T0001 – MilliNewton force sensor

Frequency bandwidth

This document gives an approximate calculation of the frequency bandwidth of the MilliNewton – and serves as a starting point for calculations on other cantilever-type sensors. The mechanical resonance frequencies of MilliNewton are high, due to the very small cantilever size. The response is in fact limited by the electrical circuit at ca. 1 kHz (MilliNewton-A) or ca. 200 Hz (MilliNewton-B).

Thomas Maeder, 14.7.2014

Table of contents

1. MECHANICAL RESONANCE ...............................................................................................................2
   1.1. SIMPLE CANTILEVER BEAM – ANALYTICAL SOLUTION .........................................................2
   1.2. SIMPLE CANTILEVER – MASS-SPRING SOLUTION ..................................................................3
   1.3. CANTILEVER – ADDITION OF A MASS ON THE BEAM ..........................................................4
   1.4. APPLICATION TO MILLINEWTON .............................................................................................4
2. ELECTRICAL BANDWIDTH .............................................................................................................5
   2.1. MEASUREMENT BRIDGE & AMPLIFIER FEEDBACK ................................................................5
   2.2. AMPLIFIER – GAIN BANDWIDTH PRODUCT ...........................................................................5
   2.3. AMPLIFIER – SLEW RATE .......................................................................................................5
3. CONCLUSION ....................................................................................................................................6

---

1 Replaces version 2004-04-19.
1. Mechanical resonance

1.1. Simple cantilever beam – analytical solution

According to Whitney\textsuperscript{1} et Strässler\textsuperscript{2}, the resonance frequencies of a constant cross section cantilever are given by:

\begin{equation}
 f_i = \frac{1}{2\pi} \sqrt{\frac{E^* \cdot I}{\rho \cdot S} \cdot (\frac{z_i}{L})^2} 
\end{equation}

- $f_i$ frequency of the $i^{th}$ mode
- $E^*$ effective elastic modulus
- $I$ flexure inertial moment of the cantilever
- $\rho$ cantilever specific mass
- $S$ cross section area
- $z_i$ constant for the $i^{th}$ mode
- $L$ cantilever free length

The approximate values of $z_i$ are given in table 1 below.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$z_i$</th>
<th>$i$</th>
<th>$z_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.8751</td>
<td>4</td>
<td>10.9955</td>
</tr>
<tr>
<td>2</td>
<td>4.6941</td>
<td>5</td>
<td>14.1372</td>
</tr>
<tr>
<td>3</td>
<td>7.8546</td>
<td>6</td>
<td>17.2786</td>
</tr>
</tbody>
</table>

Table 1. Constants $z_i$ of the cantilever resonance modes.

For a beam of rectangular cross section $b \times h$, with $b \gg h$, we have:

\begin{equation}
 E^* = \frac{E}{1 - \nu^2} 
\end{equation}

\begin{equation}
 I = \frac{b \cdot h^3}{12} 
\end{equation}

\begin{equation}
 S = b \cdot h 
\end{equation}

Combining (1)–(4), we obtain finally:

\begin{equation}
 f_i = \frac{1}{4 \cdot \sqrt{3} \cdot \pi} \cdot \frac{E}{\sqrt{(1-\nu^2) \cdot \rho \cdot L}} \cdot h \cdot (\frac{z_i}{L})^2 
\end{equation}

In general, the first mode is taken in consideration only, because it limits the working frequency of the sensor in practice. In this case:

\begin{equation}
 f_i \approx 0.16154 \cdot \frac{E}{\sqrt{(1-\nu^2) \cdot \rho \cdot L}} \cdot h 
\end{equation}
1.2. **Simple cantilever – mass-spring solution**

One seeks to express the above solution in terms of a mass-spring system, in order to be able to easily introduce an extra mass (the force centring ball). For a mass \( m \) supporter by a massless spring of stiffness \( k \), the resonance frequency is given by:

\[
f = \frac{1}{2\pi} \cdot \sqrt{\frac{k}{m^*}}
\]

(7) \( f \) resonance frequency  
\( k \) spring elastic constant  
\( m^* \) "effective" mass (see below)

For a cantilever, where the force is applied at a point \( x \leq L \):

\[
m_c = \rho \cdot S \cdot L
\]

(8) \( m_c \) cantilever mass  
\( x \) force application point

\[
k = 3 \frac{E^* \cdot I}{x^3}
\]

(9) \( k \) spring elastic constant

For a cantilever beam of rectangular cross section \( b \times h \) with \( b \gg h \) – see (2)-(4):

\[
m_c = \rho \cdot b \cdot h \cdot L
\]

(10) \( m_c \) cantilever mass  
\( u \) coefficient real mass – equivalent mass

\[
k = \frac{E \cdot b \cdot h^3}{4x^3}
\]

(11) \( k \) spring elastic constant

\[
m^* = u \cdot m_c
\]

(12) \( m^* \) "effective" mass

If one writes frequency according to (6) :

\[
f = \frac{1}{2\pi} \sqrt{\frac{E}{(1-\nu^2) \cdot \rho} \cdot \frac{h}{\sqrt{u \cdot x^3 \cdot L}}}
\]

\[\approx 0.16154 \cdot \sqrt{\frac{E}{(1-\nu^2) \cdot \rho} \cdot \frac{h}{L^2}}\]

(13) \( f \) resonance frequency  
(14) \( u \approx 0.24267 \cdot \frac{L^3}{x^3} \)
1.3. **Cantilever – addition of a mass on the beam**

If one adds a mass $m'$ at point $x$ of the beam, one gets:

\[
(15) \quad f' \approx \frac{1}{2\pi} \cdot \frac{k}{\sqrt{u \cdot m_c + m'}}
\]

\[
(16) \quad f' \approx 0.16154 \cdot \frac{E}{\sqrt{(1-v^2) \cdot \rho}} \cdot \frac{h}{L^2} \cdot \frac{1}{\sqrt{1 + \frac{m' \cdot x^3}{0.24267 \cdot \rho \cdot b \cdot h \cdot L^4}}}
\]

1.4. **Application to MilliNewton**

The parameters for MilliNewton are given in table 2 below.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>400 mN</th>
<th>1'000 mN</th>
<th>2'000 mN</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>width</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>mm</td>
</tr>
<tr>
<td>$h$</td>
<td>thickness</td>
<td>0.25</td>
<td>0.40</td>
<td>0.635</td>
<td>mm</td>
</tr>
<tr>
<td>$x$</td>
<td>ball position</td>
<td>8.0</td>
<td>8.0</td>
<td>8.0</td>
<td>mm</td>
</tr>
<tr>
<td>$L$</td>
<td>free length</td>
<td>9.3</td>
<td>9.3</td>
<td>9.3</td>
<td>mm</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of $\text{Al}_2\text{O}_3$</td>
<td>3.9</td>
<td>3.9</td>
<td>3.9</td>
<td>mg/mm$^3$</td>
</tr>
<tr>
<td>$m'$</td>
<td>masse (ball)</td>
<td>4.1</td>
<td>4.1</td>
<td>4.1</td>
<td>mg</td>
</tr>
<tr>
<td>$k$</td>
<td>stiffness</td>
<td>7.6</td>
<td>30.9</td>
<td>124</td>
<td>N/mm</td>
</tr>
<tr>
<td>$f$</td>
<td>resonance (w/o ball)</td>
<td>4.30</td>
<td>6.87</td>
<td>10.91</td>
<td>kHz</td>
</tr>
<tr>
<td>$f'$</td>
<td>resonance (with ball)</td>
<td>3.64</td>
<td>6.16</td>
<td>10.16</td>
<td>kHz</td>
</tr>
</tbody>
</table>

Table 2. Calculated mechanical resonance frequencies.

In reality, the clamping of the cantilever is not perfect. Therefore, the real mechanical resonance frequencies will be slightly lower.
2. Electrical bandwidth

2.1. Measurement bridge & amplifier feedback

The electrical bandwidth of the measurement bridge is limited by:

\[ f_j \approx \frac{1}{2\pi k_j R_j C_j} \]

where \( f_j \) is the limiting frequency of the measurement bridge, \( k_j \) is the capacitor factor, 1 (outputs) or \( \frac{1}{2} \) (ground), \( R_j \) is the measuring bridge resistance, and \( C_j \) is the measurement bridge filtering capacitor.

For MilliNewton-A, we have: \( C_j = 10 \text{nF}, k_j = 1 \) (one capacitor across bridge output) and \( R_j \approx 8 \text{k\Omega} \) (somewhat variable), therefore \( f_j \approx 2 \text{kHz} \).

For MilliNewton-B, we roughly have the same values, except \( k_j = 1/2 \) (two capacitors, one between each bridge output and ground), which yields \( f_j \approx 5 \text{kHz} \). However, there is an additional feedback capacitor \( C_f \) in parallel with an effective feedback resistor \( R_f \) (corresponding in fact to a feedback network), yielding a much lower frequency limit:

\[ f_f \approx \frac{1}{2\pi R_f C_f} \]

where \( R_f \approx \frac{1}{2} R_j z \)

With \( C_f = 220 \text{pF} \) and \( z \approx 300 \) (conservative, normally lower), we get a lower limiting frequency of \( f_f \approx 700 \text{Hz} \), which dominates the electric response of MilliNewton-B.

2.2. Amplifier – gain bandwidth product

The amplification bandwidth is limited by:

\[ f_A \approx \frac{f_1}{z} \]

where \( f_1 \) is the gain bandwidth product, \( z \) is the amplification (gain), and \( f_A \) is the amplification limiting frequency.

For MilliNewton-A, we have: \( f_1 \approx 1 \text{MHz} \) and \( z \approx 300 \), therefore \( f_A \approx 3.3 \text{kHz} \).

For MilliNewton-B, we have: \( f_1 \approx 0.8 \text{MHz} \) and \( z \approx 300 \), therefore \( f_A \approx 2.7 \text{kHz} \).

2.3. Amplifier – slew rate

At large output swings, the working frequency may also be limited by the amplifier slew rate. In this case:

\[ f_s \approx \frac{U'_{\text{max}}}{\pi U_S S} \]

where \( f_s \) is the limiting frequency due to slew rate, \( U'_{\text{max}} \) is the slew rate, \( U_S \) is the supply voltage, and \( S \) is the full scale output span (ratiometric).
For MilliNewton-A\textsuperscript{3}, the minimal value of $U'_{\text{max}}$ is 0.1 V/$\mu$s. For the nominal values $U_z = 5$ V and $S = 0.6$, one obtains $f_S \approx 10$ kHz. For MilliNewton-B\textsuperscript{4}, the higher slew rate of the amplifier ($U'_{\text{max}} = 1$ V/$\mu$s) yields a much higher limiting frequency of $f_S \approx 100$ kHz.

3. Conclusion

For MilliNewton-A, the working frequency is essentially limited by two electrical characteristics: the RC network formed by the measurement bridge and its filtering capacitor, and the gain bandwidth product of the used amplifier (LM 358). These two frequency are of the order of a few kHz. Therefore, one expects a notable alteration of the output signal beyond 1 kHz.

For MilliNewton-B, the limiting frequency is lower (~700 Hz), which arises almost exclusively from the amplifier feedback circuit. This is more in line with the specified < 10 ms response time, and would alter the signal beyond ca. 200 Hz.

The mechanical resonance frequencies vary with the cantilever thickness, and lie in general above the electrical limits, starting with > 3 kHz for the lowest force range.

\textsuperscript{1} Whitney-S, "Vibrations of cantilever beams: deflection, frequency and research uses", University of Nebraska - Lincoln (UNL), 1999.

\textsuperscript{2} Strässler-S Ryser-P, "Project VIVIGLUS (mechanical)", EPFL-LPM, 2003.
