldots,a_{2n-1}\}$, we have*

$$h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \in \frac{1}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R},$$

and for each $a_{2n} \in \partial_0 V_{\Omega_\delta}\cap\{a_1,\ldots,a_{2n-1}\}$, we have, for the natural extension of Proposition 76.

$$h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \in \frac{i}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R}.$$

Proof. If $a_{2n} \in \partial_0 V_{\Omega_\delta^m}\setminus\{a_1,\ldots,a_{2n-1}\}$, then for topological reasons, we have

$$h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) = h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}^{o_{2n}}\right),$$

where $o_{2n} = \frac{\nu_{int}(a_{2n})}{|\nu_{int}(a_{2n})|}$. Indeed, in that case, the only possibility for an admissible walk to arrive at a_{2n} is to pass through $a_{2n} + o_{2n}\frac{\delta}{2}$ and hence the result follows from the definition of $h_{\Omega_\delta}^{\{e_1^{s_1},\ldots,e_m^{s_m}\}}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}^{o_{2n}}\right)$.

If $a_{2n} \in \partial_0 V_{\Omega_\delta}\cap\{a_1,\ldots,a_{2n-1}\}$, the result follows from Propostion 76. □

6.5. Discrete Riemann-Hilbert boundary value problem

We now summarize the information obtained in the previous sections to express the unfused complex fermionic observables $h_{\Omega_\delta}(\ldots)$ as solution to Riemann-Hilbert boundary problems $(\spadesuit,\ldots)$, as defined in Section 2.6. The idea is to remove the singularities of the observables by substracting functions with the same residues. We can first formulate the two-point version of the observable in this way, which will be essential for proving convergence of it:

Proposition 80. *Let $a_1^{o_1} \in D_{\Omega_\delta^m}\setminus\partial_0 D_{\Omega_\delta^m}$ be an interior doubly-oriented medial vertex. Then we have that*

$$a_2 \mapsto h_{\Omega_\delta}(a_1^{o_1},a_2) - h_{C_\delta}(a_1^{o_1},a_2)$$

extends to an s-holomorphic function $V_{\Omega_\delta^m} \to \mathbb{C}$ and is the solution to the Riemann-Hilbert boundary value problem $\left(♠_{\Omega_\delta}, -h_{C_\delta}(a_1^{o_1}, \cdot)|_{\partial V_{\Omega_\delta^m}}\right)$.

Proof. From Proposition 77, we have that the function $h_{\Omega_\delta}(a_1^{o_1}, \cdot) - h_{C_\delta}(a_1^{o_1}, \cdot)$ extends to an s-holomorphic function $V_{\Omega_\delta^m} \to \mathbb{C}$. The fact that $h_{\Omega_\delta}(a_1^{o_1}, \cdot) - h_{C_\delta}(a_1^{o_1}, \cdot)$ solves the Riemann-Hilbert boundary value problem follows from the boundary condition of Proposition 79. □

Definition 81. We denote by $h_{\Omega_\delta}^{C_\delta}$ the function $h_{\Omega_\delta} - h_{C_\delta}$ and by $f_{\Omega_\delta}^{C_\delta} : D_{\Omega_\delta^m} \times D_{\Omega_\delta^m} \to \mathbb{R}$ the function $f_{\Omega_\delta} - f_{C_\delta}$. We call $f_{\Omega_\delta}^{C_\delta}$ the *discrete boundary effect fermionic observable*.

Let us then formulate the $2n$-point version of the observable in terms of the $2(n-1)$-point and two-point versions, which will help us obtaining the recursions that will yield the Pfaffian formulae:

Proposition 82. *Let $a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices. Then we have that the function*

$$a_{2n} \mapsto h_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) - \sum_{j=1}^{2n} (-1)^j \cdot f_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right) h_{\Omega_\delta}\left(a_j^{o_j}, a_{2n}\right)$$

extends to an s-holomorphic function $V_{\Omega_\delta^m} \to \mathbb{C}$ and is the solution of the Riemann-Hilbert boundary value problem $(♠_{\Omega_\delta}, 0)$ and hence is identically equal to 0.

The following corollary is immediate:

Corollary 83. We have that

$$f_{\Omega_\delta}(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}) = \sum_{j=1}^{2n} (-1)^j \cdot f_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right) f_{\Omega_\delta}\left(a_j^{o_j}, a_{2n}^{o_{2n}}\right).$$

Proof of Proposition 82. Let us denote by $r\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$ the function defined by

$$\begin{aligned} & r\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) \\ = \; & h_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) \\ & - \sum_{j=1}^{2n} (-1)^j \cdot f_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{j-1}^{o_{j-1}}, a_{j+1}^{o_{j+1}}, \ldots, a_{2n-1}^{o_{2n-1}}\right) h_{\Omega_\delta}\left(a_j^{o_j}, a_{2n}\right). \end{aligned}$$

The fact that $a_{2n} \mapsto r\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right)$ extends to an s-holomorphic function $V_{\Omega_\delta^m} \to \mathbb{C}$ follows from Proposition 76. The fact that is a solution of $(♠_{\Omega_\delta}, 0)$ depends on the following:

- For each $a_{2n} \in \partial_0 V_{\Omega_\delta^m} \setminus \{a_1, \ldots, a_{2n-1}\}$, by Proposition 79, we have that

$$h_{\Omega_\delta}\left(a_1^{o_1}, \ldots, a_{2n-1}^{o_{2n-1}}, a_{2n}\right) \in \frac{1}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R}$$

and

$$h_{\Omega_\delta}\left(a_j^{o_j}, a_{2n}\right) \in \frac{1}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R}.$$

Since f_{Ω_δ} is real, we obtain that

$$r\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \in \frac{1}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R} \quad \forall a_{2n}\in\partial_0 V_{\Omega_\delta^m}\setminus\{a_1,\ldots,a_{2n-1}\}.$$

- If $a_{2n}=a_k\in\partial_0 V_{\Omega_\delta^m}$ for some $k\in\{1,\ldots,2n-1\}$, we have, by Proposition 76, that

$$a_{2n} \mapsto h_{\Omega_\delta}\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)$$
$$-(-1)^k f_{\Omega_\delta}\left(a_1^{o_1},\ldots,a_{j-1}^{o_{j-1}},a_{j+1}^{o_{j+1}},\ldots,a_{2n-1}^{o_{2n-1}}\right)h_{\Omega_\delta}\left(a_j^{o_j},a_{2n}\right)$$

extends in an s-holomorphic way to a_{2n} by 0. And we deduce readily that in that case as well

$$r\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right) \in \frac{1}{\sqrt{\nu_{ext}(a_{2n})}}\mathbb{R} \quad \forall a_{2n}\in\partial_0 V_{\Omega_\delta^m}\cap\{a_1,\ldots,a_{2n-1}\}.$$

By Corollary 29, we deduce that $r\left(a_1^{o_1},\ldots,a_{2n-1}^{o_{2n-1}},a_{2n}\right)=0$. □

6.6. Pfaffian formulae and fusion of observables

From the previous section, we can directly derive a Pfaffian formula for the unfused observables:

Proposition 84. Let $a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices at distance at least δ from each other. Then we have

$$f_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = \mathrm{Pfaff}(A_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})),$$

where $A_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\in M_{2n}(\mathbb{R})$ is the antisymmetric matrix defined by

$$(A_{\Omega_\delta}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}))_{jk} = \begin{cases} f\left(a_j^{o_j},a_k^{o_k}\right) & \text{if } j\neq k,\\ 0 & \text{if } j=k.\end{cases}$$

Proof. It follows readily from the recursion formula for the Pfaffians given in Section 1.3.4 and from the recusion formula of Corollary 83. □

From the previous one, we obtain the Pfaffian formula for the fused observables:

Proposition 85. Let $e_1,\ldots,e_m\in E_{\Omega_\delta}$ be edges and $a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices such that $m(e_1),\ldots,m(e_m),a_1,\ldots,a_{2n}$ are at distance at least δ from each other. Then for each choice of orientations $q_1\in O(e_1),\ldots,q_m\in O(e_m)$ we have

$$f_{\Omega_\delta}^{[e_1,\ldots,e_m]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = \mathrm{Pfaff}\left(A_{\Omega_\delta}\left(e_1^{q_1},\ldots,e_m^{q_m},e_m^{(iq_m)^2},\ldots,e_1^{(iq_1)^2},a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right)\right),$$

where $A_{\Omega_\delta}\left(x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\right)\in M_{2p}(\mathbb{R})$ is defined for (non necessarily distinct) doubly-oriented vertices $x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\in D_{\Omega_\delta^m}$ by

$$\left(A_{\Omega_\delta}\left(x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\right)\right)_{jk} = \begin{cases} f_{\Omega_\delta}\left(x_j^{\xi_j},x_k^{\xi_k}\right) & \text{if } x_j\neq x_k,\\ f_{\Omega_\delta}^{C_\delta}\left(x_j^{\xi_j},x_k^{\xi_k}\right) & \text{if } x_j=x_k \text{ and } \xi_j\neq\xi_k,\\ 0 & \text{otherwise.}\end{cases}$$

where $f_{\Omega_\delta}^{C_\delta}=f_{\Omega_\delta}-f_{C_\delta}$ is the discrete boundary effect observable.

Proof. This follows from the Corollary 78 and Proposition 84, by induction on $m\geq 0$.

- For $m = 0$ and all values of $n \geq 0$, this is given by Proposition 84.
- Let us suppose the assertion proven for $m - 1 \geq 0$ and all values of $n \geq 0$. Then by the induction assumption, we have that for $a^o \in D_{\Omega_\delta^m}$, $f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},a^o)$ is given by

$$\mathrm{Pfaff}\left(A_{\Omega_\delta}\left(e_1^{q_1},\ldots,e_{m-1}^{q_{m-1}},e_{m-1}^{(iq_{m-1})^2},\ldots,e_1^{(iq_1)^2},a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},a^o\right)\right).$$

and that $f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,f_{C_\delta}(e_m^{q_m},a^o)$ is given by

$$\mathrm{Pfaff}\left(\tilde{A}_{\Omega_\delta}\left(e_1^{q_1},\ldots,e_{m-1}^{q_{m-1}},e_{m-1}^{(iq_{m-1})^2},\ldots,e_1^{(iq_1)^2},a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},a^o\right)\right),$$

where

$$\tilde{A}_{\Omega_\delta} = A_{\Omega_\delta}\left(e_1^{q_1},\ldots,e_{m-1}^{q_{m-1}},e_{m-1}^{(iq_{m-1})^2},\ldots,e_1^{(iq_1)^2},a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right)\oplus A_{C_\delta}(e_m^{q_m},a^o),$$

with $M_1 \oplus M_2 \in M_{p+r}(\mathbb{R})$ denoting the direct sum of the matrices $M_1 \in M_p(\mathbb{R})$ and $M_2 \in M_r(\mathbb{R})$, where the diagonal blocks are M_1 and M_2 and the off-diagonal ones are zero. By definition, we have

$$h_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},a^o) = -\frac{i}{\sqrt{o}}f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},a^o)$$

$$f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,h_{C_\delta}(e_m^{q_m},a^o) = -\frac{i}{\sqrt{o}}f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,f_{C_\delta}(e_m^{q_m},a^o)$$

and from Corollary 78, we have

$$\begin{aligned} h_{\Omega_\delta}^{[e_1,\ldots,e_m]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) &= h_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},e_m) \\ &\quad - f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,h_{C_\delta}(e_m^{q_m},e_m) \\ &= h_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}\left(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},e_m^{(i\sqrt{q_m})^2}\right) \\ &\quad - f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,h_{C_\delta}\left(e_m^{q_m},e_m^{(i\sqrt{q_m})^2}\right) \end{aligned}$$

since the right-hand side can be extended in an s-holomorphic way in the variable a^o at e_m. The second equality follows from the phase at e_m, which is also given by Corollary 78. Hence we obtain

$$\begin{aligned} f_{\Omega_\delta}^{[e_1,\ldots,e_m]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) &= f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}\left(a_1^{o_1},\ldots,a_{2n}^{o_{2n}},e_m^{q_m},e_m^{(i\sqrt{q_m})^2}\right) \\ &\quad - f_{\Omega_\delta}^{[e_1,\ldots,e_{m-1}]}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\,f_{C_\delta}\left(e_m^{q_m},e_m^{(i\sqrt{q_m})^2}\right), \end{aligned}$$

for any choice of q_m. Replacing the first and second parts of the right hand side by the Pfaffians and reordering, we obtain the desired result.

□

This is the final formula of the discrete part: it reduces all the questions concerning the observables to the computation of the two-point observable.

CHAPTER 7

Scaling Limits of the Observables and Formulae

In this chapter, we define the continuous versions of the discrete observables introduced in Chapters 5 and 6, and then use the results of Chapter 4 to show the convergence of the discrete observables to the continuous ones. More precisely, the plan of this chapter is the following:

- In Section 7.1, we define the continuous full-plane two-point fermionic observables and obtain convergence of the discrete full-plane observables to the continuous ones.
- In Section 7.2, we define the continuous two-point fermionic observables (on finitely-connected domains), give properties for them and obtain convergence of the discrete two-point fermionic observables (including the boundary effect ones), to the continuous observables.
- In Section 7.3, we define and study the continuous general n-point fused observables and give properties for them.
- In Section 7.4, we obtain the main result of this text concerning the convergence of the general discrete observables to the continuous ones.
- In Section 7.5, we give the proofs of the main theorems of the introduction.

For a domain Ω, we denote by $D_\Omega = \{a^o : a \in \Omega, o \in (\mathbb{S})^2\}$ the set of *doubly-oriented points* of Ω and by $S_\Omega = \{a^o : a \in \Omega, o \in \mathbb{S}\}$ the set of *simply-oriented points* of Ω.

7.1. Continuous full-plane two-point observable

We first define the continuous version of the full-plane two-point fermionic observables and study the convergence of their discrete versions (defined in Section 5.2.4).

Definition 86. For a doubly-oriented point $a_1^{o_1} \in D_\Omega$, we denote by $h_{\mathbb{C}}(a_1^{o_1}, \cdot) : \mathbb{C} \setminus \{a_1\} \to \mathbb{C}$ the function defined by

$$h_{\mathbb{C}}(a_1^{o_1}, a_2) = \frac{\sqrt{\overline{o_1}}}{a_2 - a_1}.$$

For a doubly-oriented points $a_1^{o_1}, a_2^{o_2} \in D_\Omega$, we define

$$\begin{aligned}
h_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) &= \mathrm{P}_{\sqrt{\frac{i}{o_2}}\cdot\mathbb{R}}[h_{\mathbb{C}}(a_1^{o_1}, a_2)] = \frac{\sqrt{\overline{o_1}}}{a_2 - a_1} - \frac{1}{\sqrt{\overline{o_1}\cdot o_2}} \cdot \frac{1}{\overline{a_2} - \overline{a_1}},\\
f_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) &= i\sqrt{\overline{o_2}} \cdot h_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) = \frac{i\sqrt{\overline{o_1}}\sqrt{\overline{o_2}}}{a_2 - a_1} - \frac{i}{\sqrt{\overline{o_1}}\sqrt{\overline{o_2}}} \cdot \frac{1}{\overline{a_2} - \overline{a_1}},\\
g_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) &= \sqrt{\overline{o_1}} \cdot h_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) = \frac{\overline{o_1}}{a_2 - a_1} - \frac{1/o_2}{\overline{a_2} - \overline{a_1}}.
\end{aligned}$$

We call $f_{\mathbb{C}}$ and $h_{\mathbb{C}}$ the **continuous real** (respectively **complex**) **full-plane fermionic observables**. We call $g_{\mathbb{C}}$ the **two-point free fermion**.

Notice that $h_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2})$ does not depend on the branch choice of o_2 and that $g_{\mathbb{C}}(a_1^{o_1}, a_2^{o_2})$ does not depend on the branch choices of o_1 and o_2.

Note also that have not introduced a discrete version of $g_{\mathbb{C}}$, although it could be properly defined without any problem. The reason is that this discrete version is not convenient from a discrete complex analysis point of view, not being s-holomorphic.

We can now state the following convergence result:

Theorem 87. *For each $\varepsilon > 0$, the (renormalized) discrete full-plane real and complex fermionic observables $(a_1^{o_1}, a_2^{o_2}) \mapsto \frac{1}{\delta} \cdot f_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2})$ and $(a_1^{o_1}, a_2^{o_2}) \mapsto \frac{1}{\delta} \cdot h_{\mathbb{C}_\delta}(a_1^{o_1}, a_2^{o_2})$ converge uniformly on $\{(a_1, a_2) \in \mathbb{C}^2 : |a_1 - a_2| \geq \varepsilon\}$ to their continuous analogues $f_{\mathbb{C}}$ and $h_{\mathbb{C}}$ as $\delta \to 0$.*

Proof. This follows directly from the construction of the full-plane observable (Theorem 22) and the convergence of the Green's function $G^{\overline{\partial}_\delta}$ to its continuous analogue (Theorem 33). □

7.2. Continuous domain two-point observable

We now turn to a central result of this paper: the convergence of the two-point discrete fermionic observables to their continuous analogues, which, thanks to the Pfaffian formulae for the more-point versions established in the previous chapters, will allow for the proof of convergence of the general observables (handled in Section 7.4). Let us first define the continuous two-point observables:

Definition 88. Let Ω be a finitely-connected domain. For a doubly-oriented point $a_1^{o_1} \in \mathcal{D}_\Omega$, we denote by $h_\Omega(a_1^{o_1}, \cdot) : \Omega \setminus \{a_1\} \to \mathbb{C}$ the unique holomorphic function solving the following boundary value problem:

$$h_\Omega(a_1^{o_1}, a_2) \sim \frac{\sqrt{o_1}}{2\pi(a_2 - a_1)}$$

$$h_\Omega(a_1^{o_1}, a_2) \parallel \frac{1}{\sqrt{\nu_{\mathrm{ext}}(a_2)}} \quad \forall a_2 \in \partial\Omega,$$

where the boundary condition is defined in the integral sense defined in Section 4.1. Equivalently, $h_\Omega(a_1^{o_1}, \cdot)$ is the unique solution to the continuous Riemann-Hilbert boundary value problem $(\spadesuit_\Omega, h_{\mathbb{C}}(a_1^{o_1}, \cdot)|_{\partial\Omega})$.

Analogously to the discrete case, we define $h_\Omega(a_1^{o_1}, a_2^{o_2})$ for a doubly-oriented point $a_1^{o_1} \in \mathcal{D}_\Omega$ and a simply-oriented point $a_2^{o_2} \in \mathcal{S}_\Omega$ by

$$h_\Omega(a_1^{o_1}, a_2^{o_2}) = \mathbb{P}_{i\sqrt{o_2}\mathbb{R}}\left[h_\Omega(a_1^{o_1}, a_2)\right].$$

For $a_1^{o_1}, a_2^{o_2} \in \mathcal{D}_\Omega$, we define $f_\Omega(a_1^{o_1}, a_2^{o_2})$ as $i \cdot \sqrt{o_2} h_\Omega(a_1^{o_1}, a_2^{o_2})$. We call h_Ω and f_Ω the **continuous (two-point) complex and real fermionic observables.**

Unfortunately, on general finitely-connected domains, we do not have an Ising-independent proof of the existence of these observable: the uniqueness is guaranteed by Proposition 48, and the existence is obtained by using the scaling limit of the discrete observables (Proposition 92). On simply connected domains, though, the continuous complex fermionic observable can be constructed explicitly in terms of the conformal mapping to the half-plane.

Proposition 89. If Ω is bounded and simply connected, the two-point continuous observables h_Ω is given, for any conformal mapping $\phi_\Omega : \Omega \to \mathbb{H}$ and any branch choice of $\sqrt{\phi_\Omega'}$, by

$$h_\Omega(a_1^{o_1}, a_2) = \sqrt{|\phi_\Omega'(a_1)|}\sqrt{\phi_\Omega'(a_2)}\, h_{\mathbb{H}}\left(\tilde{a}_1^{\tilde{o}_1}, \tilde{a}_2\right),$$

where $\tilde{a}_1, \tilde{a}_2 = \phi_\Omega(a_1), \phi_\Omega(a_2)$ and $\tilde{o}_1 = \left(\sqrt{\phi_\Omega'(a_1)}\sqrt{o_1}\right)^2 / |\phi_\Omega'(a_1)|$ and

$$h_{\mathbb{H}}(z_1^{q_1}, z_2) = \frac{\sqrt{q_1}}{z_2 - z_1} + \frac{i\sqrt{\overline{q_1}}}{z_2 - \overline{z_1}}.$$

Proof. It is straightforward that

$$h_{\mathbb{H}}(z_1^{q_1}, z_2) \sim \frac{\sqrt{q_1}}{z_2 - z_1} \text{ as } z_2 \to z_1$$

and it is easy to check that

$$h_{\mathbb{H}}(z_1^{q_1}, z_2) \parallel \frac{1}{\sqrt{\nu_{\text{ext}}(z_2)}} \quad \forall z_2 \in \mathbb{R}.$$

The only subtlety is for non-smooth boundary point at infinity, but it is easy to check that

$$\Re\left(\int_0^{\zeta} h_{\mathbb{H}}(z_1^{q_1}, z_2)^2\, dz_2\right) \underset{\zeta\to\infty}{\longrightarrow} 0$$

and hence that by change of variable formula, we have

$$\sqrt{|\phi_\Omega'(a_1)|}\sqrt{\phi_\Omega'(a_2)} \cdot h_{\mathbb{H}}\left(\tilde{a}_1^{\tilde{o}_1}, \tilde{a}_2\right) \parallel \frac{1}{\sqrt{\nu_{\text{ext}}(a_2)}} \quad \forall a_2 \in \partial\Omega,$$

in the integral sense defined in Section 4.1. By expanding as $a_2 \to a_1$, it is again straightforward that we have

$$\sqrt{|\phi_\Omega'(a_1)|}\sqrt{\phi_\Omega'(a_2)} \cdot h_{\mathbb{H}}\left(\tilde{a}_1^{\tilde{o}_1}, \tilde{a}_2\right) \sim \frac{\sqrt{o_1}}{a_2 - a_1},$$

which shows the result. □

Also analogously to the discrete case, we define the boundary effect observables.

Definition 90. We denote by $f_\Omega^{\mathbb{C}}$ and $h_\Omega^{\mathbb{C}}$ and call *continuous* (respectively *real* and *complex*) *boundary effect observables* the functions $D_\Omega \times D_\Omega \to \mathbb{C}$ defined by

$$\begin{aligned} f_\Omega^{\mathbb{C}} &= f_\Omega - f_{\mathbb{C}}, \\ h_\Omega^{\mathbb{C}} &= h_\Omega - h_{\mathbb{C}}. \end{aligned}$$

As before for $a_1^{o_1} \in D_\Omega$ and $a_2 \in \Omega$, we define

$$h_\Omega^{\mathbb{C}}(a_1^{o_1}, a_2) = h_\Omega^{\mathbb{C}}(a_1^{o_1}, a_2^{o_2}) + h_\Omega^{\mathbb{C}}\left(a_1^{o_1}, a_2^{-o_2}\right),$$

for any $o_2 \in \mathbb{S}$ (the choice thereof does not matter).

Equivalently, $h_\Omega^{\mathbb{C}}(a_1^{o_1}, \cdot)$ can be formulated as the solution to the continuous Riemann-Hilbert boundary value problem $(\spadesuit_\Omega, h_\Omega)$.

Since they depend on the continuous real and complex fermionic observables, the existence of the boudary effect real and complex observables on multiply connected domains is only guaranteed by the existence of scaling limits of the discrete versions.

Using the results of the previous chapter, we can now prove the following theorem, which is central to our strategy:

Theorem 91. Let Ω be a smooth finitely-connected domain with straight boundary parts $\partial^s\Omega \subset \partial\Omega$ and $(\Omega_\delta)_{\delta>0}$ a discretization of Ω. Then we have that the following limits exist:

$$\frac{1}{\delta} \cdot f^{C_\delta}_{\Omega_\delta}(a_1^{o_1}, a_2^{o_2}) \underset{\delta\to 0}{\longrightarrow} f^{C}_{\Omega}(a_1^{o_1}, a_2^{o_2}),$$
$$\frac{1}{\delta} \cdot h^{C_\delta}_{\Omega_\delta}(a_1^{o_1}, a_2^{o_2}) \underset{\delta\to 0}{\longrightarrow} h^{C}_{\Omega}(a_1^{o_1}, a_2^{o_2}),$$

where the convergence is uniform on the compact subsets of

$$\{(a_1, a_2) \in (\Omega \times (\Omega \cup \partial^s\Omega)) \cup ((\Omega \cup \partial^s\Omega) \times \Omega)\} \times \left\{(o_1, o_2) \in (S)^2 \times (S)^2\right\}.$$

and that the following limits exist:

$$\frac{1}{\delta} \cdot f_{\Omega_\delta}(a_1^{o_1}, a_2^{o_2}) \underset{\delta\to 0}{\longrightarrow} f_{\Omega}(a_1^{o_1}, a_2^{o_2}),$$
$$\frac{1}{\delta} \cdot h_{\Omega_\delta}(a_1^{o_1}, a_2^{o_2}) \underset{\delta\to 0}{\longrightarrow} h_{\Omega}(a_1^{o_1}, a_2^{o_2}),$$

where the convergence is uniform on the compact subsets of

$$\{(a_1, a_2) \in (\Omega \cup \partial^s\Omega) \times (\Omega \cup \partial^s\Omega) : a_1 \neq a_2\}$$
$$\times \left\{(o_1, o_2) \in (S)^2 \times (S)^2\right\}.$$

Proof. We have that

$$\frac{1}{\delta} \cdot h^{C_\delta}_{\Omega_\delta}(a_1^{o_1}, a_2) \underset{\delta\to 0}{\longrightarrow} h^{C}_{\Omega}(a_1^{o_1}, a_2)$$

uniformly on the compact subsets of

$$\{(a_1, a_2) \in \Omega \times (\Omega \cup \partial^s\Omega)\} \times \left\{o_1 \in (S)^2\right\}$$

and in particular that $h^{C}_{\Omega}(a_1^{o_1}, a_2)$ exists: it follows from Theorem 59, since $\frac{1}{\delta} \cdot h^{C_\delta}_{\Omega}(a_1^{o_1}, \cdot)$ solves the Riemann-Hilbert boundary value problem $\left(\Omega_\delta, \frac{1}{\delta} \cdot h_{C_\delta}(a_1^{o_1}, \cdot)\right)$, since for each $a_1^{o_1}$ we have

$$\frac{1}{\delta} \cdot h_{C_\delta}(a_1^{o_1}, \cdot) \underset{\delta\to 0}{\longrightarrow} h_C(a_1^{o_1}, \cdot)$$

and since the convergence (on a neighoborhood of $\partial\Omega$) is uniformly continuous with respect to $a_1^{o_1}$ (while a_1 stays away from $\partial\Omega$). We deduce that

$$\frac{1}{\delta} \cdot h_{\Omega_\delta}(a_1^{o_1}, a_2) \underset{\delta\to 0}{\longrightarrow} h_{\Omega}(a_1^{o_1}, a_2),$$

uniformly on the compact subsets of

$$\{(a_1, a_2) \in \Omega \times (\Omega \cup \partial^s\Omega) : a_1 \neq a_2\} \times \left\{o_1 \in (S)^2\right\},$$

the existence of $h_\Omega(a_1^{o_1}, a_2)$ being guaranteed by the one of $h_\Omega^C(a_1^{o_1}, a_2)$. Using the definition of h_{Ω_δ} and the antisymmetry of $f_{\Omega_\delta}(\cdot,\cdot)$, we easily obtain

$$
\begin{aligned}
h_{\Omega_\delta}(a_1^{o_1}, a_2^{o_2}) &= -\frac{\sqrt{o_1}}{\sqrt{o_2}} h_{\Omega_\delta}(a_2^{o_2}, a_1^{o_1}), \\
h_{\Omega_\delta}\left(a_1^{(i\sqrt{o_1})^2}, a_2^{o_2}\right) &= -\frac{i\sqrt{o_1}}{\sqrt{o_2}} h_{\Omega_\delta}(a_2^{o_2}, a_1^{o_1})
\end{aligned}
$$

which, given the convergence exchanging the indices 1 and 2, gives that

$$\frac{1}{\delta}\cdot h_{\Omega_\delta}(a_1^{o_1}, a_2) \underset{\delta\to 0}{\longrightarrow} h_\Omega(a_1^{o_1}, a_2)$$

uniformly on the compact subsets of

$$\{(a_1, a_2) \in (\Omega \cup \partial^s\Omega) \times \Omega : a_1 \neq a_2\} \times \left\{o_1 \in (S)^2_{\square}\right\}.$$

By Proposition 54, since we have

$$h_{\Omega_\delta}(a_1^{o_1}, a_2) \,\square\, \frac{1}{\sqrt{V_{ext}(a_2)}} \quad \forall a_2 \in \partial_0 V_{\Omega_\delta^m},$$

we can extend the convergence of $\frac{1}{\delta}\cdot h_{\Omega_\delta}(a_1^{o_1}, a_2)$ uniformly on the compact subsets of

$$\{(a_1, a_2) \in (\Omega \cup \partial^s\Omega) \times (\Omega \cup \partial^s\Omega) : a_1 \neq a_2\} \times \left\{o_1 \in (S)^2_{\square}\right\}.$$

$\square$

Hence, we finally obtain the existence of the continuous observables:

Proposition 92. The continuous observables f_Ω, h_Ω, f_Ω^C, h_Ω^C exist on any finitely-connected domain Ω and we have the following conformal covariance properties: if $\phi : \Omega \to \tilde\Omega$ is a conformal mapping, for any branch choice of $\sqrt{\phi'}$, we have

$$
\begin{aligned}
f_\Omega(a_1^{o_1}, a_2^{o_2}) &= \sqrt{|\phi'(a_1)|}\cdot\sqrt{|\phi'(a_2)|}\, f_{\tilde\Omega}\left(\tilde a_1^{\tilde o_1}, \tilde a_2^{\tilde o_2}\right), \\
h_\Omega(a_1^{o_1}, a_2) &= \sqrt{|\phi'(a_1)|}\cdot\sqrt{\phi'(a_2)}\cdot g_{\tilde\Omega}\left(\tilde a_1^{\tilde o_1}, \tilde a_2\right), \\
f_\Omega^C\left(a_1^{o_1}, a_1^{(i\sqrt{o_1})^2}\right) &= |\phi'(a_1)|\cdot f_{\tilde\Omega}^C\left(\tilde a_1^{\tilde o_1}, \tilde a_1^{(i\sqrt{\tilde o_1})^{2s}}\right), \\
h_\Omega^C(a_1^{o_1}, a_1) &= \frac{1}{\sqrt{o_1}}\cdot|\phi'(a_1)|\cdot h_{\tilde\Omega}^C\left(\tilde a_1^{\tilde o_1}, \tilde a_1\right),
\end{aligned}
$$

where $\tilde a_j = \phi(a_j)$, $\tilde o_j = \left(\sqrt{\phi'(a_j)}\cdot\sqrt{o_j}\right)^2$, for $j \in \{1, 2\}$. We have the following antisymmetry property:

$$f_\Omega(a_1^{o_1}, a_2^{o_2}) = -f_\Omega(a_2^{o_2}, a_1^{o_1}).$$

Proof. Let us first suppose that Ω is smooth. Then for any lattice double-orientation $o_1 \in (S)^2_{\square}$, by Theorem 91, we have that $h_\Omega(a_1^{o_1}, a_2)$ exists. For a general double-orientation $\tilde o_1 \in (S)^2$, we have that $\sqrt{\tilde o_1} = \cos\vartheta\sqrt{p_1} + \sin\vartheta\sqrt{q_1}$ for some angle ϑ and some lattice double-orientations $p_1, q_1 \in (S)^2_{\square}$, and hence we can construct $h_\Omega\left(a_1^{\tilde o_1}, \cdot\right)$ as $\cos\vartheta\cdot h_\Omega(a_1^{p_1}, \cdot) + \sin\vartheta\cdot h_\Omega(a_1^{q_1}, \cdot)$. If Ω is not smooth, we can conformally map it by a conformal mapping ϕ to a smooth domain $\tilde{\Omega}$. We have that $h_{\Omega}(a_1^{o_1}, a_2)$ can be constructed by

$$h_{\Omega}(a_1^{o_1}, a_2) = \sqrt{|\phi'(a_1)|} \cdot \sqrt{\phi'(a_2)} \cdot h_{\tilde{\Omega}}\left(\tilde{a}_1^{\tilde{o}_1}, \tilde{a}_2\right),$$

where $\tilde{a}_1 = \phi(a_1)$, $\tilde{a}_2 = \phi(a_2)$ and $\tilde{o}_1 = \left(\sqrt{\phi'(a_1)}\sqrt{o_1}\right)^2 / |\phi'(a_1)|$.

The right hand side does not depend on the branch choice of $\sqrt{\phi'}$ (and such a branch choice exists by Proposition 47) and $h_{\tilde{\Omega}}\left(\tilde{a}_1^{\tilde{o}_1}, \tilde{a}_2\right)$ exists since $\tilde{\Omega}$ is smooth. By Proposition 47, it is easy to check that we have

$$h_{\Omega}(a_1^{o_1}, a_2) \in \frac{1}{\sqrt{\nu_{\text{ext}}(a_2)}} \mathbb{R} \quad \forall a_2 \in \partial\Omega$$

and by expanding near a_1, that we have

$$h_{\Omega}(a_1^{o_1}, a_2) \sim \frac{\sqrt{o_1}}{a_2 - a_1}.$$

Hence $h_{\Omega}(a_1^{o_1}, a_2)$ and it follows readily from their constructions that the other observables h_{Ω}^{C}, f_{Ω} and f_{Ω}^{C} also exist and from the above discussion the conformal covariance formulae follow.

The antisymmetry of f_{Ω} for smooth Ω follows from the one of the discrete real fermioinic observable given by Proposition 67, by passing to the limit. For general domains it follows from the conformal covariance formula. □

Finally, let us define slightly rephased versions of the observables:

Definition 93. For simply-oriented points $y_1^{\zeta_1}, y_2^{\zeta_2} \in S_{\Omega}$, We define the *domain fermion* g_{Ω} by

$$g_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) = \left(\sqrt{\zeta_1} \cdot h_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right)\right) = -i \cdot \frac{\sqrt{\zeta_1}}{\sqrt{\zeta_2}} f_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right),$$

which, as can be easily checked, does not depend on the branch choices of ζ_1 or ζ_2.

The following lemma will be useful to us:

Lemma 94. *Let Ω be a finitely-connected domain any $y_1^{\zeta_1}, y_2^{\zeta_2}$ be two simply-oriented points. Then we have*

$$g_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) = \overline{g_{\Omega}\left(y_2^{\zeta_2}, y_1^{\zeta_1}\right)}. \tag{7.2.1}$$

Also, we have that

$$g_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + g_{\Omega}\left(y_1^{\zeta_1}, y_2^{-\zeta_2}\right) \tag{7.2.2}$$

does not depend on ζ_2, that

$$g_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + g_{\Omega}\left(y_1^{-\zeta_1}, y_2^{\zeta_2}\right) \tag{7.2.3}$$

does not depend on ζ_1, that

$$g_{\Omega}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + g_{\Omega}\left(y_1^{\zeta_1}, y_2^{-\zeta_2}\right) + g_{\Omega}\left(y_1^{-\zeta_1}, y_2^{\zeta_2}\right) + g_{\Omega}\left(y_1^{-\zeta_1}, y_2^{-\zeta_2}\right) \tag{7.2.4}$$

does not depend on ζ_1 or ζ_2, and that

$$\bar{\zeta}_1\left(g_{\Omega}\left(y_1^{\zeta_1}, y_2\right) - g_{\Omega}\left(y_1^{-\zeta_1}, y_2\right)\right) \tag{7.2.5}$$

does not depend on ζ_1.

Proof. Equation 7.2.1 follows from the definition of the antisymmetry of f_Ω, given by Proposition 92, and the definition of g_Ω.

The independence with respect to ζ_2 of the expression 7.2.2 follows from the definition of g_Ω and the fact that $h_\Omega\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + h_\Omega\left(y_1^{\zeta_1}, y_2^{-\zeta_2}\right) = h_\Omega\left(y_1^{\zeta_1}, y_2\right)$ does not depend on ζ_2 by definition. The independence with respect to ζ_1 of expression 7.2.3 follows from the independence of 7.2.2 and Equation 7.2.1. The independence of 7.2.4 follows from the independence of

$$\left(g_\Omega\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + g_\Omega\left(y_1^{\zeta_1}, y_2^{-\zeta_2}\right)\right) + \left(g_\Omega\left(y_1^{-\zeta_1}, y_2^{\zeta_2}\right) + g_\Omega\left(y_1^{-\zeta_1}, y_2^{-\zeta_2}\right)\right)$$

on ζ_2 by expression 7.2.2 and the independence of

$$\left(g_\Omega\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) + g_\Omega\left(y_1^{-\zeta_1}, y_2^{\zeta_2}\right)\right) + \left(g_\Omega\left(y_1^{\zeta_1}, y_2^{-\zeta_2}\right) + g_\Omega\left(y_1^{-\zeta_1}, y_2^{-\zeta_2}\right)\right)$$

on ζ_1. On the other hand, we have

$$\begin{aligned}
\overline{\zeta_1}\left(g_\Omega\left(y_1^{\zeta_1}, y_2\right) - g_\Omega\left(y_1^{-\zeta_1}, y_2\right)\right) &= \overline{\zeta_1}\left(\overline{\zeta_1} h_\Omega\left(y_1^{\zeta_1}, y_2\right) - i \cdot \overline{\zeta_1} h_\Omega\left(y_1^{(i\sqrt{\overline{\zeta_1}})^2}, y_2\right)\right)\\
&= \overline{\zeta_1}\left(\overline{\zeta_1}\,\overline{\zeta_1} h_\Omega\left(y_1^{(1)^2}, y_2\right) - i \cdot \overline{\zeta_1}\,\overline{\zeta_1} h_\Omega\left(y_1^{(i)^2}, y_2\right)\right)\\
&= h_\Omega\left(y_1^{(1)^2}, y_2\right) - i \cdot h_\Omega\left(y_1^{(i)^2}, y_2\right),
\end{aligned}$$

and hence that the expression 7.2.5 is indeed independent of ζ_1. □

Definition 95. We denote by $g_\Omega\left(y_1^{\zeta_1}, y_2\right)$, $g_\Omega\left(y_1, y_2^{\zeta_2}\right)$, $g_\Omega(y_1, y_2)$ and $g_\Omega^\times(y_1, y_2)$ the four functions defined by Equations 7.2.2, 7.2.3, 7.2.4 and 7.2.5 respectively. We denote by $g_\Omega^{\mathbb{C}}$ the difference $g_\Omega - g_{\mathbb{C}}$.

The following lemma follows from a straightforward computation:

Lemma 96. *On the upper half-plane $\mathbb{H}$, we havef*

$$\begin{aligned}
g_{\mathbb{H}}\left(y_1^{\zeta_1}, y_2^{\zeta_2}\right) &= \frac{1}{4\pi}\left(\frac{\zeta_1}{y_2 - y_1} - \frac{\overline{\zeta_2}}{\overline{y_2} - \overline{y_1}} + \frac{i}{y_2 - \overline{y_1}} - \frac{i \cdot \zeta_1 \overline{\zeta_2}}{\overline{y_2} - y_1}\right),\\
g_{\mathbb{H}}\left(y_1^{\zeta_1}, y_2\right) &= \frac{1}{2\pi}\left(\frac{\zeta_1}{y_2 - y_1} + \frac{i}{y_2 - \overline{y_1}}\right).\\
g_{\mathbb{H}}\left(y_1, y_2^{\zeta_2}\right) &= \frac{1}{2\pi}\left(\frac{\overline{\zeta_2}}{\overline{y_2} - \overline{y_1}} - \frac{i}{\overline{y_2} - y_1}\right),\\
g_{\mathbb{H}}(y_1, y_2) &= \frac{i}{\pi(y_2 - \overline{y_1})},\\
g_{\mathbb{H}}^\times(y_1 . y_2) &= \frac{1}{\pi(y_2 - y_1)}.
\end{aligned}$$

7.3. Continuous general observables

We now define the natural candidates for the scaling limits of general discrete observables, as in the discrete case, as Pfaffians of the two-point functions defined in the previous section. Let Ω be a domain and let $a_1^{o_1}, \ldots, a_{2n}^{o_{2n}} \in D_\Omega$ be doubly-oriented points. We first define the continuous analogues of the discrete observables introduced in the previous chapters.

Definition 97. We define the fermionic matrix $A_\Omega(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) \in M_{2n}(\mathbb{R})$ by

$$\left(A_\Omega\left(x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\right)\right)_{jk} = \begin{cases} f_\Omega\left(x_j^{\xi_j}, x_k^{\xi_k}\right) & \text{if } x_j \neq x_k, \\ f_\Omega^C\left(x_j^{\xi_j}, x_k^{\xi_k}\right) & \text{if } x_j = x_k \text{ and } \xi_j \neq \xi_k, \\ 0 & \text{otherwise,} \end{cases}$$

where the continuous two-point fermionic observables f_Ω and f_Ω^C are as defined in the previous section.

Definition 98. We define the continuous (unfused) real fermionic observable $f_\Omega(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$ for distinct points $a_1,\ldots,a_{2n} \in \Omega$ by

$$f_\Omega(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}) = \mathrm{Pfaff}\left(A_\Omega(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\right).$$

The following lemma allows us to canonically define the continuous fused fermionic observable:

Proposition 99. *Let Ω be a finitely-connected domain and let $a_1^{o_1},\ldots,a_{2n}^{o_{2n}} \in D_\Omega$ be distinct doubly-oriented points. Let $e_1,\ldots,e_m \in \Omega \setminus \{a_1,\ldots,a_{2n}\}$ be points and let $q_1,\ldots,q_m \in (S)^2$ be double orientations. Then the Pfaffian*

$$\mathrm{Pfaff}\left(A_\Omega\left(e_1^{q_1},\ldots,e_m^{q_m}, e_m^{(i\sqrt{q_m})^2},\ldots,e_1^{(i\sqrt{q_1})^2}, a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right)\right) \tag{7.3.1}$$

is independent of the choices $q_1,\ldots,q_m$ and is equal to

$$\frac{(2i)^{-n}\cdot i^m}{\sqrt{o_1}\cdot\ldots\cdot\sqrt{o_{2n}}}\mathrm{Pfaff}\left(X_\Omega(e_1,\ldots,e_m,a_1^{o_1},\ldots,a_{2n}^{o_{2n}})\right),$$

where, if we set $v = (e_1,\ldots,e_m,a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$, we have

$$X_\Omega(v) = \begin{pmatrix} X_\Omega^{11}(v) & X_\Omega^{12}(v) & X_\Omega^{13}(v) \\ X_\Omega^{21}(v) & X_\Omega^{22}(v) & X_\Omega^{23}(v) \\ X_\Omega^{31}(v) & X_\Omega^{32}(v) & X_\Omega^{33}(v) \end{pmatrix},$$

and where $X_\Omega^{11}(v), X_\Omega^{12}(v), X_\Omega^{21}(v), X_\Omega^{22}(v) \in M_m(\mathbb{C})$, are defined for $j,k \in \{1,\ldots,m\}$ by

$$\left(X_\Omega^{11}(v)\right)_{jk} = \begin{cases} 0 & \text{if } j = k \\ \overline{g_\Omega^\times(e_k,e_j)} & \text{if } j < k \\ -g_\Omega^\times(e_j,e_k) & \text{if } j > k \end{cases}$$

$$\left(X_\Omega^{12}(v)\right)_{jk} = \begin{cases} -i\cdot g_\Omega^C(e_j,e_j) & \text{if } j+k = m \\ -i\cdot\overline{g_\Omega(e_k,e_{m-j})} & \text{if } j+k \neq m \end{cases}$$

$$\left(X_\Omega^{21}(v)\right)_{jk} = \begin{cases} -i\cdot\overline{g_\Omega^C(e_j,e_j)} & \text{if } j+k = m \\ -i\cdot\overline{g_\Omega(e_k,e_{m-j})} & \text{if } j+k \neq m \end{cases}$$

$$\left(X_\Omega^{22}(v)\right)_{jk} = \begin{cases} 0 & \text{if } j = k \\ \overline{g_\Omega^\times(e_{m-k},e_{m-j})} & \text{if } j < k \\ -g_\Omega^\times(e_{m-k},e_{m-j}) & \text{if } j > k \end{cases}$$

where $X^{13}_{\Omega}(v), X^{23}_{\Omega}(v) \in M_{m,n}(\mathbb{C})$ are defined for $j \in \{1,\ldots m\}$ and $k \in \{1,\ldots,n\}$ by

$$\left(X^{13}_{\Omega}(v)\right)_{jk} = g_{\Omega}(a_k^{o_k}, e_j),$$
$$\left(X^{23}_{\Omega}(v)\right)_{jk} = i \cdot \overline{g^{\times}_{\Omega}(a_k^{o_k}, e_{m-j})},$$

and $X^{31}_{\Omega}(v), X^{32}_{\Omega}(v) \in M_{n,m}(\mathbb{C})$ are defined by $X^{31}_{\Omega}(v) = -\left(X^{13}_{\Omega}(v)\right)^{\dagger}$ and $X^{32}_{\Omega}(v) = -\left(X^{23}_{\Omega}(v)\right)^{\dagger}$ (where † denotes the adjoint matrix) and $X^{33}_{\Omega}(v) \in M_n(\mathbb{C})$ is defined by

$$\left(X^{33}_{\Omega}(v)\right)_{jk} = \begin{cases} 0 & \text{if } j = k \\ o_j \cdot g_{\Omega}\left(a_k^{o_k}, a_j^{o_j}\right) & \text{if } j < k \\ o_k \cdot g_{\Omega}\left(a_k^{o_k}, a_j^{o_j}\right) & \text{if } j > k. \end{cases}$$

In particular, $X_{\Omega}(v)$ is independent of the branch choices of $o_1,\ldots,o_{2n}$.

Definition 100. We denote by $f^{[e_1,\ldots,e_m]}_{\Omega}(a_1^{o_1},\ldots,a_{2n}^{o_{2n}})$ the **continuous (fused) real fermionic observable** defined by the Equation 7.3.1.

Proof of Proposition 99. To shorten the notation, set

$$p = \left(e_1^{q_1},\ldots,e_m^{q_m}, e_m^{(i\sqrt{q_m})^2},\ldots,e_1^{(i\sqrt{q_1})^2}, a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right),$$
$$q = (q_1,\ldots,q_m).$$

We can rewrite $A_{\Omega}(p)$ in terms of g_{Ω} and g^{C}_{Ω}, since we have by definition

$$f_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right) = i\sqrt{\frac{\xi_k}{\xi_j}}\, g_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right),$$
$$f^{C}_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right) = i\sqrt{\frac{\xi_k}{\xi_j}}\, g^{C}_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right).$$

Let $B(q) \in M_{2m+2n}(\mathbb{C})$ be the diagonal matrix with

$$B_{jj} = \begin{cases} \bar{\lambda}\sqrt{q_j} & \text{if } 1 \le j \le m \\ \bar{\lambda}\sqrt{iq_{m-j}} & \text{if } m+1 \le j \le 2m \\ \bar{\lambda}\sqrt{o_{j-2m}} & \text{if } 2m+1 \le j \le 2m+2n \end{cases}$$

Then we have that $B(q) A_{\Omega}(p) B(q) = \tilde{A}_{\Omega}(p)$, where $\tilde{A}_{\Omega}\left(x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\right) \in M_{2p}(\mathbb{C})$ is defined by

$$\left(\tilde{A}_{\Omega}\left(x_1^{\xi_1},\ldots,x_{2p}^{\xi_{2p}}\right)\right)_{jk} = \begin{cases} \xi_j\, g_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right) & \text{if } x_j \neq x_k, \\ \xi_j\, g^{C}_{\Omega}\left(x_j^{\xi_j}, x_k^{\xi_k}\right) & \text{if } x_j = x_k \text{ and } \xi_j \neq \xi_k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C(q) \in M_{2m+2n}(\mathbb{C})$ be the matrix defined by

$$(C(q))_{jk} = \begin{cases} \bar{o}_j & \text{if } 1 \le j = k \le m \\ -i & \text{if } m+1 \le j = k \le 2m \\ -i & \text{if } 1 \le j = 2m+1-k \le m \\ -\bar{o}_j & \text{if } m+1 \le j = 2m+1-k \le 2m \\ 1 & \text{if } 2m+1 \le j = k \le 2m+2n \\ 0 & \text{else} \end{cases}$$

Then a straightforward computation, together with the relations of Lemma 94 shows that we have

$$C(q)\,\tilde{A}_\Omega(p)\,C^T(q) = -X_\Omega\left(e_1, \ldots, e_m, a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}\right).$$

Hence, by the formula $\mathrm{Pfaff}(Q^t R Q) = \det(Q)\,\mathrm{Pfaff}(R)$, we have that

$$\mathrm{Pfaff}(X_\Omega(p)) = \det(C(q))\det(B(q))\,\mathrm{Pfaff}(A_\Omega(p)).$$

Since we have

$$\det(B(q)) = \frac{q_1 \cdot \ldots \cdot q_m \cdot \sqrt{\bar{o}_1} \cdot \ldots \cdot \sqrt{\bar{o}_{2n}}}{i^n}$$

and

$$\det(C(q)) = 2^m \frac{\bar{q}_1 \cdot \ldots \bar{q}_m}{i^m},$$

we obtain

$$\mathrm{Pfaff}(A_\Omega(p)) = \frac{(2i)^{-n} \cdot i^m}{\sqrt{\bar{o}_1} \cdot \ldots \cdot \sqrt{\bar{o}_{2n}}}\,\mathrm{Pfaff}\left(X_\Omega\left(e_1, \ldots, e_m, a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}\right)\right)$$

□

7.4. Convergence of general observables

We can now state the main convergence result for the general discrete observables to the continuous ones, which are defined in the previous section.

Theorem 101. Let Ω be a smooth finitely-connected domain with straight boundary parts $\partial_s\Omega \subset \partial\Omega$ and $(\Omega_\delta)_{\delta>0}$ a family of discrete domains discretizing it. Then for any integers $n, m \ge 0$, the renormalized discrete real fermionic observable

$$\left(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}, e_1, \ldots, e_m\right) \mapsto \frac{1}{\delta^n} \cdot f_{\Omega_\delta}^{[e_1,\ldots,e_m]}\left(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}\right)$$

converges uniformly, as $\delta \to 0$, on the compact subsets of

$$\left\{ (z_1, \ldots, z_{2n}, w_1, \ldots, w_m) \in (\Omega \cup \partial_s\Omega) \times \ldots \times (\Omega \cup \partial_s\Omega) \times \Omega \times \ldots \Omega \; : \; z_j \neq z_k, e_j \neq e_k, \forall j \neq k, z_j \neq e_k \, \forall j, k \right\}$$
$$\times \left\{ (o_1, \ldots, o_{2n}) \in (\mathbb{S})^2 \times \ldots \times (\mathbb{S})^2 \right\},$$

where to its continuous counterpart

$$f_\Omega^{[e_1,\ldots,e_m]}\left(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}}\right).$$

Proof. This follows directly from Proposition 85, which gives a Pfaffian representation of the discrete fermionic observables in terms of two-point observables, from Theorem 91, which gives convergence of the two-point observable and from Proposition 99 which allows to define canonically the continuous fused observables in terms of the continuous two-point observable. □

The conformal covariance properties and of the fused observable will allow to prove the conformal covariance of the scaling correlation functions we are interested in:

Theorem 102. *The continuous real fermionic observable f_Ω satisfies the following conformal covariance properties: for any conformal mapping $\phi : \Omega \to \tilde{\Omega}$ and any branch choice of $\sqrt{\phi'}$, we have*

$$f_\Omega^{[e_1,\ldots,e_m]}\left(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right) = \left(\prod_{j=1}^{m} |\phi'(e_j)|\right)\left(\prod_{k=1}^{2n} |\phi'(a_k)|^{\frac{1}{2}}\right) f_{\tilde{\Omega}}^{[\tilde{e}_1,\ldots\tilde{e}_m]}\left(\tilde{a}_1^{\tilde{o}_1},\ldots,\tilde{a}_{2n}^{\tilde{o}_{2n}}\right),$$

where $\tilde{e}_j = \phi(e_j)$ for all $j = 1,\ldots,m$ and $\tilde{a}_1 = \phi(a_1)$, $\tilde{o}_1 = \left(\frac{\sqrt{\phi'(o_1)}\cdot\sqrt{o_1}}{|\phi'(o_1)|}\right)^2$.

Proof. This follows from the Pfaffian representation of Proposition 99 and of the conformal covariance of the two-point function given by Proposition 92. □

And the following explicit computation of the observable on the upper half-plane, together with the conformal covariance result above, allows for the one of the scaling correlation functions in simply connected domains:

Theorem 103. *Let $a_1,\ldots,a_{2n} \in \mathbb{R}$ and $e_1,\ldots,e_m \in \mathbb{H}$ be distinct points. Then for any choices of double orientations $o_1,\ldots,o_{2n} \in (\mathbb{S})^2$ such that $o_j = (\pm\lambda)^2$ for each $j = 1,\ldots,2n$, we have*

$$\begin{aligned} f_{\mathbb{H}}^{[e_1,\ldots,e_m]}\left(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right) &= \frac{i^m}{\pi^{n+m}}\cdot\prod_{j=1}^{2n}\frac{1}{2\sqrt{o_j}} \\ &\quad\cdot \operatorname{Pfaff}\left(K\left(e_1,\ldots,e_m,\overline{e_m},\ldots\overline{e_1},a_1,\ldots,a_{2n}\right)\right), \end{aligned}$$

where the antisymmetric matrix $K(x_1,\ldots,x_{2p}) \in M_{2p}(\mathbb{C})$ is defined by

$$(K(x_1,\ldots,x_{2p}))_{jk} = \begin{cases} \frac{1}{x_j - x_k} & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

Proof. From Proposition 99, we have that $f_{\mathbb{H}}^{[e_1,\ldots,e_m]}\left(a_1^{o_1},\ldots,a_{2n}^{o_{2n}}\right)$ equals

$$\frac{1}{i^n}\left(\prod_{j=1}^{2n}\frac{1}{2\sqrt{o_j}}\right)\operatorname{Pfaff}\left(X_\Omega\left(e_1,\ldots,e_m,a_1^i,\ldots,a_{2n}^i\right)\right),$$

where we have replaced the double orientations $o_1,\ldots o_{2n}$. by the simple orientation $i \in \mathbb{S}$, which is allowed since X_Ω does not depend on their branch choices. Using Lemma 96, straightforward computations give that $X_{\mathbb{H}}\left(e_1,\ldots,e_m,a_1^i,\ldots,a_{2n}^i\right)$ is equal to $\tilde{K}\left(e_1,\ldots,e_m,\overline{e_m},\ldots,\overline{e_1} \,|\, a_1,\ldots,a_{2n}\right)$, where $\tilde{K}(x_1,\ldots,x_{2q} \,|\, y_1,\ldots,y_{2r}) \in M_{2q+2r}(\mathbb{C})$ is defined by

$$\left(\tilde{K}(x_1,\ldots,x_{2q} \,|\, y_1,\ldots,y_{2r})\right)_{jk} = \begin{cases} 0 & \text{if } j = k \\ \frac{1}{\pi(x_j - x_k)} & \text{if } 1 \leq j,k \leq 2q, j \neq k \\ \frac{i}{\pi(x_j - y_{k-2q})} & \text{if } 1 \leq j \leq 2q < k \leq 2q+2r \\ \frac{i}{\pi(y_{j-2q} - x_k)} & \text{if } 1 \leq k \leq 2q < j \leq 2q+2r \\ -\frac{1}{\pi(y_j - y_k)} & \text{if } 2q+1 \leq j,k \leq 2q+2r, j \neq k \end{cases}$$

It is easy to check that

$$D(2m,2n)\, X_H \left(e_1, \ldots, e_m, a_1^i, \ldots, a_{2n}^i\right) D(2m,2n) = K\left(e_1, \ldots, e_m, \overline{e_m}, \ldots \overline{e_1}, a_1, \ldots, a_{2n}\right),$$

where $D(2m,2n) \in M_{2m+2n}(\mathbb{C})$ is the diagonal matrix with the $2m$ first diagonal elements equal to 1 and the $2n$ last ones equal to $-i$. Using the formula $\mathrm{Pfaff}(Q^t R Q) = \det(Q)\,\mathrm{Pfaff}(R)$, the result follows. □

7.5. Proofs of the theorems of the introduction

Having established connections between the discrete correlation funnctions of the introduction and the fused fermionic observables in Chapter 5 and shown their convergence in Section 7.4, we can now prove the theorems of the introduction.

Proof of Theorem 1. We have that the correlation function $\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b$ is given in terms of the observables by:

$$\langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b = (-1)^n \cdot 2^n \cdot \frac{f_\Omega^{[a_1,\ldots,a_n]}(x_1, \ldots, x_{2k})}{f_\Omega(x_1, \ldots, x_{2k})}.$$

Indeed, by Proposition 72, if we denote by $x_{1,\delta}, \ldots, x_{2k,\delta} \in \partial_0 V_{\Omega_\delta^m}$ the medial vertices of $b_1, \ldots, b_{2p}$ we have

$$\mathbb{E}_{\Omega_\delta^*}^{(b_{1,\delta},\ldots,b_{2k,\delta})}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] = (-1)^n \cdot 2^n \cdot \frac{f_{\Omega_\delta}^{[a_{1,\delta},\ldots,a_{n,\delta}]}\left(x_{1,\delta}^{o_{1,\delta}}, \ldots, x_{2k,\delta}^{o_{2k,\delta}}\right)}{f_{\Omega_\delta}\left(x_{1,\delta}^{o_{1,\delta}}, \ldots, x_{2k,\delta}^{o_{2k,\delta}}\right)},$$

for any branch choice of orientations such that $o_{j,\delta}$ is pointing in the inward normal direction at $x_{j,\delta}$ (i.e. towards a vertex of $\partial_0 V_{\Omega_\delta}$) for each $j = 1, \ldots, 2k$. By Theorem 101, we have that

$$\frac{1}{\delta^n} \frac{f_{\Omega_\delta}^{[a_{1,\delta},\ldots,a_{n,\delta}]}\left(x_{1,\delta}^{o_{1,\delta}}, \ldots, x_{2k,\delta}^{o_{2k,\delta}}\right)}{f_{\Omega_\delta}\left(x_{1,\delta}^{o_{1,\delta}}, \ldots, x_{2k,\delta}^{o_{2k,\delta}}\right)} \underset{\delta \to 0}{\longrightarrow} \frac{f_\Omega^{[a_1,\ldots,a_n]}(x_1, \ldots, x_{2k})}{f_\Omega(x_1, \ldots, x_{2k})},$$

and hence

$$\mathbb{E}_{\Omega_\delta^*}^{(b_{1,\delta},\ldots,b_{2k,\delta})}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] \underset{\delta \to 0}{\longrightarrow} \langle \varepsilon(a_1) \cdot \ldots \cdot \varepsilon(a_n) \rangle_\Omega^b.$$

By Theorem 102, the right hand side satisfies the claimed conformal covariance properties. The explicit formula for the half-plane follows from Theorem 103. □

Proof of Corollary 2. By Proposition 64, we have that

$$\mathbb{E}_{\Omega_\delta}^{\mathrm{free}}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] = -\mathbb{E}_{\Omega_\delta^*}^{\varnothing}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right]$$

and hence the result follows from Theorem 1. □

Proof of Corollary 3. By $\sigma \mapsto -\sigma$ symmetry of the energy density ε_δ, we have that

$$\mathbb{E}_{\Omega_\delta^*}^{+}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] = \mathbb{E}_{\Omega_\delta^*}^{-}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right].$$

Hence we have

$$\mathbb{E}_{\Omega_\delta^*}^{+}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] = \mathbb{E}_{\Omega_\delta^*}^{\varnothing}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right]$$

and the result follows from Theorem 3. Similary, we have

$$\mathbb{E}_{\Omega_\delta}^{\left(b_1^+ b_2^- \ldots b_{2k-1}^+ b_{2k}^-\right)}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right] = \mathbb{E}_{\Omega_\delta}^{\left(b_1^- b_2^+ \ldots b_{2k-1}^- b_{2k}^+\right)}\left[\varepsilon_\delta(a_{1,\delta}) \cdot \ldots \cdot \varepsilon_\delta(a_{n,\delta})\right],$$

which gives

$$E_{\Omega_\delta^*}^{\left(b_1^+ b_2^- \dots b_{2k-1}^+ b_{2k}^-\right)}\left[\mu_\delta(a_{1,\delta})\cdot\ldots\cdot\mu_\delta(a_{n,\delta})\right] = E_{\Omega_\delta^*}^{\{b_1,b_2,\dots b_{2k-1},b_{2k}\}}\left[\mu_\delta(a_{1,\delta})\cdot\ldots\cdot\mu_\delta(a_{n,\delta})\right]$$

and shows the result. $\square$

Proof of Theorem 5. We have that the correlation function $\langle\sigma(v_1)\cdot\ldots\cdot\sigma(v_{2n})\rangle_\Omega^{\text{free}}$ is given by

$$\langle\sigma(v_1)\cdot\ldots\cdot\sigma(v_{2n})\rangle_\Omega^{\text{free}} = \frac{1}{\alpha^n} f_\Omega(w_1^{o_1},\ldots,w_{2n}^{o_{2n}}).$$

The conformal covariance of $\langle\sigma(v_1)\cdot\ldots\cdot\sigma(v_{2n})\rangle_\Omega^{\text{free}}$ follows from the one of f_Ω given by Theorem 102 and the explicit formula in the upper half-plane follows from the one given by Theorem 103.

By Proposition 73, if we denote by $w_{1,\delta},\ldots,w_{2n,\delta}\in\partial_0 V_{\Omega_\delta^m}$ the closest boundary medial vertices to $v_{1,\delta},\ldots,v_{2n,\delta}$, we have

$$E_{\Omega_\delta}^{\text{free}}\left[\sigma_\delta(v_{1,\delta})\cdot\ldots\cdot\sigma_\delta(v_{2n,\delta})\right] = \frac{1}{\alpha^n}\left|f_{\Omega_\delta}\left(w_{1,\delta}^{o_1},\ldots,w_{2n,\delta}^{o_{2n}}\right)\right|,$$

for any branch choices of inward-pointing doubly orientations $o_1,\ldots,o_{2n}$ at $w_{1,\delta}^{o_1},\ldots,w_{2n,\delta}^{o_{2n}}$. By Theorem 101, we have that

$$\frac{1}{\delta^n}\cdot f_{\Omega_\delta}\left(w_{1,\delta}^{o_1},\ldots,w_{2n,\delta}^{o_{2n}}\right)\underset{\delta\to 0}{\longrightarrow} f_\Omega(w_1^{o_1},\ldots,w_{2n}^{o_{2n}}).$$

Hence we obtain that

$$\frac{1}{\delta^n}\cdot E_{\Omega_\delta}^{\text{free}}\left[\sigma_\delta(v_{1,\delta})\cdot\ldots\cdot\sigma_\delta(v_{2n,\delta})\right]\underset{\delta\to 0}{\longrightarrow}\langle\sigma(v_1)\cdot\ldots\cdot\sigma(v_{2n})\rangle_\Omega^{\text{free}}.$$

$\square$

Appendix A: High-Temperature Expansion and Duality

In this appendix, we give the Proofs of Proposition 63, which is one of the standard tools of classical statistical mechanics, and of 64, which also relies on classical arguments – although we were not able to find this statement in the literature.

Proof of Proposition 63

This result, which dates back to Kramers and Wanniers can be found in [Pal07] for instance. We put it for self-containedness reasons mainly, and also because although standard in classical statistical mechanics, it appears very rarely in probability.

Proposition (Proposition 63). *Consider the Ising model on Ω_δ with free boundary condition at inverse temperature β and denote by $Z^{\text{free}}_{\Omega_\delta}$ its partition function. Then we have*

$$Z^{\text{free}}_{\Omega_\delta} = 2^{|V_{\Omega_\delta}|} (\cosh \beta)^{|E_{\Omega_\delta}|} \sum_{\omega \in C_{\Omega_\delta}} \alpha_h^{|\omega|},$$

where $\alpha_h = \tanh \beta$ and $|\omega|$ is the number of edges of ω. In particular, at the critical value $\beta_c = \frac{1}{2} \ln \left(\sqrt{2} + 1 \right)$ of β, we have $\alpha_h = \alpha = \sqrt{2} - 1$. More generally, for distinct vertices $v_1, \ldots, v_{2n} \in V_{\Omega_\delta}$, if we denote by $Z^{\text{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n})$ the partition function

$$Z^{\text{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n}) = \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \sigma_{v_1} \cdot \ldots \cdot \sigma_{v_{2n}} \cdot e^{-\beta H(\sigma)},$$

we have

$$Z^{\text{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n}) = 2^{|V_{\Omega_\delta}|} (\cosh \beta)^{|E_{\Omega_\delta}|} \sum_{\omega \in C_{\Omega_\delta}(v_1, \ldots, v_{2n})} \alpha_h^{|\omega|}.$$

In particular, we have

$$\mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_\delta(v_1) \cdot \ldots \cdot \sigma_\delta(v_{2n})] = \left(\sum_{\omega \in C_{\Omega_\delta}(v_1, \ldots, v_{2n})} \alpha_h^{|\omega|} \right) / \left(\sum_{\omega \in C_{\Omega_\delta}} \alpha_h^{|\omega|} \right).$$

In the critical case $\beta = \beta_c$, we can rewrite this latter ratio as

$$Z_{\Omega_\delta}(v_1, \ldots, v_{2n}) / Z_{\Omega_\delta},$$

where Z_{Ω_δ} is as defined in the previous paragraph.

Proof. For a configuration $\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}$, we have

$$\begin{aligned}
\exp(-\beta H(\sigma)) &= \prod_{\langle x,y\rangle \in E_{\Omega_\delta}} \exp(\beta \sigma_x \sigma_y) \\
&= \prod_{\langle x,y\rangle \in E_{\Omega_\delta}} (\cosh(\beta\sigma_x\sigma_y) + \sinh(\beta\sigma_x\sigma_y)) \\
&= \prod_{\langle x,y\rangle \in E_{\Omega_\delta}} (\cosh\beta + \sigma_x\sigma_y \sinh(\beta)) \\
&= (\cosh\beta)^{|E_{\Omega_\delta}|} \prod_{\langle x,y\rangle \in E_{\Omega_\delta}} (1 + \sigma_x\sigma_y \tanh(\beta)).
\end{aligned}$$

Let us write the expansion of $Z^{\mathrm{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n})$ (the case $n = 0$ corresponding to the first statement).

$$\begin{aligned}
& Z^{\mathrm{free}}_{\Omega_\delta}(v_1, \ldots, v_{2n}) \\
= & \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \cdot \exp(-\beta H(\sigma)) \\
= & (\cosh\beta)^{|E_{\Omega_\delta}|} \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \prod_{\langle x,y\rangle \in E_{\Omega_\delta}} (1 + \sigma_x\sigma_y \tanh(\beta)) \\
= & (\cosh\beta)^{|E_{\Omega_\delta}|} \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \sum_{\omega \subset E_{\Omega_\delta}} \prod_{\langle x,y\rangle \in \omega} \sigma_x\sigma_y \tanh(\beta) \\
= & (\cosh\beta)^{|E_{\Omega_\delta}|} \sum_{\omega \subset E_{\Omega_\delta}} \tanh(\beta)^{|E|} \sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \left(\prod_{\langle x,y\rangle \in \omega} \sigma_x\sigma_y \right) \right).
\end{aligned}$$

For each edge configuration $\omega \subset E_{\Omega_\delta}$, we have that if a spin σ_v with $v \in V_{\Omega_\delta}$ appears an odd number of times in the product $\left(\prod_{j=1}^{2n} \sigma_{v_j} \prod_{\langle x,y\rangle\in\omega} \sigma_x\sigma_y\right)$, the sum

$$\sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \prod_{\langle x,y\rangle \in \omega} \sigma_x\sigma_y$$

vanishes by symmetry. Hence the only edge configurations for which this sum does not vanish are those where each σ_v appears an even number of time in the above product. Equivalently, those are the edge configurations ω such that each vertex $v \in V_{\Omega_\delta} \setminus \{v_1, \ldots, v_{2n}\}$ belongs to an even number of edges of ω and such that each of the $v_1, \ldots, v_{2n}$ belongs to an odd number of edges of E. This set of edge configurations is exactly $C_{\Omega_\delta}(v_1, \ldots, v_{2n})$. For each $\omega \in C_{\Omega_\delta}(v_1, \ldots, v_{2n})$, we have that $\prod_{\langle x,y\rangle\in\omega} \sigma_x\sigma_y$ is equal to 1 for each spin configuration $\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}$ and hence that

$$\sum_{\sigma \in \{\pm 1\}^{V_{\Omega_\delta}}} \left(\prod_{j=1}^{2n} \sigma_{v_j} \right) \prod_{\langle x,y\rangle \in \omega} \sigma_x = 2^{|V_{\Omega_\delta}|}.$$

It follows that

$$Z^{\mathrm{free}}_{\Omega_\delta}(v_1,\ldots,v_{2n}) = 2^{|V_{\Omega_\delta}|}(\cosh\beta)^{|E_{\Omega_\delta}|}\sum_{\omega\subset C_{\Omega_\delta}(v_1,\ldots,v_{2n})}\alpha_h^{|\omega|},$$

which shows the result. The expression for $\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\sigma_\delta(v_1)\cdot\ldots\cdot\sigma_\delta(v_{2n})]$ folllows from the fact that it is equal to $Z^{\mathrm{free}}_{\Omega_\delta}(v_1,\ldots,v_{2n})/Z^{\mathrm{free}}_{\Omega_\delta}$ and from the fact that the prefactor terms simplify. ∎

Proof of Proposition 64

We now give the proof of the application of Kramers-Wannier duality to the energy density field.

Proposition (Proposition 64). *Consider both the critical Ising model on Ω^*_δ with locally monochromatic boundary condition and the critical Ising model on Ω_δ with free boundary condition. Let $e_1,\ldots,e_m\in E(\Omega_\delta)$ be distinct edges. Then we have*

$$\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}[\varepsilon_\delta(e_1)\cdot\ldots\cdot\varepsilon_\delta(e_m)] = (-1)^m\,\mathbb{E}^{\emptyset}_{\Omega^*_\delta}[\varepsilon_\delta(e^*_1)\cdot\ldots\cdot\varepsilon_\delta(e^*_m)],$$

where $\emptyset$ denotes the locally monochromatic boundary condition with no boundary changing operators.

In other words, the discrete energy field with free boundary condition is equal to minus the discrete energy field with locally monochromatic boundary condition.

Proof. Let us denote by B the set $\{0,1\}^m$, equipped with the lexicographical order. For each $b=(b_1,\ldots,b_n)\in B$, let us denote by s_b the vector of signs $(1-2b_1,\ldots,1-2b_m)\in\{\pm1\}^m$ and by p_b the subset $\{e_1,\ldots,e_m\}$ defined as $\{e_j\in\{e_1,\ldots,e_m\}: b_j=1\}$. We denote by $v_0=(1,0)$ and $v_1=(0,1)$ the canonical basis of $\mathbb{R}^2$ and we denote by $(v_b)_{b\in B}$ the canonical basis of $\left(\mathbb{R}^2\right)^{\otimes m}$, where, for each $b\in B$, v_b is defined by

$$v_{b_1}\otimes\ldots\otimes v_{b_m}\in\left(\mathbb{R}^2\right)^{\otimes m}.$$

Let us define the five vectors $x^1,\ldots,x^5\in\left(\mathbb{R}^2\right)^{\otimes m}$ by

$$\begin{aligned}
x^1 &= \sum_{b\in B}\mathbb{E}^{\emptyset}_{\Omega^*_\delta}\left[\prod_{e\in p_b}\varepsilon_\delta(e^*)\right]\cdot v_b,\\
x^2 &= \sum_{b\in B}\mathbb{E}^{\emptyset}_{\Omega^*_\delta}\left[\prod_{e\in p_b}(\varepsilon_\delta(e^*)+\mu)\right]\cdot v_b,\\
x^3 &= \sum_{b\in B}\frac{1}{Z_{\Omega_\delta}}\cdot Z_{\Omega_\delta}^{\left(e_1^{(s_b)_1},\ldots,e_m^{(s_b)_m}\right)}\cdot v_b,\\
x^4 &= \sum_{b\in B}\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}\left[\prod_{e\in p_b}(\varepsilon_\delta(e)+\mu)\right]\cdot v_b,\\
x^5 &= \sum_{b\in B}\mathbb{E}^{\mathrm{free}}_{\Omega_\delta}\left[\prod_{e\in p_b}\varepsilon_\delta(e)\right]\cdot v_b,
\end{aligned}$$

where $\mu = \frac{\sqrt{2}}{2}$ and the restricted partition functions $Z_{\Omega_\delta}^{\{\ldots\}}$ are as defined above. Let us now define the four matrices $A^{1,2}, A^{2,3}, A^{3,4}, A^{4,5} \in M_{2^m}(\mathbb{R})$ by

$$\begin{aligned} A^{1,2} &= \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}^{\otimes m}, \\ A^{2,3} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^{\otimes m}, \\ A^{3,4} &= \begin{pmatrix} 1 & 1 \\ \alpha & \alpha^{-1} \end{pmatrix}^{\otimes m}, \\ A^{4,5} &= \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}^{\otimes m}. \end{aligned}$$

It is elementary to check that the product $A^{4,5} \cdot A^{3,4} \cdot A^{2,3} \cdot A^{1,2}$ equals

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes m}.$$

It is hence sufficient to check that

$$\begin{aligned} A^{1,2}x^1 &= x^2, \\ A^{2,3}x^2 &= x^3, \\ A^{3,4}x^3 &= x^4, \\ A^{4,5}x^4 &= x^5, \end{aligned}$$

to obtain that

$$\begin{aligned} \mathbb{E}_{\Omega_\delta}^{\mathrm{free}}[\sigma_\delta(e_1) \cdot \ldots \cdot \sigma_\delta(e_m)] &= \left(x^1\right)_{(1,\ldots,1)} \\ &= (-1)^m \left(x^5\right)_{(1,\ldots,1)} \\ &= (-1)^m\, \mathbb{E}_{\Omega_\delta^*}^{\emptyset}[\sigma_\delta(e_1^*) \cdot \ldots \cdot \sigma_\delta(e_m^*)], \end{aligned}$$

which is the desired result.

The claims $A^{1,2}x^1 = x^2$ and $A^{4,5}x^4 = x^5$ can easily be proven by induction on m and by expanding the products in x^2 and x^5 (note that the matrices $A^{1,2}$ and $A^{4,5}$ are inverse of each other).

For the claim $A^{2,3}x^2 = x^3$, notice that x^2 can be rewritten as

$$\sum_{b \in B} \frac{(-1)^{\sum_{j=1}^m b_j}}{Z_{\Omega_\delta}} \cdot Z_{\Omega_\delta}^{\left\{e_1^{(s_b)_1}, \ldots, e_m^{(s_b)_m}\right\}} \cdot v_b,$$

as can be checked readily, since $\sigma_\delta(e^*) = 1$ on $Z_{\Omega_\delta}^{\{e^+\}}$ and $\sigma_\delta(e^*) = -1$ on $Z_{\Omega_\delta}^{\{e^-\}}$ for each edge $e \in E_{\Omega_\delta}$ and that hence

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes m} x^3 = x^2.$$

From there the claim follows, since $A^{2,3}$ is the inverse of that matrix.

For the claim $A^{3,4}x^3 = x^4$, notice that for each edge $e = \langle y, z\rangle \in E_{\Omega_\delta}$, we can add e to any configuration in $Z_{\Omega_\delta}^{\{e^+\}}$ and remove e to each configuration in $Z_{\Omega_\delta}^{\{e^-\}}$, hence obtaining a configuration in $C_{\Omega_\delta}(y,z)$, and that this realizes a bijection between C_{Ω_δ} and $C_{\Omega_\delta}(y,z)$. For $b \in B$, by Proposition 63, if we denote by $e_1, \dots, e_k$ the edges in p_b, we have

$$\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\prod_{j=1}^{k} \sigma_\delta(e_j)\right] = \frac{1}{Z_{\Omega_\delta}} \sum_{\omega \in C_{\Omega_\delta}(y_1,z_1,\dots,y_k,z_k)} \alpha^{|\omega|}$$

and from the above observation, it follows that

$$\begin{aligned}
Z_{\Omega_\delta}\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\prod_{j=1}^{k} \sigma_\delta(e_j)\right] &= \sum_{\omega \in C_{\Omega_\delta}(y_1,z_1,\dots,y_k,z_k)} \alpha^{|\omega|} \\
&= \sum_{(s_1,\dots,s_k)\in\{\pm 1\}^k} \alpha^{\sum_{j=1}^{k} s_j} \sum_{\tilde{\omega}\in C_{\Omega_\delta}^{e_1^{s_1},\dots,e_k^{s_k}}} \alpha^{|\tilde{\omega}|} \\
&= \sum_{(s_1,\dots,s_m)\in\{\pm 1\}^m} \alpha^{\sum_{j=1}^{k} s_j} \sum_{\omega^\dagger\in C_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}}} \alpha^{|\omega^\dagger|} \\
&= \sum_{(s_1,\dots,s_m)\in\{\pm 1\}^m} \alpha^{\sum_{j=1}^{k} s_j} \cdot Z_{\Omega_\delta}^{\{e_1^{s_1},\dots,e_m^{s_m}\}} \\
&= A^{3,4}x^3,
\end{aligned}$$

which shows the claim. □

Appendix B: Winding Phases

In this appendix, we give the proofs of Propositions 67 and 68.

Proof of Proposition 67.

Let us first recall the statement of Proposition 67 (Section 5.2), which guarantees the well-definedness of the observable.

Proposition (Proposition 67). *Let $a_1^{o_1}, \ldots, a_{2n}^{o_{2n}} \in D_{\Omega_\delta^m}$ be doubly-oriented medial vertices. Then for each configuration $\omega \in C_{\Omega_\delta}(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}})$, the winding phases $\varphi(\gamma_1, \ldots, \gamma_n, o_1, \ldots, o_{2n})$ and $\varphi(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, o_1, \ldots, o_{2n})$ of any two admissible choices of walks on ω are the same.*

We denote by $\varphi(\omega, o_1, \ldots, o_{2n})$ the winding phase of ω defined as the winding phase of its admissible choices of walks. The winding phase is antisymmetric with respect to the permutations of the indices $\{1, \ldots, 2n\}$.

Proof. We can assume that $a_1, \ldots, a_{2n}$ are midpoints of horizontal edges and that $o_1, \ldots, o_{2n}$ are equal to $(1)^2$: indeed, the oriented winding phase of an admissible walk is always an odd multiple of π, all the admissible choices of walks use the same half-edges, and the possible ambiguities depending on the choices of walks in a configurations $\omega \in C_{\Omega_\delta}$ are caused by the vertices in V_{Ω_δ} that belong to four edges or half-edges of ω.

To each configuration $C_{\Omega_\delta}\left(a_1^{(1)^2}, \ldots, a_{2n}^{(1)^2}\right)$, we add the $2n$ half-edges $h_1, \ldots, h_{2n} \in \overline{H}_{\Omega_\delta}$, with $h_j = \langle a_j, b_j \rangle$ for $j \in \{1, \ldots, 2n\}$, yielding a configuration in $C_{\Omega_\delta}(b_1, \ldots, b_{2n})$. Clearly, any admissible choice of walks can be extended by adding these $2n$ half-edges and we define naturally the winding phase of any choice of walks on $C_{\Omega_\delta}(b_1, \ldots, b_{2n})$ as the winding phase of the corresponding choice of walks on $C_{\Omega_\delta}\left(a_1^{(1)^2}, \ldots, a_{2n}^{(1)^2}\right)$ and denote it also by φ. We denote by $C_{\Omega_\delta}^{\rightarrow}(b_1, \ldots, b_{2n}) \subset C_{\Omega_\delta}(b_1, \ldots, b_{2n})$ the set of configurations $\omega \in C_{\Omega_\delta}(b_1, \ldots, b_{2n})$ such that we have $\langle b_k, b_k + \delta \rangle \in \omega$ for each $k \in \{1, \ldots, 2n\}$ (which is in bijection with $C_{\Omega_\delta}\left(a_1^{(1)^2}, \ldots, a_{2n}^{(1)^2}\right)$).

Let us first prove the $n = 2$ case:

Lemma 104. *Let $\omega \in C_{\Omega_\delta}(b_1, b_2)$ be a configuration with $b_1, b_2 \in V_{\Omega_\delta}$. Then for any two admissible extended walks $\gamma, \tilde{\gamma} : b_1 \to b_2$, we have $\varphi(\gamma) = \varphi(\tilde{\gamma})$.*

Proof of Lemma 104. The idea is to construct a bijection between edge configurations of $C_{\Omega_\delta}(b_1, b_2)$ and "twisted" spin configurations, such that for any admissible walk, its winding phase is represented as a local observable of this spin configuration. Let us denote by Θ the space

$$\left\{ \rho \in S^{\overline{V}_{\Omega_\delta^*}} : \rho_v \in \left\{ \pm \exp\left(-\frac{i}{2} (\arg(v - b_1) + \arg(v - b_2)) \right) \right\} \right\}$$

and by Θ_+ the subspace $\{\rho \in \Theta : \arg(\rho_{b_1}) \geq 0\}$ (where arg denotes the principal determination of the argument). To each configuration $\omega \in \mathcal{C}_{\Omega_\delta}(b_1, b_2)$, we associate the configuration $(\rho(\omega)_v)_{v \in \overline{V}_{\Omega_\delta^*}} \in \Theta_+$, such that

$$\Re\left(\rho(\omega)_v \overline{\rho(\omega)_w}\right) > 0 \iff \langle v, w\rangle^* \notin \omega \quad \forall \langle v, w\rangle \in \overline{E}_{\Omega_\delta^*}.$$

It is easy to see that this determines uniquely $\rho(\omega) \in \Theta_+$ and that such configuration always exists. Denote by $\gamma_1, \ldots, \gamma_L \in \vec{E}_{\Omega_\delta}$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{L}} \in \vec{E}_{\Omega_\delta}$ as the sequence of the edges of γ and $\tilde{\gamma}$ when going along them from b_1 to b_2, by $l_1, \ldots, l_L \in \overline{V}_{\Omega_\delta^*}$ and $\tilde{l}_1, \ldots, \tilde{l}_{\tilde{L}} \in \overline{V}_{\Omega_\delta^*}$ the sequence of the dual vertices on the left of $\gamma_1, \ldots, \gamma_L$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{L}}$ respectively, by $i_1, \ldots, i_L \in V_{\Omega_\delta}$ and $\tilde{i}_1, \ldots, \tilde{i}_{\tilde{L}} \in V_{\Omega_\delta}$ the initial vertices of $\gamma_1, \ldots, \gamma_L \in V_{\Omega_\delta^m}$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{L}} \in V_{\Omega_\delta^m}$, and finally by $m_1, \ldots, m_L \in V_{\Omega_\delta^m}$ and $\tilde{m}_1, \ldots, \tilde{m}_{\tilde{L}} \in V_{\Omega_\delta^m}$ the midpoints of the edges $\gamma_1, \ldots, \gamma_L$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{L}}$. As can easily be checked by induction, we have

$$\Re\left(\rho(\omega)_{l_{j-1}} \rho(\omega)_{l_j}\right) > 0$$

$$\Re\left(\rho(\omega)_{\tilde{l}_{\tilde{j}-1}} \rho(\omega)_{\tilde{l}_{\tilde{j}}}\right) > 0$$

for each $j \in \{2, \ldots, L\}$ and each $\tilde{j} \in \{2, \ldots, \tilde{L}\}$. From there, we can show by induction that for each $j \in \{1, \ldots, L\}$ and $\tilde{j} \in \{1, \ldots, \tilde{L}\}$, we have

$$\rho(\omega)_{l_j} = \rho(\omega)_{l_1} \frac{\exp\left(-\frac{i}{2}\int_{(\gamma[0,j-1])\cup[i_j,m_j]\cup[m_j,l_j]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)},$$

$$\rho(\omega)_{\tilde{l}_{\tilde{j}}} = \rho(\omega)_{\tilde{l}_1} \frac{\exp\left(-\frac{i}{2}\int_{(\tilde{\gamma}[0,\tilde{j}-1])\cup[i_{\tilde{j}},\tilde{m}_{\tilde{j}}]\cup[\tilde{m}_j,\tilde{l}_j]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)},$$

where the integration variable is ζ and where $\gamma[0, j-1]$ and $\tilde{\gamma}\left[0, \tilde{j}-1\right]$ denote the j and $\tilde{j}$ first edges of γ and $\tilde{\gamma}$ (viewed as paths from b_1 to i_j and from b_1 to $\tilde{i}_{\tilde{j}}$) and where for $x, y \in \mathbb{C}$, $[x, y] \subset \mathbb{C}$ denotes the straight segment between them. We have used that $\tilde{i}_1, \tilde{m}_1, \tilde{l}_1 = i_1, m_1, l_1$ In particular, when evaluating at $j = L$ and $\tilde{j} = \tilde{L}$, we obtain

$$\rho(\omega)_{b_2} = \rho(\omega)_{l_1} \frac{\exp\left(-\frac{i}{2}\int_{\gamma\cup[i_L,m_L]\cup[m_L,l_L]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)},$$

$$\rho(\omega)_{b_1} = \rho(\omega)_{\tilde{l}_1} \frac{\exp\left(-\frac{i}{2}\int_{\tilde{\gamma}\cup[i_L,m_L]\cup[m_L,l_L]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]} (d\arg(\zeta - b_1) + d\arg(\zeta - b_2))\right)},$$

where we have used that $\tilde{i}_{\tilde{L}}, \tilde{m}_{\tilde{L}}, \tilde{l}_{\tilde{L}} = i_L, m_L, l_L$ Now, let us notice any non-self crossing path $\gamma^* : x_1 \rightsquigarrow x_2$ with $x_1, x_2 \in V_{\Omega_\delta}$ and $\gamma^* \subset E_{\Omega_\delta}$, we have

$$\int_{\gamma^*} (d\arg(\zeta - x_1) + d\arg(\zeta - x_2)) = w(\gamma),$$

as can be shown by induction on the length of γ^* (or using argument's principle). Hence we finally obtain

$$\exp\left(-\frac{i}{2}w(\gamma)\right) = \frac{\rho(\omega)_{l_1}}{\rho(\omega)_{b_2}} \frac{\exp\left(-\frac{i}{2}\int_{[i_L,m_L]\cup[m_L,l_L]}(d\arg(\zeta-b_1)+d\arg(\zeta-b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]}(d\arg(\zeta-b_1)+d\arg(\zeta-b_2))\right)},$$

$$\exp\left(-\frac{i}{2}w(\tilde{\gamma})\right) = \frac{\rho(\omega)_{\tilde{l}_1}}{\rho(\omega)_{b_1}} \frac{\exp\left(-\frac{i}{2}\int_{[i_L,m_L]\cup[m_L,l_L]}(d\arg(\zeta-b_1)+d\arg(\zeta-b_2))\right)}{\exp\left(-\frac{i}{2}\int_{[i_1,m_1]\cup[m_1,l_1]}(d\arg(\zeta-b_1)+d\arg(\zeta-b_2))\right)},$$

which is the desired result, since

$$\varphi(\gamma) = \exp\left(-\frac{i}{2}w(\gamma)\right),$$
$$\varphi(\tilde{\gamma}) = \exp\left(-\frac{i}{2}w(\tilde{\gamma})\right).$$

□

Lemma 105. *Let $\omega \in C_{\Omega_\delta}(b_1,\ldots,b_{2n})$ be a configuration with $b_1,\ldots,b_{2n} \in V_{\Omega_\delta}$. Then for any admissible choice of extended walks $\gamma_1,\ldots,\gamma_n$, the phase*

$$\varphi(\gamma_1,\ldots,\gamma_n)$$

is antisymmetric with respect to indices permutations of $b_1,\ldots,b_{2n}$.

Proof of Lemma 105. It is sufficient to check this for the transpositions $k \leftrightarrow k+1$ for each $k \in \{1,3,\ldots,2n\}$ (with the indices taked modulo $2n$). If b_k and b_{k+1} are linked by one of the walks γ_j, then the orientation of γ_j is reversed under the change of indices and since $w(\gamma_j)$ is an odd multiple of π, the winding phase is multiplied by -1. If this b_k and b_{k+1} are not linked, then the walks $\gamma_1,\ldots,\gamma_n$ are unchanged, but the parity of the crossing number $c(\gamma_1,\ldots,\gamma_n)$ is changed, so the winding phase is multiplied by -1. □

Let $\omega \in C_{\overrightarrow{\Omega_\delta}}(b_1,\ldots,b_{2n})$ be a configuration. It is easy to see that any choice of admissible walks is determined by prescribing a "turn left" or "turn right" $q_v \in \{l,r\}$ rule at each of the vertices $v \in V_{\Omega_\delta}$: fix a set of such rules $(q_v)_v$, and start from b_1 and follow the edges and apply the rule whenever we arrive at a vertex where there is an ambiguity (that is, a vertex that belongs to four edges or half-edges of ω), until the walk arrives to an other medial vertex in $\{b_1,\ldots,b_{2n}\}$, then take the unvisited medial vertex in $\{b_1,\ldots,b_{2n}\}$ with the smallest index and follow the rules, and so on. Let $(q_v)_v$ be a set of rules $\gamma_1,\ldots,\gamma_n$ be an admissible choice of walks following $(q_v)_v$. Let $\tilde{v} \in V_{\Omega_\delta}$ be an arbitrary vertex and denote by $(\tilde{q}_v)_v$ the set of rules $(q_v)_v$ modified at $\tilde{v}$ and let be $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n$ be the admissible choice of walks determined by $(\tilde{q}_v)$. We have to show that

$$\varphi(\gamma_1,\ldots,\gamma_n) = \varphi(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n).$$

If the connection diagram induced by $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n$ is the same as the one by $\gamma_1,\ldots,\gamma_n$, then the result follows from the $n = 2$ case: we have that $\gamma_1 : b_{i_1} \to b_{t_1}$ and $\tilde{\gamma}_1 : b_{i_1} \to b_{t_1}$ are two admissible choices of walks on $\omega \oplus \gamma_2 \oplus \ldots \oplus \gamma_n \in C_{\overrightarrow{\Omega_\delta}}(b_{i_1}, b_{t_1})$ and hence

$$\exp\left(-\frac{i}{2}\cdot w(\gamma_1)\right) = \exp\left(-\frac{i}{2}\cdot w(\tilde{\gamma}_1)\right).$$

So suppose it is not the case. Then we can suppose (by changing the indices of the walks) that we have $\gamma_1 \cup \gamma_2 = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$ and that $\gamma_1 : b_{i_1} \rightsquigarrow b_{t_1}, \gamma_2 : b_{i_2} \rightsquigarrow b_{t_2}$ and $\tilde{\gamma}_1 : b_{\tilde{i}_1} \rightsquigarrow b_{\tilde{t}_1}, \tilde{\gamma}_2 : b_{\tilde{i}_2} \rightsquigarrow b_{\tilde{t}_2}$ where $(i_1, t_1, i_2, t_2) \neq (\tilde{i}_1, \tilde{t}_1, \tilde{i}_2, \tilde{t}_2)$. Since $\varphi(\gamma_1, \ldots, \gamma_n)$ and $\varphi(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)$ are both antisymmetric with respect to permutation of the indices of $b_1, \ldots, b_{2n}$ we can suppose that $i_1 = 1$, $t_1 = 2$, $i_2 = 3$, $t_2 = 4$. Then we have to show that

$$w(\gamma_1) + w(\gamma_2) = w(\tilde{\gamma}_1) + w(\tilde{\gamma}_2) \pmod{4\pi}$$

if the pair partition of $\{1, \ldots, 4\}$ induced by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ is planar and

$$w(\gamma_1) + w(\gamma_2) = w(\tilde{\gamma}_1) + w(\tilde{\gamma}_2) + 2\pi \pmod{4\pi}$$

otherwise: in that case we have $\tilde{\gamma}_1 : b_1 \rightsquigarrow b_3$ and $\tilde{\gamma}_2 : b_2 \rightsquigarrow b_4$. By exchanging if necessary the indices of b_1 and b_2, we can assume that we are in the first case, since the winding numbers of the admissible walks on ω are odd multiple of π.

So, in that case we have $\tilde{\gamma}_1 : b_1 \rightsquigarrow b_4$ and $\tilde{\gamma}_2 : b_2 \rightsquigarrow b_3$, where $\tilde{\gamma}_1$ is obtained by following γ_1 from b_1 to $\tilde{v}$ and then following γ_2 from $\tilde{v}$ to b_4 and $\tilde{\gamma}_2$ is obtained by following γ_1 backwards from b_2 to $\tilde{v}$ and then following γ_2 backwards from $\tilde{v}$ to b_3. We hence have

$$w(\tilde{\gamma}_1) - w(\tilde{\gamma}_2) = w(\gamma_1) + w(\gamma_2) + 2\pi \pmod{4\pi},$$

the 2π difference coming from the fact that γ_1, γ_2 both turn left (thus contributing for π to $w(\gamma_1) + w(\gamma_2)$) or right (thus contributing for $-\pi$ to $w(\gamma_1) + w(\gamma_2)$) at $\tilde{v}$ and that in these respective cases, $\tilde{\gamma}_1$ turns right and $\tilde{\gamma}_2$ left $\tilde{v}$ (thus contributing for $-\pi$ to $w(\tilde{\gamma}_1) - w(\tilde{\gamma}_2)$) or the converse (thus contributing for π to $w(\tilde{\gamma}_1) - w(\tilde{\gamma}_2)$). Using that the winding numbers of the admissible walks on ω are odd multiple of π, we finally obtain the result. □

Proof of Proposition 68

Let us now give the proof of Proposition 68, which allows to show that the winding phase of a configuration with boundary points is independent of the configuration, and which is used later to factor the winding phase out of the observables and to use them to derive probabilistic observations about the model. Let us first recall its statement:

Proposition (Proposition 68). *Let $a_1, \ldots, a_{2n} \in \partial_0 V_{\Omega_\delta^m}$ be boundary medial vertices such that each connected component of $\partial\Omega_\delta$ contains an even number of a_j's let and $o_1, \ldots, o_{2n} \in (S)^2_\square$ be inward-pointing double orientations at $a_1, \ldots, a_{2n}$. Then for each $\omega \in C_{\Omega_\delta}(a_1^{o_1}, \ldots, a_{2n}^{o_{2n}})$, the winding phase $\varphi(\omega)$ is the same.*

Proof. It is easy to see that we can suppose that $a_1, \ldots, a_{2n}$ are horizontal boundary medial vertices and that $o_1, \ldots, o_{2n} = (1)^2$. Then, the proposition follows from the following lemma, which defines the continuous version of the winding phase. The small difference that the discrete domains are only piecewise smooth and the paths not self-avoiding but only non-self-crossing plays no role. □

Lemma 106. *For any piecewise smooth path $\gamma : [0, 1] \to \mathbb{C}$, let us define its winding number $w(\gamma)$ as*

$$\int_0^1 d \arg(\dot{\gamma}(t)),$$

where $\dot{\gamma}$ denotes the time derivative of γ and the integration variable is t.

Let $\Omega \subset \mathbb{C}$ be a smooth domain and let $a_1, \ldots, a_{2n} \in \partial\Omega$ be boundary points such that on each connected component of $\partial\Omega$ there are an even number of a_j's and such that $\partial\Omega$ is vertical near each of the points $a_1, \ldots, a_{2n}$ and such that Ω is on the right side of $\partial\Omega$ near those points. Let us call *admissible collection of paths* a collection $\gamma_1, \ldots, \gamma_n : [0,1] \to \overline{\Omega}$ of simple, mutually avoiding paths linking pairwise the points $a_1, \ldots, a_{2n}$ with $\gamma_j\{0\} = a_{\iota_j}$ and $\gamma_j\{1\} = a_{\tau_j}$ such that $\iota_j < \tau_j$ for each $j \in \{1, \ldots, n\}$, with $\dot\gamma_1(0), \ldots, \dot\gamma_n(0) > 0$ and $\dot\gamma_1(1), \ldots, \dot\gamma_n(1) < 0$.

For each admissible collection $\gamma_1, \ldots, \gamma_n$, let us define $\varphi(\gamma_1, \ldots, \gamma_n)$ by

$$\varphi(\gamma_1, \ldots, \gamma_n) = (-1)^{c(\gamma_1, \ldots, \gamma_n)} \cdot i^n \cdot \exp\left(-\frac{i}{2}\sum_{j=1}^n w(\gamma_j)\right),$$

where $(-1)^{c(\gamma_1, \ldots, \gamma_n)}$ is the crossing signature of the pair partition of $\{1, \ldots, 2n\}$ induced by the paths $\gamma_1, \ldots, \gamma_n$.

Then for any two admissible collections of paths $\gamma_1, \ldots, \gamma_n$ and $\tilde\gamma_1, \ldots \tilde\gamma_n$, we have that

$$\varphi(\gamma_1, \ldots, \gamma_n) = \varphi(\tilde\gamma_1, \ldots, \tilde\gamma_n).$$

Proof of Lemma 106. Let us show the claim by induction on the number of points.

For $n = 1$, it is easy to see by planarity that we have

$$\exp\left(-\frac{i}{2}w(\gamma_1)\right) = \exp\left(-\frac{i}{2}(w(\widehat{a_1 a_2}) + \pi)\right),$$

where $\widehat{a_1 a_2}$ is the counterclockwise arc of $\partial\Omega$ between a_1 and a_2.

Let us show the claim for $n \geq 2$, supposing it true for $n-1$. Since $\varphi(\gamma_1, \ldots, \gamma_n)$ and $\varphi(\tilde\gamma_1, \ldots, \tilde\gamma_n)$ are both antisymmetric with respect to the permutations of the indices $\{1, \ldots, 2n\}$, by exactly the same arguments as the ones used in Lemma 105, we can assume that a_{2n-1} and a_{2n} are on the same connected component of $\partial\Omega$ and that the counterclockwise arc between a_{2n-1} and a_{2n} does not contain any of the points $a_1, \ldots, a_{2n-2}$. By reordering the paths $\gamma_1, \ldots, \gamma_n$ and $\tilde\gamma_1, \ldots, \tilde\gamma_n$, we can assume that a_{2n} is the endpoint of both γ_n and $\tilde\gamma_n$. From $\gamma_1, \ldots, \gamma_n$, let us define an admissible collection of paths $\gamma_1^\dagger, \ldots, \gamma_{n-1}^\dagger$ linking pairwise $a_1, \ldots, a_{2n-2}$ in the following way:

- If γ_n links a_{2n-1} and a_{2n}, we define $\gamma_1^\dagger, \ldots, \gamma_{n-1}^\dagger$ as $\gamma_1, \ldots, \gamma_{n-1}$. By planarity, we have that

  $$\exp\left(-\frac{i}{2}w(\gamma_n)\right) = \exp\left(-\frac{i}{2}\left(w\left(\widehat{a_{2n-1}a_{2n}}\right) + \pi\right)\right)$$

  Taking into account the crossing signature, we obtain

  $$\varphi\left(\gamma_1^\dagger, \ldots, \gamma_{n-1}^\dagger\right) = \exp\left(\frac{i}{2}w\left(\widehat{a_{2n-1}a_{2n}}\right)\right)\varphi(\gamma_1, \ldots, \gamma_n).$$

- If γ_n links a_{ι_n} to a_{2n} and γ_k links a_{ι_k} to a_{2n-1}, and $\iota_k < \iota_n$, we define $\gamma_1^\dagger, \ldots, \gamma_{n-1}^\dagger$ as $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_{n-1}, \gamma^*$, where γ^* is obtained by following first γ_k from a_{ι_k} to a_{2n-1}, making a $\pi/2$ turn, following the counterclockwise arc $\widehat{a_{2n-1}a_{2n}}$ of $\partial\Omega$ between a_{2n-1} and a_{2n}, making another

$\pi/2$ turn and finally following γ_n backwards from a_{2n} to a_{l_n}. Modulo 4π, we have that

$$\begin{aligned} w(\gamma^*) &= w(\gamma_k) + \pi + w\left(a_{2n-1}a_{2n}\right) - w(\gamma_n) \\ &= w(\gamma_k) + w(\gamma_n) + w\left(a_{2n-1}a_{2n}\right) + 3\pi, \end{aligned}$$

since $w(\gamma_n)$ is an odd multiple of π. Since we have

$$(-1)^{c\left(\gamma_1^\dagger,\ldots,\gamma_{n-1}^\dagger\right)} = -(-1)^{c(\gamma_1,\ldots,\gamma_n)},$$

we obtain

$$\varphi\left(\gamma_1^\dagger,\ldots,\gamma_{n-1}^\dagger\right) = \exp\left(\frac{i}{2}w\left(a_{2n-1}a_{2n}\right)\right)\varphi(\gamma_1,\ldots,\gamma_n).$$

- If γ_n links a_{l_n} to a_{2n} and γ_k links a_{l_k} to a_{2n-1} with $l_n < l_k$, we define $\gamma_1^\dagger,\ldots,\gamma_{n-1}^\dagger$ as $\gamma_1,\ldots,\gamma_{k-1},\gamma_{k+1},\ldots,\gamma_{n-1},\gamma^*$, where γ^* is obtained by following first γ_n from a_{l_n} to a_{2n}, making $-\pi/2$ turn, following the clockwise arc of $\partial\Omega$ between a_{2n} and a_{2n-1}, making another $-\pi/2$ turn and following γ_k backwards from a_{2n-1} to a_{l_k}. Modulo 4π, we have that

$$\begin{aligned} w(\gamma^*) &= w(\gamma_n) - \pi - w\left(a_{2n-1}a_{2n}\right) - w(\gamma_k) \\ &= w(\gamma_n) + \pi + w\left(a_{2n-1}a_{2n}\right) + w(\gamma_k), \end{aligned}$$

since $w\left(a_{2n-1}a_{2n}\right)$ is a multiple of 2π and $w(\gamma_k)$ an odd multiple of π. Taking into account the crossing signature (which is unchanged), we obtain

$$\varphi\left(\gamma_1^\dagger,\ldots,\gamma_{n-1}^\dagger\right) = \exp\left(\frac{i}{2}w\left(a_{2n-1}a_{2n}\right)\right)\varphi(\gamma_1,\ldots,\gamma_n).$$

We proceed similary to define admissible paths $\tilde{\gamma}_1^\dagger,\ldots,\tilde{\gamma}_{n-1}^\dagger$ linking $a_1,\ldots,a_{2n-2}$ from $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n$ such that we have

$$\varphi\left(\tilde{\gamma}_1^\dagger,\ldots,\tilde{\gamma}_{n-1}^\dagger\right) = \exp\left(\frac{i}{2}w\left(a_{2n-1}a_{2n}\right)\right)\varphi(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n).$$

By the above considerations and the induction hypothesis, we obtain

$$\begin{aligned} \varphi(\gamma_1,\ldots,\gamma_n) &= \exp\left(-\frac{i}{2}w\left(a_{2n-1}a_{2n}\right)\right)\varphi\left(\gamma_1^\dagger,\ldots,\gamma_{n-1}^\dagger\right) \\ &= \exp\left(-\frac{i}{2}w\left(a_{2n-1}a_{2n}\right)\right)\varphi\left(\tilde{\gamma}_1^\dagger,\ldots,\tilde{\gamma}_{n-1}^\dagger\right) \\ &= \varphi(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n), \end{aligned}$$

which is the desired result. □

Appendix C: Pfaffian for Spin Correlation

As a short by-product of our analysis of the discrete observables, let us give the following discrete proposition, that can be of independent interest. Our proof use discrete complex analysis and hence only works in the critical temperature, but it seems that they can be generalized without too much trouble at other temperatures, using massive fermionic observables.

Proposition 107. *Consider the critical Ising model on a simply connected discrete domain Ω_δ with free boundary condition and let $v_1, \ldots, v_{2n} \in \partial_0 V_{\Omega_\delta}$ be boundary vertices enumerated in counterclockwise order. Then we have that*

$$\mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_{v_1} \cdot \ldots \cdot \sigma_{v_{2n}}] = |\text{Pfaff}(Z_{\Omega_\delta})|,$$

where $Z_{\Omega_\delta} \in M_{2n}(\mathbb{R})$ is given by

$$(Z_{\Omega_\delta})_{jk} = \begin{cases} 0 & \text{if } j = k, \\ \mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_{v_j}\sigma_{v_k}\right] & \text{if } j < k, \\ -\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_{v_j}\sigma_{v_k}\right] & \text{if } j > k. \end{cases}$$

Proof. By Proposition 73, we have, if we denote by $w_1, \ldots, w_{2n} \in \partial_0 V_{\Omega^m_\delta}$ the closest boundary medial vertices to $v_1, \ldots, v_{2n}$, for any choice of inward-pointing double orientations $o_1, \ldots, o_{2n}$ at $v_1, \ldots, v_{2n}$

$$\mathbb{E}^{\text{free}}_{\Omega_\delta}[\sigma_{v_1} \cdot \ldots \cdot \sigma_{v_{2n}}] = \frac{1}{\alpha^n}\left|f_{\Omega_\delta}(w_1^{o_1}, \ldots, w_{2n}^{o_{2n}})\right|,$$

where $\alpha = \sqrt{2} - 1$ as usual.

Let us choose the branch chocies of the double orientations $o_1, \ldots, o_{2n}$ in such a way that $f_{\Omega_\delta}\left(w_j^{o_j}, w_{j+1}^{o_{j+1}}\right) > 0$ for each $j \in \{1, \ldots, 2n-1\}$, which is possible, since by Proposition 68, the winding phase of all configurations in $\mathcal{C}_{\Omega_\delta}\left(w_j^{o_j}, w_{j+1}^{o_{j+1}}\right)$ is the same.

It is easy to see that, since the points are in counterclockwise order, this implies that $f_{\Omega_\delta}\left(w_j^{o_j}, w_k^{o_k}\right) > 0$ for each $j, k \in \{1, \ldots, 2n\}$ with $j < k$: we can construct a configuration $\omega \in \mathcal{C}_{\Omega_\delta}\left(w_j^{o_j}, w_k^{o_k}\right)$ by $\omega_j \oplus \ldots \oplus \omega_{k-1}$, where $\omega_l \in \mathcal{C}_{\Omega_\delta}\left(w_l^{o_l}, w_{l+1}^{o_{l+1}}\right)$ for each $l \in \{j, \ldots, k-1\}$, and it is easy to see that $\varphi(\omega, o_j, o_k)$ is positive, and again by Proposition 68, the winding phase $\varphi(\cdot, o_j, o_k)$ of all configurations in $\mathcal{C}_{\Omega_\delta}\left(w_j^{o_j}, w_k^{o_k}\right)$ is also positive. By Proposition 73, we obtain that $f_{\Omega_\delta}\left(w_j^{o_j}, w_k^{o_k}\right) = \frac{1}{\alpha} \cdot \mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_{v_j}\sigma_{v_k}\right]$ for each $j < k$. By Proposition 84, we have that

$$f_{\Omega_\delta}(w_1^{o_1}, \ldots, w_{2n}^{o_{2n}}) = \text{Pfaff}\left(A_{\Omega_\delta}(w_1^{o_1}, \ldots, w_{2n}^{o_{2n}})\right),$$

where

$$(A_{\Omega_\delta}(w_1^{o_1}, \ldots, w_{2n}^{o_{2n}}))_{jk} = \begin{cases} f\left(w_j^{o_j}, w_k^{o_k}\right) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

Hence, the result follows readily. □

Bibliography

[Bax89] R. Baxter, Exactly solved models in statistical mechanics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989.

[BoDT09] C. Boutillier, B. de Tilière, The critical Z-invariant Ising model via dimers: locality property. arXiv:0902.1882v1.

[BoDT08] C. Boutillier, B. de Tilière, The critical Z-invariant Ising model via dimers: the periodic case. arXiv:0812.3848v1.

[BuGu93] T. Burkhardt, I. Guim, Conformal theory of the two-dimensional Ising model with homogeneous boundary conditions and with disordered boundary fields, Phys. Rev. B (1), 47:14306-14311, 1993.

[Car84] J. Cardy, Conformal invariance and Surface Critical Behavior, Nucl. Phys. B 240:514-532, 1984.

[ChSm08] D. Chelkak, S. Smirnov, Discrete complex analysis on isoradial graphs. Advances in Mathematics, to appear. arXiv:0810.2188v1.

[ChSm09] D. Chelkak, S. Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables. arXiv:0910.2045v1, 2009.

[CFL28] R. Courant, K. Friedrichs, H. Lewy, Uber die partiellen Differenzengleichungen der mathematischen Physik. Math. Ann., 100:32-74, 1928.

[DMS97] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Graduate texts in contemporary physics. Springer-Verlag New York, 1997.

[DHN09] H. Duminil-Copin, C. Hongler, P. Nolin, Connection probabilities and RSW-type bounds for the FK Ising model, arXiv:0912.4253, 2009.

[Gri06] G. Grimmett, The Random-Cluster Model. Volume 333 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2006.

[HoKy10] C. Hongler, K. Kytölä, Dipolar SLE in Ising model with plus-minus-free boundary conditions, preprint.

[HoSm10] C. Hongler, S. Smirnov, The energy density in the critical planar Ising model, preprint.

[Isi25] E. Ising, Beitrag zur Theorie des Ferromagnetismus. Zeitschrift für Physik, 31:253-258, 1925.

[KaCe71] L. Kadanoff, H. Ceva, Determination of an operator algebra for the two-dimensional Ising model. Phys. Rev. B (3), 3:3918-3939, 1971.

[Kau49] B. Kaufman, Crystal statistics. II. Partition function evaluated by spinor analysis. Phys. Rev., II. Ser., 76:1232-1243, 1949.

[KaOn49] B. Kaufman, L. Onsager, Crystal statistics. III. Short-range order in a binary Ising lattice. Phys. Rev., II. Ser., 76:1244-1252, 1949.

[KaOn50] B. Kaufman, L. Onsager, Crystal statistics. IV. Long-range order in a binary crystal. Unpublished typescript, 1950.

[Ken00] R. Kenyon, Conformal invariance of domino tiling. Ann. Probab., 28:759-795, 2000.

[Kes87] H. Kesten, Hitting probabilities of random walks on Z^d, Stochastic Processes Appl. 25:165-184, 1987.

[KrWa41] H. A. Kramers and G. H. Wannnier. Statistics of the two-dimensional ferromagnet. I. Phys. Rev. (2), 60:252-262, 1941.

[Lel55] J. Lelong-Ferrand. Représentation conforme et transformations à intégrale de Dirichlet bornée. Gauthier-Villars, Paris, 1955.

[Len20] W. Lenz, Beitrag zum Verständnis der magnetischen Eigenschaften in festen Körpern. Phys. Zeitschr., 21:613-615, 1920.

[Mer01] C. Mercat, Discrete Riemann surfaces and the Ising model. Comm. Math. Phys. 218:177-216, 2001.

[McWu73] B. M. McCoy, and T. T. Wu, The two-dimensional Ising model. Harvard University Press, Cambridge, Massachusetts, 1973.

[Ons44] L. Onsager, Crystal statistics. I. A two-dimensional model with an order-disorder transition, Phys Rev (2), 65:117-149, 1944.

[Pal07] J. Palmer, Planar Ising correlations. Birkhäuser, 2007.

[Smi06] S. Smirnov, Towards conformal invariance of 2D lattice models. Sanz-Solé, Marta (ed.) et al., Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures, 1421-1451. Zürich: European Mathematical Society (EMS), 2006.

[Smi07] S. Smirnov, Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. Annals of Math., to appear, 2007.

[Yan52] C. N. Yang, The spontaneous magnetization of a two-dimensional Ising model. Phys. Rev. (2), 85:808-816, 1952.