Homotopic Hopf-Galois extensions of commutative differential graded algebras

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Abstract

This thesis is concerned with the definition and the study of properties of homotopic Hopf-Galois extensions in the category $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$ of chain complexes over a field $\mathbb{k}$, equipped with its projective model structure.

Given a differential graded $\mathbb{k}$-Hopf algebra H of finite type, we define a *homotopic H-Hopf-Galois extension* to be a morphism $\varphi : B \to A$ of augmented H-comodule dg-$\mathbb{k}$-algebras, where B is equipped with the trivial H-coaction, for which the associated Galois functor $(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$ and the comparison functor $(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$ are Quillen equivalences. Here A^{hcoH} denotes the object of homotopy H-coinvariants of the dg-algebra A, and $\mathbf{Mod}_{A^{hcoH}}$ denotes the category of right modules over A^{hcoH} in $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$, endowed with the model category structure right-induced by the forgetful functor from $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$ (and similarly for B). The categories $\mathcal{M}_A^{W_\varphi^{can}}$ and $\mathcal{M}_A^{W_\rho}$ denote, respectively, the categories of right $A \otimes_B A$- and $A \otimes H$-comodules in the category $\mathbf{Mod}_A$, and they are equipped with the model category structures left-induced from $\mathbf{Mod}_A$ by the forgetful functor.

We investigate the behavior of homotopic Hopf-Galois extensions of *commutative* dg-$\mathbb{k}$-algebras under base change. First, we study their preservation under base change. Given a homotopic H-Hopf-Galois extension $\varphi : B \to A$, with B, A commutative, and a morphism $f : B \to B'$ of commutative dg-$\mathbb{k}$-algebras, we determine conditions on φ and f, under which the induced morphism $\overline{\varphi} : B' \to B' \otimes_B A$ is also a homotopic H-Hopf-Galois extension. Secondly, we examine the reflection of such extensions under base change. We suppose that the induced morphism $\overline{\varphi} : B' \to B' \otimes_B A$ is a homotopic H-Hopf-Galois extension, and we specify conditions on φ and f that guarantee that $\varphi : B \to A$ was a homotopic H-Hopf-Galois extension.

The main result of this thesis establishes one direction of a Hopf-Galois correspondence for homotopic Hopf-Galois extensions over *co-commutative* dg-$\mathbb{k}$-Hopf-algebras of finite type. We show that if $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension, and $g : H \to K$ is an inclusion of co-commutative dg-$\mathbb{k}$-Hopf-algebras of finite type, then $A^{hcoK} \to A$ is always a homotopic K-Hopf-Galois extension, and $B \to A^{hcoK}$ is a homotopic H^{hcoK}-Hopf-Galois extension, provided that A is semi-free as a B-module.

We end with an example, derived from the context of simplicial sets, which offers interesting possibilities of application of our main result to principal fibrations of simplicial sets.

Key words: Hopf algebra, homotopic Hopf-Galois extension, Hopf-Galois correspondence, homotopic descent, coring, category of comodules over a coring, homotopy coinvariants.

Résumé

Cette thèse élabore une définition et étudie les propriétés des extensions de Hopf-Galois homotopiques dans la catégorie $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ des complexes de chaînes sur un corps $\Bbbk$, munie de sa structure de catégorie modèle projective.

Étant donné H une $\Bbbk$-algèbre de Hopf différentielle graduée de type fini, on appelle *H-extension de Hopf-Galois homotopique* un morphisme $\varphi : B \to A$ de dg-$\Bbbk$-algèbres augmentées, où B est munie avec une H-coaction triviale, pour lequel le *foncteur de Galois* associé $(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$ et le *foncteur de comparaison* associé $(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$ sont des équivalences de Quillen. Ici, A^{hcoH} dénote l'objet des H-coinvariants homotopiques de l'algèbre A, et $\mathbf{Mod}_{A^{hcoH}}$ est la catégorie des A^{hcoH}-modules à droite dans $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, munie de sa structure de catégorie modèle induite à droite par le foncteur oubli depuis $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (et de même, pour B). On note par $\mathcal{M}_A^{W_\varphi^{can}}$ et $\mathcal{M}_A^{W_\rho}$, respectivement, la catégorie des $A \otimes_B A$- et $A \otimes H$-comodules à droite dans la catégorie $\mathbf{Mod}_A$. Elles sont toutes les deux équipées de structures de catégorie modèles induites à gauche par le foncteur oubli depuis $\mathbf{Mod}_A$.

Nous examinons le comportement des extensions de Hopf-Galois homotopiques de dg-$\Bbbk$-algèbres *commutatives* sous changement de base. Dans un premier temps, nous étudions leur préservation sous changement de base. Étant donné une H-extension de Hopf-Galois homotopique $\varphi : B \to A$, avec B, A commutatifs, et un morphisme $f : B \to B'$ de dg-$\Bbbk$-algèbres commutatives, nous déterminons les conditions sur φ et f, sous lesquelles le morphisme induit $\overline{\varphi} : B' \to B' \otimes_B A$ est aussi une H-extension de Hopf-Galois homotopique. Dans un deuxième temps, nous considérons la question de réflexion de telles extensions sous un changement de base. Nous supposons que le morphisme induit $\overline{\varphi} : B' \to B' \otimes_B A$ est une H-extension de Hopf-Galois homotopique, et nous spécifions les conditions sur φ et f qui garantissent que $\varphi : B \to A$ était une H-extension de Hopf-Galois homotopique.

Le résultat principal de cette thèse établit une direction d'une correspondance de Hopf-Galois pour des extensions de Hopf-Galois homotopiques sur des dg-$\Bbbk$-algèbres de Hopf *co-commutatives* de type fini. Nous démontrons le résultat suivant: si $\varphi : B \to A$ est une H-extension de Hopf-Galois homotopique et $g : H \to K$ est une inclusion de dg-$\Bbbk$-algèbres de Hopf co-commutatives de type fini, alors $A^{hcoK} \to A$ est toujours une K-extension de Hopf-Galois homotopique, et $B \to A^{hcoK}$ est une H^{hcoK}-extension de Hopf-Galois homotopique, à condition que A soit un module B-semi-libre.

Nous terminons avec un exemple, issu du contexte des ensembles simpliciaux, qui offre des possibilités intéressantes d'application de notre résultat principal aux fibrations principales des ensembles simpliciaux.

Mots-clés: algèbre de Hopf, extension de Hopf-Galois homotopique, correspondance de Hopf-Galois, descente homotopique, coanneau, catégorie des comodules sur un coanneau, coinvariants homotopiques.

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Contents

Notations

A^{coH}	H-coinvariants of A
A^{hcoH}	homotopy H-coinvariants of A
$\mathbf{Alg}_H^{\varepsilon}$	category of augmented H-comodule algebras
$\mathrm{Aut}_{\Bbbk}(\mathbb{E})$	group of $\Bbbk$-automorphisms of $\mathbb{E}$
$\mathcal{B}(B)$	bar construction on B
${}_B\,\mathbf{Bimod}_A$, ${}_B\,\mathbf{Mod}_A$	category of B-A-bimodules
$\mathcal{C}(X,Y)$	hom-set of morphisms from X to Y in a small category $\mathcal{C}$
$\mathbf{Ch}_{\Bbbk}^{\geq 0}$	category of non-negatively graded chain complexes over a field $\Bbbk$
$\mathbf{Coalg}_R$	category of R-coalgebras
$\mathbf{Comod}_C$	category of C-comodules
$\mathbf{grMod}_R^{\geq 0}$	category of graded R-modules
$\mathrm{Hom}_{\Bbbk}(X,Y)$	$\Bbbk$-homomorphisms from X to Y
$M \underset{C}{\square} N$	cotensor product of M and N over C
$M \tilde{\otimes} X$	M is semi-free on X
$\mathcal{M}_A^W$	category of W-comodules in the category $\mathbf{Mod}_A$
$\mathbf{Mod}_A$	category of A-modules
$\mathbf{Set}$	category of sets
$\mathbf{sSet}$	category of simplicial sets
X^c	X is a cofibrant object
X^f	X is a fibrant object
X^G	G-fixed points of X
$\Omega(A;H;\Bbbk)$	one-sided cobar construction on an H-comodule algebra A
$\Omega(A;H;H)$	two-sided cobar construction on an H-comodule algebra A
$\Omega(C)$	cobar construction on C
$\xrightarrow{\sim}$	weak equivalence
$\twoheadrightarrow$	fibration
$\rightarrowtail$	cofibration

Introduction

The origins of (homotopic) Hopf-Galois theory

It is certainly classical Galois theory that lies at the origin of (homotopic) Hopf-Galois theory. Given a finite algebraic field extension $\alpha : \mathbb{k} \hookrightarrow \mathbb{E}$, an important object associated to α is the group of automorphisms of the field $\mathbb{E}$ that fix $\mathbb{k}$, which we denote by G. The extension $\alpha : \mathbb{k} \hookrightarrow \mathbb{E}$ is called *G-Galois* if it is normal and separable. In this case, the *Galois correspondence*

$$\{\text{fields } \mathbb{M} : \mathbb{k} \subseteq \mathbb{M} \subseteq \mathbb{E}\} \longleftrightarrow \{\text{subgroups } N \leq G\}$$

holds. To a field $\mathbb{M}$ it associates the group $\mathrm{Aut}_{\mathbb{M}}(\mathbb{E})$ of automorphisms of $\mathbb{E}$ that fix $\mathbb{M}$, and to a group N it associates the field of fixed points $\mathbb{E}^N$, thus defining a bijection between the set of intermediate field extensions of $\alpha : \mathbb{k} \hookrightarrow \mathbb{E}$, and the set of subgroups of the Galois group G.

The first generalization of the Galois theory goes back to the sixties and the work of Auslander and Goldman [AG60], who were the first to generalize the notion of Galois extension to commutative rings. This theory was developed further by Chase, Harrison and Rosenberg in [CHR65], who proposed six equivalent ways of characterizing a Galois extension of commutative rings. One of these equivalent characterizations was formulated in terms of two particular maps associated to this extension. Namely, an inclusion of commutative rings $\alpha : R \hookrightarrow S$ was called *G-Galois* for $G \leq \mathrm{Aut}_R(S)$, finite, if certain maps

$$i : R \hookrightarrow S^G \qquad \text{and} \qquad \beta : S \otimes_R S \to \prod_G S$$

(see Section 2.4.2) are isomorphisms of R-algebras, where S^G is the ring of G-fixed points in S and $\prod_G S$ is the set of all G-indexed sequences of elements in S, equipped with point-wise multiplication.

In the case where S and R are fields, the definitions of Auslander and Goldman, and Chase, Harrison and Rosenberg coincide with the original definition of a finite Galois extension of fields. This not only offered a different, "zoomed-out" perspective on what it means for a finite field extension to be Galois, but it also turned out to be fruitful for proving a version of Galois correspondence for commutative rings ([CHR65]) and crucial for the future developments of Galois and (homotopic) Hopf-Galois theory in various contexts.

A dualization of the theory started with the emergence of Hopf-Galois extensions of (not necessarily commutative) algebras over a commutative ring R. This notion was first developed in a purely algebraic context, by Chase and Sweedler in [CS69] and by Kreimer and Takeuchi in [KT81]. The idea was to dualize the framework, by replacing the action of a group G by a coaction of a Hopf R-algebra H. *Hopf-Galois data* consist of a morphism of H-comodule R-algebras $\varphi : B \to A$, where B is augmented and has the trivial H-coaction. One should not be surprised to learn that φ will be an *H-Hopf-Galois extension*, if a relevant pair of maps, associated to φ, are isomorphisms. These maps are the *Galois map*

$$\beta_\varphi : A \otimes_B A \xrightarrow{A \otimes_B \rho} A \otimes_B A \otimes H \xrightarrow{\overline{\mu_A} \otimes H} A \otimes H,$$

defined in Section 2.2.2, and the *comparison map*

$$i_\varphi : B \to A^{coH}.$$

Here ρ is the H-coaction on A, and A^{coH} denotes the H-coinvariants of A.

Hopf-Galois extensions are noteworthy for numerous reasons. It turns out that a Hopf-Galois extension arises naturally from a free group action on a set, as we will explain in Example 2.4.11, and they can also be used as a tool in the study of the structure of Hopf algebras themselves ([Sch04]). Moreover, Hopf-Galois extensions have an important relation to descent theory.

The classical descent problem for rings can be informally formulated as follows. *Given an inclusion of (not necessarily commutative) rings $i : B \to A$ and an A-module M, what extra structure on M guarantees that there exists a B-module N, such that $N \otimes_B A \cong M$?*

In order to answer this question, one needs to work with the category $\mathcal{D}(i)$ of *descent data*, associated to i, which is actually isomorphic to the category $\mathcal{M}_A^{W_i^{can}}$ of right $A \otimes_B A$-comodules in $\mathbf{Mod}_A$. Given a Hopf algebra H that is flat over a commutative ring R, Schneider's structure theorem, established in [Schn90], states that $i : B \to A$ is an H-Hopf-Galois extension, such that A is faithfully flat over B, if and only if the category $\mathbf{Mod}_B$ is equivalent to the category $\mathcal{M}_A^{W_\rho}$ of right $A \otimes H$-comodules in $\mathbf{Mod}_A$ (i.e., φ satisfies effective descent).

The philosophy of homotopy theorists consists in studying objects and concepts up to homotopy. So, a *homotopic* Hopf-Galois extension, "living" in whichever category, will take into account the homotopical information contained in the model structure of this category.

The first homotopic analog of Galois theory was studied in [Rog08] by Rognes, in the category of structured ring spectra. He formulated the definition of a Galois extension of ring spectra (which mimicked the definition from [CHR65]) and studied many of their properties, such as their behavior under cobase change. Rognes also proved a full version of homotopic Galois correspondence for ring spectra.

He discovered that the unit map from the sphere spectrum $\mathbb{S}$ to the complex cobordism spectrum MU, can not be realized as a Galois extension, for any group spectrum G, but rather constitutes an example of a homotopic Hopf-Galois extension for the Hopf algebra given by $\Sigma^\infty BU_+$, the unreduced suspension spectrum of BU.

Motivated by the desire to provide a general framework in which to study homotopic Hopf-Galois extensions, Hess laid the foundations of a theory of Hopf-Galois extensions in monoidal model categories in [Hes09], generalizing both the classical case of rings and its extension to ring spectra. Just as in the algebraic case, there exists a close relation between homotopic Hopf-Galois extensions and homotopic descent theory, for which a new homotopy-theoretic framework was developed by Hess in [Hes10] for simplicially enriched categories, and by Müller in [Mul11] for categories enriched in an arbitrary model monoidal category $\mathcal{V}$. In this spirit, Berglund and Hess established in [BH12] that a homotopic analog of Schneider's result holds and allows one to view homotopic Hopf-Galois extensions of dg-$\mathbb{k}$-algebras as an interesting class of morphisms, satisfying effective descent.

In [Hes09] homotopic Hopf-Galois extensions in two particular examples of categories were briefly studied. These were the category of simplicial monoids and the category of finite-type chain algebras of $\mathbb{k}$-vector spaces. The topic of this thesis takes its roots in [Hes09] and develops in yet another category of interest, the category of chain complexes of $\mathbb{k}$-vector spaces. It also takes its inspiration from the work of Rognes [Rog08].

The goal of this thesis is to refine the definition of homotopic Hopf-Galois extensions proposed in [Hes09] (see Definition 2.4.12), which allows us to study the behavior of homotopic Hopf-Galois extensions in $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$ under cobase change and to prove successfully one direction of homotopic Hopf-Galois correspondence. We show that if $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension, and $g : H \to K$ is an inclusion of co-commutative dg-$\mathbb{k}$-Hopf-algebras of finite type, then $A^{hcoK} \to A$ is always a homotopic K-Hopf-Galois extension, and $B \to A^{hcoK}$ is a homotopic H^{hcoK}-Hopf-Galois extension, provided that A is semi-free as a B-module.

Organization of the thesis

Chapter 1 gathers the categorical, algebraic and model-theoretic background material that we will need for studying homotopic Hopf-Galois extensions. In particular, the model category section contains all the right- and left-transfer results that we will need to ensure that our categories of interest are equipped with the appropriate model structures. The left-transfer results appear in a very recent preprint [BHKKRS14].

One can formulate reasonable conditions under which a quasi-isomorphism of algebras (respectively, of corings) induces a Quillen equivalence on the categories of modules (respectively, the categories of comodules over corings). Reciprocally, one can determine when having a Quillen equivalence implies that the underlying morphism is a quasi-isomorphism. These characterizations, some of which were established in [BH12], are given in Chapter 1 and will prove extremely useful to us throughout the thesis.

The goal of Chapter 2 is to introduce our definition of homotopic Hopf-Galois extensions. This definition involves the *homotopy* coinvariants of a coaction of a Hopf algebra, so we will first need to know how to calculate the homotopy coinvariants A^{hcoH} of an H-comodule algebra A. To be able to do this, it is important to have valid models for fibrant replacements in the category $\mathbf{Alg}_H^{\varepsilon}$ of augmented H-comodule algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. They will be given by the two-sided cobar construction $\Omega(A;H;H)$, so that A^{hcoH} is modeled by the one-sided cobar construction $\Omega(A;H;\Bbbk)$. The *twisted* multiplication, defined in Corollary 3.6 [HL07], allows us to endow both $\Omega(A;H;H)$ and $\Omega(A;H;\Bbbk)$ with a natural differential graded algebra structure.

We explain the construction of the *Galois functor* $(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$ and the *comparison functor* $(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$, which are essential to our definition of a homotopic Hopf-Galois extension (Definition 2.3.1). To make connections between this definition and other concepts of (non-homotopic) (Hopf)-Galois extensions, mentioned earlier in this Introduction, we end Chapter 2 with a more detailed panorama of (homotopic) Hopf-Galois theory.

Chapter 3 explores the behavior of homotopic Hopf-Galois extensions of *commutative* algebras under cobase change. The commutativity assumption guarantees that the pushout of two algebras B' and A over an algebra B, is given by the coequalizer $B' \otimes_B A$. Given a homotopic H-Hopf-Galois extension $\varphi : B \to A$ and its pushout $\overline{\varphi} : B' \to B' \otimes_B A$ along a map f, we explain in detail how the pairs of associated Quillen equivalences $\big((\beta_\varphi)_*, (i_\varphi)^*\big)$ and $\big((\beta_{\overline{\varphi}})_*, (i_{\overline{\varphi}})^*\big)$ are related. After that, we address the question of preservation "if φ is H-Hopf-Galois, when is $\overline{\varphi}$ H-Hopf-Galois?" (Proposition 3.2.7), followed by the question of reflection "if $\overline{\varphi}$ H-Hopf-Galois, under which conditions was φ H-Hopf-Galois?" (Proposition 3.3.5). In both cases, we work under conditions such that the results from Chapter 1 on relation between quasi-isomorphisms and associated Quillen equivalences are applicable, which facilitates our task. We need primarily to impose semi-freeness and homological faithfulness assumptions on some of the algebras.

The fourth chapter of this thesis is devoted to establishing the backward direction of homotopic Hopf-Galois correspondence in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. We will need to work in the framework of "conormality", developed in [FH12] and dual to the Galois framework.

An important observation is the following. Since we are working up to homotopy, studying a homotopic H-Hopf-Galois extension φ is equivalent to studying the *normal basis extension* associated to φ, $\iota_H : \Omega(A; H; \Bbbk) \to \Omega(A; H; H)$, which is also homotopic H-Hopf-Galois. So, instead of working directly with the factorization $B \to A^{hcoK} \to A$, we work with a certain commuting diagram of cobar constructions.

Here is the exact statement of our Hopf-Galois correspondence result. We refer the reader to the body of the thesis for the necessary definitions and terminology.

Theorem 0.0.1 (Theorem 4.3.6)**.** *Let $\Bbbk$ be a field and $g : H \to K$ a morphism of co-commutative, 1-connected, degree-wise finitely generated Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, such that $K_2 = 0$. Let $\varphi : B \to A$ be a homotopic H-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ and consider the following diagram*

$$\begin{array}{ccccc} \Omega(A;H;\Bbbk) & \xhookrightarrow{\iota_H} & \Omega(A;H;H) & \xrightarrow{\simeq} & \Omega(A;K;K), \\ & {\scriptstyle\omega}\searrow & & \nearrow{\scriptstyle\iota_K} & \\ & & \Omega(A;K;\Bbbk) & & \end{array}$$

where ι_H and ι_K denote the normal basis homotopic Hopf-Galois extensions, associated to Hopf algebras H and K, respectively. If

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$; and*

(2) *$g : (H, \Delta_H, d_H) \hookrightarrow (K, \Delta_K, d_K)$ is an inclusion of differential graded $\Bbbk$-coalgebras,*

then the map

$$\omega : \Omega(A; H; \Bbbk) \longrightarrow \Omega(A; K; \Bbbk)$$

is a generalized homotopic $\Omega(H; K; \Bbbk)$-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Using results from Chapter 1 on the relation between quasi-isomorphisms and associated Quillen equivalences, the proof proceeds by establishing that the comparison map i_ω and the Galois map β_ω are quasi-isomorphisms, which is done in several steps. In particular, properties of twisted extensions and of semi-free extensions allow us to apply spectral sequence techniques to establish the existence of some of the intermediate quasi-isomorphisms.

Finally, Chapter 5 contains a few open questions.

Chapter 1

Background material

1.1 The zoo of categories

Notation 1.1.1. The following notation is used throughout this thesis. If A is an object of a small category $\mathcal{C}$, we write $A \in \mathcal{C}$. The set of morphisms from $A \in \mathcal{C}$ to $B \in \mathcal{C}$ is denoted by $\mathcal{C}(A, B)$. The identity morphism on an object A is denoted by A or Id_A.

1.1.1 Monoidal categories' language

The goal of this section is to clarify the categorical context we are working in, and also to fix notation. It starts with a brief reminder about monoidal categories, as well as related categories, such as the categories of (co)modules in a monoidal category, and then describes our categories of interest in this project.

Definition 1.1.2. A **monoidal category** $(\mathcal{M}, \otimes, \mathbb{I})$ is a category $\mathcal{M}$, together with a bifunctor $-\otimes- : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, called the **monoidal product** and an object $\mathbb{I} \in \mathcal{M}$, called the **unit**, such that $-\otimes-$ is associative and unital with respect to $\mathbb{I}$. More precisely, this means that for every triple $A, B, C \in \mathcal{M}$, there is given an isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

natural in A, B, C, and for all $A \in \mathcal{M}$, there exist isomorphisms

$$l_A : \mathbb{I} \otimes A \to A \quad \text{and} \quad r_A : A \otimes \mathbb{I} \to A,$$

natural in A, such that the diagram

$$\begin{array}{ccc} ((A\otimes B)\otimes C)\otimes D & \xrightarrow{\alpha_{A\otimes B,C,D}} & (A\otimes B)\otimes(C\otimes D) \\ \downarrow{\scriptstyle \alpha_{A,B,C}\otimes D} & & \\ (A\otimes(B\otimes C))\otimes D & & \downarrow{\scriptstyle \alpha_{A,B,C\otimes D}} \\ \downarrow{\scriptstyle \alpha_{A,B\otimes C,D}} & & \\ A\otimes((B\otimes C)\otimes D) & \xrightarrow{A\otimes\alpha_{B,C,D}} & A\otimes(B\otimes(C\otimes D)) \end{array}$$

commutes for all $A, B, C, D \in \mathcal{M}$, and the diagram

$$\begin{array}{ccc} (A\otimes\mathbb{I})\otimes B & \xrightarrow{\alpha_{A,\mathbb{I},B}} & A\otimes(\mathbb{I}\otimes B) \\ & {\scriptstyle r_A\otimes B}\searrow \quad \swarrow{\scriptstyle A\otimes l_B} & \\ & A\otimes B & \end{array}$$

commutes for all $A, B \in \mathcal{M}$.

A monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ is called **symmetric** if, for all $A, B \in \mathcal{M}$, there is given an isomorphism $tw_{A,B} : A \otimes B \to B \otimes A$, natural in A an B, appropriately compatible (see Chapter 7, §7 in [McL98]) with the associativity isomorphism α and the unit isomorphisms l and r, and such that $tw_{B,A} = tw_{A,B}^{-1}$, i.e., such that the diagram

$$\begin{array}{ccc} B\otimes A & \xrightarrow{tw_{B,A}} & A\otimes B \\ & \searrow^{=} & \downarrow{\scriptstyle tw_{A,B}} \\ & & B\otimes A \end{array}$$

commutes.

A monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ is called **closed** if for any $X \in \mathcal{M}$, the functor $-\otimes X : \mathcal{M} \to \mathcal{M}$ has a right adjoint, denoted $\mathrm{Hom}_{\mathcal{M}}(X, -) : \mathcal{M} \to \mathcal{M}$.

Notation 1.1.3. To avoid heavy notation, we will omit the names of the natural isomorphisms α, l, r and tw in the diagrams and simply write the symbol $\cong$ when necessary.

Example 1.1.4. Our main underlying closed symmetric monoidal category in this project is the category of non-negatively graded chain complexes of $\Bbbk$-modules over a field $\Bbbk$, with differential of degree -1, denoted $\big(\mathbf{Ch}_{\Bbbk}^{\geq 0}, \otimes, \Bbbk[0]\big)$. Here, $\otimes$ is the usual tensor product of chain complexes, and $\Bbbk[0]$ stands for the chain complex with value $\Bbbk$, concentrated in degree 0.

Occasionally, we will work in the category $\big(\mathbf{Ch}_{R}^{\geq 0}, \otimes, R[0]\big)$ of chain complexes over a commutative ring R.

Notation 1.1.5. For any $X \in \mathbf{Ch}_{\Bbbk}^{\geq 0}$, if $x \in X_m$, we say that the **degree of** x, denoted $\deg(x)$, is equal to m, for all $m \geq 0$. Moreover, we adopt the notation

$$X_{\leq m} := \{x \in X : \deg(x) \leq m\},$$

and will sometimes identify this set of elements with the chain complex

$$X_{\leq m} : \qquad \cdots \to X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 \to 0.$$

Remark 1.1.6. Note that, given X, $Y \in \mathbf{Ch}_{\Bbbk}^{\geq 0}$, the symmetry isomorphism is defined by

$$tw : X \otimes Y \to Y \otimes X : x \otimes y \mapsto (-1)^{\deg(x)\deg(y)} y \otimes x,$$

for all $x \in X$, $y \in Y$.

Notation 1.1.7. For any $X \in \mathbf{Ch}_{\Bbbk}^{\geq 0}$, we will denote by $\natural X$ the underlying graded $\Bbbk$-module of X.

Terminology 1.1.8. A chain complex $X \in \mathbf{Ch}_{\Bbbk}^{\geq 0}$ is **connected** if $X_0 = \Bbbk$, and is 1**-connected** if $X_0 = \Bbbk$, $X_1 = 0$.

Definition 1.1.9. Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a symmetric monoidal category.

- A **monoid** (A, μ_A, η_A) in $\mathcal{M}$ consists of an object $A \in \mathcal{M}$, equipped with two morphisms $\mu_A : A \otimes A \to A$ and $\eta_A : \mathbb{I} \to A$ such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{A \otimes \mu_A} & A \otimes A \\ {\scriptstyle \mu_A \otimes A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ A \otimes A & \xrightarrow{\mu_A} & A \end{array}$$

$$\begin{array}{ccccc} A \otimes \mathbb{I} & \xrightarrow{A \otimes \eta_A} & A \otimes A & \xleftarrow{\eta_A \otimes A} & \mathbb{I} \otimes A \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu_A} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

  The monoid A is **commutative** if the diagram

$$\begin{array}{ccc} A \otimes A & \underset{\cong}{\xrightarrow{tw_{A,A}}} & A \otimes A \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ A & =\!= & A \end{array}$$

  commutes. The category of monoids in $(\mathcal{M}, \otimes, \mathbb{I})$ will be denoted **Alg**.

Remark 1.1.10. Under suitable assumptions on $\mathcal{M}$, the forgetful functor $U : \mathbf{Alg} \to \mathcal{M}$, given on objects by $U(A, \mu_A, \eta_A) = A$, for any monoid A, admits a left adjoint $F_{\mathbf{Alg}} : \mathcal{M} \to \mathbf{Alg}$, called the free monoid functor, so that there exists an adjunction

$$\mathcal{M} \underset{U}{\overset{F_{\mathbf{Alg}}}{\rightleftarrows}} \mathbf{Alg}\,.$$

For example, if $\mathcal{M}$ has all coproducts and is closed monoidal, then for all $X \in \mathcal{M}$, $- \otimes X : \mathcal{M} \to \mathcal{M}$ preserves coproducts. In this case, $F_{\mathbf{Alg}}(X) := (\sqcup_{n \geq 0} X^{\otimes n}, \mu_X, \eta_X)$, for all $X \in \mathcal{M}$, with $X^0 := \mathbb{I}$, where μ_X is given by concatenation of tensors, and η_X is the inclusion of the summand $\mathbb{I}$.

- A **comonoid** $(C, \Delta_C, \varepsilon_C)$ in $\mathcal{M}$ consists of an object $C \in \mathcal{M}$, equipped with two morphisms $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \mathbb{I}$ such that the following diagrams commute.

$$\begin{CD} C @>{\Delta_C}>> C \otimes C \\ @V{\Delta_C}VV @VV{\Delta_C \otimes C}V \\ C \otimes C @>{C \otimes \Delta_C}>> C \otimes C \otimes C \end{CD}$$

$$\begin{array}{ccccc} C \otimes \mathbb{I} & \xleftarrow{C \otimes \varepsilon_C} & C \otimes C & \xrightarrow{\varepsilon_C \otimes C} & \mathbb{I} \otimes C \\ & \nwarrow{\scriptstyle \cong} & \uparrow{\scriptstyle \Delta_C} & {\scriptstyle \cong}\nearrow & \\ & & C & & \end{array}$$

  The comonoid C is **co-commutative** if the diagram

$$\begin{CD} C @= C \\ @V{\Delta_C}VV @VV{\Delta_C}V \\ C \otimes C @>{tw_{C,C}}>{\cong}> C \otimes C \end{CD}$$

  commutes.

- A **bimonoid** $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ in $\mathcal{M}$ consists of an object $H \in \mathcal{M}$, such that (H, μ_H, η_H) is a monoid in $\mathcal{M}$, $(H, \Delta_H, \varepsilon_H)$ is a comonoid in $\mathcal{M}$, μ_H and η_H are morphisms of comonoids, where $H \otimes H$ is equipped with the comonoid structure given by

$$\Delta_{H \otimes H} : H \otimes H \xrightarrow{\Delta_H \otimes \Delta_H} H \otimes H \otimes H \otimes H \underset{\cong}{\xrightarrow{H \otimes tw_{H,H} \otimes H}} H \otimes H \otimes H \otimes H$$

and

$$H \otimes H \xrightarrow{\varepsilon_H \otimes \varepsilon_H} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\cong} \mathbb{I}, \qquad \varepsilon_{H \otimes H} = \cong \circ (\varepsilon_H \otimes \varepsilon_H)$$

and, moreover, $\varepsilon_H \circ \eta_H = \mathrm{Id}_{\mathbb{I}}$.

Remark 1.1.11. The condition "μ_H and η_H are morphisms of comonoids" is equivalent to the condition "Δ_H and ε_H are morphisms of monoids"; one just changes the viewpoint on the corresponding commuting diagrams.

Definition 1.1.12. Let A be a monoid in $(\mathcal{M}, \otimes, \mathbb{I})$. The category $\mathbf{Mod}_A$ of **right** A**-modules** has as objects pairs (M, r), where $M \in \mathcal{M}$ and $r : M \otimes A \to M$ is a morphism that makes the following diagrams commute.

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{r \otimes A} & M \otimes A \\ {\scriptstyle M \otimes \mu_A}\downarrow & & \downarrow{\scriptstyle r} \\ M \otimes A & \xrightarrow{r} & M \end{array} \qquad \begin{array}{ccc} M \otimes \mathbb{I} & \xrightarrow{M \otimes \eta_A} & M \otimes A \\ {\scriptstyle \cong}\downarrow & \swarrow{\scriptstyle r} & \\ M & & \end{array}$$

The morphisms in $\mathbf{Mod}_A$ are morphisms in $\mathcal{M}$ that respect the structure maps r.

For any monoid $A \in \mathcal{M}$, there exists an adjunction

$$\mathcal{M} \underset{U}{\overset{- \otimes A}{\rightleftarrows}} \mathbf{Mod}_A,$$

where $- \otimes A$ is the left adjoint, which sends $X \in \mathcal{M}$ to $(X \otimes A, X \otimes \mu_A) \in \mathbf{Mod}_A$, and U denotes the forgetful functor, given by $(M, r) \mapsto M$, for all $(M, r) \in \mathbf{Mod}_A$.

One defines similarly the category ${}_A\mathbf{Mod}$ of left A-modules in $\mathcal{M}$.

Definition 1.1.13. Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a monoidal category that admits coequalizers. Given $(A, \mu_A, \eta_A) \in \mathbf{Alg}$, $(M, r) \in \mathbf{Mod}_A$ and $(N, l) \in {}_A\mathbf{Mod}$, one defines the **tensor product of** M **and** N **over** A to be the coequalizer

$$M \otimes_A N := \mathrm{coequal}\Big(M \otimes A \otimes N \underset{M \otimes l}{\overset{r \otimes N}{\rightrightarrows}} M \otimes N \Big),$$

computed in $\mathcal{M}$.

Note that if $(\mathcal{M}, \otimes, \mathbb{I})$ admits coequalizers, then one can define the category $\big({}_A\mathbf{Mod}_A, \otimes_A, A\big)$ of A-bimodules, with the monoidal structure given by the tensor product $\otimes_A$.

Definition 1.1.14. Let C be a comonoid in $(\mathcal{M}, \otimes, \mathbb{I})$. The category $\mathbf{Comod}_C$ of **right C-comodules** has as objects pairs (M, ρ), where $M \in \mathcal{M}$ and $\rho : M \to M \otimes C$ is a morphism, making the following diagrams commutative.

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle M \otimes \Delta_C} \\ M \otimes C & \xrightarrow{\rho \otimes C} & M \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ {\scriptstyle\cong}\downarrow & \swarrow{\scriptstyle M \otimes \varepsilon_C} & \\ M \otimes \mathbb{I} & & \end{array}$$

The morphisms in $\mathbf{Comod}_C$ are morphisms in $\mathcal{M}$ that respect the structure maps ρ.

For any comonoid $C \in \mathcal{M}$, there exists an adjunction

$$\mathbf{Comod}_C \underset{- \otimes C}{\overset{U}{\rightleftarrows}} \mathcal{M},$$

where $- \otimes C$ is the right adjoint, sending $X \in \mathcal{M}$ to $(X \otimes C, X \otimes \Delta_C) \in \mathbf{Comod}_C$, and U is the forgetful functor, given by $(M, \rho) \mapsto M$, for all $(M, \rho) \in \mathbf{Comod}_C$.

Definition 1.1.15. Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a monoidal category that admits equalizers. Given $(C, \Delta_C, \varepsilon_C)$ a comonoid in $\mathcal{M}$, $(M, \rho) \in \mathbf{Comod}_C$ and $(N, \sigma) \in {}_C\mathbf{Comod}$, one defines the **cotensor product of M and N over C** to be the equalizer

$$M \underset{C}{\square} N := \operatorname{equal}\left(M \otimes N \underset{M \otimes \sigma}{\overset{\rho \otimes N}{\rightrightarrows}} M \otimes C \otimes N\right),$$

computed in $\mathcal{M}$.

Definition 1.1.16. Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a symmetric monoidal category and H a bimonoid in $\mathcal{M}$. The category of **right H-comodule algebras** in $\mathcal{M}$ will be denoted $\mathbf{Alg}_H$. Its objects are monoids (A, μ_A, η_A) in $\mathcal{M}$, that are also equipped with a compatible H-comodule structure (A, ρ), i.e., such that the H-coaction $\rho : A \to A \otimes H$ is a morphism of monoids. Here, the monoid structure on $A \otimes H$ is defined by

$$\mu_{A \otimes H} : (A \otimes H) \otimes (A \otimes H) \cong A \otimes A \otimes H \otimes H \xrightarrow{\mu_A \otimes \mu_H} A \otimes H$$

and

$$\eta_{A \otimes H} : \mathbb{I} \cong \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta_A \otimes \eta_H} A \otimes H.$$

There exists a cofree-forgetful adjunction

$$\mathbf{Alg}_H \underset{- \otimes H}{\overset{U}{\rightleftarrows}} \mathbf{Alg},$$

similar to the one in Definition 1.1.14.

Definition 1.1.17. Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a monoidal category and A be a monoid in $\mathcal{M}$. An A-**coring** (W, ψ, ε) is a comonoid in $\big({}_A\mathbf{Mod}_A, \otimes_A, A\big)$. In other words, W is an object in $\mathcal{M}$, equipped with

- a left A-action $l : A \otimes W \to W$,
- a right A-action $r : W \otimes A \to W$,
- a comultiplication $\psi : W \to W \otimes_A W$,
- a counit $\varepsilon : W \to A$,

where l and r are compatible, ψ is co-associative and counital with respect to ε, and ψ and ε are both morphisms of A-bimodules.

We now introduce two examples of corings that are essential for the rest of this project.

Examples 1.1.18.

(1) Let $\varphi : B \to A$ be a morphism of monoids in $(\mathcal{M}, \otimes, \mathbb{I})$. The **canonical coring associated to** φ is denoted by W_φ^{can} and has as underlying A-bimodule $A \otimes_B A$. It is endowed with the comultiplication ψ_φ^{can} that given by the composite

$$A \underset{B}{\otimes} A \cong A \underset{B}{\otimes} B \underset{B}{\otimes} A \xrightarrow{A \underset{B}{\otimes} \varphi \underset{B}{\otimes} A} A \underset{B}{\otimes} A \underset{B}{\otimes} A \cong (A \underset{B}{\otimes} A) \underset{A}{\otimes} (A \underset{B}{\otimes} A).$$

The counit $\varepsilon_\varphi^{can}$ is the morphism

$$\overline{\mu} : A \otimes_B A \to A$$

induced by the multiplication $\mu_A : A \otimes A \to A$, using the definition of the tensor product $A \otimes_B A$ and the universal property of coequalizers. The left and right A-actions l_φ^{can} and r_φ^{can} on $A \otimes_B A$ are both induced by μ_A.

(2) Let $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bimonoid in a symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ and let (A, μ_A, η_A, ρ) be an H-comodule algebra. The tensor product $A \otimes H$ can naturally be endowed with the structure of an A-coring, denoted W_ρ, and called the **coring associated to** ρ. Its left A-module action l_ρ is given by

$$A \otimes (A \otimes H) \cong (A \otimes A) \otimes H \xrightarrow{\mu_A \otimes H} A \otimes H,$$

and its right A-module action r_ρ by

$$(A \otimes H) \otimes A \xrightarrow{A \otimes H \otimes \rho} A \otimes H \otimes A \otimes H \cong A \otimes A \otimes H \otimes H \xrightarrow{\mu_A \otimes \mu_H} A \otimes H,$$

where we have used the symmetry isomorphism in the second composite. The comultiplication ψ_ρ is

$$A \otimes H \xrightarrow{A\otimes\Delta} A \otimes H \otimes H \cong (A \otimes H) \otimes_A (A \otimes H),$$

and the counit ε_ρ is defined by

$$A \otimes H \xrightarrow{A\otimes\varepsilon_H} A \otimes \mathbb{I} \cong A.$$

Definition 1.1.19. Let A be a monoid in a symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ that has coequalizers, and let (W, ψ, ε) be an A-coring, with right A-action $r : W \otimes A \to W$. We denote by $\mathcal{M}_A^W$ the category of **right W-comodules over the coring** A, i.e., the category of right W-comodules in $\mathbf{Mod}_A$.

An object of $\mathcal{M}_A^W$ is a triple (M, γ, θ), where $M \in \mathcal{M}$, $\gamma : M \otimes A \to M$ is the right A-action on M and $\theta : M \to M \otimes_A W$ is the W-coaction on M. Recall that the object $M \otimes_A W$ is computed as a coequalizer in $\mathcal{M}$ (see Definition 1.1.13).

Depending on the context, we will use for objects of $\mathcal{M}_A^W$ the notation (M, γ, θ), or (M, θ), when the A-action γ is not relevant.

Later on in this project, adjunctions of the form

$$\mathcal{M}_A^W \underset{-\otimes_A W}{\overset{U}{\rightleftarrows}} \mathbf{Mod}_A,$$

will play an essential role (namely, in obtaining a model category structure on the category $\mathcal{M}_A^W$). Here, $-\otimes_A W$ is the right adjoint, which sends $(M, r) \in \mathbf{Mod}_A$ to $(M \otimes_A W, M \otimes_A r, M \otimes_A \psi) \in \mathcal{M}_A^W$, and U is the forgetful functor, given by $(M, \gamma, \theta) \mapsto (M, \gamma)$, for all $(M, \gamma, \theta) \in \mathcal{M}_A^W$.

Remark 1.1.20. We found helpful to end this section with a diagram offering a general view of our "zoo of categories" and adjunctions between them, with right adjoints always beneath and to the right of the corresponding left adjoints.

$$\begin{array}{ccccccc} \mathbf{Alg}_H & \underset{-\otimes H}{\overset{U}{\rightleftarrows}} & \mathbf{Alg} & & & & \\ & & F_{\mathbf{Alg}} \uparrow \downarrow U & & & & \\ \mathbf{Comod}_H & \underset{-\otimes H}{\overset{U}{\rightleftarrows}} & \mathcal{M} & \underset{U}{\overset{-\otimes A}{\rightleftarrows}} & \mathbf{Mod}_A & \underset{-\otimes_A W}{\overset{U}{\leftrightarrows}} & \mathcal{M}_A^W \end{array}$$

Here $\mathcal{M}$ is a symmetric monoidal category, (e.g., $\mathcal{M} := \mathbf{Ch}_{\Bbbk}^{\geq 0}$), A, H are, respectively, a monoid and a bimonoid in $\mathcal{M}$, and U denotes the forgetful functor.

1.1.2 Lifting adjunctions

In Section 3.1.2 of this thesis, we will need to show the existence of (right) adjoints of certain functors. Different adjoint functor theorems exist and give methods for finding adjoints. Our situation will be somewhat particular, since we will be interested in *lifting* an existing adjunction. In other words, given an adjunction (F, G) between a pair of categories $\mathcal{C}$ and $\mathcal{D}$, we will need to establish the existence of an adjunction $(\hat{F}, \hat{G})$ between another pair of categories $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ that are related to $\mathcal{C}$ and $\mathcal{D}$ in a special way. Because of the particular nature of the adjunctions and functors involved, we will be able to use the Adjoint lifting theorem for comonadic functors, presented below.

Remark 1.1.21. Let $\mathbb{K} = (K, \Delta, \varepsilon)$ be a comonad on a category $\mathcal{C}$. We denote by $\mathcal{C}_{\mathbb{K}}$ the **Eilenberg-Moore category of $\mathbb{K}$-coalgebras** in $\mathcal{C}$. Its objects are pairs (C, δ), where $C \in \mathcal{C}$ and $\delta \in \mathcal{C}(C, KC)$ is a morphism satisfying

$$K(\delta) \circ \delta = \Delta_C \circ \delta \qquad \text{and} \qquad \varepsilon_C \circ \delta = \mathrm{Id}_C .$$

A morphism in $\mathcal{C}_{\mathbb{K}}$ from (C, δ) to (C', δ') is a morphism $f : C \to C'$ in $\mathcal{C}$ such that $K(f) \circ \delta = \delta' \circ f$.

Definition 1.1.22. A functor $\Psi : \mathcal{W} \to \mathcal{C}$ is called **comonadic** if there exists a comonad $\mathbb{K} = (K, \Delta, \varepsilon)$ on $\mathcal{C}$ and an equivalence of categories $J : \mathcal{W} \to \mathcal{C}_{\mathbb{K}}$, such that the composite of functors $U_{\mathbb{K}} \circ J : \mathcal{W} \to \mathcal{C}$ is naturally isomorphic to Ψ, where $U_{\mathbb{K}} : \mathcal{C}_{\mathbb{K}} \to \mathcal{C}$ denotes the left adjoint of the cofree coalgebra functor $F_{\mathbb{K}} : \mathcal{C} \to \mathcal{C}_{\mathbb{K}}$, i.e., $U_{\mathbb{K}}$ is the forgetful functor.

Theorem 1.1.23 (Dual version of Theorem 4.5.6, [Bor94])**.** *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;Q\;} & \mathcal{B} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathcal{C} & \xrightarrow{\;L\;} & \mathcal{D} \end{array}$$

be a diagram of functors, where

(i) $L \circ U = V \circ Q$;

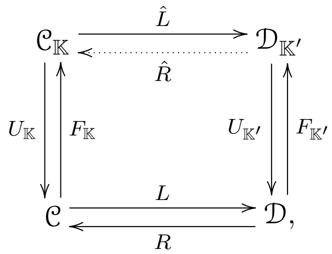
(ii) the functors U and V are comonadic;

(iii) the category $\mathcal{A}$ has all equalizers.

Then the functor Q has a right adjoint, whenever L has a right adjoint.

Francis Borceux offers in his book [Bor94] a detailed proof of the dual result about lifting a *left* adjoint in a *monadic* context. The entry "Adjoint lifting theorem" on nLab can also be enlightening, because it reformulates and highlights the key steps of Borceux's proof.

Remark 1.1.24. Observe that the forgetful functor $U_{\mathbb{K}} : \mathcal{C}_{\mathbb{K}} \to \mathcal{C}$ is comonadic, for any comonad $\mathbb{K}$ on $\mathcal{C}$. Therefore, Theorem 1.1.23 is useful, for example, if one wishes to prove the existence of a right adjoint $\hat{R}$ in the following situation



where $\mathbb{K}$, $\mathbb{K}'$ are comonads on $\mathcal{C}$ and $\mathcal{D}$, respectively, and left adjoints are displayed on top and on the left.

1.1.3 Digression on Hopf algebras

A few words should be said about the concept of a (graded) *Hopf algebra* over a field or a commutative ring, as there are a few subtleties.

The classical "algebraic" definition of a Hopf algebra over a field is the following.

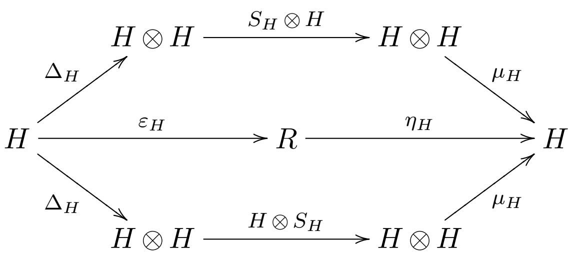
Definition 1.1.25. A **Hopf algebra** $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H, S_H)$ over a field $\Bbbk$ consists of an $\Bbbk$-vector space H, together with five $\Bbbk$-linear morphisms, such that

(1) (H, μ_H, η_H) is a $\Bbbk$-algebra;

(2) $(H, \Delta_H, \varepsilon_H)$ is a $\Bbbk$-coalgebra;

(3) the maps μ_H, η_H are morphisms of $\Bbbk$-coalgebras (or, equivalently, Δ_H, ε_H are morphisms of $\Bbbk$-algebras); and

(4) the morphism $S : H \to H$, called the **antipode**, makes the following diagram commute.



Example 1.1.26. Let G be any group and $\Bbbk$ a field. The **group ring**

$$H := \Bbbk[G] := \{\sum_{g \in G} k_g \cdot g : k_g \in \Bbbk, \text{ for all } g \in G\}$$

is a $\Bbbk$-vector space with basis given by the set of elements of G. The algebra structure on $\Bbbk[G]$ is given on basis elements by the group structure on G, and then extended $\Bbbk$-linearly to all of $\Bbbk[G]$. One defines a coalgebra structure on $\Bbbk[G]$ on basis elements by

$$\Delta_{\Bbbk[G]} : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G] : g \mapsto g \otimes g$$

and

$$\varepsilon_{\Bbbk[G]} : \Bbbk[G] \to \Bbbk : g \mapsto 1,$$

for all $g \in G$, and then extends it $\Bbbk$-linearly to all of $\Bbbk[G]$. The antipode $S : \Bbbk[G] \to \Bbbk[G]$ is given by $S(g) := g^{-1}$, for all $g \in G$.

Example 1.1.27. If the group G is *finite*, then the dual of the group ring

$$H := \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$$

is also a Hopf algebra. Its algebra structure is defined by

$$\mu : \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) \to \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) : f \otimes h \mapsto \mu(f \otimes h),$$

with $\mu(f \otimes h)(g) := \mu_{\Bbbk} \circ (f \otimes h) \circ \Delta_{\Bbbk[G]}(g)$, for all $g \in G$. Its coalgebra structure is given by the formal dual of the algebra structure:

$$\Delta : \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) \to \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) : f \mapsto \Delta(f)$$

with $\Delta(f)(g \otimes g') := f(gg')$, for all $g, g' \in G$. The antipode is

$$S : \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) \to \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) : f \mapsto S(f)$$

with $S(f)(g) := f(S_{\Bbbk}(g))$, for all $g \in G$. Note that $H = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$ is always commutative, and is co-commutative, if the group G is abelian.

This example truly is a "fundamental" example for constructing Hopf algebras, in the sense that many Hopf algebras arise by dualizing a group structure.

Example 1.1.28. A beautiful example of a Hopf algebra that is of interest to algebraic topologists is given by the direct sum $H_* = \bigoplus_{n\geq 0} H_n(X)$ of all homology groups of an H-space X. It is a *graded* Hopf algebra.

In [MM65], Milnor and Moore introduced the definition of a **graded Hopf Algebra** $H_* = (H_*, \mu_{H*}, \eta_{H_*}, \Delta_{H_*}, \varepsilon_{H_*})$ over a commutative ring R. It consists of a non-negatively graded R-module H_*, equipped with four morphisms of graded R-modules, which satisfy properties, analogous to properties $(1) - (3)$ in the Definition 1.1.25, only in the graded setting. Observe that the antipode S is not explicitly part of the definition of a graded Hopf algebra, so that H_* is actually only a bialgebra in the category of graded R-modules.

However, if a graded R-bialgebra H_* is connected (see Terminology 1.1.8), then the bimonoid structure on H_* is sufficient to define an antipode map $S : H_* \to H_*$, recursively for all $x \in H_n$ and all $n \geq 0$ (see Proposition 3.8.8 in [HGK10]).

Terminology 1.1.29. Since we will mainly work in the symmetric monoidal category $\left(\mathbf{Ch}_{\Bbbk}^{\geq 0}, \otimes, \Bbbk[0]\right)$ of differential *graded* $\Bbbk$-modules over a field $\Bbbk$, by a Hopf algebra H we will mean a bimonoid $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (which, eventually, will be taken to be connected).

More precisely, a Hopf algebra is an object $(H, d_H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$, such that

(i) (H, d_H) is a non-negatively graded chain complex, with differential $d : H \to H$ of degree -1, satisfying $d^2 = 0$;

(ii) (H, d_H, μ_H, η_H) is a unital, associative, differential graded algebra, i.e., a monoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, with μ_H and η_H of degree 0, and the **Leibniz rule**

$$d_H(hh') = d_H(h)h' + (-1)^{\deg(h)} h d_H(h')$$

holds for all $h, h' \in H$;

(iii) $(H, d_H, \Delta_H, \varepsilon_H)$ is a counital, coassociative, differential graded coalgebra, i.e., a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, with Δ_H and ε_H of degree 0, and the **"dual Leibniz" rule**

$$\Delta_H \circ d_H(h) = \left(d_H \otimes H + tw \circ (d_H \otimes H) \circ tw\right) \circ \Delta_H(h)$$

holds for all $h \in H$ (see [Tan83], Definition 0.3 (7)), i.e., d_H is a coderivation;

(iv) μ_H and η_H are morphisms of comonoids;

(v) $\varepsilon_H \circ \eta_H = \mathrm{Id}_{\Bbbk[0]}$.

Remark 1.1.30. We will see other topologically interesting examples of graded co-commutative $\Bbbk$-Hopf-algebra in Section 4.4.2.

1.2 Algebraic tools

1.2.1 Cobar construction and twisted structures

The classical cobar construction $\Omega(-)$ is a functor from the category $\mathbf{Coalg}_R^{\eta,1}$ of 1-connected, coaugmented differential coalgebras in $\mathbf{Ch}_R^{\geq 0}$ to the category $\mathbf{Alg}_R^{\varepsilon,0}$ of connected, augmented differential algebras in $\mathbf{Ch}_R^{\geq 0}$.

We will use the cobar construction in Chapter 2, in order to build valid models for fibrant replacements in the categories $\mathbf{Comod}_C$ and $\mathbf{Alg}_C$, which will allow us to define valid models for homotopy coinvariants of C-comodules and C-comodule algebras, under mild hypotheses on the coalgebra C.

Before introducing the definition of the cobar construction for coalgebras of chain complexes, we recall some preliminary constructions. Our references for this section are [Hes07], Chapter 10 in [Nei10] and Chapter 0 in [Tan83].

We fix a commutative ring R and denote by $\mathbf{grMod}_R^{\geq 0}$ the category of non-negatively graded modules over R.

Whenever we need to commute elements of a graded module past each other, or commute a morphism of graded modules past an element in its source, we will apply the **Koszul sign convention**. For instance, if X and Y are graded algebras, then for all $x \otimes y$, $x' \otimes y' \in X \otimes Y$ their product is given by

$$(x \otimes y) \cdot (x' \otimes y') = (-1)^{mn} xx' \otimes yy',$$

if $x' \in X_m$ and $y \in Y_n$. Also, if $f : X \to X'$ and $g : Y \to Y'$ are morphisms of graded modules, of degree p and q, respectively, then we have

$$(f \otimes g)(x \otimes y) = (-1)^{mq} f(x) \otimes g(y),$$

for all $x \otimes y \in X_m \otimes Y_n$.

Reminder 1.2.1.

(1) Let $X \in \mathbf{grMod}_R^{\geq 0}$. The **tensor algebra** on X, denoted $T(X)$, is defined as follows. As a graded R-module, it is given by

$$T(X) = R[0] \oplus X \oplus (X \otimes X) \oplus (X \otimes X \otimes X) \otimes \dots .$$

We write $T(X) = \bigoplus_{n \geq 0} T^n(X)$. An element of $T(X)$, coming from the summand $T^n(X)$, will be denoted by $x_1|...|x_n$. The multiplication

$$\mu : T(X) \otimes T(X) \to T(X)$$

is given on elementary tensors of length n by concatenation

$$\mu\big((x_1|\cdots|x_n) \otimes (x'_1|\cdots|x'_m)\big) = x_1|\cdots|x_n|x'_1|\cdots|x'_m$$

for all $n, m \geq 0$, $x_1|\cdots|x_n \in T^n(X)$, $x'_1|\cdots|x'_m \in T^m(X)$.

The unit

$$\eta : R[0] \to T(X)$$

is simply the inclusion. Moreover, the algebra $T(X)$ is augmented via

$$\varepsilon : T(X) \to R[0]$$

defined by $\varepsilon(r) = r$, for all $r \in R$ and $\varepsilon(v) = 0$, for all $v \in X$.

(2) For any $(X, d) \in \mathbf{Ch}_R^{\geq 0}$, the **desuspension of** X is a chain complex $(s^{-1}X, D)$ given by $(s^{-1}X)_n = X_{n+1}$, for all $n \geq 0$ and $D(s^{-1}(x)) = -s^{-1}d(x)$, for all $x \in X$.

(Note that the **suspension of** X is a chain complex (sX, D') given by $(sX)_n = X_{n-1}$, for all $n \geq 0$, where $X_{-1} = 0$ and where $D'(s(x)) = -sd(x)$, for all $x \in X$).

(3) Given a coalgebra (C, Δ, ε) in $\mathbf{Ch}_R^{\geq 0}$, the associated **reduced coalgebra** is defined to be

$$\overline{C} = \ker(\varepsilon : C \to R[0]).$$

If the coalgebra C is coaugmented, i.e., has a unit $\eta : R[0] \to C$, then $\varepsilon \circ \eta = \mathrm{Id}_{R[0]}$ and we have a direct sum decomposition

$$C \cong \overline{C} \oplus \eta(R) \cong \overline{C} \oplus R.$$

The comultiplication $\Delta : C \to C \otimes C$ can be described by

$$\Delta(1) = 1 \otimes 1 \qquad \text{and} \qquad \Delta(c) = c \otimes 1 + 1 \otimes c + \sum_i c_i \otimes c^i,$$

for all $c \in \overline{C}$, and where $c_i, c^i \in \overline{C}$. Thus, the **reduced comultiplication** $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$ is given for all $c \in \overline{C}$ by

$$\overline{\Delta}(c) = \sum_i c_i \otimes c^i.$$

(4) An element $c \in C$ of a coaugmented coalgebra C is called **primitive** if $\Delta(c) = c \otimes 1 + 1 \otimes c$. A coalgebra C is **primitively generated** if any $c \in C$ can be expressed as an R-linear combination of primitive elements in C.

Now we can state the definition of the classical cobar construction. Although this construction is defined for any coaugmented coalgebra C, we will assume our coalgebras to be 1-connected. This choice makes the cobar construction $\Omega(C)$ easier to handle in degree 0.

Definition 1.2.2. Let R be a commutative ring. The **cobar construction functor**

$$\Omega : \mathbf{Coalg}_R^{\eta,1} \to \mathbf{Alg}_R^{\varepsilon,0}$$

is defined as follows.

- For any (C, d_C) in $\mathbf{Coalg}_R^{\eta,1}$, its cobar construction

$$(\Omega C, D_{\Omega C}, \mu_{\Omega C}, \eta_{\Omega C}, \varepsilon_{\Omega C})$$

has the following description. As an augmented graded algebra,

$$\Omega C := \big(T(s^{-1}\overline{C}), \mu_{\Omega C}, \eta_{\Omega C}, \varepsilon_{\Omega C}\big)$$

i.e., it is the tensor algebra on the desuspension of the reduced coalgebra $\overline{C}$. The differential $D_{\Omega C}$ makes ΩC into a dga and is given by

$$D_{\Omega C} = d_I + d_E,$$

where d_I is the *internal* differential, specified on generators by

$$d_I(s^{-1}c) = -s^{-1}(d_C(c)), \text{ for all } c \in \overline{C},$$

and d_E is the *external* differential, specified on generators by

$$d_E(s^{-1}c) = \sum_i (-1)^{\deg c_i} s^{-1}c_i|s^{-1}c^i, \text{ for all } c \in \overline{C}.$$

- A map $f \in \mathbf{Coalg}_R^{\eta,1}(C, C')$ induces a map $\Omega f \in \mathbf{Alg}_R^{\varepsilon,0}(\Omega C, \Omega C')$, satisfying $\Omega f(s^{-1}c) = s^{-1}f(c)$.

In addition to classical cobar construction, we will need a few other ingredients for constructing fibrant replacements and homotopy coinvariants in $\mathbf{Comod}_C$ and $\mathbf{Alg}_C$ in Chapter 2.

Definition 1.2.3. Let $(C, d_C) \in \mathbf{Coalg}_R^{\eta,1}$ and $(A, d_A) \in \mathbf{Alg}_R^{\varepsilon,0}$. A **twisting morphism** from (C, d_C) to (A, d_A) is a morphism $t : C \to A$ of graded R-modules, of degree -1, such that

$$d_A \circ t + t \circ d_C = \mu_A \circ (t \otimes t) \circ \Delta_C.$$

Example 1.2.4. The **universal twisting morphism**, associated to a connected augmented chain coalgebra C is the morphism $t_\Omega : C \to \Omega C$ specified by $t_\Omega(1) = 0$ and $t_\Omega(c) = s^{-1}(c)$, for any $c \in \overline{C}$.

Twisting morphisms allow one to define twisted extensions.

Definition 1.2.5. Let $(C, d_C) \in \mathbf{Coalg}_R^{\eta,1}$, $(A, d_A) \in \mathbf{Alg}_R^{\varepsilon,0}$ and let $t : C \to A$ be a twisting morphism. Let (N, d_N) be a right C-comodule with coaction $\rho : N \to N \otimes C$ and (M, d_M) a left A-module with action $\lambda : A \otimes M \to M$.

- The **twisted extension of** N **by** M is

  $$(N, d_N) \otimes_t (M, d_M) := (N \otimes M, D_t),$$

  where

  $$D_t = d_N \otimes M + N \otimes d_M + (N \otimes \lambda) \circ (N \otimes t \otimes M) \circ (\rho \otimes M).$$

  In other words, the twisted extension of N by M has $N \otimes M$ as its underlying graded R-module, and its differential has an additional term, involving the module and the comodule structure maps and t.

- Let $\gamma : C \to C'$ be a morphism of coalgebras, and $(N, \rho) \in \mathbf{Comod}_C$, $(N', \rho') \in \mathbf{Comod}_{C'}$. Note that γ induces a C'-coaction $(N \otimes \gamma) \circ \rho$ on N. Let

  $$g : (N, (N \otimes \gamma) \circ \rho) \to (N', \rho')$$

be a morphism of C'-comodules. Let also $\alpha : A \to A'$ be a morphism of algebras, and $(M, \lambda) \in {}_A\mathbf{Mod}$, $(M', \lambda') \in {}_{A'}\mathbf{Mod}$. Similarly, α induces an A-action $\lambda' \circ (\alpha \otimes M')$ on M'. Let

$$g' : (M, \lambda) \to (M', \lambda' \circ (\alpha \otimes M'))$$

be a morphism of A-modules. Suppose now that we are given $t : C \to A$ and $t' : C' \to A'$ two twisting morphisms. If $\alpha \circ t = t' \circ \gamma$, then g and g' induce a **morphism of twisted extensions**

$$g \widetilde{\otimes} g' : (N \otimes M, D_t) \to (N' \otimes M', D_{t'}) : x \otimes y \mapsto g(x) \otimes g'(y)$$

which is a morphism of chain complexes from $(N \otimes M, D_t)$ to $(N' \otimes M', D_{t'})$.

Notation 1.2.6. To simplify notation slightly, we will write $N \otimes_t M$ instead of $(N \otimes M, D_t)$. Thus, a morphism of twisted extensions induced by g and g' as above will be written

$$g \widetilde{\otimes} g' : N \otimes_t M \to N' \otimes_{t'} M'.$$

Example 1.2.7. The twisted extension $(C, d_C) \otimes_{t_\Omega} (\Omega C, D_{\Omega C}) = C \otimes_{t_\Omega} \Omega C$, associated to a connected augmented chain coalgebra C, is called the **acyclic cobar construction**. The reason for this terminology is that there exists a contracting homotopy $c : C \otimes_{t_\Omega} \Omega C \to C \otimes_{t_\Omega} \Omega C$ of chain complexes, such that $C \otimes_{t_\Omega} \Omega C \xrightarrow{\simeq} R[0]$ is a homotopy equivalence (see Proposition 10.6.3 in [Nei10]).

Proposition 1.2.8 (Proposition 3.5(2), [HMS74])**.** *Let $(C, d_C) \in \mathbf{Coalg}_R^{\eta,1}$ and $(A, d_A) \in \mathbf{Alg}_R^{\varepsilon,0}$. There is a functorial, bijective correspondence*

$$\{\textit{twisting morphisms } t : C \to A\} \longleftrightarrow \{\textit{morphims of algebras } \theta_t : \Omega C \to A\},$$

where

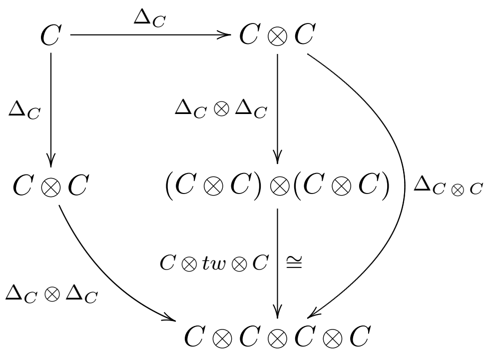
- *$t : C \to A$ gives rise to $\theta_t : \Omega C \to A$, determined by $\theta_t(s^{-1}c) = t(c)$, for all $c \in C$;*
- *$\theta_t : \Omega C \to A$ gives rise to $t_\theta : C \to A$, given by the composite*

$$C \xrightarrow{t_\Omega} \Omega C \xrightarrow{\theta} A.$$

Example 1.2.9. The morphism of algebras, associated via this correspondence to the universal twisting morphism $t_\Omega : C \to \Omega C$, is $\theta_{t_\Omega} = \mathrm{Id} : \Omega C \to \Omega C$.

Remark 1.2.10. Let $C \in \mathbf{Coalg}_R^{\eta,1}$. If the comultiplication Δ_C on C is co-commutative, then one can define a comultiplication on ΩC, which makes the cobar construction into a co-commutative Hopf algebra. We will now explain the construction.

First of all, the fact that Δ_C is co-commutative ensures that the map $\Delta_C : C \to C \otimes C$ is a map of coalgebras, because then the diagram



commutes. The desired comultiplication map

$$\Delta_{\Omega C} : \Omega C \to \Omega C \otimes \Omega C$$

is constructed as the composite

$$\Delta_{\Omega C} : \Omega C \xrightarrow{\Omega(\Delta_C)} \Omega(C \otimes C) \underset{\simeq}{\xrightarrow{m}} \Omega C \otimes \Omega C,$$

where $m : \Omega(C \otimes C) \to \Omega C \otimes \Omega C$ is a morphism of chain algebras, called the **Milgram map**. It is the map of algebras associated to the twisting morphism

$$t_\Omega * t_\Omega := t \otimes \eta_{\Omega C} \circ \varepsilon_C + \eta_{\Omega C} \circ \varepsilon_C \otimes t : C \otimes C \to \Omega C \otimes \Omega C$$

using the correspondence described in Proposition 1.2.8. Unravelling the definition, we see that m is defined on generating elements of $\Omega(C \otimes C)$ by

$$m(s^{-1}(c_1 \otimes c_2)) = \begin{cases} 0, & \text{if } \deg(c_1) > 0, \deg(c_2) > 0, \\ 1 \otimes s^{-1}c_2, & \text{if } c_1 = 1, \forall c_2 \in C, \\ s^{-1}c_1 \otimes 1, & \text{if } c_2 = 1, \forall c_1 \in C, \end{cases}$$

for all $s^{-1}(c_1 \otimes c_2) \in \Omega(C \otimes C)$.

Theorem 1.2.11 (Theorem A.1, [HPS07]). *Let C and C' be coaugmented differential graded coalgebras in $\mathbf{Ch}_R^{\geq 0}$. There exists a strong deformation retract of chain complexes*

$$\Omega C \otimes \Omega C' \underset{m}{\overset{\sigma}{\rightleftarrows}} \Omega(C \otimes C') \circlearrowleft h.$$

In particular, m is a chain homotopy equivalence.

Remark 1.2.12. Theorem 1.2.11 is an improvement of the original result of Milgram (Theorem 7.4 [Mil66]), which stated that the Milgram map m is a quasi-isomorphism of chain complexes, under the assumption that the coalgebras C and C' are 1-connected (and not necessarily coaugmented).

Finally, $\Delta_{\Omega C} : \Omega C \to \Omega C \otimes \Omega C$ is given on an arbitrary generating element $s^{-1}c \in \Omega C$ by

$$\Delta_{\Omega C}(s^{-1}c) = 1 \otimes s^{-1}c \pm s^{-1}c \otimes 1,$$

where we use that s^{-1} is additive and the expression of Δ_C from Reminder 1.2.1 (3). More generally, for any $s^{-1}c_1|\dots|s^{-1}c_n \in (s^{-1}\overline{C})^{\otimes n}$,

$$\Delta_{\Omega C}(s^{-1}c_1|\dots|s^{-1}c_n) =$$

$$\sum_{p=1}^{n-1} \sum_{\sigma \in \mathcal{S}_{p,n}} \pm(s^{-1}c_{\sigma(1)}|\dots|s^{-1}c_{\sigma(p)}) \otimes (s^{-1}c_{\sigma(p+1)}|\dots|s^{-1}c_{\sigma(n)}),$$

where $\mathcal{S}_{p,n}$ is the collection of all $(p, n-p)$ shuffles, for $n \geq 1$.

The map $\Delta_{\Omega C}$ makes ΩC into a co-commutative Hopf algebra in $\mathbf{Ch}_R^{\geq 0}$ (see [Tan83]).

An immediate consequence of the previous remark is that for any co-commutative Hopf algebra $H \in \mathbf{Ch}_R^{\geq 0}$, its cobar construction ΩH can be endowed with a co-commutative Hopf algebra structure, too.

1.2.2 Semi-free modules in $\mathbf{Ch}_R^{\geq 0}$

Definition 1.2.13 ([FHT01], Ch1, §6)**.** Let (A, d_A) be a dg-algebra in $\mathbf{Ch}_R^{\geq 0}$. A left A-module (E, d_E) in $\mathbf{Ch}_R^{\geq 0}$ is **semi-free** on a generating graded R-module Z, denoted by $E \cong A \tilde{\otimes} Z$, if the following properties hold.

(a) $Z = \bigoplus_{k \geq 0} Z(k)$, where $Z(k)$ is a graded R-free module, for all $k \geq 0$.

(b) The A-module E is equipped with a filtration

$$\{0\} = E(-1) \subset E(0) \subset \cdots \subset E(k-1) \subset E(k) \subset \cdots \subset E = \bigcup_{k \geq 0} E(k)$$

of A-submodules such that $E(k)/E(k-1) \cong (A, d_A) \otimes (Z(k), 0)$, as dg A-modules, for all $k \geq 0$. It follows that there is an isomorphism of graded R-modules

$$E(k) \cong E(k-1) \oplus \big(A \otimes Z(k)\big), \quad \text{with } d_E(A \otimes Z(k)) \subseteq E(k-1),$$

for all $k \geq 0$, so that $E \cong \bigoplus_{k \geq 0} A \otimes Z(k)$, as graded R-modules.

(c) In addition to the conditions in [FHT01], we require that the induced filtration on Z satisfy $Z(k)_n = 0$, for all $k > n$.

Lemma 1.2.14. *Let R be a commutative ring, (A, d_A) a differential graded R-algebra and (B, d_B) a chain complex over R, such that $B_0 = R$ and B is degree-wise R-free and finitely generated.*

The tensor product complex $(A \otimes B, D)$, where

$$D(a \otimes b) = d_A(a)b + (-1)^{\deg(a)} a \otimes d_B(b),$$

for all $a \in A$, $b \in B$, is semi-free as a left A-module on a generating graded R-module that is degree-wise finitely generated.

Proof. The left A-module structure on $A \otimes B$ is given by $A \otimes (A \otimes B) \cong A \otimes A \otimes B \xrightarrow{\mu_A} A \otimes B$.

The graded module, that generates $A \otimes B$ as A-semi-free, is the underlying graded module of (B, d_B). It satisfies $B = \bigoplus_{k \geq 0} B_k$, where we identify B_k with the graded R-module with value B_k in degree k and 0 everywhere else. In particular, for all $k < n$, $(B_n)_k = 0$. By hypothesis, B is degree-wise R-free and finitely generated.

Define a filtration $\mathcal{F}$ on the A-module $A \otimes B$ by setting

$$\mathcal{F}_k(A \otimes B) = A \otimes B_{\leq k},$$

for all $k \geq 0$ (recall Notation 1.1.5). Every $A \otimes B_{\leq k}$ is a graded A-submodule of $A \otimes B$, and

$$\{0\} \subset A \subset A \otimes B_{\leq 1} \subset \cdots \subset A \otimes B_{\leq k-1} \subset A \otimes B_{\leq k} \subset \cdots ,$$

so that $\bigcup_{k \geq 0} \mathcal{F}_k(A \otimes B) = A \otimes B$, as graded A-modules. The differential D satisfies $D(B_k) \subset \mathcal{F}_{k-1}(A \otimes B)$, for all $k \geq 0$, since for all $k \geq 0$ and for all $b \in B_k$,

$$D(1 \otimes b) = (-1)^k 1 \otimes d_B(b) \in A \otimes B_{k-1}.$$

Therefore, the quotient

$$\mathcal{F}_k(A \otimes B)/\mathcal{F}_{k-1}(A \otimes B) = (A \otimes B_{\leq k})/(A \otimes B_{\leq k-1})$$

is isomorphic, as a differential graded A-module, to the tensor product $(A, d_A) \otimes (B_k, 0)$, for all $k \geq 0$. □

Lemma 1.2.15 (Lemma 6.3, [FHT01])**.** *Let (T, d_T) be a differential graded algebra. Suppose a (T, d_T)-module (M, d_M) is the union of an increasing sequence*

$$\{0\} \subset M(0) \subset M(1) \subset \cdots \subset M(k-1) \subset M(k) \subset \cdots$$

of submodules, such that $M(0)$ and each $M(k)/M(k-1)$ are (T, d_T)-semi-free. Then (M, d_M) itself is (T, d_T)-semi-free.

Remark 1.2.16. The proof of Lemma 6.3 in [FHT01] ensures that conditions (a) and (b) of Definition 1.2.13 hold for (M, d_M). A careful investigation of this proof shows that the condition (c) of Definition 1.2.13 is also satisfied for (M, d_M).

1.2.3 A toolbox of spectral sequences

This section mainly assembles some spectral sequences that will be useful to us for making calculations throughout the thesis. For the basic definitions and terminology used in the world of spectral sequences, we refer the reader to [McC01].

Notations 1.2.17.

- By "a graded object" we mean "a non-negatively graded object".
- In this section it will be important to make a distinction between graded and non-graded objects. Therefore, we will take special care of notation, and decorate graded objects with subscript $_*$ when necessary.
- Given a differential graded module M_*, we will denote by $\{M(p)_*\}_p$ a filtration on M_* and by $\{F^pH_*(M_*)\}_p$ the induced filtration on its homology. These filtrations are increasing.

The following spectral sequence is useful when one works with semi-free modules.

Theorem 1.2.18 (Homological version of Theorem 2.6, [McC01])**.** *Each filtered differential graded module $(M_*, d_M, M(-)_*)$ determines a spectral sequence, $\{E^r_{*,*}, d_r\}$, $r = 1, 2, ...$ with d_r of bidegree $(-r, r-1)$, and the first page given by*

$$E^1_{p,q} \cong H_{p+q}\big(M(p)_*/M(p-1)_*\big).$$

If the filtration is bounded, i.e., for each degree n, there exist $s = s(n)$ and $t = t(n)$ so that

$$\{0\} \subset M(s)_n \subset M(s+1)_n \subset \cdots \subset M(t-1)_n \subset M(t)_n = M_n,$$

then the spectral sequence converges to $H_*(M_*, d_M)$, i.e.,

$$E^\infty_{p,q} \cong F^pH_{p+q}(M_*, d_M)/F^{p-1}H_{p+q}(M_*, d_M).$$

On many occasions, we will use Zeeman's comparison theorem, which is a standard tool in the application of spectral sequences.

Theorem 1.2.19 ([Zee57])**.** *Suppose $\{E^r_{*,*}, d_r\}$ and $\{\overline{E}^r_{*,*}, \overline{d}_r\}$ are first quadrant spectral sequences of homological type (that is, d_r and $\overline{d}_r$ are of bidegree $(-r, r-1)$), such that for all $p, q \geq 0$ there exist short exact sequences*

$$0 \longrightarrow E^2_{p,0} \otimes E^2_{0,q} \longrightarrow E^2_{p,q} \longrightarrow \mathrm{Tor}^1_R\left(E^2_{p+1,0}, E^2_{0,q}\right) \longrightarrow 0$$

and

$$0 \longrightarrow \overline{E}^2_{p,0} \otimes \overline{E}^2_{0,q} \longrightarrow \overline{E}^2_{p,q} \longrightarrow \mathrm{Tor}^1_R\left(\overline{E}^2_{p+1,0}, \overline{E}^2_{0,q}\right) \longrightarrow 0.$$

Suppose that $A = \oplus_{n=0}^{\infty} A_n$, $\overline{A} = \oplus_{n=0}^{\infty} \overline{A}_n$ *are filtered graded* R*-modules such that their associated graded* R*-modules satisfy, respectively,* $\mathrm{Gr}(A) = \oplus_{p,q\geq 0}^{\infty} E_{p,q}^{\infty}$, $\mathrm{Gr}(\overline{A}) = \oplus_{p,q\geq 0}^{\infty} \overline{E}_{p,q}^{\infty}$. *Moreover, suppose that there is a homomorphism* f *between the spectral sequences* $\{E_{*,*}^r, d_r\}$ *and* $\{\overline{E}_{*,*}^r, \overline{d}_r\}$*, i.e., there are sets of homomorphisms*

$$f_{p,q}^r : E_{p,q}^r \to \overline{E}_{p,q}^r, \qquad (r = 2, 3, \ldots, \infty, \text{ and } p, q \geq 0),$$

$$f_n^A : A_n \to \overline{A}_n, \qquad (n \geq 0),$$

$$f_p^2 : E_{p,0}^2 \to \overline{E}_{p,0}^2, \qquad (p \geq 0),$$

$$f_q^2 : E_{0,q}^2 \to \overline{E}_{0,q}^2, \qquad (q \geq 0),$$

and that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{p,0}^2 \otimes E_{0,q}^2 & \longrightarrow & E_{p,q}^2 & \longrightarrow & \mathrm{Tor}_R^1\left(E_{p+1,0}^2, E_{0,q}^2\right) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_p^2 \otimes f_q^2} & & \downarrow{\scriptstyle f_{p,q}^2} & & \downarrow{\scriptstyle \mathrm{Tor}_R^1(f_{p+1,0}^2, f_{0,q}^2)} & & \\ 0 & \longrightarrow & \overline{E}_{p,0}^2 \otimes \overline{E}_{0,q}^2 & \longrightarrow & \overline{E}_{p,q}^2 & \longrightarrow & \mathrm{Tor}_R^1\left(\overline{E}_{p+1,0}^2, \overline{E}_{0,q}^2\right) & \longrightarrow & 0 \end{array}$$

Then any two of the following conditions imply the third:

(1) $f_p^2 : E_{p,0}^2 \to \overline{E}_{p,0}^2$ *is an isomorphism for all* $p \geq 0$*;*

(2) $f_q^2 : E_{0,q}^2 \to \overline{E}_{0,q}^2$ *is an isomorphism for all* $q \geq 0$*;*

(3) $f_n^A : A_n \to \overline{A}_n$ *is an isomorphism for all* $n \geq 0$.

Theorem 1.2.18 applied to the context of twisted extensions gives the following spectral sequence. Here, we formulate a slightly more general version of it than the one described in §10.4 in [Nei10].

Theorem 1.2.20 (Generalization of §10.4, [Nei10])**.** *Let* $(C, d_C) \in \mathbf{Coalg}_R^{\eta,1}$, $(A, d_A) \in \mathbf{Alg}_R^{\varepsilon,0}$ *and let* $t : C \to A$ *be a twisting morphism. Given* (N, d_N) *a right* C*-comodule in* $\mathbf{Ch}_R^{\geq 0}$ *and* (M, d_M) *a left* A*-module in* $\mathbf{Ch}_R^{\geq 0}$*, consider the twisted extension* $N \otimes_t M$.

If N *is degree-wise flat over* R*, then the filtration*

$$\{F_k(N \otimes_t M) = N_{\leq k} \otimes M\}_{k \geq 0}$$

of $N \otimes_t M$ *by differential graded submodules defines a first quadrant homology spectral sequence* $\{E_{*,*}^r, d_r\}$*. Its initial pages are given by*

$$E_{p,q}^0 = N_p \otimes M_q, \; d^0 = \mathrm{Id} \otimes d_M,$$

$$E_{p,q}^1 = N_p \otimes H_q(M), \; d^1 = d_N \otimes \mathrm{Id},$$

$$E^2_{p,q} = H_p(N \otimes H_q(M), d_N \otimes \mathrm{Id}),$$

for all $p, q \geq 0$*, and the sequence converges to* $H_*(N \otimes_t M)$*.*

If, in addition, $H_q(M)$ *is flat over* R *for all* $q \geq 0$*, then the second page simplifies to*

$$E^2_{p,q} = H_p(N) \otimes H_q(M),$$

for all $p, q \geq 0$*.*

Remark 1.2.21. The spectral sequence given in [Nei10] is formulated in the particular case where the C-comodule N is equal to C, and the A-module M is equal to A. However, the construction of the spectral sequence requires neither having a coalgebra structure on N nor having an algebra structure on M. What really matters for defining the twisted extension $N \otimes_t M$ and for building the associated spectral sequence is having a structure of a differential C-comodule on N and a structure of a differential A-module on M.

Remark 1.2.22. Note that the spectral sequence of Theorem 1.2.20 admits a short exact sequence of the form required in Theorem 1.2.19, by the Universal Coefficient Theorem.

Finally, Zeeman's comparison theorem gives the following result for twisted tensor products.

Proposition 1.2.23 (Generalization of Proposition 10.4.6, [Nei10])**.** *Let* $\gamma : C \to C'$ *be a morphism of coalgebras,* $N \in \mathbf{Comod}_C$*,* $N' \in \mathbf{Comod}_{C'}$*,* $\alpha : A \to A'$ *a morphism of algebras,* $M \in {}_A\mathbf{Mod}$ *and* $M' \in {}_{A'}\mathbf{Mod}$*. In addition, let* $t : C \to A$ *and* $t' : C' \to A'$ *be two twisting morphisms and* $g \,\widetilde{\otimes}\, g' : N \otimes_t M \to N' \otimes_{t'} M'$ *be a morphism of twisted extensions. Suppose that* N *and* N' *are degree-wise flat as graded modules over* R*.*

If two out of g*,* g'*,* $g \,\widetilde{\otimes}\, g'$*, are quasi-isomorphisms, then so is the third.*

Remark 1.2.24. In the context of twisted tensor products, a morphism $g \,\widetilde{\otimes}\, g' : N \otimes_t M \to N' \otimes_{t'} M'$ of twisted extensions is a morphism of *filtered objects*, respecting the filtrations $\{F_k(N \otimes_t M)\}_{k \geq 0}$, $\{F'_k(N' \otimes_t M')\}_{k \geq 0}$. Therefore, $g \,\widetilde{\otimes}\, g'$ will induce a homomorphism of the associated spectral sequences, described in Theorem 1.2.20. In particular, it will induce homomorphisms on the homology groups

$$f^A_n = H_n(g \,\widetilde{\otimes}\, g') : A_n = H_n(M \otimes_t N) \to H_n(M' \otimes_t N') = \overline{A}_n,$$

for all $n \geq 0$, using notation of Theorem 1.2.19.

1.2.4 Homologically faithful modules in $\mathbf{Ch}_R^{\geq 0}$

Definition 1.2.25. Let B be a dg-algebra in $\mathbf{Ch}_R^{\geq 0}$. A left B-module M is **homologically faithful (over** B**)** if for any right B-module N such that $H_n(N \otimes_B M) = 0$ for all $n \geq 0$, we have $H_n(N) = 0$ for all $n \geq 0$.

Examples 1.2.26.

- If $M \cong B \otimes X$, i.e., M is free as a left B-module on a generating graded R-module that is degree-wise R-free and finitely generated, then M is homologically faithful over B.

  Indeed, if $H_n(N \otimes_B M) \cong H_n(N \otimes X) = 0$, for all $n \geq 0$, then the Künneth formula implies that $H_n(N) = 0$ for all $n \geq 0$.

- Let $\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B$ denote the *acyclic bar construction*, defined for any dg-algebra B. Here, $\mathcal{B}(B)$ denotes the *bar construction* on B and $t_{\mathcal{B}} : \mathcal{B}(B) \to B$ denotes the *couniversal twisting cochain*. All these objects and constructions are dual to the ones that we described in Section 1.2.1 (see Section 6.3 in [Nei10] for their definitions).

  If $M = B \otimes_{t_{\mathcal{B}}} \mathcal{B}\, B \otimes_{t_{\mathcal{B}}} B$, then $N \otimes_B M \cong N \otimes_{t_{\mathcal{B}}} \mathcal{B}\, B \otimes_{t_{\mathcal{B}}} B$, and we have $N \otimes_{t_{\mathcal{B}}} \mathcal{B}\, B \otimes_{t_{\mathcal{B}}} B \xrightarrow{\simeq} N$, since $N \otimes_{t_{\mathcal{B}}} \mathcal{B}\, B \otimes_{t_{\mathcal{B}}} B$ is a proper projective resolution of N over B, by Proposition 7.8 in [McC01]. Thus, $H_*(N \otimes_B M) = 0$ if and only if $H_*(N) = 0$, i.e., M is homologically faithful over B.

We mention the following standard fact from homological algebra.

Lemma 1.2.27 ([Rot88], Chapter 9)**.** *A morphism $g : X \to Y$ in $\mathbf{Ch}_R^{\geq 0}$ is a quasi-isomorphism if and only if its associated mapping cone $C(g)$ is acyclic.*

Lemma 1.2.28. *Let B, B' be two dg-algebras in $\mathbf{Ch}_R^{\geq 0}$. Suppose that B' is a left B-module that is homologically faithful over B and is also semi-free on a generating graded R-module Z that is degree-wise finitely generated. Then, for any morphism $g : X \to Y$ of right B-modules, the map $g \otimes_B B' : X \otimes_B B' \to Y \otimes_B B'$ is a quasi-isomorphism in $\mathbf{Mod}_{B'}$ if and only if $g : X \to Y$ is a quasi-isomorphism in $\mathbf{Mod}_B$.*

Proof. Suppose that $g : X \to Y$ is a quasi-isomorphism in $\mathbf{Ch}_R^{\geq 0}$. By Lemma 1.2.27, the mapping cone $C(g)$ is then acyclic. Consider the short exact sequence of chain complexes

$$0 \to Y \to C(g) \to s(X) \to 0,$$

where $s(X)$ denotes the suspension of X (see Reminder 1.2.1). It is easy to check that $C(g) \otimes_B B' \cong C\big(g \otimes_B B'\big)$, so if we apply $- \otimes_B B'$ to the sequence above, we obtain the sequence

$$0 \to Y \otimes_B B' \to C\big(g \otimes_B B'\big) \to s(X) \otimes_B B' \to 0$$

that is short exact, because B' is semi-free, hence flat over B. It remains to show that $H_n\big(C\big(g \otimes_B B'\big)\big) \cong H_n\big(C(g) \otimes_B B'\big) = 0$ for all $n \geq 0$.

By assumption, the dg-algebra $(B', d_{B'})$ is semi-free as a left B-module on a generating graded R-module Z, and Z is degree-wise finitely generated. Using the notation $B' \cong B \widetilde{\otimes} Z$, we have

$$H_n\big(C(g) \otimes_B B'\big) \cong H_n\big(C(g) \otimes_B B \widetilde{\otimes} Z\big) \cong H_n\big(C(g) \widetilde{\otimes} Z\big),$$

for all $n \geq 0$. To conclude that all the homology groups $H_n\big(C(g) \widetilde{\otimes} Z\big)$ vanish, we will apply Theorem 1.2.18.

The graded module $M := C(g) \widetilde{\otimes} Z$ inherits a filtration of R-submodules

$$\{0\} = M(-1) \subset M(0) \subset \cdots \subset M(k-1) \subset M(k) \subset \cdots \subset M = \cup_{k \geq 0} M(k)$$

such that

$$M(k)/M(k-1) \cong (C(g), \bar{d}) \otimes (Z(k), 0) \text{ for all } k \geq 0.$$

Since Z_n is finitely generated for all $n \geq 0$, the filtration on M is bounded. Moreover, the differential on M satisfies $d_M : M(k) \to M(k-1)$, for all $k \geq 0$.

Applying Theorem 1.2.18, the first page of the spectral sequence associated to the filtration of (M, d) is given for all $p, q \geq 0$ by

$$\begin{aligned}
E^1_{p,q} &\cong H_{p+q}\big(M(p)/M(p-1)\big) \\
&\cong H_{p+q}\big(C(g) \otimes Z(p), \bar{d} \otimes \mathrm{Id} + \mathrm{Id} \otimes 0\big) \\
&\cong \bigoplus_{\gamma = p+q} H_\gamma(C(g)) \otimes Z(p), \text{ using that } (Z(p), 0) \text{ is degree-wise } R\text{-free,} \\
&\cong \bigoplus_{\gamma \geq p} H_\gamma(C(g)) \otimes Z(p), \text{ since } \gamma, p, q \geq 0, \\
&\cong 0, \text{ since } C(g) \text{ is acyclic.}
\end{aligned}$$

The first page being null, $E^\infty_{p,q} = 0$ as well for all $p, q \geq 0$, and therefore $H_n(M, d) = H_n\big(C(g) \widetilde{\otimes} Z\big) \cong 0$, for all $n \geq 0$, as desired. So, we can conclude that the map $g \otimes_B B'$ is a quasi-isomorphism, by Lemma 1.2.27.

Reciprocally, suppose that $g \otimes_B B' : X \otimes_B B' \to Y \otimes_B B'$ is a quasi-isomorphism in $\mathbf{Mod}_{B'}$. Lemma 1.2.27 tells us that in that case for all $n \geq 0$,

$$H_n\big(C\big(g \otimes_B B'\big)\big) \cong H_n\big(C(g) \otimes_B B'\big) = 0.$$

Since by assumption B' is homologically faithful as a B-module, this implies that $H_n(C(g)) = 0$, for all $n \geq 0$, which means that g is a quasi-isomorphism. □

Remark 1.2.29. We observe that Lemma 4.3.2 (a) in [Rog08], analogous to Lemma 1.2.28 in the case of ring spectra, also needs the (homotopical) faithfulness condition on B', but does not require the semi-freeness condition (or, at least, it is not mentioned explicitly). A possible reason for this may be that Rognes assumes an implicit cofibrancy of all objects. (Therefore, given a quasi-isomorphism $g : X^c \to Y^c$ between cofibrant objects, the fact that the map $g \otimes_B B' : X^c \otimes_B B' \to Y^c \otimes_B B'$ is a quasi-isomorphism, too, will follow by Ken Brown's lemma, whenever the functor $- \otimes_B B' : \mathbf{Mod} : B \to \mathbf{Mod}_{B'}$ is left Quillen.)

1.3 Model category theory

We assume that the reader is familiar with the definition of a model category and the definition of a Quillen pair between model categories. Otherwise, we invite him or her to look in [Hes02] for a good introduction to the basics of model categories.

Notation 1.3.1. Given a model category $\mathcal{M}$, we denote by $WE_{\mathcal{M}}$, $Fib_{\mathcal{M}}$ and $Cof_{\mathcal{M}}$, respectively, the classes of weak equivalences, fibrations, and cofibrations. In terms of arrows, $\xrightarrow{\sim}$ denotes a weak equivalence, $\twoheadrightarrow$ denotes a fibration and $\rightarrowtail$ denotes a cofibration.

1.3.1 Recognizing Quillen equivalences

Why is recognizing Quillen equivalences particularly important for us in this thesis? It is because our main objects of study - homotopic Hopf-Galois extensions - will be characterized in terms of pairs of Quillen equivalences that are induced by two special maps associated to them. Consequently, the more tools we have for identifying Quillen equivalences, the better.

Let

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be a Quillen pair between model categories $\mathcal{C}$ and $\mathcal{D}$. We denote by

$$\mathbb{L}\, F : \mathrm{Ho}\, \mathcal{C} \rightleftarrows \mathrm{Ho}\, \mathcal{D} : \mathbb{R}\, G$$

the associated derived adjunction on the homotopy categories.

Definition 1.3.2. A Quillen pair (F, G) is called a **Quillen equivalence** if for all cofibrant $X^c \in \mathcal{C}$ and all fibrant $Y^f \in \mathcal{D}$, a morphism $f : FX^c \to Y^f$ is a weak equivalence in $\mathcal{D}$ if and only if its transpose $f^\sharp : X^c \to GY^f$ is a weak equivalence in $\mathcal{D}$.

The key property of a Quillen equivalence (F, G) is that it induces an equivalence of categories $(\mathbb{L}\, F, \mathbb{R}\, G)$ on the homotopy categories. We will now see two useful criteria that allow one to recognize Quillen equivalences.

Let X^c be a cofibrant object in $\mathcal{C}$ and let $r : FX^c \xrightarrow{\sim} R(FX^c)$ be a fibrant replacement of FX^c in $\mathcal{D}$. The map $\tilde{\eta}_{X^c} : X^c \to GRFX^c$, which is the transpose of r, represents the homotopy unit, i.e., the unit of the derived adjunction $(\mathbb{L}\, F, \mathbb{R}\, G)$.

Dually, let Y^f be a fibrant object in $\mathcal{D}$ and let $q : Q(GY^f) \xrightarrow{\sim} GY^f$ be a cofibrant replacement of GY^f in $\mathcal{C}$. The map $\tilde{\varepsilon}_{Y^f} : FQGY^f \to Y^f$, which is the transpose of q, represents the homotopy counit, i.e., the counit of the derived adjunction $(\mathbb{L}\, F, \mathbb{R}\, G)$.

Proposition 1.3.3 (Corollary 1.3.16, [Hov99]). *A Quillen pair*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is a Quillen equivalence if

(1) the homotopy unit $\tilde{\eta}_{X^c} : X^c \to GRFX^c$ is a weak equivalence in $\mathcal{C}$ for all cofibrant $X^c \in \mathcal{C}$, and

(2) the functor G reflects weak equivalences between fibrant objects;

or, dually, if

(1') the homotopy counit $\tilde{\varepsilon}_{Y^f} : FQGY^f \to Y^f$ is a weak equivalence for all fibrant $Y^f \in \mathcal{D}$, and

(2') the functor F reflects weak equivalences between cofibrant objects.

1.3.2 Some relevant model structures

The model structure we will equip the category $\mathbf{Ch}_R^{\geq 0}$ with is the **projective model structure**. As shown in [DS95], it is specified by

$WE_{\mathbf{Ch}_R^{\geq 0}}$ = quasi-isomorphisms;

$Fib_{\mathbf{Ch}_R^{\geq 0}}$ = degree-wise surjections in strictly positive degrees;

$Cof_{\mathbf{Ch}_R^{\geq 0}}$ = degree-wise monomorphisms with degree-wise projective cokernel.

Convention 1.3.4. Throughout this project, the category $\mathbf{Ch}_R^{\geq 0}$ is equipped with the projective model structure, described above. Similarly, when $\Bbbk$ is a field, the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ is equipped with the projective model structure

Right-induced model structures on categories of modules and algebras

Given a model category $\mathcal{M}$ and an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{D} : G$, we say that $\mathcal{D}$ inherits a **right-induced model structure** if there exists a model structure on $\mathcal{D}$ such that the right adjoint G creates the weak equivalencies and fibrations in $\mathcal{D}$, i.e., if $WE_{\mathcal{D}} := G^{-1}(WE_{\mathcal{M}})$ and $Fib_{\mathcal{D}} := G^{-1}(Fib_{\mathcal{M}})$. In this case, (F, G) becomes a Quillen pair with respect to these model structures on $\mathcal{M}$ and $\mathcal{D}$.

Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a cofibrantly generated monoidal model category. In [SS00], S. Schwede and B. Shipley have established conditions under which the category of modules over a monoid in $\mathcal{M}$, and the category of monoids in $\mathcal{M}$ both inherit a right-induced model structure from $\mathcal{M}$. The following two theorems state these results in the special case where $\mathcal{M} := \mathbf{Ch}_R^{\geq 0}$.

Theorem 1.3.5 (follows from Theorem 4.1(1), [SS00])**.** *Let R be a commutative ring. For any differential graded algebra A in $\mathbf{Ch}_R^{\geq 0}$, there is a cofibrantly generated model structure on the category $\mathbf{Mod}_A$, obtained by right transfer of the projective structure on $\mathbf{Ch}_R^{\geq 0}$ via the adjunction*

$$\mathbf{Ch}_R^{\geq 0} \underset{U}{\overset{-\otimes A}{\rightleftarrows}} \mathbf{Mod}_A,$$

where U denotes the forgetful functor.

Theorem 1.3.6 (follows from Theorem 4.1(3), [SS00])**.** *Let R be a commutative ring. There is a cofibrantly generated model structure on the category* $\mathbf{Alg}$ *of differential graded R-algebras, obtained by right transfer of the projective structure on $\mathbf{Ch}_R^{\geq 0}$ via the adjunction*

$$\mathbf{Ch}_R^{\geq 0} \underset{U_{\mathbf{Alg}}}{\overset{F_{\mathbf{Alg}}}{\rightleftarrows}} \mathbf{Alg},$$

where $F_{\mathbf{Alg}}$ denotes the free monoid functor and $U_{\mathbf{Alg}}$ denotes the forgetful functor.

Remark 1.3.7. In early October 2013, Tobias Barthel, J.P. May and Emily Riehl submitted a new article to the arXiv (see [BMR13]), in which they describe six projective-type model category structures on the category of dg-modules over a dg-algebra A over a commutative ring R, which "offer interesting alternatives to the model structures in common use". We will need one of these, namely the r-model structure, to prove the existence of a model category structure on the categories $\mathcal{M}_A^W$ and $\mathbf{Alg}_H^{\varepsilon}$, where the latter denotes the category of *augmented* H-comodule algebras.

Left-induced model structures on categories of comodules

Let us dualize the setting above. Given a model category $\mathcal{M}$ and an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{M} : G$, we say that $\mathcal{C}$ inherits a **left-induced model structure** if there exists a model structure on $\mathcal{C}$ such that the left adjoint F creates the weak equivalences and cofibrations in $\mathcal{C}$, i.e., if $WE_{\mathcal{C}} := F^{-1}(WE_{\mathcal{M}})$ and $Cof_{\mathcal{C}} := F^{-1}(Cof_{\mathcal{M}})$. In this case, (F, G) becomes a Quillen pair with respect to these model structures on $\mathcal{M}$ and $\mathcal{C}$.

To determine conditions under which there exist left-induced model structures on the categories that are relevant to us, we will need two crucial left-transfer results from [BHKKRS14].

In order to understand their statements, we need first to introduce some terminology.

Definition 1.3.8. A category $\mathcal{M}$ is **locally presentable** if it is locally small, has all small colimits and there exists a set $S \subset Ob\,\mathcal{M}$ that generates $\mathcal{M}$ under colimits (i.e., every object $M \in \mathcal{M}$ can be written as a colimit over a diagram with objects in S).

Definition 1.3.9. A model structure on a category $\mathcal{M}$ is **combinatorial** if it is cofibrantly generated and the underlying category is locally presentable.

Convention 1.3.10. From now and until the beginning of Section 1.4, we work over a **field** $\Bbbk$ (see Remark 1.3.14).

(1) Model structure on the category of comodules over a coring in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$

Theorem 1.3.11 ([Hes])**.** *Let $\Bbbk$ be a field and A an augmented differential graded $\Bbbk$-algebra. Endow the category $\mathbf{Mod}_A$ with the model category structure right-induced from the projective model structure on $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (see Theorem 1.3.5). If V is an A-coring that is semi-free as a left A-module on a graded $\Bbbk$-vector space X of finite type, then the adjunction*

$$\mathcal{M}_A^V \underset{-\otimes_A V}{\overset{U}{\rightleftarrows}} \mathbf{Mod}_A$$

left-induces a combinatorial model category structure on the category $\mathcal{M}_A^V$ of V-comodules in $\mathbf{Mod}_A$.

(2) Model structure on the category of comodules over a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$

Observe that if $A := \Bbbk[0]$ and $V := H$, a Hopf algebra, the category $\mathcal{M}_A^V$ is the category $\mathbf{Comod}_H$. So Theorem 1.3.11 gives the following result.

Theorem 1.3.12 ([Hes]). *Let $\Bbbk$ be a field and H a differential graded $\Bbbk$-Hopf algebra that is finite-dimensional in each degree. Endow the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ with the projective model category structure. The adjunction*

$$\mathbf{Comod}_H \underset{-\otimes H}{\overset{U}{\rightleftarrows}} \mathbf{Ch}_{\Bbbk}^{\geq 0}$$

then left-induces a combinatorial model category structure on the category $\mathbf{Comod}_H$ of H-comodules.

(3) Model structure on the category of augmented H-comodule algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$

Let $\mathbf{Alg}^{\varepsilon}$ denote the category of augmented differential graded $\Bbbk$-algebras, over a field $\Bbbk$. Equip it with the model category structure right-induced from the projective structure on $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (see Theorem 1.3.6).

Theorem 1.3.13 (Theorem 3.8, [BHKKRS14]). *If $\Bbbk$ is a field, and H is a differential graded $\Bbbk$-Hopf algebra that is finite-dimensional in each degree, then the adjunction*

$$\mathbf{Alg}_H^{\varepsilon} \underset{-\otimes H}{\overset{U}{\rightleftarrows}} \mathbf{Alg}^{\varepsilon}$$

left-induces a combinatorial model category structure on the category $\mathbf{Alg}_H^{\varepsilon}$ of augmented right H-comodule algebras.

Remark 1.3.14. It is highly likely that the left transfer results 1.3.11, 1.3.12 and 1.3.13 can be generalized to an arbitrary **commutative ring** R, but to do this one needs an enriched version of Theorem 2.21 in [BHKKRS14].

1.4 Quillen pairs and Quillen equivalences between categories of modules

Remark 1.4.1. In Section 1.4 we work over a commutative ring R.

1.4.1 Quillen pairs induced by morphisms of algebras

Let $\alpha : C \to E$ be a morphism of dg-algebras in $\mathbf{Ch}_R^{\geq 0}$. It induces an extension-restriction of scalars adjunction $- \otimes_C E : \mathbf{Mod}_C \rightleftarrows \mathbf{Mod}_E : \alpha^*$. When is $(- \otimes_C E, \alpha^*)$ a Quillen pair?

Lemma 1.4.2. *Let R be a commutative ring and suppose that the category $\mathbf{Ch}_R^{\geq 0}$ is equipped with the projective model structure. Let $\alpha : C \to E$ be a morphism of dg-algebras in $\mathbf{Ch}_R^{\geq 0}$. Then the adjunction*

$$\mathbf{Mod}_C \underset{\alpha^*}{\overset{-\otimes_C E}{\rightleftarrows}} \mathbf{Mod}_E$$

is always a Quillen pair with respect to the model structures on $\mathbf{Mod}_C$ *and* $\mathbf{Mod}_E$*, right-induced from* $\mathbf{Ch}_R^{\geq 0}$*.*

Proof. It is easy to see that the right adjoint α^* preserves fibrations and acyclic fibrations. Indeed, by the definition of the right-induced model structure on $\mathbf{Mod}_E$ and $\mathbf{Mod}_C$, $f \in Fib_{\mathbf{Mod}_E}$ if and only if its underlying morphism is a fibration of chain complexes if and only if $\alpha^*(f) \in Fib_{\mathbf{Mod}_C}$. Similarly, $f \in WE_{\mathbf{Mod}_E}$ if and only if its underlying morphism is a quasi-isomorpism of chain complexes if and only if $\alpha^*(f) \in WE_{\mathbf{Mod}_C}$. □

1.4.2 Quillen equivalences induced by quasi-isomorphisms of algebras

In this section we would like to understand the relationship between the situation where a morphism of algebras is a quasi-isomorphism and the situation where the extension/restriction of scalars adjunction induced by this morphism on the categories of modules is a Quillen equivalence.

Knowing more about this relationship will allow us to investigate questions about the behavior of the comparison functor $(i_\varphi)^*$, which plays an essential role in the definition of a homotopic Hopf-Galois extension φ, at least in a particular case where the underlying comparison map i_φ happens to be a quasi-isomorphism.

The next Proposition comes from [BH12].

Proposition 1.4.3. *Let* R *be a commutative ring. If* $g : B \to A$ *is a morphism of augmented dg-*R*-algebras in* $\mathbf{Ch}_R^{\geq 0}$ *such that*

(1) $A \cong B \tilde{\otimes} X$ *is semi-free as a left* B*-module on a generating graded* R*-module* X *that is degree-wise finitely generated, and*

(2) g *is a quasi-isomorphism,*

then the adjunction

$$\mathbf{Mod}_B \underset{g^*}{\overset{-\otimes_B A}{\rightleftarrows}} \mathbf{Mod}_A$$

is a Quillen equivalence with respect to the model structures on $\mathbf{Mod}_A$ *and* $\mathbf{Mod}_B$*, right-induced from the projective model structure on* $\mathbf{Ch}_R^{\geq 0}$*, as in Theorem 1.3.5.*

Sketch of the proof: One can use the criteria (1) and (2) from Proposition 1.3.3. The restriction of scalars functor $g^* : \mathbf{Mod}_A \to \mathbf{Mod}_B$ obviously preserves and reflects weak equivalences. Since all objects in $\mathbf{Mod}_B$ are fibrant, it remains to show that the homotopy unit

$$\tilde{\eta}_{M^c} : M^c \to \varphi^*(M^c \otimes_B A)$$

is a weak equivalence in $\mathbf{Mod}_A$ for all cofibrant $M^c \in \mathbf{Mod}_A$, which is equivalent to showing that

$$M^c \to \varphi^*(M^c \tilde{\otimes} X)$$

is a weak equivalence in $\mathbf{Mod}_A$, using assumption (1). The idea is to use a few spectral sequence arguments, involving the *acyclic bar construction* $\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B$ (see Example 1.2.26). Here are the main steps of the argument. There exists a homotopy equivalence $\eta_{\mathcal{B}(B)} \otimes \eta_B : R[0] \to \mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B$ (see Proposition 10.6.1 in [Nei10]), so

$$H_*(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \cong R[0] \cong H_*((\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes_B B).$$

Since $g : B \to B \tilde{\otimes} X$ is a quasi-isomorphism by assumption, and since the acyclic bar construction $\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B$ is right B-semi-free, it follows from Lemma 1.2.28 that the map

$$\mathrm{Id} \otimes_B g : (\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes_B B \to (\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes_B (B \tilde{\otimes} X)$$

is a quasi-isomorphism, too.

By Theorem 1.2.18, a filtration of the domain $(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes_B B \cong \mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B$ of $\mathrm{Id} \otimes_B g$ by the length in the bar construction $\mathcal{B}(B)$ will induce a spectral sequence $\{E^r_{*,*}\}$, converging to $H_*(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B)$. Similarly, a filtration of the codomain $(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes_B (B \tilde{\otimes} X) \cong \mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B \tilde{\otimes} X$ of $\mathrm{Id} \otimes_B g$ by the length in $\mathcal{B}(B)$ will induce a spectral sequence $\{\overline{E}^r_{*,*}\}$, converging to $H_*(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B \tilde{\otimes} X)$, with the second page given by $\overline{E}^2_{p,q} = H_p(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B) \otimes H_q(X)$, for all $p, q \geq 0$. One can then use the Zeeman comparison Theorem to conclude that $H_*(\mathcal{B}(B) \otimes_{t_{\mathcal{B}}} B \tilde{\otimes} X) \cong H_*(X) \cong R[0]$.

Now, consider the map

$$M^c \cong M^c \otimes_B B \xrightarrow{M^c \underset{B}{\otimes} g} M^c \otimes_B B \tilde{\otimes} X \cong M^c \tilde{\otimes} X.$$

Because the graded R-module $M^c \tilde{\otimes} X$ is a semi-free extension of M by X, which is degree-wise finitely generated, it is equipped with a filtration. Theorem 1.2.18 tells us that there exists a spectral sequence $\hat{E}^r_{*,*}$ converging to $H_*(M^c \tilde{\otimes} X)$, of which the second page will simplify to $\hat{E}^2_{p,q} = H_p(M^c) \otimes H_q(X)$, for all $p, q \geq 0$. Similarly, a filtration of M^c by degree will give rise to a spectral sequence converging to $H_*(M^c)$. Using that $H_*(X) \cong R[0]$ and the Zeeman comparison Theorem once again, one will be able to conclude that the map $M^c \underset{B}{\otimes} g$ is a quasi-isomorphism, as desired. □

Here is the converse result.

Proposition 1.4.4. *Let R be a commutative ring and let $g : A \to B$ be a morphism of dg algebras in $\mathbf{Ch}_R^{\geq 0}$. If the induced functor $g^* : \mathbf{Mod}_B \to \mathbf{Mod}_A$ is a Quillen equivalence with respect to the model structures on $\mathbf{Mod}_B$ and $\mathbf{Mod}_A$ that are right-induced from the projective model structure on $\mathbf{Ch}_R^{\geq 0}$, as in Theorem 1.3.5, then $g : A \to B$ is a quasi-isomorphism.*

Proof. By Lemma 1.4.2, the extension/restriction of scalars adjunction $\big(- \otimes_A B, g^*\big)$ is a Quillen pair. By assumption, this is a Quillen equivalence, so condition (1) from Proposition 1.3.3 holds, i.e., the homotopy unit

$$\tilde{\eta}_{M^c} : M^c \to g^*\big(R(M^c \otimes_A B)\big)$$

is a weak equivalence in $\mathbf{Mod}_A$, for all cofibrant $M^c \in \mathbf{Mod}_A$. Here $R(-)$ denotes the fibrant replacement functor. Because of the definition of the right-induced model structure on the categories $\mathbf{Mod}_B$ and $\mathbf{Mod}_A$, this condition is equivalent to saying that the maps

$$\tilde{\eta}_{M^c} : M^c \to g^*\big(M^c \otimes_A B\big)$$

are quasi-isomorphisms of chain complexes, for all $M^c \in \mathbf{Mod}_A$, since all objects are fibrant in $\mathbf{Mod}_A$.

Now, A is cofibrant as a A-module in $\mathbf{Mod}_A$, so $A^c = A$ and the (homotopy) unit evaluated at A factors in $\mathbf{Mod}_A$ as

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{\eta_A} & g^*\big(A \otimes_A B\big) \\ & \searrow^{g} & \downarrow \cong \\ & & g^*(B), \end{array}$$

so g is a quasi-isomorphism, as desired. ☐

1.5 Adjunctions between categories of comodules over corings

1.5.1 Quillen adjunctions induced by bimodules

This section is based on [BH12]. It briefly introduces the concept of adjunctions between categories of comodules over corings, *induced* by adjunctions between categories of modules. This type of situation will be relevant later on, because it will allow us to justify the existence of certain Quillen pairs between certain model categories of interest.

Let A and B be monoids in a monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$. It follows from [BH12] that, up to isomorphism, every adjunction

$$\mathbf{Mod}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Mod}_B$$

is induced by a bimodule ${}_AX_B \in {}_A\,\mathbf{Bimod}_B$ in $\mathcal{M}$, and is of the form

$$\mathbf{Mod}_A \underset{\mathrm{Hom}_B(X,-)}{\overset{-\otimes_A X}{\rightleftarrows}} \mathbf{Mod}_B\,.$$

Here, for any $Y \in \mathbf{Mod}_B$, the right A-module structure on the set of left B-module morphisms $\mathrm{Hom}_B(X,Y)$ is given by

$$\xi : \mathrm{Hom}_B(X,Y) \otimes A \to \mathrm{Hom}_B(X,Y) : f \otimes a \mapsto (f \cdot a),$$

where $(f \cdot a) : X \to Y$ is given by $(f \cdot a)(x) := f(ax)$, for all $x \in X$.

Let $(V, \psi_V, \varepsilon_V)$ and $(W, \psi_W, \varepsilon_W)$ be corings in the categories ${}_A\,\mathbf{Mod}_A$ and ${}_B\,\mathbf{Mod}_B$, respectively. A fixed adjunction $\mathbf{Mod}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Mod}_B$ can be lifted to a *relative adjunction* $\mathcal{M}_A^V \underset{\tilde{G}}{\overset{\tilde{F}}{\rightleftarrows}} \mathcal{M}_B^W$ between the categories of comodules over corings, in the sense that the diagram of left adjoints (or, equivalently, the diagram of right adjoints) in

$$\begin{array}{ccc}
\mathcal{M}_A^V & \underset{\tilde{G}}{\overset{\tilde{F}}{\rightleftarrows}} & \mathcal{M}_B^W \\
U_A \downarrow \uparrow -\otimes_A V & & U_B \downarrow \uparrow -\otimes_B W \\
\mathbf{Mod}_A & \underset{G}{\overset{F}{\rightleftarrows}} & \mathbf{Mod}_B
\end{array}$$

commutes, up to natural isomorphism.

Definition 1.5.1. A **braided bimodule** is a pair $({}_AX_B, T_X^{V,W})$, where ${}_AX_B$ is an A-B-bimodule in $\mathcal{M}$ and

$$T_X^{V,W} : V \otimes_A X \to X \otimes_B W$$

is a morphism of A-B-bimodules such that the diagrams

(Pentagon axiom)

$$\begin{array}{ccccc}
V \otimes_A X & & \xrightarrow{\quad T_X^{V,W} \quad} & & X \otimes_B W \\
\downarrow{\scriptstyle \psi_V \otimes_A X} & & & & \downarrow{\scriptstyle X \otimes_B \psi_W} \\
V \otimes_A V \otimes_A X & & & & X \otimes_B W \otimes_B W \\
& \searrow{\scriptstyle V \otimes_A T_X^{V,W}} & & \nearrow{\scriptstyle T_X^{V,W} \otimes_B W} & \\
& & V \otimes_A X \otimes_B W & &
\end{array}$$

(Counit axiom)

$$\begin{array}{ccccc} V\otimes_A X & & \xrightarrow{\quad T_X^{V,W}\quad} & & X\otimes_B W \\ {\scriptstyle \varepsilon_V\otimes_A X}\downarrow & & & & \downarrow{\scriptstyle X\otimes_B \varepsilon_W} \\ A\otimes_A X & \xrightarrow{\cong} & X & \xleftarrow{\cong} & X\otimes_B B \end{array}$$

commute. The map $T_X^{V,W}$ will sometimes be referred to as a **V-W-braiding morphism**.

Braided bimodules classify relative adjunctions, in the sense of the next Proposition, which follows form Proposition 2.5 in [BH12].

Proposition 1.5.2 ([BH12])**.** *Using the notation above, if the monoid A admits a V-comodule structure, e.g., if V is coaugmented, then, up to isomorphism, every adjunction $(\tilde{F},\tilde{G})$ between $\mathcal{M}_A^V$ and $\mathcal{M}_B^W$, relative to a given adjunction (F,G) between $\mathbf{Mod}_A$ and $\mathbf{Mod}_B$, is given by a braided bimodule ${}_AX_B$ in $\mathcal{M}$.*

The following Proposition holds for $\mathcal{M} := \mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Proposition 1.5.3. *Consider the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, endowed with the projective model structure. Let (A,V) be a coring in ${}_A\mathbf{Mod}_A$ and (B,W) be a coring in ${}_B\mathbf{Mod}_B$, such that*

(i) *there exist model structures on the categories $\mathbf{Mod}_A$ and $\mathbf{Mod}_B$, right-induced from $\mathcal{M}$, as in Theorem 1.3.5;*

(ii) *there exist model structures on the categories $\mathcal{M}_A^V$ and $\mathcal{M}_B^W$, left-induced from $\mathbf{Mod}_A$, respectively, $\mathbf{Mod}_B$, as in Theorem 1.3.11.*

Let $({}_AX_B, T_X^{V,W})$ be a braided bimodule in $\mathcal{M}$. If X is cofibrant as right B-module, then the adjunction

$$\mathcal{M}_A^V \underset{\widetilde{R_X}}{\overset{\widetilde{-\otimes_A X}}{\rightleftarrows}} \mathcal{M}_B^W,$$

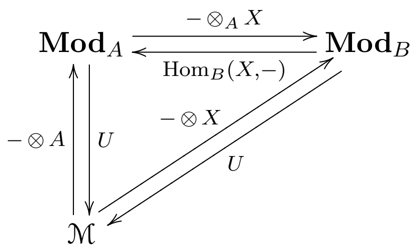
is a Quillen pair.

Proof. In the diagram

$$\begin{array}{ccc} \mathcal{M}_A^V & \underset{\widetilde{R_X}}{\overset{\widetilde{-\otimes_A X}}{\rightleftarrows}} & \mathcal{M}_B^W \\ {\scriptstyle U_A}\downarrow\uparrow{\scriptstyle -\otimes_A V} & & {\scriptstyle U_B}\downarrow\uparrow{\scriptstyle -\otimes_B W} \\ \mathbf{Mod}_A & \underset{\mathrm{Hom}_B(X,-)}{\overset{-\otimes_A X}{\rightleftarrows}} & \mathbf{Mod}_B \end{array}$$

the left adjoints are displayed on top and on the left and they commute. Moreover, the model category structures on $\mathcal{M}_A^V$ and $\mathcal{M}_B^W$ are left induced via U_A and U_B, respectively. So, $\widetilde{- \otimes_A X} : \mathcal{M}_A^V \to \mathcal{M}_B^W$ is a left Quillen functor if $- \otimes_A X : \mathbf{Mod}_A \to \mathbf{Mod}_B$ is.

Now, observe that the diagram of adjunctions



commutes, where left adjoint are displayed on top and on the left. Since the model structure on $\mathbf{Mod}_A$ is right induced from $\mathcal{M}$ via the vertical adjunction, the horizontal adjunction is a Quillen adjunction if and only if the diagonal one is. By definition of a left Quillen functor, this happens if and only if for every (acyclic) cofibration $i : K \overset{\simeq}{\rightarrowtail} L$ in $\mathcal{M}$, the induced map $i \otimes X : K \otimes X \overset{\simeq}{\rightarrowtail} L \otimes X$ is a (acyclic) cofibration in $\mathbf{Mod}_B$. By assumption, X is cofibrant as right B-module, so this condition is satisfied. Indeed, the category $\mathbf{Mod}_B$ is model monoidal, because $\mathcal{M}$ is (see Theorem 1.3.5). Hence, the push-out product axiom holds in $\mathbf{Mod}_B$, which implies that $i \otimes X$ is an acyclic cofibration. □

1.5.2 Quillen equivalences induced by quasi-isomorphisms of corings

In this section we work in the underlying category $\mathcal{M} := \mathbf{Ch}_{\Bbbk}^{\geq 0}$, where $\Bbbk$ is a field. We would like to understand the relationship between the situation where a morphism of corings is a quasi-isomorphism and the situation where the functor induced by this morphism on the categories of comodules over corresponding corings is a Quillen equivalence.

Knowing more about this relationship will allow us to investigate questions about the behavior of the Galois functor $(\beta_\varphi)_*$, which plays an essential role in the definition of a homotopic Hopf-Galois extension φ, at least in a particular case where the underlying Galois map β_φ happens to be a quasi-isomorphism.

Remark 1.5.4. Throughout this section we suppose that $\Bbbk$ is a field, A is an augmented dg $\Bbbk$-algebra and $g : V \to W$ is a morphism of A-corings in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Moreover, we assume that V and W satisfy the semi-freeness hypothesis of Theorem 1.3.11, so that the categories $\mathcal{M}_A^V$ and $\mathcal{M}_A^W$ inherit the left-induced model structure from $\mathbf{Mod}_A$, as described in Theorem 1.3.11.

Remark 1.5.5. Let $g : V \to W$ be a morphism of A-corings as in Remark 1.5.4. The induced functor $g_* : \mathcal{M}_A^V \to \mathcal{M}_A^W$ preserves and reflects all weak equivalences, hence it automatically satisfies condition (2′) in Proposition 1.3.3.

The next Proposition comes from [BH12].

Proposition 1.5.6. *Suppose that $g : V \to W$ is a quasi-isomorphism of A-corings (i.e., a quasi-isomorphism of underlying chain complexes), where V and W are semi-free as left A-modules on generating graded $\Bbbk$-modules that are degree-wise finitely generated. Then the adjunction*

$$\mathcal{M}_A^V \underset{-\underset{W}{\square} g_*(V)}{\overset{g_*}{\rightleftarrows}} \mathcal{M}_A^W$$

is a Quillen equivalence, whenever the categories $\mathcal{M}_A^V$ and $\mathcal{M}_A^W$ are equipped with the model structure left-induced from $\mathbf{Mod}_A$, as in Theorem 1.3.11.

How much can be said about the converse? In other words, if the induced functor $g_* : \mathcal{M}_A^V \to \mathcal{M}_A^W$ is a Quillen equivalence, can one find reasonable conditions that would guarantee that g was a quasi-isomorphism of A-corings? It turns out that conditions of Remark 1.5.4 are sufficient to guarantee this.

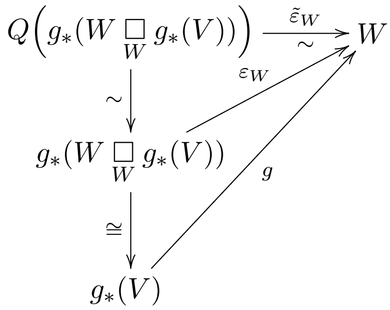
Proposition 1.5.7. *Let $\Bbbk$, A, V and W be as in Remark 1.5.4 and let $g : V \to W$ be a morphism of A-corings. Suppose that the functor $g_* : \mathcal{M}_A^V \to \mathcal{M}_A^W$ is a Quillen equivalence. Then $g : V \to W$ is a quasi-isomorphism of A-corings.*

Proof. Since g_* is a Quillen equivalence, criterion (1′) from Proposition 1.3.3 implies that for all fibrant $N^f \in \mathcal{M}_A^W$ the homotopy counit

$$\tilde{\varepsilon}_{N^f} : Q\Big(g_*\big(N^f \underset{W}{\square} g_*(V)\big)\Big) \to N^f$$

is a weak equivalence in $\mathcal{M}_W^A$, i.e., a quasi-isomorphism of underlying chain complexes, where $Q(-)$ stands for the cofibrant replacement functor.

Note that all objects are fibrant in the right-induced model structure on $\mathbf{Mod}_A$. In particular, the dg-algebra A is fibrant as a right A-module. Now, the functor $- \otimes_A W : \mathbf{Mod}_A \to \mathcal{M}_A^W$ is the right member of a Quillen pair, by Theorem 1.3.11, so it preserves fibrant objects. Therefore, the object $A \otimes_A W \cong W$ is fibrant in $\mathcal{M}_A^W$, and the associated homotopy counit $\tilde{\varepsilon}_W$ is a quasi-isomorphism. We now look at the commutative diagram



and conclude that g is a quasi-isomorphism, as desired. □

Propositions 1.5.6 and 1.5.7 combine together to give the following corollary.

Corollary 1.5.8. *Let $g : V \to W$ be a morphism of A-corings and let $g_* : M_A^V \to M_A^W$ denote the induced functor. Suppose that V and W are semi-free as left A-modules on generating graded $\Bbbk$-modules that are degree-wise finitely generated. Then $g : V \to W$ is a quasi-isomorphism of A-corings if and only if the functor $g_* : \mathcal{M}_A^V \to \mathcal{M}_A^W$ is a Quillen equivalence.*

Chapter 2

Foundations of homotopic Hopf-Galois extensions

2.1 (Homotopy) C-coinvariants

Convention 2.1.1. In this section, we will work over a field $\Bbbk$.

To understand the definition of a homotopic H-Hopf-Galois extension $\varphi : B \to A$ of H-comodule algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, for a given bimonoid H, it is important to know how to calculate the object of homotopy coinvariants A^{hcoH} of A.

Actually, in the most general case, (homotopy) coinvariants of a coaction are defined for coactions by a coaugmented comonoid C. Therefore, this section starts with the definition of the (non-homotopy) C-coinvariants of C-comodules, for a coaugmented comonoid C, and then considers their homotopic analog.

2.1.1 Calculating C-coinvariants

Definition 2.1.2. Let $(C, \Delta_C, \varepsilon_C, \eta_C)$ be a coaugmented coalgebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Given a left C-comodule (X, ρ) in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, the **object of C-coinvariants of X** is defined to be the cotensor product $X^{coC} := X \underset{C}{\square} \Bbbk[0]$, calculated in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (see Definition 1.1.15).

The previous definition is functorial, so that one can define the **coinvariants functor**

$$\mathrm{Coinv} : \mathbf{Comod}_C \to \mathbf{Ch}_{\Bbbk}^{\geq 0} : (X, \rho) \mapsto X^{coC}$$

and its left adjoint, the **trivial coaction functor**

$$\mathrm{Triv} : \mathbf{Ch}_{\Bbbk}^{\geq 0} \to \mathbf{Comod}_C : X \mapsto \mathrm{Triv}(X) = (X, X \otimes \eta_C),$$

where $X \otimes \eta_C$ is the composite $X \cong X \otimes \Bbbk[0] \xrightarrow{X \otimes \eta_C} X \otimes C$.

However, things become slightly more complicated when it comes to calculating *homotopy* coinvariants of C-comodules. To be able to do this, it is important to have valid models for fibrant replacements in $\mathbf{Comod}_C$.

2.1.2 Homotopy C-coinvariants in $\mathbf{Comod}_C$

Convention 2.1.3. In this section, we suppose that C is a coaugmented, 1-connected comonoid in the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, i.e., that $C \in \mathbf{Coalg}_{\Bbbk}^{\eta,1}$, and that C is also degree-wise finitely generated.

In this situation, Theorem 1.3.12 guarantees that there exists a left-induced model category structure on $\mathbf{Comod}_C$, coming from the adjunction

$$\mathbf{Comod}_C \underset{-\otimes C}{\overset{U}{\rightleftarrows}} \mathbf{Ch}_{\Bbbk}^{\geq 0},$$

such that the weak equivalences and the cofibrations in $\mathbf{Comod}_C$ are the same as in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, equipped with the projective model structure. In this situation, it is easy to see that the adjunction

$$\mathbf{Ch}_{\Bbbk}^{\geq 0} \underset{\mathrm{Coinv}}{\overset{\mathrm{Triv}}{\rightleftarrows}} \mathbf{Comod}_C$$

is a Quillen pair. It turns out that a good model for fibrant replacements in $\mathbf{Comod}_C$ is given by two-sided cobar constructions.

Definition 2.1.4. Let $\Bbbk$ be a field and C a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.3. Let $(X, \rho_C) \in \mathbf{Comod}_C$. The **two-sided cobar construction on** X is an object in $\mathbf{Comod}_C$, defined by the following twisted tensor product of chain complexes

$$\Omega(X;C;C) := (X \otimes \Omega C \otimes C, D_{t_\Omega \otimes t_\Omega})$$

(see Definitions 1.2.2, 1.2.3 and 1.2.5 and Example 1.2.4). The differential $D_{t_\Omega \otimes t_\Omega}$ is given by

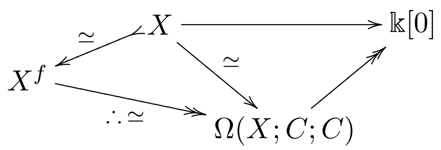
$$\begin{aligned} D_{t_\Omega \otimes t_\Omega} &= d_X \otimes \Omega C \otimes C + X \otimes D_{\Omega C} \otimes C + X \otimes \Omega C \otimes d_C \\ &+ \big((X \otimes \mu_{\Omega C} \otimes C) \circ (X \otimes t_\Omega \otimes \Omega C \otimes C) \circ (\rho_C \otimes \Omega C \otimes C)\big) \\ &- \big((X \otimes \mu_{\Omega C} \otimes C) \circ (X \otimes \Omega C \otimes t_\Omega \otimes C) \circ (X \otimes \Omega C \otimes \Delta_C)\big) \end{aligned}$$

(see also Definition 7.6 in [HS12]). The right C-coaction on $\Omega(X;C;C)$ is given on the underlying graded module by

$$\rho_{\Omega(X;C;C)} : X \otimes \Omega C \otimes C \xrightarrow{X \otimes \Omega C \otimes \Delta_C} X \otimes \Omega C \otimes (C \otimes C) \cong (X \otimes \Omega C \otimes C) \otimes C.$$

Notation 2.1.5. We will often omit the differential $D_{t_\Omega \otimes t_\Omega}$, but keep specifying the twisting morphisms. So, the two-sided cobar construction on X will be denoted either $\Omega(X; C; C)$ or $X \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} C$.

Remark 2.1.6. The homotopy equivalence $X \xrightarrow{\simeq} \Omega(X; C; C)$ is not a cofibration in the model structure in $\mathbf{Comod}_C$ we are working with. However, it suffices for computing right derived functors. In the commutative diagram



X^f denotes a fibrant replacement of a C-comodule X. It is obtained by factoring the map $X \to \Omega(X; C; C)$ as a trivial cofibration, followed by a fibration that is necessarily trivial. Since the map $X^f \to \Omega(X; C; C)$ is a weak equivalence between fibrant objects, for any right Quillen functor $G : \mathbf{Comod}_C \to \mathcal{C}$, the map $G(X^f) \to G(\Omega(X; C; C))$ is a weak equivalence in $\mathcal{C}$.

Lemma 2.1.7. *Let $\Bbbk$ be a field and C a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.3. For any $(X, \rho) \in \mathbf{Comod}_C$, the two-sided cobar construction $\Omega(X; C; C)$ is a fibrant replacement of (X, ρ) in the left-induced model structure on $\mathbf{Comod}_C$, as described in Theorem 1.3.12.*

Proof. This Lemma follows from §7 in [HS12] and from [BHKKRS14]. □

Using Definition 2.1.4 and Notation 2.1.5, for any $(X, \rho) \in \mathbf{Comod}_C$ we have

$$\begin{aligned} \Omega(X; C; C) \mathop{\square}_{C} \Bbbk[0] &:= (X \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} C) \mathop{\square}_{C} \Bbbk[0] \\ &\overset{(1)}{\cong} X \otimes_{t_\Omega} \Omega C \otimes_{t_\Omega} \big(C \mathop{\square}_{C} \Bbbk[0]\big) \\ &\overset{(2)}{\cong} X \otimes_{t_\Omega} \Omega C \mathop{\otimes}_{t_\Omega \circ \eta_C} \Bbbk[0] \\ &\overset{(3)}{\cong} X \otimes_{t_\Omega} \Omega C, \end{aligned}$$

where (1) uses that C is degree-wise $\Bbbk$-flat, (2) follows from Chapter 1 in [HMS74], and (3) holds because $\Omega C \otimes \Bbbk[0] \cong \Omega C$ as chain complexes and $t_\Omega \circ \eta_C = 0$.

Remark 2.1.8. Note that the chain complex $X \otimes_{t_\Omega} \Omega C$ is actually the **one-sided cobar construction** $(X \otimes \Omega C, D_{t_\Omega})$ on (X, ρ_C). The differential D_{t_Ω} is given by

$$D_{t_\Omega} = d_X \otimes \Omega C + X \otimes D_{\Omega C} + \big((X \otimes \mu_{\Omega C}) \circ (X \otimes t_\Omega \otimes \Omega C) \circ (\rho_C \otimes \Omega C)\big).$$

Notation 2.1.9. We will also use the notation $X \otimes_{t_\Omega} \Omega C := \Omega(X; C; \Bbbk)$.

Remark 2.1.10. Given $(X, \rho) \in {}_C\mathbf{Comod}$, a *left* C-comodule, one defines symmetrically the **two-sided cobar construction** $\Omega(C; C; X)$, and the **one-sided cobar construction** $\Omega(\Bbbk; C; X)$.

We are finally ready to define the homotopy coinvariants functor on the category of C-comodules.

Definition 2.1.11. Let $\Bbbk$ be a field and C a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.3. The **homotopy coinvariants functor** on $\mathbf{Comod}_C$ can be explicitly defined by

$$(-)^{hcoC} : \mathbf{Comod}_C \to \mathbf{Ch}_{\Bbbk}^{\geq 0}$$

$$(X, \rho) \mapsto X^{hcoC} = \Omega(X; C; \Bbbk)$$

for all $(X, \rho) \in \mathbf{Comod}_C$.

2.1.3 Homotopy H-coinvariants in $\mathbf{Alg}_H^{\varepsilon}$

Convention 2.1.12. From now on, we suppose that H is a 1-connected Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, such that H_n is finitely generated for all $n \geq 0$.

By Theorem 1.3.13, if H satisfies the conditions of Convention 2.1.12, then there exists a model structure on the category $\mathbf{Alg}_H^{\varepsilon}$, left-induced from the category of augmented monoids $\mathbf{Alg}^{\varepsilon}$ via the adjunction

$$\mathbf{Alg}_H^{\varepsilon} \underset{-\otimes H}{\overset{U}{\rightleftarrows}} \mathbf{Alg}^{\varepsilon},$$

such that the weak equivalences and the cofibrations in $\mathbf{Alg}_H^{\varepsilon}$ are the same as in $\mathbf{Alg}^{\varepsilon}$, equipped with the right-induced model structure from $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (see Theorem 1.3.6). In this situation, it is easy to see that the adjunction

$$\mathbf{Alg}^{\varepsilon} \underset{\mathbf{Coinv}}{\overset{\mathbf{Triv}}{\rightleftarrows}} \mathbf{Alg}_H^{\varepsilon}$$

is a Quillen pair. Using the free-forgetful adjunctions on both source and target categories, the previous adjunction fits into the following diagram of Quillen adjunctions (left adjoints are on top and on the left).

$$\begin{array}{ccc}
\mathbf{Alg}_H^{\varepsilon} & \underset{\mathbf{Coinv}}{\overset{\mathbf{Triv}}{\leftrightarrows}} & \mathbf{Alg}^{\varepsilon} \\
F_{\mathbf{Alg},H} \uparrow \downarrow U_{\mathbf{Alg},H} & & F_{\mathbf{Alg}} \uparrow \downarrow U_{\mathbf{Alg}} \\
\mathbf{Comod}_H & \underset{\mathrm{Coinv}}{\overset{\mathrm{Triv}}{\leftrightarrows}} & \mathbf{Ch}_{\Bbbk}^{\geq 0}
\end{array}$$

This diagram commutes, because all limits of algebras are created in the underlying category, and then equipped with a multiplicative structure. So, for any $(A,\rho) \in \mathbf{Alg}^{\varepsilon}_{H}$ we have

$$U_{\mathbf{Alg}}(\mathbf{Coinv}(A,\rho)) = U_{\mathbf{Alg}}(A^{coH}) = \mathrm{Coinv}(U_{\mathbf{Alg},H}(A,\rho)).$$

In particular, given an augmented H-comodule algebra (A,ρ), the associated object of H-coinvariants A^{coH}, can be calculated in the underlying category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ and then equipped with an algebra structure.

We now turn to a discussion on how to calculate the *homotopy* coinvariants of an augmented H-comodule algebra $(A,\rho) \in \mathbf{Alg}^{\varepsilon}_{H}$. If we forget the multiplication, (UA,ρ) is an H-comodule, and we have seen in the previous section that the one-sided cobar construction $\Omega(UA;H;\Bbbk)$ gives a good model for homotopy coinvariants of (UA,ρ) in $\mathbf{Comod}_H$.

On the other hand, Corollary 3.6 in [HL07], shows how the free right ΩH-module structure on $\Omega(A;H;\Bbbk)$ can be extended to a natural differential algebra structure. In other words, there exists a functor $\mathbf{Cobar} : \mathbf{Alg}^{\varepsilon}_{H} \to \mathbf{Alg}^{\varepsilon}$ that makes the following diagram commute.

$$\begin{array}{ccc} \mathbf{Alg}^{\varepsilon}_{H} & \xrightarrow{\mathbf{Cobar}} & \mathbf{Alg}^{\varepsilon} \\ \downarrow{\scriptstyle U_{\mathbf{Alg},H}} & & \downarrow{\scriptstyle U_{\mathbf{Alg}}} \\ \mathbf{Comod}_H & \xrightarrow{\mathrm{Cobar}} & \mathbf{Ch}_{\Bbbk}^{\geq 0} \end{array}$$

Remark 2.1.13. The idea used in [HL07] for defining the multiplication $\mu_{A\otimes_{t_\Omega}\Omega H}$ on $A\otimes_{t_\Omega}\Omega H$ is the following. Suppose that one wants to determine the product

$$(a_1 \otimes s^{-1}h_1|\cdots|s^{-1}h_m)(a'_1 \otimes s^{-1}h'_1|\cdots|s^{-1}h'_n)$$

of two arbitrary elements $a_1 \otimes s^{-1}h_1|\cdots|s^{-1}h_m,\ a'_1 \otimes s^{-1}h'_1|\cdots|s^{-1}h'_n \in A\otimes_{t_\Omega}\Omega H$. Using the fact that the multiplication on ΩH is free and associative, one can rewrite this product as a product of simpler terms of the form $a\otimes 1$ and $1\otimes s^{-1}h$, for some $a \in A$, $s^{-1}h \in \Omega H$. The key observation now is that it is sufficient to know how to multiply two terms $(1\otimes s^{-1}h)(a\otimes 1)$ to obtain the expression of the initial product.

Moreover, the partial product $(1\otimes s^{-1}h)(a\otimes 1)$ must be compatible with the differential $D_{t_\Omega} : A\otimes\Omega H \to A\otimes\Omega H$, if one wants it to induce a differential algebra structure on $A\otimes_{t_\Omega}\Omega H$.

A careful investigation of the Leibniz rule then shows that one must set

$$\begin{aligned} (1\otimes s^{-1}h)(a\otimes 1) \;:=\;& (-1)^{\deg(h+1)\deg(a)} a\otimes s^{-1}h \\ &+\; (-1)^{\deg(h')} 1\otimes s^{-1}(h\cdot h') \\ &+\; (-1)^{\deg(a)+\deg(h)\deg(h'^i)} \sum_i a_i \otimes s^{-1}(h\cdot h'^i), \end{aligned}$$

for all $a \in A$, $h \in H$, where $\rho_A(a) = a \otimes 1 + 1 \otimes h' + \sum_i a_i \otimes h'^i$, and also require that

$$(a \otimes 1)(a' \otimes 1) = a \cdot a' \otimes 1,$$

for all $a, a' \in A$ and that

$$(a \otimes 1)(1 \otimes s^{-1}h) = a \otimes s^{-1}h,$$

for all $a \in A$, $h \in H$.

An analogous construction holds for any left H-comodule A, and defines a differential algebra structure $\mu_{\Omega H \otimes_{t_\Omega} A}$ on $\Omega H \otimes_{t_\Omega} A$.

It is also possible to endow the two-sided cobar construction $\Omega(A; H; H)$ with a multiplication $\mu_{\Omega(A;H;H)}$, making it into a differential H-comodule algebra. Observe that

$$\Omega(A; H; H) = A \otimes_{t_\Omega} \Omega H \otimes_{t_\Omega} H \cong (A \otimes_{t_\Omega} \Omega H) \underset{\Omega H}{\otimes} (\Omega H \otimes_{t_\Omega} H),$$

and that Remark 2.1.13 tells us how to define multiplications $\mu_{A \otimes_{t_\Omega} \Omega H}$ and $\mu_{\Omega H \otimes_{t_\Omega} A}$.

Given two elements $a \otimes s^{-1}h_1 \otimes h'$, $a' \otimes s^{-1}h_2 \otimes h''$ in $\Omega(A; H; H)$, their product can be written as

$$(a \otimes s^{-1}h_1 \otimes h')(a' \otimes s^{-1}h_2 \otimes h'') = \\ (a \otimes 1 \otimes 1)(1 \otimes s^{-1}h_1 \otimes h')(a \otimes s^{-1}h_2 \otimes 1)(1 \otimes 1 \otimes h''),$$

using the fact that the multiplication on ΩH is free and the conditions required for $\mu_{A \otimes_{t_\Omega} \Omega H}$ and $\mu_{\Omega H \otimes_{t_\Omega} A}$. Applying this decomposition and taking into account all conditions on multiplications required by Remark 2.1.13, one defines a partial multiplication on $\Omega(A; H; H)$ by setting

$$(1 \otimes s^{-1}h_1 \otimes h')(a \otimes s^{-1}h_2 \otimes 1) = \\ \big((1 \otimes s^{-1}h_1)(a \otimes 1) \otimes 1\big)\big(1 \otimes (1 \otimes h')(s^{-1}h_2 \otimes 1)\big),$$

$$(a \otimes 1 \otimes 1)(a' \otimes s^{-1}h_1 \otimes h') = aa' \otimes s^{-1}h_1 \otimes h',$$

$$(a \otimes s^{-1}h_1 \otimes h')(1 \otimes 1 \otimes h'') = a \otimes s^{-1}h_1 \otimes h'h'',$$

for all $a, a' \in A$, $h, h', h_1, h_2 \in H$. This partial product satisfies the Leibniz rule, because partial multiplications on $A \otimes_{t_\Omega} \Omega H$ and $\Omega H \otimes_{t_\Omega} A$ do, and therefore generates the multiplication $\mu_{\Omega(A;H;H)}$ on $\Omega(A; H; H)$.

Lemma 2.1.14. *Let $\Bbbk$ be a field and H a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. For any $(A, \rho) \in \mathbf{Alg}_H^{\varepsilon}$, the two-sided cobar construction $\Omega(A; H; H)$ is a fibrant replacement of (A, ρ) in the left-induced model structure on $\mathbf{Alg}_H^{\varepsilon}$, as described in Theorem 1.3.13.*

Proof. This follows from Example 7.9 in [HS12] and from [BHKKRS14]. $\square$

We can finally give a definition of the homotopy coinvariants functor on the category of augmented H-comodule algebras.

Definition 2.1.15. Let $\Bbbk$ be a field, and H a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. The **homotopy coinvariants functor** on $\mathbf{Alg}_H^{\varepsilon}$ can be explicitly defined by

$$(-)^{hcoH} : \mathbf{Alg}_H^{\varepsilon} \to \mathbf{Alg}^{\varepsilon}$$
$$(A, \rho) \mapsto A^{hcoH} = \Omega(A; H; \Bbbk),$$

for all $(A, \rho) \in \mathbf{Alg}_H^{\varepsilon}$.

2.2 Special maps associated to a morphism of augmented H-comodule algebras φ

Let $\Bbbk$ be a field and let H be a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. Let (A, ρ) be an augmented H-comodule algebra.

2.2.1 The comparison map i_φ

Let $\varphi : \mathrm{Triv}(B) \to A$ be a morphism of augmented H-comodule algebras. Observe that, by definition, φ induces the map ξ in the equalizer

$$\begin{array}{ccccc} A^{coH} & \longrightarrow & A & \overset{\rho\otimes\Bbbk}{\underset{A\otimes\eta_H}{\rightrightarrows}} & A\otimes H\otimes\Bbbk, \\ {\scriptstyle \exists!}\uparrow{\scriptstyle \xi} & \nearrow{\scriptstyle \varphi} & & & \\ \mathrm{Triv}(B) & & & & \end{array}$$

computed in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. The **comparison map** associated to φ is the morphism of augmented H-comodule algebras

$$i_\varphi : \mathrm{Triv}(B) \to A^{hcoH}$$

that fits into the following commutative diagram in $\mathbf{Alg}_H^{\varepsilon}$.

$$\begin{array}{ccc} \mathrm{Triv}(B) & \xrightarrow{\varphi} & A \\ \downarrow{\scriptstyle \xi} & & \| \\ A^{coH} & \hookrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ (RA)^{coH} = A^{hcoH} & \hookrightarrow & RA \end{array} \qquad (i_\varphi : \mathrm{Triv}(B) \to (RA)^{coH})$$

Here, $R(-)$ stands for the fibrant replacement functor in the model category $\mathbf{Alg}^{\varepsilon}_{H}$. Note that the two-sided cobar constructions $\Omega(A; H; \Bbbk)$ and $\Omega(A; H; H)$ can be chosen as explicit models for A^{hcoH} and RA, respectively, as explained in Definition 2.1.15 and Lemma 2.1.14.

2.2.2 The Galois map β_φ

Recall from Examples 1.1.18 the definition of A-corings $(W^{can}_{\varphi}, l^{can}_{\varphi}, r^{can}_{\varphi}, \psi^{can}_{\varphi}, \varepsilon^{can}_{\varphi})$ and $(W_\rho, l_\rho, r_\rho, \psi_\rho, \varepsilon_\rho)$.

The **Galois map** $\beta_\varphi : W^{can}_{\varphi} \to W_\rho$ is a morphism of A-corings, given explicitly by the composite

$$A \otimes_B A \xrightarrow{A \otimes_B \rho} A \otimes_B A \otimes H \xrightarrow{\overline{\mu_A} \otimes H} A \otimes H, \qquad \beta_\varphi : A \otimes_B A \to A \otimes H$$

where $\overline{\mu_A}$ denotes the map induced in the coequalizer by the multiplication μ_A on A.

Remark 2.2.1. To check that β_φ is indeed a morphism of A-corings, one needs to show that it respects all the structure maps, i.e., that the following four diagrams commute.

$$\begin{array}{ccc} W^{can}_{\varphi} & \xrightarrow{\beta_\varphi} & W_\rho \\ \downarrow{\scriptstyle \psi_\varphi} & & \downarrow{\scriptstyle \psi_\rho} \\ W^{can}_{\varphi} \otimes_A W^{can}_{\varphi} & \xrightarrow{\beta_\varphi \otimes_A \beta_\varphi} & W_\rho \otimes_A W_\rho \end{array} \qquad \begin{array}{ccc} W^{can}_{\varphi} & \xrightarrow{\beta_\varphi} & W_\rho \\ & {\scriptstyle \varepsilon_\varphi} \searrow \quad \swarrow {\scriptstyle \varepsilon_\rho} & \\ & A & \end{array}$$

$$\begin{array}{ccc} A \otimes W^{can}_{\varphi} & \xrightarrow{A \otimes \beta_\varphi} & A \otimes W_\rho \\ \downarrow{\scriptstyle l_\varphi} & & \downarrow{\scriptstyle l_\rho} \\ W^{can}_{\varphi} & \xrightarrow{\beta_\varphi} & W_\rho \end{array} \qquad \begin{array}{ccc} W^{can}_{\varphi} \otimes A & \xrightarrow{\beta_\varphi \otimes A} & W_\rho \otimes A \\ \downarrow{\scriptstyle r_\varphi} & & \downarrow{\scriptstyle r_\rho} \\ W^{can}_{\varphi} & \xrightarrow{\beta_\varphi} & W_\rho \end{array}$$

While verifying compatibility of β_φ with the left and right A-actions is quite straightforward, establishing its compatibility with the coactions ψ_φ, ψ_ρ and counits ε_φ, ε_ρ turns out to be a rather tedious and long exercise, especially if one decides to check this property in an arbitrary monoidal model category $(\mathcal{M}, \otimes, \mathbb{I})$ (i.e., without the possibility of "taking elements").

One difficulty is that many of objects and maps involved here have tensor products *over* A *and over* B in their definition. So, breaking the above diagrams into smaller pieces to check their commutativity, one immediately has to deal with a lot of coequalizers and induced maps, and the number of intermediate diagrams increases quickly.

On the other hand, taking $\mathcal{M} := \mathbf{Ch}^{\geq 0}_{\Bbbk}$, "element arguments" will quickly allow one to see that the four diagrams above commute.

2.3 The definition of homotopic Hopf-Galois extensions

Before giving the definition of homotopic Hopf-Galois extensions, we need to introduce two functors, induced by the two special maps from above.

The **comparison functor**

$$(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$$

is the right adjoint of the extension/restriction of scalars adjunction

$$\mathbf{Mod}_B \underset{(i_\varphi)^*}{\overset{-\otimes_B A^{hcoH}}{\rightleftarrows}} \mathbf{Mod}_{A^{hcoH}},$$

induced by the morphism $i_\varphi : \mathrm{Triv}(B) \to A^{hcoH}$ in $\mathbf{Alg}_H^\varepsilon$.

The **Galois functor**

$$(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$$

is induced by the morphism of A-corings $\beta_\varphi : W_\varphi^{can} \to W_\rho$, and is given on objects by

$$(M, \theta_\varphi) \mapsto \Big(M, (M \otimes_A \beta_\varphi) \circ \theta_\varphi\Big),$$

for all $(M, \theta_\varphi) \in \mathcal{M}_A^{W_\varphi^{can}}$. Observe that $(\beta_\varphi)_*$ does not change the underlying A-module M, but only equips it with a new coaction.

Definition 2.3.1. Let $\Bbbk$ be a field, and let H be a Hopf algebra in $\mathbf{Ch}_\Bbbk^{\geq 0}$, satisfying Convention 2.1.12. A morphism $\varphi : \mathrm{Triv}(B) \to A$ of augmented H-comodule algebras in $\mathbf{Ch}_\Bbbk^{\geq 0}$ is called a **homotopic H-Hopf-Galois extension** if both the Galois functor

$$(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$$

and the comparison functor

$$(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$$

are Quillen equivalences with respect to the model structures given by Theorems 1.3.5 and 1.3.11.

Remark 2.3.2. To make the definition of a homotopic H-Hopf-Galois extension φ meaningful, it is important that the extension/restriction of scalars adjunction

$$\mathbf{Mod}_B \underset{\varphi^*}{\overset{-\otimes_B A}{\rightleftarrows}} \mathbf{Mod}_A,$$

induced by φ, be a Quillen pair. Lemma 1.4.2 actually tells us that this is the case whenever the categories of modules are equipped with the model structures right-induced from the projective model structure on $\mathbf{Ch}_\Bbbk^{\geq 0}$ (as in Theorem 1.3.5).

Example 2.3.3. Let $\Bbbk$ be a field, H a Hopf algebra satisfying Convention 2.1.12, and $A \in \mathbf{Alg}^{\varepsilon}_{H}$. The map of H-coalgebras

$$\iota_H : \Omega(A; H; \Bbbk) \to \Omega(A; H; H)$$

is a homotopic H-Hopf-Galois extension, called the **normal basis extension**. Here, $\Omega(A; H; \Bbbk)$ is equipped with a trivial H-comodule structure, and the H-coaction on $\Omega(A; H; H)$ is defined similarly to the coaction described in Definition 2.1.4. We explained in Section 2.1.3 how to equip the objects $\Omega(A; H; \Bbbk)$ and $\Omega(A; H; H)$ with algebra structures.

Now, observe that

$$\begin{aligned}
\Omega(A; H; H) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A; H; H) &= \big(A \underset{t_\Omega}{\otimes} \Omega H \underset{t_\Omega}{\otimes} H\big) \underset{\substack{A \otimes \Omega H \\ t_\Omega}}{\otimes} \big(A \underset{t_\Omega}{\otimes} \Omega H \underset{t_\Omega}{\otimes} H\big) \\
&\cong \big(A \underset{t_\Omega}{\otimes} \Omega H \underset{t_\Omega}{\otimes} H\big) \otimes H \\
&= \Omega(A; H; H) \otimes H,
\end{aligned}$$

so the Galois map

$$\beta_{\iota_H} : \Omega(A; H; H) \otimes_{\Omega(A;H;\Bbbk)} \Omega(A; H; H) \to \Omega(A; H; H) \otimes H$$

is an isomorphism in $\mathbf{Ch}^{\geq 0}_{\Bbbk}$. This implies that the Galois functor $(\beta_{\iota_H})_*$ is an equivalence of categories.

On the other hand, since $\Omega(A; H; H)$ is fibrant in $\mathbf{Alg}^{\varepsilon}_{H}$ (see Lemma 2.1.14), we can write

$$\begin{aligned}
\Omega(A; H; H)^{hcoH} &= \Omega(A; H; H)^{coH} \\
&:= \Omega(A; H; H) \underset{H}{\square} \Bbbk[0] \\
&\cong \Omega(A; H; \Bbbk),
\end{aligned}$$

which shows that the comparison map

$$i_{\iota_H} : \Omega(A; H; \Bbbk) \to \Omega(A; H; H)^{hcoH}$$

is an isomorphism in $\mathbf{Ch}^{\geq 0}_{\Bbbk}$. This implies that the comparison functor $(i_{\iota_H})_*$ is an equivalence of categories, as well.

2.4 Connections to other works

In the previous section, the definition of homotopic Hopf-Galois extensions, as it will be used in this thesis, was given. It is a good moment to make connections between this definition and the concepts of (non-homotopic) (Hopf)-Galois extensions, which have been widely studied in other contexts and which are at the origin of Definition 2.3.1.

2.4.1 Brief reminder of Galois extensions of fields

Our reference book for this subsection is [Cox04]. Recall that an extension of fields $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ is an inclusion $\Bbbk \subseteq \mathbb{E}$, where $\Bbbk$ is a sub-vector space of $\mathbb{E}$. We will assume that all our field extensions are *finite*. This will guarantee, in particular, that they are algebraic. i.e., that for every $e \in \mathbb{E}$, there exists a non-zero polynomial $p \in \Bbbk[X]$, such that $p(e) = 0$.

Definitions 2.4.1. Let $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ be a finite field extension.

- The extension α is called **normal** if the minimal polynomial $\min(e; \Bbbk) \in \Bbbk[X]$ of every element $e \in \mathbb{E}$ splits completely over the field $\mathbb{E}$.
- Suppose that the extension α is normal. Then it is called **separable** if for all $e \in \mathbb{E}$, its minimal polynomial $\min(e; \Bbbk) \in \Bbbk[X]$ is non-constant and all its roots are simple in $\mathbb{E}$.
- The extension α is called a **Galois extension** if it is normal and separable.

Given a finite field extension $\alpha : \Bbbk \hookrightarrow \mathbb{E}$, one can associate to it the corresponding **Galois group** defined by

$$\mathrm{Gal}(\mathbb{E}; \Bbbk) = \big\{ f : \mathbb{E} \to \mathbb{E} : f \text{ is an automorphism and } f(k) = k, \text{ for all } k \in \Bbbk \big\}.$$

This is a finite group, where the group law is given by composition, and the neutral element is $\mathrm{Id}_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$.

2.4.2 Galois extensions of commutative rings

Auslander and Goldman were the first to define the notion of Galois extensions of commutative rings in [AG60]. We will need to introduce some terminology and notation before we can state their original definition, formulated in terms of two particular maps associated to such an extension.

An inclusion $\alpha : R \hookrightarrow S$ of commutative rings is an *extension* if R is a subring of S. Observe that α endows S with the structure of an R-algebra. Denote by $\mathrm{Aut}_R(S)$ the group of automorphisms of S that fix R. For any finite subgroup $G \leq \mathrm{Aut}_R(S)$,

$$S^G := \{ s \in S : gs = s, \text{ for all } g \in G \}$$

is the ring of G-fixed points in S. Note that $i : R \hookrightarrow S^G$ is an inclusion of rings.

Define the **twisted group ring** by setting

$$S \langle G \rangle := \{ \sum_{g \in G} s_g \cdot g : s_g \in S, \text{ for all } g \in G \},$$

with at most finitely many $s_g \neq 0$. It is an S-algebra, where the addition and the scalar multiplication are given component-wise, and where the multiplication is "twisted" and defined by

$$\Big(\sum_{g\in G} s_g \cdot g\Big)\Big(\sum_{g'\in G} s'_{g'} \cdot g'\Big) = \sum_{g,g'\in G} (s_g \cdot g(s'_{g'})) \cdot gg',$$

for all $g, g' \in G$, $s_g, s'_{g'} \in S$.

On the other hand, consider the endomorphism ring $\mathrm{End}_R(S)$ with its usual S-algebra structure, defined component-wise. The map

$$\delta : S\langle G\rangle \to \mathrm{End}_R(S) : s_g \cdot g \mapsto (\delta_{s_g\cdot g} : S \to S),$$

where

$$\delta_{s_g\cdot g}(s') := s_g \cdot g(s')$$

is a homomorphism of S-algebras, for all $g \in G$, $s_g, s' \in S$.

Definition 2.4.2 ([AG60])**.** An extension of commutative rings $\alpha : R \hookrightarrow S$ is G**-Galois** for a finite subgroup $G \leq \mathrm{Aut}_R(S)$ if S is a finitely generated projective R-module and the maps

$$i : R \hookrightarrow S^G \qquad \text{and} \qquad \delta : S\langle G\rangle \to \mathrm{End}_R(S)$$

are isomorphisms of rings.

Let us define another map, β, as follows. Consider the tensor product $S \otimes_R S$, endowed with an S-algebra structure by multiplication on the left. Let $\prod_G S := \{(s_g)_{g\in G}\}$ be the set of all G-indexed sequences of elements in S. It is also an S-algebra, where the addition, multiplication and scalar multiplication by S are all defined component-wise. The map

$$\beta : S \otimes_R S \to \prod_G S : s \otimes s' \mapsto (s \cdot g(s'))_{g\in G},$$

is then a homomorphism of S-algebras, for all $s, s' \in S$.

Remark 2.4.3. Observe that the map $\beta : S \otimes_R S \to \prod_G S$ is an isomorphism of rings if and only if $\delta : S\langle G\rangle \to \mathrm{End}_R(S)$ is an isomorphism of rings and S is finitely generated and projective as an R-module. Indeed, using that G is a finite group, and that

$$S \otimes_R - : \mathbf{Mod}_S \rightleftarrows \mathbf{Mod}_S : \mathrm{Hom}_R(-, S)$$

is an adjunction, one checks that

$$\begin{array}{ccc} \mathrm{Hom}_S\Big(\prod_G S, S\Big) & \xrightarrow{\mathrm{Hom}_S(\beta,S)} & \mathrm{Hom}_S(S \otimes_R S, S) \\ \cong\downarrow & & \downarrow\cong \\ S\langle G\rangle & \xrightarrow{\delta} & \mathrm{Hom}_S(S, \mathrm{Hom}_R(S, S)) \cong \mathrm{End}_R(S), \end{array}$$

where $\mathrm{Hom}_S(A, S)$ denotes the set of all homomorphisms of S-modules from any S-algebra A into S.

Definition 2.4.4 (follows from Definition 2.4.2 and Remark 2.4.3)**.** An extension of commutative rings $\alpha : R \hookrightarrow S$ is **G-Galois** for a finite subgroup $G \leq \mathrm{Aut}_R(S)$ if the maps

$$i : R \hookrightarrow S^G \qquad \text{and} \qquad \beta : S \otimes_R S \to \prod_G S$$

are isomorphisms of rings.

Remark 2.4.5. Definition 2.4.4 was proposed by Chase, Harrison and Rosenberg in [CHR65] as one of the six equivalent ways of defining a Galois extension of commutative rings (see [CHR65], Definition 1.4). In this paper, the theory of Galois extensions of commutative rings was developed further and a version of Galois correspondence was proven.

In the case where S and R are fields, Definition 2.4.2 coincides with the original definition of a finite G-Galois extension of fields $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ (see [AG60], p.396 and Remark 1.5 in [CHR65]). Thus, it gives a way of characterizing α in terms of the isomorphisms of rings $i : \Bbbk \hookrightarrow \mathbb{E}^G$ and $\beta : \mathbb{E} \otimes_{\Bbbk} \mathbb{E} \to \prod_G \mathbb{E}$.

Proposition 2.4.6 (Proposition 1.2, [Gre92])**.** *For any finite Galois extension of fields $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ with Galois group G, the maps*

$$i : \Bbbk \hookrightarrow \mathbb{E}^G \qquad \textit{and} \qquad \delta : \mathbb{E}\langle G \rangle \to \mathrm{End}_{\Bbbk}(\mathbb{E})$$

are isomorphisms of rings.

2.4.3 Hopf-Galois extensions of algebras

The theory of classical (i.e., non-homotopic) Hopf-Galois extensions was developed by Chase and Sweedler in [CS69] (who considered coactions of Hopf algebras on *commutative* $\Bbbk$-algebras, for a fixed commutative ring R), and by Kreimer and Takeuchi in [KT81] (who considered coactions of *finite-dimensional* Hopf-algebras on $\Bbbk$-algebras, for a fixed commutative ring R). It offers a generalization of the Galois theory of fields and commutative rings, by studying coactions of a Hopf (or bi-) algebra H on an algebra over a commutative ring R.

Notation 2.4.7. Throughout this thesis we will sometimes use the notation $A^{\circlearrowleft H}$ to mean that A is an H-comodule algebra with a non-trivial H-coaction.

The definition of a classical Hopf-Galois extension requires the following data:

- R, a commutative ring,
- H, an R-bialgebra,
- B, an augmented R-algebra, endowed with trivial H-coaction,
- A, an H-comodule algebra with coaction $\rho : A \to A \otimes H$,
- $\varphi : B \to A^{\circlearrowleft H}$ a morphism of H-comodule algebras.

Moreover, two associated homomorphisms are important, namely, the **Galois map**

$$\beta_\varphi : A \otimes_B A \xrightarrow{A \otimes_B \rho} A \otimes_B A \otimes H \xrightarrow{\overline{\mu_A} \otimes H} A \otimes H,$$

defined exactly as in Section 2.2.2, and the **comparison map**

$$i_\varphi : B \to A^{coH} := \{a \in A : \rho(a) = a \otimes 1\}.$$

Definition 2.4.8. Using the notation above, the H-comodule algebra morphism $\varphi : B \to A$ is an **H-Hopf-Galois extension** if both β_φ and i_φ are isomorphisms .

Remark 2.4.9. Observe that the Definition 2.3.1 of a homotopic Hopf-Galois extension is "homotopified" (homotopy coinvariants, rather than simple coinvariants) and also "categorified" (Quillen equivalences, rather than isomorphisms) in comparison to Definition 2.4.8 of a classical Hopf-Galois extension.

Example 2.4.10 (Example 2.3, [Mon09]). A finite G-Galois extension of fields $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ is an H-Hopf-Galois extension for the Hopf algebra $H := \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$ (see Example 1.1.27), where $\Bbbk[G]$ denotes the usual group ring on G.

Example 2.4.11 (Example 2.11, [Mon09]). Here is another motivating example, which explains how a Hopf-Galois extension arises from a free group action on a set.

Let G be a finite group acting on a finite set X via $r : X \times G \to X : (x, g) \mapsto x \cdot g$. Let $Y := X_G$ be the set of G-orbits of X and denote the quotient map by $q : X \to Y$. Given a commutative ring R, consider the dual objets

- $H := \mathrm{Hom}_R(R[G], R)$, which is a Hopf algebra, by Example 1.1.27;
- $A := \mathbf{Set}(X, R)$, the R-algebra of functions from X to R;
- $B := \mathbf{Set}(Y, R)$, the R-algebra of functions from Y to R (i.e., the functions from X to R that have constant value on G-orbits).

Both A and B are equipped with point-wise addition and multiplication. The right G-action on X induces a left G-action on A given by

$$r_* : G \times \mathbf{Set}(X,R) \to \mathbf{Set}(X,R) : (g,f) \mapsto r_* f : X \to R,$$

with $r_*(f)(x) := f(x \cdot g)$, for all $x \in X$, $g \in G$. In turn, r_* induces a right H-coaction on A

$$\rho : \mathbf{Set}(X,R) \to \mathbf{Set}(X,R) \otimes \mathrm{Hom}_R(R[G],R) : f \mapsto \rho(f),$$

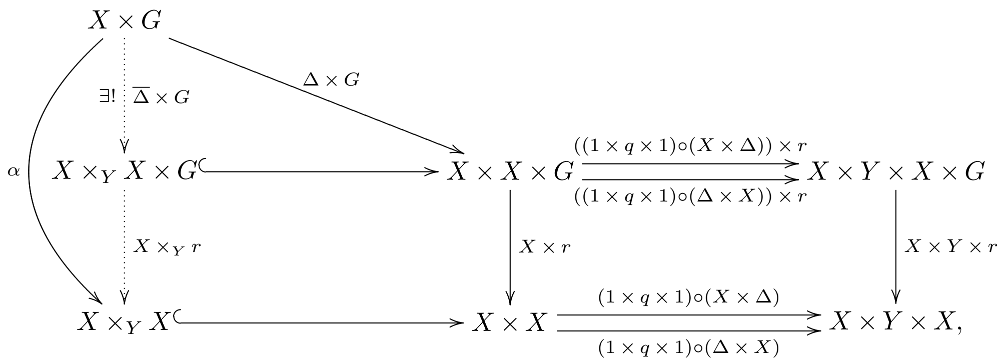
with $\rho(f)(x \otimes g) := f(x \cdot g)$, for all $x \in X$, $g \in G$.

The map q induces then a morphism of H-comodule algebras

$$q^* : B = \mathbf{Set}(Y,R) \to \mathbf{Set}(X,R) = A,$$

where B has a trivial H-coaction and, moreover, $i_{q^*} : B \xrightarrow{\cong} A^{coH}$ is an isomorphism.

Now, consider the following diagram



where $\Delta \times G : X \times G \to X \times X \times G$ satisfies the equalizer condition, since Δ is coassociative. One can check that the dual of the map α is actually the Galois map associated to $q^* : \mathbf{Set}(Y,R) \to \mathbf{Set}(X,R)$, i.e.,

$$\alpha^* = \beta_{q^*} : \mathbf{Set}(X,R) \otimes_{\mathbf{Set}(Y,R)} \mathbf{Set}(X,R) \to \mathbf{Set}(X,R) \otimes \mathrm{Hom}_R(R[G],R).$$

It follows that q^* is a $\mathrm{Hom}_R(R[G],R)$-Hopf-Galois extension if and only if β_{q^*} is an isomorphism. I.e., if and only if α^* is an isomorphism, which is equivalent to $\alpha : X \times G \to X \times_Y X$ being an isomorphism. This happens if the G-action r is free.

A few more examples of Hopf-Galois extensions can be found in [Mon09], which is a very good survey paper on (classical) Hopf-Galois theory by Susan Montgomery.

2.4.4 Homotopifying (Hopf-)Galois extensions

"Brave new" Galois and Hopf-Galois extensions

Homotopic Hopf-Galois extensions were first introduced by John Rognes in his monograph on Galois extensions of structured ring spectra [Rog08].

Among other things, Rognes formulated in [Rog08] the definition of Galois extensions of spectra, investigated the behavior of Galois extensions under cobase change (which inspired Chapter 3 of this thesis) and was able to establish a full version of Galois correspondence for ring spectra (which inspired Chapter 4 of this thesis).

He also observed that the unit map $\eta : \mathbb{S} \to MU$ from the sphere spectrum $\mathbb{S}$ to the complex cobordism spectrum MU was a Hopf-Galois extension in a homotopical sense, for the Hopf algebra spectrum $\Sigma^\infty BU_+$, the unreduced suspension spectrum of BU (Proposition 12.2.1, [Rog08]). Rognes noticed that η could not be a G-Galois extension for any G (Remark 12.2.2 [Rog08]).

Foundations of homotopic Hopf-Galois theory

Motivated by the desire to provide a general framework in which to study homotopic Hopf-Galois extensions, Kathryn Hess laid the foundations of a theory of Hopf-Galois extensions in monoidal model categories in [Hes09], generalizing both the classical case of rings and its extension to ring spectra. This article of K. Hess is at the origin of this thesis.

We use the same notation as in Definition 2.3.1.

Definition 2.4.12 (Definition 3.2, [Hes09]). Let $(\mathcal{M}, \otimes \mathbb{I})$ be a monoidal model category and let H be a bimonoid in $\mathcal{M}$. A map $\varphi : \mathrm{Triv}(B) \to A$ of H-comodule algebras is a **homotopic H-Hopf-Galois extension** if

(1) the associated Galois map $\beta_\varphi : A \otimes_B A \to A \otimes H$ is a weak equivalence in $\mathcal{M}$, and

(2) there is a choice of fibrant replacement $j : A \overset{\simeq}{\rightarrowtail} A'$ in $\mathbf{Alg}_H$ such that the comparison map $i_\varphi : B \to A^{hcoH}$ induces a Quillen equivalence

$$- \otimes_B A^{hcoH} : \mathbf{Mod}_B \rightleftarrows \mathbf{Mod}_{A^{hcoH}} : i_\varphi^*.$$

Observe that, in the spirit of Remark 4.22 from [Hes09], condition (1) of the definition above was "categorified" in Definition 2.3.1 and transformed into the requirement for the Galois functor $(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$ to be a Quillen equivalence (at least, as far as one works in the case where $\mathcal{M} := \mathbf{Ch}_{\mathbb{k}}^{\geq 0}$).

In [Hes09], the conditions for having model structures on the categories $\mathbf{Comod}_H$, $\mathbf{Alg}_H$, $\mathcal{M}_A^W$ that are left-induced from a suitable category $\mathcal{M}$ with

Postnikov presentation (X, Z) were already under investigation (Theorems 1.13, 1.17, 4.10 in [Hes09]) The difficult problem was to prove the existence of required factorizations in $\mathbf{Comod}_H$, $\mathbf{Alg}_H$, $\mathcal{M}_A^W$, as well as to characterize fibrant replacements therein. More progress on these questions was done in [HS12], and also in [BHKKRS14] during the Banff project in August 2013.

However, the existence of required factorizations and the form of fibrant replacements were well-understood in a number of particular cases, which made it possible in [Hes09] to study examples of homotopic Hopf-Galois extensions in the categories of simplicial monoids and of finite-type chain algebras of $\Bbbk$-vector spaces.

K. Hess also conjectured in [Hes09] that a homotopic version of Schneider's theorem (see Theorem 2.4.14 below) should hold for homotopic Hopf-Galois extensions, which turned out to be true, at least in the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

2.4.5 Relation to (homotopic) Grothendieck descent

There is an interesting relation between the theory of (homotopic) Hopf-Galois extensions and the Grothendieck descent theory. We will only give a brief sketch of it here, and invite the reader to find more details in the mini-course "Homotopic Hopf-Galois extensions and Descent" [Hes13], given by K. Hess in September 2013 in Louvain-la-Neuve.

Any ring homomorphism $\varphi : B \to A$ induces an adjunction

$$\mathbf{Mod}_B \underset{\varphi^*}{\overset{-\otimes_B A}{\rightleftarrows}} \mathbf{Mod}_A\,.$$

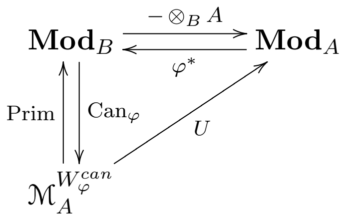
In this situation, the classical descent problem for modules over (commutative) rings tries to answer the following two questions.

- Given an A-module M, what extra structure on M guarantees that there exists a B-module N, such that $N \otimes_B A \cong M$?

- Given $f : N \otimes_B A \to N' \otimes_B A$, what extra structure on the B-modules N, N' guarantees that there exists a map of B-modules $g : N \to N'$, such that $f = g \otimes_B A$?

In order to formalize answers to these questions, one needs to work with the **category of descent data**, associated to φ. This category is sometimes denoted $\mathcal{D}(\varphi)$ and is actually isomorphic to the category $\mathcal{M}_A^{W_\varphi^{can}}$ of W_φ^{can}-comodules in $\mathbf{Mod}_A$, where W_φ^{can} is the canonical coring associated to φ

(see Examples 1.1.18, (1)). There is a commutative diagram of functors



where $\mathrm{Can}_\varphi : \mathbf{Mod}_B \to \mathcal{M}_A^{W_\varphi^{can}}$ is the **canonical functor**, defined for all $N \in \mathbf{Mod}_B$ by $\mathrm{Can}_\varphi(N) = (N \otimes_B A, \rho_N)$, with ρ_N given by the composite

$$N \otimes_B A \cong N \otimes_B B \otimes_B A \xrightarrow{1 \underset{B}{\otimes} \varphi \underset{B}{\otimes} 1} N \otimes_B A \otimes_B A \cong (N \otimes_B A) \otimes_A (A \otimes_B A).$$

See Definition 4.15 in [Hes09] for the definition of the right adjoint Prim to Can_φ

This allows one to formulate the following definition for classical descent.

Definition 2.4.13. A morphism of rings $\varphi : B \to A$ satisfies **descent** if the functor Can_φ is fully faithful, and satisfies **effective descent** if the functor Can_φ is an equivalence of categories.

For example, a homomorphism of rings φ will satisfy descent if A is faithfully flat as a B-module, i.e., if $M \otimes_B A = 0 \Leftrightarrow M = 0$, for all $M \in \mathbf{Mod}_B$ (see Theorem 4.16, [Hes09]).

Schneider's structure theorem given below relates H-Hopf-Galois extensions of rings $\varphi : B \to A$ to the category $\mathcal{M}_A^{W_\rho}$, where, $W_\rho = A \otimes H$, see Examples 1.1.18, (2). Peter Schauenburg provides in [Sch04] a proof of this theorem, based on the characterization of faithfully flat ring extensions in terms of descent.

Theorem 2.4.14 ([Schn90]). *Let R be a commutative ring, and let H be an R-flat Hopf algebra. The following are equivalent for any H-comodule algebra A, with coinvariant algebra $B = A^{coH}$.*

1. *The inclusion $i : B \hookrightarrow A$ is an H-Hopf-Galois extension, and A is a faithfully flat B-module.*

2. *The categories $\mathbf{Mod}_B$ and $\mathcal{M}_A^{W_\rho}$ are equivalent via the adjunction*

$$\mathbf{Mod}_B \underset{\mathrm{Prim}}{\overset{\mathrm{Can}_\varphi}{\rightleftarrows}} \mathcal{M}_A^{W_\varphi^{can}},$$

i.e., φ satisfies effective descent.

Once all the categories in the triangular diagram above are endowed with "appropriate" model structures, the classical definition of effective descent can be "homotopified" as follows.

Definition 2.4.15. A morphism $\varphi : B \to A$ of algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ satisfies **effective homotopic descent** if the adjunction

$$\mathbf{Mod}_A \underset{\text{Prim}}{\overset{\text{Can}}{\rightleftarrows}} \mathcal{M}_A^{W_{\varphi}^{can}}$$

is a Quillen equivalence, where the category $\mathbf{Mod}_A$ is endowed with a model structure, right-induced from $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ and $\mathcal{M}_A^{W_{\varphi}^{can}}$ is endowed with a model structure, left-induced from $\mathbf{Mod}_A$.

The following result establishes a homotopic version of Schneider's result and allows us to view homotopic Hopf-Galois extensions as an interesting class of morphisms of differential graded algebras, satisfying effective descent.

Theorem 2.4.16 ([BH12])**.** *Let $\Bbbk$ be a field and H a 1-connected dg-$\Bbbk$-Hopf algebra of finite type. Let $\varphi : B \to A$ be a morphism of augmented H-comodule algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where B is endowed with a trivial H-coaction. If the functor $i_{\varphi}^{*} : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$ is a Quillen equivalence, then φ is a homotopic H-Hopf-Galois extension if and only if φ satisfies effective homotopic descent.*

2.4.6 Bujard's Master Thesis

We end our panorama with a short comment on Chapter 5 of the Master thesis of Cédric Bujard [Buj06]. His work took place in a general cofibrantly generated symmetric monoidal model category $\mathcal{C}$. However, he made a lot of assumptions and conjectures throughout his project, and many gaps remained to be filled in.

Given a morphism $\varphi : \text{Triv}(B) \to A^{\circlearrowleft H}$ of commutative monoids in $\mathcal{C}$, where A has a coaction of a commutative bimonoid H, Bujard used two models for the object of homotopy coinvariants A^{hcoH} and worked under the assumption that suitable (functorial) fibrant replacements existed.

One of them was given by the totalization of a fibrant replacement of the Hopf cobar complex

$$C(H; A) := \text{tot}(RC^{\bullet}(H; A)),$$

under the assumption that such fibrant replacements exist in the category of cosimplicial commutative B-algebras (see Definition 5.1.6 in [Buj06]). The other model was given by the totalization of a fibrant replacement of the Amitsur complex

$$C(A/B) := \text{tot}(RC^{\bullet}(A/B)),$$

also assuming that such fibrant replacements are given in the category of cosimplicial commutative B-algebras (see Definition 5.1.12 in [Buj06]).

A morphism $\varphi : \mathrm{Triv}(B) \to A^{\circlearrowleft H}$ was defined to be a homotopic H-Hopf-Galois extension if both the Galois map $\beta_\varphi : A \otimes_B A \to A \otimes H$ and the comparison map $i_\varphi : B \to C(H; A)$ were weak equivalences in $\mathcal{C}$.

Working under assumptions, Cédric Bujard attempted to characterize homotopic Hopf-Galois extensions in terms of faithfulness and dualizability (Theorem 5.2.18 in [Buj06]), and also to investigate their behavior under cobase change (Propositions 5.3.1, 5.3.2, Theorem 5.3.4 in [Buj06]), which leads us smoothly to the subject of our next chapter.

Chapter 3

Behavior of homotopic Hopf-Galois extensions under base change

Remark 3.0.17. In this chapter, we work over a field $\mathbb{k}$. All dg-$\mathbb{k}$-algebras are assumed to be **commutative**, except the algebra underlying the bialgebra H. The commutativity assumption on dg-algebras ensures that the pushout of two commutative dg-algebras B' and A over a commutative dg-algebra B, is given by the coequalizer $B' \otimes_B A$.

3.1 The context

This chapter is inspired by results in Section 1 of Chapter 7 in [Rog08] on preservation and reflection of (faithful) G-Galois extensions of commutative ring spectra under base change along arbitrary maps. Our goal is to see whether and how these results translate into our context.

Let H be a dg-$\mathbb{k}$-Hopf algebra in $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$ and consider a pushout of commutative augmented H-comodule algebras in $\mathbf{Ch}_{\mathbb{k}}^{\geq 0}$

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A'. \end{array} \qquad (\blacksquare)$$

Remark 3.1.1. Observe that the category $\mathbf{Alg}_H^{\varepsilon}$ is the Eilenberg-Moore category of coalgebras over the comonad $- \otimes H : \mathbf{Alg}^{\varepsilon} \to \mathbf{Alg}^{\varepsilon}$ (see Remark 1.1.21). It has all colimits that exist in $\mathbf{Alg}^{\varepsilon}$, and they are created by the forgetful functor $U : \mathbf{Alg}_H^{\varepsilon} \to \mathbf{Alg}^{\varepsilon}$. The same is true in the commutative case, so the pushout (■) is actually a pushout in the category of commutative augmented H-comodule algebras.

Firstly, we will suppose that the map φ is a homotopic H-Hopf-Galois extension, and we will investigate under which conditions on the initial data in the pushout (■) the map $\overline{\varphi}$ is again a homotopic H-Hopf-Galois extension. This is the question of *preservation* of Hopf-Galois extensions under base change.

Secondly, we will assume that the map $\overline{\varphi}$ is a homotopic H-Hopf-Galois extension, and we will find conditions on the initial data in the pushout (■) that guarantee that the map φ was initially a homotopic H-Hopf-Galois extension. This is the problem of *reflection* of Hopf-Galois extensions under base change.

Remark 3.1.2. In [AH86], Avramov and Halperin provide an existence result for a semi-free replacement of a morphism of commutative differential graded Γ-algebras (i.e., commutative dga's equipped with an assigned system of *divided powers* (see Definition 1.3 in [AH86])). More specifically, if follows from the "Existence Property" and Lemma 2.2(i) in [AH86] that every morphism of dg-Γ-algebras $f : B \to A$ admits a factorization

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & A \\ & \searrow \quad \nearrow_{\simeq} & \\ & B\langle X \rangle & \end{array}$$

in the category of dg-Γ-algebras, where the graded B-module, underlying $B\langle X \rangle$ is B-semi-free.

3.1.1 Some comments on the comparison maps i_φ, $i_{\overline{\varphi}}$ and their induced functors

We first make a useful observation on the relation between the comparison maps $i_\varphi : B \to A^{hcoH}$ and $i_{\overline{\varphi}} : B' \to (A')^{hcoH}$.

Remark 3.1.3. Consider the pushout of the comparison map i_φ along f:

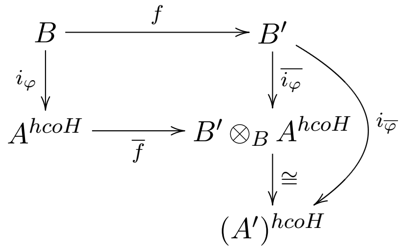
$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & B' \\ {\scriptstyle i_\varphi}\downarrow & & \downarrow{\scriptstyle \overline{i_\varphi}} \\ A^{hcoH} & \xrightarrow[\quad \overline{f} \quad]{} & B' \otimes_B A^{hcoH}. \end{array}$$

Recall from Section 2.1.3 that if H satisfies conditions of Convention 2.1.12, then the homotopy H-coinvariants of A can be modeled via the cobar construction by $A^{hcoH} = A \otimes_{t_\Omega} \Omega H$. Therefore, one can write

$$B' \otimes_B A^{hcoH} \cong B' \otimes_B (A \otimes_{t_\Omega} \Omega H) \cong (B' \otimes_B A) \otimes_{t_\Omega} \Omega H$$

$$\cong (B' \otimes_B A)^{hcoH} = (A')^{hcoH}.$$

In other words, the previous pushout square becomes



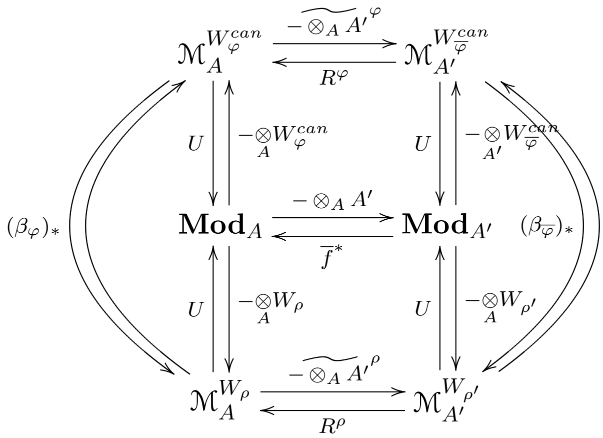
and shows that the comparison map $i_{\overline{\varphi}} : B' \to (A')^{hcoH}$ is obtained from i_φ by pushout along f. In particular, since $B' \cong B' \otimes_B B$, the two comparison maps are related via $i_{\overline{\varphi}} = B' \otimes_B i_\varphi$.

The induced comparison functors $(i_\varphi)^*$ and $(i_{\overline{\varphi}})^*$ are right adjoints of two extension/restriction of scalars adjunctions, which will be discussed in more detail in Section 3.2.1.

3.1.2 The context in which the Galois functors $(\beta_\varphi)_*$ and $(\beta_{\overline{\varphi}})_*$ arise

The Galois functors $(\beta_\varphi)_*$ and $(\beta_{\overline{\varphi}})_*$ arise in a slightly complicated context. It will be useful to have a good understanding of the situation before studying the reflection and preservation of homotopic Hopf-Galois extensions.

The commutative diagram below gives a general picture and helps to understand what the categories of interest are, when studying the Galois functors, and also how these categories are related. The left adjoints are displayed on top and on the left.



(♦)

Remark 3.1.4. Recall from Examples 1.1.18 that $W_\varphi^{can} = A \otimes_B A$ and $W_\rho = A \otimes H$. Also, it follows from the pushout (■) that $W_{\overline{\varphi}}^{can} = A' \otimes_{B'} A'$ and $W_{\overline{\rho}} = A' \otimes H$, where $A' = B' \otimes_B A$.

We will now explain how all the functors involved in the diagram (⧫) are defined.

The definition of functors in the diagram (⧫)

- The Galois functor $(\beta_\varphi)_*$, described in Section 2.3, admits a right adjoint
$$- \underset{W_\rho}{\square} (\beta_\varphi)_*(W_\varphi^{can}) : \mathcal{M}_A^{W_\rho} \to \mathcal{M}_A^{W_\varphi^{can}} .$$
For all $(M', \theta'_\rho) \in \mathcal{M}_A^{W_\rho}$, its value $(M', \theta'_\rho) \underset{W_\rho}{\square} (\beta_\varphi)_*(W_\varphi^{can})$ is given by the equalizer
$$\text{equal}\Big(M' \underset{A}{\otimes} W_\varphi^{can} \overset{\theta'_\rho \underset{A}{\otimes} W_\varphi^{can}}{\underset{(M' \underset{A}{\otimes} \beta_\varphi \underset{A}{\otimes} W_\varphi^{can}) \circ (M' \underset{A}{\otimes} \psi)}{\rightrightarrows}} M' \underset{A}{\otimes} W_\rho \underset{A}{\otimes} W_\varphi^{can} \Big),$$
computed in $\mathcal{M}_A^{W_\varphi^{can}}$ (see [Hes09], Remark 4.7). Since we will be working under conditions that guarantee the existence of a model structure on the category $\mathcal{M}_A^{W_\varphi^{can}}$, all such equalizers will exist.
- The definition of the adjoint pair $\big((\beta_{\overline{\varphi}})_*, - \underset{W_{\rho'}}{\square} (\beta_{\overline{\varphi}})_*(W_{\overline{\varphi}}^{can})\big)$ is similar.

The top and the bottom adjunctions in the diagram (⧫) are relative to the central adjunction $\big(- \otimes_A A', \overline{f}^* \big)$, in the sense that in both squares, the diagrams of left adjoints (or equivalently, the diagrams of right adjoints) commute up to natural isomorphism.

- More precisely, the left adjoint $\widetilde{- \otimes_A A'}^{\varphi} : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}}$ is defined for all $(X, \gamma, \theta_\varphi) \in \mathcal{M}_A^{W_\varphi^{can}}$ by
$$\widetilde{- \otimes_A A'}^{\varphi}(X, \gamma, \theta_\varphi) = \Big(X \otimes_A A', X \otimes_A \mu_{A'}, (X \otimes_A T_{A'}^{W_\varphi^{can}, W_{\overline{\varphi}}^{can}}) \circ (\theta_\varphi \otimes_A A') \Big).$$
The map
$$T_{A'}^{W_\varphi^{can}, W_{\overline{\varphi}}^{can}} : W_\varphi^{can} \otimes_A A' \to A' \otimes_{A'} W_{\overline{\varphi}}^{can}$$
is the $(W_\varphi^{can}, W_{\overline{\varphi}}^{can})$–braiding morphism, associated to $A' \in {}_A\mathbf{Bimod}_{A'}$ (see Definition 1.5.1). Use Remark 3.1.4 to see that
$$W_\varphi^{can} \otimes_A A' = (A \otimes_B A) \otimes_A A' \cong A \otimes_B A'$$
and
$$A' \otimes_{A'} W_{\overline{\varphi}}^{can} = A' \otimes_{A'} (A' \otimes_{B'} A') \cong A' \otimes_{B'} A' \cong A' \otimes_{B'} (B' \otimes_B A) \cong A' \otimes_B A.$$

One can check that the symmetry isomorphism

$$T_{A'}^{W_{\varphi}^{can}, W_{\overline{\varphi}}^{can}} : A \otimes_B A' \xrightarrow{\cong} A' \otimes_B A$$

satisfies the axioms of Definition 1.5.1 and gives a braiding morphism.

- On the other hand, the left adjoint $\widetilde{- \otimes_A A'}^{\rho} : \mathcal{M}_A^{W_\rho} \to \mathcal{M}_{A'}^{W_{\rho'}}$ is defined for all $(X, \gamma, \theta_\rho) \in \mathcal{M}_A^{W_\rho}$ by

  $$\widetilde{- \otimes_A A'}^{\rho}(X, \gamma, \theta_\rho) = \Big(X \otimes_A A', X \otimes_A \mu_{A'}, (X \otimes_A T_{A'}^{W_\rho, W_{\rho'}}) \circ (\theta_\rho \otimes_A A')\Big).$$

  Here the map

  $$T_{A'}^{W_\rho, W_{\rho'}} : W_\rho \otimes_A A' \to A' \otimes_{A'} W_{\rho'}$$

  is the $(W_\rho, W_{\rho'})$–braiding morphism, associated to $A' \in {}_A\mathbf{Bimod}_{A'}$. Use Remark 3.1.4 to see that

  $$W_\rho \otimes_A A' = (A \otimes H) \otimes_A A' \text{ and } A' \otimes_{A'} W_{\rho'} = A' \otimes_{A'} (A' \otimes H) \cong A' \otimes H.$$

  The braiding morphism

  $$T_{A'}^{W_\rho, W_{\rho'}} : (A \otimes H) \otimes_A A' \to A' \otimes H$$

  is induced in the coequalizer

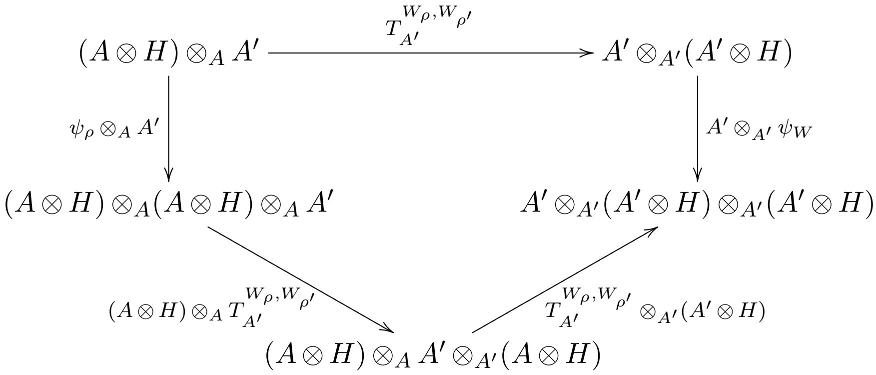
  $$\begin{array}{ccccc} (A \otimes H) \otimes A \otimes A' & \underset{(1 \otimes 1 \otimes \mu_{A'}) \circ (1 \otimes 1 \otimes \overline{f} \otimes 1)}{\overset{(\mu_A \otimes \mu_H \otimes 1) \circ (1 \otimes 1 \otimes \rho \otimes 1)}{\rightrightarrows}} & (A \otimes H) \otimes A' & \longrightarrow & (A \otimes H) \otimes_A A' \\ & & & \searrow^{\xi} & \downarrow T_{A'}^{W_\rho, W_{\rho'}} \\ & & & & A' \otimes H, \end{array}$$

  where ξ is the composite

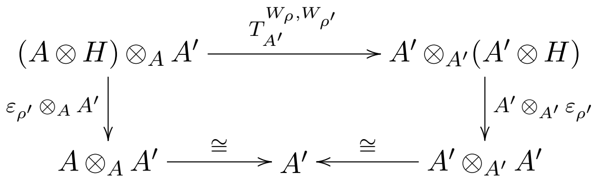
  $$\begin{array}{c} A \otimes H \otimes A' \xrightarrow{1 \otimes 1 \otimes \rho'} A \otimes H \otimes A' \otimes H \xrightarrow{\cong} A \otimes A' \otimes H \otimes H \xrightarrow{1 \otimes 1 \otimes \mu_H} A \otimes A' \otimes H \\ \downarrow \cong \\ A' \otimes A \otimes H \\ \downarrow 1 \otimes \overline{f} \otimes 1 \\ A' \otimes A' \otimes H \\ \downarrow \mu_{A'} \otimes 1 \\ A' \otimes H. \end{array}$$

  When showing that ξ satisfies the coequalizer condition the crucial facts are that ρ and $\overline{f}$ are morphisms of algebras, and that the multiplication $\mu_{A'}$ is commutative.

To see that the map $T_{A'}^{W_\rho, W_{\rho'}}$ defined above is indeed a braiding morphism, one needs to check that the associated diagrams for Pentagon axiom



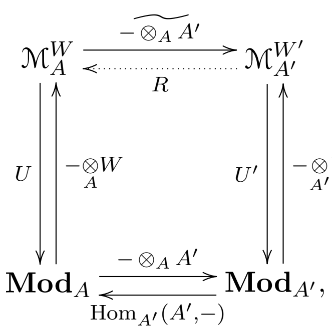
and for Counit axiom



commute. The Pentagon axiom is quite quick to check. It holds, thanks to the coassociativity of ρ' and the compatibility of μ_H and Δ_H. The counit axiom is slightly more tedious to verify; and it uses the fact that A' is an *augmented* H-comodule algebra, i.e., the compatibility between $\varepsilon_{A'}$ and both $\mu_{A'}$ and ρ' (the explicit definition of $\varepsilon_{A'}$ is given in the proof of Proposition 3.1.8).

The remaining right adjoints R^φ and R^ρ in the diagram (♦)

Recall that adjunctions induced by bimodules were introduced in Section 1.5.1. Consider ${}_A A'_{A'} \in {}_A\mathbf{Bimod}_{A'}$ with the right A-action on A' induced by $\overline{f} : A \to A'$ and the left A'-action induced by multiplication $\mu_{A'}$. We will discuss the existence and sketch the definition of the right adjoint R in the following situation



(◊)

assuming some conditions on the algebras A, A' and on the corings W, W'. The existence and the definition of right adjoints $R^{\varphi} : \mathcal{M}_{A'}^{W^{can}_{\overline{\varphi}}} \to \mathcal{M}_{A}^{W^{can}_{\varphi}}$ and $R^{\rho} : \mathcal{M}_{A'}^{W_{\rho'}} \to \mathcal{M}_{A}^{W_{\rho}}$ in the diagram (♦) will then follow as two special cases, by setting $W' := W^{can}_{\overline{\varphi}}, W_{\rho'}$ and $W := W^{can}_{\varphi}, W_{\rho}$.

Lemma 3.1.5. *Let $\overline{f} : A \to A'$ be the map of algebras as in the pushout (■). Let $W = (W, \psi, \varepsilon)$ be an A-coring and $W' = (W', \psi', \varepsilon')$ be an A'-coring. Consider the diagram of categories and functors*

$$\begin{array}{ccc}
\mathcal{M}_A^W & \underset{R}{\overset{\widetilde{-\otimes_A A'}}{\rightleftarrows}} & \mathcal{M}_{A'}^{W'} \\
U \downarrow \uparrow -\underset{A}{\otimes} W & & U' \downarrow \uparrow -\underset{A'}{\otimes} W' \\
\mathbf{Mod}_A & \underset{\mathrm{Hom}_{A'}(A',-)}{\overset{-\otimes_A A'}{\rightleftarrows}} & \mathbf{Mod}_{A'},
\end{array} \qquad (\lozenge)$$

where the left adjoints are displayed on top and on the left, and where we have $U' \circ (\widetilde{-\otimes_A A'}) = (-\otimes_A A') \circ U$. Assume moreover that the A-coring W is semi-free as a left A-module on a generating graded $\Bbbk$-module that is degree-wise finitely generated. Then the right adjoint $R : \mathcal{M}_{A'}^{W'} \to \mathcal{M}_A^W$ exists.

Proof. Observe that we are in the situation described in Remark 1.1.24. Indeed, it is easy to check that the category $\mathcal{M}_A^W$ of W-comodules in $\mathbf{Mod}_A$ is isomorphic to the category $\big(\mathbf{Mod}_A\big)_{\mathbb{K}_W}$ of $\mathbb{K}_W$-coalgebras in $\mathbf{Mod}_A$, for the comonad $\mathbb{K}_W := \big(-\otimes_A W, -\otimes_A \psi, -\otimes_A \varepsilon\big)$. The fact that W is assumed to be A-semi-free allows us to apply Lemma 6.8 from [HS12] to establish that all limits in $\mathcal{M}_A^W$ are created in $\mathbf{Mod}_A$. Since $\mathbf{Mod}_A$ is complete, it follows in particular that $\mathcal{M}_A^W$ admits all equalizers. We can therefore use Theorem 1.1.23 and conclude that the right adjoint $R : \mathcal{M}_{A'}^{W'} \to \mathcal{M}_A^W$ exists.

Dualizing carefully Borceux's proof, one can see that R is defined on any $(X', \theta') \in \mathcal{M}_{A'}^{W'}$ by the equalizer

$$R(X') \to \mathrm{Hom}_{A'}(A', X') \otimes_A W \underset{②}{\overset{①}{\rightrightarrows}} \mathrm{Hom}_{A'}(A', X' \otimes_{A'} W') \otimes_A W.$$

Here

$$① := (\theta')_* \otimes_A W : \mathrm{Hom}_{A'}(A', X') \otimes_A W \to \mathrm{Hom}_{A'}(A', X' \otimes_{A'} W') \otimes_A W,$$

and the map ② is the composite

$$\mathrm{Hom}_{A'}(A', X') \otimes_A W \xrightarrow{(-\otimes_{A'} W')\otimes_A W} \mathrm{Hom}_{A'}(A' \otimes_{A'} W', X' \otimes_{A'} W') \otimes_A W$$

$$\xrightarrow{(T^{W,W'}_{A'})^* \otimes_A W} \mathrm{Hom}_{A'}(W \otimes_A A', X' \otimes_{A'} W') \otimes_A W$$

$$\xrightarrow{\mathrm{Id} \otimes_A \psi} \mathrm{Hom}_{A'}(W \otimes_A A', X' \otimes_{A'} W') \otimes_A W \otimes_A W$$

$$\xrightarrow{ev^\sharp \otimes_A W} \mathrm{Hom}_{A'}(A', X' \otimes_{A'} W') \otimes_A W,$$

where $T^{W,W'}_{A'} : W \otimes_A A' \to A' \otimes_A W'$ is the (W, W')-braiding morphism associated to $A' \in {}_A\mathbf{Bimod}_{A'}$ and

$$ev^\sharp : \mathrm{Hom}_{A'}(W \otimes_A A', X' \otimes_{A'} W') \otimes_A W \to \mathrm{Hom}_{A'}(A', X' \otimes_{A'} W')$$

is the adjoint of the evaluation map

$$ev : \mathrm{Hom}_{A'}(W \otimes_A A', X' \otimes_{A'} W') \otimes_A W \otimes_A A' \to X' \otimes_{A'} W'.$$

The important facts for establishing the definition of R are the following.

- Every object $(X', \gamma', \theta') \in \mathcal{M}^{W'}_{A'}$ can be written as a split equalizer of cofree W'-comodules (see [BW05]).
- Since the diagram $(\Diamond)$ must commute, R must satisfy

  $$R(- \otimes_{A'} W') = \mathrm{Hom}_{A'}(A', -) \otimes_A W$$

  and preserve limits.
- At this point, one will be able to define a collection of functions $R : Ob(\mathcal{M}^{W'}_{A'}) \to Ob(\mathcal{M}^{W}_{A})$. In order to complete this family into a *functor*, having $- \otimes_A A' : \mathcal{M}^{W}_{A} \to \mathcal{M}^{W'}_{A'}$ as a left adjoint, one uses Theorem IV.1.2 (iv) in [McL98] and constructs a family of $- \otimes_A A'$-couniversal arrows.

□

Remark 3.1.6. Our results on the behavior of homotopic Hopf-Galois extensions in Sections 3.2 and 3.3 will all be formulated under the assumption that the algebra A is semi-free (as a left B-module) on a generating graded $\Bbbk$-module X that is degree-wise finitely generated. This hypothesis will imply (see proof of Proposition 3.1.7) that the A-corings W^{can}_{φ} and W_ρ satisfy the semi-freeness condition of Lemma 3.1.5, which will give us the existence and construction of right adjoints $R^\varphi : \mathcal{M}^{W^{can}_{\overline{\varphi}}}_{A'} \to \mathcal{M}^{W^{can}_{\varphi}}_{A}$ and $R^\rho : \mathcal{M}^{W_{\rho'}}_{A'} \to \mathcal{M}^{W_\rho}_{A}$.

Existence of model structures and Quillen pairs in the diagram (♦) (p.79)

We will see that, under reasonable conditions on the initial data given by the pushout (■), all adjunctions in the diagram (♦) are Quillen pairs with respect to suitable model structures.

Recall that the categories of modules $\mathbf{Mod}_A$ and $\mathbf{Mod}_{A'}$ are both equipped with the right-induced model structure from $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ (see Theorem 1.3.5).

The fact that the adjoint pair

$$\mathbf{Mod}_A \underset{f^*}{\overset{-\otimes_A A'}{\rightleftarrows}} \mathbf{Mod}_{A'}$$

is a Quillen pair follows directly from Lemma 1.4.2.

To determine conditions under which there exist induced model structures on the categories $\mathcal{M}_A^{W_\varphi^{can}}$, $\mathcal{M}_A^{W_\rho}$, $\mathcal{M}_{A'}^{W_\varphi^{can}}$ and $\mathcal{M}_{A'}^{W_{\rho'}}$, we apply Theorem 1.3.11.

We first concentrate on the adjunction $\big((\beta_\varphi)_*, - \underset{W_\rho}{\square} (\beta_\varphi)_*(W_\varphi^{can})\big)$ from the diagram (♦).

Proposition 3.1.7. *Let $\Bbbk$ be a field, and let H be a Hopf algebra satisfying Convention 2.1.12. Suppose that*

(1) *A is an augmented $\Bbbk$-algebra; and*

(2) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$.*

Under these assumptions the categories $\mathcal{M}_A^{W_\varphi^{can}}$ and $\mathcal{M}_A^{W_\rho}$ admit model structures, left-induced from the category $\mathbf{Mod}_A$ by the forgetful functor. Moreover, the adjoint pair

$$\mathcal{M}_A^{W_\varphi^{can}} \underset{- \underset{W_\rho}{\square} (\beta_\varphi)_*(W_\varphi^{can})}{\overset{(\beta_\varphi)_*}{\rightleftarrows}} \mathcal{M}_A^{W_\rho}$$

is then a Quillen pair.

Proof. Using assumption (2), one can write $A \cong B \widetilde{\otimes} X$. So, the A-coring

$$W_\varphi^{can} = A \otimes_B A \cong A \otimes_B (B \widetilde{\otimes} X) \cong A \widetilde{\otimes} X$$

is semi-free as a left A-module on X. It follows from Theorem 1.3.11 that there exists a model structure on the category $\mathcal{M}_A^{W_\varphi^{can}}$, left-induced by the forgetful functor U from $\mathbf{Mod}_A$.

On the other hand, by hypothesis the A-coring $W_\rho = A \otimes H$ is free as a left A-module on the generating graded $\Bbbk$-module H, satisfying Convention 2.1.12, and thus satisfying the conditions of Theorem 1.3.11. Hence, there exists a model structure on the category $\mathcal{M}_A^{W_\rho}$, left-induced by the forgetful functor U from $\mathbf{Mod}_A$.

It remains to show that the adjunction $\big((\beta_\varphi)_*, - \underset{W_\rho}{\square} (\beta_\varphi)_*(W_\varphi^{can})\big)$ is a Quillen pair with respect to these model structures. Let $(M, \gamma, \theta_\varphi) \in \mathcal{M}_A^{W_\varphi^{can}}$. From the definition of $(\beta_\varphi)_*$, it follows that applying this functor to M does not change the underlying A-module (M, γ), but only modifies its comodule structure.

Therefore, if $j : M \rightarrowtail N$ is a cofibration in $\mathcal{M}_A^{W_\varphi^{can}}$, i.e., a cofibration in $\mathbf{Mod}_A$, the map of A-modules underlying $(\beta_\varphi)_*(j)$, remains the same and is a cofibration in $\mathcal{M}_A^{W_\rho}$, by Theorem 1.3.11. Similarly, if $j : M \xrightarrow{\sim} N$ is a weak-equivalence in $\mathcal{M}_A^{W_\varphi^{can}}$, i.e., a quasi-isomorphism of chain complexes, the map of A-modules underlying $(\beta_\varphi)_*(j)$, remains a quasi-isomorphism of chain complexes and is a weak-equivalence in $\mathcal{M}_A^{W_\rho}$, again by Theorem 1.3.11. Consequently, the functor $(\beta_\varphi)_*$ preserves cofibrations and acyclic cofibrations, so it is left Quillen. □

Now we study conditions under which the adjunction $\big((\beta_{\overline{\varphi}})_*, - \underset{W_{\rho'}}{\square} (\beta_{\overline{\varphi}})_*(W_{\overline{\varphi}}^{can})\big)$ from the diagram (♦) is a Quillen pair.

Proposition 3.1.8. *Let $\Bbbk$ be a field, and let H be a Hopf algebra satisfying Convention 2.1.12. Suppose that*

(1) *A is an augmented $\Bbbk$-algebra;*

(2) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$;*

(3) *B' is an augmented $\Bbbk$-algebra.*

Under these assumptions the categories $\mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}}$ and $\mathcal{M}_{A'}^{W_{\rho'}}$ admit model structures, left-induced from the category $\mathbf{Mod}_{A'}$ by the forgetful functor. Moreover, the adjoint pair

$$\mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}} \underset{- \underset{W_{\rho'}}{\square} (\beta_{\overline{\varphi}})_*(W_{\overline{\varphi}}^{can})}{\overset{(\beta_{\overline{\varphi}})_*}{\rightleftarrows}} \mathcal{M}_{A'}^{W_{\rho'}}$$

is then a Quillen pair.

Proof. We show this result by applying Proposition 3.1.7 to A', $W_{\overline{\varphi}}^{can}$ and $W_{\rho'}$. By assumption (2), one can write $A \cong B \mathbin{\widetilde{\otimes}} X$, so that

$$A' = B' \otimes_B A \cong B' \otimes_B B \mathbin{\widetilde{\otimes}} X \cong B' \mathbin{\widetilde{\otimes}} X,$$

i.e., A' is semi-free as a left B'-module on a generating graded $\Bbbk$-module X, where X_n is finitely generated for all $n \geq 0$. This gives us condition (2) of Proposition 3.1.7 for A'.

Since both algebras A and B' are augmented by hypotheses (1) and (3), the pushout algebra A' is augmented via the map $\varepsilon_{A'}$ induced in the coequalizer

$$\begin{array}{ccccc} B' \otimes B \otimes A & \underset{(1\otimes\mu_A)\circ(1\otimes\varphi\otimes 1)}{\overset{(\mu_{B'}\otimes 1)\circ(1\otimes f\otimes 1)}{\rightrightarrows}} & B' \otimes A & \longrightarrow & B' \otimes_B A \\ & & & \searrow^{\varepsilon_{B'}\otimes\varepsilon_A} & \downarrow{\scriptstyle \varepsilon_{A'}} \\ & & & & R. \end{array}$$

Therefore, condition (1) of Proposition 3.1.7 is satisfied for A' and it follows that there exist model structures on the categories $\mathcal{M}_{A'}^{W^{can}_{\widetilde{\varphi}}}$ and $\mathcal{M}_{A'}^{W_{\rho'}}$, left-induced from $\mathbf{Mod}_{A'}$ by the forgetful functor U.

It is not difficult to show that the adjunction $\big((\beta_{\widetilde{\varphi}})_*, - \underset{W_{\rho'}}{\square} (\beta_{\widetilde{\varphi}})_*(W^{can}_{\widetilde{\varphi}})\big)$ is a Quillen pair with respect to these model structures. To do this, one proceeds in exactly the same way as in the proof of Proposition 3.1.7 for the adjunction $\big((\beta_\varphi)_*, - \underset{W_\rho}{\square} (\beta_\varphi)_*(W^{can}_\varphi)\big)$. □

It remains now to see when the top and the bottom adjunctions in the diagram (♦) are Quillen pairs.

Lemma 3.1.9. *Under the conditions that guarantee the existence of the left-induced model structures on the categories $\mathcal{M}_A^{W^{can}_\varphi}$, $\mathcal{M}_{A'}^{W^{can}_{\widetilde{\varphi}}}$, $\mathcal{M}_A^{W_\rho}$ and $\mathcal{M}_{A'}^{W_{\rho'}}$ (for example, under hypotheses of Proposition 3.1.8) both adjunctions*

$$\mathcal{M}_A^{W^{can}_\varphi} \underset{R^\varphi}{\overset{\widetilde{-\otimes_A A'}^{\varphi}}{\rightleftarrows}} \mathcal{M}_{A'}^{W^{can}_{\widetilde{\varphi}}} \qquad \textit{and} \qquad \mathcal{M}_A^{W_\rho} \underset{R^\rho}{\overset{\widetilde{-\otimes_A A'}^{\rho}}{\rightleftarrows}} \mathcal{M}_{A'}^{W_{\rho'}}$$

are Quillen pairs.

Proof. Since A' is cofibrant as a right A'-module in the right-induced model structure on $\mathbf{Mod}_{A'}$, Proposition 1.5.3 applies and allows us to deduce that both adjunctions $\big(\widetilde{-\otimes_A A'}^{\varphi}, R^\varphi\big)$ and $\big(\widetilde{-\otimes_A A'}^{\rho}, R^\rho\big)$ give rise to Quillen pairs. □

3.2 Preservation of homotopic Hopf-Galois extensions under base change

We recall the following result in [Rog08] on preservation of faithful G-Galois extensions of commutative ring spectra under base change along arbitrary maps.

Lemma 3.2.1 (Lemma 7.1.1, [Rog08])**.** *Let $f : A \to B$ be a map of commutative S-algebras and $\varphi : A \to C$ be a faithful G-Galois extension. Then the induced map $\overline{\varphi} : B \to B \wedge_A C$ in the pushout is also a faithful G-Galois extension.*

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ C & \xrightarrow[\;\overline{f}\;]{} & B \wedge_A C \end{array}$$

Our goal is to prove an analog of this result for homotopic H-Hopf-Galois extensions in the closed symmetric monoidal model category $\big(\mathbf{Ch}_{\Bbbk}^{\geq 0}, \otimes, \Bbbk[0]\big)$. See Proposition 3.2.7 for the final result.

3.2.1 The behavior of the comparison functor $(i_{\overline{\varphi}})^*$

As Definition 2.3.1 shows, the comparison functor $(i_\varphi)^*$ is crucial for defining a homotopic H-Hopf-Galois extension $\varphi : B \to A$. Let us first determine conditions under which the adjunctions $\big(- \otimes_B A^{hcoH}, (i_\varphi)^*\big)$ and $\big(- \otimes_{B'} (A')^{hcoH}, (i_{\overline{\varphi}})^*\big)$ are Quillen pairs.

Lemma 3.2.2. *Let $\Bbbk$ be a field. If the category $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ is equipped with the projective model structure, then both adjunctions*

$$\mathbf{Mod}_B \underset{(i_\varphi)^*}{\overset{-\otimes_B A^{hcoH}}{\rightleftarrows}} \mathbf{Mod}_{A^{hcoH}}$$

and

$$\mathbf{Mod}_{B'} \underset{(i_{\overline{\varphi}})^*}{\overset{-\otimes_{B'} (A')^{hcoH}}{\rightleftarrows}} \mathbf{Mod}_{(A')^{hcoH}}$$

are Quillen pairs with respect to the right-induced model structures from $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ on, respectively, $\mathbf{Mod}_B$, $\mathbf{Mod}_{A^{hcoH}}$, and $\mathbf{Mod}_{B'}$, $\mathbf{Mod}_{(A')^{hcoH}}$, for any models A^{hcoH}, $(A')^{hcoH}$ of the homotopy coinvaraints.

Proof. This is a direct consequence of Lemma 1.4.2. □

We now state conditions under which $(i_{\overline{\varphi}})^*$ is a Quillen equivalence.

Proposition 3.2.3. *Let $\Bbbk$ be a field and H a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. Consider the following pushout of commutative objects in $\mathbf{Alg}_H^{\varepsilon}$, where B and B' have trivial H-coactions, and where $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension.*

$$\begin{array}{ccc} B & \xrightarrow{\;f\;} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\;\overline{f}\;]{} & B' \otimes_B A := A' \end{array} \qquad (\blacksquare)$$

If

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$; and*

(2) *B' is semi-free as a right B-module on a generating graded $\Bbbk$-module Z, such that Z_n is finitely generated for all $n \geq 0$,*

then the comparison functor associated to $\overline{\varphi}$,

$$(i_{\overline{\varphi}})^* : \mathbf{Mod}_{(A')^{hcoH}} \to \mathbf{Mod}_{B'},$$

is a Quillen equivalence.

Proof. The strategy is to apply the criterion from Proposition 1.3.3 to the adjunction $\big(- \otimes_{B'} (A')^{hcoH}, (i_{\overline{\varphi}})^* \big)$. We must show that

(a) the homotopy counit

$$\tilde{\varepsilon}'_{M^f} : Q_{B'}\big((i_{\overline{\varphi}})^*(M^f)\big) \otimes_{B'} (A')^{hcoH} \to M^f$$

is a weak equivalence in $\mathbf{Mod}_{(A')^{hcoH}}$ for all fibrant $M^f \in \mathbf{Mod}_{(A')^{hcoH}}$, where $Q_{B'}(-)$ stands for the cofibrant replacement functor in $\mathbf{Mod}_{B'}$; and that

(b) the functor $- \otimes_{B'} (A')^{hcoH} : \mathbf{Mod}_{B'} \to \mathbf{Mod}_{(A')^{hcoH}}$ reflects weak equivalences between cofibrant objects.

In view of the model structure on $\mathbf{Mod}_{(A')^{hcoH}}$, every object in $\mathbf{Mod}_{(A')^{hcoH}}$ is fibrant. So, point (a) amounts to showing that

$$\tilde{\varepsilon}'_M : Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big) \otimes_{B'} (A')^{hcoH} \to M$$

is a quasi-isomorphism of chain complexes, for any $(A')^{hcoH}$-module M.

Using Remark 3.1.3, we have $(A')^{hcoH} \cong B' \otimes_B A \otimes_{t_\Omega} \Omega H$, and therefore

$$\begin{array}{ccc} Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big) \otimes_{B'} (A')^{hcoH} & \xrightarrow{\tilde{\varepsilon}'_M} & M \\ \downarrow \cong & & \\ Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big) \otimes_{B'} B' \otimes_B A \otimes_{t_\Omega} \Omega H & & \uparrow \tilde{\varepsilon}'_M \\ \downarrow \cong & & \\ f^*\Big(Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big)\Big) \otimes_B A \otimes_{t_\Omega} \Omega H, & & \end{array}$$

where $f^* : \mathbf{Mod}_{B'} \to \mathbf{Mod}_B$ equips $Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big) \in \mathbf{Mod}_{B'}$ with a B-action. It follows from assumption (2) that B' is cofibrant as a B-module, which implies that $f^*\Big(Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big)\Big)$ is cofibrant in $\mathbf{Mod}_B$.

Now, observe that the composite of functors

$$f^* \circ (i_{\overline{\varphi}})^* : \mathbf{Mod}_{(A')^{hcoH}} \to \mathbf{Mod}_B$$

is equal to the composite

$$(i_\varphi)^* \circ \overline{f}^* : \mathbf{Mod}_{(A')^{hcoH}} \to \mathbf{Mod}_B,$$

since $\overline{f} \circ i_\varphi = i_{\overline{\varphi}} \circ f$ (see Remark 3.1.3). Therefore, $f^*\Big(Q_{B'}\big((i_{\overline{\varphi}})^*(M)\big)\Big)$ gives a cofibrant replacement of $(i_\varphi)^*(\overline{f}^*(M))$, as a B-module. Thus, the homotopy counit $\tilde{\varepsilon'}_M$ can be rewritten as

$$\tilde{\varepsilon'}_M : Q_B\Big((i_\varphi)^*(\overline{f}^*(M))\Big) \otimes_B A \otimes_{t_\Omega} \Omega H \to M,$$

where $Q_B(-)$ stands for the cofibrant replacement functor in $\mathbf{Mod}_B$.

Now, by hypothesis, $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension, so its comparison functor $(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$ is Quillen equivalence. This means that the homotopy counit

$$\tilde{\varepsilon}_N : Q_B\Big((i_\varphi)^*(N)\Big) \otimes_B A \otimes_{t_\Omega} \Omega H \to N$$

of the adjunction $\big(- \otimes_B A^{hcoH}, (i_\varphi)^*\big)$ is a quasi-isomorphism of chain complexes, for all $N \in \mathbf{Mod}_{A^{hcoH}}$. By setting $N := \overline{f}^*(M)$, the fact that $\tilde{\varepsilon'}_M$ is a quasi-isomorphism follows.

It remains to show point (b). Let $g : M \to M'$ be a map of B'-modules and suppose that the morphism

$$g \otimes_{B'} (A')^{hcoH} : M \otimes_{B'} (A')^{hcoH} \to M' \otimes_{B'} (A')^{hcoH}$$

is a weak equivalence (i.e., a quasi-isomorphism of underlying chain complexes), with both $M \otimes_{B'} (A')^{hcoH}$ and $M' \otimes_{B'} (A')^{hcoH}$ cofibrant in $\mathbf{Mod}_{(A')^{hcoH}}$. We want to show that $g : M \to M'$ is a quasi-isomorphism, too.

Using the definition of A', the explicit model for the coinvariants $(A')^{hcoH}$ and the hypothesis that A is semi-free as a B-module on a generating graded $\Bbbk$-module X, we can write

$$M \otimes_{B'} (A')^{hcoH} \cong M \otimes_{B'} B' \otimes_B (B \tilde{\otimes} X) \otimes_{t_\Omega} \Omega H \cong M \tilde{\otimes} X \otimes_{t_\Omega} \Omega H,$$

and, similarly, $M' \otimes_{B'} (A')^{hcoH} \cong M' \tilde{\otimes} X \otimes_{t_\Omega} \Omega H$. So we have a quasi-isomorphism

$$g \tilde{\otimes} X \otimes_{t_\Omega} \Omega H : M \tilde{\otimes} X \otimes_{t_\Omega} \Omega H \to M' \tilde{\otimes} X \otimes_{t_\Omega} \Omega H.$$

Observe that the following square commutes in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$

$$\begin{array}{ccc} M \tilde{\otimes} X \otimes_{t_\Omega} \Omega H \otimes_{t_\Omega} H & \xrightarrow{\simeq} & M \tilde{\otimes} X \\ {\scriptstyle g \tilde{\otimes} X \otimes_{t_\Omega} \Omega H \otimes_{t_\Omega} H} \downarrow \simeq & & \dot{\simeq} \downarrow {\scriptstyle g \tilde{\otimes} X} \\ M' \tilde{\otimes} X \otimes_{t_\Omega} \Omega H \otimes_{t_\Omega} H & \xrightarrow{\simeq} & M' \tilde{\otimes} X. \end{array}$$

The left vertical map is a quasi-isomorphism, because H is degree-wise $\Bbbk$-free, and the horizontal maps are quasi-isomorphisms induced by the homotopy equivalence $\Omega H \otimes_{t_\Omega} H \xrightarrow{\sim} \Bbbk[0]$ (see Proposition 10.6.3 in [Nei10] and Proposition 7.8 in [McC01]). So, by 2-out of-3 property, the right-hand map $g \tilde{\otimes} X$ is also a quasi-isomorphism.

We will use Theorem 1.2.18 and the Zeeman comparison Theorem 1.2.19, to conclude that g is a quasi-isomorphism, too.

Because the graded $\Bbbk$-module $N = M \tilde{\otimes} X$ is semi-free on X, which is degree-wise finitely generated, it is equipped with a bounded filtration. Theorem 1.2.18 then tells us that there exists a spectral sequence $\{E^r_{*,*}\}$ converging to $H_*(M \tilde{\otimes} X)$. For similar reasons, for $N' = M' \tilde{\otimes} X$ there exists a spectral sequence $\{\overline{E}^r_{*,*}\}$ converging to $H_*(M' \tilde{\otimes} X)$.

By definition, the quasi-isomorphism $g \tilde{\otimes} X : M \tilde{\otimes} X \to M' \tilde{\otimes} X$ induces isomorphisms on all homology groups

$$H_n(g \tilde{\otimes} X) : H_*(M \tilde{\otimes} X) \to H_*(M' \tilde{\otimes} X),$$

for all $n \geq 0$. Because both spectral sequences converge, the collection of isomorphisms $\{H_n(g \tilde{\otimes} X)\}_{n \geq 0}$ gives the isomorphism

$$(g \tilde{\otimes} X)^\infty : E^\infty_{p,q} \to \overline{E}^\infty_{p,q}.$$

On the second pages, we have $E^2_{p,0} \cong H_p(X) = \overline{E}^2_{p,0}$ and $E^2_{0,q} = H_q(M)$, $\overline{E}^2_{0,q} = H_q(M')$, for all $p, q \geq 0$. it follows form the Universal Coefficient Theorem that $\overline{E}^2_{p,q} = \overline{E}^2_{p,0} \otimes \overline{E}^2_{0,q}$, and the Tor part in the diagram of Theorem 1.2.19 is zero, since we are working over a field $\Bbbk$. Therefore, we can conclude that $E^2_{0,q} = H_q(M) \to H_q(M') = \overline{E}^2_{0,q}$ is an isomorphism for all $q \geq 0$. □

3.2.2 The behavior of the Galois functor $(\beta_{\overline{\varphi}})_*$

Our goal in this section is to determine when the Galois functor $(\beta_{\overline{\varphi}})_*$ associated to $\overline{\varphi}$ is a Quillen equivalence, assuming that the Galois functor $(\beta_\varphi)_*$ associated to φ is a Quillen equivalence, by hypothesis.

To begin with, we observe that Proposition 1.5.6 allows us to formulate the following Corollary.

Corollary 3.2.4. *Let $\Bbbk$ be a field, and let H be a Hopf algebra satisfying Convention 2.1.12. Let $\varphi : B \to A$ be a map of commutative H-comodule algebras, where*

(1) *A is an augmented $\Bbbk$-algebra;*

(2) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$.*

Suppose that the map $\beta_\varphi : W_\varphi^{can} \to W_\rho$ is a quasi-isomorphism of A-corings and that the categories $\mathcal{M}_A^{W_\varphi^{can}}$ and $\mathcal{M}_A^{W_\rho}$ are equipped with the induced model structures, as in Proposition 3.1.7. Then the functor

$$(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$$

is a Quillen equivalence.

Proof. As was already observed in the proof of Proposition 3.1.7, assumption (2) implies that the A-coring $W_\varphi^{can} \cong A \widetilde{\otimes} X$ is semi-free as a left A-module on X. Also, the A-coring $W_\rho = A \otimes H$ is free as a left A-module on H. By Proposition 1.5.6, it follows that the functor $(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$ is a Quillen equivalence. $\square$

Remark 3.2.5. Note that this Corollary works for a morphism φ that is not assumed to be a homotopic H-Hopf-Galois extension. This is crucial if one wants to apply it to $\overline{\varphi}$ in Proposition 3.2.6.

Suppose now that $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension. The next result gives conditions under which $\beta_\varphi : W_\varphi^{can} \to W_\rho$ will be a quasi-isomorphism and $(\beta_{\overline{\varphi}})_*$ will be a Quillen equivalence.

Proposition 3.2.6. *Let $\Bbbk$ be a field, and let H be a Hopf algebra satisfying Convention 2.1.12. Consider the following pushout of commutative objects in $\mathbf{Alg}_H^\varepsilon$, where B and B' have trivial H-coactions, and where $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension.*

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A' \end{array} \qquad (\blacksquare)$$

Suppose that

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$;*

(2) *B' is semi-free as a right B-module on a generating graded $\Bbbk$-module Z, such that Z_n is finitely generated for all $n \geq 0$,*

then

- *the Galois map $\beta_\varphi : W_\varphi^{can} \to W_\rho$ is a quasi-isomorphism, and*
- *the functor*
$$(\beta_{\overline{\varphi}})_* : \mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}} \to \mathcal{M}_{A'}^{W_{\rho'}}$$
is a Quillen equivalence.

Proof. Since $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension, the functor $(\beta_\varphi)_*$ is a Quillen equivalence, by Definition 2.3.1. As was observed in the proof of Proposition 3.1.7, assumption (1) also implies that the A-coring W_φ^{can} is semi-free as a left A-module on a generating graded $\Bbbk$-module X that is degree-wise finitely generated. The coring $W_\rho = A \otimes H$ is free as a left A-module. Therefore, the assumptions of Proposition 1.5.7 hold for A, W_φ^{can}, W_ρ, and we can deduce that the Galois map $\beta_\varphi : W_\varphi^{can} \to W_\rho$ is a quasi-isomorphism, which shows the first statement in this Proposition.

We now prove the second statement. By assumption (1), we know that A' is semi-free as a left B'-module on a generating graded module X that is degree-wise finitely generated, as we saw in the proof of Proposition 3.1.8. It follows from the proof of Corollary 3.2.4 that the A'-corings $W_{\overline{\varphi}}^{can}$ and $W_{\rho'}$ are then semi-free as left A'-modules.

On the other hand, we have

$$W_{\overline{\varphi}}^{can} \cong (B' \otimes_B A) \otimes_{B'} (B' \otimes_B A) \cong (B' \otimes_B A) \otimes_B A \cong B' \otimes_B (A \otimes_B A) \cong B' \otimes_B \varphi^*(W_\varphi^{can})$$

and

$$W_{\rho'} \cong (B' \otimes_B A) \otimes H \cong B' \otimes_B (A \otimes H) = B' \otimes_B \varphi^*(W_\rho),$$

where we used the fact that the tensor product $- \otimes_B -$ is associative, and the functor φ^* to equip the A-corings with left B-actions, being careful with notation. Therefore, the two Galois maps associated to φ and $\overline{\varphi}$ are related via $\beta_{\overline{\varphi}} = B' \otimes_B \varphi^*(\beta_\varphi)$. Since the morphism β_φ is a quasi-isomorphism and in view of assumption (2), we can apply Lemma 1.2.28 to deduce that $\beta_{\overline{\varphi}}$ is a quasi-isomorphism, too.

Assumption (1) guarantees that the categories $\mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}}$ and $\mathcal{M}_{A'}^{W_{\rho'}}$ are equipped with the induced model structures as in Proposition 3.1.8. To conclude that the induced functor $(\beta_{\overline{\varphi}})_* : \mathcal{M}_{A'}^{W_{\overline{\varphi}}^{can}} \to \mathcal{M}_{A'}^{W_{\rho'}}$ is a Quillen equivalence, we can apply Corollary 3.2.4 to $\overline{\varphi} : B' \to A'$, since all required hypotheses for doing so have been fulfilled. □

In conclusion, the following Proposition gives the conditions under which a homotopic H-Hopf-Galois extension φ will be preserved under base change.

Proposition 3.2.7. *Let $\Bbbk$ be a field and H a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. Consider the following pushout of commutative objects in $\mathbf{Alg}_{H}^{\varepsilon}$, where B and B' have trivial H-coactions.*

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A' \end{array} \qquad (\blacksquare)$$

If $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, and

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$;*

(2) *B' is semi-free as a left B-module on a generating graded $\Bbbk$-module Z, such that Z_n is finitely generated for all $n \geq 0$,*

then $\overline{\varphi} : B' \to A'$ is also a homotopic H-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Proof. This follows from Proposition 3.2.3 and Proposition 3.2.6. □

3.3 Reflection of homotopic Hopf-Galois extensions under base change

We return to the pushout of commutative augmented H-comodule algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A'. \end{array} \qquad (\blacksquare)$$

This time, we assume that the map $\overline{\varphi}$ is a homotopic H-Hopf-Galois extension, and would like to find conditions on the initial data in the pushout ($\blacksquare$) that guarantee that the map φ was a homotopic H-Hopf-Galois extension.

We are inspired by the following result in [Rog08] on reflection of (faithful) G-Galois extensions of commutative ring spectra under base change along arbitrary maps.

Lemma 3.3.1 (Lemma 7.1.4., [Rog08])**.** *Let $\varphi : A \to C$ and $f : A \to B$ be maps of commutative S-algebras, with B a faithful and dualizable A-module,* *and let G be a stably dualizable group acting on C through A-algebra maps. Consider the following pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ C & \xrightarrow[\overline{f}]{} & B \wedge_A C. \end{array}$$

(a) If $\overline{\varphi} : B \to B \wedge_A C$ is a G-Galois extension, then $\varphi : A \to C$ is a G-Galois extension;

(b) If $\overline{\varphi} : B \to B \wedge_A C$ is a faithful G-Galois extension, then $\varphi : A \to C$ is a faithful G-Galois extension.

Just as we did in the previous section, we will split our investigation into two parts and analyze separately the behavior of the Galois functor and the behavior of the comparison functor "under reflection".

3.3.1 The behavior of the Galois functor $(\beta_\varphi)_*$

Proposition 3.3.2. *Let $\Bbbk$ be a field, and let H be a Hopf algebra satisfying Convention 2.1.12. Consider the following pushout of commutative objects in $\mathbf{Alg}_H^{\varepsilon}$, where B and B' have trivial H-coactions.*

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A'. \end{array} \qquad (\blacksquare)$$

Suppose that $\overline{\varphi} : B' \to A'$ is a homotopic H-Hopf-Galois extension. If

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$;*

(2) *B' is homologically faithful as a B-module,*

then the functor

$$(\beta_\varphi)_* : \mathcal{M}_A^{W_\varphi^{can}} \to \mathcal{M}_A^{W_\rho}$$

is a Quillen equivalence.

Proof. Since $\overline{\varphi} : B' \to A'$ is a homotopic H-Hopf-Galois extension, the functor $(\beta_{\overline{\varphi}})_*$ is a Quillen equivalence, by Definition 2.3.1. Note that assumption (1) implies that the A-corings W_φ^{can} and W_ρ are A-semi-free. In this situation, Corollary 1.5.8 tells us that showing that the functor $(\beta_\varphi)_*$ is a Quillen equivalence is equivalent to showing that the map β_φ is a quasi-isomorphism of A-corings. This is what we will do.

It also follows from assumption (1) that the A'-corings $W^{can}_{\overline{\varphi}}$ and $W_{\rho'}$ are A'-semi-free. Since $(\beta_{\overline{\varphi}})_*$ is a Quillen equivalence, Proposition 1.5.7 tells us that the map $\beta_{\overline{\varphi}}$ is a quasi-isomorphism. By definition, the Galois maps are related via $\beta_{\overline{\varphi}} = B' \otimes_B \varphi^*(\beta_\varphi)$ (where the functor $\varphi^*(-)$ permits to view β_φ as a map in $\mathbf{Mod}_B$), so assumption (2) allows us to conclude that β_φ is a quasi-isomorphism, as desired, using Lemma 1.2.28. □

3.3.2 The behavior of the comparison functor $(i_\varphi)^*$

Remark 3.3.3. Under conditions of Lemma 3.2.2, all categories of modules $\mathbf{Mod}_B$, $\mathbf{Mod}_{A^{hcoH}}$, $\mathbf{Mod}_{B'}$ and $\mathbf{Mod}_{(A')^{hcoH}}$ in adjunctions

$$\mathbf{Mod}_B \underset{(i_\varphi)^*}{\overset{-\otimes_B A^{hcoH}}{\rightleftarrows}} \mathbf{Mod}_{A^{hcoH}}$$

and

$$\mathbf{Mod}_{B'} \underset{(i_{\overline{\varphi}})^*}{\overset{-\otimes_{B'} (A')^{hcoH}}{\rightleftarrows}} \mathbf{Mod}_{(A')^{hcoH}}$$

are equipped with the right-induced model structures from $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where the weak equivalences are quasi-isomorphisms, i.e., quasi-isomorphisms of underlying chain complexes. In this situation, both comparison functors $(i_\varphi)^*$ and $(i_{\overline{\varphi}})^*$ preserve and reflect all weak equivalences. It follows that they both automatically satisfy condition (2) in Proposition 1.3.3.

We obtain the following "reflection result" for the comparison functor. Proposition 1.4.3 is crucial here.

Proposition 3.3.4. *Let $\Bbbk$ be a field and H a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 2.1.12. Consider the following pushout of commutative objects in $\mathbf{Alg}_H^{\varepsilon}$, where B and B' have trivial H-coactions.*

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\overline{f}]{} & B' \otimes_B A := A'. \end{array} \qquad (\blacksquare)$$

Suppose that $\overline{\varphi} : B' \to A'$ a homotopic H-Hopf-Galois extension. If

(1) *A is semi-free as a B-module on a generating graded $\Bbbk$-module X that is degree-wise finitely generated;*

(2) *B' is homologically faithful as a B-module,* *then the comparison functor associated to* φ,

$$(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B,$$

is a Quillen equivalence.

Proof. The adjunction $\big(-\otimes_{B'}(A')^{hcoH}, (i_{\overline{\varphi}})^*\big)$ is a Quillen pair, as explained in Lemma 3.2.2. Because $\overline{\varphi}$ is a homotopic H-Hopf-Galois extension, its associated comparison functor $(i_{\overline{\varphi}})^* : \mathbf{Mod}_{(A')^{hcoH}} \to \mathbf{Mod}_{B'}$ is a Quillen equivalence, by definition. It then follows that the map $i_{\overline{\varphi}}$ is a quasi-isomorphism, according to Proposition 1.4.4.

Since the coinvariants maps associated to φ and $\overline{\varphi}$ are related via $i_{\overline{\varphi}} = B' \otimes_B (i_\varphi)$ (as observed in Remark 3.1.3), and since B' is homologically faithful over B, Lemma 1.2.28 implies that i_φ is a quasi-isomorphism, as well.

To conclude that the functor $(i_\varphi)^*$ is a Quillen equivalence, we can apply Proposition 1.4.3. Indeed, it follows from assumption (1) that $A^{hcoH} \cong B \tilde{\otimes} X \otimes_{t_\Omega} \Omega H$ is B-semi-free. The underlying graded $\Bbbk$-module $X \otimes \Omega H$ is degree-wise finitely generated, because for all $n \geq 0$

$$(X \otimes \Omega H)_n = \bigoplus_{p+q=n} X_p \otimes (\Omega H)_q$$

has a finite number of summands (since the underlying chain complexes are bounded below), with X_p finitely generated for all $p \geq 0$ by hypothesis, and $(\Omega H)_q = T(s^{-1}H_{>0})$ finitely generated for all $q \geq 0$, because H satisfies Convention 2.1.12. □

Summarizing the results above, the following Proposition gives conditions under which a homotopic Hopf-Galois extension $\overline{\varphi}$ is reflected under base change.

Proposition 3.3.5. *Let* $\Bbbk$ *be a field and* H *a Hopf algebra in* $\mathbf{Ch}_\Bbbk^{\geq 0}$, *satisfying Convention 2.1.12. Consider the following pushout of commutative objects in* $\mathbf{Alg}_H^{\varepsilon}$, *where* B *and* B' *have trivial* H*-coactions.*

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\overline{\varphi}} \\ A & \xrightarrow[\ \overline{f}\ ]{} & B' \otimes_B A := A' \end{array} \qquad (\blacksquare)$$

Assume that $\overline{\varphi} : B' \to A'$ *is a homotopic* H*-Hopf-Galois extension. If*

(1) A *is semi-free as a left* B*-module on a generating graded* $\Bbbk$*-module* X*, such that* X_n *is finitely generated for all* $n \geq 0$;

(2) B' *is homologically faithful as a* B*-module,*

then $\varphi : B \to A$ *is also a homotopic* H*-Hopf-Galois extension in* $\mathbf{Ch}_\Bbbk^{\geq 0}$.

Proof. This follows from Propositions 3.3.2 and 3.3.4. □

Chapter 4

One direction of the homotopic Hopf-Galois correspondence

4.1 Generalized situation

The notion of a homotopic Hopf-Galois extension can be seen from a slightly more general perspective, for which we need the following setup. This generalized setup will be important in our Main Theorem (Theorem 4.3.6).

Let $\gamma : H' \to H$ be a morphism of dg-$\Bbbk$-Hopf algebras. Let $B \in \mathbf{Alg}_{H'}$, $A \in \mathbf{Alg}_H$, and let $\varphi : B \to A$ be a morphism of dg-$\Bbbk$-algebras, such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow{\scriptstyle \rho_B} & & \downarrow{\scriptstyle \rho_A} \\ B \otimes H' & \xrightarrow{\varphi \otimes \gamma} & A \otimes H \end{array}$$

commutes in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Remark 4.1.1. The morphism φ underlying a homotopic H-Hopf-Galois extension $\varphi : \mathrm{Triv}(B) \to A$ in the sense of Definition 2.3.1, can be seen as part of the general setup above, in the special case where $H' := \Bbbk[0]$ and $(\gamma : H' \to H) = (\eta_H : \Bbbk[0] \to H)$.

The morphisms $\varphi : B \to A$ and $\gamma : H' \to H$ induce a morphism of dg-algebras $\varphi^{co\gamma} : B^{coH'} \to A^{coH}$ between the objects of coinvariants, which arises from the diagram

$$\begin{array}{ccccc} B^{coH'} & \hookrightarrow & B & \overset{\rho_B}{\underset{B \otimes \eta_{H'}}{\rightrightarrows}} & B \otimes H' \\ {\scriptstyle \varphi^{co\gamma}} \vdots {\scriptstyle \exists !} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \otimes \gamma} \\ A^{coH} & \hookrightarrow & A & \overset{\rho_A}{\underset{A \otimes \eta_H}{\rightrightarrows}} & A \otimes H. \end{array}$$

On the other hand, to identify the map $\varphi^{hco\gamma} : B^{hcoH'} \to A^{hcoH}$ induced by $\varphi : B \to A$ and $\gamma : H' \to H$ on the objects of *homotopy coinvariants* can be more difficult, in general. However, if one works under conditions allowing the use of the particular models $A^{hcoH} \cong \Omega(A; H; \Bbbk)$ and $B^{hcoH'} \cong \Omega(B; H'; \Bbbk)$ for the homotopy coinvariants (see Section 2.1.3), then the map $\varphi^{hco\gamma}$ is given precisely by

$$\Omega(\varphi; \gamma; \Bbbk) : \Omega(B; H'; \Bbbk) \to \Omega(A; H; \Bbbk).$$

Remark 4.1.2. Observe that in the special case of Remark 4.1.1, the map

$$\varphi^{hco\eta_H} : B^{hco\,\Bbbk[0]} \cong B \longrightarrow A^{hcoH}$$

is the comparison map i_φ, associated to the homotopic H-Hopf-Galois extension $\varphi : \mathrm{Triv}(B) \to A$.

4.2 A brief reminder of Galois correspondence for fields

We follow the notation and terminology of Section 2.4.1.

Let $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ be a finite field extension, and let $G := \mathrm{Gal}(\mathbb{E}; \Bbbk)$ denote its Galois group. For all subgroups $N \leq G$, one defines the object of fixed points in $\mathbb{E}$ by

$$\mathbb{E}^N := \{e \in \mathbb{E} : n(e) = e, \text{ for all } n \in N\},$$

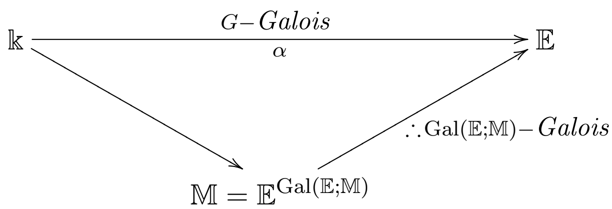
called **the fixed field of** N. It is easy to show that $\mathbb{E}^N$ is a subfield of $\mathbb{E}$ (see [Cox04], Section 7.1).

The following result is known as the Fundamental Theorem of the Galois Theory. It recovers the classical Galois correspondence in the case of finite fields.

Theorem 4.2.1 (Theorems 7.3.1 and 7.3.2, [Cox04])**.** *Let $\alpha : \Bbbk \hookrightarrow \mathbb{E}$ be a finite Galois extension of fields and denote by $G := \mathrm{Gal}(\mathbb{E}; \Bbbk)$ its associated Galois group.*

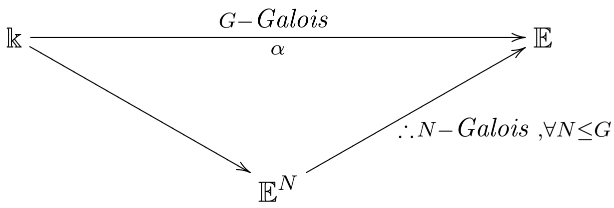
(a) *For any sub-extension $\Bbbk \subset \mathbb{M} \subset \mathbb{E}$, we have*

$$\mathrm{Gal}(\mathbb{E}; \mathbb{M}) \leq G \qquad \textit{and} \qquad \mathbb{E}^{\mathrm{Gal}(\mathbb{E};\mathbb{M})} = \mathbb{M}\,.$$



(b) *For any subgroup $N \leq G$, we have*

$$\Bbbk \subseteq \mathbb{E}^N \subseteq \mathbb{E} \qquad \textit{and} \qquad \mathrm{Gal}(\mathbb{E}; \mathbb{E}^N) = N.$$



(c) *The correspondence given by*

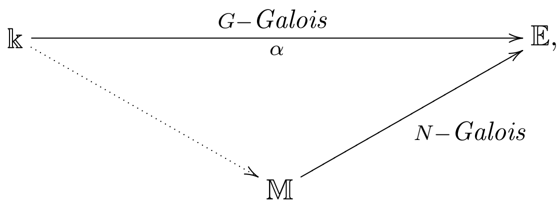
$$\{\textit{fields } \mathbb{M} : \Bbbk \subseteq \mathbb{M} \subseteq \mathbb{E}\} \longleftrightarrow \{\textit{subgroups } N \leq G\}$$

$$\mathbb{M} \longmapsto \mathrm{Gal}(\mathbb{E}; \mathbb{M})$$

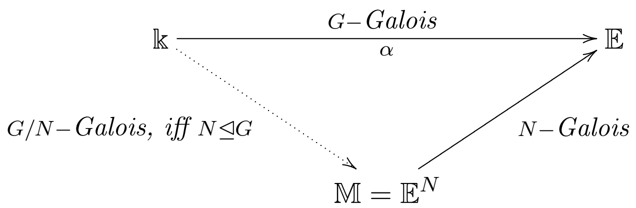
$$\mathbb{E}^N \longleftarrow\!\shortmid N$$

is a bijection of sets. Both maps reverse the order of inclusions.

(d) *Suppose we are given a diagram*



where the field $\mathbb{M}$ *and the group* N *are related via* $\mathrm{Gal}(\mathbb{E}; \mathbb{M}) = N$ *(or, equivalently, via* $\mathbb{M} := \mathbb{E}^N$*, by (c)). The field extension* $\Bbbk \subseteq \mathbb{M}$ *is then a Galois extension if and only if* N *is a normal subgroup of* G*. In this case,* $\mathrm{Gal}(\mathbb{M}; \Bbbk) = G/N$*.*



$(*)$

Terminology 4.2.2. Sometimes authors speak about the "forward" and the "backward" parts of the Galois correspondence. According to our choice of notation in (c) of Theorem 4.2.1 (i.e., the set of field extensions is on the left, the set of subgroups is on the right), the directions are interpreted as follows.

Given a Galois extension $\Bbbk \hookrightarrow \mathbb{E}$ and an intermediate sub-field $\mathbb{M}$, the "forward" part consists of identifying the sub-group of the Galois group $\mathrm{Gal}(\mathbb{E};\Bbbk)$ that corresponds to $\mathbb{M}$ under (c) of Theorem 4.2.1.

Reciprocally, given a Galois extension $\Bbbk \hookrightarrow \mathbb{E}$ and a sub-group N of the Galois group $\mathrm{Gal}(\mathbb{E};\Bbbk)$, the "backward" part consists of identifying the sub-field extension between $\Bbbk$ and $\mathbb{E}$ that corresponds to H under (c) of Theorem 4.2.1.

Warning: the choice of notation for these directions in [Rog08] is opposite!

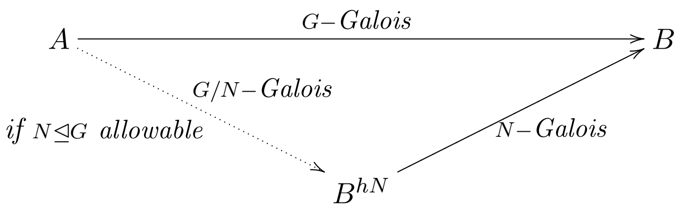
4.3 One direction of homotopic Hopf-Galois correspondence

Remark 4.3.1. In this Section, we will sometimes omit the word "homotopic" when talking about homotopic Hopf-Galois extensions, especially in diagrams. Without doubt, we will always mean "homotopic" implicitly, and we kindly ask the reader to keep this in mind.

Here is the backward part of the Galois correspondence for E-local commutative S-algebras, established in [Rog08] by John Rognes. The object B^{hG} denotes the homotopy fixed points of a spectrum B under the action of a stably dualizable group G.

Theorem 4.3.2 (Theorem 7.2.3, [Rog08])**.** *Let $A \to B$ be a faithful G-Galois extension and $N \subset G$ any allowable subgroup. Then $B^{hN} \to B$ is a faithful N-Galois extension.*

If furthermore $N \subset G$ is an allowable normal subgroup, then $A \to B^{hN}$ is a faithful G/N-Galois extension.



Theorem 4.3.2 inspired us to investigate and provide an answer to the question: how could one formulate the backward part of a homotopic Hopf-Galois correspondence problem, and what are the conditions under which this problem be solved?

4.3.1 The setting

Let $\Bbbk$ be a field and $g : H \to K$ a map of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Consider a homotopic H-Hopf-Galois extension $\varphi : \mathrm{Triv}(B) \to A$ and recall from

Section 2.2 that the diagram of augmented H-comodule algebras

$$\begin{array}{ccc} B & \xrightarrow{\ \varphi\ } & A \\ {\scriptstyle i_\varphi}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\ A^{hcoH} & \hookrightarrow & RA \end{array}$$

commutes. Here, the comparison map $i_\varphi : B \to A^{hcoH}$ is a quasi-isomorphism, by Proposition 1.4.4, because the functor $(i_\varphi)^* : \mathbf{Mod}_{A^{hcoH}} \to \mathbf{Mod}_B$ is a Quillen equivalence, by definition of a homotopic Hopf-Galois extension.

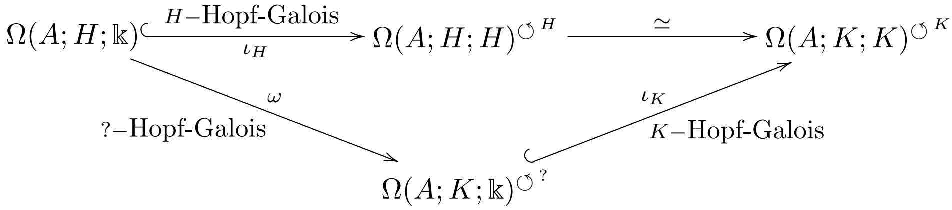
Choosing explicit models for A^{hcoH} and RA, given respectively by two-sided cobar constructions $\Omega(A; H; \Bbbk)$ and $\Omega(A; H; H)$ (see Section 2.1.3), and adding to this diagram the normal basis extension ι_K associated to the Hopf algebra K (see Example 2.3.3), we obtain the following commutative diagram.

$$\begin{array}{ccccc} B & \xrightarrow{\qquad\varphi\qquad} & A & \searrow^{\simeq} & \\ {\scriptstyle i_\varphi}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} & & \\ \Omega(A;H;\Bbbk) & \xhookrightarrow{\ \iota_H\ } & \Omega(A;H;H) & \xrightarrow[\therefore \simeq]{\Omega(A;g;g)} & \Omega(A;K;K) \quad (**) \\ & {\scriptstyle \Omega(A;g;R) = \omega}\searrow & & \nearrow{\scriptstyle \iota_K} & \\ & & \Omega(A;K;\Bbbk) & & \end{array}$$

Let us make a few comments.

- Recall from Section 2.1.3 that $\Omega(A; H; \Bbbk)$, $\Omega(A; H; H)$, $\Omega(A; K; \Bbbk)$ and $\Omega(A; K; K)$ can all be endowed with a structure of a dga.
- We consider this diagram in the category $\mathbf{Alg}_K^\varepsilon$ of augmented K-comodule algebras, using the map $g : H \to K$ to endow each of the augmented H-comodule algebras A, B, $\Omega(A; H; \Bbbk)$ and $\Omega(A; H; H)$ with K-coactions, by post-compostion with g.
- The map $\Omega(A; g; g)$, induced by g on the two-sided cobar constructions, is a weak equivalence, by 2-out-of-3 property.
- Homotopically speaking, this diagram indicates that, up to homotopy, studying the homotopic H-Hopf-Galois extension φ is the same as studying the normal basis extension ι_H.

The last point above allows us to reformulate the homotopic Hopf-Galois correspondence problem. Recall Notation 2.4.7 and consider the lower sub-diagram of $(**)$ in $\mathbf{Alg}_K^{\varepsilon}$.



It was shown in Example 2.3.3 that the normal basis extensions ι_H and ι_K are homotopic H- and K-Hopf-Galois, respectively. The goal of the subsequent sections is to find the appropriate candidate for the Hopf algebra ? and to formulate conditions on the Hopf algebras H, K, and on the algebra A, under which the map

$$\omega : \Omega(A;H;\Bbbk) \to \Omega(A;K;\Bbbk)^{\circlearrowleft ?}$$

will be a homotopic ?-Hopf-Galois extension.

4.3.2 The candidate Hopf algebra and the Main Theorem

Convention 4.3.3. From now on, we suppose that both H and K are co-commutative, 1-connected, degree-wise finitely generated Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Moreover, we assume that $K_2 = 0$.

We remind the reader that the co-commutativity assumption on H and K guarantees that their respective cobar constructions ΩH and ΩK both inherit (co-commutative) comultiplications (see Remark 1.2.10).

Let $g : H \to K$ be a morphism of Hopf algebras. It endows H with a right K-comodule structure

$$\rho_{H,K} : H \xrightarrow{\Delta_H} H \otimes H \xrightarrow{H \otimes g} H \otimes K,$$

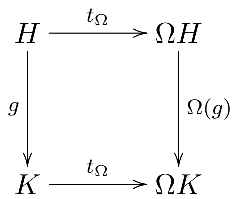
and therefore makes it meaningful to consider the one-sided cobar construction $\Omega(H;K;\Bbbk)$ on $(H, \rho_{H,K}) \in \mathbf{Comod}_K$. We use Remark 2.1.8 to see that

$$\Omega(H;K;\Bbbk) := (H \otimes \Omega K, D_{t_\Omega}) \in \mathbf{Ch}_{\Bbbk}^{\geq 0}.$$

The differential D_{t_Ω} is given by

$$D_{t_\Omega} = d_H \otimes \Omega K + H \otimes D_{\Omega K} + \big((H \otimes \mu_{\Omega K}) \circ (H \otimes t_\Omega \otimes \Omega K) \circ (\rho_{H,K} \otimes \Omega K)\big).$$

Remark 4.3.4. Observe that the diagram



commutes, by definition of the maps t_Ω and $\Omega(g)$. Therefore, there exists another way of writing the differential $D_{t_\Omega} : H \otimes \Omega K \to H \otimes \Omega K$ that is

$$D_{t_\Omega} = d_H \otimes \Omega K + H \otimes D_{\Omega K} \\ + \big((H \otimes \mu_{\Omega K}) \circ (H \otimes \Omega(g) \otimes \Omega K) \circ (H \otimes t_\Omega \otimes \Omega K) \circ (\Delta_H \otimes \Omega K)\big),$$

where

$$\Omega(g) \circ t_\Omega : H \xrightarrow{t_\Omega} \Omega H \xrightarrow{\Omega(g)} \Omega K$$

is the new twisting morphism.

At this point, we know from Chapter 2 only that $\Omega(H; K; \Bbbk)$ is a chain complex. Let us see how it can be endowed with a Hopf algebra structure.

Lemma 4.3.5. *If H and K satisfy Convention 4.3.3, the one-sided cobar construction $\Omega(H; K; \Bbbk)$ is a Hopf algebra.*

Proof. The multiplication

$$\mu_{H \otimes_{t_\Omega} \Omega K} : H \otimes_{t_\Omega} \Omega K \otimes H \otimes_{t_\Omega} \Omega K \to H \otimes_{t_\Omega} \Omega K$$

is the twisted multiplication, defined in Corollary 3.6 in [HL07] and explained in Remark 2.1.13.

At the level of the underlying graded modules, we define morphisms

$$\eta_{H \otimes_{t_\Omega} \Omega K} : \Bbbk[0] \cong \Bbbk[0] \otimes \Bbbk[0] \xrightarrow{\eta_H \otimes \eta_{\Omega K}} H \otimes \Omega K,$$

$$\Delta_{H \otimes_{t_\Omega} \Omega K} : H \otimes \Omega K \xrightarrow{\Delta_H \otimes \Delta_{\Omega K}} H \otimes H \otimes \Omega K \otimes \Omega K \xrightarrow{\cong} (H \otimes \Omega K) \otimes (H \otimes \Omega K),$$

$$\varepsilon_{H \otimes_{t_\Omega} \Omega K} : H \otimes \Omega K \xrightarrow{\varepsilon_H \otimes \varepsilon_{\Omega K}} \Bbbk[0] \otimes \Bbbk[0] \cong \Bbbk[0],$$

(see Reminder 1.2.1, Definition 1.2.2 and Remark 1.2.10 for the expressions of $\eta_{\Omega K}$, $\Delta_{\Omega K}$ and $\varepsilon_{\Omega K}$). We observe that

(1) the associativity of $\mu_{H \otimes_{t_\Omega} \Omega K}$ follows from the fact that the multiplication $\mu_{\Omega K}$ is associative, and the definition of $\mu_{H \otimes_{t_\Omega} \Omega K}$ was designed in order to make it satisfy the Leibniz rule with respect to $D_{t_{\Omega_K}}$ (see Remark 2.1.13 and Corollary 3.6 in [HL07] for details).

(2) an easy calculation shows that $\mu_{H \otimes_{t_\Omega} \Omega K}$ is unital with respect to $\eta_{H \otimes_{t_\Omega} \Omega K}$;

(3) the comultiplication $\Delta_{H \otimes_{t_\Omega} \Omega K}$ is co-associative, because both Δ_H and $\Delta_{\Omega K}$ are coassociative, and it is counital with respect to ε, because Δ_H and $\Delta_{\Omega K}$ are counital regarding ε_H and $\varepsilon_{\Omega K}$, respectively;

(4) the comultiplication $\Delta_{H \otimes_{t_\Omega} \Omega K}$ satisfies the "dual Leibniz" rule with respect to D_{t_Ω}, because both (H, d_H, Δ_H) and $(\Omega K, D_{\Omega K}, \Delta_{\Omega K})$ are differential coalgebras, and Δ_H is coassociative and co-commutative;

(5) to see that $\eta_{H\otimes_{t_\Omega}\Omega K}$ is a morphism of comonoids (i.e., is appropriately compatible with $\Delta_{H\otimes_{t_\Omega}\Omega K}$ and $\varepsilon_{H\otimes_{t_\Omega}\Omega K}$) one uses the compatibility between η_H and Δ_H, ε_H on the one hand, and the compatibility between $\eta_{\Omega K}$ and $\Delta_{\Omega K}$, $\varepsilon_{\Omega K}$ on the other hand;

(6) an easy calculation shows that $\mu_{H\otimes_{t_\Omega}\Omega K}$ is counital with respect to $\varepsilon_{H\otimes_{t_\Omega}\Omega K}$;

(7) $\mu_{H\otimes_{t_\Omega}\Omega K}$ is compatible with $\Delta_{H\otimes_{t_\Omega}\Omega K}$ because Δ_H is coassociative and co-commutative;

(8) $\varepsilon_{H\otimes_{t_\Omega}\Omega K}\circ\eta_{H\otimes_{t_\Omega}\Omega K} = \mathrm{Id}_{\Bbbk[0]}$, because $\varepsilon_H\circ\eta_H = \mathrm{Id}_{\Bbbk[0]}$, $\varepsilon_{\Omega K}\circ\eta_{\Omega K} = \mathrm{Id}_{\Bbbk[0]}$ and by bifonctoriality of $-\otimes-$.

To summarize, (1) and (2) imply that $\big(H\otimes\Omega K, D_{t_\Omega}, \mu_{H\otimes_{t_\Omega}\Omega K}, \eta_{H\otimes_{t_\Omega}\Omega K}\big)$ is a monoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$; (3) and (4) imply that $\big(H\otimes\Omega K, D_{t_\Omega}, \Delta_{H\otimes_{t_\Omega}\Omega K}, \varepsilon_{H\otimes_{t_\Omega}\Omega K}\big)$ is a comonoid in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$; finally, having (5), (6), (7) and (8) allow us to conclude that

$$\Omega(H;K;\Bbbk) = \big(H\otimes\Omega K, D_{t_\Omega}, \mu_{H\otimes_{t_\Omega}\Omega K}, \eta_{H\otimes_{t_\Omega}\Omega K}, \Delta_{H\otimes_{t_\Omega}\Omega K}, \varepsilon_{H\otimes_{t_\Omega}\Omega K}\big)$$

is a Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, as desired. □

We now state our Main Theorem, which establishes one direction of homotopic Hopf-Galois correspondence for homotopic Hopf-Galois extensions of chain complexes. This theorem will be proved in Section 4.3.4

Theorem 4.3.6. *Let $\Bbbk$ be a field and $g : H \to K$ a morphism of co-commutative, 1-connected, degree-wise finitely generated Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, such that $K_2 = 0$. Let $\varphi : B \to A$ be a homotopic H-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ and consider the following diagram*

$$\begin{array}{ccccc} \Omega(A;H;\Bbbk) & \xhookrightarrow{\ \iota_H\ } & \Omega(A;H;H)^{\circlearrowleft H} & \xrightarrow{\ \simeq\ } & \Omega(A;K;K)^{\circlearrowleft K}, \\ & \searrow^{\omega} & & \nearrow_{\iota_K} & \\ & & \Omega(A;K;\Bbbk)^{\circlearrowleft \Omega(H;K;\Bbbk)} & & \end{array}$$

where ι_H and ι_K denote the normal basis homotopic Hopf-Galois extensions, associated to Hopf algebras H and K, respectively. If

(1) *A is semi-free as a left B-module on a generating graded $\Bbbk$-module X, such that X_n is finitely generated for all $n \geq 0$; and*

(2) *$g : (H, \Delta_H, d_H) \hookrightarrow (K, \Delta_K, d_K)$ is an inclusion of differential graded $\Bbbk$-coalgebras,* *then the map*

$$\omega : \Omega(A; H; \Bbbk) \longrightarrow \Omega(A; K; \Bbbk)$$

is a generalized homotopic $\Omega(H; K; \Bbbk)$*-Hopf-Galois extension in* $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Remark 4.3.7. We use the terminology "generalized" to indicate that the map

$$\omega : \Omega(A; H; \Bbbk) \longrightarrow \Omega(A; K; \Bbbk)$$

has the special property that its codomain $\Omega(A; H; \Bbbk)$ is equipped with a trivial $\Bbbk[0]$-coaction, **up to homotopy**. This follows from putting the map of dg-algebras $\omega : \Omega(A; H; \Bbbk) \to \Omega(A; K; \Bbbk)$ into the general perspective, described in Section 4.1.

More specifically, the morphism of dg-Hopf algebras $g : H \to K$ induces a morphism of dg-Hopf algebras $\Omega(H; g; \Bbbk) : \Omega(H; H; \Bbbk) \to \Omega(H; K; \Bbbk)$ (see Lemma 4.3.5, which also holds for $K = H$), and ω fits into the commutative diagram

$$\begin{array}{ccc}
\Omega(A;H;\Bbbk) & \xrightarrow{\omega=\Omega(A;g;\Bbbk)} & \Omega(A;K;\Bbbk) \\
\downarrow \rho & & \downarrow \rho_\omega \\
\Omega(A;H;\Bbbk)\otimes\Omega(H;H;\Bbbk) & \xrightarrow{\Omega(A;g;\Bbbk)\otimes\Omega(H;g;\Bbbk)} & \Omega(A;K;\Bbbk)\otimes\Omega(H;K;\Bbbk). \\
\downarrow \simeq & & \\
\Omega(A;H;\Bbbk)\otimes\Bbbk[0] & &
\end{array}$$

The coaction maps $\rho : \Omega(A; H; \Bbbk) \to \Omega(A; H; \Bbbk) \otimes \Omega(H; H; \Bbbk)$ and ρ_ω : $\Omega(A; K; \Bbbk) \to \Omega(A; K; \Bbbk) \otimes \Omega(H; K; \Bbbk)$ will be defined in Lemma 4.3.10.

The comparison map induced by ω and $\Omega(H; g; \Bbbk)$ between the objects of homotopy coinvariants is denoted by

$$\omega^{hco\Omega(H;g;\Bbbk)} : \Omega(A; H; \Bbbk)^{hco(\Omega(H;H;\Bbbk))} \to \Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))}$$

and will be defined in Section 4.3.4.

We need a series of preliminary results before we can prove Theorem 4.3.6.

4.3.3 Technical preliminaries

Recall from diagram $(*)$ and Theorem 4.2.1 that in the Galois case, the sub-extension of fields $\Bbbk \to \mathbb{E}^N$ is G/N-Galois if N is a normal subgroup of G. So, the meaningful algebraic object, associated to $\Bbbk \to \mathbb{E}^N$ is the group quotient G/N.

Why is $\Omega(H; K; \Bbbk)$ an appropriate object to consider in the situation of diagram $(**)$? I.e., why can the Hopf algebra $\Omega(H; K; \Bbbk)$ be seen as the right analog of the quotient group G/N from the dual Galois context?

Our situation is not only dual to (∗), but it also needs to be homotopically coherent.

In the case of a group G acting on a set X, a good model for the *homotopy quotient* of the action of G on X, denoted X_{hG}, and also called the *homotopy orbit space*, is given by the Borel construction. It can be modelled by the coequalizer

$$X_{hG} := \text{coequal}\left(X \times G \times EG \underset{X \times \lambda}{\overset{\gamma \times EG}{\rightrightarrows}} X \times EG\right),$$

Here EG is a contractible space with free G-action λ and is cofibrant (because it can be built as a colimit of CW-complexes), and $\gamma : X \times G \to X$ is the G-action on X.

We are interested in the dual situation. We have already observed at the beginning of this Section that a morphism of Hopf algebras $g : H \to K$ endows H with a K-coaction $\rho_{H,K} : H \to H \otimes K$. In this situation, the object of *homotopy K-coinvariants* of H, which can also be seen as *homotopy fixed points* of the K-coaction on H, is given by the equalizer

$$H^{hcoK} := \text{equal}\left(RH \otimes \Bbbk[0] \underset{RH \otimes \sigma}{\overset{\rho \otimes \Bbbk[0]}{\rightrightarrows}} RH \otimes K \otimes \Bbbk[0]\right),$$

where RH denotes a fibrant replacement of H. It followed from Lemma 2.1.7 and Definition 2.1.11 that a good, homotopically meaningful model for H^{hcoK} is precisely given by the one-sided cobar construction $\Omega(H; K; R)$.

Remark 4.3.8. In the discussion above the crucial relation between the groups N and G is codified via "normality". Given a subgroup inclusion $N \hookrightarrow G$, the quotient G/N has a group structure (and thus, makes it meaningful to talk about G/N-Galois extensions), if and only if N is a normal subgroup of G.

In [FH12], Emmanuel Farjoun and Kathryn Hess defined and studied homotopy-invariant and dual notions of *normality* for maps of monoids, and of *conormality* for maps of comonoids within a twisted homotopical category $\mathcal{M}$.

In particular, if a map of comonoids $g : C' \to C$ is *(homotopy) conormal* (see Definition 2.4, [FH12]), then the *Borel kernel* of g, $C \backslash\backslash C'$, which models the homotopy kernel of g (see Definition B.16, [FH12]), is weakly equivalent to a comonoid. This codifies the desired situation in a context, dual to the context of groups. (Moreover, and reciprocally, they also provided conditions under which a comonoid structure on $C \backslash\backslash C'$ implies conormality of g, see Lemma 2.8, [FH12]).

We emphasize that in the case $\mathcal{M} := \mathbf{Ch}_{\Bbbk}^{\geq 0}$, the co-commutativity of H and K as coalgebras turns out to be crucial for establishing the conormality of the map $g : H \to K$ (see Proposition 2.20 [FH12]).

Remark 4.3.9. Let $g : H \to K$ be a map of co-commutative Hopf algebras, satisfying Convention 4.3.3. Recall that in Lemma 4.3.5 we defined a Hopf algebra structure on $\Omega(H; K; \Bbbk)$.

In view of our choice of models for fibrant replacements in $\mathbf{Alg}^{\varepsilon}_{K}$, discussed in Chapter 2, the Borel kernel of g is given exactly by $K \setminus\setminus H = \Omega(H; K; \Bbbk)$. So, one could apply Proposition 2.20 in [FH12] and conclude directly that g is conormal, and that the comultiplication $\Delta_{H \otimes_{t_\Omega} \Omega K}$, defined in Lemma 4.3.5, endows $\Omega(H; K; \Bbbk)$ with a coassociative, counital coalgebra structure. However, this argument does not suffice to prove that $\Delta_{H \otimes_{t_\Omega} \Omega K}$ is appropriately compatible with the multiplication $\mu_{H \otimes_{t_\Omega} \Omega K}$ to give a Hopf algebra structure on $\Omega(H; K; \Bbbk)$. The corresponding verifications were done in Lemma 4.3.5.

The next lemma shows that the algebra $\Omega(A; K; \Bbbk)$ has an $\Omega(H; K; \Bbbk)$-comodule structure.

Lemma 4.3.10. *If H and K satisfy Convention 4.3.3, then the differential algebra $\Omega(A; K; \Bbbk)$ admits a $\Omega(H; K; \Bbbk)$-coaction*

$$\rho_\omega : \Omega(A; K; \Bbbk) \to \Omega(A; K; \Bbbk) \otimes \Omega(H; K; \Bbbk).$$

This coaction is compatible with the algebra structure and therefore makes $\Omega(A; K; \Bbbk)$ into an $\Omega(H; K; \Bbbk)$-comodule algebra.

Proof. The coaction $\rho_\omega : \Omega(A; K; \Bbbk) \to \Omega(A; K; \Bbbk) \otimes \Omega(H; K; \Bbbk)$ is given on the underlying graded modules by the composite

$$\rho_\omega : A \otimes \Omega K \xrightarrow{\rho_A \otimes \Delta_{\Omega K}} A \otimes H \otimes \Omega K \otimes \Omega K \xrightarrow{A \otimes tw \otimes \Omega K} A \otimes \Omega K \otimes H \otimes \Omega K,$$

where tw is the twisting morphism. One checks by calculation that ρ_ω is

- coassociative with respect to Δ (because ρ_A and $\Delta_{\Omega K}$ are both coassociative and because Δ_K is co-commutative);
- counital with respect to ε (because ρ_A and $\Delta_{\Omega K}$ are both counital);
- compatible with the unit $\eta_{A \otimes \Omega K}$ (because both ρ_A and $\Delta_{\Omega K}$ are morphisms of algebras, so, in particular, they are compatible with respective units);
- compatible with the twisted multiplication $\mu_{A \otimes_{t_\Omega} \Omega K}$, defined in Remark 2.1.13. To see this, it is sufficient to check the compatibility of ρ_ω with the partial multiplication $(1 \otimes s^{-1}k)(a \otimes 1)$, for all $k \in K$, $a \in A$. This works, because ρ_A and $\Delta_{\Omega K}$ are both coassociative and because g is a morphism of Hopf algebras;
- compatible with respect to the differential $D_{t_\Omega} : A \otimes \Omega K \to A \otimes \Omega K$. This is a consequence of the fact that ρ_A and $\Delta_{\Omega K}$ are compatible with the differentials d_A and $D_{\Omega K}$, respectively.

Consequently, $\big(\Omega(A;K;\Bbbk),\rho_\omega\big)$ is a $\Omega(H;K;\Bbbk)$-comodule algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. $\square$

Most of the subsequent results have to do with the semi-freeness of certain objects. This will be an important factor in the proof of the main theorem.

Recall the definition of a semi-free module from Definition 1.2.13. The following proposition, of a general flavor, will be extremely useful for detecting semi-freeness of various cobar constructions.

Proposition 4.3.11. *Let R be a commutative ring and let V,W be graded R-modules, that are degree-wise R-free and finitely generated, and such that $V_0 = W_0 = 0$. Equip the free algebras TV and $T(V \oplus W)$ with differentials d and d', respectively, such that $d'|_{TV} = d$ and such that the extension of differential graded algebras $(TV, d) \hookrightarrow (T(V \oplus W), d')$ is a free extension of TV. Then*

(a) *$(T(V \oplus W), d')$ is semi-free as a left (TV, d)-module; and*

(b) *$(T(V \oplus W), d')$ is cofibrant in the category ${}_{TV}\mathbf{Mod}$, equipped with the model structure right-induced from the projective model structure on $\mathbf{Ch}_R^{\geq 0}$, as in Theorem 1.3.5.*

Proof. **Proof of (a):** For all words $b \in (T(V \oplus W), d')$ denote by

$$\ell_w(b) = \text{ number of } w \in W \text{ in the word } b.$$

Define an increasing filtration on $(T(V \oplus W), d')$ by "word-length in W", setting

$$\mathcal{F}_0(T(V \oplus W)) = TV$$

and

$$\mathcal{F}_k(T(V \oplus W)) = \{b \in T(V \oplus W) : \ell_k(b) \leq k\}$$

for all $k \geq 1$. Every $\mathcal{F}_k(T(V \oplus W))$ is a left TV-submodule of $T(V \oplus W)$, the action being given by multiplication in TV. Moreover, $\bigoplus_{k\geq 0} \mathcal{F}_k(T(V \oplus W)) = T(V \oplus W)$, as TV-modules. Observe that

$$\begin{aligned}
\mathcal{F}_{k+1}(T(V \oplus W)) &:= \{b \in T(V \oplus W) : \ell_k(b) \leq k+1\} \\
&= \{b \in T(V \oplus W) : \ell_k(b) \leq k\} \\
&\qquad \oplus \{b \in T(V \oplus W) : \ell_k(b) = k+1\} \\
&= \mathcal{F}_k(T(V \oplus W)) \\
&\qquad \oplus \{a_0 w_1 a_1 \cdots w_{k+1} a_{k+1} : a_i \in TV, w_i \in W\} \\
&= \mathcal{F}_k(T(V \oplus W)) \\
&\qquad \oplus TV \otimes \underbrace{\{w_1 a_1 \cdots w_{k+1} a_{k+1} : w_i \in W, a_i \in TV\}}_{=:Z(k+1)} \\
&= \mathcal{F}_k(T(V \oplus W)) \oplus TV \otimes Z(k+1),
\end{aligned}$$

where $Z(k+1)$ is degree-wise R-free for all $k \geq 0$, because V and W are degree-wise R-free by assumption. To be able to conclude that $T(V \oplus W)$ is semi-free as a left TV-module on $Z = \bigoplus_{k \geq 0} Z(k)$, we should check that the differential $d' : T(V \oplus W) \to T(V \oplus W)$ respects the filtration, i.e., that $d' : Z(k+1) \to \mathcal{F}_k(T(V \oplus W))$, for all $k \geq 0$. This verification is quite technical, and we distinguish several cases.

Given $b = w_1 a_1 \cdots w_{k+1} a_{k+1} \in Z(k+1)$, we have $\ell_w(b) = k+1$ and

$$\begin{aligned} d'(b) &= \sum_{i=1}^{k+1} \pm (w_1 a_1 \cdots w_i d(a_i) \cdots w_{k+1} a_{k+1}) \\ &\qquad + \sum_{i=1}^{k+1} \pm (w_1 a_1 \cdots d'(w_i) a_i \cdots w_{k+1} a_{k+1}), \end{aligned}$$

where the signs are given by the Koszul sign rule.

Case 1: $d'(w) \in TV$, for all $w \in W$.

- If $d(a) = 0$, for all $a \in TV$, then

$$d'(b) = \sum_{i=1}^{k+1} \pm (w_1 a_1 \cdots \underbrace{d'(w_i) a_i}_{\in TV} \cdots w_{k+1} a_{k+1}),$$

  thus $\ell_w(d'(b)) = k$ and $d'(b) \in \mathcal{F}_k(T(V \oplus W))$.

- Suppose now that $d'|_{TV} = d \neq 0$.

  Fix $k \geq 0$, and filter $Z(k+1)$ by degree of elements in TV by setting

$$F_m Z(k+1) = \{w_1 a_1 \cdots w_{k+1} a_{k+1} \in Z_{k+1} : \sum_{i=1}^{k+1} \deg(a_i) \leq m\},$$

  for all $m \geq 0$. Clearly, this filtration satisfies $F_m Z_{k+1} \subset F_{m+1} Z_{k+1}$ and $\bigoplus_{m \geq 0} F_m Z_{k+1} = Z_{k+1}$.

  There exists an $m \geq 0$ such that $b \in F_{m+1} Z(k+1)$ and we have

$$\begin{aligned} d'(b) &= \sum_{i=1}^{k+1} \pm (\underbrace{w_1 a_1 \cdots w_i d(a_i) \cdots w_{k+1} a_{k+1}}_{\in F_m Z(k+1)}) \\ &\qquad + \sum_{i=1}^{k+1} \underbrace{\pm (w_1 a_1 \cdots \underbrace{d'(w_i)}_{\in TV} a_i \cdots w_{k+1} a_{k+1}),}_{\ell_w = k} \end{aligned}$$

  so $d'(F_{m+1} Z(k+1)) \subset \mathcal{F}_k(T(V \oplus W)) \oplus TV \otimes F_m Z(k+1)$.

In other words, for any fixed $k \geq 0$, the quotient

$$\mathcal{F}_{k+1}(T(V \oplus W))/\mathcal{F}_k(T(V \oplus W))$$

is semi-free as a left TV-module on the free graded R-module $Q = \bigoplus_{m \geq 0} Q(m)$, where

$$Q(m) = \{a_1 w_1 \cdots a_{k+1} w_{k+1} \in Z_{k+1} : \sum_{i=1}^{k+1} \deg(a_i) = m\}$$

is degree-wise free on R for all $m \geq 0$. We can therefore apply Lemma 1.2.15 to conclude that $T(V \oplus W)$ is itself semi-free as a left TV-module.

Case 2: $d'(W) \nsubseteq TV$, i.e., the image of any $w \in W$ by d' might have factors in both TV and TW.

Write $W = W_{\geq N_{\min}}$, where $N_{\min} \geq 0$ is such that $W_n = 0$ for all $n < N_{\min}$. Set

$$T(V \oplus W_{\leq(N_{\min}+l)}) := \{w_\alpha a_\alpha w_\beta a_\beta \cdots : \deg(w_i) \leq (N_{\min} + l)\},$$

for all $l \geq 0$, with $TV = T(V \oplus W_{\leq(N_{\min}+0)})$.

For every $l \geq 0$, the inclusion

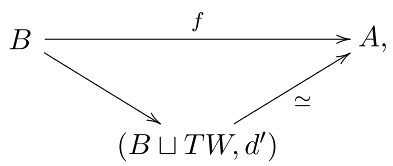
$$(T(V \oplus W_{\leq(N_{\min}+l)}), d') \hookrightarrow (T(V \oplus W_{\leq(N_{\min}+l+1)}), d')$$

is an extension of dg-algebras, where the differential d' sends a $w \in W_{\leq(N_{\min}+l+1)}$ to $d'(w) \in T(V \oplus W_{\leq(N_{\min}+l)})$, and, therefore, satisfies the situation of **Case 1**. To show that this extension is TV-semi-free, one can apply the previous strategy, first filtering elements of $T(V \oplus W_{\leq(N_{\min}+l+1)})$ by the number of w's in $W_{N_{\min}+l+1}$ and then treating both cases where $d = 0$ and where $d \neq 0$.

Finally, since $T(V \oplus W) \cong \operatorname{colim}_{l \geq 0} T(V \oplus W_{\leq(N_{\min}+l)})$, as differential graded algebras, we conclude that the algebra $T(V \oplus W)$ is semi-free as a left TV-module, by Lemma 1.2.15.

Proof of (b): This follows from §9.2 in [BMR13]. □

Remark 4.3.12. By [FHT95], every morphism of differential graded algebras $f : B \to A$ admits a factorization



where $(B \sqcup TW, d')$ is a free algebra over B. Since the proof of Proposition 4.3.11 easily generalizes to the case of a free extension of dg-algebras of the form $(B, d_B) \hookrightarrow (B \sqcup TW, d')$, it follows that every morphism of differential graded algebras $f : B \to A$ admits a semi-free replacement, up to homotopy.

Corollary 4.3.13. *Let $g : H \to K$ be a morphism of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where H and K satisfy Convention 4.3.3. Suppose that $(H, \Delta_H, d_H) \hookrightarrow (K \cong H \oplus Z, \Delta_K, d_K)$ is an inclusion of differential graded coalgebras.*

The algebra $(\Omega K, D_{t_{\Omega K}})$ is then semi-free as a left module over ΩH on a generating graded $\Bbbk$-module that is degree-wise finitely generated.

Proof. Since we are working over a field, the inclusion $H \hookrightarrow K$ splits, and we have $K \cong H \oplus Z$, as graded $\Bbbk$-modules, where Z is $\Bbbk$-free. The counit $\varepsilon_K : K \to \Bbbk[0]$ is induced by $\varepsilon_H + \varepsilon_Z : H \oplus Z \to \Bbbk[0]$, and therefore $\overline{K} = \ker(\varepsilon_K) = \ker(\varepsilon_H) \oplus \ker(\varepsilon_Z) = \overline{H} \oplus \overline{Z}$, whence, $s^{-1}\overline{K} = s^{-1}\overline{H} \oplus s^{-1}\overline{Z}$. Therefore,

$$\begin{aligned} \Omega(K) := (T(s^{-1}\overline{K}), D_{\Omega K}) &\cong (T(s^{-1}\overline{H} \oplus s^{-1}\overline{Z}), D_{\Omega K}) \\ &\cong (T(s^{-1}\overline{H}) \sqcup T(s^{-1}\overline{Z}), D_{\Omega K}) \\ &= (\Omega H \sqcup \Omega Z, D_{\Omega K}), \end{aligned}$$

where the differential satisfies

$$D_{\Omega K}(s^{-1}h) = \underbrace{-s^{-1}d_K(h) \pm \sum_i s^{-1}h_i | s^{-1}h^i,}_{\in (\Omega H)_n}$$

for all $s^{-1}h \in (\Omega H)_n$, and

$$D_{\Omega K}(s^{-1}z) = \underbrace{-s^{-1}d_K(z)}_{\in (\Omega H)_{n-1} \sqcup (\Omega Z)_{n-1}} \pm \underbrace{\sum_i s^{-1}z_i | s^{-1}z^i,}_{\in (\Omega H \oplus Z)_{n-1}}$$

for all $s^{-1}z \in (\Omega Z)_n$, for all $n \geq 1$. We can apply Proposition 4.3.11 to conclude that the differential graded algebra ΩK is semi-free as a left ΩH-module. □

Lemma 4.3.14. *Let $g : H \to K$ be a morphism of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where H and K satisfy Convention 4.3.3. Suppose that $(H, \Delta_H, d_H) \hookrightarrow (K \cong H \oplus Z, \Delta_K, d_K)$ is an inclusion of differential graded coalgebras.*

The chain complex $\Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))}$ is then semi-free as a left $\Omega(A; H; \Bbbk)$-module on a generating graded $\Bbbk$-module that is degree-wise finitely generated.

Proof. Observe that

$$\begin{aligned} \Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))} &= \Omega\big(\Omega(A; K; \Bbbk); \Omega(H; K; \Bbbk); \Omega(H; K; \Bbbk)\big) \underset{\Omega(H;K;\Bbbk)}{\square} \Bbbk[0] \\ &\cong \Omega\big(\Omega(A; K; \Bbbk); \Omega(H; K; \Bbbk); \Bbbk[0]\big) \\ &:= (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{t}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K), \end{aligned}$$

where t denotes the universal twisting morphism

$$t : H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K \to \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K).$$

Since $A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K \cong (A \underset{t_\Omega}{\otimes} \Omega H) \underset{\Omega H}{\otimes} \Omega K$, and since ΩK is semi-free as a left module over ΩH by Corollary 4.3.13, we have that $A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K$ is semi-free as a left module over $A \underset{t_\Omega}{\otimes} \Omega H$, which implies that $(A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{t}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$ is semi-free over $A \underset{t_\Omega}{\otimes} \Omega H$. ☐

Remark 4.3.15. Lemma 4.3.14, applied to the case $K = H$ and $g = \mathrm{Id}_H$ gives us an isomorphism of chain complexes

$$\Omega(A; H; \Bbbk)^{hco(\Omega(H;H;\Bbbk))} \quad \cong \quad (A \underset{t_\Omega}{\otimes} \Omega H) \underset{s}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H),$$

where s denotes the universal twisting morphism

$$s : H \underset{t_\Omega}{\otimes} \Omega H \to \Omega(H \underset{t_\Omega}{\otimes} \Omega H).$$

Lemma 4.3.16. *Let $g : H \to K$ be a morphism of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where H and K satisfy Convention 4.3.3. Suppose that $(H, \Delta_H, d_H) \hookrightarrow (K \cong H \oplus Z, \Delta_K, d_K)$ is an inclusion of differential graded coalgebras.*

The tensor product of chain complexes $\Omega(A; K; \Bbbk) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A; K; \Bbbk)$ is then semi-free as a left $\Omega(A; K; \Bbbk)$-module on a generating graded $\Bbbk$-module that is degree-wise finitely generated.

Proof. Observe that

$$\begin{aligned} \Omega(A; K; \Bbbk) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A; K; \Bbbk) &= (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{A \underset{t_\Omega}{\otimes} \Omega H}{\otimes} (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \\ &\cong (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{A \underset{t_\Omega}{\otimes} \Omega H}{\otimes} ((A \underset{t_\Omega}{\otimes} \Omega H) \underset{\Omega H}{\otimes} \Omega K) \\ &\cong (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{\Omega H}{\otimes} \Omega K, \end{aligned}$$

where the first isomorphism uses properties of twisted extensions (see Chapter 1 in [HMS74]), specifically, the fact that the K-comodule structure on A is created by the morphism $g : H \to K$, which also induces a morphism of algebras $\Omega(g) : \Omega H \to \Omega K$.

By Corollary 4.3.13, ΩK is semi-free as a left module over ΩH. Thus, $(A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{\Omega H}{\otimes} \Omega K$ is semi-free as a left module over $A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K$. ☐

Lemma 4.3.17. *Let $g : H \to K$ be a morphism of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where H and K satisfy Convention 4.3.3.*

The tensor product of chain complexes $\Omega(A;K;\Bbbk)\otimes\Omega(H;K;\Bbbk)$ is then semi-free as a left $\Omega(A;K;\Bbbk)$-module on a generating graded $\Bbbk$-module that is degree-wise finitely generated.

Proof. It follows from Convention 4.3.3 that the underlying graded $\Bbbk$-module of $\Omega(H;K;\Bbbk)$, $H\otimes\Omega K$, is degree-wise finitely generated over $\Bbbk$, and that $(H\otimes\Omega K)_0=\Bbbk$. So, the proof is a direct application of Lemma 1.2.14. □

We need to have one more Lemma in our toolbox before proving Theorem 4.3.6.

Lemma 4.3.18. *Let H be a 1-connected Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ and let $H\underset{t_\Omega}{\otimes}\Omega H$ be the associated acyclic cobar construction. The contracting homotopy*

$$c: H\otimes_{t_\Omega}\Omega H\to H\otimes_{t_\Omega}\Omega H$$

defined by

$$h\otimes s^{-1}h_1|\cdots|s^{-1}h_n\mapsto 0,\ \textit{if}\ \deg(h)>0$$

and

$$1\otimes s^{-1}h_1|\cdots|s^{-1}h_n\mapsto h_1\otimes s^{-1}h_2|\cdots|s^{-1}h_n,$$

for all $h\in H$, $s^{-1}h_1|\cdots|s^{-1}h_n\in\Omega H$, is a morphism of right ΩH-modules.

Proof. Proposition 10.6.3 in [Nei10] shows that c is indeed a contracting homotopy of chain complexes. We must prove that c respects the right ΩH-actions, i.e., that the following diagram commutes in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

$$\begin{array}{ccc}
H\otimes_{t_\Omega}\Omega H\otimes\Omega H & \xrightarrow{c\otimes\Omega H} & H\otimes_{t_\Omega}\Omega H\otimes\Omega H\\
\downarrow{\scriptstyle H\otimes\mu_{\Omega H}} & & \downarrow{\scriptstyle H\otimes\mu_{\Omega H}}\\
H\otimes_{t_\Omega}\Omega H & \xrightarrow{c} & H\otimes_{t_\Omega}\Omega H
\end{array}$$

Let $s^{-1}h_1|\cdots|s^{-1}h_n\in\Omega^n(H)$, $s^{-1}h'_1|\cdots|s^{-1}h'_m\in\Omega^m(H)$, $h\in H$. If $\deg(h)>0$, we have

$$\begin{aligned}
&(H\otimes\mu_{\Omega H})\circ(c_n\otimes\Omega H_m)(h\otimes s^{-1}h_1|\cdots|s^{-1}h_n\otimes s^{-1}h'_1|\cdots|s^{-1}h'_m)\\
&\quad= 0\\
&\quad= c_{n+m}(h\otimes s^{-1}h_1|\cdots|s^{-1}h_n|s^{-1}h'_1|\cdots|s^{-1}h'_m)\\
&\quad= c_{n+m}\circ(H\otimes\mu_{\Omega H})(h\otimes s^{-1}h_1|\cdots|s^{-1}h_n\otimes s^{-1}h'_1|\cdots|s^{-1}h'_m).
\end{aligned}$$

If $h=1$, then

$$\begin{aligned}
&(H\otimes\mu_{\Omega H})\circ(c_n\otimes\Omega H_m)(1\otimes s^{-1}h_1|\cdots|s^{-1}h_n\otimes s^{-1}h'_1|\cdots|s^{-1}h'_m)\\
&\quad= (H\otimes\mu_{\Omega H})(h_1\otimes s^{-1}h_2|\cdots|s^{-1}h_n\otimes s^{-1}h'_1|\cdots|s^{-1}h'_m)\\
&\quad= h_1\otimes s^{-1}h_2|\cdots|s^{-1}h_n|s^{-1}h'_1|\cdots|s^{-1}h'_m\\
&\quad= c_{n+m}(1\otimes s^{-1}h_1|\cdots|s^{-1}h_n|s^{-1}h'_1|\cdots|s^{-1}h'_m)\\
&\quad= c_{n+m}\circ(H\otimes\mu_{\Omega H})(1\otimes s^{-1}h_1|\cdots|s^{-1}h_n\otimes s^{-1}h'_1|\cdots|s^{-1}h'_m).
\end{aligned}$$

□

4.3.4 Proof of the Main Theorem (Theorem 4.3.6)

Proof. Since $\varphi : B \to A$ is a homotopic H-Hopf-Galois extension, A is an augmented algebra, by definition. This implies that the domain of ω, $\Omega(A; H; \Bbbk)$, and its codomain, $\Omega(A; K; \Bbbk)$, are augmented algebras, too, as required by Definition 2.3.1.

Recall from Definition 2.3.1 and Remark 4.3.7 that to prove that the map $\omega : \Omega(A; H; \Bbbk) \to \Omega(A; K; \Bbbk)$ is a generalized homotopic $\Omega(H; K; \Bbbk)$-Hopf-Galois extension, we need to show that both the Galois functor

$$(\beta_\omega)_* : \mathcal{M}^{W_\omega}_{\Omega(A;K;\Bbbk)} \longrightarrow \mathcal{M}^{W_{\rho\omega}}_{\Omega(A;K;\Bbbk)}$$

and the comparison functor

$$(\omega^{hco\Omega(H;g;\Bbbk)})^* : \mathbf{Mod}_{\Omega(A;K;\Bbbk)^{hco(\Omega(H;K;\Bbbk))}} \longrightarrow \mathbf{Mod}_{\Omega(A;H;\Bbbk)^{hco(\Omega(H;H;\Bbbk))}}$$

are Quillen equivalences. To do this, we will first show that, under the conditions of this theorem, the maps

$$\beta_\omega : \Omega(A; K; \Bbbk) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A; K; \Bbbk) \to \Omega(A; K; \Bbbk) \otimes \Omega(H; K; \Bbbk)$$

and

$$\omega^{hco\Omega(H;g;\Bbbk)} : \Omega(A; H; \Bbbk)^{hco(\Omega(H;H;\Bbbk))} \to \Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))}$$

are quasi-isomorphisms of chain complexes, and then use criteria from Proposition 1.5.6 and Proposition 1.4.3 to deduce that they induce Quillen equivalences.

Part I. We begin by studying the comparison map

$$\omega^{hco\Omega(H;g;\Bbbk)} : \Omega(A; H; \Bbbk)^{hco(\Omega(H;H;\Bbbk))} \to \Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))}.$$

Recall from the proof of Lemma 4.3.14 that

$$\Omega(A; K; \Bbbk)^{hco(\Omega(H;K;\Bbbk))} \cong (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{t}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K),$$

as chain complexes, where t is the universal twisting morphism

$$t : H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K \to \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K).$$

Similarly, recall from Remark 4.3.15 that

$$\Omega(A; H; \Bbbk)^{hco(\Omega(H;H;\Bbbk))} \cong (A \underset{t_\Omega}{\otimes} \Omega H) \underset{s}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H),$$

as chain complexes, where s denotes the universal twisting morphism

$$s : H \underset{t_\Omega}{\otimes} \Omega H \to \Omega(H \underset{t_\Omega}{\otimes} \Omega H).$$

Therefore, the comparison map $\omega^{hco\Omega(H;g;\Bbbk)}$ is the map

$$(A \underset{t_\Omega}{\otimes} \Omega H) \underset{s}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H) \xrightarrow{\bigstar = A \otimes \Omega(g) \otimes \Omega(H \otimes \Omega(g))} (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{t}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K).$$

Claim. To show that $\bigstar$ is a quasi-isomorphism in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, it suffices to show that

$$\Omega H \underset{t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H) \xrightarrow{\bigstar\bigstar = \Omega(g) \otimes \Omega(H \otimes \Omega(g))} \Omega K \underset{t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$$

is a quasi-isomorphism in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, where $\Omega H \in \mathbf{Comod}_{\Omega H}$, $\Omega K \in \mathbf{Comod}_{\Omega K}$, $\Omega(H \underset{t_\Omega}{\otimes} \Omega H)$ and $\Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$ are seen as left modules over themselves, and

$$t_\Omega \circ (\eta_H \otimes \Omega H) : \Omega H \cong \Bbbk[0] \otimes \Omega H \to H \otimes \Omega H \to \Omega(H \otimes \Omega H),$$

$$t_\Omega \circ (\eta_H \otimes \Omega K) : \Omega K \cong \Bbbk[0] \otimes \Omega K \to H \otimes \Omega K \to \Omega(H \otimes \Omega K)$$

are twisting morphisms.

Proof of the Claim: We use a version of Zeeman's comparison theorem for twisted extensions. Consider the map $\bigstar$ and, using notation of Proposition 1.2.23, set $N = N' := A$, $M := \Omega H \underset{t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H)$, $M' := \Omega K \underset{t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$. The maps $\gamma := \mathrm{Id}_H$ and $\alpha := \Omega(g)$ are appropriately compatible (see Definition 1.2.5) with twisting morphisms $t_\Omega : H \to \Omega H$ and $\Omega(g) \circ t_\Omega : H \to \Omega K$, because

$$\Omega(g) \circ t_\Omega = \Omega(g) \circ t_\Omega \circ \mathrm{Id}_H .$$

Since A is degree-wise $\Bbbk$-flat, Proposition 1.2.23 guarantees that the morphism of twisted extensions

$$(A \underset{t_\Omega}{\otimes} \Omega H) \underset{s}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H) \xrightarrow{\bigstar} (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{t}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$$

is a quasi-isomorphism in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ if and only if the morphism

$$\Omega H \underset{t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H) \xrightarrow{\bigstar\bigstar} \Omega K \underset{t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K)$$

is a quasi-isomorphism in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. $\square$

The proof of the fact that ★★ is a quasi-isomorphism breaks down into several steps, explained in the following commutative diagram of chain complexes.

$$
\begin{array}{ccc}
\Omega H \underset{t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} \Omega(H \underset{t_\Omega}{\otimes} \Omega H) & \xrightarrow{\star\star} & \Omega K \underset{t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \\
\text{①} \; \cong \downarrow \Omega H \otimes \Omega(\tau) & & \text{❶} \; \cong \downarrow \Omega K \otimes \Omega(\tau) \\
\Omega H \underset{\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} \Omega(\Omega H \underset{t_\Omega}{\otimes} H) & \longrightarrow & \Omega K \underset{\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} \Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H) \\
\text{②} \; \simeq \downarrow \Omega H \otimes m & & \text{❷} \; \simeq \downarrow \Omega K \otimes m \\
\Omega H \underset{m\circ\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega H)}{\otimes} (\Omega^2 H \underset{t_\Omega}{\otimes} \Omega H) & \longrightarrow & \Omega K \underset{m\circ\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega K)}{\otimes} (\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H) \\
\text{③} \; \cong \downarrow \tau' \otimes \Omega^2 H \otimes \Omega H & & \text{❸} \; \cong \downarrow \tau' \otimes \Omega^2 K \otimes \Omega H \\
\Omega H \underset{m\circ\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega H)\circ\tau'}{\otimes} (\Omega^2 H \underset{t_\Omega}{\otimes} \Omega H) & \longrightarrow & \Omega K \underset{m\circ\Omega(\tau)\circ t_\Omega \circ (\eta_H \otimes \Omega K)\circ\tau'}{\otimes} (\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H) \\
\text{④} \; \Vert & & \text{❹} \; \Vert \\
\Omega H \underset{t_\Omega}{\otimes} \Omega^2 H \underset{t_\Omega}{\otimes} \Omega H & \longrightarrow & \Omega K \underset{t_\Omega}{\otimes} \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H \\
\text{⑤} \; \simeq \downarrow & & \text{❺} \; \simeq \downarrow \\
\Omega H & \xrightarrow{=} & \Omega H
\end{array}
$$

If we show Steps ❶ through ❺, and Steps ① through ⑤, then the equality in the bottom row and the 2-out-of-3 property will imply that ★★ is a quasi-isomorphism, using the commutativity of this diagram.

Step ❶:

Lemma 4.3.19. *Let K be a co-commutative Hopf algebra. The map*

$$\tau' : \Omega K \to \Omega K$$

given by

$$s^{-1}k_1|\cdots|s^{-1}k_n \mapsto s^{-1}k_n|\cdots|s^{-1}k_1,$$

for all $k_i \in K$, $1 \leq i \leq n$, for all $n \geq 0$, is an isomorphism of differential graded coalgebras.

Proof. The inverse of τ' clearly sends $s^{-1}k_n|\cdots|s^{-1}k_1$ to $s^{-1}k_1|\cdots|s^{-1}k_n$, for all $k_i \in K$, $1 \leq i \leq n$, for all $n \geq 0$.

The compatibility of τ' and τ'^{-1} with the comultiplication $\Delta_{\Omega K} : \Omega K \to \Omega K \otimes \Omega K$ is checked using the definition of $\Delta_{\Omega K}$ on n-fold elements of ΩK, involving $(p, n-p)$-shuffles (see Remark 1.2.10). □

Lemma 4.3.20. *Let $g : H \to K$ be a morphism of co-commutative Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. The map*

$$\tau : H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K \to \Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H$$

$$h \otimes s^{-1}k_1|\cdots|s^{-1}k_n \mapsto \pm s^{-1}k_n|\cdots|s^{-1}k_1 \otimes h$$

is an isomorphism of differential graded coalgebras.

Proof. Recall that the existence of a differential graded coalgebra structure on $H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K$ was shown in Lemma 4.3.5. The proof of this fact for $\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H$ is the same, up to a symmetry.

The co-commutativity assumptions on both H and K play a crucial role in establishing that τ commutes with the differentials. Let D denote the differential on $H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K$ and D' denote the differential on $\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H$. For all $k_j \in K$, $1 \leq j \leq n$, we use the notation $\Delta_K(k_j) = \sum_l k_{jl} \otimes k_j^l$. Also, for all $h \in H$, we use the notation $\Delta_H(h) = \sum_i h_i \otimes h^i$.

We have

$$\begin{aligned}
\tau \circ D(h \otimes s^{-1}k_1|\cdots|s^{-1}k_n) \quad = \quad & \tau\Big(d_H(h) \otimes s^{-1}k_1|\cdots|s^{-1}k_n \\
& + h \otimes \sum_{j=1}^{n} \pm s^{-1}k_1|\cdots|s^{-1}d_k(k_j)|\cdots|s^{-1}k_n \\
& + h \otimes \sum_{l}\sum_{j=1}^{n} \pm s^{-1}k_1|\cdots|s^{-1}k_{jl}|s^{-1}k_j^l|\cdots|s^{-1}k_n \\
& + \sum_i \pm h_i \otimes s^{-1}g(h^i)|s^{-1}k_1|\cdots|s^{-1}k_n\Big) \\
= \quad & \pm s^{-1}k_n|\cdots|s^{-1}k_1 \otimes d_H(h) \\
& + \sum_{j=1}^{n} \pm s^{-1}k_n|\cdots|s^{-1}d_k(k_j)|\cdots|s^{-1}k_1 \otimes h \\
& + \sum_{l}\sum_{j=1}^{n} \pm s^{-1}k_n|\cdots|s^{-1}k_j^l|\ s^{-1}k_{jl}|\cdots|s^{-1}k_1 \otimes h \\
& + \sum_i \pm s^{-1}k_n|\cdots|s^{-1}k_1|s^{-1}g(h^i) \otimes h_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \pm s^{-1}k_n|\cdots|s^{-1}d_k(k_j)|\cdots|s^{-1}k_1\otimes h\\
&\quad \sum_{l}\sum_{j=1}^{n} \pm s^{-1}k_n|\cdots|s^{-1}k_{jl}|s^{-1}k_j^l|\cdots|s^{-1}k_1\otimes h\\
&\quad \pm s^{-1}k_n|\cdots|s^{-1}k_1\otimes d_H(h)\\
&\quad + \sum_i \pm s^{-1}k_n|\cdots|s^{-1}k_1|s^{-1}g(h_i)\otimes h^i\\
&= D'(s^{-1}k_n|\cdots|s^{-1}k_1\otimes h)\\
&= D'(\tau(h\otimes s^{-1}k_1|\cdots|s^{-1}k_n)),
\end{aligned}
$$

for all $h \in H$, $k_i \in K$, $1 \leq i \leq n$, and the signs are given by the Koszul rule. Here we used that $k_{jl}\otimes k_j^l = \pm k_j^l \otimes k_{jl}$, for all j, l, since $\Delta_K : K \to K\otimes K$ is co-commutative, that $h_i \otimes h^i = \pm h^i \otimes h_i$, for all i, since $\Delta_H : H \to H \otimes H$ is co-commutative, and also that $g : H \to K$ is a morphism of coalgebras.

Finally, τ is an *isomorphism* of differential graded coalgebras, because τ' is so, by Lemma 4.3.19. □

It follows from Lemma 4.3.20 that

$$
\Omega(\tau) : \Omega(H \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \to \Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H)
$$

is an isomorphism of differential graded algebras, and this completes Step ❶.

Step ❷:

Lemma 4.3.21. *Let $g : H \to K$ be a morphism of Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$, satisfying Convention 4.3.3. The Milgram map (see Remark 1.2.10)*

$$
m : \Omega(\Omega K \otimes H) \xrightarrow{\simeq} \Omega^2 K \otimes \Omega H
$$

commutes with the differentials on $\Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H)$ and on $\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$. Therefore, it induces a morphism of differential graded algebras

$$
m : \Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H) \xrightarrow{\simeq} \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H,
$$

which is a quasi-isomorphism of the underlying chain complexes. Here, the codomain algebra $\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$ is equipped with the multiplication defined in Corollary 3.6 in [HL07] (see Remark 2.1.13).

Proof. Recall that the Milgram map is given on generating elements of $\Omega(\Omega K \otimes H)$ by

$$
m(s^{-1}(w\otimes h)) = \begin{cases} 0, & \text{if } \deg(w) > 0, \deg(h) > 0,\\ 1\otimes s^{-1}h, & \text{if } w = 1, \forall h \in H,\\ s^{-1}w\otimes 1, & \text{if } h = 1, \forall w \in \Omega K,\end{cases}
$$

for all $s^{-1}(w\otimes h)\in\Omega(\Omega K\otimes H)$.

We use the notation $\Delta_H(h)=\sum_i h_i\otimes h^i$, for all $h\in H$ and $\Delta_{\Omega K}(w)=\sum_j w_j\otimes w^j$ for all $w\in\Omega K$. The differential on $\Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H)$, denoted by D_1, is given on $s^{-1}(w\otimes h)\in\Omega(\Omega K\otimes H)$ by

$$\begin{aligned} D_1(s^{-1}(w\otimes h)) &= -s^{-1}(D_{\Omega K}(w)\otimes h)\pm s^{-1}(w\otimes d_H(h)) \\ &\quad +\sum_i \pm s^{-1}\big(w|s^{-1}g(h_i)\otimes h^i\big) \\ &\quad \sum_i\sum_j \pm s^{-1}(w_j\otimes h_i)|s^{-1}(w^j\otimes h^i). \end{aligned}$$

On the other hand, the differential on $\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$, denoted by D_2, is given on $s^{-1}w\otimes s^{-1}h\in\Omega^2K\otimes\Omega H$ by

$$\begin{aligned} D_2(s^{-1}w\otimes s^{-1}h) &= -s^{-1}(D_{\Omega K}(w))\otimes s^{-1}h\pm\sum_j s^{-1}w_j|s^{-1}w^j\otimes s^{-1}h \\ &\quad \pm s^{-1}w\otimes s^{-1}d_H(h)+s^{-1}w\otimes\sum_i \pm s^{-1}h_i|s^{-1}h^i \\ &\quad \pm s^{-1}w|s^{-1}(s^{-1}g(h))\otimes 1. \end{aligned}$$

In view of the definition of the Milgram map, it suffices to check the compatibility of m, D_1, D_2 in the following three cases.

- If $w=1$, we have for all $h\in H$

$$\begin{aligned} m\circ D_1(s^{-1}(1\otimes h)) &= m\Big((-1)s^{-1}(1\otimes d_H(h)) \\ &\quad +\sum_i(-1)s^{-1}\big(s^{-1}g(h_i)\otimes h^i\big) \\ &\quad +\sum_i(-1)^{\deg(h_i)}s^{-1}(1\otimes h_i)|s^{-1}(1\otimes h^i)\Big) \\ &= -(1\otimes s^{-1}d_H(h))-(s^{-1}(s^{-1}(g(h)))\otimes 1) \\ &\quad +\sum_i(-1)^{\deg(h_i)}1\otimes s^{-1}h_i|s^{-1}h^i, \end{aligned}$$

  and

$$\begin{aligned} D_2\circ m(s^{-1}(1\otimes h)) &= D_2(1\otimes s^{-1}h) \\ &= -(1\otimes s^{-1}d_H(h)) \\ &\quad +\sum_i(-1)^{\deg(h_i)}1\otimes s^{-1}h_i|s^{-1}h^i \\ &\quad -(s^{-1}(s^{-1}(g(h)))\otimes 1). \end{aligned}$$

- If $h = 1$, we have for all $w \in \Omega K$

$$\begin{aligned}
m \circ D_1(s^{-1}(w \otimes 1)) &= m\Big(- s^{-1}(D_{\Omega K}(w) \otimes 1) \\
&\quad + \sum_j (-1)^{\deg(w_j)} s^{-1}(w_j \otimes 1)|s^{-1}(w^j \otimes 1)\Big) \\
&= -(s^{-1} D_{\Omega K}(w) \otimes 1) \\
&\quad + \sum_j (-1)^{\deg(w_j)} s^{-1} w_j | s^{-1} w^j \otimes 1,
\end{aligned}$$

and

$$\begin{aligned}
D_2 \circ m(s^{-1}(w \otimes 1)) &= D_2(s^{-1} w \otimes 1) \\
&= -(s^{-1} D_{\Omega K}(w) \otimes 1) \\
&\quad + \sum_j (-1)^{\deg(w_j)} s^{-1} w_j | s^{-1} w^j \otimes 1.
\end{aligned}$$

- If $\deg(w) \geq 1$, and $\deg(h) \geq 1$, then we can actually suppose that $\deg(w) > 1$ and $\deg(h) > 1$, because it follows from Convention 4.3.3 that $H_1 = (\Omega K)_1 = 0$. We then have $D_2 \circ m(s^{-1}(w \otimes h)) = 0$ and

$$\begin{aligned}
m \circ D_1(s^{-1}(w \otimes h)) &= m\Big(- s^{-1}(D_{\Omega K}(w) \otimes h) + (-1)^{|v|+1} s^{-1}(w \otimes d_H(h)) \\
&\quad + \sum_i (-1)^{-|v|+1} s^{-1}\big(w | s^{-1} g(h_i) \otimes h^i\big) \\
&\quad + \sum_{i,j} (-1)^{|w_j| + |h_i| + |h_i||w^j|} s^{-1}(w_j \otimes h_i)|s^{-1}(w^j \otimes h^i)\Big) \\
&= (-1)^{-|v|+1} s^{-1}\big(w | s^{-1} g(h)\big) \otimes 1 \\
&\quad + (-1)^{|h| + |h||v|}(1 \otimes s^{-1} h) \cdot (s^{-1} w \otimes 1) \\
&\quad + (-1)^{|v|}(s^{-1} w \otimes 1) \cdot (1 \otimes s^{-1} h) \\
&= (-1)^{-|v|+1} s^{-1}\big(w | s^{-1} g(h)\big) \otimes 1 \\
&\quad + (-1)^{|h| + |h||v| + (|v|+1)(|h|-1)} s^{-1} w \otimes s^{-1} h \\
&\quad + (-1)^{|v|-1} s^{-1}(w | s^{-1} g(h)) \otimes 1 \\
&\quad + (-1)^{|v|}(s^{-1} w \otimes s^{-1} h) \\
&= (-1)^{|v|+1} s^{-1} w \otimes s^{-1} h + (-1)^{|v|} s^{-1} w \otimes s^{-1} h \\
&= 0,
\end{aligned}$$

  where we wrote $|x|$ for $\deg(x)$ for space reasons, and where one should be careful to use the twisted multiplication $\cdot$ on $\Omega^2 K \underset{\Omega^2(g) \circ t_\Omega}{\otimes} \Omega H$.

This finishes the proof that

$$m : \Omega(\Omega K \underset{\Omega(g) \circ t_\Omega}{\otimes} H) \xrightarrow{\simeq} \Omega^2 K \underset{\Omega^2(g) \circ t_\Omega}{\otimes} \Omega H$$

is a quasi-isomorphism of differential graded algebras. $\square$

To complete Step ❷, we apply a version of Zeeman's comparison theorem for twisted extensions. Using notation of Proposition 1.2.23, set $N = N' := \Omega K$, $M := \Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H)$, $M' := \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$. The maps $\gamma := \mathrm{Id}_{\Omega K}$ and $\alpha := m$ are appropriately compatible (see Definition 1.2.5) with the twisting morphisms by definition. It follows from Convention 4.3.3 that ΩK is degree-wise $\Bbbk$-flat, so Proposition 1.2.23 guarantees that the morphism of twisted extensions

$$\Omega K \underset{\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)}{\otimes} \Omega(\Omega K \underset{\Omega(g)\circ t_\Omega}{\otimes} H) \to \Omega K \underset{m\circ\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)}{\otimes} (\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H)$$

is a quasi-isomorphism in $\mathbf{Ch}_\Bbbk^{\geq 0}$, because m is a quasi-isomorphism, by Lemma 4.3.21.

Step ❸: This step is immediate because τ' is an isomorphism of coalgebras by Lemma 4.3.19.

Step ❹:

Lemma 4.3.22. *There is an equality of twisted tensor products of chain complexes*

$$\Omega K \underset{m\circ\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)\circ\tau'}{\otimes} (\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H) = \Omega K \underset{t_\Omega}{\otimes} \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H.$$

Proof. A careful observation of twisted structures shows that the chain complex $\Omega K \underset{t_\Omega}{\otimes} \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$ is identical to the chain complex

$$\Omega K \underset{(\Omega^2 K\otimes\eta_{\Omega H})\circ t_\Omega}{\otimes} \big(\Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H\big).$$

Secondly, for all $k_i \in K$, $1 \leq i \leq n$, $n \geq 1$, we have

$$\begin{aligned}
&m\circ\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)\circ\tau'(s^{-1}k_1|\cdots|s^{-1}k_n) = \\
&\quad= \pm m\circ\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)(s^{-1}k_n|\cdots|s^{-1}k_1)\\
&\quad= \pm m\circ\Omega(\tau)\circ t_\Omega(1\otimes s^{-1}k_n|\cdots|s^{-1}k_1)\\
&\quad= \pm m\circ\Omega(\tau)(s^{-1}(1\otimes s^{-1}k_n|\cdots|s^{-1}k_1))\\
&\quad= m(s^{-1}(s^{-1}k_1|\cdots|s^{-1}k_n\otimes 1))\\
&\quad= s^{-1}(s^{-1}k_1|\cdots|s^{-1}k_n)\otimes 1\\
&\quad= ((\Omega^2 K\otimes\eta_{\Omega H})\circ t_\Omega)(s^{-1}k_1|\cdots|s^{-1}k_n).
\end{aligned}$$

This shows that for any element $s^{-1}k_1|\cdots|s^{-1}k_n$ of length n in ΩK, the twisting morphism

$$m\circ\Omega(\tau)\circ t_\Omega\circ(\eta_H\otimes\Omega K)\circ\tau' : \Omega K \to \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$$

is equal to the twisting morphism

$$(\Omega^2 K \otimes \eta_{\Omega H}) \circ t_\Omega : \Omega K \to \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H$$

and preserves the internal order of the $s^{-1}k_i$'s, which finishes the proof. $\square$

Step ❺: The homotopy equivalence

$$\varepsilon_{\Omega K} \otimes \varepsilon_{\Omega^2 K} : \Omega K \underset{t_\Omega}{\otimes} \Omega^2 K \longrightarrow \Bbbk[0] \otimes \Bbbk[0] \cong \Bbbk[0]$$

is a homotopy equivalence of right $\Omega^2 K$-modules, where $\Bbbk[0]$ is equipped with a trivial $\Omega^2 K$-action. The associated contracting homotopy $c : \Omega K \otimes \Omega^n(\Omega K) \to \Omega K \otimes \Omega^{n-1}(\Omega K)$ is also a homotopy of right $\Omega^2 K$-modules, by Lemma 4.3.18 (note that ΩK is a 1-connected Hopf algebra, by Convention 4.3.3). Therefore,

$$\varepsilon_{\Omega K} \otimes \varepsilon_{\Omega^2 K} \otimes \Omega H : \Omega K \underset{t_\Omega}{\otimes} \Omega^2 K \underset{\Omega^2(g)\circ t_\Omega}{\otimes} \Omega H = \Omega(\Omega K; \Omega K; \Omega H) \xrightarrow{\simeq} \Omega H$$

is a quasi-isomorphism by the dual of Proposition 7.8 in [McC01].

Steps ① through ⑤ follow from Steps ❶ through ❺, respectively, by setting $K := H$ and $g := \mathrm{Id}_H$.

At this point, we have established that $\omega^{hco\Omega(H;g;\Bbbk)}$ is a quasi-isomorphism. Use Proposition 1.4.3 to show that the induced comparison functor $(\omega^{hco\Omega(H;g;\Bbbk)})^*$ is a Quillen equivalence. By Lemma 4.3.14, the chain complex $\Omega(A;K;\Bbbk)^{hco(\Omega(H;K;\Bbbk))}$ is semi-free as a left $\Omega(A;H;\Bbbk)$-module on a generating graded $\Bbbk$-module that is degree-wise finitely generated. We can therefore conclude that $(\omega^{hco\Omega(H;g;\Bbbk)})^*$ is a Quillen equivalence, as desired.

Part II. We now study the Galois map

$$\beta_\omega : \Omega(A;K;\Bbbk) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A;K;\Bbbk) \to \Omega(A;K;\Bbbk) \otimes \Omega(H;K;\Bbbk).$$

From Lemma 4.3.16 and Corollary 4.3.13 it follows that the domain of β_ω is given by

$$\begin{aligned} \Omega(A;K;\Bbbk) \underset{\Omega(A;H;\Bbbk)}{\otimes} \Omega(A;K;\Bbbk) &\cong (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \underset{\Omega H}{\otimes} \Omega K \\ &\cong (A \underset{\Omega(g)\circ t_\Omega}{\otimes} \Omega K) \tilde{\otimes} Y, \end{aligned}$$

where Y is a graded $\Bbbk$-module of finite type. On the other hand, the codomain of β_ω is

$$\begin{aligned}\Omega(A;K;\Bbbk)\otimes\Omega(H;K;\Bbbk) &= (A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes(H\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\\ &\cong (A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes H\underset{t_\Omega}{\otimes}(\Omega H\,\widetilde{\otimes}\,Y)\\ &\cong (A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes(H\underset{t_\Omega}{\otimes}\Omega H)\,\widetilde{\otimes}\,Y,\end{aligned}$$

where we use again that ΩK is ΩH-semi-free on a graded $\Bbbk$-module Y of finite type, by Corollary 4.3.13. To show that

$$\beta_\omega : (A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\,\widetilde{\otimes}\,Y\to(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes(H\underset{t_\Omega}{\otimes}\Omega H)\,\widetilde{\otimes}\,Y$$

is a quasi-isomorphism of chain complexes, one proceeds in two steps.

Observe that $H\underset{t_\Omega}{\otimes}\Omega H\xrightarrow{\simeq}\Bbbk[0]$ and both A and ΩK are degree-wise $\Bbbk$-free. Thus, $(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes(H\underset{t_\Omega}{\otimes}\Omega H)\xrightarrow{\simeq}A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K$ is a quasi-isomorphism of chain complexes, so that

$$H_*(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\overset{(*)}{\cong}H_*\big((A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\otimes(H\underset{t_\Omega}{\otimes}\Omega H)\big).$$

In the second step one uses that the graded modules $(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\,\widetilde{\otimes}\,Y$ and $(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K\otimes(H\underset{t_\Omega}{\otimes}\Omega H))\,\widetilde{\otimes}\,Y$ are filtered, as they are semi-free on Y, which is degree-wise finitely generated. Theorem 1.2.18 gives us two spectral sequences, converging, respectively, to $H_*(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K)\,\widetilde{\otimes}\,Y$ and to $H_*(A\underset{\Omega(g)\circ t_\Omega}{\otimes}\Omega K\otimes(H\underset{t_\Omega}{\otimes}\Omega H)\,\widetilde{\otimes}\,Y$. Using $(*)$, the fact that Y is degree-wise $\Bbbk$-free, and the Zeeman Comparison Theorem 1.2.19, one can conclude that β_ω is a quasi-isomorphism.

Since we know that β_ω is a quasi-isomorphism of chain complexes, we can apply Proposition 1.5.6 to show that the induced Galois functor $(\beta_\omega)_*$ is a Quillen equivalence. Indeed, assumption (3) and Convention 4.3.3 allow us to use Lemmas 4.3.16 and 4.3.17 to see that the chain complexes $\Omega(A;K;\Bbbk)\underset{\Omega(A;H;\Bbbk)}{\otimes}\Omega(A;K;\Bbbk)$ and $\Omega(A;K;\Bbbk)\otimes\Omega(H;K;\Bbbk)$ are both semi-free as left $\Omega(A;K;\Bbbk)$-modules on generating graded $\Bbbk$-modules that are degree-wise finitely generated. We can therefore conclude that $(\beta_\omega)_*$ is a Quillen equivalence, as desired. □

4.4 An example

The purpose of this section is to illustrate the context where our Main Theorem can be applied, through an example coming from the world of simplicial sets. Let $\Bbbk$ be a field. Given a simplicial set, one can associate to it a dg-$\Bbbk$-algebra, in a natural way, using the cobar construction on the normalized chain complex of X. If one puts an extra assumption on X, one can even obtain a co-commutative dg-Hopf algebra from X. In the same vein, a map of simplicial sets will induce a morphism of dg-$\Bbbk$-algebras. Adding (trivial) coactions of dg-Hopf algebras to the picture will give us data tying together into a homotopic Hopf-Galois extension of chain complexes over $\Bbbk$ and allow to apply our Main Theorem in this context.

Since this section serves mainly as an illustration of differential graded Hopf-Galois extensions, we chose to be brief "simplicially speaking". This means that we will neither reproduce all of the definitions, nor prove all of the results concerning simplicial sets and associated constructions here. For details, we refer the reader to the classical reference [May67]. Another good reference is [HPS07], which offers a concise reminder on the relevant simplicial constructions, as well as establishes some facts that we will use.

We will denote by $C_* : \mathbf{sSet} \to \mathbf{Ch}_{\Bbbk}^{\geq 0}$ the normalized chain complex functor.

4.4.1 The simplicial context

Let $(\mathbf{sSet}, \times, *)$ be the monoidal category of simplicial sets, where the monoidal product is the categorical product, and the unit is the terminal object.

Any object $X \in \mathbf{sSet}$ is naturally a comonoid, with comultiplication given by the diagonal $\Delta_X : X \to X \times X$ and the counit given by the unique map to the terminal object $\varepsilon : X \to *$. Consequently, any map $f \in \mathbf{sSet}(X, Y)$ endows X with a right Y-comodule structure

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{X \times f} X \times Y, \qquad \rho = (X \times f)\circ \Delta_X : X \to X \times Y.$$

Let $Y \in \mathbf{sSet}$. It is easy to show that the category $\mathbf{Comod}_Y$ is equivalent to the slice category $\mathbf{sSet}/Y$ of simplicial sets over Y (see [Hes09], Section 1.2.1). Recall that objects of $\mathbf{sSet}/Y$ are morphisms of simplicial sets with codomain Y, and a morphism from $(f : X \to Y) \in \mathbf{sSet}/Y$ to $(g : Z \to Y) \in \mathbf{sSet}/Y$ is a morphism of simplicial sets $a : X \to Z$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{a} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

commutes.

The next lemma is our first step towards dg-algebras.

Lemma 4.4.1. *Let $\Bbbk$ be a field. For any $X \in \mathbf{sSet}$, the normalized chain complex $C_*(X)$ has a structure of a differential graded $\Bbbk$-coalgebra.*

Sketch of the proof: Let $X \in \mathbf{sSet}$. The composite

$$C_*(X) \xrightarrow{C_*(\Delta_X)} C_*(X \times X) \xrightarrow{AW_{X,X}} C_*(X) \otimes C_*(X), \qquad \Delta_{C_*(X)} = AW_{X,X} \circ C_*(\Delta_X)$$

where $AW_{X,X} : C_*(X \times X) \to C_*(X) \otimes C_*(X)$ denotes the Alexander-Whitney map, endows the normalized chain complex $C_*(X)$ with a natural, coassociative, and counital comultiplication (see [May67]). The unique simplicial morphism $\varepsilon : X \to *$ induces a morphism of chain complexes $C_*(\varepsilon) : C_*(X) \to C_*(*) \cong R$ which gives the counit map

$$\varepsilon_{C_*(X)} : C_*(X) \to \Bbbk[0]$$

compatible with $\Delta_{C_*(X)}$. □

For any *pointed* simplicial set $X \in \mathbf{sSet}_*$, we denote by $\mathbb{E}(X)$ the *simplicial suspension* of X (see [May67]). The idea is that the simplicial set $\mathbb{E}(X)$ is a "shifted version" of X, where a non-degenerate point was added in degree 0. A crucial property of this construction is the existence, for any $X \in \mathbf{sSet}_*$, of a homeomorphism $\Sigma(|X|) \approx |\mathbb{E}(X)|$ between the topological reduced suspension of the geometric realization of X and the geometric realization of $\mathbb{E}(X)$.

4.4.2 Obtaining a co-commutative Hopf algebra from a simplicial set

Remark 4.4.2. Henceforth, we will work in the subcategory of $\mathbf{sSet}_*$ consisting of $(0-)$*reduced* pointed simplicial sets, denoted by $\mathbf{sSet}_0$. The objects of $\mathbf{sSet}_0$ are pointed simplicial sets X such that $X_0 = \{x_0\}$ (i.e., they have only one 0-simplex), and the morphisms are simplicial maps between them. This restriction on the nature of simplicial sets will guarantee that an important connectivity condition holds for associated cobar constructions.

Remark 4.4.3. More generally, a simplicial set X is called r-**reduced** if $X_i = \{x_i\}$, for $0 \leq i \leq r$, for any $r \geq 0$. Observe that if X is r-reduced then its suspension $\mathbb{E}\, X$ is $(r+1)$-reduced.

For any simplicial set X that is 1-reduced, the chain coalgebra $C_*(X)$ is 1-connected, so its cobar construction $\Omega(C_*(X))$ is well-defined and is a coaugmented dg-$\Bbbk$-algebra (see Definition 1.2.2). In [HPST06], a *coalgebra* structure on $\Omega(C_*(X))$ was defined, for any $X \in \mathbf{sSet}_0$. It is given by the composite

$$\psi_X := m \circ \varphi_{AW} \circ \Omega(C_*(\Delta_X)) : \Omega(C_*(X)) \to \Omega(C_*(X)) \otimes \Omega(C_*(X)),$$

and extended to an algebra morphism. Here φ_{AW} is the natural quasi-isomorphism of chain algebras realizing the strongly homotopy coalgebra map structure of the Alexander-Whitney map

$$AW_{X,X} : C_*(X \times X) \to C_*(X) \otimes C_*(X),$$

and m denotes the Milgram map (see Remark 1.2.10).

This coalgebra structure is compatible with the algebra structure on $\Omega(C_*(X))$ and makes $\Omega(C_*(X))$ into a Hopf algebra. Let $\mathbb{G}$ denote the Kan loop group functor (see [Kan58]). Theorem 4.4 in [HPST06] establishes that there exists a quasi-isomorphism $\Omega(C_*(X)) \xrightarrow{\simeq} C_*(\mathbb{G}\, X)$, which is a map of differential graded algebras, and a map of differential graded coalgebras up to strong homotopy.

Remark 4.4.4. Let $\Bbbk$ be a field. Lemma 3.1 in [HPS07] shows that for any $X \in \mathbf{sSet}_0$, the differential graded $\Bbbk$-coalgebra $C_* \mathbb{E}(X)$ is primitively generated (see Reminder 1.2.1), and thus is co-commutative.

Remark 4.4.5. Theorem 4.9 in [HPS07], applied in the case $K := \mathbb{E}\, X$, and Remark 4.4.4 imply that the cobar construction $\Omega(C_* \mathbb{E}^2(X))$ on the normalized chains of the *double* suspension of X will have a primitively generated coalgebra structure. Being primitively generated, $\Omega(C_* \mathbb{E}^2(X))$ is a *co-commutative* Hopf algebra in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Therefore, the above construction gives us an example of how graded co-commutative Hopf algebras can arise, starting from a reduced simplicial set.

It follows that any morphism of reduced simplicial sets $f : X \to Y$ induces a morphism of co-commutative Hopf algebras

$$\Omega(C_* \mathbb{E}^2(f)) : \Omega(C_* \mathbb{E}^2(X)) \to \Omega(C_* \mathbb{E}^2(Y)).$$

Moreover, since X and Y are reduced, the coalgebras, underlying the associated cobar constructions $\Omega(C_* \mathbb{E}^2(X))$ and $\Omega(C_* \mathbb{E}^2(Y))$, are 1-connected. It then follows from Proposition 2.20 in [FH12] that the colagebra map, underlying the map of Hopf algebras $\Omega(C_* \mathbb{E}^2(f)) : \Omega(C_* \mathbb{E}^2(X)) \to \Omega(C_* \mathbb{E}^2(Y))$, is *conormal* (see Remarks 4.3.8 and 4.3.9).

4.4.3 An example of application of Theorem 4.3.6

We start with the following simplicial data, organized into a commutative diagram in $\mathbf{sSet}_0$

$$\begin{array}{ccc} W & \xrightarrow{\zeta} & V \\ & {\scriptstyle *}\searrow & \downarrow{\scriptstyle \xi} \\ & & \mathbb{E}^2(X) \xrightarrow{\mathbb{E}^2(f)} \mathbb{E}^2(Y), \end{array} \qquad (\square)$$

where $* : W \to \mathbb{E}^2(X)$ denotes the constant map, sending all $w \in W_n$ to the basepoint of $\mathbb{E}^2(X)$. More precisely, $f : X \to Y$ is a map of reduced simplicial sets, inducing a map on double suspensions $\mathbb{E}^2(f) : \mathbb{E}^2(X) \to \mathbb{E}^2(Y)$. The map $\zeta : W \to V$ is a morphism in the slice category $\mathbf{sSet}_0 \,/\, \mathbb{E}^2(X)$, between $* : W \to \mathbb{E}^2(X)$ and $\xi : V \to \mathbb{E}^2(X)$. As observed in Section 4.4.1, this implies that $V \in \mathbf{sSet}_0$ is a $\mathbb{E}^2(X)$-comodule with coaction

$$V \xrightarrow{\Delta_V} V \times V \xrightarrow{V \times \xi} V \times \mathbb{E}^2(X), \qquad \rho = (V \times \xi)\circ \Delta_V$$

and that $W \in \mathbf{sSet}_0$ has a trivial $\mathbb{E}^2(X)$-comodule structure

$$W \xrightarrow{\Delta_W} W \times W \xrightarrow{W \times *} W \times \mathbb{E}^2(X). \qquad \rho = (W \times *)\circ \Delta_W$$

We now apply Theorem 4.3.6 to this situation.

Corollary 4.4.6. *Let $\Bbbk$ be a field. Consider the following commutative diagram in $\mathbf{sSet}_0$.*

$$\begin{array}{ccc} W & \xrightarrow{\zeta} & V \\ & {\scriptstyle *}\searrow & \downarrow{\scriptstyle \xi} \\ & & \mathbb{E}^2(X) \xrightarrow{\mathbb{E}^2(f)} \mathbb{E}^2(Y) \end{array} \qquad (\square)$$

Suppose that

(a) *the map $f : X \hookrightarrow Y$ is an inclusion of simplicial sets, where X is reduced, and Y is 1-reduced, such that for all $n \geq 0$, $(Y/X)_n$ has finite number of non-degenerate simplices;*

(b) *the map $\zeta : W \hookrightarrow V$ is an inclusion of reduced simplicial sets, such that for all $n \geq 0$, V_n has finitely many non-degenerate simplices;*

(c) *the map $\Omega(C_*(\zeta)) : \Omega(C_*(W)) \to \Omega(C_*(V))$ is a homotopic $\Omega(C_* \mathbb{E}^2(X))$-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.*

Then the map

$$\Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 f : \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 X \to \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 Y$$

in the commutative diagram

$$\begin{array}{ccc}
\Omega C_* W & \xrightarrow{\Omega C_* \zeta} & \Omega C_* V \\
\downarrow \simeq & & \downarrow \simeq \\
\Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 X & \xhookrightarrow{\iota_{\Omega C_* \mathbb{E}^2 X}} & \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 X \underset{t_\Omega}{\otimes} \Omega C_* \mathbb{E}^2 X \\
\downarrow {\scriptstyle \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 f} & & \downarrow {\scriptstyle \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 f \underset{t_\Omega}{\otimes} \Omega C_* \mathbb{E}^2 f} \ \simeq \\
\Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 Y & \xhookrightarrow[\iota_{\Omega C_* \mathbb{E}^2 Y}]{} & \Omega C_* V \underset{t_\Omega}{\otimes} \Omega^2 C_* \mathbb{E}^2 Y \underset{t_\Omega}{\otimes} \Omega C_* \mathbb{E}^2 Y
\end{array}$$

(where we chose to omit all parentheses for space reasons) is a homotopic $\Omega(C_* \mathbb{E}^2(X)) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2(Y))$-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Proof. We use the notation of Theorem 4.3.6 (however, note that the letter X has a different role now). It will become clear that the diagram in the statement of this theorem is of the same form as the diagram $(**)$ in Section 4.3.1.

Set
$$H := \Omega(C_* \mathbb{E}^2(X)) \qquad \text{and} \qquad K := \Omega(C_* \mathbb{E}^2(Y)).$$

From Remark 4.4.5 it follows that the map $f : X \to Y$ induces a morphism

$$g := \Omega(C_* \mathbb{E}^2(f)) : \Omega(C_* \mathbb{E}^2(X)) \to \Omega(C_* \mathbb{E}^2(Y))$$

of co-commutative Hopf algebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$ that are

- 1-connected, because X and Y are reduced;
- degree-wise finitely generated, by assumption (a);
- and $\Omega(C_* \mathbb{E}^2(Y))_2 = 0$, since Y is 1-reduced.

This shows that the Hopf algebras H and K satisfy Convention 4.3.3. Observe that the inclusion $f : X \hookrightarrow Y$ induces an inclusion

$$(g : H \to K) \qquad = \qquad \Omega(C_* \mathbb{E}^2(f)) : \Omega(C_* \mathbb{E}^2(X)) \to \Omega(C_* \mathbb{E}^2(Y))$$

of dg-coalgebras in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$. Thus, condition (2) of Theorem 4.3.6 holds.

Furthermore, set

$$B := \Omega(C_* W) \qquad \text{and} \qquad A := \Omega(C_* V).$$

The map $\zeta : W \to V$ induces a morphism

$$(\varphi : B \to A) \qquad = \qquad \Omega(C_*(\zeta)) : \Omega(C_*(W)) \to \Omega(C_*(V))$$

of differential graded $\Bbbk$-algebras, where $\Omega(C_*(W))$ has a trivial $\Omega(C_* \mathbb{E}^2(X))$-structure and $\Omega(C_*(V))$ has an $\Omega(C_* \mathbb{E}^2(X))$-structure ρ given by the composite

$$\Omega(C_*(V)) \xrightarrow{\psi_V} \Omega(C_*(V)) \otimes \Omega(C_*(V)) \xrightarrow{\mathrm{Id} \otimes \Omega(C_* \xi)} \Omega(C_*(V)) \otimes \Omega(C_* \mathbb{E}^2(X)).$$

Recall Notation 1.1.7. It follows from assumption (b) that there is an isomorphism $\natural C_*(V) \cong \natural C_*(W) \oplus \natural C_*(V/W)$ of graded $\Bbbk$-modules, and that $\natural C_*(W)$, $\natural C_*(V)$ are degree-wise $\Bbbk$-free and finitely-generated. Moreover, the differentials on the free algebras $\Omega(C_*(V)) \cong T(s^{-1}\natural C_*(W) \oplus s^{-1}\natural C_*(V/W))$ and $\Omega(C_*(W)) = T(s^{-1}\natural C_*(W))$ satisfy the condition of Proposition 4.3.11. So, $\Omega(C_*(V))$ is semi-free as a left $\Omega(C_*(W))$-module, on a generating graded $\Bbbk$-module that is degree-wise finitely generated. Thus, condition (1) of Theorem 4.3.6 holds, as well.

Finally, since $H^{hcoK} := \Omega(H; K; \Bbbk) \cong \Omega(C_* \mathbb{E}^2(X)) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2(Y))$, it follows from Theorem 4.3.6 that the map

$$\Omega(C_* V) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2(f)) : \Omega(C_* V) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2 X) \to \Omega(C_* V) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2 Y)$$

is a homotopic $\Omega(C_* \mathbb{E}^2(X)) \underset{t_\Omega}{\otimes} \Omega^2(C_* \mathbb{E}^2(Y))$-Hopf-Galois extension of chain complexes. □

We now give an example of reasonable conditions on the simplicial data in the diagram (□) that will imply hypothesis (c) of Corollary 4.4.6.

Example 4.4.7. Suppose we are given the following data:

- G, a simplicial group,
- $W \in \mathbf{sSet}_0$ of finite-type, equipped with a right G-action $r : W \times G \to W$,
- $Z \in \mathbf{sSet}_0$ of finite type, and

- $\tau : Z \to G$, a twisting function.

Denote by $\overline{W}G$ the *Kan classifying space* of G (see [May67]) and consider the pullback

$$\begin{array}{ccc} G \underset{\tau}{\times} Z & \longrightarrow & G \underset{\nu_G}{\times} \overline{W}G \\ \big\downarrow & \lrcorner & \big\downarrow \\ Z & \xrightarrow[\overline{\tau}]{} & \overline{W}G \end{array}$$

of the universal G-bundle $G \underset{\nu_G}{\times} \overline{W}G \to \overline{W}G$ along the map $\overline{\tau} : Z \to \overline{W}G$, induced by the couniversal twisting function $\nu_G : \overline{W}G \to G$ in the commutative diagram

$$\begin{array}{ccc} & & Z \\ & \swarrow^{\exists!\,\overline{\tau}} & \big\downarrow \tau \\ \overline{W}G & \xrightarrow{\nu_G} & G. \end{array}$$

Now, set $V = W \times_\tau Z := W \underset{G}{\otimes} (G \underset{\tau}{\times} Z)$, i.e.,

$$V := \operatorname{coequal}\Big(W \times G \times G \underset{\tau}{\times} Z \underset{W \times m_G \times Z}{\overset{r \times G \times Z}{\rightrightarrows}} W \times G \underset{\tau}{\times} Z \Big).$$

We then obtain a commutative diagram of reduced simplicial sets

$$\begin{array}{ccc} W & \overset{\zeta}{\hookrightarrow} & V = W \times_\tau Z \\ & {}_{*}\searrow & \big\downarrow pr_Z \\ & & Z, \end{array}$$

where both W and V are of finite type. By Corollary 4.3.13 it follows that $\Omega C_*V \cong \Omega C_*W \tilde{\otimes} U$ is semi-free as a left ΩC_*W-module on a generating graded module U of finite type. One can then show that the induced map of dg-$\Bbbk$-algebras

$$\Omega C_*(\zeta) : \Omega C_*W \to \Omega C_*V$$

is a homotopic ΩC_*Z-Hopf-Galois extension in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$.

Let us briefly sketch the main steps of the argument. To show that $\Omega C_*(\zeta)$ is a homotopic ΩC_*Z-Hopf-Galois extension, it suffices to show that $i_{\Omega C_*(\zeta)} : \Omega C_*W \to (\Omega C_*V)^{hco(\Omega C_*Z)}$ and $\beta_{\Omega C_*(\zeta)} : \Omega C_*V \otimes_{\Omega C_*W} \Omega C_*V \to \Omega C_*V \otimes \Omega C_*Z$ are quasi-isomorphisms, using the semi-freeness hypothesis on ΩC_*V. Note also that the existence of a Hopf algebra structure on ΩC_*Z is guaranteed because $Z \in \mathbf{sSet}_0$ (see Section 4.4.2).

We have

$$\begin{aligned}(\Omega C_*V)^{hco(\Omega C_*Z)} &\cong \Omega C_*V \otimes_{t_\Omega} \Omega^2 C_*Z\\ &\simeq \Omega C_*W \otimes_{t_\Omega} \Omega C_*Z \otimes_{t_\Omega} \Omega^2 C_*Z\\ &\simeq \Omega C_*W.\end{aligned}$$

using that

$$\Omega C_*V = \Omega C_*(W \underset{\tau}{\times} Z) \xrightarrow{\simeq} \Omega C_*W \underset{t_\Omega}{\otimes} \Omega C_*Z,$$

which is a consequence of Thm 3.15 and a generalization of Thm 3.16 in [?]. This implies that the comparison map is $i_{\Omega C_*(\zeta)}$ is a quasi-isomorphism.

Because $\Omega C_*V \cong \Omega C_*W \tilde{\otimes} U$ is semi-free as a left ΩC_*W-module, there is a commuting diagram of ΩC_*W-modules

$$\begin{array}{ccc} & & \Omega C_*W \otimes_{t_\Omega} \Omega C_*Z \\ & \nearrow & \downarrow \simeq \\ \Omega C_*W & & \\ & \searrow & \\ & & \Omega C_*W \tilde{\otimes} U. \end{array}$$

Using semi-freeness over ΩC_*W, we will have

$$(\Omega C_*W \otimes_{t_\Omega} \Omega C_*Z) \underset{\Omega C_*W}{\otimes} (\Omega C_*W \otimes_{t_\Omega} \Omega C_*Z) \xrightarrow{\simeq} (\Omega C_*W \tilde{\otimes} U) \underset{\Omega C_*W}{\otimes} (\Omega C_*W \tilde{\otimes} U),$$

i.e.,

$$\begin{array}{ccc} \Omega C_*W \otimes_{t_\Omega} (\Omega C_*Z \otimes \Omega C_*Z) & \xrightarrow{\simeq} & \Omega C_*W \tilde{\otimes}(U \otimes U) \\ \simeq \downarrow & & \\ \Omega C_*V \otimes \Omega C_*Z & & \end{array}$$

which will imply that the Galois map $\beta_{\Omega C_*(\zeta)}$ is a quasi-isomorphism, too.

Remark 4.4.8. To apply Corollary 4.4.6 to the simplicial context of the previous example, one needs to restrict to $Z := \mathbb{E}^2 X$ and to consider a map of simplicial sets $\mathbb{E}^2 f : \mathbb{E}^2 X \to \mathbb{E}^2 Y$, where both X, $Y \in \mathbf{sSet}_0$ are of finite type.

Remark 4.4.9. We started this chapter with a review on the classical Galois correspondence for fields, and John Rognes's analog of it for ring spectra was our motivating starting point. We will end it with a very brief remark on (Hopf)-Galois correspondence results in other contexts.

The Galois correspondence holds for Galois extensions of commutative rings, if the codomain ring is *connected* (i.e., if its only idempotents are 0 and 1). More precisely, given a G-Galois extension of commutative rings $\alpha : R \hookrightarrow S$, with S connected, there is a bijection of sets

$$\{\text{separable sub-}R\text{-algebras of } S\} \longleftrightarrow \{\text{subgroups of } G\},$$

(see Theorem 2.3, [CHR65]). (An R-algebra A is called *separable* if it is projective as a module over $A \otimes_R A$, with action given by $(a \otimes a') \cdot x := axa'$, for all $a, a,' x \in A$).

A detailed discussion on various attempts to establish a Hopf-Galois correspondence for classical Hopf-Galois extensions of algebras can be found in §6 in [Mon09].

Chapter 5

Perspectives

As a follow-up to this work, one could consider the following open questions.

- Can one obtain an analog of Theorem 4.3.6 if the hypotheses on the Hopf algebras H and/or K are relaxed, e.g., if they are not assumed to be co-commutative?

- How one should proceed if one desires to establish the other direction of the homotopic Hopf-Galois correspondence in $\mathbf{Ch}_{\Bbbk}^{\geq 0}$? More precisely, the question to answer would be the following.

  If $\varphi : \mathrm{Triv}(B) \to A$ is a homotopic H-Hopf-Galois extension, such that the underlying morphism factors as $B \to C \xrightarrow{\varphi'} A$, when is φ' a homotopic H'-Hopf-Galois extension, and what the associated Hopf algebra H' is exactly?

  Note that in the classical Galois case, as well as in the Hopf-Galois case, this direction of the correspondence is more difficult to establish (see Chapter 6 in [Mon09]).

- Another interesting topic will be to explore the relation of homotopic Hopf-Galois extensions with localization and Quillen homology.

- It will be interesting to see whether and how it is possible to establish results similar to Propositions 3.2.7 and 3.3.5 and to Theorem 4.3.6 in other contexts, e.g., working in a simplicial monoidal category, in a monoidal dg-category, in a monoidal category enriched over spectra, or in a monoidal category enriched over a model monoidal category $\mathcal{V}$.

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Supervised three Semester Projects (Bachelor and Master level) and six Mini-projects (*Travaux pratiques de mathématiques*, Bachelor level)

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Teaching and promotion of mathematics to young people

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March - April 2010 and 2011 — Co-organized the Swiss (semi-)finals of *Jeux Mathématiques et Logiques*

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Conferences and workshops organized

2011 and 2013 — Co-organized two editions of annual international conferences for young researchers in mathematics "Young Topologists' Meeting"

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