Comment on "Recent advances on solving the three-parameter infiltration equation" by Prabhata K. Swamee, Pushpa N. Rathie, Luan Carlos de S.M. Ozelim and André L.B. Cavalcante, Journal of Hydrology 509 (2014) 188-192

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Summary

A recent approximation to the three-parameter infiltration was compared with an existing approximation. The new approximation has a minimum relative error that is two orders of magnitude greater than the maximum relative error of the existing approximation.

The three-parameter equation of Parlange et al. (1982) interpolates between the infiltration formulas of Green and Ampt (1911) and Talsma and Parlange (1972). The three parameters are the sorptivity, S [LT$^{-1/2}$], the hydraulic conductivity, K [LT^{-1}] and the interpolation parameter, α, where $0 \le \alpha \le 1$. The two limits of α, 0 and 1, correspond to the Green-Ampt and Talsma-Parlange limits, respectively. S and K are used to non-dimensionalise the three parameter equation as (Parlange et al., 2002):

$$t_* = F_* - (1-\alpha)^{-1}\ln[\alpha^{-1} - (\alpha^{-1} - 1)\exp(-\alpha F_*)], \quad (1)$$

where t_* is dimensionless time and F_* is dimensionless infiltration.

Equation (1) is implicit in F_*, and so for practical applications it is useful to provide approximations to give $F_*(t_*)$ explicitly, making use of simple functions. Such an approximation, denoted by $F_{*,P}$, was given by Parlange et al. (2002).

The error of an approximation can be compared using different metrics. An oft-used metric is the maximum relative error. Because α is fixed for a given soil, but t_* is variable, the relative error, E_{rel}, is specified for given α as:

$$E_{rel}(F_{*,A}) = \max_{t_* \ge 0}\left|1 - \frac{F_{*,A}}{F_*}\right|, \quad (2)$$

where $F_{*,A}$ is an approximation to F_*. An alternative is the absolute error, E_{abs}, given by:

$$E_{abs}(F_{*,A}) = \max_{t_* \ge 0}\left|F_* - F_{*,A}\right|. \quad (3)$$

Because an approximation should be applicable for any soil, it is also useful to calculate the relative error of F_* for any α in the range $0 \le \alpha \le 1$ and for all t_*. Thus, the global (subscript g) relative and absolute errors are, from Eqs. (2) and (3), respectively,

$$E_{rel,g}(F_{*,A}) = \max_{\substack{t_* \geq 0 \\ 0 \leq \alpha \leq 1}} \left|1 - \frac{F_{*,A}}{F_*}\right|, \tag{4}$$

and

$$E_{abs,g}(F_{*,A}) = \max_{\substack{t_* \geq 0 \\ 0 \leq \alpha \leq 1}} \left|F_* - F_{*,A}\right|. \tag{5}$$

Swamee et al. (2014) provided a new approximation to F_* (denoted $F_{*,S}$). The short-time limit of F_* is given by Parlange et al. (1982) and that for $F_{*,S}$ by Swamee et al. (2014). Using these expressions, the relative error in this limit is:

$$\lim_{t_* \to 0} \left|1 - \frac{F_{*,S}}{F_*}\right| = \left|1 - \frac{25}{39\sqrt{2}} \exp\left(\frac{29}{20}\alpha^{1/4}\right) \ln\left[\alpha^{(\alpha-1)^{-1}} - \alpha^{-1}\right]\right|. \tag{6}$$

In the small α limit, the right side of Eq. (6) diverges according to $\lim_{\alpha \to 0} \ln[-\ln(\alpha)]$. Thus, $E_{rel,g}(F_{*,S}) = \infty$. Numerical experiments show that, for given $\alpha > 0$, the maximum of $E_{rel}(F_{*,S})$ is given by Eq. (6). Further, since the right side of Eq. (6) decreases monotonically, the $\alpha = 1$ limit gives the minimum value of $E_{rel}(F_{*,S})$. In this limit, the right side of Eq. (6) is $\left|1 - 25\exp(29/20)\ln[\exp(1) - 1]/(39\sqrt{2})\right| \approx 0.04606$. In contrast, for the approximation of Parlange et al. (2002) we have $E_{rel,g}(F_{*,P}) < 0.00048$. Thus, the *minimum* relative error of $F_{*,S}$ is 100 times greater than the *maximum* relative error of $F_{*,P}$.

Of course, the absolute error could be used rather than the relative error as the metric for comparing the two approximations, $F_{*,P}$ and $F_{*,S}$. Indeed, this was the metric chosen by Swamee et al. (2014). Figure 2 of Swamee et al. (2014) contains the caption: "Logarithm of the absolute value of the difference between 30 digit precision exact solutions and approximations, revealing the number of correct decimal digits of the solutions." This latter phrase cannot hold in general. Consider, for example, if the $Exact$ value is 10^6 and the $Approximation$ is $10^6 + 1$. Then, the $Approximation$ is correct to six significant digits. However, $-\log_{10}[(|Exact - Approximation|) = 0$. To get the number of correct significant digits of the $Approximation$, which is a key metric to assess its quality, one must instead use the relative error, i.e., $-\log_{10}(|1 - Approximation/Exact|) = -\log_{10}(E_{rel})$. This point was also made by Barry et al. (2012). For the just-given example, this formula gives six correct significant digits. In brief, because the relative error is a proxy for the number of correct significant digits of an approximation, we consider it as an appropriate metric to assess the quality of an approximation.

In comparing $F_{*,S}$ and $F_{*,P}$, Swamee et al. (2014) state “only near $\alpha = 1/2$ the solution by Parlange et al. (2002) is more accurate”. Unfortunately, Swamee et al. (2014) apparently did not use the correct form of $F_{*,P}$, specifically Eq. (18) of Parlange et al. (2002). Swamee et al. (2014) give this expression in their Eq. (4), but they did not include the final power two inside the exponential. We made the same omission in our comment (although the computations were done using the correct expression) on the previous approximation to Eq. (1) of Swamee et al. (2012); see Eq. (6) of Barry et al. (2012). Swamee et al. (2014) also noted that they could not reproduce the absolute error plots given by Barry et al. (2012), i.e., they remark that “a different precision was attested by Barry et al. (2012) for Parlange et al. (2002) solution”. This error was due the taking the value as a percentage and dividing by 100. The points plotted are correct, but are shifted by two log units.

In Fig. 1, we plot the corrected version of Fig. 2 of Swamee et al. (2014). This figure shows $E_{abs}(F_{*,P})$ and $E_{abs}(F_{*,S})$ for various α. As was the case for the relative error, based on this metric $F_{*,P}$ is more accurate than $F_{*,S}$. Also shown in Fig. 1 are the results (triangles) of omitting the final power two in Eq. (18) of Parlange et al. (2002). These points are, visually at least, identical to those plotted in Fig. 2 of Swamee et al. (2014). For ease of checking, we include a table specifying the data in Fig. 1 in the supplementary material.

In conclusion, the approximation of Parlange et al. (2002) for the three-parameter infiltration equation (Parlange et al., 1982) is significantly more accurate than that of Swamee et al. (2014). Considering the relative error, which is closely related to the number of significant digits given by the approximation, the short-time limit of the new approximation of Swamee et al. (2014) has a *minimum* relative error of about 4.6%, which occurs at $\alpha = 1$. Further, the relative error of this new approximation increases monotonically as α is reduced from 1 to 0, and becomes infinite in the $\alpha = 0$ limit. Thus, this approximation is far less accurate than the approximation of Parlange et al. (2002), which has a *maximum* relative error of less than 0.048%. A similar conclusion is reached if the absolute error is used instead of the relative error to compare these approximations.

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Figure

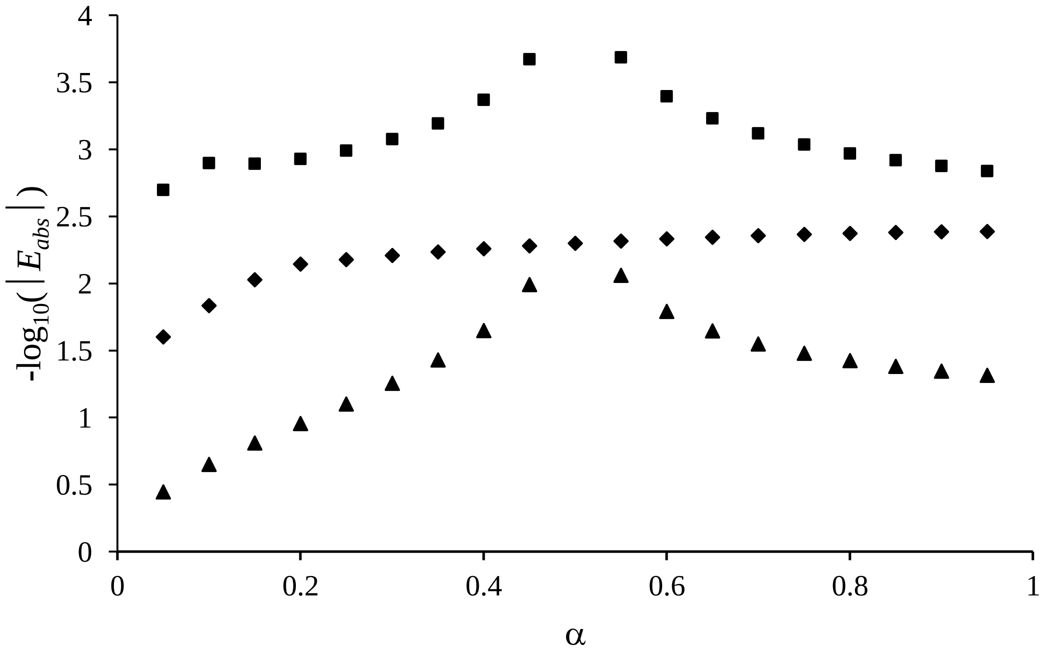

Figure 1. Plot of the absolute error of different approximations for various α in the range $0 \leq \alpha \leq 1$ (analogous to Fig. 2 of Swamee et al. (2014)). The squares are for $F_{*,P}$ and diamonds for $F_{*,S}$. The triangles for the erroneous version of $F_{*,P}$ used by Swamee et al. (2014). The squares and triangles do not show any point plotted for $\alpha = ½$, since the approximations are exact there, so the absolute error is zero.