Stabilizing bisets and expansive subgroups

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Abstract

The research of this thesis belongs to the representation theory of groups. One purpose in representation theory is to try to describe representations of a finite group via information about those of a subgroup of order as small as possible. A way to do so is to use stabilizing bisets. Indeed, let k be a field, G a finite group, U a (G,G)-biset and L a kG-module. Then U is said to stabilize L if $U(L) := kU \otimes_{kG} L$ is isomorphic to L. If we suppose that L is indecomposable, then one can show that U is of the form $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ for some subgroups and an isomorphism $\phi : C/D \to A/B$. In particular, this means that L can be constructed from a representation of A/B. Given an indecomposable module it is not easy in general to find explicitly a proper stabilizing biset. In [3] it is proved that a good example of stabilizing bisets arises from expansive subgroups.

Indeed, for a finite group G, it is shown that if V is a simple kG-module then there exist a genetic subgroup T of G and a faithful simple $k(N_G(T)/T)$-module M such that $V \cong \mathrm{Indinf}_{N_G(T)/T}^{G}(M)$ and V is stabilized by the biset $U = \mathrm{Indinf}_{N_G(T)/T}^{G} \mathrm{Defres}_{N_G(T)/T}^{G}$. However, it is possible that T is trivial. As $N_G(T)/T$ is Roquette, T could only be trivial if G is Roquette.

This raises the question of proving the existence, or non-existence, of stabilizing bisets for Roquette groups. To do so, one will use two approaches. The first one is to improve the theorem and find some genetic subgroups in Roquette groups. The second one is to find stabilizing bisets for Roquette groups without the use of genetic subgroups.

The purpose of this thesis is to investigate these two directions, also to try to generalize the theory of stabilizing bisets to n-stabilizing bisets, i.e. bisets U such that $U(L) \cong nL$.

Key words: stabilizing biset, indecomposable module, Roquette group, genetic and expansive subgroup.

Résumé

Cette thèse s'inscrit dans la théorie des représentations de groupes finis. L'un des buts de cette théorie est de décrire les représentations d'un groupe donné G par celles de sous-groupes d'ordres aussi petits que possible. Une des manières de le faire est d'utiliser les bi-ensembles stabilisants. En effet, soient k un corps, U un (G, G)-bi-ensemble et L un kG-module indécomposable. Alors U stabilise L si $U(L) := kU \otimes_{kG} L$ est isomorphe à L. On peut montrer que dans ce cas U est de la forme $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ et donc L provient d'une représentation de A/B. Dans l'article [3], il est montré que des exemples d'une telle situation proviennent de sous-groupes expansifs.

En effet, il est montré que si V est un kG-module simple, alors il existe un sous-groupe génétique T tel que $\mathrm{Indinf}_{N_G(T)/T}^{G} \mathrm{Defres}_{N_G(T)/T}^{G}$ stabilise L. Toutefois, T peut être trivial et donc le bi-ensemble réduit à l'identité. Comme $N_G(T)/T$ doit être Roquette cela n'est possible que lorsque G est Roquette. Pour contrer ce problème nous avons deux solutions. La première est d'améliorer ce théorème pour les groupes de Roquette et montrer l'existence d'un tel T non-trivial. La deuxième est de trouver un bi-ensemble stabilisant L en utilisant d'autres types de sous-groupes que les sous-groupes génétiques.

Le but de cette thèse est tout d'abord d'examiner ces deux options et dans un deuxième temps d'étudier le cas des n-stabilisations, c'est-à-dire lorsque $U(L) \cong nL$.

Mots clés: bi-ensemble stabilisant, module indécomposable, groupe de Roquette, sous-groupe génétique et expansif.

Quant à parler à des non-spécialistes de mes recherches ou de toute autre recherche mathématique, autant vaudrait, il me semble, expliquer une symphonie à un sourd. Cela peut se faire; on emploie des images, on parle de thèmes qui se poursuivent, qui s'entrelacent, qui se marient ou divorcent; d'harmonies tristes ou de dissonances triomphantes: mais qu'a-t-on fait quand on a

fini? Des phrases, ou tout au plus un poème, bon ou mauvais, sans rapport avec ce qu'il prétendait décrire. La mathématique de ce point de vue n'est pas autre chose qu'un art, une espèce de sculpture dans une matière extrêmement dure et résistante.

André Weil (1906-1998), France.

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Four years on the same project is quite a long time. More than a thesis on a specific subject it is a journey on which people embark. I could not mention all of them in this brief section but their impact as well as my acknowledgment to them is undoubted.

Finally this thesis, as well as me, would not have seen the light of the day without my parents and their unswerving support. To you and my little twin brother, I dedicate this thesis.

Contents

List of Notation

Throughout this report, we will try to use as much as possible some standard notation. Even if the term "standard" has not really a definition. In our case it will mean that we follow the notations of our references.

Let k be a field. The symbol $\mathrm{Sp}(V)$ denotes the symplectic group on the k-vector space V of dimension $2n$. Formally, one should write $\mathrm{Sp}_{2n}(k)$ when a basis of the vector space is chosen. But for a better understanding, we will, by abuse of notation, continue to write $\mathrm{Sp}(V)$.

Finally, we shall use this notation:

$G_1 \circ G_2$	The central product of the groups G_1 and G_2.
χ_V	The character associated to the module V.
V^*	The dual of the module V.
$H \leq G$	H is a subgroup of G.
$Z(G)$	The center of a group G.
$\|G\|$	The order of G.
${}^g s$	The element gsg^{-1}.
$[K\backslash G/H]$	A set of representatives of (K, H)-double cosets.
S_d	The symmetric group of order $d!$.
$Irr(G)$	The set of isomorphism classes of irreducible $\mathbb{C}G$-modules.
A^t	The transpose of the matrix A.

Introduction

The research of this thesis belongs to the general framework of group theory. It lies between the theory of representations, with the study of stabilizing bisets and pure group theory with the study of expansive subgroups in a Roquette group. The notion of stabilizing bisets is introduced in the article of Serge Bouc and Jacques Thévenaz "Stabilizing bisets" referred as [3] in the bibliography. Let k be a field. A (G, G)-biset U is said to stabilize a kG-module L if $U(L) := kU \otimes_{kG} L$ is isomorphic to L. One can actually reduce the study to bisets U of the form $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ for some subgroups of G and an isomorphism $\phi : C/D \to A/B$. The first goal is to find a way to construct examples of such a situation. The first method developed in [3] is the use of idempotents bisets, which are bisets such that $U^2 \cong U$. Therefore if $L := U(V)$ for any kG-module V then $U(L) \cong L$. These bisets are completely classified in [3]. They correspond to idempotents in the double Burnside ring. The only problem with this method is that one cannot assure the indecomposability of L. The second method consists in using expansive and genetic subgroups. We first recall the definition of such subgroups.

(i) A subgroup T of a finite group G is called expansive in G if, for every $g \notin N_G(T)$, the $N_G(T)$-core of the subgroup $\big({}^{g}T \cap N_G(T)\big)T$ contains properly T.

(ii) A finite group H is said to be a Roquette group if all its normal abelian subgroups are cyclic.

(iii) A subgroup T of a finite group G is called a genetic subgroup if T is an expansive subgroup of G and $N_G(T)/T$ is a Roquette group.

It is proved in [3] that if T is an expansive subgroup of G and M is a faithful simple $k[N_G(T)/T]$-module, then $L := \mathrm{Indinf}_{N_G(T)/T}^G(M)$ is indecomposable and $U = \mathrm{Indinf}_{N_G(T)/T}^G \mathrm{Defres}_{N_G(T)/T}^G$ stabilizes L. This time, one has the indecomposability of L but in general, as one can see from the definition, it is not easy to find expansive subgroups.

The second goal is to find a theorem of existence of stabilizing bisets. In [3] it is shown that if V is a simple kG-module then there exist a genetic subgroup T of G and a faithful simple $k[N_G(T)/T]$-module M such that $V \cong \mathrm{Indinf}_{N_G(T)/T}^G(M)$ and V is stabilized by the biset

$$U = \mathrm{Indinf}_{N_G(T)/T}^G \mathrm{Defres}_{N_G(T)/T}^G .$$

The only issue is that if $T = 1$ we obtain a trivial biset. This situation can only arise if G is Roquette as $N_G(T)/T$ is Roquette by assumption. The main purpose of this thesis is to investigate the existence of stabilizing bisets for Roquette groups. Also, discuss the minimality of the stabilizing bisets and the generalization to the theory of n-stabilizing bisets, i.e. bisets U such that $U(L) \cong nL$.

Our willingness in the order of the presentation of the results is to go form the general to the particular. The reason is to present the general results before some more specialized ones that are only relevant to particular situations. This is why, after a first introducing chapter on basic notions, one starts and introduces the notion of n-stabilizing bisets. This generalizes the notion of stabilizing bisets introduced in [3]. One develops the first general properties, following the results of [3] which are in the case $n = 1$. One also study in depth the notion of minimality and n-idempotents bisets. Finally, one introduces the notion of n-expansive subgroups in order to recover the existing link between stabilizing bisets and expansive subgroups for n greater than one.

As we did not generalize all the results of stabilizing bisets to n-stabilizing bisets we state these results in the third chapter, especially as some of these additional results are needed in order to treat the examples in the last two chapters. Indeed, the fifth chapter is devoted finding expansive subgroups in certain Roquette groups such as Roquette p-groups, simple groups, groups with Fitting subgroups containing cyclic or extraspecial groups. Finally the last chapter is the study of the existence of n-stabilizing bisets in the same examples as chapter five.

Chapter 1

Basics

This first chapter is dedicated to a brief review of a few of the most fundamental properties of group theory, representation theory and biset theory which are going to be used in the following chapters.

1.1 Background of group theory

In this section, one recalls some elementary results in group theory and representation theory that one uses in this thesis. We refer to the wide literature for the proofs.

Theorem 1.1. ***Schur-Zassenhaus Theorem***

Let G be a finite group, and N be a normal subgroup whose order is coprime to the order of the quotient group G/N, then N has a complement in G.

Proposition 1.2. *Lemma* 1.1 *page* 353, *[12]*

Let G_1 and G_2 be two groups. Let S be a subgroup of $G_1 \times G_2$. For $i = 1, 2$, define $k_i(S) := S \cap G_i$ and let $p_i(S)$ be the projection of S on G_i. Then S is determined by a subquotient $p_1(S)/k_1(S)$ of G_1, a subquotient $p_2(S)/k_2(S)$ of G_2 and an isomorphism $\phi : p_1(S)/k_1(S) \to p_2(S)/k_2(S)$. Specifically, S is the inverse image $\pi^{-1}(\Delta_\phi)$, where Δ_ϕ is the graph of ϕ and $\pi : p_1(S) \times p_2(S) \to p_1(S)/k_1(S) \times p_2(S)/k_2(S)$ is the quotient map.

Proposition 1.3. *Let q be a prime number. Let A be a q'-group of automor-*

phisms of the abelian q-group Q. Then we have

$$Q = C_Q(A) \times [Q, A].$$

Proof. See Theorem 2.3 of [5] on page 177. $\square$

Proposition 1.4. *Let A be a group of automorphisms of the cyclic group C such that $(|A|, |C|) = 1$. Then we have*

$$C = C_C(A) \times [C, A].$$

Proof. Write C as the product of cyclic groups, $C = C_1 \times \cdots \times C_r$ where C_i is a q_i-group, for a prime number q_i. By Proposition 1.3 we have, for all integers i,

$$C_i = C_{C_i}(A) \times [C_i, A].$$

Using the definition of the commutator it's easy to check that $[C_i \times C_j, A] = [C_i, A] \times [C_j, A]$ via $[(x, y), a] \mapsto ([x, a], [y, a])$. Moreover, by definition, one has $C_{C_i \times C_j}(A) = C_{C_i}(A) \times C_{C_j}(A)$. This shows that

$$C = \prod_i C_i = \prod_i \big(C_{C_i}(A) \times [C_i, A]\big) = C_{\prod_i C_i}(A) \times [\prod_i C_i, A] = C_C(A) \times [C, A].$$

$\square$

Theorem 1.5. ***Krull-Schmidt Theorem***

Let k be a field, G a finite group and M a kG-module. Then M is expressible as a finite direct sum of indecomposable submodules. Furthermore, the decomposition is unique up to isomorphism. In other words if $M = \oplus_{i=1}^r M_i$ and $M = \oplus_{j=1}^s N_j$ are two decompositions then $r = s$ and there is a permutation σ in S_n such that $M_i \cong N_{\sigma(i)}$ for every integer $1 \leq i \leq r$.

Theorem 1.6. ***Clifford's Theorem***

Let k be a field, N be a normal subgroup of a finite group G and M a simple kG-module. Then one has the following decomposition

$$\mathrm{Res}_N^G(M) \cong \bigoplus_{\bar{g} \in G/I} m\, {}^g V,$$

where V is an irreducible kN-module, $I := \{h \in G \mid {}^h V \cong V\}$ and m divides $|I : N|$.

Proposition 1.7. ***Mackey Decomposition Formula***

Let k be a field, H and K be subgroups of a finite group G and L be a kH-module. Then, there exists an isomorphism of kK-modules

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H(L) \cong \bigoplus_{x\in[K\backslash G/H]} \operatorname{Ind}^K_{{}^xH\cap K} \operatorname{Res}^{{}^xH}_{{}^xH\cap K}({}^xL).$$

Theorem 1.8. ***Green's Indecomposability Theorem***

Let k be an algebraically closed field of characteristic p. Let N be a normal subgroup of a finite group G. Suppose G/N is a p-group. If M is an indecomposable kN-module, then the kG-module $\operatorname{Ind}^G_N(M)$ is indecomposable.

1.2 Symplectic Groups

In the last two chapters one will work with symplectic groups thus we need to present the usual facts concerning these groups.

Definitions 1.9.

(i) Let k be a field and b be a bilinear form on a k-vector space V. The form b is *skew symmetric* if $b(x,y) = -b(y,x)$ for all x, y in V. Moreover, the form is said to be *symplectic* if b is nondegenerate and skew symmetric, and in addition when $\operatorname{char}(k) = 2$ we must have $b(x,x) = 0$ for all x in V.

(ii) Given a symplectic form b on V, we define the *symplectic group* $\operatorname{Sp}(V)$ as the elements of $\operatorname{GL}(V)$ which preserve b, in other words

$$\operatorname{Sp}(V) := \{T \in \operatorname{GL}(V) \mid b(Tx,Ty) = b(x,y) \text{ for all } x,y \in V\}.$$

Remark 1.10. Let $v_1 \in V$ be non-zero and w_1 such that $b(v_1, w_1) = 1$. Since $b_{|\langle v_1,w_1\rangle}$ is nondegenerate we know that

$$V = \langle v_1, w_1\rangle \oplus \langle v_1, w_1\rangle^{\perp}.$$

Similarly, if we consider the space $\langle v_1, w_1\rangle^{\perp}$ of dimension $\dim(V)-2$ equipped with the symplectic form $b_{|\langle v_1,w_1\rangle^{\perp}}$ we obtain v_2, w_2 in $\langle v_1, w_1\rangle^{\perp}$ such that

$$V = \langle v_1, w_1\rangle \oplus \langle v_2, w_2\rangle \oplus \langle v_2, w_2\rangle^{\perp}.$$

Continuing in this fashion, we obtain a basis $\{v_1, w_1, v_2, w_2, \ldots, v_m, w_m\}$, called *a symplectic basis*. The matrix of b in the basis is

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Furthermore, the matrix of b in the basis $\{v_1, v_2, \ldots v_m, w_1, w_2, \ldots, w_m\}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_m \\ -\mathrm{id}_m & 0 \end{pmatrix}.$$

Finally, the matrix of b in the basis $\{v_1, v_2, \ldots v_m, w_m, w_{m-1}, \ldots, w_1\}$ is

$$\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \text{ where } K = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

In term of matrices,

$$\mathrm{Sp}(V) := \{A \in \mathrm{GL}(V) \mid {}^tAJA = J\},$$

where J denotes the matrix of the symplectic form. The shape of an element A of $\mathrm{Sp}(V)$ depends on J. In this thesis one always equips the vector space V with a symplectic basis. Nevertheless, one rearranges the order of the vectors in the basis depending on the situation and thus the shape of A could vary.

1.3 A little bit of bisets

In this section one gives a short introduction to the theory of bisets and in particular to their actions on modules.

Definitions 1.11.

(i) *A section* of a group G is a pair (A, B) of subgroups of G such that B is a normal subgroup of A.

(ii) Two sections (A, B) and (C, D) of a group G are *linked* if

$$(A \cap C)B = A, (A \cap C)D = C \text{ and } A \cap D = C \cap B.$$

Remark 1.12. If (A, B) and (C, D) are linked then $A/B \cong C/D$, indeed, using the second isomorphism theorem, one has

$$\begin{aligned} A/B &= \big((A \cap C)B\big)/B \cong (A \cap C)/(B \cap C) = (A \cap C)/(A \cap D) \\ &\cong \big((A \cap C)D\big)/D = C/D. \end{aligned}$$

This isomorphism between A/B and C/D is called *the isomorphism induced by the linking*.

Definition 1.13. Let G and H denote two finite groups. A (G, H)-*biset* U is a set which is both left G-set and right H-set such that

$$(gu)h = g(uh), \text{ for all } g \in G, h \in H \text{ and } u \in U.$$

Let k be a field, then kU denotes the k-vector space with basis U. It's also a (kG, kH)-bimodule.

Examples 1.14. We give a list of basic bisets which play an essential role in this thesis. Let (A, B) be a section of a finite group G. The action on each of the following bisets is just the group multiplication.

(i) The inflation is a $(A, A/B)$-biset defined as $\mathrm{Inf}_{A/B}^{A} := A/B$.

(ii) The induction is a (G, A)-biset defined as $\mathrm{Ind}_{A}^{G} := G$.

(iii) The deflation is a $(A/B, A)$-biset defined as $\mathrm{Def}_{A/B}^{A} := A\backslash B$.

(iv) The restriction is a (A, G)-biset defined as $\mathrm{Res}_{A}^{G} := A$.

(v) Given an isomorphism $\phi : H \to G$, the isomorphism is a (G, H)-biset defined as $\mathrm{Iso}_{\phi} := H$ with left action of G via ϕ^{-1}.

(vi) An element $g \in G$ induces an isomorphism $c_g : G \to G$ defined by $c_g(x) = gxg^{-1}$ and $\mathrm{Conj}_g := \mathrm{Iso}_{c_g}$ denotes the corresponding (G, G)-biset.

Definition 1.15. Let G, H, K be finite groups, U a (G,H)-biset and U' a (H,K)-biset. Then *the product* $U \times_H U'$ denotes the (G,K)-biset defined by

$$U \times_H U' := (U \times V)/\sim,$$

where $\sim$ is the equivalence relation defined by $(uh, v) \sim (u, hv)$ for all $u \in U, v \in V$ and $h \in H$. The left action of G on $U \times_H U'$ is induced by the left action of G on U and the right action of K is induced by the right action of K on U'. We write simply UU' instead of $U \times_H U'$.

Examples 1.16. We use the previous examples. Let (A,B) be a section of a finite group G.

(i) The $(G, A/B)$-biset $\mathrm{Indinf}_{A/B}^G$ is defined as $\mathrm{Ind}_A^G \mathrm{Inf}_{A/B}^A = G \times_A A/B$.

(ii) The $(A/B, G)$-biset $\mathrm{Defres}_{A/B}^G$ is defined as $\mathrm{Def}_{A/B}^A \mathrm{Res}_A^G = A/B \times_A G$.

Proposition 1.17. *Relations* 1.1.3 *page* 2*, [2]*

Let (A,B) and (C,D) be two sections of a finite group G. Let N and K be normal subgroups of G and H a finite group. Let $\phi : G \to H$ be a group isomorphism, then

(*i*)

$$\begin{array}{rcl} \mathrm{Iso}_{\phi'} \mathrm{Res}_A^G & \cong & \mathrm{Res}_{\phi(A)}^H \mathrm{Iso}_\phi \\ \mathrm{Iso}_\phi \mathrm{Ind}_A^G & \cong & \mathrm{Ind}_{\phi(A)}^H \mathrm{Iso}_{\phi'} \end{array}$$

where $\phi' : A \to \phi(A)$ is the restriction of ϕ to A.

(*ii*)

$$\begin{array}{rcl} \mathrm{Iso}_{\phi''} \mathrm{Def}_{G/N}^G & \cong & \mathrm{Def}_{H/\phi(N)}^H \mathrm{Iso}_\phi \\ \mathrm{Iso}_\phi \mathrm{Inf}_{G/N}^G & \cong & \mathrm{Inf}_{H/\phi(N)}^H \mathrm{Iso}_{\phi''} \end{array}$$

where $\phi'' : G/N \to H/\phi(H)$ is the isomorphism induced by ϕ.

(*iii*)

$$\mathrm{Def}_{G/N}^G \mathrm{Inf}_{G/K}^G \cong \mathrm{Inf}_{G/(NK)}^{G/N} \mathrm{Def}_{G/(NK)}^{G/K}.$$

(iv)

$$\begin{aligned}\mathrm{Def}^G_{G/N}\,\mathrm{Ind}^G_K &\cong \mathrm{Ind}^{G/N}_{KN/N}\,\mathrm{Iso}_\psi\,\mathrm{Def}^K_{K/K\cap N}\\ \mathrm{Res}^G_K\,\mathrm{Inf}^G_{G/N} &\cong \mathrm{Inf}^K_{K/K\cap N}\,\mathrm{Iso}_{\psi^{-1}}\,\mathrm{Res}^{G/N}_{KN/N},\end{aligned}$$

where $\psi : K/K\cap N \to KN/N$ is the isomorphism given by the second isomorphism theorem.

(v) Suppose $N \leq K$ then

$$\begin{aligned}\mathrm{Res}^{G/N}_{K/N}\,\mathrm{Def}^G_{G/N} &\cong \mathrm{Def}^K_{K/N}\,\mathrm{Res}^G_K\\ \mathrm{Ind}^G_K\,\mathrm{Inf}^K_{K/N} &\cong \mathrm{Inf}^G_{G/N}\,\mathrm{Ind}^{G/N}_{K/N}.\end{aligned}$$

Lemma 1.18. *Lemma* 2.1 *page* 1613*, [3]*
Let U be a transitive (G,H)-biset. Then there exist a section (A,B) of G, a section (C,D) of H and an isomorphism $\phi : C/D \to A/B$ such that

$$U \cong \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}.$$

Moreover, the triple $\big((A,B),(C,D),\phi\big)$ is unique up to conjugation.

Lemma 1.19. ***Zassenhaus****. Lemma* 2.3 *page* 1614*, [3]*
Let G be a finite group and let (A,B) and (C,D) be two sections of G. Then the subsection $\big((A\cap C)B,(A\cap D)B\big)$ of (A,B) is linked to the subsection $\big((A\cap C)D,(B\cap C)D\big)$ of (C,D). The isomorphism corresponding to the linking is the composite

$$(A\cap C)D/(B\cap C)D \to (A\cap C)/(B\cap C)(A\cap D) \to (A\cap C)B/(A\cap D)B.$$

Definition 1.20. Let (A,B) and (C,D) be two sections of a group G. *The butterfly* associated to (A,B) and (C,D) is the $(A/B,C/D)$-biset defined as follows

$$\mathrm{Btf}(A,B,C,D) := \mathrm{Indinf}^{A/B}_{(A\cap C)B/(A\cap D)B}\,\mathrm{Iso}_\psi\,\mathrm{Defres}^{C/D}_{(A\cap C)D/(B\cap D)D},$$

where ψ is the isomorphism of the Zassenhaus Lemma (see Lemma 1.19).

Remark 1.21. The Butterfly is reduced to an isomorphism if and only if the sections are linked.

Proposition 1.22. ***Generalized Mackey Formula.*** *Lemma* 2.5 *page* 1615, *[3]*

Let (A, B) and (C, D) be two sections of a finite group G. Then there is the following decomposition as a disjoint union of bisets

$$\mathrm{Defres}^G_{C/D}\,\mathrm{Indinf}^G_{A/B} \cong \bigcup_{x\in[C\backslash G/A]} \mathrm{Btf}(C, D, {}^xA, {}^xB)\,\mathrm{Conj}_x\,.$$

Definition 1.23. Let k be a field. Let U be a (G, H)-biset and L be a left kH-module. Then U acts on L as follows

$$U(L) := kU \otimes_{kH} L.$$

This is a kG-module. We say that U is *applied* to L.

Remark 1.24.

(i) If U is one of the inflation, induction, restriction, deflation or isomorphism bisets, then $U(L)$ is obtained from L by applying the corresponding operation with the same name.

(ii) If U is the disjoint union of two (G, H)-bisets U_1 and U_2 then

$$U(L) \cong U_1(L) \oplus U_2(L).$$

(iii) If U' is a (K, G)-biset, U is a (G, H)-biset, and M is a kH-module, then

$$\begin{aligned} U'\big(U(M)\big) &:= kU' \otimes_{kG} (kU \otimes_{kH} M) \cong (kU' \otimes_{kG} kU) \otimes_{kH} M \\ &\cong k[U' \times_G U] \otimes_{kH} M \cong (U' \times_G U)(M). \end{aligned}$$

Chapter 2

n-Stabilizing Bisets

In the previous chapter one has seen how a biset can act on a module. In this chapter one introduces the notion of stabilizing bisets and more generally n-stabilizing bisets.

In the first section one finds some properties and characterisations of this situation. In the second section, one looks at n-stabilizing bisets and strong minimality. Then one looks at means of obtaining n-stabilizing bisets. One discusses one way with the help of n-idempotent bisets. Finally, in the last section one generalises section 6 of [3] by introducing a notion of n-expansive subgroups. This is another way to construct examples of n-stabilization.

2.1 Some elementary properties

In this section, one generalizes section 3 of [3]. Indeed, Theorem 2.12 is a generalization of Corollary 3.4 of [3] from the stabilization case to the n-stabilization.

Definition 2.1. Let k be a field. Let U be a (G, G)-biset, let n be an integer and let L be a kG-module for a field k. Then U is said to *n-stabilize* L if $U(L) \cong nL$. In the case $n = 1$, U is said to *stabilize* L.

Example 2.2. One refers to the last section of [3] for examples with $n = 1$. Here are examples with $n > 1$. Let k be an algebraically closed field of characteristic p, let P be a p-group. Let (A, B) be a normal section of P. Define L as $\mathrm{Ind}_A^P(k)$. By Green's indecomposability theorem L is indecomposable and then it's easy to see that $U(L) = |P : A|L$ for $U := \operatorname{Indinf}_{A/B}^{P} \operatorname{Defres}_{A/B}^{P}$. Indeed, $(A, B) = ({}^{g}A, {}^{g}B)$ for all g in P because both A and B are normal therefore using the generalized Mackey formula one has

$$\begin{aligned} U(L) &= U\big(\operatorname{Ind}_{A}^{P}(k)\big) = \bigoplus_{g\in[A\backslash P/A]} \operatorname{Indinf}_{A/B}^{P} \operatorname{Btf}(A, B, {}^{g}A, {}^{g}B)(k) \\ &= \bigoplus_{g\in[A\backslash P/A]} \operatorname{Indinf}_{A/B}^{P}(k) = |P : A|L. \end{aligned}$$

For example one can apply this to an extraspecial group P with $B := Z(P)$ and $A := N_P(\langle x\rangle)$ where x a non-central element of order p or also to P the dihedral group D_8 of order 8 with $A = \langle r\rangle$ and $B = \langle r^2\rangle$ where r is the rotation by an angle of $\pi/2$.

Remark 2.3.

- We will focus our interest on indecomposable modules. If $U = \cup_{i=1}^{r} U_i$ is a decomposition of U as disjoint union of transitive bisets and if U n-stabilizes an indecomposable module L then

  $$nL \cong U(L) \cong \bigoplus_{i=1}^{r} U_i(L).$$

  Therefore by Krull-Schmidt Theorem one has for every $1 \leq i \leq r$ that

  $$U_i(L) = k_i L$$

  for an integer k_i. For this reason, we shall assume that the biset U is transitive, hence of the form

  $$U = \operatorname{Indinf}_{A/B}^{G} \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}.$$

- Suppose $U(L) = nL$ with $U = \operatorname{Indinf}_{C/D}^{G} \operatorname{Defres}_{C/D}^{G}$. Set $M := \operatorname{Defres}_{C/D}^{G}(L)$. Then, the first thing to note is that by adjunction properties of induction and inflation, we have

  $$\begin{aligned} n \operatorname{Hom}_{kG}(L, L) &\cong \operatorname{Hom}_{kG}(L, nL) \\ &\cong \operatorname{Hom}_{kG}(L, \operatorname{Indinf}_{C/D}^{G}(M)) \\ &\cong \operatorname{Hom}_{k[C/D]}(\operatorname{Defres}_{C/D}^{G}(L), M) \\ &\cong \operatorname{Hom}_{k[C/D]}(M, M) \end{aligned}$$

as k-vector spaces. As a result one can see that for $k=\mathbb{C}$, the module M is decomposable, except when $n=1$ and L is indecomposable. Moreover if $M=\oplus_{i=1}^{r} a_i M_i$ with M_i indecomposable non-isomorphic modules, then

$$\mathrm{Hom}_{k[C/D]}(M,M)\cong \bigoplus_{i,j}\mathrm{Hom}(a_iM_i,a_jM_j)=\bigoplus_{i=1}^{r}\mathrm{Hom}(a_iM_i,a_iM_i).$$

As L is indecomposable only if $\dim \mathrm{Hom}_{kG}(L,L)=1$ this happens only if $\sum_{i=1}^{r} a_i^2=n$. In the case $r=n$ this implies that $a_i=1$ for all $1\le i\le r$.

Proposition 2.4. *Let $U:=\mathrm{Indinf}_{A/B}^{G}\mathrm{Iso}_{\phi}\mathrm{Defres}_{C/D}^{G}$ be an n-stabilizing biset for a module L. Let $M:=\mathrm{Defres}_{C/D}^{G}(L)$. Then n equals $\frac{|G:A|\dim M}{\dim L}$. In particular, n is smaller than the order of G.*

Proof. By taking the dimension of $U(L)\cong nL$ one has

$$n\dim L=|G:A|\dim \mathrm{Defres}_{C/D}^{G}(L).$$

Therefore one has $n=\frac{|G:A|\dim M}{\dim L}$. As $\dim M$ is smaller than $\dim L$, the integer n is smaller than $|G:A|$ and in particuler smaller than $|G|$. □

Definition 2.5. Let $U=\mathrm{Indinf}_{A/B}^{G}\mathrm{Iso}_{\phi}\mathrm{Defres}_{C/D}^{G}$ be a biset n-stabilizing a kG-module L.

(i) The biset U is said to be *minimal* if, for any transitive biset $U'=\mathrm{Indinf}_{A'/B'}^{G}\mathrm{Iso}_{\phi'}\mathrm{Defres}_{C'/D'}^{G}$ n-stabilizing L, we have $|C/D|\le|C'/D'|$.

(ii) The biset U is said to be *strongly minimal* if, for any transitive biset $U'=\mathrm{Indinf}_{A'/B'}^{G}\mathrm{Iso}_{\phi'}\mathrm{Defres}_{C'/D'}^{G}$ m-stabilizing L, for some integer $m\ge 1$, we have $|C/D|\le|C'/D'|$.

Remark 2.6. Note that in the second definition the integer m could be different from or equal to n. Therefore, the strong minimality of a biset U implies its minimality.

Lemma 2.7. *Let $U:=\mathrm{Indinf}_{A/B}^{G}\mathrm{Iso}_{\phi}\mathrm{Defres}_{C/D}^{G}$ be an n-stabilizing biset for a non-trivial simple module L. If $|A/B|=p$, where p is the smallest prime dividing $|G|$, then U is strongly minimal.*

Proof. Suppose U is not strongly minimal. Let

$$U' = \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{C'/D'}^{G}$$

be an m-stabilizing biset such that $|A'/B'| < |A/B| = p$. Then one has $1 = |A'/B'| = |C'/D'|$ and so U can be written as $\mathrm{Ind}_{A'}^{G} \mathrm{Inf}_{1'}^{A'} \mathrm{Iso}_{\phi'} \mathrm{Def}_{1}^{C'} \mathrm{Res}_{C'}^{G}$. The module $\mathrm{Inf}_{1'}^{A'} \mathrm{Iso}_{\phi'} \mathrm{Def}_{1}^{C'} \mathrm{Res}_{C'}^{G}(L)$ is isomorphic to copies of the trivial module k thus $nL = \nu \,\mathrm{Ind}_{A'}^{G}(k)$ for an integer $\nu \geq 1$. But the trivial kG-module is always a submodule of $\mathrm{Ind}_{A'}^{G}(k)$ which contradicts the assumption that L is not the trivial module. Therefore such U' cannot exist and U is strongly minimal. $\square$

Example 2.8. In chapter 6 one will find examples of minimal but not strongly minimal bisets. Indeed, for $G = A_5, A_6, \mathrm{PSL}_2(\mathbb{F}_{11})$ the only stabilizing bisets $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ for these groups are reduced to an isomorphism and therefore minimal with $(A, B) = (C, D) = (G, 1)$ and $|A/B| = |G|$. Nevertheless, one can find examples of 2-stabilizing bisets $U' := \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{C'/D'}^{G}$ with $|A'/B'| < |G| = |A/B|$.

Theorem 2.9. *Consider two transitive (G, G)-bisets*

$$U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G} \text{ and } U' = \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{C'/D'}^{G}.$$

Let L be an indecomposable kG-module such that $U(L) \cong nL$ and $U'(L) \cong mL$ for $n, m \in \mathbb{N}$. Let $M = \mathrm{Defres}_{C/D}^{G}(L)$ and suppose U is strongly minimal. Let g be an element of G. Then only two cases are possible.

- *The module $\mathrm{Btf}(C', D', {}^{g}A, {}^{g}B) \mathrm{Conj}_g \mathrm{Iso}_{\phi}(M)$ is zero and the section $({}^{g}A, {}^{g}B)$ is not linked to $((C' \cap {}^{g}A)D', (C' \cap {}^{g}B)D')$.*
- *The biset $\mathrm{Btf}(C', D', {}^{g}A, {}^{g}B)$ is reduced to $\mathrm{Indinf}_{(C' \cap {}^{g}A)D'/(C' \cap {}^{g}B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^{g}A, {}^{g}B)$ and $((C' \cap {}^{g}A)D', (C' \cap {}^{g}B)D')$.*

Proof. Applying successively U and U' one obtains

$$\begin{aligned} U'(U(L)) &\cong \bigoplus_{g \in [C' \backslash G/A]} \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Btf}(C', D', {}^{g}A, {}^{g}B) \mathrm{Conj}_g \mathrm{Iso}_{\phi}(M) \\ &\cong mnL. \end{aligned}$$

Therefore, by Krull-Schmidt theorem, one has, for all $g \in [C'\backslash G/A]$,

$$\mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B) \,\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M) \cong k_g L.$$

In other words, one has a k_g-stabilizing biset for L, for a certain $k_g \in \mathbb{N}$. If k_g is not zero and because U is strongly minimal, the biset $\mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B)$ must be reduced to $\mathrm{Indinf}_{(C' \cap {}^{g}\!A)D'/(C' \cap {}^{g}\!B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^{g}\!A, {}^{g}\!B)$ and $((C' \cap {}^{g}\!A)D', (C' \cap {}^{g}\!B)D')$. Indeed, otherwise $\mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B)$ would go through a subsection of (A, B) which is a contradiction with the fact that U is strongly minimal. If k_g is zero then the module $\mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B) \,\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M)$ is zero as the operation $\mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'}$ cannot annihilate a module. For such g, the section $({}^{g}\!A, {}^{g}\!B)$ is not linked to $((C' \cap {}^{g}\!A)D', (C' \cap {}^{g}\!B)D')$ as otherwise the biset $\mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B)$ would have been reduced to

$$\mathrm{Indinf}_{(C' \cap {}^{g}\!A)D'/(C' \cap {}^{g}\!B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)}\,.$$

But the latter does not annihilate $\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M)$. □

Remark 2.10. Let M' be the module $\mathrm{Defres}_{C'/D'}^{G}(L)$. Using the same notations, observe that one has

$$\begin{aligned} nM' &= \mathrm{Defres}_{C'/D'}^{G}(nL) \cong \mathrm{Defres}_{C'/D'}^{G} \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L) \\ &\cong \bigoplus_{g \in [C'\backslash G/A]} \mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B) \,\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M) \\ &\cong \bigoplus_{\substack{g \in [C'\backslash G/A] \\ k_g \neq 0}} \mathrm{Indinf}_{(C' \cap {}^{g}\!A)D'/(C' \cap {}^{g}\!B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M). \end{aligned}$$

Theorem 2.11. *Consider two transitive (G, G)-bisets*

$$U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G} \ \textit{ and } U' = \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{C'/D'}^{G}.$$

Let L be an indecomposable kG-module such that $U(L) \cong nL$ and $U'(L) \cong mL$ for $n, m \in \mathbb{N}$. Let $M = \mathrm{Defres}_{C/D}^{G}(L)$ and $M' = \mathrm{Defres}_{C'/D'}^{G}(L)$ and suppose U and U' are strongly minimal. Let g be an element of G.

1. *Only two cases are possible.*
   - *The module* $\mathrm{Btf}(C', D', {}^{g}\!A, {}^{g}\!B) \,\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M)$ *is zero and the section* $({}^{g}\!A, {}^{g}\!B)$ *is not linked to* $((C' \cap {}^{g}\!A)D', (C' \cap {}^{g}\!B)D')$.

- *The biset* $\mathrm{Btf}(C, D, {}^gA, {}^gB)$ *is reduced to* $\mathrm{Iso}_{\beta(g)}$*, where* $\beta(g)$ *is the isomorphism corresponding to the linking between the sections* $({}^gA, {}^gB)$ *and* (C', D')*.*

Let $\mathscr{M}$ *be the set of elements of* $[C'\backslash G/A]$ *such that we are in the second case above and let d be the cardinal of* $\mathscr{M}$*.*

2. *There exists an isomorphism between* nM' *and* $\bigoplus_{g\in\mathscr{M}} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M)$*.*

3. *One has the following equality* $nm = dd'$*, where* d' *is the number of double cosets* ChA' *such that*

$$\mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Btf}(C, D, {}^hA', {}^hB') \mathrm{Conj}_h \mathrm{Iso}_{\phi'}(M') \neq \{0\}.$$

Proof. One uses the same argument as in the proof of Theorem 2.9. Suppose now that U' is strongly minimal. Applying successively U and U' one obtains again that

$$\begin{aligned} U'(U(L)) &\cong \bigoplus_{g\in[C'\backslash G/A]} \mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Btf}(C', D', {}^gA, {}^gB) \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \\ &\cong mnL. \end{aligned}$$

Again, for all $g \in [C'\backslash G/A]$, one obtains that

$$\mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Btf}(C', D', {}^gA, {}^gB) \mathrm{Conj}_g \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$$

is a k_g-stabilizing biset for L, for a certain $k_g \in \mathbb{N}$. With the same argument as in Theorem 2.9, one deduces that $\mathrm{Btf}(C', D', {}^gA, {}^gB)$ is reduced to an isomorphism if k_g is not zero, because U and U' are strongly minimal. This means that, if k_g is not zero,

$$\mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \cong k_g L.$$

In particular if k_g is not zero, the dimension on the right hand side does not depend on g, because on the left of the isomorphism it does not. Therefore all non-zero k_g are equal. The previous isomorphism becomes

$$mnL \cong U'(U(L)) \cong \bigoplus_{\substack{g\in[C'\backslash G/A] \\ k_g\neq 0}} \mathrm{Indinf}^G_{A'/B'} \mathrm{Iso}_{\phi'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M).$$

By looking at the dimension in this equality, one obtains that

$$mn \dim L = dk_g \dim L$$

where d is the number of double cosets $C'gA$ such that $k_g \neq 0$.

Exchanging the roles of U and U' in the previous argument one has $mn = k'_h d'$ where d' is the number of double cosets ChA' such that $k'_h \neq 0$ and k'_h is such that $\mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Btf}(C, D, {}^hA', {}^hB') \mathrm{Conj}_h \mathrm{Iso}_{\phi'}(M')$ is isomorphic to $k'_h L$.

Furthermore, using Remark 2.10, one has

$$nM' = \bigoplus_{\substack{g \in [C' \backslash G/A] \\ k_g \neq 0}} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M).$$

By looking at the dimension one obtains that $n \dim M' = d \dim M$. Exchanging the roles of U and U' in the previous argument one has $m \dim M = d' \dim M'$. Finally, using these two equations, one obtains that $mn = dd'$ and that $k_g = d'$ and $k'_h = d$ whenever k_g and k'_h are non-zero. □

Theorem 2.12. *Let $U = \mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$ be a strongly minimal n-stabilizing biset for an indecomposable kG-module L. Let $M = \mathrm{Defres}^G_{C/D}(L)$. Then, there exist n double cosets CgA such that*

1. $\mathrm{Btf}(C, D, {}^gA, {}^gB) \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \neq \{0\}$,
2. *the sections (C, D) and $({}^gA, {}^gB)$ are linked,*
3. *the module M is invariant under $\beta(g)c_g\phi$ where $\beta(g)$ is the isomorphism corresponding to the linking between the sections (C, D) and $({}^gA, {}^gB)$,*
4. *if $h \in G$ does not belong to one of these cosets, the section $({}^hA, {}^hB)$ is not linked to (C, D).*

Proof. Using the part 3 of Theorem 2.11 with $U' = U$, $m = n$ and $d' = d$, one obtains that $n = d$. Therefore by the first part, there exist exactly n double cosets CgA such that $\mathrm{Btf}(C, D, {}^gA, {}^gB) \mathrm{Conj}_g \mathrm{Iso}_\phi(M) \neq \{0\}$. For these double cosets one knows that $\mathrm{Btf}(C, D, {}^gA, {}^gB)$ is reduced to $\mathrm{Iso}_{\beta(g)}$, where $\beta(g)$ is the isomorphism corresponding to the linking between the sections $({}^gA, {}^gB)$ and (C, D). In particular, the sections (C, D) and $({}^gA, {}^gB)$ are linked. If $h \in G$ does not belong to one of these cosets, the section

$({}^hA, {}^hB)$ cannot be linked to (C, D). Otherwise we would have another non-zero module of the form $\mathrm{Btf}(C, D, {}^hA, {}^hB)\,\mathrm{Conj}_h \mathrm{Iso}_\phi(M)$.

Finally one proves *3*. By Krull-Schmidt Theorem write M as

$$a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)}),$$

where the M_{jr_j}'s are indecomposable and pairwise non-isomorphic, $f(j)$ is an integer depending on j and $a_j < a_{j+1}$ for all j. Using the second part of Theorem 2.11 and the fact that $n = d = |\mathscr{M}|$, one has

$$nM \cong \bigoplus_{g \in \mathscr{M}} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_\phi(M) = \bigoplus_{i=1}^{n} \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M),$$

for some $g_1, \ldots g_n$ in $\mathscr{M}$. Using the decomposition of M one obtains

$$\begin{aligned}
nM &\cong na_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus na_k(M_{k1} \oplus \cdots \oplus M_{kf(k)}) \\
&\cong \bigoplus_{i=1}^{n} \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M) \\
&\cong \mathrm{Iso}_{\beta(g_1)c_{g_1}\phi}\big(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\big) \\
&\oplus \mathrm{Iso}_{\beta(g_2)c_{g_2}\phi}\big(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\big) \\
&\vdots \\
&\oplus \mathrm{Iso}_{\beta(g_n)c_{g_n}\phi}\big(a_1(M_{11} \oplus \cdots \oplus M_{1f(1)}) \oplus \cdots \oplus a_k(M_{k1} \oplus \cdots \oplus M_{kf(k)})\big).
\end{aligned}$$

Note that M_{11} appears in the decomposition of $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$ for all $i = 1, \ldots, n$. Indeed, $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}$ sends an indecomposable module to an indecomposable module and if $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_1r_{j_1}}) \cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_2r_{j_2}})$ then $M_{j_1r_{j_1}} \cong M_{j_2r_{j_2}}$ by applying $\mathrm{Iso}_{(\beta(g_i)c_{g_i}\phi)^{-1}}$ on both sides. As the M_{jr_j} are all pairwise non-isomorphic this means that there is the same number of indecomposable modules in M than in $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$ and that the indecomposable modules in the decomposition are the same. Denote by m_i the multiplicity of M_{11} in $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M)$, then $m_i \geq a_1$ for all $i = 1, \ldots, n$ as for all i the module M_{11} corresponds to $\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M_{j_ir_i})$ for some $M_{j_ir_i}$ which appears $a_{j_i} \geq a_1$ for all i. Moreover, looking at the two decompositions of nM one has

$$\sum_{i=1}^{n} m_i = na_1$$

and so $m_i = a_1$ for all i. Applying this argument to all the modules M_{1r_1} one obtains that, for all i,

$$\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\big(a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\big)\cong a_1(M_{11}\oplus\cdots\oplus M_{1f(1)}).$$

Using this result, the same argument proves that

$$\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\big(a_2(M_{21}\oplus\cdots\oplus M_{2f(1)})\big)\cong a_2(M_{21}\oplus\cdots\oplus M_{2f(1)}).$$

Finally, continuing like this, one has, for all i

$$\begin{aligned}
\mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}(M) &\cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\Big(a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\oplus\dots\\
&\qquad \cdots\oplus a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\Big)\\
&\cong \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\big(a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\big)\oplus\dots\\
&\qquad \cdots\oplus \mathrm{Iso}_{\beta(g_i)c_{g_i}\phi}\big(a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\big)\\
&\cong a_1(M_{11}\oplus\cdots\oplus M_{1f(1)})\oplus\cdots\oplus a_k(M_{k1}\oplus\cdots\oplus M_{kf(k)})\\
&\cong M.
\end{aligned}$$

□

Corollary 2.13. *Let $U = \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi\,\mathrm{Defres}^G_{C/D}$ be a strongly minimal n-stabilizing biset for an indecomposable kG-module L. Then there exists a section $(\tilde{A},\tilde{B})$ linked to (C,D) by σ such that L is n-stabilized by*

$$\tilde{U} := \mathrm{Indinf}^G_{\tilde{A}/\tilde{B}}\,\mathrm{Iso}_\sigma\,\mathrm{Defres}^G_{C/D}\,.$$

Proof. Let $M = \mathrm{Defres}^G_{C/D}(L)$ and let CgA be one of the n double cosets as given by Theorem 2.12. Let $(\tilde{A},\tilde{B}) = ({}^gA, {}^gB)$ and σ the linking isomorphism. One knows, by the third part of Theorem 2.12, that M is invariant under $\sigma^{-1}c_g\phi$, therefore one has

$$\begin{aligned}
\tilde{U}(L) &\cong \mathrm{Indinf}^G_{\tilde{A}/\tilde{B}}\,\mathrm{Iso}_\sigma(M)\cong \mathrm{Indinf}^G_{\tilde{A}/\tilde{B}}\,\mathrm{Iso}_\sigma\,\mathrm{Iso}_{\sigma^{-1}c_g\phi}(M)\\
&\cong \mathrm{Indinf}^G_{{}^gA/{}^gB}\,\mathrm{Iso}_{c_g\phi}(M)\cong \mathrm{Indinf}^G_{A/B}\,\mathrm{Iso}_\phi(M) = U(L)\cong nL.
\end{aligned}$$

□

Remark 2.14. If $n = 1$, it is sufficient to suppose in the above Corollary that U is minimal. See Corollary 3.5 of [3].

Proposition 2.15. *Let $U := \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$ be a minimal biset n-stabilizing a module L and let $M := \operatorname{Defres}_{C/D}^G(L)$. Then M is a faithful module.*

Proof. Let N/D be the kernel of the action of C/D on M. Then

$$M \cong \operatorname{Inf}_{C/N}^{C/D} \operatorname{Def}_{C/N}^{C/D}(M)$$

and therefore L is n-stabilized by

$$\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Inf}_{C/N}^{C/D} \operatorname{Def}_{C/N}^{C/D} \operatorname{Defres}_{C/D}^G \cong \operatorname{Indinf}_\star^G \operatorname{Iso}_\star \operatorname{Defres}_{C/N}^G .$$

By minimality of U, one must have $|C/D| = |C/N|$ and so $N = D$. $\square$

Proposition 2.16. *Let $U = \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$ be a (G,G)-biset n-stabilizing a simple kG-module L and let $M = \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L)$. If M is the trivial $k[A/B]$-module then $n = 1$, the kG-module L is trivial and $A = G$.*

Proof. If M is the trivial module then $nL \cong \operatorname{Ind}_A^G(k)$. Since the trivial kG-module is always a submodule of $\operatorname{Ind}_A^G(k)$, this module can only be the sum of n copies of a simple module if it is a trivial module. Indeed, nL does not have a trivial submodule except if it is trivial. But then L is a trivial module too. As $\operatorname{Ind}_A^G(k)$ is isomorphic to $k[G/A]$, this module is trivial only if $A = G$ and this implies that $n = 1$ as $\operatorname{Ind}_A^G(k) \cong k$. $\square$

Definition 2.17. Let G be a group and B a subgroup of G. *The G-core* of B is the largest normal subgroup of G contained in B, that is, the intersection of all the G-conjugates of B.

Proposition 2.18. *Let G be a group and L a faithful $k[G]$-module such that L is n-stabilized by $\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$. Then the G-core of B is trivial.*

Proof. Let M be the module $\operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L)$, so nL is $\operatorname{Indinf}_{A/B}^G(M)$, which has the following kernel $\cap_{g\in G} {}^g\operatorname{Ker}(\operatorname{Inf}_{A/B}^A(M))$. Obviously B is contained in $\operatorname{Ker}(\operatorname{Inf}_{A/B}^A(M))$ and so $\cap_{g\in G} {}^gB$ is contained in $\cap_{g\in G} {}^g\operatorname{Ker}(\operatorname{Inf}_{A/B}^A(M))$. As nL is faithful, the latter is trivial and so is the G-core of B. $\square$

Proposition 2.19. *Let G be a group and L a faithful simple $k[G]$-module such that L is n-stabilized by $\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$. Then the G-core of D is trivial.*

Proof. Let N be the G-core of D. It is a normal subgroup of G contained in D. One has

$$\mathrm{Defres}^{G/N}_{C/D}\,\mathrm{Def}^{G}_{G/N}(L) = \mathrm{Defres}^{G}_{C/D}(L) \neq 0$$

and thus $\mathrm{Def}^{G}_{G/N}(L) \neq 0$. But $\mathrm{Def}^{G}_{G/N}(L)$ is a quotient of L and N acts trivially on it. Since L is simple and faithful one must have $N = 1$. □

Proposition 2.20. *Let k be a field and let $U = \mathrm{Indinf}^{G}_{A/B}\,\mathrm{Iso}_{\phi}\,\mathrm{Defres}^{G}_{C/D}$ be a biset n-stabilizing a simple kG-module L. Then $n|A| \geq |N_G(D)|$ and in particular $n|A| \geq |C|$.*

Proof. By definition of the deflation map we have a surjective homomorphism

$$\psi : \mathrm{Res}^{G}_{N_G(D)}(L) \to \mathrm{Defres}^{G}_{N_G(D)/D}(L),$$

where $\mathrm{Defres}^{G}_{N_G(D)/D}(L)$ is viewed as a module for $N_G(D)$ by inflation. It follows that there is a non-zero homomorphism of kG-modules

$$\tilde{\psi} : L \to \mathrm{Ind}^{G}_{N_G(D)}\big(\mathrm{Defres}^{G}_{N_G(D)/D}(L)\big).$$

This is injective by simplicity of L and so

$$\dim L \leq |G : N_G(D)| \dim \mathrm{Defres}^{G}_{N_G(D)/D}(L).$$

By Lemma 2.4, one has $n \dim L = |G : A| \dim \mathrm{Defres}^{G}_{C/D}(L)$. Moreover, $\dim \mathrm{Defres}^{G}_{N_G(D)/D}(L)$ is equal to $\dim \mathrm{Defres}^{G}_{C/D}(L)$ as it only depends on the action of D on L. Therefore

$$\frac{|G : A| \dim \mathrm{Defres}^{G}_{N_G(D)/D}(L)}{n} \leq |G : N_G(D)| \dim \mathrm{Defres}^{G}_{N_G(D)/D}(L)$$

and the result follows.

□

2.2 n-stabilizing bisets and strong minimality

In this section one treats the question of strong minimality and existence of strongly minimal n-stabilizing bisets. We treat the case $n = 1$ in the next chapter.

Proposition 2.21. *Let G be a finite group, U be a n_U-stabilizing biset of the form* $\mathrm{Indinf}_{A/B}^{G} V \,\mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}$ *for a kG-module L and V a strongly minimal n_V-stabilizing biset for* $M := \mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}(L)$. *Moreover suppose that M is indecomposable. Then U is strongly minimal.*

Proof. Set $V := \mathrm{Indinf}_{H/J}^{A/B} \mathrm{Iso}_\sigma \,\mathrm{Defres}_{S/T}^{A/B}$ and let W be a n_W-stabilizing biset for L. Set $W := \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \,\mathrm{Defres}_{C'/D'}^{G}$. We have to show that $|H/J| \leq |A'/B'|$. Using these settings, one has

$$\begin{aligned}\mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G} W \,\mathrm{Indinf}_{A/B}^{G} V(M) &\cong \mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G} W(n_U L)\\ &\cong n_U n_W \,\mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}(L)\\ &\cong n_U n_W M.\end{aligned}$$

Using generalized Mackey formula, the left hand side becomes

$$\oplus_{g,h} \mathrm{Iso}_\phi \,\mathrm{Btf}(C, D, {}^gA', {}^gB') \,\mathrm{Conj}_g \,\mathrm{Iso}_{\phi'} \,\mathrm{Btf}(C', D', {}^hH, {}^hJ) \,\mathrm{Conj}_h \,\mathrm{Iso}_\sigma \,\mathrm{Defres}_{S/T}^{A/B}(M),$$

where the sum is taken over $g \in [C\backslash G/A']$ and $h \in [C'\backslash G/H]$. Because M is indecomposable, this implies that for each summand there exists a certain $k_{g,h}$ such that

$$\mathrm{Iso}_\phi \,\mathrm{Btf}(C, D, {}^gA', {}^gB') \,\mathrm{Conj}_g \,\mathrm{Iso}_{\phi'} \,\mathrm{Btf}(C', D', {}^hH, {}^hJ) \,\mathrm{Conj}_h \,\mathrm{Iso}_\sigma \,\mathrm{Defres}_{S/T}^{A/B}(M) \cong k_{g,h} M.$$

Note that $k_{g,h}$ is not equal to zero for at least one pair (g, h). The biset V is strongly minimal therefore the biset $\mathrm{Btf}(C', D', {}^hH, {}^hJ)$ has to be reduced to

$$\mathrm{Indinf}_{(C'\cap {}^hH)D'/(C'\cap {}^hJ)D'}^{C'/D'} \mathrm{Iso}_\psi,$$

when $k_{g,h} \neq 0$, which means that $({}^hH, {}^hJ)$ is linked to a subsection of (C', D'). In particular $|H/J| \leq |C'/D'| = |A'/B'|$ which proves the strong minimality of U. □

Proposition 2.22. *Let G be a finite group,* $U := \mathrm{Indinf}_{A/B}^{G} V \,\mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}$ *a strongly minimal n_U-stabilizing biset for an indecomposable kG-module L where V n_V-stabilizes* $M := \mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}(L)$. *Then V is strongly minimal.*

Proof. Set $V := \mathrm{Indinf}_{H/J}^{A/B} \mathrm{Iso}_\sigma \,\mathrm{Defres}_{S/T}^{A/B}$ and let W be a n_W-stabilizing biset for M. Set $W := \mathrm{Indinf}_{H'/J'}^{A/B} \mathrm{Iso}_{\sigma'} \,\mathrm{Defres}_{S'/T'}^{A/B}$, then

$$\begin{aligned}\mathrm{Indinf}_{A/B}^{G} VW \,\mathrm{Iso}_\phi \,\mathrm{Defres}_{C/D}^{G}(L) &\cong \mathrm{Indinf}_{A/B}^{G} VW(M)\\ &\cong n_W \,\mathrm{Indinf}_{A/B}^{G} V(M)\\ &\cong n_W n_U L.\end{aligned}$$

Using Mackey formula, the first term on the left becomes

$$\oplus_g \operatorname{Indinf}^G_{H/J} \operatorname{Iso}_\sigma \operatorname{Btf}(S, T, {}^gH', {}^gJ') \operatorname{Conj}_g \operatorname{Iso}_{\sigma'} \operatorname{Defres}^{A/B}_{S'/T'} \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}(L) \cong n_U n_W L.$$

Because L is indecomposable, this implies that for each summand there exists a certain k_g such that

$$\operatorname{Indinf}^G_{H/J} \operatorname{Iso}_\sigma \operatorname{Btf}(S, T, {}^gH', {}^gJ') \operatorname{Conj}_g \operatorname{Iso}_{\sigma'} \operatorname{Defres}^{A/B}_{S'/T'} \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}(L) \cong k_g L,$$

and k_g is not zero for at least one g. By strongly minimality of U the biset $\operatorname{Btf}(S, T, {}^gH', {}^gJ')$ must, at least, be reduced to $\operatorname{Iso}_\psi \operatorname{Defres}^{{}^gH'/{}^gJ'}_{(S\cap {}^gH')\,{}^gJ'/(T\cap {}^gH')\,{}^gJ'}$ which means that (S, T) is linked to a subsection of $({}^gH', {}^gJ')$. In particular $|H/J| = |S/T| \leq |H'/J'|$ which proves the strongly minimality of V. □

Proposition 2.23. *Let G be a finite group, $U := \operatorname{Indinf}^G_{A/B} \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}$ and L a kG-module n_U-stabilized by U. Suppose $M := \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}(L)$ is indecomposable. Then there exists a biset V, n_V-stabilizing M, such that $W := \operatorname{Indinf}^G_{A/B} V \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}$ is strongly minimal for L. Moreover V is strongly minimal for M.*

Proof. One proves this by induction on $|G|$. If G is of order 1 then the trivial biset is strongly minimal. Now suppose the statement is true for groups of order less than $|G|$. If U is strongly minimal then $V = \mathrm{Id}$. Suppose U is not strongly minimal. Moreover suppose $|A/B| < |G|$ and apply the induction on the indecomposable module M with the identity as stabilizing biset. So one obtains a strongly minimal biset $V := \operatorname{Indinf}^{A/B}_\star \operatorname{Iso}_\star \operatorname{Defres}^{A/B}_\star$ such that $V(M) \cong n_V M$. By Proposition 2.21 the biset

$$W := \operatorname{Indinf}^G_{A/B} V \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}$$

is strongly minimal for L.

One needs to treat the case $|A/B| = |G|$. This implies that $U = \operatorname{Iso}_\phi$ but U is not strongly minimal by assumption, therefore there exists a proper biset V_1, i.e. not reduced to an isomorphism, such that $V_1(L) \cong n_{V_1} L$. Replacing U by V_1 in the argument of the first case, one obtains a strongly minimal n_V-stabilizing biset V for the module L and therefore $W = V \operatorname{Iso}_\phi$ is strongly minimal for L. □

Remark 2.24. Note that W is a $n_U n_V$-stabilizing biset for L and not simply a n_U-stabilizing biset.

2.3 n-idempotent bisets

This generalizes section 5 of [3] on idempotent bisets to n-idempotent bisets for $n > 1$. It gives also examples of idempotents in the double Burnside ring $\mathbb{Q}B(G, G)$. One gives here a complete classification of such bisets.

Definition 2.25. Let U be a (G, G)-biset, then U is *an n-idempotent biset* if $U^2 \cong nU$.

Theorem 2.26. *Let $U = \operatorname{Indinf}_{A/B}^{G} \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}$ be a (G, G)-biset. Then $U^2 \cong nU$ if and only if the following three conditions hold:*

1. *There are n (C, A)-double cosets.*
2. *The sections (C, D) and $({}^{g}A, {}^{g}B)$ are linked for all g.*
3. *For every $g \in G$, there exist $x \in N_G({}^{g}A, {}^{g}B)$ and $y \in N_G(C, D)$ such that*
$$\phi\beta(g)^{-1} \operatorname{Conj}_g \phi = \operatorname{Conj}_x \phi \operatorname{Conj}_y^{-1},$$
where $\beta(g) : C/D \to {}^{g}A/{}^{g}B$ is the isomorphism induced by the linking.

Proof. By Mackey formula, one has

$$U^2 \cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Indinf}_{A/B}^{G} \operatorname{Iso}_{\phi} \operatorname{Btf}(C, D, {}^{g}A, {}^{g}B) \operatorname{Conj}_g \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}.$$

Suppose $U^2 \cong nU$. Because on both sides it is the union of disjoint transitive bisets, one must have the same number of transitive bisets as one transitive biset goes to another via an isomorphism. So there are n (C, A)-double cosets. Moreover, for every $g \in G$, we have

$$U \cong \operatorname{Indinf}_{A/B}^{G} \operatorname{Iso}_{\phi} \operatorname{Btf}(C, D, {}^{g}A, {}^{g}B) \operatorname{Conj}_g \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}.$$

Now the argument used in the proof of Proposition 5.1 of [3] works. Indeed, the butterfly factorizes through a subsection of (C, D), which is a contradiction with the isomorphism with U unless the subsection is the whole of (C, D). Indeed, U can be uniquely written, up to conjugation, see Lemma 1.18. Therefore the sections have to be linked. We are left with

$$U \cong \operatorname{Indinf}_{A/B}^{G} \operatorname{Iso}_{\phi} \operatorname{Iso}_{\beta(g)^{-1}} \operatorname{Conj}_g \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}.$$

Again this is exactly the same situation as in Proposition 5.1 of [3]. Since two transitive bisets are isomorphic if and only if the corresponding stabilizers in $G \times G$ are conjugate, this isomorphism implies the existence of $(x, y) \in G \times G$ conjugating one stabilizer into the other. Here, x must normalize A and B and y must normalize C and D, while the isomorphism $\phi\beta(g)^{-1}\operatorname{Conj}_g \phi$ must differ from ϕ by the two conjugations Conj_x and $\operatorname{Conj}_y^{-1}$. So the third condition follows.

Conversely, assume $1), 2), 3)$ hold and compute U^2 with these three conditions. One has

$$\begin{aligned}
U^2 &\cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Btf}(C, D, {}^gA, {}^gB) \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G \\
&\cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Iso}_{\beta(g)^{-1}} \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G \\
&\cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_{\operatorname{Conj}_x \phi \operatorname{Conj}_y^{-1}} \operatorname{Defres}_{C/D}^G \\
&\cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Conj}_x \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G \operatorname{Conj}_y^{-1} \\
&\cong \bigsqcup_{g\in[C\backslash G/A]} \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G = nU
\end{aligned}$$

where the second isomorphism holds because of 2), the third by 3) and the last one by 1). □

Proposition 2.27. *Let U be an n-idempotent (G, G)-biset. For any kG-module L', the kG-module $L := U(L')$ is n-stabilized by U.*

Remark 2.28. Note that in general L need not be indecomposable.

Examples 2.29.

- An example can be found in A_5. Let U be $\operatorname{Indinf}_{D_{10}/C_5}^{A_5} \operatorname{Defres}_{D_{10}/C_5}^{A_5}$ where D_{10} denotes $\langle(1, 2, 3, 4, 5), (2, 5)(3, 4)\rangle$ and $C_5 = \langle(1, 2, 3, 4, 5)\rangle$. An easy calculation, which can be made by GAP, see [4], gives 2 double (D_{10}, D_{10})-cosets in A_5 and the section (D_{10}, C_5) is linked via the conjugation to its conjugate. By taking $x = 1$ and $y = 1$ in the last conditions of Theorem 2.26 one can see that U is a 2-idempotent biset.

- If A and B are normal subgroups of G and $U := \operatorname{Indinf}_{A/B}^G \operatorname{Defres}_{A/B}^G$ then U is $|G:A|$-idempotent. Indeed, one has $|G:A|$ (A,A)-double cosets. By normality the sections are trivially linked and by taking $x = y = 1$ the third condition is also fulfilled. This is the case, in particular, of Example 2.2.

Proposition 2.30. *Let G be a group. Let $U := \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$ be a minimal n-stabilizing biset for an indecomposable module L. Then, for all $g \in G$,*

$$\dim \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Btf}(C, D, {}^gA, {}^gB) \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L) = n \dim L$$

if and only if the following two conditions are fulfilled

1. *there are n (C, A)-double cosets,*

2. *the sections (C, D) and $({}^gA, {}^gB)$ are linked for all g.*

Proof. Suppose first that the dimensions are equal. Then because $U(L) \cong nL$ one has $U^2(L) = n^2L$. In other words

$$\bigoplus_{g \in [C\backslash G/A]} \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Btf}(C, D, {}^gA, {}^gB) \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L) \cong n^2L$$

and therefore, looking at the dimensions, the number of summands must be equal to n. Therefore there are n (C, A)-double cosets. Moreover, looking again at the hypothesis on the dimensions and using Krull-Schmidt Theorem, one concludes that

$$\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Btf}(C, D, {}^gA, {}^gB) \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L) \cong nL.$$

By minimality of U, the bisets $\operatorname{Btf}(C, D, {}^gA, {}^gB)$ have to be reduced to isomorphisms, which means that the sections are linked.

Conversely suppose 1) and 2). Then, because for all g in G the sections are linked via $\beta(g)$, one has

$$\begin{aligned}&\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Btf}(C, D, {}^gA, {}^gB) \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L)\\ &\cong \operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Iso}_{\beta(g)} \operatorname{Conj}_g \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G(L)\end{aligned}$$

which means that all these modules have the same dimension as these dimensions do not depend on g. But

$$\begin{aligned} n^2 L &\cong \bigoplus_{g \in [C\backslash G/A]} \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Btf}(C, D, {}^{g}\!A, {}^{g}\!B) \mathrm{Conj}_g \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L) \\ &\cong \bigoplus_{g \in [C\backslash G/A]} \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_g \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L) \end{aligned}$$

and therefore looking at the dimensions one has, for all $g \in G$,

$$\dim \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Btf}(C, D, {}^{g}\!A, {}^{g}\!B) \mathrm{Conj}_g \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L) = n \dim L.$$

□

Remark 2.31. Suppose that the linking between (C, D) and $({}^{g}\!A, {}^{g}\!B)$ is just the composition of the conjugation map by g and the linking between (C, D) and (A, B), then conditions 1) and 2) are equivalent to saying that $U^2 \cong nU$ as a biset, as one can take $x = 1$ and $y = 1$ in Theorem 2.26 to fulfil the third condition.

2.4 n-expansivity

In this section one introduces a type of subgroup called n-expansive. It will be a useful notion to find n-stabilizing bisets.

Definition 2.32. Let n be an integer. A subgroup T of a group G is called (S, n)-*expansive relatively to* (A, B) if

(i) The pairs (A, B) and (S, T) are sections of G.

(ii) The sections (A, B) and (S, T) are linked via ϕ.

(iii) The composition of ϕ with the conjugation map, $\phi \circ c_g$, links the sections (A^g, B^g) and (S, T) for exactly n elements g in $[A\backslash G/S]$. For the other elements g in $[A\backslash G/S]$ the S-core of the subgroup $(B^g \cap S)T$ contains T properly.

Remark 2.33.

- One will mainly use this notion with $S = N_G(T)$ and $(A, B) = (S, T)$. In this case the subgroup T is simply called n-expansive. If moreover $n = 1$ one says that T is expansive as defined in Chapter 6 of [3].
- By assumption (A, B) is linked to (S, T) and therefore the first part of the condition (iv) is fulfilled at least for $g = 1$ in $[A\backslash G/S]$.

Lemma 2.34. *Let (A, B) be a section of a finite group G. Let M be a faithful simple $k[A/B]$-module. Then $\mathrm{Def}^{A/B}_{A/N}(M) = \{0\}$ for any non-trivial normal subgroup N/B of A/B.*

Proof. Since M is simple and faithful, the largest quotient of M with trivial action of N/B must be zero and therefore $\mathrm{Def}^{A/B}_{A/N}(M) = \{0\}$. □

Proposition 2.35. *Let T be (S, n)-expansive relatively to (A, B). Let ϕ be the link between (A, B) and (S, T). Suppose that M is a faithful simple $k[A/B]$-module. Let $L := \mathrm{Indinf}^G_{S/T} \mathrm{Iso}_\phi(M)$. Then,*

(*i*) $\mathrm{Defres}^G_{A/B}(L) \cong nM$.

(*ii*) *The biset $U := \mathrm{Indinf}^G_{S/T} \mathrm{Iso}_\phi \mathrm{Defres}^G_{A/B}$ n-stabilizes L.*

(*iii*) *If $n = 1$ and $(S, T) = (A, B)$, the module L is indecomposable. In particular, if k is a field of characteristic prime to $|G|$, then L is simple if and only if M is simple.*

Proof. Let's decompose $\mathrm{Defres}^G_{A/B}(L)$ using generalized Mackey formula, see Proposition 1.22,

$$\begin{aligned}
\mathrm{Defres}^G_{A/B}(L) &= \mathrm{Defres}^G_{A/B} \mathrm{Indinf}^G_{S/T} \mathrm{Iso}_\phi(M) \\
&\cong \bigoplus_{x\in[A\backslash G/S]} \mathrm{Btf}(A, B, {}^xS, {}^xT) \mathrm{Conj}_x \mathrm{Iso}_\phi(M) \\
&\cong \bigoplus_{x\in[A\backslash G/S]} \mathrm{Conj}_x \mathrm{Btf}(A^x, B^x, S, T) \mathrm{Iso}_\phi(M).
\end{aligned}$$

Now one looks closely at $\mathrm{Btf}(A^x, B^x, S, T) \mathrm{Iso}_\phi(M)$. By definition one has

$$\mathrm{Btf}(A^x, B^x, S, T) = \mathrm{Indinf}^{A^x/B^x}_{(A^x\cap S)B^x/(A^x\cap T)B^x} \mathrm{Iso}_\psi \mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T}.$$

Since T is (S,n)-expansive the S-core N_x of the subgroup $(B^x \cap S)T$ contains T properly, except for exactly n elements x in $[A\backslash G/S]$. In other words, except for these n elements, N_x/T is a non-trivial subgroup of S/T contained in $(B^x \cap S)T$. As

$$\mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T} = \mathrm{Defres}^{S/N_x}_{(A^x\cap S)T/(B^x\cap S)T}\,\mathrm{Def}^{S/T}_{S/N_x}$$

one has, by Lemma 2.34 applied to $\mathrm{Iso}_\phi(M)$, that

$$\mathrm{Defres}^{S/T}_{(A^x\cap S)T/(B^x\cap S)T}\,\mathrm{Iso}_\phi(M) = \{0\}$$

for all x except n elements. Theses n elements have the property that the composition of ϕ with the conjugation map links the sections (A^x, B^x) and (S,T), which implies that

$$\mathrm{Conj}_x\,\mathrm{Btf}(A^x, B^x, S, T)\,\mathrm{Iso}_\phi(M) \cong M.$$

As this occurs exactly n times, one concludes that

$$\mathrm{Defres}^{G}_{A/B}(L) \cong nM.$$

The second point follows from the first and the definition of L. Finally the last point has been proved in Proposition 6.2 of [3] on page 1624. $\square$

Examples 2.36. Here is an example of n-expansivity in S_6.

- First, consider $T := \langle(1,2,3)\rangle \times \langle(4,5,6),(5,6)\rangle$ which is isomorphic to $C_3 \times S_3$. Its normalizer S is $T \rtimes \langle(2,3)(4,6)\rangle$. There are four (S,S)-double cosets in S_6. Here is a list of representatives:

  $$\{\mathrm{id}, (3,4), (2,4)(3,5), (1,4)(2,5)(3,6)\}.$$

  The first two elements satisfy the first part of (iv) in definition 2.32 and the last two elements satisfy the second part of the definition. Therefore T is an example of a 2-expansive subgroup in S_6. Setting M to be the sign representation of S/T one obtains an example of a 2-stabilizing biset. However the module $L := \mathrm{Indinf}^{S_6}_{S/T}(M)$ is not an indecomposable module for S_6 over $\mathbb{C}$.

- Now consider $T := \langle(5,6)\rangle \times \langle(1,2)(3,4),(1,3)(2,4),(2,3,4)\rangle$ which is isomorphic to $C_2 \times A_4$. Its normalizer S is $T \rtimes \langle(3,4)\rangle$. There are three (S,S)-double cosets in S_6. Here is a list of representatives:

$$\{\mathrm{id},(4,5),(3,5)(4,6)\}.$$

  The second one satisfies the second part of definition 2.32 and the two others the first part. Therefore T is another example of a 2-expansive subgroup in S_6. Again, setting M to be the sign representation of S/T one obtains an example of a 2-stabilizing biset but the module $L := \mathrm{Indinf}_{S/T}^{S_6}(M)$ is not indecomposable over $\mathbb{C}$.

Remark 2.37. This definition of n-expansivity only involves conditions on subgroups but rises to examples of n-stabilizing bisets. However, one could not assure that the module L given in Proposition 2.35 is indecomposable except for $n = 1$. Another way to define the notion is the following. Let n be an integer. Let M be a faithful indecomposable $k[A/B]$-module and $L := \mathrm{Indinf}_{S/T}^{G} \mathrm{Iso}_\phi(M)$. The conditions on the pairs (S,T) and (A,B) would be that

(i) The pairs (A,B) and (S,T) are sections of G.

(ii) The sections (A,B) and (S,T) are linked via ϕ.

(iii) The sections (A^g,B^g) are linked via $\beta(g)$ with (S,T) for exactly n elements g in $[A\backslash G/S]$. Moreover $\mathrm{Iso}_{\beta(g)\circ c_g\circ\phi}(M)$ is not isomorphic to $\mathrm{Iso}_\phi M$ but $\mathrm{Indinf}_{S/T}^{G} \mathrm{Iso}_\phi(M)$ is isomorphic to $\mathrm{Indinf}_{S/T}^{G} \mathrm{Iso}_{c_g\circ\beta(g)\circ\phi}(M)$ for all of these g. For the other elements g in $[A\backslash G/S]$ the S-core of the subgroup $(B^g \cap S)T$ contains T properly.

With this definition, we would not have the property that $\mathrm{Defres}_{A/B}^{G}(L) \cong nM$ but $U := \mathrm{Indinf}_{S/T}^{G} \mathrm{Iso}_\phi \mathrm{Defres}_{A/B}^{G}$ would be a n-stabilizing biset for L. The module L would be indecomposable under certain additional assumptions, for example if $k = \mathbb{C}$, $(A,B) = (C,D)$ and $\phi = \mathrm{id}$ as noticed in Remark 2.3. Nevertheless this is not a definition involving only group properties. In our approach, we try to generalize the group theory notion of expansivity, that was introduced in [3] to obtain examples of stabilizing bisets, to a group theory notion of n-expansivity to obtain n-stabilizing bisets.

Chapter 3

Stabilizing Bisets

In the previous chapter, one has seen general theorems about n-stabilization. In this chapter one focus on the case $n = 1$. Therefore, as Theorem 3.3 shows, one can obtain sometimes the same results but with weaker hypothesis. One also has results that are not yet generalized as for example Theorem 3.5. One ends this chapter with a study of the minimality for stabilizing bisets. But first, one starts this chapter with some useful definitions.

Definitions 3.1.

(i) A finite group G is called a *Roquette group* if all its normal abelian subgroups are cyclic. In other words, for any prime p, any normal elementary abelian p-subgroup of G has order 1 or p.

(ii) A subgroup T of a finite group G is called a *genetic subgroup* if T is an expansive subgroup of G and $N_G(T)/T$ is a Roquette group.

Remark 3.2. Note that definition 2.32 introduced the notion of expansive subgroup in a more general setting. Recall a subgroup T of G is called *expansive* in G if, for every $g \notin N_G(T)$, the $N_G(T)$-core of the subgroup $({}^{g}T \cap N_G(T))T$ contains T properly.

3.1 Stabilizing bisets

In this section one highlights the important results of [3]. One refers to this article for the proofs.

Theorem 3.3. *Theorem* 3.3 *page* 1616*, [3]*

Consider two transitive (G, G)*-bisets*

$$U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G} \text{ and } U' = \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{C'/D'}^{G}.$$

Let L *be an indecomposable* kG*-module such that* $U(L) \cong L$ *and* $U'(L) \cong L$*. Let* $M = \mathrm{Defres}_{C/D}^{G}(L)$ *and* $M' = \mathrm{Defres}_{C'/D'}^{G}(L)$*.*

1. *Then, there exists a unique double coset* CgA *such that*

$$\mathrm{Btf}(C', D', {}^{g}A, {}^{g}B) \,\mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M) \neq \{0\}.$$

2. *Suppose that* U *is minimal. Let* g *belong to the unique double coset of part* (1)*. Then*

   (i) *the subsection* $((C' \cap {}^{g}A)D', (C' \cap {}^{g}B)D')$ *is linked to the section* $({}^{g}A, {}^{g}B)$*.*

   (ii) *The biset* $\mathrm{Btf}(C, D, {}^{g}A, {}^{g}B)$ *is reduced to*

$$\mathrm{Indinf}_{(C' \cap {}^{g}A)D'/(C' \cap {}^{g}B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)},$$

   where $\beta(g)$ *is the isomorphism corresponding to the linking of (i).*

   (iii) $M' \cong \mathrm{Indinf}_{(C' \cap {}^{g}A)D'/(C' \cap {}^{g}B)D'}^{C'/D'} \mathrm{Iso}_{\beta(g)} \mathrm{Conj}_{g} \mathrm{Iso}_{\phi}(M)$*.*

   (iv) *If* $h \in G$ *does not belong to one of these cosets, the section* $({}^{h}A, {}^{h}B)$ *is not linked to a subsection of* (C', D')*.*

3. *Suppose that* U *and* U' *are both minimal bisets. Let* g *belong to the unique double coset of part* (1)*. Then:*

   (i) *the section* $({}^{g}A, {}^{g}B)$ *is linked to* (C', D')*.*

   (ii) *The biset* $\mathrm{Btf}(C, D, {}^{g}A, {}^{g}B)$ *is reduced to* $\mathrm{Iso}_{\beta(g)}$*, where* $\beta(g)$ *is the isomorphism corresponding to the linking between the sections* $({}^{g}A, {}^{g}B)$ *and* (C', D')*.*

   (iii) $M' \cong \mathrm{Iso}_{\beta(g)c_{g}\phi}(M)$*.*

   (iv) *If* $h \in G$ *does not belong to one of these cosets, the section* $({}^{h}A, {}^{h}B)$ *is not linked to* (C', D')*.*

Corollary 3.4. *Corollary* 3.4 *page* 1619, *[3]*

Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ *be a minimal stabilizing biset for an indecomposable* kG*-module* L*. Let* $M = \mathrm{Defres}_{C/D}^{G}(L)$*. Then, there exists a unique double coset* CgA *such that*

1. $\mathrm{Btf}(C, D, {}^{g}A, {}^{g}B) \, \mathrm{Conj}_g \mathrm{Iso}_{\phi}(M) \neq \{0\}$,
2. *the sections* (C, D) *and* $({}^{g}A, {}^{g}B)$ *are linked,*
3. *the module* M *is invariant under* $\beta(g)c_g\phi$ *where* $\beta(g)$ *is the isomorphism corresponding to the linking between the sections* (C, D) *and* $({}^{g}A, {}^{g}B)$,
4. *if* $h \in G$ *does not belong to the same double coset as* g*, the section* $({}^{h}A, {}^{h}B)$ *is not linked to* (C, D)*.*

Theorem 3.5. *Theorem* 7.3 *page* 1626, *[3]*

Let k *be a field and let* G *be a finite group. If* L *is a simple* kG*-module, then there exist a genetic subgroup* T *of* G *and a faithful simple* $k[N_G(T)/T]$*-module* M *such that*

$$L \cong \mathrm{Indinf}_{N_G(T)/T}^{G}(M).$$

One also has $M \cong \mathrm{Defres}_{N_G(T)/T}^{G}(L)$*, so that* L *is stabilized by the biset*

$$U = \mathrm{Indinf}_{N_G(T)/T}^{G} \mathrm{Defres}_{N_G(T)/T}^{G}.$$

Moreover $\mathrm{End}_{kG}(L) \cong \mathrm{End}_{k[N_G(T)/T]}(M)$ *as* k*-algebras.*

Remark 3.6. This theorem proves the existence of stabilizing bisets for simple modules. It is possible that this biset is trivial, i.e. it is reduced to an isomorphism. This could only be the case if G is Roquette as one should have $(G, 1) = (N_G(T), T)$ and $N_G(T)/T$ is Roquette by assumption. Therefore the only case to treat is the case of Roquette groups.

This raises the question of proving the existence, or non-existence, of non-trivial stabilizing bisets for Roquette groups. To do so, one will use two approaches. The first one is to improve the theorem and find some genetic subgroups in Roquette groups, and then investigate whether it gives us non-trivial stabilizing bisets. The second one is to find stabilizing bisets for Roquette groups without the use of genetic subgroups. The first approach will be presented for certain types of Roquette groups in Chapter 5. Then the second will be presented for certain types of Roquette groups in Chapter 6.

Finally, remark that if L is not faithful then $L \cong \mathrm{Inf}_{G/\ker L}^{G} \mathrm{Def}_{G/\ker L}^{G}(L)$ and so one can find non-trivial stabilizing bisets also if G is Roquette. This is the reason why our focus will be on stabilizing bisets for faithful modules and thus, by Proposition 2.18, on the study of expansive subgroups with trivial G-core.

Corollary 3.7. *Let G be a finite group. Suppose there exists a faithful simple module but G is without non-trivial expansive subgroups with trivial G-core, then G is Roquette.*

Proof. This is equivalent to proving that a non-Roquette group has non-trivial expansive subgroups with trivial G-core, by Theorem 3.5 this is the case as there exists a faitful simple module for G. □

Corollary 3.8. *Corollary* 7.8 *page* 1630*, [3]*

Let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ be a minimal (G,G)-biset stabilizing a simple kG-module L. Then C/D is a Roquette group.

Proposition 3.9. *Proposition* 8.4 *page* 1632*, [3]*

Let k be a field and let $U = \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ be a minimal biset stabilizing a simple kG-module L. If B is a normal subgroup of G, then $A = G$. In other words, A is the normalizer of B in G.

Theorem 3.10. *Theorem* 9.2 *page* 1633*, [3]*

Let P be a Roquette p-group, where p is a prime number. Let k be a field and L be a simple faithful kP-module. If L is stabilized by a biset $U = \mathrm{Indinf}_{A/B}^{P} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{P}$, then $(A,B) = (C,D) = (P,1)$.

Remark 3.11. This theorem indicates us that one may not be able to find stabilizing bisets for Roquette groups. In other words, we will be more tempted to prove the non-existence than the existence of stabilizing bisets for Roquette groups.

3.2 Stabilizing bisets and minimality

In this section one treats the question of minimality and existence of minimal stabilizing bisets. These are the analogue results as in section 2.2 for the case $n = 1$, except the assumptions are weaker. Indeed, we only need minimality and not strong minimality.

Moreover, Proposition 3.14 completes a missing argument in the early version of the proof of Theorem 9.3 of [3].

Proposition 3.12. *Let G be a finite group, $U := \operatorname{Indinf}_{A/B}^{G} V \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}$ a stabilizing biset for the indecomposable kG-module L and V a minimal stabilizing biset for $M := \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}(L)$, then U is minimal.*

Proof. Set $V := \operatorname{Indinf}_{H/J}^{A/B} \operatorname{Iso}_{\sigma} \operatorname{Defres}_{S/T}^{A/B}$ and let U' be a stabilizing biset for L. Set $U' := \operatorname{Indinf}_{A'/B'}^{G} \operatorname{Iso}_{\phi'} \operatorname{Defres}_{C'/D'}^{G}$ then

$$\begin{aligned}
\operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G} U' \operatorname{Indinf}_{A/B}^{G} V(M) &\cong \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G} U' \operatorname{Indinf}_{A/B}^{G}(M) \\
&\cong \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G} U'(L) \\
&\cong \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}(L) \\
&\cong M.
\end{aligned}$$

Because L is indecomposable, M is indecomposable. Using this fact and Mackey formula, the first biset becomes, for some $g \in [C\backslash G/A']$ and some $h \in [C'\backslash G/H]$

$$\operatorname{Iso}_{\phi} \operatorname{Btf}(C, D, {}^{g}A', {}^{g}B') \operatorname{Conj}_{g} \operatorname{Iso}_{\phi'} \operatorname{Btf}(C', D', {}^{h}H, {}^{h}J) \operatorname{Conj}_{h} \operatorname{Iso}_{\sigma} \operatorname{Defres}_{S/T}^{A/B}.$$

As a stabilizing biset for M one has, by minimality of V, that the biset $\operatorname{Btf}(C', D', {}^{h}H, {}^{h}J)$ has to be reduced to

$$\operatorname{Indinf}_{(C'\cap {}^{h}H)D'/(C'\cap {}^{h}J)D'}^{C'/D'} \operatorname{Iso}_{\psi},$$

which means that $({}^{h}H, {}^{h}J)$ is linked to a subsection of (C', D'). In particular $|H/J| \leq |C'/D'| = |A'/B'|$ which proves the minimality of U. □

Proposition 3.13. *Let G be a finite group, $U := \operatorname{Indinf}_{A/B}^{G} V \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}$ a minimal stabilizing biset for the indecomposable kG-module L where V stabilizes $M := \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}(L)$. Then V is minimal.*

Proof. Let $V := \operatorname{Indinf}_{H/J}^{A/B} \operatorname{Iso}_{\sigma} \operatorname{Defres}_{S/T}^{A/B}$ and $V' := \operatorname{Indinf}_{H'/J'}^{A/B} \operatorname{Iso}_{\sigma'} \operatorname{Defres}_{S'/T'}^{A/B}$ be a stabilizing biset for M. Then

$$\begin{aligned}
\operatorname{Indinf}_{A/B}^{G} VV' \operatorname{Iso}_{\phi} \operatorname{Defres}_{C/D}^{G}(L) &\cong \operatorname{Indinf}_{A/B}^{G} VV'(M) \\
&\cong \operatorname{Indinf}_{A/B}^{G} V(M) \\
&\cong L.
\end{aligned}$$

Using this fact and generalized Mackey formula, one has for some $g \in [S\backslash G/H']$,

$$\mathrm{Indinf}_{H/J}^{G} \mathrm{Iso}_{\sigma} \mathrm{Btf}(S, T, {}^{g}H', {}^{g}J') \mathrm{Conj}_{g} \mathrm{Iso}_{\sigma'} \mathrm{Defres}_{S'/T'}^{A/B} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L) \cong L.$$

By minimality of U the biset $\mathrm{Btf}(S, T, {}^{g}H', {}^{g}J')$ must, at least, be reduced to $\mathrm{Iso}_{\psi} \mathrm{Defres}_{(S\cap {}^{g}H')\,{}^{g}J'/(T\cap {}^{g}H')\,{}^{g}J'}^{{}^{g}H'/{}^{g}J'}$ which means that (S, T) is linked to a subsection of $({}^{g}H', {}^{g}J')$. In particular $|H/J| = |S/T| \leq |H'/J'|$ which proves the minimality of V. □

Proposition 3.14. *Let G be a finite group, $U := \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ and L an indecomposable kG-module stabilized by U. Then there exists a biset V such that $U' := \mathrm{Indinf}_{A/B}^{G} V \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ is minimal for L. Moreover V is minimal for $M := \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}(L)$.*

Proof. One proves this by induction on $|G|$. If G is of order 1 then the trivial biset is minimal. Now suppose the statement is true for groups of order less than $|G|$. If U is minimal then $V = \mathrm{Id}$. Suppose U is not minimal. Moreover suppose $|A/B| < |G|$ and apply induction to the indecomposable module M with the identity as stabilizing biset. So one obtains a minimal biset $V := \mathrm{Indinf}_{\star}^{A/B} \mathrm{Iso}_{\star} \mathrm{Defres}_{\star}^{A/B}$ such that $V(M) \cong M$. By Proposition 3.12 the biset

$$U' := \mathrm{Indinf}_{A/B}^{G} V \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$$

is minimal for L.

Finally, one needs to treat the case $|A/B| = |G|$. This implies that $U = \mathrm{Iso}_{\phi}$ but U is not minimal by assumption, therefore there exists a proper biset V_1 such that $V_1(L) \cong L$. Replacing U by V_1 in the argument of the first case, one obtains a minimal stabilizing biset V for the module L and therefore $U' = V \mathrm{Iso}_{\phi}$ is minimal for L. □

Proposition 3.15. *Let L be a faithful simple $k[G]$-module. Suppose that whenever $U(L) \cong L$ for U a minimal biset then U is reduced to an isomorphism. Then, for an arbitrary biset $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ stabilizing L one has $(A, B) = (C, D) = (G, 1)$.*

Proof. By proposition 3.14 there exist subgroups H and J with J a normal subgroup of H and with $B \leq H \leq A$ and $B \leq J \leq A$ such that

$$\mathrm{Indinf}_{A/B}^{G} \mathrm{Indinf}_{H/J}^{A/B} \mathrm{Iso}_{\sigma} \mathrm{Defres}_{S/T}^{A/B} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G} \cong \mathrm{Indinf}_{H/J}^{G} \mathrm{Iso}_{\sigma\phi} \mathrm{Defres}_{\phi^{-1}(S/T)}^{G}$$

is minimal for L. As a minimal stabilizing bisets one has, by hypothesis, that $J = 1$ and $H = G$ and so in particular $B = 1$ and $A = G$. $\square$

Chapter 4

On Roquette Groups

As seen in the previous chapter, we are going to concentrate our attention on stabilizing bisets for Roquette groups. For this reason, one gives a brief description of Roquette groups. One finishes this chapter with a useful way to obtain Roquette groups as extension of known groups.

Definition 4.1.

(i) For a prime p, the *p-core of a finite group* G, denoted $O_p(G)$, is defined to be its largest normal p-subgroup.

(ii) *The Fitting subgroup of* G, denoted $F(G)$, is the product of the normal subgroups $O_p(G)$ for all primes p.

(iii) *The generalized Fitting subgroup of* G, denoted $F^\star(G)$, is the product of $F(G)$ and all quasisimple subnormal subgroups of G.

Definition 4.2. Let G be a group and $(G_i, 1 \leq i \leq m)$ a family of subgroups of G for some integer m. Then G is said to be a *central product* of the groups G_i, if

(i) $G = \langle G_i \mid 1 \leq i \leq m \rangle$,

(ii) $[G_i, G_j] = 1$ for $i \neq j$.

In this case, we will write $G = G_1 \circ \cdots \circ G_m$.

Remarks 4.3.

- This definition implies in fact that all G_i are normal subgroups of G. Indeed, let $x \in G_k$ and $y \in G$. By the first and the second conditions $y = y_1 \dots y_m$ with $y_i \in G_i$ for all i and by the second one we have $[x, y] = [x, y_1 \dots y_m] = [x, y_k] \in G_k$. Therefore we have shown that $[G_k, G] \leq G_k$ which is equivalent to the statement.

- Moreover, we have $G_i \cap G_j \subseteq Z(G)$ for $i \neq j$. Effectively, let x be an element of $G_i \cap G_j$ and $y = y_1 \dots y_m$ an arbitrary element of G as in the first point. Then $[x, y] = [x, y_1 \dots y_m] = [x, y_i]$ because $x \in G_i$, but x is also in G_j so $[x, y_i] = 1$, which shows that $G_i \cap G_j \subseteq Z(G)$.

- Let G_1 and G_2 be groups and let $Z(G_1)$, respectively $Z(G_2)$, be the center of G_1, respectively G_2. Suppose that the two subgroups $Z(G_1)$ and $Z(G_2)$ are isomorphic. Given an isomorphism $\theta : Z(G_1) \to Z(G_2)$ we construct a central product $G_1 \circ G_2 := (G_1 \times G_2)/N$, where N is the normal subgroup generated by the elements $\{(z, \theta(z^{-1})) \mid z \in Z(G_1)\}$.

Let G be a Roquette group and denote by $F(G)$ the Fitting subgroup of G, which is the product of the normal subgroups $O_p(G)$ for all prime numbers p dividing the order of G. As G is Roquette each $O_p(G)$ does not contain a characteristic abelian subgroup which is not cyclic. By theorem 4.9 of [5] on page 198, such groups are known. More precisely, each subgroup $O_p(G)$ is the central product of an extraspecial group (possibly trivial) with a Roquette p-group. Such a group is called *quasi-Roquette*. This description helps us to give a characterization of those groups.

Proposition 4.4. *A finite group G is Roquette if and only if the Fitting subgroup is the direct product of quasi-Roquette groups and the action of G on $F(G)$ does not fix an elementary abelian normal subgroup of rank at least* 2.

Proof. If G is Roquette, then it is straightforward that the Fitting subgroup is the direct product of quasi-Roquette groups and the action of G on $F(G)$ does not fix an elementary abelian normal subgroup of rank at least 2.

Suppose the Fitting subgroup is the direct product of quasi-Roquette groups and the action of G on $F(G)$ does not fix an elementary abelian normal subgroup of rank at least 2. Let N be a normal elementary abelian p-subgroup of G. One has to prove that N is actually cyclic. Remark that

N is contained in $O_p(G)$ and so $N = N \cap O_p(G) \leq N \cap F(G) \leq F(G)$. Therefore N is a normal subgroup of $F(G)$ fixed by the action of G, as N is a normal subgroup of G. By assumption N has to be cyclic. $\square$

One also presents a way to have Roquette groups as an extension of known groups. As one has described a Roquette group with its Fitting subgroup, one continues in this fashion.

Suppose G is solvable. It is a well known fact that in this case $C_G(F(G)) \leq F(G)$. Therefore $G/F(G)$ injects into $\mathrm{Out}(F(G))$. Indeed consider the map $\phi : G \to \mathrm{Out}(F(G))$ sending an element g to the class of the conjugation map c_g. Then $\ker\phi = \{g \in G \mid c_g \in \mathrm{Inn}(F(G))\}$ which means that if $g \in \ker\phi$ there exists $h \in F(G)$ such that $c_g = c_h$. This is equivalent to saying that ${}^{h^{-1}g}x = x$ for all $x \in F(G)$, in other words $h^{-1}g$ belongs to $C_G(F(G))$ and therefore g to $F(G)C_G(F(G)) = F(G)$. This shows that $\ker\phi \leq F(G)$ and the other inclusion is trivial. So one has the following exact sequence

$$1 \longrightarrow F(G) \longrightarrow G \longrightarrow S \longrightarrow 1$$

where S is a subgroup of $\mathrm{Out}(F(G))$.

In the case where G is not solvable one has to replace $F(G)$ by the generalized Fitting group $F^\star(G)$ to have $C_G(F^\star(G)) \leq F^\star(G)$ and therefore $G/F^\star(G)$ injects into $\mathrm{Out}(F^\star(G))$.

Chapter 5

Expansivity and Roquette Groups

This chapter is motivated by Remark 3.6. Indeed in the previous chapters we saw that we have to focus our interest on stabilizing bisets for Roquette groups. As one way to obtain stabilizing bisets uses expansive subgroups, see Proposition 2.35, one looks at expansive subgroups in Roquette groups.

One separates our study, motivated by Chapter 4, in four types of Roquette groups.

- Roquette p-groups.
- Some simple groups.
- Groups with cyclic Fitting subgroup.
- Groups with extraspecial groups in the Fitting subgroup.

Recall that a subgroup T of G is called *expansive* in G if, for every $g \notin N_G(T)$, the $N_G(T)$-core of the subgroup $\big({}^gT \cap N_G(T)\big)T$ contains T properly. Motivated by the result of Theorem 3.10, the general idea is to prove that for the majority of these groups G there is no non-trivial expansive subgroup with trivial G-core. To do so, for an arbitrary subgroup T of G with trivial G-core one finds a specific element g in G which is not in $N_G(T)$ such that $({}^gT \cap N_G(T))$ is contained in T. Thus, the $N_G(T)$-core of the subgroup $({}^gT \cap N_G(T))T$ is T and so T is not expansive. Remark also that this is equivalent to proving that T is not expansive or ${}^hT$ is not expansive for a $h \in G$.

We also discuss, if possible, the case of n-expansivity for $n > 1$.

Note that expansive subgroups and more precisely genetic subgroups appear in the study of biset functors given in [2]. They are used to define rational biset functors. This kind of biset functors has a wide variety of applications such as units of Burnside Rings, the Dade Group and the kernel of the linearization morphism. Although there is a classification of Roquette p-groups (see Chapter 5, Section 4 of [5]), there are not many results on the existence of expansive or genetic subgroups. Therefore this chapter has interest beyond the scope of this thesis.

5.1 Roquette p-groups

In this section one looks at expansive subgroups in Roquette p-groups for p a prime number. Let P be such a group. One knows from 3.10 that if U is a stabilizing biset for a faithful simple kP-module, then U has to be reduced to an isomorphism. Therefore there is no hope to use expansive subgroups to find a stabilizing biset. Nevertheless, as mentioned above we have an interest in understanding this notion. In this first case, an important ingredient is the classification of all Roquette p-groups, which we first recall.

Lemma 5.1. *Let p be a prime and let P be a Roquette p-group of order p^n.*

1. *If p is odd, then P is cyclic.*
2. *If $p = 2$, then P is cyclic, generalized quaternion (with $n \geq 3$), dihedral (with $n \geq 4$), or semi-dihedral (with $n \geq 4$).*
3. *If P is cyclic or generalized quaternion, there is a unique subgroup Z of order p. Any non-trivial subgroup contains Z.*
4. *If P is dihedral and $Z = Z(P)$, then any non-trivial subgroup contains Z, except for two conjugacy classes of non-central subgroups of order 2. If T is a non-central subgroup of order 2, then $S = N_P(T) = TZ$ is a Klein 4-group and $N_P(S)$ is a (dihedral) group of order 8.*
5. *If P is semi-dihedral and $Z = Z(P)$, then any non-trivial subgroup contains Z, except for one conjugacy class of non-central subgroups of order 2. If T is a non-central subgroup of order 2, then $S = N_P(T) = TZ$ is a Klein 4-group and $N_P(S)$ is a (dihedral) group of order 8.*

Proof. See Chapter 5, Section 4, in [5]. □

Using this classification we are able to prove the non-existence of expansive subgroups with trivial P-core in a Roquette p-group.

Theorem 5.2. *Let p be a prime number and let P be a Roquette p-group. Then P has no non-trivial expansive subgroup with trivial P-core.*

Proof. Let T be a non-trivial subgroup with trivial P-core. Then $T \cap Z(P)$ has to be trivial, otherwise $T \cap Z(P)$ would be contained in the P-core of T. It follows from Lemma 5.1 that T is trivial, except possibly if $p = 2$, P is dihedral or semi-dihedral, and T is a non-central subgroup of order 2. To prove that such T is not expansive one wants to look at ${}^{g}T \cap N_P(T)$ for a suitable element g where $g \notin N_P(T)$.

In both cases $S = N_P(T)$ is a Klein group. Moreover, since $N_P(S)$ is (dihedral) of order 8 and P has order at least 16, we can choose $g \notin N_P(S)$. Using such g one has ${}^{g}T \cap N_P(T) = {}^{g}T \cap TZ = 1$ which proves that T is not expansive as the $N_P(T)$-core of the subgroup $({}^{g}T \cap N_P(T))T$ is exactly T and so does not contain T properly. $\square$

5.2 Some simple groups

In [3], it is shown that no non-trivial expansive subgroup exists in the simple groups A_5, A_6, A_7 and $\mathrm{PSL}_2(\mathbb{F}_{11})$. Even so genetic subgroups appear in A_8, M_{11} and $\mathrm{PSL}_2(\mathbb{F}_7)$. Using GAP one can see that there is no non-trivial n-expansive subgroup in A_5, A_6 and $\mathrm{PSL}_2(\mathbb{F}_{11})$ but there is a 3-expansive subgroup in A_7. Indeed, let $T = \langle(5,6,7)\rangle \times A_4$ and $S = N_{A_7}(T) = T \rtimes \langle(2,4)(6,7)\rangle$. There are four (S,S)-double cosets in A_7. One of them satisfies the second part of (iv) in definition 2.32 and the three others the first part.

5.3 Expansive subgroups in a group with cyclic Fitting subgroup

In this section one wants to investigate groups G such that the Fitting subgroup $F(G)$ is cyclic of order n. One wants to know if expansive subgroups with trivial G-core exist in such groups. In this section, one assumes that G is solvable. One will prove that such a group G has no non-trivial expansive subgroup with trivial G-core. First note that, by Chapter 4, one has the following exact sequence

$$1 \longrightarrow C_n \longrightarrow G \longrightarrow S \longrightarrow 1$$

where S is a subgroup of $\mathrm{Out}(C_n) = \mathrm{Aut}(C_n)$. The map $\iota : C_n \to G$ is the inclusion map. The map $\pi : G \to S$ sends an element g to the conjugation map c_g. One says that G is an *extension* of S by C_n. Note that this is not enough to ensure that G is Roquette at this stage. One discusses this issue later on, see Theorem 5.9.

Suppose $n = p_1^{k_1} \dots p_m^{k_m}$ for some primes p_i and integers k_i, so $C_n = \prod_{i=1}^m C_{p_i^{k_i}}$. It's a well-known result that $\mathrm{Aut}(C_n) \cong \prod_{i=1}^m \mathrm{Aut}(C_{p_i^{k_i}})$. Recall also that $\mathrm{Aut}(C_{2^k}) \cong C_2 \times C_{2^{k-2}}$ and for an odd prime p_i one has $\mathrm{Aut}(C_{p_i^{k_i}}) \cong C_{p_i-1} \times C_{p_i^{k_i-1}}$. Let g_{p_i} be a generator of $C_{p_i^{k_i}}$ in C_n and define α_{p_i} an element of $\mathrm{Aut}(C_n)$ by

$$\alpha_{p_i} : g_{p_i} \mapsto g_{p_i}^{1+p_i^{k_i-1}}$$

and $\alpha_{p_i}(g_{p_j}) = g_{p_j}$ if $j \neq i$. The map α_{p_i} is an element of order p_i if $k_i > 1$, otherwise it is the identity map.

Lemma 5.3. *Let p be a prime number dividing n, then*

$$H^1\big(\langle \alpha_p \rangle, C_n\big) = \begin{cases} 1 & \text{if } p \text{ is odd or } p = 2 \text{ and } k > 2, \\ C_2 & \text{if } p = 2 \text{ and } k = 2. \end{cases}$$

Proof. Decompose n as $n = p^k \cdot n/p^k$ such that p does not divide n/p^k and let g be a generator of C_{p^k} in C_n. Note that

$$H^1\big(\langle \alpha_p \rangle, C_n\big) \cong H^1\big(\langle \alpha_p \rangle, C_{p^k}\big) \times H^1\big(\langle \alpha_p \rangle, C_{n/p^k}\big)$$

but $H^1\big(\langle \alpha_p \rangle, C_{n/p^k}\big)$ is trivial because the order of $\langle \alpha_p \rangle$ and the order of C_{n/p^k} are coprime. Therefore $H^1\big(\langle \alpha_p \rangle, C_n\big)$ is equal to $H^1\big(\langle \alpha_p \rangle, C_{p^k}\big)$. Moreover, recall that in the cyclic case $H^1\big(\langle \alpha_p \rangle, C_{p^k}\big) = \mathrm{Ker}(t)/\mathrm{Im}(\alpha_p \cdot \upsilon)$, where $t = \prod_{i=0}^{p-1} \alpha_p^i$, $\upsilon \in \mathrm{Aut}(C_{p^k})$ sends g to g^{-1} and $(\alpha_p \cdot \upsilon)(g) := \alpha_p(g)\upsilon(g)$. Let's start to describe the action of t on C_{p^k}.

$$\begin{aligned} t(g^j) &= \prod_{i=0}^{p-1} \alpha_p^i(g)^j = \prod_{i=0}^{p-1} g^{j(1+p^{k-1})^i} = \prod_{i=0}^{p-1} g^{j+jip^{k-1}+\dots} \\ &= \prod_{i=0}^{p-1} g^{j+jip^{k-1}} = g^{pj+jp^{k-1}\frac{p(p-1)}{2}}. \end{aligned}$$

The equality between the first and the second line holds because the power of p for the rest of the terms is bigger than k. If p is odd the last term is equal to g^{pj}. Therefore t sends g to g^p, its kernel is $\left\langle g^{p^{k-1}} \right\rangle$. But the image of $\alpha_p \cdot \upsilon$ is also $\left\langle g^{p^{k-1}} \right\rangle$ as $(\alpha_p \cdot \upsilon)(g) = g^{1+p^{k-1}} g^{-1} = g^{p^{k-1}}$. If $p = 2$, the map t sends g^j to $g^{j(p+p^{k-1})}$. The kernel is again $\left\langle g^{p^{k-1}} \right\rangle$ if $k > 2$ and $\left\langle g^{p^{k-2}} \right\rangle$ if $k = 2$. As the kernel is a subgroup of the cyclic group C_{2^k} it's easy to check it by hand. The image of $\alpha_p \cdot \upsilon$ is also $\left\langle g^{p^{k-1}} \right\rangle$ as $(\alpha_p \cdot \upsilon)(g) = g^{1+p^{k-1}} g^{-1} = g^{p^{k-1}}$. This leads us to the conclusion. □

Lemma 5.4. *Let p be a prime number dividing n, then*

$$H^2\big(\langle \alpha_p \rangle , C_n \big) = \begin{cases} 1 & \text{if } p \text{ is odd or } p = 2 \text{ and } k > 2, \\ C_2 & \text{if } p = 2 \text{ and } k = 2. \end{cases}$$

Proof. Again, decompose n as $n = p^k \cdot n/p^k$ such that p does not divide n/p^k and let g be a generator of C_{p^k} in C_n. Using the same argument, one has $H^2\big(\langle \alpha_p \rangle , C_n \big) = H^2\big(\langle \alpha_p \rangle , C_{p^k} \big)$. In the cyclic case $H^2\big(\langle \alpha_p \rangle , C_{p^k} \big) = C_{C_{p^k}}(\langle \alpha_p \rangle) / \operatorname{Im}(t)$, where $t = \prod_i \alpha_p^i$. First, note that $C_{C_{p^k}}(\langle \alpha_p \rangle) = \langle g^p \rangle$. Indeed, it is easy to check that $C_{C_{p^k}}(\langle \alpha_p \rangle) \geq \langle g^p \rangle$ but $\langle g^p \rangle$ is a maximal subgroup of $\langle g \rangle$ and g is not stabilized by α_p therefore the other inclusion follows. Secondly, using the description of the action of t in Lemma 5.3, one has also that $\operatorname{Im}(t) = \langle g^p \rangle$ if p is odd or $p = 2$ and $k > 2$ and therefore $H^2\big(\langle \alpha_p \rangle , C_n \big)$ is trivial for these cases. Nevertheless if $p = 2$ and $k = 2$ then $\operatorname{Im}(t) = 1$ and so $H^2\big(\langle \alpha_p \rangle , C_n \big) = C_2$. □

Corollary 5.5. *Let p be an odd prime or $p = 2$ and $k > 2$. Suppose $\langle \alpha_p \rangle$ is a subgroup of S. Then there exists a subgroup D of G such that $\pi(D) = \langle \alpha_p \rangle$ and $D \cap C_n = 1$.*

Proof. By Lemma 5.4, there exists only one class of extensions of $\langle \alpha_p \rangle$ by C_n. Therefore the extension $\pi^{-1}(\langle \alpha_p \rangle)$ is the semi-direct product of C_n by a cyclic group of order p, which is the subgroup D that we are looking for. □

Lemma 5.6. *Let G be an extension of S by C_n as above. Let D be a subgroup of G such that $D \cap C_n = 1$, then $N_{C_n}(D) = C_{C_n}(D) = C_{C_n}(\pi(D))$.*

Proof. For the first equality, let x be an element of $N_{C_n}(D)$. Then, for all $d \in D$ one has $xdx^{-1} \in D$. But $xdx^{-1} = x\, {}^d x^{-1} d$ which belongs to D if, and only if, $x\,{}^{d}x^{-1} = 1$ which means that $x = {}^{d}x$. This implies that x is an element of $C_{C_n}(D)$. The other inclusion is trivial.

For the second equality, note that the action of D on C_n is the same as the action of $\pi(D)$ on C_n by definition of the map π. $\square$

Lemma 5.7. *Let H be a subgroup of S and H_i the i^{th}-projection of H on* $\mathrm{Aut}(C_n)$*. Then*

$$C_{C_n}(H) = \prod_{i=1}^{m} C_{C_{p_i^{k_i}}}(H_i).$$

Proof. Recall that $C_n = \prod_{i=1}^{m} C_{p_i^{k_i}}$. Now this is just a calculation :

$$\begin{aligned}
C_{C_n}(H) &= \{c = (c_1, \ldots, c_m) \in C_n \mid {}^{h}c = c \text{ for all } h \in H\} \\
&= \{c = (c_1, \ldots, c_m) \in C_n \mid {}^{h}c_i = c_i \text{ for all } i \text{ and for all } h \in H\} \\
&= \prod_{i=1}^{m} \{c_i \in C_{p_i^{k_i}} \mid {}^{h}c_i = c_i \text{ for all } i \text{ and for all } h \in H\} \\
&= \prod_{i=1}^{m} \{c_i \in C_{p_i^{k_i}} \mid {}^{h_i}c_i = c_i \text{ for all } i \text{ and for all } h_i \in H_i\} \\
&= \prod_{i=1}^{m} C_{C_{p_i^{k_i}}}(H_i).
\end{aligned}$$

$\square$

Lemma 5.8. *Let G be the group $C_{2^k} \rtimes C_2$ with $k > 2$, where C_2 is generated by either $\beta_1 : g \mapsto g^{-1}$ or $\beta_2 : g \mapsto g^{-1+2^{k-1}}$ where g is a generator of C_{2^k}. Let b be an element of C_2. If the element $g\,{}^{b}g^{-1}$ belongs to $C_{C_{2^k}}(C_2)$ then the only possibility is that $b = 1$.*

Proof. Note that in both cases $C_{C_{2^k}}(C_2) = \{c \in C_{2^k} \mid c^2 = 1\}$. Suppose now that $g\,{}^{b}g^{-1}$ belongs to $C_{C_{2^k}}(C_2)$, where b is an element of C_2. So actually, except being the identity, b could only be β_1 or β_2 depending in which case we are. One shows that it must imply $b = 1$ anyway. Indeed, remark that $g\beta_1(g^{-1}) = g^2$ is not an element of $C_{C_{2^k}}(C_2)$ because $k > 2$. Similary for $g\beta_2(g^{-1}) = g^{2+2^{k-1}}$. Therefore in both cases one must have $b = 1$. $\square$

Theorem 5.9. *Let G be a group such that there is an exact sequence*

$$1 \longrightarrow C_n \longrightarrow G \longrightarrow S \longrightarrow 1$$

where S is a subgroup of $\mathrm{Aut}(C_n)$*, the map $\iota : C_n \to G$ is the inclusion map, the map $\pi : G \to S$ sends an element g to the conjugation map c_g and the power of the prime* 2 *in the decomposition of n is different from* 2*. Then*

1. *if G is a Roquette group then S does not contain any subgroup $\langle \alpha_p \rangle$ for a prime p dividing n.*

2. *If S_{p_j} does not contain a subgroup $\langle \alpha_{p_j} \rangle$ for all prime p_j dividing n then G has no non-trivial expansive subgroup with trivial G-core, where S_{p_j} denotes the j^{th}-projection of S on* $\mathrm{Aut}(C_n)$.

3. *If G has no non-trivial expansive subgroup with trivial G-core then G is Roquette.*

Proof. One proves 1 by proving the converse. Suppose there exists a prime p such that $\langle \alpha_p \rangle$ is a subgroup of S. Decompose n as $n = p^k \cdot n/p^k$ such that p does not divide n/p^k and let g_p be a generator of C_{p^k} in C_n. By Corollary 5.5, let D be a subgroup of G such that $\pi(D) = \langle \alpha_p \rangle$ and $D \cap C_n = 1$. By Lemma 5.6 and a quick calculation, one has $N_{C_n}(D) = C_{C_n}(\langle \alpha_p \rangle) = \langle (g_p)^p \rangle \times C_{n/p^k}$. Indeed, it is easy to check that $C_{C_n}(\langle \alpha_p \rangle) \geq \langle (g_p)^p \rangle \times C_{n/p^k}$ but $\langle (g_p)^p \rangle$ is a maximal subgroup of $\langle (g_p) \rangle$ and g_p is not stabilized by α_p therefore the other inclusion follows. Define the subgroup $E := \left\langle (g_p)^{p^{k-1}} \right\rangle \times D$, which is an elementary abelian p-group. One shows that E is a normal subgroup of G. Indeed, let h be an element of G, then ${}^hE = {}^h\left\langle (g_p)^{p^{k-1}} \right\rangle \times {}^hD = \left\langle (g_p)^{p^{k-1}} \right\rangle \times {}^hD$ as C_n is a normal subgroup of G. Moreover, by Lemma 5.3, one has ${}^hD = {}^cD$ for some $c \in C_{p^k}$, because ${}^hD$ is a complement of C_n in $C_n \rtimes D$ and the elements of D act trivially on C_{n/p^k}. Indeed one has ${}^hD \cap C_n = {}^h(D \cap {}^{h^{-1}}C_n) = {}^h(D \cap C_n) = 1$ and ${}^hD \subset \pi^{-1}\pi({}^hD) = \pi^{-1}({}^{\pi(h)}\pi(D)) = \pi^{-1}(\pi(D)) = C_n \rtimes D$. As α_p acts trivially on C_{n/p^k} so does D and thus one can restrict the conjugation to an element $c \in C_{p^k}$ instead of C_n. Now one looks at ${}^cD$. Let x be an element of ${}^cD$, then $x = c\,{}^dc^{-1}d$ for a d in D. Write c as g_p^i and $\pi(d)$ as α_p^j. Recall that the action of D on C_n is the same as $\langle \alpha_p \rangle$. Therefore $c\,{}^dc^{-1} = c\,{}^{\alpha_p^j}c^{-1} = (g_p)^{-ijp^{k-1}}$ which implies that ${}^cD$ is included in $\left\langle (g_p)^{p^{k-1}} \right\rangle \times D$ and so E is normal. Therefore G is not Roquette. This proves 1.

For 2, as before one has $n = p_1^{k_1} \dots p_m^{k_m}$ and moreover to simplify the notation suppose here that $p_1^{k_1} = 2^k$. Suppose that S_{p_j} does not contain a subgroup $\langle \alpha_{p_j} \rangle$ for all prime p_j dividing n and prove G has no non-trivial expansive subgroup with trivial G-core. Let A be a non-trivial subgroup of G with trivial G-core. Then $A \cap C_n = 1$ otherwise $A \cap C_n$ would be included in the G-core of A. So π induces an isomorphism between A and a subgroup H of S.

The subgroup H is included in $\prod_{i=1}^m H_i$ where H_i is the i^{th}-projection of H on $\mathrm{Aut}(C_n)$. As A is not trivial, so is H and therefore there exists an integer j such that H_j is not trivial. Now one looks at the expansivity condition. Let e be an element in $C_n = \prod_{i=1}^m C_{p_i^{k_i}}$ but not in $C_{C_n}(H)$ and write e as $\prod_{i=1}^m e_{p_i}$ where $e_{p_i} \in C_{p_i^{k_i}}$. More precisely, if p_j is odd, then take e_{p_i} in $C_{C_n}(H)$ and only e_{p_j} not in $C_{C_n}(H)$. If $p_j = 2$ take the element e_2 to be the generator g of C_{2^k} and again e_{p_i} in $C_{C_n}(H)$ if p_i is not equal to p_j. Note that if $p_j = 2$, then $k > 2$ as $k = 2$ is excluded by assumption and $k = 1$ forces H_2 to be trivial. As $N_{C_n}(A) = C_{C_n}(H)$ by Lemma 5.6, the element e is not in $N_G(A)$. Let b be an element of A. Then ${}^{e}b = e\,{}^{b}e^{-1}b$ is an element of $N_G(A)$ if and only if $e\,{}^{b}e^{-1}$ is an element of $N_G(A) \cap C_n = C_{C_n}(H)$, by Lemma 5.6. Our purpose is to show that $e\,{}^{b}e^{-1} = 1$ and therefore ${}^{e}A \cap N_G(A) \leq A$. Write $b = \prod_i b_i$ such that $\pi(b_i)$ belongs to H_i. This is possible because π induces an isomorphism between A and H. Then, using Lemma 5.7, one has that $e\,{}^{b}e^{-1} = \prod_i e_{p_i}\,{}^{b_i}e_{p_i}^{-1} \in C_{C_n}(H) = \prod_i C_{C_{p_i}^{k_i}}(H_i)$, which means that for all i one has $e_{p_i}\,{}^{b_i}e_{p_i}^{-1} \in C_{C_{p_i}}(H_i)$. But by definition, for all i, one has $e_{p_i}\,{}^{b_i}e_{p_i}^{-1} \in [C_{p_i^{k_i}}, H_i]$ and therefore $e_{p_i}\,{}^{b_i}e_{p_i}^{-1} \in [C_{p^{k_i}}, H_i] \cap C_{C_{p_i}^{k_i}}(H_i)$. If p_i is different from p_j then ${}^{b_i}e_{p_i} = e_{p_i}$, as the element e_{p_i} belongs to $C_{C_n}(H)$. By proposition 1.4, $[C_{p_j^{k_j}}, H_j] \cap C_{C_{p_j}}(H_j)$ is trivial if p_j is different from 2. The reason is that H_j has order prime to p_j because S_{p_j} does not contain a subgroup $\langle \alpha_{p_j} \rangle$. For the case $p = 2$ one uses Lemma 5.8 for $k > 2$. Indeed, in this case one has shown that if $g\,{}^{b_2}g^{-1}$ belongs to $C_{C_{2^k}}(C_2)$ then $b_2 = 1$. To sum up one has shown that ${}^{b_i}e_{p_i} = e_{p_i}$ for all i. This implies that if ${}^{e}b$ belongs to $N_G(A)$ then ${}^{e}b = e\,{}^{b}e^{-1}b = b$. Finally one concludes that ${}^{e}A \cap N_G(A) \leq A$ and therefore A is not expansive. As A was an arbitrary non-trivial subgroup of G with trivial G-core, this concludes the proof.

The third fact is a general result, see Corollary 3.7. One has just to prove the existence of a faithful simple module. Let L be $\mathrm{Ind}_{C_n}^{G}(\xi)$ where ξ is a primitive nth root of unity. This module is irreducible as the conjugate representations of ξ by the action of G/C_n are not isomorphic as it acts by automorphism on C_n. Moreover, the condition of primitivity on the root ensures the faithfulness of the induced module so L is the wanted module. □

Corollary 5.10. *Let G be a solvable group with $F(G) = C_n$. Then G is Roquette and of the form of Theorem 5.9 and so S does not contain a subgroup $\langle\alpha_p\rangle$ for a prime p dividing n.*

Proof. If $F(G) = C_n$ then G is Roquette by Proposition 4.4, as $F(G)$ does not contain an elementary abelian subgroup of rank at least 2. The group G is also of the form of Theorem 5.9 by the last argument in Chapter 4. Therefore Theorem 5.9 gives the result. □

Remark 5.11.

- To ensure the non-existence of expansive subgroups with trivial G-core in such groups one needs, not only that S does not contain a subgroup $\langle\alpha_p\rangle$ for a prime p dividing n but also that S_{p_j} does not contain a subgroup $\langle\alpha_{p_j}\rangle$ for the primes p_j dividing n. Which means that we do not want a diagonal subgroup of $\prod_{i=1}^{m}\mathrm{Aut}(C_{p_i^{k_i}})$ in S, see Proposition 1.2. However note that, depending on G, for example if there is no prime p_i dividing p_j-1, the two conditions can be equivalent.

- Using GAP, one can find examples of n-expansive subgroups for $n > 1$. For example in $G := C_{105} \rtimes \mathrm{Aut}(C_{105})$, the subgroup $T := C_{12} \times C_2$ in $\mathrm{Aut}(C_{105}) \cong C_{48}$ is 6-expansive with a number of $(N_G(T), N_G(T))$-double cosets of 8.

5.4 p-hyper-elementary groups

Let p be a prime number. Let G be $C_n \rtimes P$ where P is a p-group and C_n is a cyclic group of order prime to p. There is an action map $\psi : P \to \mathrm{Aut}(C_n)$. Such a group is called *a p-hyper-elementary group*. Let B be a subgroup of G with trivial core. So $B \cap C_n = 1$ otherwise $B \cap C_n$ would be contained in the core of B. In particular, up to conjugation, B is a subgroup of P as p is prime to n.

Theorem 5.12. *If p is odd then a p-hyper-elementary group is Roquette if and only if the kernel of ψ is cyclic. If $p = 2$, then G is Roquette if and only if the kernel of ψ is cyclic, quaternionic, semidihedral or dihedral.*

Proof. See Theorem 3.A.6 of [6]. □

Remark 5.13. If G is a Roquette p-hyper-elementary group, with p an odd prime, then we are again in the situation of a cyclic Fitting subgroup.

Lemma 5.14. *Let G be $C_n \rtimes P$, a p-hyper-elementary group. Then $Z(G) = C_{C_n}(P)\big(\operatorname{Ker}\psi \cap Z(P)\big)$.*

Proof. Let x be an element of $Z(G)$. Write x as em where e belongs to C_n and m to P. Then $x(ch)x^{-1} = ch$ for all c in C_n and h in P, which means that $e\,{}^{m}c\,{}^{mh}e^{-1}\,{}^{m}h = ch$. Identifying the elements of C_n and P one has ${}^{m}h = h$ for all h in P and so $m \in Z(P)$, moreover one has $c = e\,{}^{m}c\,{}^{mh}e^{-1} = e\,{}^{m}c\,{}^{h}e^{-1}$. Since this must be true for all $h \in P$ one can take $h = 1$ and so $c = {}^{m}c$ for all $c \in C_n$ which means that $m \in \operatorname{Ker}\psi$ and so $m \in \operatorname{Ker}\psi \cap Z(P)$. Finally one has $e = {}^{h}e$ for all $h \in P$ and so $e \in C_{C_n}(P)$.

This proves that $Z(G) \leq C_{C_n}(P)\big(\operatorname{Ker}\psi \cap Z(P)\big)$. The other inclusion is trivial. □

Lemma 5.15. *Let G be $C_n \rtimes P$, a p-hyper-elementary group and B a subgroup of P. Then $N_G(B) = C_{C_n}(B) \rtimes N_P(B)$.*

Proof. Let x be an element of $N_G(B)$. Write x as em where e belongs to C_n and m to P. Then $xhx^{-1} \in B$ for all h in B, which means that $e\,{}^{mh}e^{-1}\,{}^{m}h \in B$. Identifying the elements of C_n and P one has ${}^{m}h \in B$ for all h in B and so $m \in N_P(B)$, moreover $e\,{}^{mh}e^{-1} = 1$ so $e = {}^{mh}e$ for all h in B and therefore $e \in C_{C_n}(B)$.

This proves that $N_G(B) \leq C_{C_n}(B) \rtimes N_P(B)$. The other inclusion is trivial. □

Lemma 5.16. *Let G be $C_n \rtimes P$, a p-hyper-elementary Roquette group for p an odd prime and B be a subgroup of G with trivial core. If B is not trivial then $C_{C_n}(B) \lneqq C_n$.*

Proof. Let B be such a non-trivial subgroup. As noticed B can be chosen, up to conjugation by an element of C_n, to be a subgroup of P. As the conjugation does not change $C_{C_n}(B)$ one can suppose that B is a subgroup of P. Suppose

$C_{C_n}(B) = C_n$, which means that B is a subgroup of $\ker \psi$. But, by Theorem 5.12, $\ker \psi$ is cyclic and so B is a normal subgroup of P. But because B has trivial G-core one has $B \cap Z(G) = 1$ and so $1 = B \cap (\operatorname{Ker} \psi \cap Z(P)) = B \cap Z(P)$ which is a contradiction with the normality of B in a p-group. $\square$

Theorem 5.17. *Let G be $C_n \rtimes P$, a p-hyper-elementary group for p an odd prime. Suppose G is a Roquette group. Then G has no non-trivial expansive subgroup with trivial G-core.*

Proof. Let B be such a non-trivial subgroup. As noticed B can be chosen to be a subgroup of P. Let e be an element of C_n not contained in $C_{C_n}(B)$. Such an element exists by Lemma 5.16. Let ${}^{e}b$ be an element of ${}^{e}B \cap N_G(B)$ where b belongs to B. Then ${}^{e}b = e\,{}^{b}e^{-1} b$ which is an element of $N_G(B)$ if and only if $e\,{}^{b}e^{-1} \in C_{C_n}(B)$ by Lemma 5.15. But $e\,{}^{b}e^{-1}$ is an element of $[C_n, B]$ and so $e\,{}^{b}e^{-1} \in [C_n, B] \cap C_{C_n}(B)$. By Proposition 1.4, $[C_n, B] \cap C_{C_n}(B)$ is trivial so ${}^{b}e = e$. Therefore ${}^{e}b = b$ and ${}^{e}B \cap N_G(B) \leq B$ which proves that B is not expansive. $\square$

5.5 Groups with extraspecial groups in the Fitting subgroup

In this section one wants to investigate groups G such that the Fitting subgroup $F(G)$ contains an extraspecial subgroup. The goal is to determine if expansive subgroups with trivial G-core exist in such groups. We were not able to completely treat this case as in the previous section with $F(G) = C_n$. So, one looks at particular examples. One starts with a 2-extraspecial Q_8 contained in $F(G)$ with $G := Q_8 \rtimes SL_2(2)$. Then one establishes partial results for $G := E \rtimes \operatorname{Sp}(E/Z)$ with E an extraspecial group of order p^{1+2n} for an odd prime p. For $n = 1$, one could prove that such a group G has no non-trivial expansive subgroup with trivial G-core. One finishes this section with a discussion and partial results on the case $G := \big(E_p \rtimes \operatorname{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \operatorname{SL}(E_q/Z_q)\big)$.

5.5.1 $Q_8 \rtimes \operatorname{SL}_2(2)$

We start with the group $Q_8 \rtimes S_3$. It is a Roquette group. The Fitting subgroup is Q_8, an extraspecial 2-group. In [3] it is shown that S_3 is an expansive subgroup. With GAP, one can see that the subgroups of order 2 in S_3 are examples of 6-expansive subgroups, with a number of double classes of 7.

5.5.2 $E \rtimes \mathrm{Sp}(E/Z)$

Let E be an extraspecial group of order p^{1+2n} and exponent p for an odd prime p. In this section Z refers to the center $Z(E)$ of E. It is also the center of $G := E \rtimes \mathrm{Sp}(E/Z)$. A general result about extraspecial groups states that E is the central product of r non-abelian subgroups of order p^3. If T_i is a non-central p-subgroup of E of order p such that T_i is contained in an E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3 then T_i^c denotes a choice of a complement of order p of T_i in E_i. That is to say that $\langle T_i, T_i^c\rangle = E_i$ and $T_i \cap T_i^c = 1$.

One introduces this notation for a fluidity in the reading, but remark that even if one writes T_i^c, the subgroup is not unique. One has to make an arbitrary choice of complement but it will not affect the arguments where the notation intervenes. For this reason, one allows this abusive notation.

Now regard Z as the field of integers modulo p and E/Z as a vector space over Z and define $\beta : E/Z \times E/Z \to Z$ by $\beta(\bar{x}, \bar{y}) = [x, y]$. It is a well-known result that β is a symplectic form on E/Z. This implies an action of $\mathrm{Sp}(E/Z)$ on E/Z. But one needs to define the action of $\mathrm{Sp}(E/Z)$ on E. To do so, define the following subgroup of $\mathrm{Sp}_{2n+2}(p)$ equipped with the symplectic form $J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$ where $K = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$:

$$L := \begin{pmatrix} 1 & & \\ & \mathrm{Sp}_{2n}(p) & \\ & & 1 \end{pmatrix}$$

and U the set of matrices of the form

$$\begin{pmatrix} 1 & v & z \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix}$$

where $v = (a_1, \dots, a_{2n})$, ${}^t w = {}^t(J\,{}^t v) = (a_{2n}, \dots a_{n+1}, -a_n, \dots, -a_1)$ for $a_i \in \mathbb{F}_p$ and $z \in \mathbb{F}_p$. It's easy to see that $L \cong \mathrm{Sp}(E/Z)$ and $U \cong E$ and that L acts by conjugation on U. A short calculation, shows that L acts by conjugation on $U/Z(U)$ as the natural action of $\mathrm{Sp}(E/Z)$ on E/Z. So one can define the action of $\mathrm{Sp}(E/Z)$ on E as the action by conjugation of L on U.

Lemma 5.18. *Let s be an element of $\mathrm{Sp}(E/Z)$ and e an element of E. If s acts trivially on $\bar{e}$ in E/Z then ${}^{s}e = e$.*

Proof. Let $u \in U$ be $\begin{pmatrix} 1 & v & z \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix}$ and suppose an element A of L acts trivially on $\bar{u}$ in $U/Z(U)$, which means that there exists $t \in \mathbb{F}_p$ such that

$$\begin{aligned} \begin{pmatrix} 1 & v & t \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & v & z \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & A^{-1} & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & vA^{-1} & z \\ 0 & \mathrm{Id}_{2n} & Aw \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore $Aw = w$ and $vA^{-1} = v$ and then $t = z$. Using this, one has

$$\begin{aligned} {}^{A}u &= \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & v & z \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & A^{-1} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & vA^{-1} & z \\ 0 & \mathrm{Id}_{2n} & Aw \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & v & z \\ 0 & \mathrm{Id}_{2n} & w \\ 0 & 0 & 1 \end{pmatrix} = u. \end{aligned}$$

□

Note that given a subgroup H of Econtaining Z, there exists a decomposition of E such that H can be written as

$$\prod_{i=1}^{k} T_i \times E_{k+1} \circ \cdots \circ E_r$$

for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i for a choice of decomposition of E. Indeed, let $\overline{H}$ be H/Z and let $\{v_1, \dots, v_k\}$ be a basis of $\overline{H} \cap \overline{H}^{\perp}$ and complete it with a symplectic basis

$\{v_{k+1}, w_{k+1}, \ldots, v_r, w_r\}$ of $\overline{H}/(\overline{H} \cap \overline{H}^{\perp})$ to obtain a basis of $\overline{H}$. This is possible because the restriction of the symplectic form on $\overline{H}/(\overline{H} \cap \overline{H}^{\perp})$ is nondegenerate. Let $\{w_1, \ldots, w_k\}$ be elements of E/Z such that $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ is part of a symplectic basis with $[v_i, w_i] = 1$ for $1 \leq i \leq k$. Denote by $\bar{E}_i = \langle v_i, w_i \rangle$ the subspace of dimension 2. By definition of the symplectic basis all $\bar{E}_i$ and $\bar{E}_j$ are orthogonal for $i \neq j$. Let $E_1, \ldots, E_r$ be preimages of $\bar{E}_1 \ldots \bar{E}_r$ in E. Then $E_1, \ldots, E_r$ are non-abelian groups of order p^3, such that $[E_i, E_j] = 1$ if $i \neq j$, which gives by definition the central product $E_1 \circ \cdots \circ E_r$. Now the preimage of H/Z, which has basis $\{v_1, \ldots, v_k, v_{k+1}, w_{k+1}, \ldots, v_r, w_r\}$, in this central product is $\prod_{i=1}^{k} T_i \times E_{k+1} \circ \cdots \circ E_r$ for T_i non-central p-subgroups of E_i of order p if r is not equal to 0, otherwise it is $Z \times \prod_{i=1}^{k} T_i$. Remark also that $C_E(H) = \prod_{i=1}^{k} T_i \times E_{r+1} \circ \cdots \circ E_n$ as $C_E(H)/Z$ corresponds, by the definition of the symplectic form β, to the orthogonal of H/Z in E/Z.

Lemma 5.19. *Let H be a subgroup of $\mathrm{Sp}(E/Z)$, e in E and $G := E \rtimes \mathrm{Sp}(E/Z)$. Then H acts trivially on e if and only if e belongs to $N_G(H)$.*

Proof. Let h be an element of H. We have $ehe^{-1} = e\, {}^{h}e^{-1} h$ so by the uniqueness of the decomposition in elements of E and $\mathrm{Sp}(E/Z)$ the element ehe^{-1} belongs to H if and only if $e\, {}^{h}e^{-1} = 1$ which means that h acts trivially on e. This must be satisfied for all h in H and the result follows. $\square$

Corollary 5.20. *Let $G := E \rtimes \mathrm{Sp}(E/Z)$ and S be a subgroup of G of the following form:*

$$S = \{\varphi(h)h \mid h \in H\} \text{ where } \varphi : H \to E$$

with $\varphi(hk) = \varphi(h)\, {}^{h}\varphi(k)$ for all $h, k \in H$ and H is a non-trivial subgroup of $\mathrm{Sp}(E/Z)$. Then E is not contained in $N_G(S)$.

Proof. Suppose $E \leq N_G(S)$ then E normalizes S and S normalizes E. Because $S \cap E = 1$ one can see that E centralizes S which means that $ses^{-1} = e$ for all $s \in S$ and $e \in E$. Now write s as $\varphi(h)h$ for h in $\mathrm{Sp}(E/Z)$. Then, using the equality above, we have ${}^{h}e = {}^{\varphi(h)^{-1}}e = ez$ for some z in Z and therefore $h(\bar{e}) = \bar{e}$ in E/Z. By Lemma 5.18 this would imply that h acts trivially on E for all h in H which is a contradiction. $\square$

Lemma 5.21. *Let $G := E \rtimes \mathrm{Sp}(E/Z)$, $x := em$ with e in E and m in $\mathrm{Sp}(E/Z)$ and S be a subgroup of the following form:*

$$\{\varphi(h)h \mid h \in H\} \text{ where } \varphi : H \to E$$

with $\varphi(hk) = \varphi(h)\,{}^h\varphi(k)$ *for all* $h, k \in H$ *and* $H \leq \mathrm{Sp}(E/Z)$. *Then* x *belongs to* $N_G(S)$ *if and only if* $m \in N_{\mathrm{Sp}(E/Z)}(H)$ *and* $e\,{}^m\varphi(h)\ {}^{mh}e^{-1} = \varphi({}^mh)$ *for all* $h \in H$.

Proof. This is a straightforward calculation. Let s be an element of S, $s = \varphi(h)h$ for some h in H. We have

$${}^{em}s = {}^{em}\big(\varphi(h)h\big) = {}^{em}\varphi(h)\,{}^{em}h = {}^{em}\varphi(h)e\ {}^{mh}e^{-1}\,{}^mh = e\,{}^m\varphi(h)\ {}^{mh}e^{-1}\,{}^mh.$$

If $em \in N_G(S)$ then ${}^{em}s = \varphi(h_2)h_2$ for some h_2 in H. The unique decomposition in elements of E and $\mathrm{Sp}(E/Z)$ implies that $e\,{}^m\varphi(h)\ {}^{mh}e^{-1} = \varphi(h_2)$ and ${}^mh = h_2$. This holds for all $h \in H$.

Conversely, if $m \in N_{\mathrm{Sp}(E/Z)}(H)$ and $e\,{}^m\varphi(h)\ {}^{mh}e^{-1} = \varphi({}^mh)$ for all $h \in H$ then ${}^{em}s = \varphi({}^mh)\,{}^mh \in S$ and this holds for all $s \in S$. □

Lemma 5.22. *Let* $G := E \rtimes \mathrm{Sp}(E/Z)$ *and* H *be a subgroup of* $\mathrm{Sp}(E/Z)$, *then*

$$N_G(H) = N_E(H) \rtimes N_{\mathrm{Sp}(E/Z)}(H).$$

Proof. Let $x := em$ be an element of $N_G(H)$ with e in E and m in $\mathrm{Sp}(E/Z)$. The same calculation as above, with $\varphi(h) = 1$ for all h in H, shows that m belongs to $N_{\mathrm{Sp}(E/Z)}(H)$. Thus ${}^{em}H = {}^eH$. But this must be equal to H by hypothesis and so e belongs to $N_E(H)$. Hence, $N_G(H) \leq N_E(H) \rtimes N_{\mathrm{Sp}(E/Z)}(H)$ and the other inclusion is obvious. □

Lemma 5.23. *Let* $G := E \rtimes \mathrm{Sp}(E/Z)$ *and* S *be a subgroup of the following form:*

$$\big\{\varphi(h)h \mid h \in H\big\} \text{ where } \varphi : H \to E$$

with $\varphi(hk) = \varphi(h)\,{}^h\varphi(k)$ *for all* $h, k \in H$ *and* $H \leq \mathrm{Sp}(E/Z)$. *Then*

$$N_E(S) = C_E(S) \leq C_E(H) = N_E(H).$$

Proof. First one proves that $N_E(S) = C_E(S)$. Let e be an element of $N_E(S)$. As above one has, for all $h \in H$,

$${}^es = {}^e\big(\varphi(h)h\big) = {}^e\varphi(h)\,{}^eh = {}^e\varphi(h)e\,{}^he^{-1}h = e\varphi(h)\,{}^he^{-1}h.$$

The unique decomposition in elements of E and $\mathrm{Sp}(E/Z)$ implies that

$$e\varphi(h)\,{}^he^{-1} = \varphi(h).$$

This equality implies that $h(\bar{e}) = \bar{e}$ in E/Z, because E/Z is abelian, and so by Lemma 5.18 $e = {}^{h}e$, for all $h \in H$, and thus e belongs to $C_E(H)$. Using this in the equality above one has ${}^{e}\varphi(h) = \varphi(h)$. This shows that $N_E(S) \leq C_E(S)$. The other inclusion is trivial and thus $N_E(S) = C_E(S)$. The same argument with $\varphi = 1$ shows that $C_E(H) = N_E(H)$.

Finally, one proves that $C_E(S)$ is a subgroup of $C_E(H)$. Indeed let e be an element of $C_E(S) = N_E(S)$. By the argument above e belongs to $C_E(H)$ and the result follows. □

Notation 5.24. Let $T := \prod_{i=1}^{k} T_i$ and let $\{v_1, \dots, v_k\}$ be a basis of $(T \times Z)/Z$ and $\{w_1, \dots, w_k\}$ the corresponding elements in order to obtain a symplectic basis, see Remark 1.10. One completes with $\{v_{k+1}, \dots, v_n, w_n \dots w_{k+1}\}$ in order to obtain a symplectic basis of E/Z. One refers to the basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n, w_n \dots w_{k+1}, w_1, \dots, w_k\}$ as a T-basis of E/Z. In this basis the symplectic form has the following matrix

$$J = \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \tilde{K} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix} \text{ where } \tilde{K} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

Lemma 5.25. *Let $G := E \rtimes \mathrm{Sp}(E/Z)$ and $T = \prod_{i=1}^{k} T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3, and k is smaller than or equal to n. Then we have*

$$N_G(T) = N_E(T) \rtimes N_{\mathrm{Sp}(E/Z)}(T) = (E_{k+1} \circ \cdots \circ E_n \times T) \rtimes N_{\mathrm{Sp}(E/Z)}(T).$$

Moreover, one has

$$N_{\mathrm{Sp}(E/Z)}(T) \leq \left\{ \begin{pmatrix} A & & \star \\ & B & \\ 0 & & A^{-t} \end{pmatrix} \mid A \in \mathrm{GL}_k(p) \text{ and } B \in \mathrm{Sp}_{2n-2k}(p) \right\}.$$

Proof. Obviously $N_E(T) \rtimes N_{\mathrm{Sp}(E/Z)}(T) \leq N_G(T)$ and if $x := em \in N_G(T)$ with $e \in E$ and $m \in \mathrm{SL}(E/Z)$ then $em \in N_G(Z \times T)$ so $m \in e^{-1}N_G(Z \times T)$. As $E \leq N_G(Z \times T)$ one has $e^{-1}N_G(Z \times T) = N_G(Z \times T)$ which means that m belongs to $N_G(Z \times T) \cap \mathrm{Sp}(E/Z)$ which is exactly $N_{\mathrm{Sp}(E/Z)}(T)$. Therefore we can conclude with the fact that $e = xm^{-1} \in N_G(T) \cap E = N_E(T)$.

Finally, let M be an element of $N_{\mathrm{Sp}(E/Z)}(T)$. Let $\{v_1, \ldots, v_k\}$ be a basis of T and $\{w_1, \ldots, w_k\}$ the corresponding elements in order to obtain a symplectic basis, see Remark 1.10. Let $\{v_{k+1}, \ldots, v_n, w_n \ldots w_{k+1}\}$ be a symplectic basis of $E_{k+1} \circ \cdots \circ E_n$, then M has the form

$$\begin{pmatrix} C & & \star \\ & D & \\ 0 & & E \end{pmatrix}$$

in the basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n, w_n \ldots w_{k+1}, w_1, \ldots, w_k\}$ because M normalizes $(E_{k+1} \circ \cdots \circ E_n \times T)$ and T. In this basis the symplectic form has the following matrix

$$J = \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \tilde{K} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix} \text{ where } \tilde{K} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

The matrix M is symplectic if ${}^tMJM = J$. This calculation implies in particular that $E = C^{-t}$ and D is a symplectic matrix and so the result follows. $\square$

We return to the question of the existence of expansive subgroups in $E \rtimes \mathrm{Sp}(E/Z)$. If S is an expansive subgroup of G with trivial G-core, one must have $S \cap Z = 1$, otherwise Z would be contained in the G-core of S. So only two cases are possible:

1. $S \cap E = 1$ or
2. $S \cap E = \prod_{i=1}^k T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3, and k is smaller than or equal to n.

In the first case, one can check that S is a subgroup of the following form:

$$\big\{\varphi(h)h \mid h \in H\big\} \text{ where } \varphi : H \to E,$$

with $\varphi(hk) = \varphi(h)\,{}^h\varphi(k)$ for all $h, k \in H$ and $H \leq \mathrm{Sp}(E/Z)$. The next proposition is the special case where $\varphi = 1$.

Proposition 5.26. *Let G be $E \rtimes \mathrm{Sp}(E/Z)$ and S be a subgroup of $\mathrm{Sp}(E/Z)$. Then S is not expansive.*

Proof. There exists a decomposition of E such that one can write $N_E(S)$ as $E_1 \circ \cdots \circ E_r \times \prod_{i=1}^{k} T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i. One knows, by Lemma 5.22, that $N_G(S) = N_E(S) \rtimes N_{\mathrm{Sp}(E/Z)}(S)$, so $N_E(S)$ is invariant by $N_{\mathrm{Sp}(E/Z)}(S)$ and so is $C_E(N_E(S))$, as the latter corresponds to the orthogonal complement of $N_E(S)$ in the quotient E/Z. Therefore $Z \times \prod_{i=1}^{k} T_i = C_E(N_E(S)) \cap N_E(S)$ is also invariant by $N_{\mathrm{Sp}(E/Z)}(S)$. This implies that the image of $Z \times \prod_{i=1}^{k} T_i$ in E/Z is a totally isotropic subspace of dimension k invariant by $N_{\mathrm{Sp}(E/Z)}(S)$. Thus if k is not equal to zero, the subgroup $N_{\mathrm{Sp}(E/Z)}(S)$ is contained in a parabolic subgroup of $\mathrm{Sp}(E/Z)$. More precisely one has, up to conjugation,

$$N_{\mathrm{Sp}(E/Z)}(S) \leq \left\{ \begin{pmatrix} A & & \star \\ & B & \\ 0 & & A^{-t} \end{pmatrix} \mid A \in \mathrm{GL}_k(p) \text{ and } B \in \mathrm{Sp}_{2n-2k}(p) \right\},$$

where the first k elements of the basis are in $N_E(S)/Z$ and then the $2n-2k$ next vectors complete a basis of $C_E(N_E(S))/Z$ to obtain a standard T-basis, see Notation 5.24. Moreover, S is a subgroup of $N_{\mathrm{Sp}(E/Z)}(S)$ acting trivially on $N_E(S)$, so

$$S \leq \left\{ \begin{pmatrix} \mathrm{Id} & & \star \\ & B & \\ 0 & & \mathrm{Id} \end{pmatrix} \mid B \in \mathrm{Sp}_{2n-2k}(p) \right\}.$$

In order to prove that S is not expansive, let g be the following element of $\mathrm{Sp}(E/Z)$ but not in $N_{\mathrm{Sp}(E/Z)}(S)$

$$\begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix}.$$

Let ${}^g s$ be an element of ${}^g S \cap N_G(S)$. One will show now that ${}^g S \cap N_G(S) \leq S$. The element ${}^g s$ of $\mathrm{Sp}(E/Z)$ belongs to $N_G(S)$ only if ${}^g s$ is an element $\tilde{s}$ of $N_{\mathrm{Sp}(E/Z)}(S)$ and so is of the following form

$$\begin{pmatrix} \tilde{A} & \tilde{C} & \tilde{D} \\ & \tilde{B} & \tilde{E} \\ 0 & & \tilde{A^{-t}} \end{pmatrix}.$$

Let s be $\begin{pmatrix} \mathrm{Id} & C & D \\ & B & E \\ 0 & & \mathrm{Id} \end{pmatrix}$, then $gs = \tilde{s}g$ only if

$$\begin{aligned} gs &= \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & C & D \\ & B & E \\ 0 & & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & B & E \\ -\mathrm{Id} & -C & -D \end{pmatrix} \\ &= \tilde{s}g = \begin{pmatrix} \tilde{A} & \tilde{C} & \tilde{D} \\ & \tilde{B} & \tilde{E} \\ 0 & & \tilde{A^{-t}} \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\tilde{D} & \tilde{C} & \tilde{A} \\ -\tilde{E} & \tilde{B} & 0 \\ -\tilde{A^{-t}} & 0 & 0 \end{pmatrix}. \end{aligned}$$

In other words $C = D = E = 0 = \tilde{C} = \tilde{D} = \tilde{E}$ and $\tilde{B} = B$ and $\tilde{A} = \mathrm{Id}$.

One has shown that ${}^{g}s$ belongs to $N_G(S)$ if and only if ${}^{g}s = s = \begin{pmatrix} \mathrm{Id} & & 0 \\ & B & \\ 0 & & \mathrm{Id} \end{pmatrix}$. Therefore, one has ${}^{g}S \cap N_G(S) \leq S$, which shows that S is not expansive.

Suppose now that $k = 0$ which means that $N_G(S) = (E_1 \circ \cdots \circ E_r) \rtimes N_{\mathrm{Sp}(E/Z)}(S)$. Take the first $2r$ elements of the basis of E/Z in $N_E(S)/Z$ and complete by a basis of $C_E(N_E(S))/Z$. Because $N_E(S)$ and $C_E(N_E(S))$ are invariant by $N_{\mathrm{Sp}(E/Z)}(S)$ and S is a subgroup of $N_{\mathrm{Sp}(E/Z)}(S)$ acting trivially on $N_E(S)$ one has

$$N_{\mathrm{Sp}(E/Z)}(S) \leq \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathrm{Sp}_{2r}(p), B \in \mathrm{Sp}_{2n-2r}(p) \right\}$$

$$\text{and } S \leq \left\{ \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & B \end{pmatrix} \mid B \in \mathrm{Sp}_{2n-2r}(p) \right\}.$$

Let g be a non-central element of $C_E\big(N_E(S)\big)$, such an element exists because of Corollary 5.20. Using Lemma 5.22, one has

$$\begin{aligned} {}^{g}S \cap N_G(S) &= \{ {}^{g}s \mid {}^{g}s \in N_G(S) \text{ and } s \in S\} \\ &= \{g\, {}^{s}g^{-1}s \mid g\, {}^{s}g^{-1}s \in N_G(S) \text{ and } s \in S\} \\ &= \{g\, {}^{s}g^{-1}s \mid g\, {}^{s}g^{-1} \in N_E(S) \text{ and } s \in S\}. \end{aligned}$$

The condition $g\, {}^{s}g^{-1} \in N_E(S)$ implies that in E/Z the element s sends $\bar{g}$ to $\bar{g}\bar{n}$ for an $n \in N_E(S)$. But g belongs to $C_E\big(N_E(S)\big)$ so, because of the form of elements of S above, the element s sends $\bar{g}$ to an element of $C_E\big(N_E(S)\big)/Z$ which shows that $\bar{n} = 1$ and s acts trivially on $\bar{g}$. By Lemma 5.18 it follows that ${}^{s}g = g$. One concludes that ${}^{g}S \cap N_G(S) \leq S$ and S is not expansive. $\square$

The next proposition treats the case where $\varphi \neq 1$ with an added assumption, as we were not able to prove it in generality. There exists a decomposition of E such that one can write $N_E(S)$ as $E_1 \circ \cdots \circ E_r \times \prod_{i=1}^{k} T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i.

Proposition 5.27. *Let G be $E \rtimes \mathrm{Sp}(E/Z)$ and S such that $S \cap E = 1$. Suppose moreover that $\{\varphi(h) \mid h \in H\} \subseteq E_1 \circ \cdots \circ E_r$ for the choice of the decomposition above. Then S is not expansive.*

Proof. First, using the notation as preceding the statement, suppose that k is not equal to 0. As $N_E(S)$ is a subgroup of $N_E(H) = C_E(H)$ by Lemma 5.23 one knows that H acts trivially on $N_E(S)$. In particular H normalizes $\prod_{i=1}^{k} T_i$ so by Lemma 5.25,

$$H \leq \left\{ \begin{pmatrix} \mathrm{Id} & & \star \\ & B & \\ 0 & & \mathrm{Id} \end{pmatrix} \mid B \in \mathrm{Sp}_{2n-2k}(p) \right\},$$

if we choose a basis with the first k elements being a basis of $\prod_{i=1}^{k} T_i$ and then the $2n-2k$ next vectors complete a basis of $C_E(N_E(S))/Z$. Finally one completes the list with k vectors of $T^c := \prod_{i=1}^{k} T_i^c$ to form a T-basis of E/Z, see 5.24. Let g be the following element of $\mathrm{Sp}(E/Z)$ but not in $N_{\mathrm{Sp}(E/Z)}(S)$

$$\begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix}.$$

Let $x := {}^g s = {}^g(\varphi(h)h)$ be an element of ${}^g S \cap N_G(S)$. An easy calculation similar to the one in the proof of Proposition 5.26, shows that x belongs to $N_G(S)$ if and only if ${}^g h = h = \begin{pmatrix} \mathrm{Id} & & 0 \\ & B & \\ 0 & & \mathrm{Id} \end{pmatrix}$. As $\{\varphi(h) \mid h \in H\} \subseteq E_1 \circ \cdots \circ E_r$, the element g acts trivially on $\varphi(h)$ and therefore ${}^g\varphi(h) = \varphi(h)$. Therefore, the fact that the element $x = {}^g s = {}^g(\varphi(h)h)$ belongs to ${}^g S \cap N_G(S)$ implies that ${}^g(\varphi(h)h) = (\varphi(h)h)$ and thus ${}^g S \cap N_G(S) \leq S$.

Now suppose $k = 0$ so $N_E(S) = E_1 \circ \cdots \circ E_r$. Again, as $N_E(S)$ is a subgroup of $N_E(H)$ one knows that H acts trivially on $N_E(S)$, so up to conjugation

$$H \leq \left\{ \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & B \end{pmatrix} \mid B \in \mathrm{Sp}_{2(n-r)}(p) \right\},$$

if we choose a basis with the first $2r$ elements taken in $N_E(S)$ and then $2n-2r$ elements of $C_E(N_E(S))$. Let g be an element of $C_E(E_1 \circ \cdots \circ E_r) = E_{r+1} \circ \cdots \circ E_n$. Then for a $z \in Z$ one has

$$^g(\varphi(h)h) = g\varphi(h)hg^{-1} = g\varphi(h)\,{}^hg^{-1}h = zg\,{}^hg^{-1}\varphi(h)h.$$

This is an element of $N_G(S)$ only if $g\,{}^hg^{-1}$ belongs to $N_E(S)$. Therefore, we have ${}^hg = gn$ for an element n of $N_E(S)$. By the form of H the element ${}^hg$ belongs to $C_E(E_1 \circ \cdots \circ E_r)$, as does g, and so n must belong to $C_E(E_1 \circ \cdots \circ E_r)$ too. As $C_E(E_1 \circ \cdots \circ E_r) \cap N_E(S) = Z$ the element n must be in the center but then $h(\bar{g}) = \bar{g}$ in E/Z which implies, by Lemma 5.18 that ${}^hg = g$ and so actually n is trivial . Therefore h acts trivially on g. Since $\{\varphi(h) \,|\, h \in H\} \subseteq E_1 \circ \cdots \circ E_r = C_E(C_E(E_1 \circ \cdots \circ E_r))$ the elements g and $\varphi(h)$ commute, so the element z above is 1. Finally, one has $^g(\varphi(h)h) = \varphi(h)h$ and so ${}^gS \cap N_G(S) \leq S$. $\square$

Remark 5.28. For the second case, i.e. when S is a subgroup of G such that $S \cap E = T$, with $T = \prod_{i=1}^k T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3, and k is smaller than or equal to n, remark that $N_G(S) \leq N_G(T)$. Indeed, if $k \in N_G(S)$ then

$${}^kT = {}^k(S \cap E) = {}^kS \cap {}^kE = S \cap {}^kE = S \cap E = T.$$

Therefore, using Lemma 5.25, one obtains

$$T \leq S \leq N_G(S) \leq N_G(T) \leq (E_{k+1} \circ \cdots \circ E_n \times T) \rtimes P_k$$

with

$$P_k = \left\{ \begin{pmatrix} A & & \star \\ & B & \\ 0 & & A^{-t} \end{pmatrix} \mid A \in \mathrm{GL}_k(p) \text{ and } B \in \mathrm{Sp}_{2n-2k}(p) \right\}.$$

So an element s of S can be decomposed as em where e is an element of $E_{k+1} \circ \cdots \circ E_n \times T$ and m an element of P_k.

Here is one way to try to solve this case. Let $x := {}^gs = {}^g(em)$ be an element of ${}^gS \cap N_G(S)$ where g is the following element of $\mathrm{Sp}(E/Z)$ but not in $N_{\mathrm{Sp}(E/Z)}(S)$

$$\begin{pmatrix} 0 & 0 & \mathrm{Id} \\ 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \end{pmatrix}.$$

Because g sends e to T^c the fact that $x = {}^g(em)$ belongs to $N_G(S) \leq (E_{k+1} \circ \cdots \circ E_n \times T) \rtimes P_k$ forces the element e to be reduced to 1. Then, an easy calculation, as in the previous cases, shows that the fact that x belongs to $N_G(S)$ implies that

$$m = \begin{pmatrix} A & & 0 \\ & B & \\ 0 & & A^{-t} \end{pmatrix} \quad \text{and} \quad {}^g m = \begin{pmatrix} A^{-t} & & 0 \\ & B & \\ 0 & & A \end{pmatrix}.$$

Now the remaining goal would be to prove that ${}^g m$ belongs to S but at the moment this has not been done.

5.5.3 $(E \circ C_{p^i}) \rtimes \mathrm{SL}(P/Z)$

Let p be an odd prime and E denotes an extraspecial group of order p^3 and exponent p. Let $P := (E \circ C_{p^i})$ be a central product of E and a cyclic group C_{p^i} over $Z(E)$ for $i \geq 1$ and G be $P \rtimes \mathrm{SL}(P/Z)$. Then one has $Z(G) = Z(P) = C_{p^i}$ and $P/Z(P)$ is elementary abelian of rank 2. In this section Z refer to the center $Z(P)$ of P. Note that with $i = 1$ one recovers the case $E \rtimes \mathrm{SL}(P/Z)$. First one has to understand the action of $\mathrm{SL}(P/Z)$ on P. One gives here a concrete definition of this action in term of generators. Let

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

be two generators of $\mathrm{SL}(P/Z)$ acting on the vector space $P/Z(P)$ with basis $\{\bar{f}_1, \bar{f}_2\}$. Let f_1 and f_2 be two representatives in P and z a generator of Z The action is defined as follows on P

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \quad f_1 \mapsto f_1, \quad f_2 \mapsto f_1 f_2, \quad z \mapsto z$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \quad f_1 \mapsto f_1 f_2, \quad f_2 \mapsto f_2, \quad z \mapsto z.$$

It is the action of $\mathrm{SL}(E/Z)$ on E as defined before with trivial action on C_{p^i}. So in the case $i = 1$, one obtains the same action as defined in the previous section of $\mathrm{Sp}(E/Z)$ on E.

Lemma 5.29. *Let s be an element of $\mathrm{SL}(P/Z)$ and e an element of P. If s acts trivially on $\bar{e}$ in P/Z then ${}^s e = e$.*

Proof. Suppose $\bar{e}$ is not in $\langle \bar{f}_1 \rangle$ or $\langle \bar{f}_2 \rangle$ otherwise the result is straightforward from the definition of the action of $\mathrm{SL}(P/Z)$ on P. By assumption there exists an element h in $\mathrm{SL}(P/Z)$ such that ${}^{h}s$ is the matrix

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{F}_p.$$

The element h corresponds to the change of basis from $(\bar{f}_1, \bar{f}_2)$ to $(\bar{e}, \bar{f}_2)$. By the definition of the action of $\mathrm{SL}(P/Z)$ on P we have ${}^{{}^{h}s}f_1 = f_1$ but the left hand side is $hsh^{-1}f_1hs^{-1}h^{-1}$ and therefore ${}^{s}(h^{-1}f_1h) = h^{-1}f_1h$. Using the fact that ${}^{h}\bar{e} = \bar{f}_1$ and so ${}^{h}e = zf_1$ for a z in Z we have

$${}^{s}e = {}^{s}(zh^{-1}f_1h) = z\,{}^{s}(h^{-1}f_1h) = zh^{-1}f_1h = e.$$

$\square$

Lemma 5.30. *Let S be a subgroup of* $\mathrm{SL}(P/Z)$*, then one has*

$$N_G(S) = N_P(S) \rtimes N_{\mathrm{SL}(P/Z)}(S).$$

Proof. Let $x = ym$ be an element of $N_G(S)$, with $y \in P$ and $m \in \mathrm{SL}(P/Z)$. Let s be an element of S. Then the fact that ${}^{x}s = y\,{}^{({}^{m}s)}y^{-1}\,{}^{m}s$ belongs to S implies that ${}^{m}s \in S$ and therefore $m \in N_{\mathrm{SL}(P/Z)}(S) \leq N_G(S)$. So $xm^{-1} = y$ is an element of $N_G(S) \cap P = N_P(S)$, the product of two elements of $N_G(S)$. The other inclusion is straightforward. $\square$

Lemma 5.31. *Let S be a subgroup of G, then one has*

$$N_P(S) = N_E(S)Z.$$

Proof. This is just a straightforward verification

$$\begin{aligned} N_P(S) &= \{ez \in P \mid e \in E, z \in Z \text{ and } {}^{ez}S = S\} \\ &= \{ez \in P \mid {}^{e}S = S\} \\ &= \{ez \in P \mid e \in N_E(S)\} \\ &= N_E(S)Z. \end{aligned}$$

$\square$

Remark 5.32. Because the action of $\mathrm{SL}(P/Z)$ on P/Z is similar to the action of $\mathrm{Sp}(E/Z)$ on E/Z, one can verify that Lemmas 5.19, 5.21, 5.23, 5.25 are also satisfied for P replacing E by P and elements of E by elements of P. In this section, we refer to and use these Lemmas for P even if they are only stated for E.

Lemma 5.33. *Let p be an odd prime number, then*

$$H^1\Big(\,\mathrm{SL}\left(E/Z(E)\right), E/Z(E)\Big) = 1.$$

Proof. Recall that the group $\mathrm{SL}(E/Z(E))$ has $p+1$ simple $\mathbb{F}_p\,\mathrm{SL}(E/Z(E))$-modules, denoted by $V_1, \ldots, V_p$, where V_i has dimension i, see [1] page 15 for more details. The action of $\mathrm{SL}(E/Z(E))$ on $E/Z(E)$ has no trivial submodule and so $V_2 = E/Z(E)$. Let P_1 be the indecomposable projective cover of V_1 and $\pi : P_1 \to V_1$ the corresponding homomorphism. As p is odd, one can show that P_1 is uniserial, with three composition factors which occur as V_1, V_{p-2} and V_1, see [1] page 48 for more details. Recall that $H^1\big(\mathrm{SL}(E/Z(E)), E/Z(E)\big) = \mathrm{Ext}^1_{\mathbb{F}_p\,\mathrm{SL}(E/Z(E))}(\mathbb{F}_p, E/Z(E))$. The latter is trivial if and only if any exact sequence

$$0 \longrightarrow E/Z(E) \longrightarrow W \longrightarrow \mathbb{F}_p \longrightarrow 0$$

splits where W is an $\mathbb{F}_p\,\mathrm{SL}(E/Z(E))$-module. Suppose that such an exact sequence does not split for some W and call g the homomorphism from W to $V_1 = \mathbb{F}_p$. As $0 \subset V_2 \subset W$ with W/V_2 simple we conclude by Jordan-Hölder Theorem on composition series that V_2 is the unique non-zero proper submodule of W. Indeed, the only other possibility for the composition series would be $0 \subset V_1 \subset W$, but then the previous exact sequence would be split, which is excluded by assumption. The module P_1 being projective there exists a homomorphism $f : P_1 \to W$ such that $gf = \pi$. By construction $g(\mathrm{Im}(f)) = V_1$ so $\mathrm{Im}(f)$ is not contained in $\mathrm{Ker}(g) = V_2$. Therefore $\mathrm{Im}(f) = W$ because V_2 is the unique non-zero proper submodule of W. But this means that W is isomorphic to a quotient of P_1 and thus V_2 occurs as a composition factor of P_1, which is a contradiction. This shows that every exact sequence above splits and $H^1\big(\mathrm{SL}(E/Z(E)), E/Z(E)\big) = 1$. □

Remark 5.34. Let G be a group and A a $\mathbb{Z}[G]$-module. One can show that

$$H^1(G, A) = \mathcal{C}/\mathcal{P}$$

where

$$\mathcal{C} := \{f : G \to A \mid f(gh) = f(g)\, {}^{g}f(h) \quad \forall g, h \in G\}$$

and

$$\mathcal{P} = \{f : G \to A \mid \text{ there exists } a \in A \text{ with } f(g) = a^{-1}\, {}^{g}a\}.$$

The elements of $\mathcal{P}$ are called the *principal crossed homomorphisms* or 1-*coboundaries* and the elements of $\mathcal{C}$ the *crossed homomorphisms* or 1-*cocyles*. One refers to [11] for a more developed presentation of group cohomology.

Lemma 5.35. *Let S be a subgroup of $P \rtimes \mathrm{SL}(P/Z)$ such that $S \cap P = 1$ and $N_P(S) = Z$. Let g be an element of E. Then, elements of ${}^{g}S \cap N_G(S)$ are of the form ${}^{g}(\varphi(h)h)$ where h acts trivially on g and h is an element of $\mathrm{SL}(E/Z)$.*

Proof. Let $s = \varphi(h)h$ be an element of S, and g an element of E. One needs to know when ${}^{g}s$ belongs to $N_G(S)$. Using the following calculation,

$${}^{g}s = g\varphi(h)hg^{-1} = g\varphi(h)\, {}^{h}g^{-1}h = zg\, {}^{h}g^{-1}\varphi(h)h \quad \text{for some } z \in Z(P),$$

where the last equality holds because $[P, P] = Z(P)$, one remarks that ${}^{g}s \in N_G(S)$ if and only if $g\, {}^{h}g^{-1} \in N_G(S)$ because z and $\varphi(h)h$ belong to $N_G(S)$. This holds only if $g\, {}^{h}g^{-1} \in Z$ as $g\, {}^{h}g^{-1} \in P$ and $N_G(S) \cap P = Z$ by assumption. But this implies that $h(\overline{g}) = \overline{g}$ in $P/Z(P)$. Therefore h acts trivially on g by Lemma 5.29. □

Theorem 5.36. *Let $G := P \rtimes \mathrm{SL}(P/Z)$, then G has no non-trivial expansive subgroup with trivial G-core. Moreover, if S is a subgroup of $E \rtimes \mathrm{SL}(E/Z)$ such that $S \cap E = 1$, then there exists $g \in E \rtimes \mathrm{SL}(E/Z)$ but not in $N_G(S)$ such that if ${}^{g}s$ belongs to ${}^{g}S \cap N_G(S)$ then ${}^{g}s = s$.*

Proof. Let S be a non-trivial expansive subgroup of G with trivial G-core. We must have $S \cap Z = 1$, otherwise Z would be contained in the G-core of S. So only two cases are possible:

1. $S \cap P = 1$ or
2. $S \cap P = T$ for T a non-central p-subgroup of E of order p.

In the first case, one can check that S is a subgroup of the following form :

$$\big\{\varphi(h)h \mid h \in H\big\} \text{ where } \varphi : H \to P,$$

for $H \leq \mathrm{SL}(P/Z)$ and with $\varphi(hk) = \varphi(h)\,{}^{h}\varphi(k)$ for all $h, k \in H$, i.e. φ is a 1-cocycle.

Assume first that $\varphi = 1$ and so $S = H$. By Lemma 5.31, $N_P(S) = N_E(S)Z$, so only two cases are possible $N_P(S) = Z$ or $N_P(S) = Z \times Q$ for Q a non-central p-subgroup of E. Indeed, by the structure of subgroups of E, the subgroup $N_E(S)$ could only be $Z(E)$ or $Z(E) \times Q$. To start, suppose that $Z = N_P(S)$. Let g be an element of E but not in $N_G(S)$. So we can write ${}^{g}S$ as $\{g\,{}^{s}g^{-1}s \mid s \in S\}$ and using Lemma 5.30 we have

$$ {}^{g}S \cap N_G(S) = \{g\,{}^{s}g^{-1}s \mid g\,{}^{s}g^{-1} \in Z \text{ and } s \in S\}. $$

If $g\,{}^{s}g^{-1} \in Z$ then s acts trivially on $\overline{g}$ in P/Z, which means that s acts trivially on g by Lemma 5.29. So the fact that $g\,{}^{s}g^{-1}$ belongs to Z implies that $g\,{}^{s}g^{-1} = 1$ as well as ${}^{g}s = s$ and thus ${}^{g}S \cap N_G(S) \leq S$. This shows that S is not expansive.

Now suppose that $N_P(S) = Z \times Q$ for Q a non-central p-subgroup of E, which implies that S acts trivially on Q and so, up to conjugation, we have

$$ S \leq \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\}. $$

Moreover, $S = 1$ is excluded by assumption, so we have equality. Let s be an element of S and g be

$$ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Then

$$ {}^{g}s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha_s & 1 \end{pmatrix} $$

which is an element of $N_{\mathrm{SL}(P/Z)}(S)$ only if $\alpha_s = 0$. Therefore one has ${}^{g}s = 1 = s$ and ${}^{g}S \cap N_{\mathrm{SL}(P/Z)}(S) \leq S$.

This shows that S is not expansive if $\varphi = 1$.

Assume now that $\varphi \neq 1$. Again, by Lemma 5.31, one has only two possibilities for $N_P(S)$, either $N_P(S) = QZ$ or $N_P(S) = Z$. In the first case, by Lemmas 5.19 and 5.23 and Remark 5.32, H acts trivially on Q and therefore, up to conjugation, H is

$$ \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\}. $$

Let $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $s = \varphi(h)h$ an element of S. If ${}^g s = {}^g\varphi(h)\,{}^g h$ belongs to ${}^g S \cap N_G(S)$ then ${}^g h \in N_{\mathrm{SL}(P/Z)}(H)$ by Lemma 5.21. Write h as $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ then

$$ {}^g h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} $$

which is an element of $N_{\mathrm{SL}(P/Z)}(H)$ only if $\alpha = 0$ and so ${}^g h = h = 1$. Thus $s = 1$ and therefore we conclude ${}^g S \cap N_G(S) = 1$ and ${}^g s = s$.

Next we suppose that $N_P(S)$ is equal to Z. Recall one has fixed a basis $\langle \bar{f}_1, \bar{f}_2 \rangle$ of $E/Z(E)$. Let $s = \varphi(h)h$ be an element of S, and $g = f_1$ a representative of $\bar{f}_1$ in E. One needs to know when ${}^g s$ belongs to $N_G(S)$. By Lemma 5.35, one knows that $h(\overline{g}) = \overline{g}$ in P/Z. The same argument works for $g = f_2$. Therefore, using Lemma 5.29, one has

$$ {}^{f_1}S \cap N_G(S) \leq \{\, {}^{f_1}(\varphi(h)h) \mid h \in H \text{ and } {}^h f_1 = f_1\} \text{ and} $$

$$ {}^{f_2}S \cap N_G(S) \leq \{\, {}^{f_2}(\varphi(k)k) \mid k \in H \text{ and } {}^k f_2 = f_2\}. $$

These sets are isomorphic to a unipotent group of order p as they act trivially on f_1 respectively f_2. Moreover, either one of these intersections is trivial and then one takes respectively g to be f_1 or f_2, or both intersections are not trivial. In the latter case, $H = \mathrm{SL}(P/Z)$ since H contains two different transvections, the one which acts trivially on f_1 and the one which acts trivially on f_2. By Lemma 5.33 , $H^1\big(\mathrm{SL}(P/Z), P/Z\big) = 1$ and so there exists $\overline{a} \in P/Z$ such that $\overline{\varphi(h)} = \overline{a}^{-1}h(\overline{a})$ for all $h \in \mathrm{SL}(P/Z)$, see Remark 5.34. The element $\bar{a}$ is not trivial otherwise it is the case where $\varphi = 1$ that has been treated before. Thus a does not belongs to $Z = N_E(S)$. Then, for a fixed h in $\mathrm{SL}(P/Z)$ there exists an element $z_h \in Z$ such that $\varphi(h) = z_h a^{-1}\,{}^h a$. Let g be equal to a which is as mentioned not an element of $N_G(S)$. One looks at ${}^a S \cap N_G(S)$. Let s be an element of S, then $s = z_h a^{-1}\,{}^h a h$ and the fact that ${}^a s$ has to belong to $N_G(S)$ implies that h acts trivially on a. This is the same reasoning as above for ${}^{f_1}s$. So $s = z_h h$ and ${}^a s = z_h h = s \in S$, which proves that ${}^a S \cap N_G(S) \leq S$.

For the second case, namely $S \cap P = T$ for T a non-central p-subgroup of E of order p, remark that by Lemma 5.25, one has up to conjugation

$$ N_G(S) \leq N_G(T) = (T \times Z) \rtimes \{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \text{ and } \alpha \in \mathbb{F}_p\}, $$

where the basis is chosen with the first element in T. Let s be an element of S and $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One can write s as $tz \begin{pmatrix} \lambda_s & \alpha_s \\ 0 & \lambda_s^{-1} \end{pmatrix}$ for some t in T and z in Z. Then, the fact that the element ${}^g s$ belongs to $N_G(S)$ implies that $\alpha_s = 0$ and $t = 1$. Indeed, ${}^g tz\, {}^g\begin{pmatrix} \lambda_s & \alpha_s \\ 0 & \lambda_s^{-1} \end{pmatrix}$ belongs to $N_G(S)$ only if ${}^g tz$ belongs to $T \times Z$ and ${}^g\begin{pmatrix} \lambda_s & \alpha_s \\ 0 & \lambda_s^{-1} \end{pmatrix}$ to $\{\begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \text{ and } \alpha \in \mathbb{F}_p\}$. With the following calculation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_s & \alpha_s \\ 0 & \lambda_s^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_s^{-1} & 0 \\ -\alpha_s & \lambda_s \end{pmatrix}$$

one sees that the latter occurs only if $\alpha_s = 0$. Moreover ${}^g tz$ belongs to $T \times Z$ only if $t = 1$ as g sends t to T^c.

Therefore $s = zm := z \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix}$ and so, because $o(z)$ divides p^i and $\lambda_s^p = \lambda_s$, the element $s^{o(z)} = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix}$ belongs to S. This implies that $sm^{-1} = z$ belongs to S as a product of elements of S. Therefore $z = 1$ because one has $S \cap P = T$. Finally, we have $s = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_s^{-1} \end{pmatrix}$ and ${}^g s = \begin{pmatrix} \lambda_s^{-1} & 0 \\ 0 & \lambda_s \end{pmatrix} = s^{-1}$ which is an element of S and so ${}^g S \cap N_G(S) \leq S$. $\square$

Remark 5.37. In the case $\varphi \neq 1$ and $N_P(S) = Z$, one has to notice that one used ${}^{f_1}S \cap N_G(S)$ and ${}^{f_2}S \cap N_G(S)$ to obtain information on S. But at the end one proves the non-expansivity of S by looking at ${}^a S \cap N_G(S)$.

Theorem 5.38. *Let $G := P \rtimes K$, with K a subgroup of $\mathrm{SL}(P/Z)$. Then G has no non-trivial expansive subgroup with trivial G-core if and only if K is not contained in a Borel subgroup of $\mathrm{SL}(P/Z)$.*

Proof. Suppose first that K is contained in a Borel subgroup of $\mathrm{SL}(P/Z)$. Let T be the p-subgroup of E of order p normalized by the Borel subgroup. Then, the normalizer $N_G(T)$ is $Z \times T \rtimes K$ and for all g in G but not in $N_G(T)$ we have

$$(N_G(T) \cap {}^g T)T = {}^g TT = Z \times T$$

because ${}^g T$ is contained in $Z \times T$ but not equal to T. Then the $N_G(T)$-core of $Z \times T$ is $Z \times T$ and so T is an expansive subgroup with trivial G-core.

Conversely, suppose that K is not contained in a Borel subgroup of $\mathrm{SL}(P/Z)$. If p divides $|K|$ then $K = \mathrm{SL}(P/Z)$. Indeed, the number of p-Sylow subgroups of K is either 1 or $p+1$. In the first case, the p-Sylow subgroup, denoted by U, is normal. We would have, up to conjugation, that

$$U = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\} \leq K \leq N_{\mathrm{SL}(P/Z)}(U).$$

So K would be contained in a Borel subgroup which is impossible by assumption. Moreover, if the number of p-Sylow subgroups is $p+1$, then K contains all the transvections which generate $\mathrm{SL}(P/Z)$. The case $K = \mathrm{SL}(P/Z)$ has already been treated therefore we can assume that p doesn't divide the order of K.

Let S be a non-trivial subgroup of G with trivial G-core. We must have $S \cap Z(P) = 1$, otherwise $Z(P)$ would be contained in the G-core of S. So only two cases are possible:

1. $S \cap P = 1$ or
2. $S \cap P = T$ for T a non-central p-subgroup of E of order p.

We start with $S \cap P = 1$. As p does not divide the order of K then $H^1(S, E) = 1$ and so up to conjugation S is a subgroup of K. Obviously, we know that $Z(P) \leq N_P(S) \lneqq P$. By Lemma 5.31, $N_P(S) = N_E(S)Z(P)$, so only two cases are possible $N_P(S) = Z(P)$ or $N_P(S) = Z(P) \times Q$ for Q a non-central p-subgroup of E. Indeed, by the structure of subgroups of E, the subgroup $N_E(S)$ could only be $Z(E)$ or $Z(E) \times Q$. To start suppose that $Z(P) = N_P(S)$. Let g be an element of E but not in $N_G(S)$. So we can write ${}^gS$ as $\left\{ g\, {}^sg^{-1}s \mid s \in S \right\}$ and using Lemma 5.30 we have

$${}^gS \cap N_G(S) = \left\{ g\, {}^sg^{-1}s \mid g\, {}^sg^{-1} \in Z(P) \text{ and } s \in S \right\}.$$

If $g\, {}^sg^{-1} \in Z(P)$ then s acts trivially on $\overline{g}$ in $P/Z(P)$, which means that s acts trivially on g by Lemma 5.29. So the fact that $g\, {}^sg^{-1}$ belongs to $Z(P)$ implies that $g\, {}^sg^{-1} = 1$ and thus ${}^gS \cap N_G(S) \leq S$. This shows that S is not expansive.

Now suppose that $N_P(S) = Z(P) \times Q$ for Q a non-central p-subgroup of E, which implies that S acts trivially on Q and so, up to conjugation, we have

$$S \leq \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\}.$$

Moreover, $S = 1$ is excluded by assumption, so we have equality. But then p divides the order of S and so of K, which is impossible. So this case can actually not occur.

For the second case, namely $S \cap P = T$, one observes that

$$\begin{aligned} N_G(S) &\leq N_G(T) = (T \times Z(P)) \rtimes C \\ &:= (T \times Z(P)) \rtimes \{\begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix} \mid \beta \in \mathbb{F}_p^* \text{ and } \alpha \in \mathbb{F}_p\} \cap K \\ &\leq (T \times Z(P)) \rtimes \{\begin{pmatrix} \beta & (\beta^{-1} - \beta)\gamma \\ 0 & \beta^{-1} \end{pmatrix} \mid \beta \in \mathbb{F}_p^*\}. \end{aligned}$$

The last inclusion holds for a fixed γ because p does not divide the order of K. Note that here the first vector of the basis belongs to T. Moreover by Schur-Zassenhaus lemma, S is of the form $T \rtimes D$ where, up to conjugation by an element of $N_G(T)$, D is a subgroup of C. As Z acts trivially on D and T is contained in S, one can assume that D is a subgroup of C. Let's prove that S is not expansive. Let s be an element of S and g an element of K but not in $N_G(T \times Z)$. This element exists because K is not contained in a Borel subgroup. One can write s as $tm := t \begin{pmatrix} \lambda & (\lambda^{-1} - \lambda)\gamma \\ 0 & \lambda^{-1} \end{pmatrix}$ for some t in T and $\lambda \in \mathbb{F}_p^*$. Then, the fact that the element ${}^g s$ belongs to $N_G(S)$ implies that $t = 1$. Indeed, ${}^g t$ belongs to $T \times Z(P)$ is only possible if $t = 1$ as g does not belong to $N_G(T \times Z)$.

Therefore ${}^g s$ is reduced to ${}^g m$ which belongs to C as it belongs to $N_G(S)$. But m belongs to C as well as it belongs to D. Finally, since C is cyclic we conclude that if the element ${}^g m$ belongs to C then it must actually belong to D, by simply looking at its order, as it is the same as the order of m. Thus ${}^g s$ belongs to S. Finally, one concludes that ${}^g S \cap N_G(S) \leq S$ and therefore S is not expansive. □

5.5.4 $E \rtimes \mathrm{SL}(E/Z)$

Again, let p be an odd prime and E denote an extraspecial group of order p^3 and exponent p. From the previous section one knows that $G := E \rtimes \mathrm{SL}(E/Z)$ has no non-trivial expansive subgroup with trivial G-core. One gives here more information about the structure of subgroups with trivial G-core.

Proposition 5.39. *Let S be a subgroup of $E \rtimes \mathrm{SL}(E/Z)$ such that $S \cap E = T$. Then either*

- *$S = T \rtimes {}^bA$ with A a subgroup of* $\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} | \lambda \in \mathbb{F}_p^* \}$ *and b an element of $N_{\mathrm{SL}(E/Z)}(T)$ with $N_G(S) = (Z \times T) \rtimes C$, where C is conjugate to a subgroup of the group of diagonal matrices, or*

- *$S = (T \times V) \rtimes {}^bA$, for b an element of $N_{\mathrm{SL}(E/Z)}(T)$ and where*

  $$V = \{\rho(u)u \mid u \in U\} \text{ and } \rho : U \to Z$$

  is a homomorphism with $U = \{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \}$ *and A is a subgroup of the diagonal matrices. In this case one has $N_G(S) = (Z \times T \times U) \rtimes C$, where again C is conjugate to a subgroup of the group of diagonal matrices.*

Proof. Remark that $N_G(S) \leq N_G(T)$ because if $k \in N_G(S)$ then

$${}^kT = {}^k(S \cap E) = {}^kS \cap {}^kE = S \cap {}^kE = S \cap E = T.$$

Moreover, by the third isomorphism theorem we have

$$N_G(S)/\big(E \cap N_G(S)\big) \cong EN_G(S)/E.$$

Looking at the action of $EN_G(S)$ on E by conjugation we have an injection from $EN_G(S)/E$ to $\mathrm{Aut}_C(E)/\mathrm{Inn}(E)$ the latter being isomorphic to $\mathrm{Sp}_2(p) = \mathrm{SL}(E/Z)$, where $\mathrm{Aut}_C(E)$ is the group of automorphisms of E fixing the center. Remark that $E \cap N_G(S) = Z \times T$, indeed it's clear that $E \cap N_G(S) \geq Z \times T$ and if $E \cap N_G(S) \nleq Z \times T$ then the only possibility is that $E \cap N_G(S) = E$. But in this case T would be a normal subgroup of E because we would have $E \leq N_G(S)$ and so ${}^eT = {}^e(S \cap E) = {}^eS \cap {}^eE = S \cap E = T$, for e in E. This is a contradiction as T is not normal in E and so we can conclude that $E \cap N_G(S) = Z \times T$. This leads us to an injection from $N_G(S)/(Z \times T)$ to $\mathrm{SL}(E/Z)$. We can be even more precise by noticing that $N_G(S) \leq N_G(Z \times T)$. Indeed, let em be an element of $N_G(S)$ with $e \in E$ and $m \in \mathrm{SL}(E/Z)$ then ${}^{em}T \leq S$ as $T \leq S$ and $e\,{}^mTe^{-1} \leq e\,{}^mEe^{-1} \leq eEe^{-1} \leq E$ so ${}^{em}T \leq S \cap E = T$. Using the previous isomorphisms and the remark above one can see that $N_G(S)/(Z \times T)$ must fix the line corresponding to $Z \times T$ in the quotient $E/Z(E)$, and therefore must inject into a Borel subgroup of $\mathrm{SL}(E/Z)$. To summarize we have seen that

$$N_G(S)/(Z \times T) \hookrightarrow \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \text{ and } \alpha \in \mathbb{F}_p \right\} = C_p \rtimes C_{p-1}$$

and the order of $N_G(S)$ could only be p^2d or p^3d where d divides $p-1$. One treats these two cases.

If the order of $N_G(S)$ is p^2d one deduces by Schur-Zassenhaus' lemma that $N_G(S) = (Z \times T) \rtimes C$ where C is a complement of $Z \times T$ in $N_G(S)$ and $N_G(T) = \big(T \times Z\big) \rtimes N_{\mathrm{SL}(E/Z)}(T)$ by Lemma 5.25. By Schur-Zassenhaus' lemma again, C is conjugate, by an element of $N_G(T)$, to a subgroup of $N_{\mathrm{SL}(E/Z)}(T)$ and so, by looking at the order, C is conjugate to a subgroup of the group of diagonal matrices. Thus C is cyclic of order d. Now we want to describe S. First, we look at its p-Sylow subgroup S_p. As $S \cap E = T$ and $S \cap Z = 1$ we have

$$T \leq S_p = \big(T \times Z\big) \cap S \leq E \cap S = T.$$

So again by Schur-Zassenhaus' lemma we have $S = T \rtimes F$ where F is a complement of T in $N_G(S)$. This complement is conjugate, by an element of $N_G(S)$, to a subgroup of C, so F is conjugate by an element k of $N_G(T)$ to a cyclic subgroup A of the group of diagonal matrices and we can write S as $T \rtimes {}^kA$. As $k \in N_G(T)$ one can write k as ztb where $z \in Z$, $t \in T$ and b belongs to the Borel subgroup, by Lemma 5.25. So one concludes that $S = T \rtimes {}^{ztb}A = T \rtimes {}^{tb}A = T \rtimes {}^{b}A$ because $t \in S$.

If the order of $N_G(S)$ is p^3d one can look at the unique p-Sylow subgroup $N_G(S)_p$ of $N_G(S)$. Its order is p^3 and using Lemma 5.25 we find

$$\begin{aligned} N_G(S)_p &= N_G(T)_p \cap N_G(S) = N_G(T)_p \\ &= T \times Z \times \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\} =: T \times Z \times U. \end{aligned}$$

By Schur-Zassenhaus' lemma we have $N_G(S) = (Z \times T \times U) \rtimes C$ where C is a complement of $Z \times T \times U$ in $N_G(S)$. By Schur-Zassenhaus' lemma again, C is conjugate, by an element of $N_G(T)$, to a subgroup of $N_{\mathrm{SL}(E/Z)}(T)$ and so, by looking at its order, C is conjugate to a subgroup of the group of diagonal matrices. As $S \cap E = T$ and $S \cap Z = 1$ the only possibilities for the order of S are p^2e or pe where e divides d. One can see that the equality $|S| = pe$ is impossible because it would imply that $S = T \rtimes F$ where F is a complement of T in $N_G(S)$ but this S is not normalize by $(Z \times T \times U) \rtimes C$. Suppose now that $|S| = p^2e$. Again by Schur-Zassenhaus' lemma we deduce that $S = T \times V \rtimes {}^bA$, where $V = \{\rho(u)u \mid u \in U\}$ with $\rho : U \to Z$ a homomorphism and A could be taken as a subgroup of the diagonal matrices by the same argument as in the preceding case. □

Remark 5.40. One has seen that there is no non-trivial expansive subgroup with trivial G-core for $G := E \rtimes \mathrm{SL}(E/Z)$. However, using GAP, one can find examples of n-expansive subgroups for $n > 1$. For example with $p = 3$, the subgroup $T := C_3 \times C_3$ is 2-expansive with a number of $(N_G(T), N_G(T))$-double cosets of 3.

5.5.5 $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$

Let p and q be two different prime numbers and E_p, respectively E_q, denotes an extraspecial group of order p^3, respectively q^3 and exponent p, respectively q. Let G_1 be $E_p \rtimes \mathrm{SL}(E_p/Z_p)$ and G_2 be $E_q \rtimes \mathrm{SL}(E_q/Z_q)$. Finally let G be the direct product of G_1 and G_2. In proposition 1.2 we recall the form of the subgroups of G. Let S be a subgroup of G. Define $k_i(S) := S \cap G_i$ and let $p_i(S)$ be the projection of S on G_i. In particular one has

$$k_1(S) \times k_2(S) \leq S \leq p_1(S) \times p_2(S).$$

Moreover, using the form of S one can see that $N_G(S)$ is a subgroup of $N_{G_p}(k_1(S)) \times N_{G_q}(k_2(S))$ as well as a subgroup of $N_{G_p}(p_1(S)) \times N_{G_q}(p_2(S))$.

In this section E will refer to $E_p \times E_q$, the subgroup Z_p to the center of E_p and Z_q to the center of E_q so that Z, the center of E, is $Z_p \times Z_q$. One can also see G as $E \rtimes (\mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q))$ with the action of $\mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q)$ defined on E componentwise. This allows us to extend Lemma 5.29.

Lemma 5.41. *Let s be an element of $\mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q)$ and e an element of E. If s acts trivially on $\overline{e}$ in E/Z then ${}^{s}e = e$.*

One wants to look at the existence of expansive subgroups in G. If S is an expansive subgroup of G with trivial G-core, one must have $S \cap Z = 1$, otherwise Z would be contained in the G-core of S. So, up to a permutation between p and q, only three cases are possible:

1. $S \cap E = 1$ or
2. $S \cap E = T_p \times T_q$ for T_p a non-central p-subgroup of E_p of order p and T_q a non-central q-subgroup of E_q of order q or
3. $S \cap E = T_p$ for T_p a non-central p-subgroup of E_p of order p.

One investigates each case separately and starts with $S \cap E = 1$.

$\mathbf{S} \cap (\mathbf{E_p} \times \mathbf{E_q}) = \mathbf{1}$

Throughout this section, let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$ and S be a subgroup of G with $S \cap (E_p \times E_q) = 1$. Such a subgroup is of the form:

$$\big\{\varphi(h)h \mid h \in H\big\} \text{ where } \varphi : H \to E_p \times E_q,$$

with $\varphi(hk) = \varphi(h)\,{}^{h}\varphi(k)$ for all $h, k \in H$ and $H \leq \mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q)$. Write $\varphi(h) = (\varphi_1(h), \varphi_2(h))$ where $\varphi_1 : H \to E_p$ and $\varphi_2 : H \to E_q$.

Lemma 5.42. *Let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$ and S be a subgroup of G with $S \cap (E_p \times E_q) = 1$. Let h, v be elements of $k_1(H) \times k_2(H)$. Write h as (h_1, h_2) with h_i in $k_i(H)$. Then, one has $\varphi_i(hv) = \varphi_i(h)\,{}^{h_i}\varphi_i(v)$ so φ_i is a cocycle on $k_i(H)$ and a homomorphism on $k_j(H)$ with j different from i. Furthermore $\varphi_i(h) = \varphi_i(h_1)\varphi_i(h_2)$.*

Proof. Write v as (v_1, v_2). One has

$$\begin{aligned}
(\varphi_1(hv), \varphi_2(hv)) &= \varphi(hv) = \varphi(h)\,{}^{h}\varphi(v) \\
&= (\varphi_1(h), \varphi_2(h))\,{}^{h}(\varphi_1(v), \varphi_2(v)) \\
&= (\varphi_1(h)\,{}^{h_1}\varphi_1(v), \varphi_2(h)\,{}^{h_2}\varphi_2(v)).
\end{aligned}$$

Therefore one has

$$\varphi_1((h_1, 1)(v_1, 1)) = \varphi_1(h_1)\,{}^{h_1}\varphi_1(v_1)$$

and

$$\varphi_1((1, h_2)(1, v_2)) = \varphi_1(h_2)\,{}^{1}\varphi_1(v_2).$$

The same argument holds for φ_2 and so φ_i is a cocycle on $k_i(H)$ and a homomorphism on $k_j(H)$ with j different from i. Finally, if h belongs to $k_1(H) \times k_2(H)$, one has

$$\varphi_1(h) = \varphi_1((h_1, h_2)) = \varphi_1((h_1, 1)(1, h_2)) = \varphi_1(h_1)\,{}^{h_1}\varphi_1(h_2),$$

on the other hand one has $\varphi_1((h_1, h_2)) = \varphi_1((1, h_2)(h_1, 1)) = \varphi_1(h_2)\varphi_1(h_1)$. Therefore $\varphi_1(h_2)\varphi_1(h_1) = \varphi_1(h_1)\,{}^{h_1}\varphi_1(h_2)$ and so ${}^{h_1}\overline{\varphi_1(h_2)} = \overline{\varphi_1(h_2)}$ in E/Z. By Lemma 5.18, the element h_1 acts trivially on $\varphi_1(h_2)$ and thus one can conclude that $\varphi_1(h) = \varphi_1(h_1)\varphi_1(h_2)$. The same argument holds for φ_2. $\square$

Proposition 5.43. *Let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$, then G has no non-trivial expansive subgroup S with $S \cap (E_p \times E_q) = 1$.*

Proof. Up to a permutation between p and q, the normalizer $N_E(S)$ is one of the following:

1. $E_p \times E_q$
2. $E_p \times T_q \times Z_q$
3. $E_p \times Z_q$
4. $T_p \times Z_p \times T_q \times Z_q$
5. $T_p \times Z_p \times Z_q$
6. $Z_p \times Z_q$

for T_p a non-central p-subgroup of E_p of order p and T_q a non-central q-subgroup of E_q of order q.

Remark that $N_E(S) = C_E(S)$ is a subgroup of $N_E(H)$, as seen in Lemma 5.23 for $E_p \rtimes \mathrm{SL}(E_p/Z_p)$ but the action being component by component the argument works for $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$ too. So one knows that H acts trivially on $N_E(S)$, therefore in the first three cases H acts trivially on E_p so $p_1(H) = 1$ and thus $H \le \mathrm{SL}(E_q/Z_q)$. So an element s of S is of the form $(\varphi_1(h_2), \varphi_2(h_2)h_2)$ for an $h_2 \in \mathrm{SL}(E_q/Z_q)$. By the Proposition 5.36, there exists $g_2 \in E_q \rtimes \mathrm{SL}(E_q/Z_q)$ but not in $N_G(S)$ such that ${}^{g_2}\big(\varphi_2(h_2)h_2\big) = \varphi_2(h_2)h_2$ if ${}^{g_2}\big(\varphi_2(h_2)h_2\big)$ belongs to $N_{G_q}(p_2(S))$. Let g be $(1, g_2)$, then ${}^{g}s = (\varphi_1(h_2), {}^{g_2}(\varphi_2(h_2)h_2)) \in N_G(S)$ implies that ${}^{g}s$ belongs to $N_{G_p}\big(p_1(S)\big) \times N_{G_q}\big(p_2(S)\big)$ as $N_G(S) \le N_{G_p}\big(p_1(S)\big) \times N_{G_q}\big(p_2(S)\big)$. Therefore we are in the situation of Proposition 5.36 and because of our choice of g one has ${}^{g}s = (\varphi_1(h_2), {}^{g_2}(\varphi_2(h_2)h_2)) = (\varphi_1(h_2), \varphi_2(h_2)h_2) = s$.

In the fourth case, namely $N_E(S) = T_p \times Z_p \times T_q \times Z_q$, because H acts trivially on T_p and T_q, one has, up to conjugation, that

$$p_1(H) \le \Big\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \Big\} \text{ and } p_2(H) \le \Big\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in \mathbb{F}_q \Big\}.$$

By the form of subgroups of G, see Proposition 1.2, the only non-trivial possibilities for H are $H = p_1(H)$, $H = p_2(H)$ and $H = p_1(H) \times p_2(H)$ as $p_1(H)$ is included in a group of order p and $p_2(H)$ of order q. In the first case take $g = (g_1, 1) = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1)$. Let s be an element of S then $s = (\varphi_1(h_1)h_1, \varphi_2(h_1))$. One looks at the implication of ${}^{g}s$ belonging to

$N_G(S)$. This implies that ${}^gs$ belongs to $N_{G_p}\big(p_1(S)\big)\times N_{G_q}\big(p_2(S)\big)$ as $N_G(S)\le N_{G_p}\big(p_1(S)\big)\times N_{G_q}\big(p_2(S)\big)$. In particular ${}^{g_1}(\varphi_1(h_1)h_1)$ belongs to $N_{G_p}\big(p_1(S)\big)$. By Lemma 5.21, one has therefore that

$$ {}^{g_1}h_1 \in N_{\mathrm{SL}(E_p/Z_p)}(p_1(H)) = \Big\{ \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix} \mid \alpha\in\mathbb{F}_p \text{ and } \mu\in\mathbb{F}_p^* \Big\}. $$

As ${}^{g_1}h_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$. This belongs to $N_{G_p}\big(p_1(S)\big)$ only when $\alpha=0$ and therefore $s=1={}^gs$. To sum up, ${}^gs\in N_G(S)$ implies that $s=1$ and so ${}^gS\cap N_G(S)=1$.

In the second case take $g=(1,\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ and applying the same argument one has again that ${}^gS\cap N_G(S)=1$.

For the third one take $g=(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. An element s is of the form $(\varphi_1(h)h_1,\varphi_2(h)h_2)$ for an h in H and as above the fact that ${}^gs$ belongs to $N_G(S)$ implies that ${}^{g_1}(\varphi_1(h)h_1)$ belongs to $N_{G_p}\big(p_1(S)\big)$ and ${}^{g_2}(\varphi_2(h)h_2)$ belongs to $N_{G_q}\big(p_2(S)\big)$. Those conditions imply $h_1=1$ and $h_2=1$ with the same calculation as above and therefore $s=1={}^gs$. This concludes the three possibilities for this fourth case as one showed that ${}^gS\cap N_G(S)=1$ and thus S is not expansive.

If $N_E(S)=T_p\times Z_p\times Z_q$ then, as above, $p_1(H)\le\Big\{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\mid\alpha\in\mathbb{F}_p\Big\}$. Let g be $g=(g_1,g_2)=(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},g_2)$ with g_2 a non central element of E_q and $s=(\varphi_1(h)h_1,\varphi_2(h)h_2)$ an element of S. Then the fact that ${}^gs=({}^{g_1}(\varphi_1(h)h_1),{}^{g_2}(\varphi_2(h)h_2))$ belongs to $N_G(S)$ implies that ${}^gs\in N_{G_p}\big(p_1(S)\big)\times N_{G_q}\big(p_2(S)\big)$. Looking at the first component, with exactly the same argument as in the previous case, one must have $h_1=1$. So $s=(\varphi_1(h_2)),\varphi_2(h_2)h_2)$ and for some $z\in Z_q=[E_p,E_p]$ one has

$$ \begin{aligned} {}^gs &= ({}^{g_1}\varphi_1(h_2),{}^{g_2}(\varphi_2(h_2)h_2)) = ({}^{g_1}\varphi_1(h_2),g_2\varphi_2(h_2)\,{}^{h_2}g_2^{-1}h_2) \\ &= ({}^{g_1}\varphi_1(h_2),zg_2\,{}^{h_2}g_2^{-1}\varphi_2(h_2)h_2) \\ &= ({}^{g_1}\varphi_1(h_2)\varphi_1(h_2)^{-1},g_2\,{}^{h_2}g_2^{-1})(1,z)(\varphi_1(h_2),\varphi_2(h_2)h_2). \end{aligned} $$

As $(1,z)$ and $(\varphi_1(h_2),\varphi_2(h_2)h_2)$ are element of $N_G(S)$ because z is in Z_q and $(\varphi_1(h_2),\varphi_2(h_2)h_2)$ in S, the element ${}^gs$ belongs to $N_G(S)$ if and only if the element $({}^{g_1}\varphi_1(h_2)\varphi_1(h_2)^{-1}, g_2\,{}^{h_2}g_2^{-1})$ belongs to $N_G(S) \cap E = T_p \times Z_p \times Z_q$. Thus $g_2\,{}^{h_2}g_2^{-1}$ belongs to Z_q and so $h_2(\overline{g_2}) = \overline{g_2}$ in E_q/Z_q. By Lemma 5.18, this implies that ${}^{h_2}g_2 = g_2$. Therefore h_2 is contained, up to conjugation, in

$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in \mathbb{F}_q \right\}.$$

Thus if the element h_2 is not trivial its order is q and so is $\varphi_1(h_2)$ in E_p as φ_1 is a homomorphism on $k_2(H)$. But elements in E_p have order p and p is different from q so $\varphi_1(h_2) = 1$. Therefore, one has

$${}^{g}S \cap N_G(S) \leq \{(1, {}^{g_2}\varphi_2(h_2)h_2) \mid h_2 \in k_2(H) \text{ and } {}^{h_2}g_2 = g_2\}.$$

This set is isomorphic to a unipotent group of order q. Taking two generators f_1 and f_2 of E_q and letting $g_2 = f_1$ and $g_2 = f_2$, then either one finds an element g_2 such that ${}^{g}S \cap N_G(S) = 1$ or

$$1 \neq {}^{g}S \cap N_G(S) = \{(1, {}^{g_2}\varphi_2(h_2)h_2) \mid h_2 \in k_2(H) \text{ and } {}^{h_2}g_2 = g_2\}$$

in both cases and so $k_2(H)$ contains two transvections, one acting trivially on f_1 the other one on f_2. This implies that $k_2(H) = \mathrm{SL}(E_q/Z_q)$ and therefore one has to treat the case $H \leq \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\} \times \mathrm{SL}(E_q/Z_q)$. By Lemma 5.33, $H^1\big(\mathrm{SL}(E_q/Z_q), E_q/Z_q\big) = 1$ and so, by definition, there exists $\overline{a} \in E_q/Z_q$ such that $\overline{\varphi_2(h)} = \overline{a}^{-1}h(\overline{a})$ for all $h \in \mathrm{SL}(E_q/Z_q)$. Then, for some $z_{h_2} \in Z_q$ we have $\varphi_2(h_2) = z_{h_2}a^{-1}\,{}^{h_2}a$ for all $h_2 \in \mathrm{SL}(E_q/Z_q)$. Now let $g = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a)$ and s be an element of S. Write s as $(\varphi_1(h)h_1, \varphi_2(h)h_2)$. Again if ${}^{g}s$ belongs to $N_G(S)$ then $h_1 = 1$ and h_2 acts trivially on a. So s is reduced to $(\varphi_1(h_2), \varphi_2(h_2)h_2)$. Moreover $\varphi_1(h_2)$ is of order q as h_2 is of order q. As an element of E_p, the order of $\varphi_1(h_2)$ is also p and therefore the only possibility is that $\varphi_1(h_2) = 1$ as p is different from q. Finally, one has

$$s = (1, \varphi_2(h_2)h_2) = (1, z_{h_2}a^{-1}\,{}^{h_2}ah_2) = (1, z_{h_2}a^{-1}ah_2) = (1, z_{h_2}h_2)$$

and thus

$${}^{g}s = (1, {}^{a}(z_{h_2}h_2)) = (1, z_{h_2}a\,{}^{h_2}a^{-1}h_2) = (1, z_{h_2}h_2) = s.$$

This shows that ${}^{g}S \cap N_G(S) \leq S$ and therefore S is not expansive.

Finally, if $N_E(S) = Z_p \times Z_q$ one takes $g = (g_1, g_2)$ where g_1 and g_2 are non-central elements of, respectively, E_p and E_q. Write s as $(\varphi_1(h)h_1, \varphi_2(h)h_2)$ for an $h = (h_1, h_2)$ in H. One has, for some $z_1 \in Z_p = [E_p, E_p]$ and $z_2 \in Z_q = [E_q, E_q]$,

$$\begin{aligned} {}^{g}s &= ({}^{g_1}(\varphi_1(h)h_1), {}^{g_2}(\varphi_2(h)h_2)) = (g_1\varphi_1(h)\, {}^{h_1}g_1^{-1}h_1, g_2\varphi_2(h)\, {}^{h_2}g_2^{-1}h_2) \\ &= (z_1 g_1\, {}^{h_1}g_1^{-1}\varphi_1(h)h_1, z_2 g_2\, {}^{h_2}g_2^{-1}\varphi_2(h)h_2) = (g_1\, {}^{h_1}g_1^{-1}, g_2\, {}^{h_2}g_2^{-1})(z_1, z_2)s. \end{aligned}$$

As (z_1, z_2) and s are element of $N_G(S)$ the element ${}^{g}s$ belongs to $N_G(S)$ if and only if the element $(g_1\, {}^{h_1}g_1^{-1}, g_2\, {}^{h_2}g_2^{-1})$ belongs to $N_G(S) \cap E = Z_p \times Z_p$. Thus $g_1\, {}^{h_1}g_1^{-1}$ belongs to Z_p and so $h_1(\overline{g_1}) = \overline{g_1}$ in E_p/Z_p. By Lemma 5.18, this implies that ${}^{h_1}g_1 = g_1$. Likewise one obtains that ${}^{h_2}g_2 = g_2$. Then,

$$ {}^{g}S \cap N_G(S) \leq \{ {}^{g}(\varphi(h)h) \mid h \in H \text{ and } {}^{h}g = g \}.$$

Take two generators f_1 and f_2 of E_p and two generators f_3 and f_4 of E_q. Letting g_1 be f_1 or f_2 and g_2 be f_3 or f_4 one obtains four possibilities for $g_{ij} = (f_i, f_j)$ and so four intersections ${}^{g_{ij}}S \cap N_G(S)$. These intersections are contained in subgroups $C_p \times C_q$ in G, where C_p is generated by a transvection that acts trivially on f_1 or f_2 and C_q is generated by a transvection that acts trivially on f_3 or f_4. Indeed,

$$\begin{aligned} H &\leq \left\{ \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \mid \alpha_1 \in \mathbb{F}_p \right\} \times \left\{ \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \mid \beta_1 \in \mathbb{F}_q \right\} \text{ for } g_{13}, \\ H &\leq \left\{ \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \mid \alpha_1 \in \mathbb{F}_p \right\} \times \left\{ \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix} \mid \beta_2 \in \mathbb{F}_q \right\} \text{ for } g_{14}, \\ H &\leq \left\{ \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} \mid \alpha_2 \in \mathbb{F}_p \right\} \times \left\{ \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \mid \beta_1 \in \mathbb{F}_q \right\} \text{ for } g_{23}, \\ H &\leq \left\{ \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} \mid \alpha_2 \in \mathbb{F}_p \right\} \times \left\{ \begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix} \mid \beta_2 \in \mathbb{F}_q \right\} \text{ for } g_{24}. \end{aligned}$$

Either ${}^{g_{ij}}S \cap N_G(S)$ is trivial for one of these g_{ij}, and in this case S is not expansive, or the intersections are not trivial for the four elements g_{ij}. Looking at the different possibilities for the non trivial intersections, it implies that H must contains at least two different transvections generating $\mathrm{SL}(E_p/Z_p)$ or $\mathrm{SL}(E_q/Z_q)$, as each intersection must contain at least a subgroup of order p or q. Without lost of generality, one may assume that $H = \mathrm{SL}(E_p/Z_p) \times H_2$ for H_2 a subgroup of $\mathrm{SL}(E_q/Z_q)$. By Lemma

5.33, $H^1\big(\mathrm{SL}(E_p/Z_p), E_p/Z_p\big) = 1$ and so there exists $\overline{a_1} \in E_p/Z_p$ such that $\overline{\varphi_1(h_1)} = \overline{a_1}^{-1} h_1(\overline{a_1})$ for all $h_1 \in \mathrm{SL}(E_p/Z_p)$. Then, for a $z_{h_1} \in Z_p$ we have $\varphi_1(h_1) = z_{h_1} a_1^{-1}\,{}^{h_1}a_1$ for all $h_1 \in \mathrm{SL}(E_p/Z_p)$. If H_2 is not $\mathrm{SL}(E_q/Z_q)$, then there exists g_2 equals to f_3 or f_4 such that ${}^{(a,g_2)}S \cap N_G(S)$ does not contain a copy of C_q. Indeed, otherwise ${}^{(a,f_3)}S \cap N_G(S)$ and ${}^{(a,f_4)}S \cap N_G(S)$ would contain a cyclic group of order q and so two different transvections generating $\mathrm{SL}(E_q/Z_q)$. So for $g = (a_1, g_2)$,

$$ {}^{g}S \cap N_G(S) = \{\, {}^{(a_1,g_2)}(\varphi(h)h) \mid h \in H \text{ and } {}^{h}(a_1,g_2) = (a_1,g_2)\} \cong C_p. $$

In other words $h_2 = 1$ and so

$$ \begin{aligned} {}^{(a_1,g_2)}(\varphi(h)h) &= {}^{(a_1,g_2)}(\varphi_1(h_1)h_1, \varphi_2(h_1)) \\ &= {}^{(a_1,g_2)}(z_{h_1} a_1^{-1}\,{}^{h_1}a_1 h_1, \varphi_2(h_1)). \end{aligned} $$

Remark that h_1 acts trivially on a_1. Moreover h_1 is of order 1 or p so is $\varphi_2(h_1)$ by Lemma 5.42 then $\varphi_2(h_1) = 1$ because $\varphi_2(h_1)$ is an element of E_q. So $s = (z_{h_1}h_1, 1)$ and

$$ {}^{(a_1,g_2)}s = {}^{(a_1,g_2)}(\varphi(h)h) = (z_{h_1}h_1, 1) = s, $$

which implies that ${}^{g}S \cap N_G(S) \leq S$. One has still to deal with the case $H = \mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q)$. First remark that this case is only possible if $k_1(H) \times k_2(H) = H = p_1(H) \times p_2(H)$. By Lemma 5.42, one knows that $\varphi_i(h) = \varphi_i(h_1)\varphi_i(h_2)$ if h belongs to $k_1(H) \times k_2(H)$. Moreover, using the fact that $H^1\big(\mathrm{SL}(E_q/Z_q), E_q/Z_q\big) = 1$, there exists $\overline{a_2} \in E_q/Z_q$ such that $\overline{\varphi_2(h_2)} = \overline{a_2}^{-1} h_2(\overline{a_2})$ for all $h_2 \in \mathrm{SL}(E_q/Z_q)$. Recall that one has seen above that

$$ {}^{g}S \cap N_G(S) \leq \{\, {}^{g}(\varphi(h)h) \mid h \in H \text{ and } {}^{h}g = g\}. $$

Now take $g = (a_1, a_2)$ and let s be an element of S. One can write s as

$$ \big(\varphi_1(h)h_1, \varphi_2(h)h_2\big) = \big(\varphi_1(h_1)\varphi_1(h_2)h_1, \varphi_2(h_1)\varphi_2(h_2)h_2\big). $$

Suppose moreover that φ_1 or φ_2 are not trivial so that g is not an element of $N_G(S)$. If ${}^{g}s$ belongs to $N_G(S)$ then h acts trivially on $g = (a_1, a_2)$ and thus, up to conjugation, h is contained in $\big\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \big\} \times \big\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in \mathbb{F}_q \big\}$. So h_1 is of order a divisor of q as well as $\varphi_2(h_1)$ in E_q. Again as p is different from q this forces $\varphi_2(h_1) = 1$. Likewise, the element $\varphi_1(h_2)$ is trivial. So, replacing $\varphi_i(h_i)$ by $z_{h_i} a_i^{-1}\,{}^{h_i}a_i = z_{h_i} a_i^{-1} a_i = z_{h_i}$ the element s is reduced to $(z_{h_1}h_1, z_{h_2}h_2)$ and ${}^g s = s$ which implies that ${}^g S \cap N_G(S) \leq S$. Suppose now that $\varphi_1 = 1$ as well as φ_2. Then $H = S = \mathrm{SL}(E_p/Z_p) \times \mathrm{SL}(E_q/Z_q)$. Let's take $g = (g_1, g_2)$ where g_1 and g_2 are non central elements of E_p and E_q respectively. Let ${}^g h$ be an element of ${}^g H \cap N_G(H)$. As ${}^g h = g\,{}^h g^{-1} h$ belongs to $N_G(H)$ if and only if $g\,{}^h g^{-1}$ belongs to $N_E(H) = Z_p \times Z_q$, one has, by Lemma 5.35, that $g\,{}^h g^{-1} = 1$. So, ${}^g h$ belongs to ${}^g H \cap N_G(H)$ implies that ${}^g h = h$ and so finally ${}^g H \cap N_G(H) \leq H$, which shows that S is not expansive. $\square$

$\mathbf{S \cap (E_p \times E_q) = T_p \times T_q}$

Such a subgroup is such that

$$T_p \times T_q \leq S \leq N_G(S) \leq N_G(T_p \times T_q) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(T_p) \times N_{E_q \rtimes \mathrm{SL}(E_q/Z_q)}(T_q).$$

Moreover $k_1(S) \cap E_p = S \cap (E_p \rtimes \mathrm{SL}(E_p/Z_p)) \cap E_p = S \cap E_p = T_p$, therefore by Proposition 5.39 one knows that either $k_1(S) = T_p \rtimes {}^b A_1$ for b an element of $N_{\mathrm{SL}(E_p/Z_p)}(T)$ with A_1 a subgroup of $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \right\}$ or $k_1(S) = (T_p \times V_1) \rtimes {}^b A_1$, for b an element of $N_{\mathrm{SL}(E_p/Z_p)}(T)$, where $V_1 = \{\rho(u)u \mid u \in U_p\}$ and $\rho : U_p \to Z_p$ is a homomorphism for $U_p = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\}$ and A_1 could again be taken as a subgroup of the diagonal matrices. In this section, one only considers the case $b = 1$.

Lemma 5.44. *Let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$, then G has no non-trivial expansive subgroup S with $S \cap (E_p \times E_q) = T_p \times T_q$ and $k_1(S) = T_p \rtimes A_1$ or $k_2(S) = T_q \rtimes A_2$.*

Proof. Suppose that $k_1(S) = T_p \rtimes A_1$. First remark that the normalizer of $T_p \rtimes A_1$ in $E_p \rtimes \mathrm{SL}(E_p/Z_p)$ is $(Z_p \times T_p) \rtimes \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \right\}$, except if $A_1 \leq \{\pm \mathrm{id}\}$. Let $g_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Let $s = (s_1, s_2)$ be an element of S. Then one can write s_1 as $z_p^i t_p^j \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some $1 \leq i, j \leq p$, $z_p \in Z_p$, $t_p \in T_p$ and $\lambda \in \mathbb{F}_p$ as $N_G(S) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S)) \times N_{E_q \rtimes \mathrm{SL}(E_q/Z_q)}(k_2(S))$. Then the fact that ${}^{g_1}s_1 = z_p^i\, {}^{g_1}t_p^j \begin{pmatrix} \lambda^{-1} & \lambda-\lambda^{-1} \\ 0 & \lambda \end{pmatrix}$ belongs to $N_{E_p\rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S))$ implies that $t_p^j = 1$ as well as $\lambda-\lambda^{-1}=0$ if A_1 is not a subgroup of $\{\pm \mathrm{id}\}$ and therefore $\lambda=\pm 1$. But if A_1 is a subgroup of $\{\pm \mathrm{id}\}$ then one also has that $\lambda = \pm 1$. So, in both cases, $s_1 = z_p^i(\pm \mathrm{Id})$ and ${}^{g_1}s_1 = s_1$. Therefore with $g=(g_1,1)$ if an element ${}^gs$ belongs to ${}^gS\cap N_G(S)$ then ${}^gs = s$.

Suppose that $k_2(S) = T_q \rtimes A_2$. A similar argument with $g = (1, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix})$ shows again that ${}^gS \cap N_G(S) \leq S$ $\square$

Next suppose that $k_1(S) = (T_p\times V_1)\rtimes A_1$ and $k_2(S) = (T_q\times V_2)\rtimes A_2$. Then one has,

$$p_1(S) \leq N_{E_p\rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S)) = Z_p\times T_p \rtimes \{\begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda\in \mathbb{F}_p^*, \alpha\in\mathbb{F}_p\}$$

as $p_1(S)$ normalizes $k_1(S)$ and the equality comes from Proposition 5.39. Similarly $p_2(S)$ is a subgroup of $N_{E_q\rtimes \mathrm{SL}(E_q/Z_q)}(k_2(S))$. First note that either Z_p is not contained in $p_1(S)$ or Z_q is not contained in $p_2(S)$. Indeed, let $s=(s_1,s_2)$ be an element of S. Then, by Lemma 1.2, there exists an isomorphism $\phi : p_1(S)/k_1(S) \to p_2(S)/k_2(S)$ such that $\phi(\overline{s_1}) = \overline{s_2}$, where $\overline{s_i}$ denotes the image of s_i in $p_i(S)/k_i(S)$. But $\overline{s_1} = \overline{z_p^i \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}}$ is of order a divisor of $p(p-1)$. Similarly $\overline{s_2}$ is of order a divisor of $q(q-1)$. As p and q are two different prime numbers, this implies that either Z_p is not contained in $p_1(S)$ or Z_q is not contained in $p_2(S)$. Otherwise p divides $q-1$ and q divides $p-1$, which is impossible. Without lost of generality, one can suppose that Z_p is not contained in $p_1(S)$. For a better understanding, we will put it as an assumption in the following lemmas even if it is not a restriction to a more specific case.

Lemma 5.45. *Let G be $\big(E_p\rtimes \mathrm{SL}(E_p/Z_p)\big)\times\big(E_q\rtimes \mathrm{SL}(E_q/Z_q)\big)$, then G has no non-trivial expansive subgroup S with $S\cap(E_p\times E_q) = T_p\times T_q$, $k_1(S) = (T_p\times V_1)\rtimes A_1$, $k_2(S) = (T_q\times V_2)\rtimes A_2$, $Z_p \nleq p_1(S)$ and $Z_q\nleq p_2(S)$.*

Proof. Take $g = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. With the assumptions, an element s of S is of the form $(t_p^j\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}, t_q^i\begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix})$ where $\mu,\lambda\in\mathbb{F}_p, t_p^j\in T_p$

and $\beta, \alpha \in \mathbb{F}_q, t_q^i \in T_q$. As

$$N_G(S) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S)) \times N_{E_q \rtimes \mathrm{SL}(E_q/Z_q)}(k_2(S)),$$

the fact that the element ${}^g s$ belongs to ${}^g S \cap N_G(S)$ implies

$${}^g\!\left(t_p^j \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}, t_q^i \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix}\right) = \left({}^{g_1}t_p^j \begin{pmatrix} \lambda^{-1} & 0 \\ \mu & \lambda \end{pmatrix}, {}^{g_2}t_q^i \begin{pmatrix} \beta^{-1} & 0 \\ \alpha & \beta \end{pmatrix}\right)$$

belongs to

$$\big(Z_p \times T_p \rtimes \{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} | \lambda \in \mathbb{F}_p^*, \mu \in \mathbb{F}_p\}\big) \times \big(Z_q \times T_q \rtimes \{ \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix} | \beta \in \mathbb{F}_p^*, \alpha \in \mathbb{F}_p\}\big).$$

To fulfill this condition, one must have $\alpha = \mu = i = j = 0$ and so ${}^g s = s^{-1}$ which belongs to S and therefore implies that ${}^g S \cap N_G(S) \leq S$. $\square$

Lemma 5.46. *Let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$, then G has no non-trivial expansive subgroup S with $S \cap (E_p \times E_q) = T_p \times T_q$, $k_1(S) = (T_p \times V_1) \rtimes A_1$, $k_2(S) = (T_q \times V_2) \rtimes A_2$, $Z_p \not\leq p_1(S)$ and*

$$\Big(p_2(S) \cap \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \}\Big)/A_2$$

is trivial or of exponent 2.

Proof. Let $\phi : p_1(S)/k_1(S) \to p_2(S)/k_2(S)$ be the isomorphism given in Lemma 1.2. Take $g = (1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. With the assumptions, an element s of S is of the form $(s_1, z_q^k t_q^i \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix})$ where $\beta, \alpha \in \mathbb{F}_q, t_q^i \in T_q$. As $N_G(S) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S)) \times N_{E_q \rtimes \mathrm{SL}(E_q/Z_q)}(k_2(S))$, the fact that the element ${}^g s$ belongs to ${}^g S \cap N_G(S)$ implies, with the same calculation as the proof of Lemma 5.45, that $\alpha = i = 0$ therefore $s = (s_1, z_q^k \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix})$ and ${}^g s = (s_1, z_q^k \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix})$. The element ${}^g s$ belongs to S if and only if $\overline{z_q^k \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix}} = \phi(\overline{s_1})$, which is equal to $\overline{z_q^k \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}}$ and so ${}^g s$ belongs to

S if and only if $\overline{\begin{pmatrix}\beta^{-1} & 0\\ 0 & \beta\end{pmatrix}} = \overline{\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}}$ in $p_2(S)/k_2(S)$, which is the case if $\overline{\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}}^2 = 1$ in $p_2(S)/k_2(S)$. This is fulfilled as

$$\Big(p_2(S) \cap \{ \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \mid \lambda \in \mathbb{F}_p^*\}\Big)/A_2$$

is of exponent 2. □

Remark 5.47. The remaining cases are when $k_1(S) = (T_p \times V_1) \rtimes A_1$, $k_2(S) = (T_q \times V_2) \rtimes A_2$, $Z_q \leq p_2(S)$ and $(p_2(S) \cap \{ \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \mid \lambda \in \mathbb{F}_p^*\})/A_2$ is not trivial or of exponent different from 2 and afterwards one has to deal with the case where b is not trivial.

$\mathbf{S \cap (E_p \times E_q) = T_p}$

Such a subgroup is such that

$$T_p \leq S \leq N_G(S) \leq N_G(T_p) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(T_p) \times (E_q \rtimes \mathrm{SL}(E_q/Z_q)).$$

Moreover $k_1(S) \cap E_p = S \cap (E_p \rtimes \mathrm{SL}(E_p/Z_p)) \cap E_p = S \cap E_p = T_p$, therefore by Proposition 5.39 one knows that either $k_1(S) = T_p \rtimes {}^bA_1$ for b an element of $N_{\mathrm{SL}(E_p/Z_p)}(T)$ with A_1 a subgroup of $\{ \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \mid \lambda \in \mathbb{F}_p^*\}$ or $k_1(S) = (T_p \times V_1) \rtimes {}^bA_1$ for b an element of $N_{\mathrm{SL}(E_p/Z_p)}(T)$, where

$$V_1 = \{\rho(u)u \mid u \in U_p\}$$

and $\rho : U_p \to Z_p$ is a homomorphism for $U_p = \{\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix} \mid \alpha \in \mathbb{F}_p\}$ and A_1 could be taken as a subgroup of the diagonal matrices. Because $S \cap E_q = 1$ one has that $k_2(S) = \{\varphi(h)h \mid h \in H\}$ where $\varphi : H \to E_q$, with $\varphi(hk) = \varphi(h)\,{}^h\varphi(k)$ for all $h, k \in H$ and $H \leq SL(E_q/Z_q)$.

Lemma 5.48. *Let G be $\big(E_p \rtimes \mathrm{SL}(E_p/Z_p)\big) \times \big(E_q \rtimes \mathrm{SL}(E_q/Z_q)\big)$, then G has no non-trivial expansive subgroup S with $S \cap (E_p \times E_q) = T_p$ and $k_1(S) = T_p \rtimes A_1$.*

Proof. The same argument, as in the proof of Lemma 5.44, works with $g = (\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, 1)$ knowing that

$$N_G(S) \leq N_{E_p \rtimes \mathrm{SL}(E_p/Z_p)}(k_1(S)) \times (E_q \rtimes \mathrm{SL}(E_q/Z_q)).$$

□

Remark 5.49. To complete this case, one has still to deal with $k_1(S) = (T_p \times V_1) \rtimes {}^bA_1$ and also, if b is not trivial with $k_1(S) = T_p \rtimes {}^bA_1$.

Chapter 6

Stabilizing Bisets and Roquette Groups

In this chapter, our goal is to know if there exists a non-trivial biset U n-stabilizing a simple faithful module for Roquette groups. One treats the same examples as in the previous chapter. Namely,

- Roquette p-groups.
- Some simple groups.
- Groups with cyclic Fitting subgroup.
- Groups with extraspecial groups in the Fitting subgroup.

6.1 Roquette p-groups

The case of Roquette p-groups has already been studied in [3]. It is shown that if U is a stabilizing biset for a faithful simple module, then U has to be reduced to an isomorphism, see Theorem 3.10. One will discuss the case of n-stabilizing bisets for $n > 1$. First we recall the character tables of generalized quaternion, dihedral and semi-dihedral groups.

One starts with the character table of $Q_{2^{k+1}}$ and $D_{2^{k+1}}$.

	1	s	sr	r	$\dots$	$r^j(j < 2^{k-1})$	$\dots$	$r^{2^{k-1}}$
χ_1	1	1	1	1	$\dots$	1	$\dots$	1
χ_2	1	1	-1	-1	$\dots$	$(-1)^j$	$\dots$	1
χ_3	1	-1	1	-1	$\dots$	$(-1)^j$	$\dots$	1
χ_4	1	-1	-1	1	$\dots$	1	$\dots$	1
χ_{D_h}	2	0	0	$2\cos h\theta$	$\dots$	$2\cos hj\theta$	$\dots$	$2(-1)^h$

where $\theta = 2\pi/2^k$ and $1 \leq h \leq 2^{k-1} - 1$.

Finally here is the character table of $SD_{2^{k+1}}$, for $k \geq 3$,

	1	s	sr	r	$\dots$	$r^j(j < 2^{k-1})$	$\dots$	$r^{2^{k-1}}$
χ_1	1	1	1	1	$\dots$	1	$\dots$	1
χ_2	1	1	-1	-1	$\dots$	1	$\dots$	1
χ_3	1	-1	1	-1	$\dots$	$(-1)^j$	$\dots$	1
χ_4	1	-1	-1	1	$\dots$	$(-1)^j$	$\dots$	1
χ_{D_h}	2	0	0	α	$\dots$	β	$\dots$	$2(-1)^h$

where $1 \leq h \leq 2^{k-1} - 1$ and α and β are non-zero elements. For more details see page 18 of [10].

Theorem 6.1. *Let p be a prime number and let P be a Roquette p-group of order p^{k+1}. Let $U := \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$ be a n-stabilizing biset for L where L is a simple faithful $\mathbb{C}P$-module. Then one has $B = D = 1$.*

Proof. First note that by 2.18 and 2.19, the P-cores of B and D are trivial. In particular, $B \cap Z(P)$ and $D \cap Z(P)$ have to be trivial, as these intersections are contained in the P-core of, respectively, B and D. It follows from Lemma 5.1 that B and D are trivial, except possibly if $p = 2$, P is dihedral or semi-dihedral, and B and D are non-central subgroups of order 2. Therefore one has four cases to treat

- B and D are non-central subgroups of order 2,
- B is a non-central subgroup of order 2 and $D = 1$,
- $B = 1$ and D is a non-central subgroup of order 2,
- $B = 1$ and $D = 1$.

One starts with a general remark on the first three cases that occur only if P is dihedral (with $k \geq 3$), or semi-dihedral (with $k \geq 3$). As L is a simple faithful module, by looking at the character tables of $D_{2^{k+1}}$ and $SD_{2^{k+1}}$, one sees that the character of L is χ_{D_h} for h odd and $1 \leq h \leq 2^{k-1}-1$. Also the character of $\mathrm{Res}^P_{C_2\times Z(P)}(L)$, for C_2 a non-central subgroup of order 2, is the following

	1	c	cz	z
$\chi_{\mathrm{Res}^P_{C_2\times Z(P)}(L)}$	2	0	0	-2

where c generates C_2 and z generates $Z(P)$. Thus the module $\chi_{\mathrm{Res}^P_{C_2\times Z(P)}(L)}$ splits in the sum of the following two characters of degree one

	1	c	cz	z
	1	1	-1	-1
	1	-1	1	$-1.$

Therefore, $\mathrm{Defres}^P_{C_2\times Z(P)/C_2}(L)$ is the sign representation.

One proves now that the first three cases are impossible. Consider first the case where B is a non-central subgroup of order 2 without assumption on D. By Lemma 5.1, one knows that $N_P(B) = B \times Z(P)$. This fact forces us to have $A = N_P(B)$, otherwise the A/B-module $M = \mathrm{Iso}_\phi \mathrm{Defres}^P_{C/D}(L)$ would be trivial and by Proposition 2.16 the module L would be trivial as well but this contradicts the fact that L is faithful. As A/B is of order 2, the module M is therefore forced to be copies of the sign representation M_1. As L is of dimension 2, either $M = M_1$ or $M = 2M_1$. We would like to know if $\mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))$ is a sum of copies of L. To do so one uses the powerful scalar product on characters and Frobenius reciprocity

$$\left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M_1))\right\rangle = \left\langle \mathrm{Res}^P_A(L), \mathrm{Inf}^A_{A/B}(M_1)\right\rangle = 1.$$

The latter equality holds because, as described in the general remarks above, $\mathrm{Res}^P_A(L)$ is the sum of two non-isomorphic represention of degree 1. It is easy to check that one of them is $\mathrm{Inf}^A_{A/B}(M_1)$. Thus at most two copies of L are in the decomposition of $\mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))$, which has dimension $2^{k-1}\dim M$. As $k \geq 3$ one has

$$\dim \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M)) = 2^{k-1}\dim M > \left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))\right\rangle \dim L.$$

Indeed, if $k > 3$, or $k = 3$ but $\dim M = 2$, then $2^{k-1}\dim M > 4 \geq \left\langle L, \mathrm{Ind}^P_A(\mathrm{Inf}^A_{A/B}(M))\right\rangle \dim L$ and if $k = 3$ and $\dim M = 1$ then $2^{k-1}\dim M =$

$4 > 2 = \langle L, \mathrm{Ind}_A^P(\mathrm{Inf}_{A/B}^A(M)) \rangle \dim L$. So $\mathrm{Ind}_A^P(\mathrm{Inf}_{A/B}^A(M))$ contains other modules, non-isomorphic to L, in its decomposition which implies that it cannot be the sum of n copies of L.

Assume now that $B = 1$ and D is a non-central subgroup of order 2. As above one has $C = N_P(D) = D \times Z(P)$ and M is the sign representation. Moreover the subgroup A is of order 2 as A is isomorphic to C/D. We would like to know if $\mathrm{Ind}_A^P(M)$ is a sum of copies of L. Again using the scalar product one has

$$\langle L, \mathrm{Ind}_A^P(M) \rangle = \langle \mathrm{Res}_A^P(L), M \rangle \leq 2.$$

The latter inequality comes because L is of dimension 2 and therefore the sign representation can only occur twice. In fact, it is easy to see that it is equal to 2 if $A = Z(P)$ and 1 otherwise. In any case one has

$$\dim \mathrm{Ind}_A^P(M) = 2^k > 4 = 2 \dim L \geq \langle L, \mathrm{Ind}_A^P(M) \rangle \dim L.$$

This means again that $\mathrm{Ind}_A^P(M)$ contains other modules, non-isomorphic to L, in its decomposition and so it cannot be the sum of n copies of L.

Finally we are restricted to the last case, namely $B = 1$ and $D = 1$ and the result follows. □

We are therefore reduced to $U := \mathrm{Ind}_A^P \mathrm{Iso}_\phi \mathrm{Res}_C^P$. In this case n must be equal to $|P : A|$ as the restriction does not change the dimension of the module. Now, if we suppose that the n-stabilizing biset is strongly minimal, then this implies that $A = C$ and A is a normal subgroup of P. Indeed, by Corollary 2.13, one can suppose that $(A, 1)$ and $(C, 1)$ are linked, which implies that $A = C$ and by Theorem 2.12, there are n double (A, A)-cosets in P and as $n = |P : A|$ this forces A to be a normal subgroup of P.

This is why we focus on that situation and completly describe it in the following theorem.

Theorem 6.2. *Let p be a prime number and let P be a Roquette p-group of order p^{k+1}. Let A be a normal subgroup of P, $U := \mathrm{Ind}_A^P \mathrm{Iso}_\phi \mathrm{Res}_A^P$ and $n = |P : A|$. Then the following conditions are equivalent*

1. *P is generalized quaternion (with $k \geq 2$), dihedral (with $k \geq 3$), or semi-dihedral (with $k \geq 3$) and A is the maximal cyclic subgroup of order p^k. In particular, n and p are equal to 2.*

2. *$U(L) \cong nL$ for all faithful $\mathbb{C}P$-modules L.*

3. *$U(L) \cong nL$ for a faithful $\mathbb{C}P$-module L.*

Proof. Throughout the proof we denote by M the module $\operatorname{Res}_A^P(L)$. First suppose that the first condition holds and prove *2*. Let L be an arbitrary faithful $\mathbb{C}P$-module. By Clifford's Theorem, one has $\operatorname{Res}_A^P(L) \cong V \oplus {}^gV$, for V a representation of dimension 1 of A. So

$$\operatorname{Ind}_A^P \operatorname{Iso}_\phi \operatorname{Res}_A^P(L) \cong \operatorname{Ind}_A^P \operatorname{Iso}_\phi(V) \oplus \operatorname{Ind}_A^P \operatorname{Iso}_\phi({}^gV)$$

and using Proposition 1.17 and the fact that A is normal one has

$$\operatorname{Ind}_A^P \operatorname{Iso}_\phi({}^gV) \cong \operatorname{Ind}_A^P \operatorname{Iso}_\phi(V).$$

Thus, one obtains that $U(L) \cong 2\operatorname{Ind}_A^P \operatorname{Iso}_\phi(V)$. Moreover, using Frobenius reciprocity one has $U(L) \cong L \oplus \big(L \otimes \operatorname{Inf}_{P/A}^P(M_1)\big)$, where M_1 is the sign represention for P/A. So

$$2\operatorname{Ind}_A^P \operatorname{Iso}_\phi(V) \cong L \oplus \big(L \otimes \operatorname{Inf}_{P/A}^P(M_1)\big)$$

and by Krull Schmidt theorem one deduces that $\operatorname{Ind}_A^P \operatorname{Iso}_\phi(V) \cong L$ and therefore $U(L) \cong 2L$.

The fact that *2* implies *3* is obvious.

Prove now that *3* implies *1* by proving the contrapositive. Suppose first that P is a cyclic group. Then by Clifford's Theorem $\operatorname{Res}_A^P(L) = V$ where V is a representation of dimension 1 of A. But then one has

$$\big\langle L, \operatorname{Ind}_A^P \operatorname{Iso}_\phi(V)\big\rangle = \big\langle \operatorname{Res}_A^P(L), \operatorname{Iso}_\phi V\big\rangle \leq 1.$$

Yet, the dimension of $\operatorname{Ind}_A^P(V)$ is $|P : A|$ which is stricly bigger than one and so other modules than L appear in the decomposition of $\operatorname{Ind}_A^P(V)$ which means that it cannot be a sum of copies of L.

Suppose that P is not cyclic. One starts with A a maximal non-cyclic subgroup of P. One knows that in this case $|P : A| = 2$. Using again Frobenius reciprocity one has $U(L) = L \oplus \big(L \otimes \operatorname{Inf}_{P/A}^P(M_1)\big)$ where M_1 is the sign representation of P/A. In order to have n-stabilization one needs $L \otimes \operatorname{Inf}_{P/A}^P(M_1)$ to be isomorphic to L. In terms of characters one must have $\chi_L(g) = 0$ for all g which are not in A, as these elements act on $\operatorname{Inf}_{P/A}^P(M_1)$ as -1. Looking at the character tables of non-cyclic Roquette p-groups one can check that this does not occur if A is a maximal non-cyclic subgroup of P. So U does not n-stabilize L. As a consequence, one deduces that $\mathrm{Res}^P_A(L)$ is irreducible. Indeed, if not then by Clifford's Theorem one could decompose $\mathrm{Res}^P_A(L)$ as the sum of two conjugate modules and using the same argument as above it would give us an example of 2-stabilization. As $\mathrm{Res}^P_A(L)$ is irreducible, one can actually see that every irreducible A-module can be written in this manner. The reason is that $\mathrm{Res}^P_A(\mathbb{C}P) = \mathbb{C}A \oplus \mathbb{C}A$. Furthermore, by the argument above, note that this implies that if V is an irreducible A-module, then $\mathrm{Ind}^P_A(V) \cong \mathrm{Ind}^P_A \mathrm{Res}^P_A(L) \cong L \oplus \big(L \otimes \mathrm{Inf}^P_{P/A}(M_1)\big) \cong L_1 \oplus L_2$ for L_1 and L_2 two non-isomorphic irreducible $\mathbb{C}P$-modules.

Finally, suppose that P is not cyclic and A is not maximal. Then, there exists a non-cyclic maximal subgroup H containing A and

$$\mathrm{Ind}^P_A(M) \cong \mathrm{Ind}^P_H \mathrm{Ind}^H_A(M).$$

Decompose $\mathrm{Ind}^H_A(M)$ as the sum of irreducible H-modules V_i and using the remark above on the induction on modules from a maximal subgroup, one obtains that

$$\mathrm{Ind}^P_A(M) \cong \mathrm{Ind}^P_H(\oplus_i V_i) \cong \oplus_i (L_{i1} \oplus L_{i2})$$

with, for all i, L_{i1} and L_{i2} two non-isomorphic irreducible P-modules. Thus the module $\mathrm{Ind}^P_A(M)$ cannot be only n copies of a module L. □

6.2 Some simple groups

In this section, one treats examples of simple groups. One looks at the existence of n-stabilizing bisets. To do so, one uses the existing descriptions of the subgroups and simple modules of these groups. One also uses GAP for the calculations.

6.2.1 A_5

Theorem 6.3. *Let* $U := \mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$ *be a stabilizing biset for* L *a simple faithful* $\mathbb{C}A_5$*-module. Then one has* $(A, B) = (C, D) = (A_5, 1)$.

Proof. By Proposition 3.15 one can suppose that U is minimal. Looking at the simple faithful A_5-modules, the dimension of L can only be 3,4 or 5. If $U(L) \cong L$ then

$$\dim L = |G : A| \dim \mathrm{Defres}^G_{C/D}(L).$$

As L is faithful and A_5 is simple if $A = A_5$ then $B = 1$ and moreover there is no subgroup of index $2, 3$ or 4 in A_5. Therefore, the only other possibility is that $\dim \mathrm{Defres}^G_{C/D}(L) = 1$ and $\dim L = |G : A| = 5$. In this case A is one of the copies of A_4 in A_5. The only possibilities for B are $1, V_4$ or A_4. By Proposition 3.9 if $B = 1$ then $A = G$ and if $B = A$ then $\mathrm{Defres}^G_{C/D}(L)$ has to be the trivial module but by Proposition 8.5 of [3] this implies that $A = G$. Both cases are impossible so this means that $B = V_4$. By Proposition 2.20 one has $|C| \leq |A|$ and $|D| \leq |B|$. Therefore, looking at the subgroups of A_5, either (C, D) is conjugate to (A_4, V_4) or $(C, D) = (C_3, 1)$. The latter case is impossible because it would imply the following equality

$$1 = \dim \mathrm{Defres}^G_{C_3/1}(L) = \dim \mathrm{Res}^G_{C_3}(L) = \dim L = 5.$$

Finally the only case to treat is $U = \mathrm{Indinf}^G_{A_4/V_4} \mathrm{Iso}\, \mathrm{Defres}^G_{A_4/V_4}$ and L is the simple module of dimension 5. An easy calculation, which can be made by GAP, shows that $\mathrm{Res}^G_{A_4}(L)$ is the sum of the three non-trivial simple representation of A_4. As V_4 acts trivially on both representation of dimension one but not on the one of dimension three, one concludes that $\mathrm{Defres}^G_{A_4/V_4}(L)$ is of dimension two. Therefore U cannot stabilize L. $\square$

Remark 6.4. Let $U = \mathrm{Indinf}^G_{A_4/V_4} \mathrm{Defres}^G_{A_4/V_4}$ and take L to be the simple module of dimension five. Then one has $U(L) \cong 2L$. Indeed, as seen in the previous proof $\mathrm{Defres}^G_{A_4/V_4}(L)$ is the sum of two non-trivial modules of dimension one. A quick calculation, which can be made with GAP, shows that if you apply $\mathrm{Indinf}^G_{A_4/V_4}$ to these modules you end up with two copies of L.

6.2.2 A_6

Theorem 6.5. *Let $U := \mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$ be a stabilizing biset for L a simple faithful $\mathbb{C}A_6$-module. Then one has $(A, B) = (C, D) = (A_6, 1)$.*

Proof. By Proposition 3.15 one can suppose that U is minimal. The dimension of L can only be 5,8,9 or 10. If $U(L) \cong L$ then

$$\dim L = |G : A| \dim \mathrm{Defres}^G_{C/D}(L).$$

As L is faithful and A_6 is simple if $A = A_6$ then $B = 1$ and moreover there is no subgroup of index $2, 3, 4$ or 5 in A_6. Therefore, the only possibility is that $\dim \mathrm{Defres}^G_{C/D}(L) = 1$ and $\dim L = |G : A| = 10$. In this case A is a maximal subgroup of A_6 of order 36 and is of the form $(C_3 \times C_3) \rtimes C_4$. As A is maximal, one has $A = N_G(B)$. The only possibilities for B are $1, C_3 \times C_3, (C_3 \times C_3) \rtimes C_2$ or A. By Proposition 3.9 if $B = 1$ then $A = G$ and if $B = A$ then $\mathrm{Defres}^G_{C/D}(L)$ has to be the trivial module but by Proposition 8.5 of [3] this implies that $A = G$. If $B = (C_3 \times C_3) \rtimes C_2$ then $\mathrm{Defres}^G_{C/D}(L)$ has to be the sign representation and using GAP one shows that if one applies $\mathrm{Indinf}^G_{A/(C_3\times C_3)\rtimes C_2}$ to the sign representation one ends up with a reducible representation. Therefore the only possibility is that $B = C_3 \times C_3$ and $A/B \cong C_4$. By Proposition 2.20 one has $|C| \leq |A|$ and $|D| \leq |B|$. Therefore, looking at the subgroups of A_6, either (C, D) is conjugate to (A, B) or $(C, D) = (C_4, 1)$. The latter case is impossible because it would imply the following equality

$$1 = \dim \mathrm{Defres}^G_{C_4/1}(L) = \dim \mathrm{Res}^G_{C_4}(L) = \dim L = 10.$$

Finally the only case to treat is

$$U = \mathrm{Indinf}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3} \mathrm{Iso}\, \mathrm{Defres}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3}$$

and L is the simple module of dimension 10. An easy calculation shows that $\mathrm{Res}^G_{(C_3\times C_3)\rtimes C_4}(L)$ is the sum of four non-trivial simple representations of $(C_3 \times C_3) \rtimes C_4$, two of dimension one and two of dimension four. As $C_3 \times C_3$ acts trivially on both representations of dimension 1 but not on the ones of dimension four we conclude that $\mathrm{Defres}^G_{A_4/V_4}$ is of dimension two. Therefore U cannot stabilize L. $\square$

Remark 6.6. Let $U = \mathrm{Indinf}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3} \mathrm{Defres}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3}$ and take L the be the simple module of dimension 10. Then one has $U(L) \cong 2L$. Indeed, as seen in the previous proof $\mathrm{Defres}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3}(L)$ is the sum of two non-trivial modules of dimension one. A quick calculation, which can be made with GAP, shows that if you apply $\mathrm{Indinf}^G_{(C_3\times C_3)\rtimes C_4/C_3\times C_3}$ to these modules you end up with two copies of L.

6.2.3 A_7

Theorem 6.7. *Let $U := \mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$ be an n-stabilizing biset for L a simple faithful $\mathbb{C}A_7$-module. Then one has $(A, B) = (C, D) = (A_7, 1)$.*

Proof. First one treats the case $n = 1$, which means that $U(L) \cong L$ and therefore L can be obtained by an induction from a subgroup A of A_7. As L is simple it must be induced from a simple module. One uses GAP to see that every induction of a simple module from a subgroup of A_7 is reducible. Therefore no such U can exist if $A < A_7$.

Let's treat the case $n > 1$. Let M be $\operatorname{Defres}^G_{C/D}(L)$ and write M as the sum of simple modules $M = \oplus_i M_i$. Then $nL = \oplus_i \operatorname{Indinf}^G_{A/B} \operatorname{Iso}_\phi(M_i)$ and so $\operatorname{Indinf}^G_{A/B} \operatorname{Iso}_\phi(M_i) = mL$ for an $m \leq n$. This means that by inducing a simple $\mathbb{C}A$-module $\operatorname{Inf}^B_{A/B}(M_i)$ one should obtain m copies of L. Using GAP, one can actually see that the induction of a simple module from a subgroup of A_7 never gives several copies of the same module L. □

Remark 6.8. One can see in this example that even if one has 3-expansive subgroups in A_7, as seen in the previous chapter see section 5.2, there is no n-stabilizing biset for simple faithful $\mathbb{C}A_7$-modules.

6.2.4 $\mathrm{PSL}_2(\mathbb{F}_{11})$

Theorem 6.9. *Let $U := \operatorname{Indinf}^G_{A/B} \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}$ be a stabilizing biset for a simple faithful $\mathbb{C}\,\mathrm{PSL}_2(\mathbb{F}_{11})$-module L. Then one has $(A, B) = (C, D) = (\mathrm{PSL}_2(\mathbb{F}_{11}), 1)$.*

Proof. By Proposition 3.15 one can suppose that U is minimal. The dimension of L can only be 5,10,11 or 12. If $U(L) \cong L$ then

$$\dim L = |G : A| \dim \operatorname{Defres}^G_{C/D}(L).$$

As L is faithful and $\mathrm{PSL}_2(\mathbb{F}_{11})$ is simple, if $A = \mathrm{PSL}_2(\mathbb{F}_{11})$ then $B = 1$. Moreover there is no subgroup in $\mathrm{PSL}_2(\mathbb{F}_{11})$ of index smaller than 11. Therefore, the only possibilities are that $\dim \operatorname{Defres}^G_{C/D}(L) = 1$ and $\dim L = |G : A| =$ 11 or 12. In the first case case A is one of the copies of A_5 in $\mathrm{PSL}_2(\mathbb{F}_{11})$ so $B = 1$ or $B = A_5$ but by Proposition 3.9 if $B = 1$ then $A = G$ and if $B = A_5$ then $\operatorname{Defres}^G_{C/D}(L)$ has to be the trivial module. But by Proposition 8.5 of [3] this implies that $A = G$. Both cases are impossible so this means that A is of index 12 and of the form $C_{11} \rtimes C_5$. Again B could only be 1, C_{11} or A, but the first and the last case have to be eliminated for the same reasons as above. Therefore $B = C_{11}$ and one has $A/B \cong C_5$. By Proposition 2.20 one has $|C| \leq |A|$ and $|D| \leq |B|$. Therefore, looking at the subgroups of

$\mathrm{PSL}_2(\mathbb{F}_{11})$, either (C,D) is conjugate to (A,B) or $(C,D) = (C_5,1)$. The latter case is impossible because it would imply the following equality

$$1 = \dim \mathrm{Defres}^G_{C_5/1}(L) = \dim \mathrm{Res}^G_{C_5}(L) = \dim L = 12.$$

Finally the only case to treat is $U = \mathrm{Indinf}^G_{(C_{11}\rtimes C_5)/C_{11}} \mathrm{Iso}\, \mathrm{Defres}^G_{(C_{11}\rtimes C_5)/C_{11}}$ and L is one of the simple modules of dimension 12. An easy calculation shows that $\mathrm{Res}^G_{C_{11}\rtimes C_5}(L)$ is the sum of four non-trivial simple representations of $C_{11} \rtimes C_5$, two of dimension one and two of dimension five. As C_{11} acts trivially on both representations of dimension one but not on the ones of dimension five one concludes that $\mathrm{Defres}^G_{(C_{11}\rtimes C_5)/C_{11}}$ is of dimension two. Therefore U cannot stabilize L. □

Remark 6.10. Let $U = \mathrm{Indinf}^G_{(C_{11}\rtimes C_5)/C_{11}} \mathrm{Defres}^G_{(C_{11}\rtimes C_5)/C_{11}}$ and take L the be the simple module of dimension twelve. Then applying $\mathrm{Indinf}^G_{(C_{11}\rtimes C_5)/C_{11}}$ to these modules one ends up with two copies of L.

Remark 6.11. The group $C_{11} \rtimes C_5$ has four non-trivial simple representations of dimension one. The group $\mathrm{PSL}_2(\mathbb{F}_{11})$ has two simple modules of dimension 12. The argument in the previous proof works for both of these modules but the decomposition of $\mathrm{Res}^G_{C_{11}\rtimes C_5}(L)$ is not the same. The two modules of dimension five appearing in the decomposition are the same. But the pair of modules of dimension 1 are disjoint. However C_{11} acts trivially on them and if one applies $\mathrm{Indinf}^G_{(C_{11}\rtimes C_5)/C_{11}}$ to these modules one ends up with two copies of L.

6.3 Groups with cyclic Fitting subgroup

In this section one proves that if G is a solvable group such that $F(G) = C_n = \prod_i C_{p_i^{k_i}}$ and U is a stabilizing biset for a simple faithful $\mathbb{C}G$-module, then U has to be reduced to an isomorphism. Then one describes the case of ν-stabilizing bisets as one did for Roquette p-groups where ν is an integer. Suppose $n = 2^k p_1^{k_1} \dots p_m^{k_m}$ for some distinct odd primes p_i and integer k_i, so $C_n = C_{2^k} \times \prod_{i=1}^m C_{p_i^{k_i}}$. In section 5.3 Corollary 5.10 one has seen that such a group G is Roquette. Also, one has the following exact sequence

$$1 \longrightarrow C_n \longrightarrow G \longrightarrow S \longrightarrow 1$$

where S is a subgroup of $\mathrm{Aut}(C_n)$. Suppose moreover that S is a subgroup of $C_2 \times \prod_i C_{p_i-1}$ where C_2 is either generated by $\beta_1 : g \mapsto g^{-1}$ or $\beta_2 : g \mapsto g^{-1+2^{k-1}}$ where g is a generator of C_{2^k} with $k > 2$, or $S \leq \prod_i C_{p_i-1}$ if $k \leq 2$. One starts with a general lemma.

Lemma 6.12. *Let C_{2^k} be a cyclic group of order 2^k and C_2 its subgroup of order 2. Denote by T_+ and T_- the trivial and the sign $\mathbb{C}$-representation of dimension 1 of C_2. Then the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ decomposes as the sum of all non-faithful representations of C_{2^k} and the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-)$ decomposes as the sum of all faithful representations of C_{2^k}.*

Proof. Observe that

$$\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-) \oplus \mathrm{Ind}_{C_2}^{C_{2^k}}(T_+) = \mathrm{Ind}_{C_2}^{C_{2^k}}(T_- \oplus T_+) = \mathrm{Ind}_{C_2}^{C_{2^k}}(\mathbb{C}C_2) = \mathbb{C}C_{2^k}.$$

But $\mathbb{C}C_{2^k}$ decomposes as the sum of all simple $\mathbb{C}C_{2^k}$-modules. Using Krull-Schmidt Theorem and the fact that $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ is not faithful as C_2 is in its kernel, one can conclude that $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_+)$ decomposes as the sum of all non-faithful representations of C_{2^k}. Therefore the module $\mathrm{Ind}_{C_2}^{C_{2^k}}(T_-)$ has to decompose as the sum of all faithful representations of C_{2^k}. □

Theorem 6.13. *Let G be a Roquette group with $F(G) = C_n$. Let $U := \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^{G}$ be a ν-stabilizing biset for L where L is a simple faithful $\mathbb{C}G$-module. Then $B = 1$ and A contains $C_2 \dots C_{p_m}$.*

Proof. The idea of this proof is to restrict the module L to certain well-chosen subgroups using once Clifford Theory and then Mackey's formula as νL can be written as $U(L)$. Then one finds information by the fact that these two decompositions should be isomorphic.

By Proposition 2.18, one knows that B has a trivial G-core. Therefore $B \cap C_n = 1$. Denote by $\tilde{M}$ the A-module $\mathrm{Inf}_{A/B}^{A} \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^{G}(L)$ and by H the product $C_2 \dots C_{p_m}$. Using Clifford Theory one has

$$\mathrm{Res}_H^G(\nu L) \cong \nu \,\mathrm{Res}_H^G(L) \cong \nu \oplus_{g \in G/I} \mu\, {}^{g}V$$

where V is a simple H-module and $I := \{g \in G |\, {}^{g}V \cong V\}$. As L is faithful the module $\mathrm{Res}_H^G(L)$ is also faithful and so is V, because $\ker({}^{g}V) = {}^{g}\ker(V) =$

$\ker(V)$, as the subgroups of H are characteristic. Now by Mackey formula one has

$$\mathrm{Res}^G_H(\nu L) \;=\; \mathrm{Res}^G_H(\mathrm{Ind}^G_A(\tilde{M})) \cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\ {}^g\big(\mathrm{Res}^A_{H\cap A}(\tilde{M})\big).$$

Let Q be a complement of $H\cap A$ in H. Such a complement exists because $H\cap A \leq C_2 \dots C_{p_m}$ and so $Q = C_{|H|/|H\cap A|}$. Now one extends $\mathrm{Res}^A_{H\cap A}(\tilde{M})$ to an H-module N by saying that Q acts trivially on N. Therefore one has $\mathrm{Res}^H_{H\cap A}(N) = \mathrm{Res}^A_{H\cap A}(\tilde{M})$. Using this in the previous equation one has:

$$\begin{aligned}
\mathrm{Res}^G_H(\nu L) &\cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\ {}^g\big(\mathrm{Res}^A_{H\cap A}(\tilde{M})\big)\\
&\cong \bigoplus_{g\in[H\backslash G/A]} \mathrm{Ind}^H_{H\cap {}^gA}\ {}^g\big(\mathrm{Res}^H_{H\cap A}(N)\big)\\
&\cong \mathrm{Ind}^H_{H\cap A}\mathrm{Res}^H_{H\cap A}(N) \oplus \bigoplus_{\substack{g\in[H\backslash G/A],\\ g\neq 1}} \mathrm{Ind}^H_{H\cap {}^gA}\ {}^g\big(\mathrm{Res}^H_{H\cap A}(N)\big)\\
&\cong N \oplus (N\otimes \mathrm{Ir}_2) \oplus \dots \oplus (N\otimes \mathrm{Ir}_f)\\
&\quad \oplus \bigoplus_{\substack{g\in[H\backslash G/A],\\ g\neq 1}} \mathrm{Ind}^H_{H\cap {}^gA}\ {}^g\big(\mathrm{Res}^H_{H\cap A}(N)\big),
\end{aligned}$$

where $\{\mathrm{Ir}_j\}$ is a set of isomorphism classes of simple $\mathbb{C}[H/H\cap A]$-modules for $1 \leq j \leq f$, with $f = |H : H\cap A|$. The kernel of N is Q but, as mentioned before, $\mathrm{Res}^G_H(L)$ is a sum of faithful modules, therefore $Q = 1$ and so $H\cap A = H$. This implies that H is a subgroup of A and therefore normalizes B, because B is normal in A. This implies that B acts trivially on H by Lemma 5.6. Therefore B is either trivial or $\pi(B)$ is generated by β_1 or β_2, where π denotes the homomorphism from G to S. Suppose the latter holds, so $k > 2$. By Clifford Theory

$$\nu\,\mathrm{Res}^G_{C_{2^k}}(L) = \nu \bigoplus_{g\in G/I_1} m_1\, {}^gL_1,$$

where L_1 is a simple C_{2^k}-module and $I_1 := \{g\in G \mid {}^gL_1 \cong L_1\}$. By definition C_n is a subgroup of I_1. As $\prod_i C_{p_i-1}$ acts trivially on C_{2^k}, it is a subgroup of I_1/C_n and so the order of G/I_1 is at most 2. This implies that there are at most 2 non-isomorphic modules appearing in $\mathrm{Res}^G_{C_{2^k}}(L)$.

On the other hand, let's use Mackey's formula, but first notice that

$$C_2 = H \cap C_{2^k} \leq A \cap C_{2^k} \leq N_G(B) \cap C_{2^k} = N_{C_{2^k}}(B) = C_2$$

where the last equality holds because either for β_1 or β_2 one has $C_{2^k}(\langle \beta_i \rangle) = \{c \in C_{2^k} \mid c^2 = 1\} = C_2$. Using this remark and Mackey's formula, let's restrict L to C_{2^k}:

$$\begin{aligned}\operatorname{Res}^G_{C_{2^k}}(\nu L) &\cong \bigoplus_{g \in [C_{2^k} \backslash G/A]} \operatorname{Ind}^{C_{2^k}}_{C_{2^k} \cap {}^gA} {}^g\big(\operatorname{Res}^A_{C_{2^k} \cap A}(\tilde{M})\big) \\ &\cong \operatorname{Ind}^{C_{2^k}}_{C_2} \operatorname{Res}^A_{C_2}(\tilde{M}) \oplus \bigoplus_{\substack{g \in [C_{2^k} \backslash G/A] \\ g \neq 1}} \operatorname{Ind}^{C_{2^k}}_{C_{2^k} \cap {}^gA} {}^g\big(\operatorname{Res}^A_{C_{2^k} \cap A}(\tilde{M})\big).\end{aligned}$$

Remark that $\operatorname{Res}^A_{C_2}(\tilde{M})$ decomposes as a sum of representations which are either the trivial or the sign representation. But the trivial cannot occur. Indeed suppose the trivial representation T_+ appears in the decomposition of $\operatorname{Res}^A_{C_2}(\tilde{M})$. Then $\operatorname{Ind}^{C_{2^k}}_{C_2}(T_+)$ is not a faithful representation as C_2 is in its kernel. This is a contradiction with the fact that $\operatorname{Res}^G_{C_{2^k}}(L)$ is faithful. Therefore $\operatorname{Res}^A_{C_2}(\tilde{M})$ is a sum of copies of the sign representation T_- and $\operatorname{Ind}^{C_{2^k}}_{C_2} \operatorname{Res}^A_{C_2}(\tilde{M}) = \oplus \operatorname{Ind}^{C_{2^k}}_{C_2}(T_-)$. But $\operatorname{Ind}^{C_{2^k}}_{C_2}(T_-)$ decomposes as the sum of all faithful representations of C_{2^k}, by Lemma 6.12. There are 2^{k-1} such non-isomorphic representations. So the module $\operatorname{Res}^G_{C_{2^k}}(L)$ decomposes with at least 2^{k-1} non-isomorphic representations. As $k > 2$ one has $2^{k-1} > 2$ and so this implies a contradiction with the decomposition using Clifford Theory. Therefore the only possibility is that $B = 1$. □

Theorem 6.14. *Let G be a Roquette group with $F(G) = C_n$. Let $U := \operatorname{Indinf}^G_{A/B} \operatorname{Iso}_\phi \operatorname{Defres}^G_{C/D}$ be a stabilizing biset for L where L is a simple faithful $\mathbb{C}G$-module. Then one has $(A, B) = (C, D) = (G, 1)$.*

Proof. By Proposition 3.15 it is sufficient to look at minimal stabilizing bisets. If U is minimal, one knows that if $B = 1$ then $A = G$ by Proposition 3.9. But Theorem 6.13 shows that $B = 1$ and so the results follows. □

One continues our investigation of ν-stabilizing bisets for $\nu > 1$. One reduces our study to strongly minimal bisets.

Theorem 6.15. *Let G be a Roquette group with $F(G) = C_n$. Let $U := \mathrm{Ind}_A^G \mathrm{Iso}_\phi \mathrm{Defres}_{C/D}^G$ be a strongly minimal ν-stabilizing biset for L where L is a simple faithful $\mathbb{C}G$-module. Then $D = 1$ and $A = C$ is a normal subgroup of G.*

Proof. First recall that by Proposition 2.19, one knows that D has a trivial G-core. Therefore $D \cap C_n = 1$. By Corollary 2.13, one can suppose that $(A, 1)$ and (C, D) are linked, which implies that $A \cap C = A$ and so A is a subgroup of C, therefore A normalizes D. As A contains $C_2 \dots C_{p_m}$, by Theorem 6.13, this implies that D acts trivially on $C_2 \dots C_{p_m}$, by Lemma 5.6. Therefore D is either trivial or $\pi(D)$ is generated by β_1 or β_2, where π denotes the homomorphism from G to S. As in the proof of Theorem 6.13, one restricts L to C_{2^k} using first Clifford Theory and secondly Mackey's formula to obtain with exactly the same arguments that $D = 1$. The key ingredient is that $A \cap C_{2^k}$ is again equal to C_2 as A normalizes D.

Finally, as the sections are linked and $D = 1$ one obtains that $A = C$. Moreover, by Theorem 2.12, there are ν double (A, A)-cosets in G but also $\nu = |G : A|$ which forces A to be a normal subgroup of G. □

One finishes our study by completely describing the the remaining case.

Theorem 6.16. *Let G be a Roquette group with $F(G) = C_n$. Let A be a normal subgroup of G, $U := \mathrm{Ind}_A^G \mathrm{Res}_A^G$ and $\nu = |G : A|$. Then the following conditions are equivalent*

1. *A contains $F(G)$.*
2. *$U(L) \cong \nu L$ for all faithful $\mathbb{C}G$-modules L.*
3. *$U(L) \cong \nu L$ for a faithful $\mathbb{C}G$-module L.*

Proof. One first proves that *1* implies *2*. Suppose first that A contains $F(G)$ and prove then that $U := \mathrm{Ind}_A^G \mathrm{Res}_A^G$ is a $|G : A|$-stabilizing biset for an arbitrary faithful $\mathbb{C}G$-module L. First note that L can be written as $\mathrm{Ind}_{F(G)}^G(\xi)$ where ξ is a primitive nth root of unity. Indeed, every irreducible $\mathbb{C}G$-module comes from a summand of an induction from $F(G)$, but the module $\mathrm{Ind}_{F(G)}^G(\xi)$ is irreducible as the conjugate representations of ξ by the action of $G/F(G)$ are not isomorphic. The condition of primitivity on the root is to ensure the faithfulness of the induced module. Furthermore, as A contains $F(G)$, then $L \cong \mathrm{Ind}_A^G(V)$ where $V := \mathrm{Ind}_{F(G)}^A(\xi)$. The A-module V

is irreducible because $\operatorname{Ind}_{F(G)}^{G}(\xi)$ is. Therefore, using Mackey's formula, one has

$$\begin{aligned} U(L) &= \operatorname{Ind}_A^G \operatorname{Res}_A^G(L) \cong \operatorname{Ind}_A^G \operatorname{Res}_A^G \operatorname{Ind}_A^G(V) \cong \bigoplus_{g\in G/A} \operatorname{Ind}_A^G({}^{g}V) \\ &= |G:A| \operatorname{Ind}_A^G(V) \cong |G:A| L, \end{aligned}$$

where the isomorphism between the first and the second line holds because A is normal. As L was arbitrarily chosen, this holds for any faithful $\mathbb{C}G$-modules L.

The fact that *2* implies *3* is obvious.

Prove now that *3* implies *1* by proving the contrapositive. Let A be a normal subgroup of G such that $A\cap F(G)$ is not equal to $F(G)$. Recall that by Theorem 6.13, one knows that A contains $C_2 \dots C_{p_m}$, so this intersection is not trivial. One shows that it is not possible to ν-stabilize L for all faithful $\mathbb{C}G$-modules L. One knows that $L \cong \operatorname{Ind}_{F(G)}^{G}(\xi)$ where ξ is a primitive nth root of unity. Then, by Mackey's formula, one has

$$\begin{aligned} U(L) &\cong \operatorname{Ind}_A^G \operatorname{Res}_A^G \operatorname{Ind}_{F(G)}^G(\xi) \cong \bigoplus_{g\in[A\backslash G/F(G)]} \operatorname{Ind}_A^G \operatorname{Ind}_{A\cap F(G)}^A \operatorname{Res}_{A\cap F(G)}^{F(G)}({}^{g}\xi) \\ &\cong \bigoplus_{g\in[A\backslash G/F(G)]} \operatorname{Ind}_{A\cap F(G)}^G \operatorname{Res}_{A\cap F(G)}^{F(G)}({}^{g}\xi) \\ &\cong |A\backslash G/F(G)| \operatorname{Ind}_{A\cap F(G)}^G \operatorname{Res}_{A\cap F(G)}^{F(G)}(\xi) \\ &\cong |A\backslash G/F(G)| \operatorname{Ind}_{F(G)}^G \operatorname{Ind}_{A\cap F(G)}^{F(G)} \operatorname{Res}_{A\cap F(G)}^{F(G)}(\xi). \end{aligned}$$

Using Frobenius reciprocity one has $\operatorname{Ind}_{A\cap F(G)}^{F(G)} \operatorname{Res}_{A\cap F(G)}^{F(G)}(\xi) \cong \oplus_j \xi\otimes \mathrm{Ir}_j$ where $\{\mathrm{Ir}_j\}$ is a set of isomorphism classes of simple $\mathbb{C}[F(G)/(F(G)\cap A)]$-modules. The sum is not reduced to one module as $A\cap F(G)$ is not equal to $F(G)$ by assumption. This means that $U(L)$ is isomorphic to

$$\bigoplus_j |A\backslash G/F(G)| \operatorname{Ind}_{F(G)}^G(\xi\otimes \mathrm{Ir}_j).$$

Thus our purpose is to show that $\operatorname{Ind}_{F(G)}^G(\xi\otimes\mathrm{Ir}_j)$ is not isomorphic to $L = \operatorname{Ind}_{F(G)}^G(\xi)$ for at least one representation Ir_j. To do so, one proves that $\xi\otimes\mathrm{Ir}$ is not conjugate, by an element of $G/F(G)$ to ξ where Ir denotes a non-trivial $\mathbb{C}[F(G)/(F(G)\cap A)]$-module. We specify which Ir is taken later on.

Let p be a prime dividing $|F(G) : A \cap F(G)|$ and note i its highest power dividing $|F(G) : A \cap F(G)|$. Choose p such that p^i is strictly smaller that p^k where k is the highest power of p such that p^k divides n. As $F(G)$ is cyclic, one decomposes Ir as the tensor product of a representation θ of C_{p^i} and a representation θ^c of its complement in $F(G)/(F(G) \cap A)$, ie $\mathrm{Ir} = \theta \otimes \theta^c$. Note that θ is a p^ith root of unity. In the same fashion $\xi = \xi_1 \otimes \xi_2$, where ξ_1 is a p^kth root of unity and ξ_2 is a representation for C_{n/p^k}. Then one has

$$\xi \otimes \mathrm{Ir} \cong \xi_1 \otimes \theta \otimes \xi_2 \otimes \theta^c.$$

One now sets Ir such that $\theta = \xi_1^{p^{k-i}}$ then one has $\xi_1 \otimes \theta = \xi_1^{1+p^{k-i}}$. Because of the assumption on S made at the beginning of the section, this representation cannot be conjugate to the representation ξ_1 by an element of $G/F(G)$. Indeed, such an element would have order a divisor of p^i as such an element must be of the following form

$$\alpha : \xi_1 \mapsto \xi_1^{1+p^{k-i}}.$$

Moreover, it is easy to check that $\alpha^\delta(\xi_1) = \xi_1^{1+\delta p^{k-i}}$ and so $\alpha^{p^i} = \mathrm{id}$. So $\xi \otimes \mathrm{Ir}$ is not conjugate to ξ. Finally one has proved that $\mathrm{Ind}_{F(G)}^G(\xi \otimes \mathrm{Ir})$ is not isomorphic to $L = \mathrm{Ind}_{F(G)}^G(\xi)$ and therefore other modules than L appear in the decomposition of $U(L)$. $\square$

6.4 p-hyper-elementary groups

Let p be a prime number. Let G be $C_n \rtimes P$ where P is a p-group and C_n is a cyclic group of order prime to p. There is an action map $\psi : P \to \mathrm{Aut}(C_n)$. Such a group is called a p-hyper-elementary group. If G is a Roquette p-hyper-elementary group, with p an odd prime, then we are again in the situation of cyclic Fitting subgroup as mentioned in Remark 5.13. Even so one gives results in this section with more flexibility on the field of our representations. Nevertheless, for $p = 2$, we are not in the situation of cyclic Fitting subgroup by Theorem 5.12. Note also that Corollary 6.23 is a generalisation of Theorem 9.6.1 page 171 of [2]. In this section, let k be a field where the irreducible representations of C_n are of degree one. For example one can take $k = \mathbb{C}$. As usual we are interested in faithful modules. Note that if the characteristic of k divides n, there is no faithful irreducible representation and therefore some results are trivially satisfied.

Theorem 6.17. *Suppose $G := C_n \rtimes P$ is Roquette for p an odd prime. Let L be a faithful simple $k[G]$-module stabilized by the biset $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$, then $(A, B) = (C, D) = (G, 1)$*

Proof. First one can assume U to be minimal by Proposition 3.15. By Clifford theory one has

$$\mathrm{Res}_{C_n}^{G}(L) \cong \bigoplus_{g \in G/I} m\, {}^{g}V,$$

where $I := \{g \in G \mid {}^{g}V \cong V\}$, m divides $|I : C_n|$ and V is an irreducible $k[C_n]$-module. By assumption the dimension of V is one so the dimension of L is $m|G : I|$. By Theorem 1.6, both m and $|G : I|$ are powers of p as I contains C_n. Therefore the dimension of L is a power of p.

Remark that A is a subgroup of $N_G(B)$ and so $|N_G(B)| = |A|d$ for some $d \in \mathbb{N}$. By proposition 2.18 one knows that the G-core of B is trivial and therefore B is conjugate to a subgroup of P. As the following argument is based on the order of $N_G(B)$ one can assume without lost of generality that B is a subgroup of P. By Lemma 5.15 one has $N_G(B) = C_{C_n}(B) \rtimes N_P(B)$. Let's compute the dimension of L.

$$\begin{aligned}
\dim L &= \dim \mathrm{Defres}_{C/D}^{G}(L)|G : A| = \dim \mathrm{Defres}_{C/D}^{G}(L)|G : N_G(B)|d \\
&= \dim \mathrm{Defres}_{C/D}^{G}(L)|P : N_P(B)||C_n : C_{C_n}(B)|d.
\end{aligned}$$

As the dimension of L is a power of p and $|C_n : C_{C_n}(B)|$ is prime to p one must have, by Lemma 5.16, that B is trivial. By Proposition 8.4 of [3] this implies, because the biset is minimal, that $A = G$ and the conclusion follows. □

Theorem 6.18. *Let H be a a p-hyper-elementary Roquette group. Then H has a unique faithful irreducible rational representation Φ_H.*

Proof. See Lemma 2.10 and Theorem 2.11 of [6]. □

Definition 6.19. Let G be $C_n \rtimes P$, a p-hyper-elementary group. Let T be a genetic subgroup of G. One defines a $\mathbb{Q}G$-module as follows

$$V(T) := \mathrm{Indinf}_{N_G(T)/T}^{G} \left(\Phi_{N_G(T)/T}\right).$$

It is a a simple module, see for example Corollary 6.3 of [3].

Definition 6.20. Let G be a group and k a field. A kG-module L is *primitive* if L is not induced from a proper subgroup.

Proposition 6.21. *Let G be $C_n \rtimes P$, a p-hyper-elementary group. Suppose G is Roquette. Then Φ_G is primitive.*

Proof. See Corollary 2.16 of [6]. □

Corollary 6.22. *Let G be $C_n \rtimes P$, a p-hyper-elementary group and suppose G is Roquette. Let L be a faithful simple $\mathbb{Q}[G]$-module stabilized by $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$, then $(A, B) = (C, D) = (G, 1)$.*

Proof. The module L could only be Φ_G, which is primitive so $A = G$ and therefore B is a normal subgroup of G. But as L is faithful the G-core of B is trivial and so $B = 1$. □

Corollary 6.23. *Let G be $C_n \rtimes P$, a p-hyper-elementary group. Let S and T be genetic subgroups of G. Suppose $V(S) \cong V(T)$, then there exists a unique g in $N_G(S)\backslash G/N_G(T)$ such that $(N_G(S), S)$ is linked to $({}^{g}N_G(T), {}^{g}T)$.*

Proof. By Corollary 6.3 of [3], one has $\Phi_{N_G(S)/S} \cong \mathrm{Defres}_{N_G(S)/S}^{G} V(S)$ and therefore, because $V(S) \cong V(T)$,

$$\Phi_{N_G(S)/S} \cong \mathrm{Defres}_{N_G(S)/S}^{G} \mathrm{Indinf}_{N_G(T)/T}^{G} \Phi_{N_G(T)/T}.$$

Using Mackey formula and the fact that $\Phi_{N_G(S)/S}$ is simple, one has

$$\Phi_{N_G(S)/S} \cong \mathrm{Btf}(N_G(S), S, {}^{g}N_G(T), {}^{g}T)\, \mathrm{Conj}_g \Phi_{N_G(T)/T},$$

for a unique $g \in [N_G(S)\backslash G/N_G(T)]$. By Proposition 6.21 the module $\Phi_{N_G(S)/S}$ cannot be obtained by induction and the fact that $\Phi_{N_G(S)/S}$ is faithful implies that it cannot be obtained by inflation. So the butterfly is reduced to

$$\mathrm{Iso}_{\psi} \mathrm{Defres}_{(N_G(S)\cap {}^{g}N_G(T))\,{}^{g}T/(S\cap {}^{g}N_G(T))\,{}^{g}T}^{{}^{g}N_G(T)/\,{}^{g}T}.$$

This means that $(N_G(S), S)$ is linked to the subsection

$$\big((N_G(S) \cap {}^{g}N_G(T))\,{}^{g}T, (S \cap {}^{g}N_G(T))\,{}^{g}T\big)$$

of $({}^{g}N_G(T), {}^{g}T)$. Interchanging the role of $\Phi_{N_G(S)/S}$ and $\Phi_{N_G(T)/T}$ one finds $(N_G(T), T)$ is linked, for a unique $h \in [N_G(T)\backslash G/N_G(S)]$, to a subsection of $({}^{h}N_G(S), {}^{h}S)$. By looking at the order of those sections, this implies actually that $(N_G(S), S)$ is linked to $({}^{g}N_G(T), {}^{g}T)$.

□

6.5 Groups with extraspecial subgroups in the Fitting subgroup.

In this section one wants to investigate groups G such that the Fitting subgroup $F(G)$ contains an extraspecial subgroup. One wants to know if there exists a non-trivial biset U stabilizing a simple faithful module for such groups. We were not able to completely settle this case as in the previous section with $F(G) = C_n$. So, one looks at particular examples. Starting with a 2-extraspecial Q_8 contained in $F(G)$ with $G := Q_8 \rtimes \mathrm{SL}_2(2)$. Then one establishes partial results for $G := E \rtimes \mathrm{Sp}(E/Z)$ with E an extraspecial group of order p^{1+2n} for an odd prime p. For $G := E \rtimes C_{p+1}$ one could prove that no such U exists, where here E has order p^3.

6.5.1 $Q_8 \rtimes S_3$

We start with the group $G := Q_8 \rtimes S_3$. It's a Roquette group. The Fitting subgroup is Q_8, an extraspecial 2-group. In [3] it is shown that S_3 is an expansive subgroup with $N_G(S_3) = S_3 \times Z(Q_8)$. Let M be the sign representation of $N_G(S_3)/S_3$. Then $L := \mathrm{Indinf}^G_{N_G(S_3)/S_3}(M)$ is stabilized by $\mathrm{Indinf}^G_{N_G(S_3)/S_3} \mathrm{Defres}^G_{N_G(S_3)/S_3}$ by Proposition 2.35. However, using GAP, one can check that there is no n-stabilizing biset for a simple module over $\mathbb{C}$ with $n > 1$.

6.5.2 $E \rtimes \mathrm{Sp}(E/Z)$

Let p be an odd prime. Let E be an extraspecial group of order p^{2r+1} and exponent p. One starts with some general results about the representations of E, $\mathrm{Sp}(E/Z)$ and $E \rtimes \mathrm{Sp}(E/Z)$. Here is a classification of simple $\mathbb{C}E$-modules.

Theorem 6.24. *Let E be an extraspecial p-group of order p^{2r+1} and z a generator for $Z(E)$. Then*

(i) E has exactly $p^{2r} + p - 1$ irreducible representations over $\mathbb{C}$.

(ii) E has p^{2r} irreducible linear representations.

(iii) E has $p - 1$ faithful irreducible representations $\phi_1, \dots, \phi_{p-1}$. Notation can be chosen so that $\phi_i(z)$ acts via the scalar ω^i on the representation module V_i of ϕ_i, where ω is some fixed primitive pth root of unity in $\mathbb{C}$.

(*iv*) *ϕ_i is of degree p^r for all $1 \leq i \leq p-1$.*

Proposition 6.25. *Let V be an irreducible $k[E]$-module of dimension p^n. Then one can extend V to an irreducible faithful $k[E \rtimes \mathrm{Sp}(E/Z)]$-module.*

Proof. See the introduction of section 5 of [8]. $\square$

We now define the Weil modules for $\mathrm{Sp}(E/Z)$ over $\mathbb{C}$. Let M be the unique irreducible $\mathbb{C}[E]$-module of dimension p^n where $Z(E)$ acts via χ, for a fixed character χ of $Z(E)$. By Proposition 6.25 one can extend M to an irreducible faithful $\mathbb{C}[E \rtimes \mathrm{Sp}(E/Z)]$-module. Now we consider M as a $\mathbb{C}[\mathrm{Sp}(E/Z)]$-module, which in fact is no longer irreducible. Indeed, denote by z the central involution in $\mathrm{Sp}(E/Z)$ and recall the following lemma.

Lemma 6.26. *Let $A, B \in \mathrm{GL}_m(\mathbb{C})$ such that $BA = BA$. Then if $E_1, \dots, E_k$ are eigenspaces for A we have $BE_i \subset E_i$ for $1 \leq i \leq k$.*

Proof. Let $v \in E_i$ and λ_i the eigenvalues for E_i. Then

$$A(Bv) = (AB)v = B(Av) = \lambda_i B(v).$$

Thus $Bv \in E_i$ and the result follows. $\square$

So if we show that z has two eigenvalues on M then with the above lemma we can conclude that M is reducible as $\mathbb{C}[\mathrm{Sp}(E/Z)]$-module. Suppose by contradiction that z acts as a scalar, so because M is a faithful $\mathbb{C}[E \rtimes \mathrm{Sp}(E/Z)]$-module, we obtain that $ze = ez$ for all $e \in E$, i.e. $zez^{-1} = e$ for all $e \in E$, which means that z acts trivially on E but since $\mathrm{Sp}(E/Z)$ acts on E as a group of automorphisms, it's a contradiction. And thus z has at least two eigenvalues. As z has order 2 this implies that z has in fact exactly two eigenvalues, namely 1 and -1. Finally, we obtain the following decomposition as $\mathbb{C}[\mathrm{Sp}(E/Z)]$-module

$$M = V_1 \oplus V_{-1} = C_M(z) \oplus [z, M].$$

We call these $\mathbb{C}[\mathrm{Sp}(E/Z)]$-submodules the *Weil modules*. Here are two fundamental properties of these modules.

Proposition 6.27.

(*i*) *The Weil modules are irreducible of dimensions $\frac{(p^n \pm 1)}{2}$.*

(*ii*) *The Weil modules are self-dual if and only if* $p \equiv 1 \mod 4$.

Proof. See the introduction of section 5 of [8] and See Proposition 3.2 of [9] for a complete proof. □

Proposition 6.28. *Let L be a faithful simple $\mathbb{C}[E \rtimes H]$-module for an extraspecial group E of order p^{2n+1} and H a subgroup of $\mathrm{Sp}(E/Z)$. Then the dimension of L is ςp^n for $n \in \mathbb{N}$ and ς is the dimension of an irreducible module of H.*

Proof. Let G be $E \rtimes H$. By Clifford theory one has

$$\mathrm{Res}^G_E(L) \cong \bigoplus_{g \in G/I} m\, {}^g\!V,$$

where $m \in \mathbb{N}$, $I := \{g \in G \mid {}^g\!V \cong V\}$ and V is an irreducible $k[E]$-module. As L is a faithful module so is V. Therefore, by the classification of the irreducible $k[E]$-modules, one knows that V is entirely characterized by the action of the center and the dimension of V is p^n. Remark that the inertial subgroup I is actually G. Indeed, the action of the center is the same on ${}^g\!V$ and V, because for an element z in Z one has $z(g \otimes V) = g \otimes zV$ as z commutes with g for all g in G. This shows that $\mathrm{Res}^G_E(L) \cong mV$.

Moreover, with M denoting the $\mathbb{C}[E \rtimes H]$-module which extends V, one has

$$\mathrm{Ind}^G_E(V) = \mathrm{Ind}^G_E \mathrm{Res}^G_E(M) \cong M \otimes \mathrm{Ind}^G_E(k) = \oplus_{U \in Irr(G/E)} M \otimes U$$

and

$$\mathrm{Ind}^G_E(mV) = \mathrm{Ind}^G_E \mathrm{Res}^G_E(L) = \oplus_{U \in Irr(G/E)} L \otimes U$$

but then

$$\mathrm{Ind}^G_E(mV) = m\, \mathrm{Ind}^G_E(V) = m(\oplus_{U \in Irr(G/E)} M \otimes U).$$

Using the trivial representation and Krull-Schmidt theorem this shows that L is isomorphic to $M \otimes W_1$ for a certain irreducible module W_1 of H. So the dimension of L is equal to the dimension of M, which is p^n, times the dimension of an irreducible module of H. □

Now one goes back to the study of stabilizing bisets.

Proposition 6.29. *Let L be a faithful simple $\mathbb{C}[E \rtimes \mathrm{Sp}(E/Z)]$-module of dimension p^n stabilized by $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$. Suppose $n > 1$, then $(A, B) = (C, D) = (E \rtimes \mathrm{Sp}(E/Z), 1)$.*

Proof. Let G be $E \rtimes \mathrm{Sp}(E/Z)$. By Proposition 2.18 we know that the G-core of B is trivial. We must have $B \cap Z = 1$, otherwise Z would be contained in the G-core of B. So only two cases are possible:

1. $B \cap E = 1$ or
2. $B \cap E = \prod_{i=1}^{k} T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3, and k is smaller than or equal to n.

Suppose that B is not trivial. Our strategy is to lower bound $|G : N_G(B)|$ by p^n. Indeed we have

$$p^n = \dim L \geq |G : A| \geq |G : N_G(B)|.$$

So if we have $|G : N_G(B)| > p^n$ then one has a contradiction and so B must be trivial and the biset reduced to an isomorphism.

In the first case, one can check that B is a subgroup of the following form:

$$\left\{ \varphi(h)h \mid h \in H \right\} \text{ where } \varphi : H \to E,$$

with $\varphi(hv) = \varphi(h)\, {}^{h}\varphi(v)$ for all $h, v \in H$ and $H \leq \mathrm{Sp}(E/Z)$.

Assume first that $\varphi = 1$ so $B = H$ and $N_G(H) = N_E(H) \rtimes N_{\mathrm{Sp}(E/Z)}(H)$. Suppose $N_E(H) = E_1 \circ \cdots \circ E_r \times \prod_{i=1}^{k} T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i. The order of $N_E(H)$ is p^{2r+k+1}. In this case one has

$$p^n = \dim L \geq |G : N_G(H)| = |E : N_E(H)||\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)|.$$

Therefore one needs to know $|\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)|$ and so one wants to find representatives of $\mathrm{Sp}(E/Z)/N_{\mathrm{Sp}(E/Z)}(H)$. The subgroup $N_E(H)$ is invariant under $N_{\mathrm{Sp}(E/Z)}(H)$ and so choosing a basis with the $2r + k$ first vectors made of elements of $N_E(H)/Z(E)$, the elements of $N_{\mathrm{Sp}(E/Z)}(H)$ are of the form

$$\begin{pmatrix} & & & \star & & & \\ & & & & & & \\ & & & & & & \\ 0 & \cdots & 0 & \star & \cdots & \star \end{pmatrix}.$$

At least the last row is made of $2r + k$ zeros. By Corollary 5.20, E is not contained in $N_E(H)$ and so r is strictly smaller than n.

Suppose first that $2r + k$ is not equal to zero. In order to find representatives, let $M(\lambda) = M(\lambda_1, \ldots, \lambda_{2n-1})$ be the following matrix

$$\begin{pmatrix} 1 & & & & -\lambda_2 & \\ & \ddots & & & -\lambda_1 & \\ & & \ddots & & -\lambda_4 & \\ & & & \ddots & \vdots & \\ & & & & -\lambda_{2n-2} & \\ & & & & 1 & \\ \lambda_1 & \cdots & \cdots & \cdots & \cdots \lambda_{2n-1} & 1 \end{pmatrix}.$$

It's easy to check that $M(\lambda)$ is an element of $\mathrm{Sp}(E/Z)$ using the following symplectic form :

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Also, $M(\lambda)^{-1}$ is the following matrix

$$\begin{pmatrix} 1 & & & & \lambda_2 & \\ & \ddots & & & \lambda_1 & \\ & & \ddots & & \lambda_4 & \\ & & & \ddots & \vdots & \\ & & & & \lambda_{2n-2} & \\ & & & & 1 & \\ -\lambda_1 & \cdots & \cdots & -\lambda_{2n-2} & \alpha & 1 \end{pmatrix},$$

where $\alpha = -\lambda_{2n-1} - 2\sum_{i=1}^{2n-3} \lambda_i\lambda_{i+1}$. An elementary calculation shows that $M(\lambda)M(\mu)^{-1} = M(\lambda_1 - \mu_1, \ldots, \lambda_{2n-2} - \mu_{2n-2}, \star)$ which is not in $N_{\mathrm{Sp}(E/Z)}(H)$ if λ is not equal to μ. As one has $p^{2r+k} - 1$ choices of λ that give elements which are not in $N_{\mathrm{Sp}(E/Z)}(H)$, one has just found at least p^{2r+k-1} different classes in $\mathrm{Sp}(E/Z)/N_{\mathrm{Sp}(E/Z)}(H)$ so that $|\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| \geq p^{2r+k-1}$. This leads us to

$$\begin{aligned} p^n &= \dim L \geq |G : N_G(H)| = |E : N_E(H)||\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| \\ &\geq \frac{p^{2n+1}}{p^{2r+k+1}} p^{2r+k-1} = p^{2n-1} > p^n. \end{aligned}$$

The last inequality holds because $n > 1$.

Now if $2r + k = 0$ this means that $N_E(H) = Z$ and so

$$\begin{aligned} p^n &= \dim L \geq |G : N_G(H)| = |E : N_E(H)||\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| \\ &\geq p^{2n}|\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| > p^n. \end{aligned}$$

In both cases, one obtains a contradiction and so B is trivial. This shows that the biset must be reduced to an isomorphism by Proposition 3.9.

Assume now that $\varphi \neq 1$ and $N_E(B) = E_1 \circ \cdots \circ E_r \times \prod_{i=1}^k T_i$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i. The order of $N_E(B)$ is p^{2r+k+1}. First recall that $C_E(B)$ is a subgroup of $C_E(H)$ by Lemma 5.23. Using exactly the same argument as for $\varphi = 1$ and the fact that $E_1 \circ \cdots \circ E_r \times \prod_{i=1}^k T_i = N_E(B) = C_E(B) \leq C_E(H) = N_E(H)$ one shows that $|\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| \geq p^{2r+k-1}$. Using the fact that $N_G(EB) = N_G(EH) = EN_G(H)$, we can conclude that

$$\begin{aligned} p^n &= \dim L \geq |G : N_G(B)| \\ &= |G : N_G(EB)||N_G(EB) : EN_G(B)||EN_G(B) : N_G(B)| \\ &\geq |G : N_G(EB)||EN_G(B) : N_G(B)| = |G : N_G(EB)||E : N_E(B)| \\ &= |G : N_G(EB)|p^{2n+1-(2r+k+1)} = p^{2n+1-(2r+k+1)}|G : EN_G(H)| \\ &= p^{2n+1-(2r+k+1)}|\mathrm{Sp}(E/Z) : N_{\mathrm{Sp}(E/Z)}(H)| \geq p^{2n+1-(2r+k+1)}p^{2r+k-1} \\ &= p^{2n-1} > p^n, \end{aligned}$$

as $n > 1$.

For the second case, namely $B \cap E = \prod_{i=1}^k T_i := T$ for T_i non-central p-subgroups of E of order p such that each T_i is contained in a different E_i, for some choice of decomposition of E as the central product of n extraspecial groups E_i of order p^3 and with the integer k smaller than or equal to n. Recall that $N_G(B) \leq N_G(T)$ because if $k \in N_G(B)$ then

$$ {}^kT = {}^k(B \cap E) = {}^kB \cap {}^kE = B \cap {}^kE = B \cap E = T. $$

Therefore, using Lemma 5.25, one obtains, in a T-basis,

$$ T \leq B \leq N_G(B) \leq N_G(T) = (E_{k+1} \circ \cdots \circ E_n \times T) \rtimes P_k $$

with

$$ P_k \leq \left\{ \begin{pmatrix} A & & \star \\ & B & \\ 0 & & A^{-t} \end{pmatrix} \mid A \in \mathrm{GL}_k(p) \text{ and } B \in \mathrm{Sp}_{2n-2k}(p) \right\}. $$

Thus, using the same argument to find representatives as before, one has $|\operatorname{Sp}(E/Z) : P_k| \geq p^{2n-k-1}$ and so one concludes

$$ \begin{aligned} p^n &= \dim L \geq |G : N_G(B)| \geq \frac{|G|}{|N_G(T)|} = \frac{|G|}{|\prod_{i=1}^k T_i \times E_{k+1} \circ \cdots \circ E_n||P_k|} \\ &= \frac{|G|}{p^{2n+1-k}|P_k|} = p^k |\operatorname{Sp}(E/Z) : P_k| \geq p^k p^{2n-k-1} = p^{2n-1} > p^n \end{aligned} $$

as $n > 1$. □

Remark 6.30. One has supposed here $n > 1$ but one will treat the case $n = 1$ in a more general framework in the next section.

6.5.3 $E \rtimes \mathrm{SL}(E/Z)$

Let p be an odd prime. Let E be an extraspecial group of order p^3 and exponent p.

Theorem 6.31. *Let L be a faithful simple $\mathbb{C}[E \rtimes \mathrm{SL}(E/Z)]$-module stabilized by $\operatorname{Indinf}_{A/B}^G \operatorname{Iso}_\phi \operatorname{Defres}_{C/D}^G$. Then $(A, B) = (C, D) = (E \rtimes \mathrm{SL}(E/Z), 1)$, except maybe if all the following conditions hold:*

1. *$B \cap E = T$ for T a non-central p-subgroup of E of order p.*
2. *$B = (T \times V) \rtimes {}^bS$, for b an element of $N_{\mathrm{SL}(E/Z)}(T)$, with S a subgroup of the diagonal, $V = \{\rho(u)u \mid u \in U\}$ and $\rho : U \to Z$ a homomorphism and U a unipotent subgroup of $\mathrm{SL}(E/Z)$.*

3. $A = N_G(B) = (T \times Z) \rtimes N_{\mathrm{SL}(E/Z)}(U \rtimes S)$.
4. *The dimension of* L *is* $p(p+1)$.
5. *The module* $\mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}(L)$ *is a non-trivial faithful module of dimension* 1.

Proof. By Proposition 3.15 , it suffices to check the result for an arbitrary minimal stabilizing biset. Let G be $E \rtimes \mathrm{SL}(E/Z)$ and M be the module $\mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}(L)$. Suppose the sections (A, B) and (C, D) are not trivial. By Propositions 6.28 and 2.18 we know that if such a module L exists its dimension is $p\varsigma$, where ς is the dimension of an irreducible module of $\mathrm{SL}(E/Z)$ and also the G-core of B is trivial. By [7] page 30 one has the possible dimensions for an irreducible representation of $\mathrm{SL}(E/Z)$. Therefore $\dim L$ can only be

$$p, p^2, p(p+1), p(p-1), p\frac{(p+1)}{2} \text{ or } p\frac{(p-1)}{2}.$$

Because the G-core of B is trivial, we must have $B \cap Z = 1$, otherwise Z would be contained in the G-core of B. So only two cases are possible:

1. $B \cap E = 1$ or
2. $B \cap E = T$ for T a non-central p-subgroup of E of order p.

In the first case, one can check that B is a subgroup of the following form:

$$\left\{\varphi(h)h \mid h \in H\right\} \text{ where } \varphi : H \to E,$$

with $\varphi(hk) = \varphi(h)\,{}^h\varphi(k)$ for all $h, k \in H$ and $H \leq \mathrm{SL}(E/Z)$. Because L is stabilized by $\mathrm{Indinf}^G_{A/B} \mathrm{Iso}_\phi \mathrm{Defres}^G_{C/D}$ one has

$$\begin{aligned}
\dim L &= \dim M|G : A| \\
&\geq \dim M|G : N_G(B)| \\
&= \dim M|G : N_G(EB)||N_G(EB) : EN_G(B)||EN_G(B) : N_G(B)| \\
&= \dim M|\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)||N_G(EB) : EN_G(B)||E : N_E(B)|.
\end{aligned}$$

The last equality holds because $N_G(EB) = N_G(EH) = E \rtimes N_{\mathrm{SL}(E/Z)}(H)$. One could actually say more. Indeed, because $|G : N_G(B)|$ divides $|G : A|$, one deduces that

$$\dim M|\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)||N_G(EB) : EN_G(B)||E : N_E(B)|$$

must divide $\dim L$.

By Corollary 5.20, one knows that E is not contained in $N_E(B)$. The only two possibilities are thus $N_E(B) = Z$ and $N_E(B) = Z \times Q$ for Q a non-central p-subgroup of E of order p. Suppose first that $N_E(B) = Z \times Q$. Then H stabilizes the image of Q in E/Z because $N_E(B) \leq N_E(H) = C_E(H)$ and so in this case

$$H = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_p \right\},$$

where the first element of the basis is an element of Q. It's a well-known result that

$$N_{\mathrm{SL}(E/Z)}(H) = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{F}_p^* \text{ and } \alpha \in \mathbb{F}_p \right\}.$$

Therefore in this case

$$\begin{aligned} \dim L &\geq \dim M |\,\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)||N_G(EB) : EN_G(B)||E : N_E(B)| \\ &= \dim M(p+1)|N_G(EB) : EN_G(B)|p. \end{aligned}$$

Looking at the possible dimensions for L listed above, one has to have $\dim L = p(p+1)$ and $\dim M = 1$ as well as $A = N_G(B)$. As $N_E(B) = Z \times Q$ one knows now that A contains the normal subgroup $Z \times Q$ and therefore A/B contains the image of $Z \times Q$ in A/B which is a normal p-elementary abelian subgroup of rank 2. This contradicts Corollary 3.8 which states that A/B has to be a Roquette group.

Suppose now that $N_E(B) = Z$. In this case one has, for an integer δ,

$$\begin{aligned} \dim L &= \delta \dim M |\,\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)||N_G(EB) : EN_G(B)||E : N_E(B)| \\ &= \delta \dim M |\,\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)||N_G(EB) : EN_G(B)|p^2. \end{aligned}$$

Looking at the possible dimensions for L listed above, one has to have $\delta = 1$ so $\dim L = p^2$ and $\dim M = 1$ as well as $A = N_G(B)$ and $|\,\mathrm{SL}(E/Z) : N_{\mathrm{SL}(E/Z)}(H)| = 1$. In other words H is a normal subgroup of $\mathrm{SL}(E/Z)$. Suppose first $p > 3$, then this means that H is either $\mathrm{SL}(E/Z)$ or its center or H is trivial. Finally one has $|G : N_G(EB)| = |G : EN_G(B)| = 1$ and so $N_G(B)/Z$ is a complement of E/Z in G/Z. This implies, as $H^1\big(\mathrm{SL}(E/Z), E/Z(E)\big) = 1$ by Lemma 5.31, that $N_G(B)/Z$ is conjugate to $(\mathrm{SL}(E/Z) \times Z)/Z$ and so $A = N_G(B) = Z \times {}^x\mathrm{SL}(E/Z)$ for an element x of G.

As $L = \mathrm{Indinf}_{A/B}^{G}(M)$, one has, by Mackey's formula,

$$\begin{aligned}
\mathrm{Res}^{G}_{{}^{x}\mathrm{SL}(E/Z)}(L) &\cong \bigoplus_{g\in[\,{}^{x}\mathrm{SL}(E/Z)\backslash G/B]} \mathrm{Btf}\left({}^{x}\mathrm{SL}(E/Z), 1, {}^{g}\!A, {}^{g}\!B\right) {}^{g}\!M \\
&\cong \mathrm{Indinf}^{\,{}^{x}\mathrm{SL}(E/Z)}_{\,{}^{x}\mathrm{SL}(E/Z)\cap A/\,{}^{x}\mathrm{SL}(E/Z)\cap B} \mathrm{Iso}_{\psi} \mathrm{Defres}^{A/B}_{(\,{}^{x}\mathrm{SL}(E/Z)\cap A)B/B}(M) \\
&\oplus \bigoplus_{\substack{g\in[\,{}^{x}\mathrm{SL}(E/Z)\backslash G/B],\\ g\neq 1}} \mathrm{Btf}\left({}^{x}\mathrm{SL}(E/Z), 1, {}^{g}\!A, {}^{g}\!B\right) {}^{g}\!M.
\end{aligned}$$

then because

$$\mathrm{Indinf}^{\,{}^{x}\mathrm{SL}(E/Z)}_{\,{}^{x}\mathrm{SL}(E/Z)\cap A/\,{}^{x}\mathrm{SL}(E/Z)\cap B} \mathrm{Iso}_{\psi} \mathrm{Defres}^{A/B}_{(\,{}^{x}\mathrm{SL}(E/Z)\cap A)B/B}(M)$$

is reduced to

$$\mathrm{Inf}^{\,{}^{x}\mathrm{SL}(E/Z)}_{\,{}^{x}\mathrm{SL}(E/Z)/\,{}^{x}\mathrm{SL}(E/Z)\cap B} \mathrm{Iso}_{\psi} \mathrm{Res}^{A/B}_{(\,{}^{x}\mathrm{SL}(E/Z)B)/B}(M)$$

which is an irreducible representation of dimension 1 of $\mathrm{SL}(E/Z)$, it is the trivial representation as $p > 3$.

However,

$$\mathrm{Res}^{G}_{{}^{x}\mathrm{SL}(E/Z)}(L) = \mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)} \mathrm{Conj}_{x^{-1}}(L) = \mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)}(L)$$

does not contain the trivial representation. Indeed, by the description of the representations of G one has $L = V \otimes \psi$ for V the G-module extended from a faithful irreducible module of E and ψ an irreducible $\mathrm{SL}(E/Z)$-module of dimension p with trivial action of E. Moreover, one has

$$\begin{aligned}
\mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)}(L) &= \mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)}(V) \otimes \mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)}(\psi) \\
&= \mathrm{Conj}_x(V_1 \oplus V_{-1}) \otimes \mathrm{Conj}_x \psi \\
&= \mathrm{Conj}_x(V_1 \otimes \psi) \oplus \mathrm{Conj}_x(V_{-1} \otimes \psi),
\end{aligned}$$

where V_1 and V_{-1} denote the Weil modules. They are irreducible $\mathrm{SL}(E/Z)$-modules of dimension $\frac{p\pm1}{2}$ as seen previously. So just by looking at the dimensions one knows that ψ is not isomorphic to the dual of V_1 or V_{-1} and therefore the trivial representation is not an irreducible component of $\mathrm{Conj}_x(V_1 \otimes \psi)$ or $\mathrm{Conj}_x(V_{-1} \otimes \psi)$ and so of $\mathrm{Conj}_x \mathrm{Res}^{G}_{\mathrm{SL}(E/Z)}(L)$. This shows that in both cases L cannot be of the form $\mathrm{Indinf}^{G}_{A/B}(M)$. This ends the case $B \cap E = 1$ if $p > 3$. If $p = 3$ one uses GAP to see that a faithful simple module L cannot be stabilized.

For the second case, namely $B \cap E = T$, one has seen in Proposition 5.39 that $|N_G(B)| \leq p^3(p-1)$ and therefore

$$\dim L \geq |G : N_G(B)| \geq \frac{p^3 p(p-1)(p+1)}{p^3(p-1)} = p(p+1).$$

Again, the only possibility is to have equality with $\dim L = p(p+1), \dim M = 1$ and $A = N_G(B)$ with order $|N_G(B)| = p^3(p-1)$. By Proposition 5.39 one has $|N_G(B)| = p^3(p-1)$ only if $B = (T \times V) \rtimes {}^bS$ with S a subgroup of the diagonal and b an element of $N_{\mathrm{SL}(E/Z)}(T)$, also $V = \{\rho(u)u \mid u \in U\}$ and $\rho : U \to Z$ is a homomorphism where U is a unipotent subgroup of $\mathrm{SL}(E/Z)$ and $A = N_G(B) = (T \times Z) \rtimes N_{\mathrm{SL}(E/Z)}(U \rtimes S)$, where $U \rtimes S$ is a Borel subgroup of $\mathrm{SL}(E/Z)$. Note that the module M has to be faithful by Proposition 4.3 of [3]. This gives us that if the five conditions of the Theorem 6.31 are not fulfilled then $(A, B) = (C, D) = (E \rtimes \mathrm{SL}(E/Z), 1)$ in other words L cannot be stabilized by a non-trivial biset. $\square$

Remark 6.32.

1. One still has to check what happens if the five conditions are fulfilled. For $p = 3$ and $p = 5$ the result is the same: $(A, B) = (C, D) = (E \rtimes \mathrm{SL}(E/Z), 1)$. This can be done by GAP.

2. This also finishes the proof of Theorem 6.29 for $n = 1$ as the case of dimension of L equal to p is completely treated.

6.5.4 $E \rtimes C_{p+1}$

In this section E denotes an extraspecial group of order p^3 and exponent p where p is an odd prime number. The group C_{p+1} is viewed as a subgroup of $\mathrm{SL}(E/Z)$ and its action is the restricted action of $\mathrm{SL}(E/Z)$. In fact there are $p+1$ lines in E/Z and C_{p+1} acts on them without fixing a line. To view it, let α be a generator of $\mathbb{F}_{p^2}^*$. It is an element of order $p^2 - 1$ and viewing $\mathbb{F}_{p^2}$ as an $\mathbb{F}_p$-vector space of dimension 2 it is also an element of $\mathrm{GL}_2(p)$. Let $\beta := \alpha^{p-1}$. It is an element of $\mathrm{SL}(E/Z)$ of order $p+1$. Its determinant is one because $\det(\beta) = \det(\alpha)^{p-1}$ and $\det(\alpha)$ belongs to $\mathbb{F}_p^*$. Now let x be an element of $\mathbb{F}_{p^2}^*$ and denote by $[x]$ the lines defined by x and 0. Then $\beta[x] = [\beta x]$ and β fixes $[x]$ if and only if $\beta x = \lambda x$ for a $\lambda \in \mathbb{F}_p^*$, i.e. $\beta = \lambda$. But β has order $p+1$ and λ is a divisor of $p-1$. So such a λ cannot exist and thus β does not fix a line. In particular, C_{p+1} does not fix pointwise a line.

Proposition 6.33. *Let L be a faithful simple $\mathbb{C}[E \rtimes C_{p+1}]$-module stabilized by $\mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{C/D}^{G}$, then $(A, B) = (C, D) = (E \rtimes C_{p+1}, 1)$.*

Proof. Let G be $E \rtimes C_{p+1}$ and suppose the sections are not trivial. By Propositions 6.28 and 2.18 we know that the dimension of such a module L is p, as the dimension of an irreducible C_{p+1}-module over $\mathbb{C}$ is one, and the G-core of B is trivial. We must have $B \cap Z = 1$, otherwise Z would be contained in the G-core of B. So only two cases are possible:

1. $B \cap E = 1$ or
2. $B \cap E = T$ for T non-central p-subgroup of E of order p.

Suppose B is not trivial. In the first case, as the order of C_{p+1} is prime to the order of E, the subgroup B is conjugate to a subgroup of C_{p+1} and its normalizer is conjugate to $C_{p+1} \times Z$. Indeed, by Lemma 5.22, its normalizer is either $C_{p+1} \times Z$ or $(Z \times Q) \rtimes C_{p+1}$ for Q a non-central p-group of E. By Lemma 5.19, we know that, for e an element of E, the group B acts trivially on e if and only if e belongs to $N_G(B)$. By Lemma 5.23, $N_E(B) = C_E(B)$ and since B does not act pointwise trivially on any non-central p-group of E, we must have $N_G(B) = C_{p+1} \times Z$. So one has

$$p = \dim L \geq |G : N_G(B)| = \frac{p^3(p+1)}{p(p+1)} = p^2 > p,$$

which is a contradiction and so $B = 1$.

In the second case, recall that $N_G(B) \leq N_G(T)$ and also note that $N_G(T) = N_{E \rtimes \mathrm{SL}(E/Z)}(T) \cap G$. So the order of $N_G(B)$ is $p^2 d$ where d divides $p+1$ and $p-1$ and therefore could only be equal to 1 or 2. If $d = 1$ then $Z \times T = N_G(B) \geq Z \times B \geq Z \times T$ and so $|B| = |T|$ and $B = T$ as $T \leq B$. If $d = 2$ then either $B = T$, which is the first case, or $|B : T| = 2$. In both cases one has $N_G(B) = Z \times B$. Therefore one obtains

$$p = \dim L \geq |G : N_G(B)| \geq \frac{p^3(p+1)}{2p^2} = \frac{p(p+1)}{2} > p,$$

which is again a contradiction. So one can conclude that, in all the cases, B is trivial and so U has to be reduced to an isomorphism by Proposition 3.9. □

Bibliography

[1] J.L. Alperin. *Local Representation Theory.* Cambridge Studies in Advanced Mathematics, 1993.

[2] S. Bouc. *Biset Functors for finite Groups.* Springer Lecture Notes in Mathematics, 2010.

[3] S. Bouc and J. Thévenaz. *Stabilizing bisets.* Advances in Mathematics, Volume 229, Pages 1610-1639, 2012.

[4] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.4.* `http://www.gap-system.org`, 2014.

[5] D. Gorenstein. *Finite Groups.* Chelsea publishing company, 1980.

[6] I. Hambleton, L. Taylor and B. Williams. *Detection theorems for K-theory and L-theory.* Journal of pure algebra and applied algebra, Volume 63, Pages 247-299, 1990.

[7] J. E. Humphreys. *Representations of* $\mathrm{SL}(2, p)$*.* The American Mathematical Monthly, Volume 82, Pages 21-39, 1975.

[8] Robert M. Guralnick, Kay Magaard, Jan Saxl, Pham Huu Tiep. *Cross characteristic representations of odd characteristic symplectic groups and unitary.* Journal of Algebra, Volume 257, Pages 291-347, 2002.

[9] A. Monnard. *Tensor Products of Weil Modules.* Master thesis. `http://tinyurl.com/MonnardPDM`, 2010.

[10] Kijti Rodtes. *The connective K theory of semidihedral groups.* Phd thesis. `http://tinyurl.com/Rodtes`, 2010.

[11] Jean-Pierre Serre. *Galois Cohomology*. Springer, 2002.

[12] J. Thévenaz. *Maximal subgroups of direct products.* Journal of Algebra, Volume 198, Pages 352-361, 1997.

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