GLOBAL REGULARITY FOR THE YANG–MILLS EQUATIONS ON HIGH DIMENSIONAL MINKOWSKI SPACE

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Abstract. We study here the global Cauchy problem for the Yang-Mills equations on $(6+1)$ and higher dimensional Minkowski space, when the initial data sets are small in the critical gauge covariant Sobolev space $\dot{H}_A^{(n-4)/2}$. Regularity is obtained through a certain a "microlocal geometric renormalization" of the equations which is constructed via a family of approximate Coulomb gauge–null Crönstrom gauge transformations. The proof is then reduced to controlling Hodge systems and degenerate elliptic equations on high index and non-isotropic L^p spaces, as well as the proof of some bilinear estimates in auxiliary square-function spaces.

1. Introduction

In this work we investigate the global in time regularity properties of the Yang-Mills equations on high dimensional Minkowski space with compact semi-simple gauge group G. Specifically, we show that if a certain gauge covariant Sobolev norm is small, the so called *critical* regularity $\dot{H}_A^{\frac{n-2}{2}}$, and the dimension satisfies $6 \leqslant n$, then a global solution exists and remains regular for all times given that the initial data is regular. This is in the same spirit as the recent result [8] for the Maxwell-Klein-Gordon system, as well as earlier results for high dimensional wave-maps (see [11], [6], [9], and [7]). Our approach shares many similarities with those works, whose underlying philosophy in basically the same. That is, to introduce Coulomb type gauges in order to treat a specific potential term as a quadratic error. In our setup, we use a non-abelian variant of the remarkable parametrix construction contained in [8], in conjunction with a version of the Uhlenbeck lemma [13] on the existence of global Coulomb gauges. This latter result has been used for high dimensional wave-maps to globally "renormalize" the equation so that the existence theory can be treated directly through Strichartz estimates applied to multi-linear expressions. In the present situation, as was the case with the Maxwell-Klein-Gordon system, the corresponding renormalization procedure is necessarily more involved because it needs to be done separately for each distinct direction in phase space. That is, we provide a renormalization of the Yang-Mills equations through the construction of a Fourier integral operator with G-valued phase. The construction and estimation of such an object relies heavily on elliptic-Coulomb theory, primary due to the difficulty one faces in that the G-valued phase function cannot be localized within a neighborhood of any given point on the group due to the critical nature of the problem (if you like, there is a logarithmic "twisting" of the group element as one moves around in physical space; fortunately the group is compact so this doesn't ruin things).

To get things started, we now give a simple gauge covariant description of the equations we are considering. The (hyperbolic) Yang-Mills equations arise as the evolution equations for a connection on the bundle $V = \mathcal{M}^n \times \mathfrak{g}$, where $\mathcal{M}^n$ is some n (spatial) dimensional Minkowski space, with metric $g := (-1, 1, \ldots, 1)$ in inertial coordinates (x^0, x^i), and $\mathfrak{g}$ is the Lie algebra of some compact semi-simple Lie group G. Here we are considering V with the $Ad(G)$ gauge structure. If ϕ is any section to V over $\mathcal{M}$, then a connection assigns to every vector-field X on the base $\mathcal{M}^n$, a derivative which we denote as D_X, such that the following Leibniz rule is satisfied for every scalar field f:

$$D_X(f\phi) \ = \ X(f)\phi + fD_X\phi \ .$$

In this setup, we assume that V is equipped with an $Ad(G)$ invariant metric $\langle \cdot, \cdot \rangle$ which respects the action of D. That is, one has the formula:

$$d\langle \phi, \psi \rangle \ = \ \langle D\phi, \psi \rangle + \langle \phi, D\psi \rangle \ . \tag{1}$$

In the present situation we will take $\langle \cdot, \cdot \rangle$ to be the Killing form on $\mathfrak{g}$. The curvature associated to D is the $\mathfrak{g}$ valued two-form F which arises from the commutation of covariant derivatives and is defined via the formula:

$$D_X D_Y \phi - D_Y D_X \phi - D_{[X,Y]}\phi \ = \ [F(X,Y), \phi] \ . \tag{2}$$

We say that the connection D satisfies the *Yang-Mills* equations if its curvature is a (formal) local minima of the following Maxwell type functional:

$$\mathcal{L}[F] \ = \ -\frac{1}{4} \int_{\mathcal{M}^n} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle \ DV_{\mathcal{M}^n} \ . \tag{3}$$

The Euler-Lagrange equations of (3) read:

$$D^\beta F_{\alpha\beta} \ = \ 0 \ . \tag{4}$$

Also, from the fact that F arises as the curvature of some connection, we have that the following identity known as "Bianchi" is satisfied:

$$D_{[\alpha} F_{\beta\gamma]} \ = \ 0. \tag{5}$$

From now on we will refer to the system (4)–(5) as the first order Yang–Mills equations (FYM).

As we have already mentioned, our aim is to study the regularity properties of the Cauchy problem for the (FYM) system. To describe this in a geometrically invariant way, we make use of the following splitting of the connection-curvature pair (F, D): Foliating $\mathcal{M}$ into the standard Cauchy hypersurfaces $t = const.$, we decompose:

$$(F, D) \ = \ (\underline{F}, \underline{D}) \oplus (E, D_0) \ , \tag{6}$$

where $(\underline{F}, \underline{D})$ denotes the portion of (F, D) which is tangent to the surfaces $t = const.$ (i.e. the induced connection), and (E, D_0) denotes respectively the interior product of F with the foliation generator $T = \partial_t$, and the normal portion of D. In inertial coordinates we have:

$$E_i \ = \ F_{0i} \ .$$

On the initial Cauchy hypersurface $t = 0$ we call a set $(\underline{F}(0), \underline{D}(0), E(0))$ *admissible Cauchy data*[1] if it satisfies the following compatibility condition:

$$\underline{D}^i E_i(0) \ = \ 0 \ .$$

We define the *Cauchy problem* for the Yang-Mills equation to be the task of construction a connection (F, D) which solves (4), and has Cauchy data equal to $(\underline{F}(0), \underline{D}(0), E(0))$.

In order to understand what the appropriate condition on the initial data should be (and what we would like it to be!), it is necessary to consider the following two basic mathematical features of the system (4)–(5). The first is *conservation*. From the Lagrangian nature of the field equations (4)–(5), we have the tensorial conservation law:

$$\begin{aligned} Q_{\alpha\beta}[F] \ &= \ \langle F_{\alpha\gamma}, F_\beta{}^\gamma \rangle - \frac{1}{4} g_{\alpha\beta} \langle F_{\gamma\delta}, F^{\gamma\delta} \rangle \ , \\ \nabla^\alpha Q_{\alpha\beta}[F] \ &= \ 0 \ , \end{aligned}$$

where ∇ is the covariant derivative on $\mathcal{M}^n$. In particular, contracting Q with the vector-field $T = \partial_t$, we arrive at the following constant of motion for the system (4)–(5):

$$\int_{\mathbb{R}^n} Q_{00} \, dx \ = \ \frac{1}{2} \int_{\mathbb{R}^n} \left(|E|^2 + |\underline{F}|^2 \right) dx \ . \tag{7}$$

The second main aspect of the system (4)–(5) is that of *scaling*. If we perform the transformation:

$$(x^0, x^i) \ \rightsquigarrow \ (\lambda x^0, \lambda x^i) \ , \tag{8}$$

on $\mathcal{M}^n$, then an easy calculation shows that:

$$D \ \rightsquigarrow \ \lambda D \ , \qquad\qquad F \ \rightsquigarrow \ \lambda^2 F \ . \tag{9}$$

If we now define the gauge covariant (integer) Sobolev spaces:

$$\| F \|_{\dot{H}^s_A}^2 \ := \ \sum_{|I|=s} \| \underline{D}^I F \|_{L^2(\mathbb{R}^n)}^2 \ , \tag{10}$$

where for each multiindex $I = (i_1, \ldots, i_s)$ we have that $D^I = D_{\partial_{i_1}} \ldots D_{\partial_{i_s}}$ is the repeated covariant differentiation with respect to the translation invariant spatial vector-fields $\{\partial_1, \ldots, \partial_n\}$, then for even[2] spatial dimensions, the norm $\dot{H}_A^{\frac{n-4}{2}}$ is invariant with respect to the scaling transformation (9). In particular, the conserved

[1] Of course, this set is overdetermined as the curvature $\underline{F}$ depends completely on the connection $\underline{D}$. Also, it is perhaps not completely obvious at first that the set $(\underline{F}(0), \underline{D}(0), E(0))$ determined *uniquely* a solution (F, D) to (4)–(5). For example, the initial normal derivative $D_0(0)$ does not need to be specified. We will show this is the case in the sequel (in particular see Proposition 5.3).

[2] For odd spatial dimensions, the above discussion needs to be modified somewhat because we will not make an attempt here to define fractional powers of the spaces $\dot{H}^s_A$. Instead, what one should do is to simply put things in a Coulomb gauge and then use the usual fractional Sobolev spaces. This later approach is what we will take in the sequel, although for sake of concreteness we will only discuss the case of even dimensions. We have opted for the covariant approach in the introduction because it makes stating our main result a bit easier, and has an appealing simplicity. Also, since we shall need many specifics on how Coulomb gauges are constructed in order to create and control our parametrix, we will explain how the Coulomb gauge relates to the Cauchy problem in detail in the following two sections.

quantity (7) is invariant when $n = 4$ and this is called the *critical* dimension.

Now, based on numerical evidence as well as analytical arguments, it is not suspected that in general the Cauchy problem for (4)–(5) with smooth initial data will not be well behaved without size control of the critical regularities $s_c = \frac{n-4}{2}$ in high dimensions. What we will take this statement to mean here is simply that if $4 \leqslant n$ and the $\dot{H}^{s_c}_A$ norm of the initial data is not sufficiently small, then one can expect the existence of regular (i.e. C^∞_A) sets $(\underline{F}(0), \underline{D}(0), E(0))$ such that the corresponding solution to (4)–(5) will develop a singularity in finite time. By singularity development, we mean that some higher norm of the type (10) will fail to be bounded at a later time, given that it was initially; or even more specifically, that the L^∞ norm of the curvature F will blow up in finite time for some regular initial data sets. Since these norms are gauge covariant, this type of singularity development would correspond to an intrinsic geometric breakdown of the equations, and could not be an artifact of poorly chosen local coordinates (gauge) on V. This has been rigorously demonstrated in the equivariant category for the *supercritical* dimensions $5 \leqslant n$ (see [3]). In the critical dimension, things are much less clear, although there is numerical evidence that there is still in fact blowup for large data (see [2]). This is thought to be connected with the existence of large static solutions (instantons). One possible conjecture is that there is global regularity when the norm (7) is below the ground state energy.

Going in the other direction, it is expected that if the critical norm $\dot{H}^{\frac{n-2}{2}}_A$ is sufficiently small, then regular initial data will remain regular for all times. This can be seen as an easier preliminary step toward understanding in detail the issue of large data for dimension $n = 4$, and is furthermore an interesting problem in its own right. A central difficulty in the demonstration of this conjecture is to construct a stable set coordinates on the bundle V such that the Christoffel symbols of D are well behaved in the sense that they obey the natural range of estimates one expects for this type of problem. This is precisely what we shall do in dimensions $6 \leqslant n$ through the well known procedure of using (spatial) Coulomb gauges. Unfortunately, this preliminary gauge construction is far from sufficient to close the regularity argument, and it will in fact be necessary for us to go much further and control infinitely many Coulomb gauges, each of which correspond to a distinct polarized plane wave solution to the usual (flat) wave equation $\Box = \nabla^\alpha \nabla_\alpha$. However, this does not effect the statement of our main result which is in fact quite simple:

Theorem 1.1 (Critical regularity for high dimensional Yang-Mills)**.** *Let the number of spatial dimensions be even and such that $6 \leqslant n$. Then there exists fixed constants $0 < \varepsilon_0, C$ such that if $(\underline{F}(0), \underline{D}(0), E(0))$ is an admissible data set which satisfies the smallness condition:*

$$\| (\underline{F}(0), E(0)) \|_{\dot{H}^{\frac{n-4}{2}}_A} \ \leqslant \ \varepsilon_0 \ , \tag{11}$$

and there exists constants $M_k < \infty$, $\frac{n-4}{2} < k \in \mathbb{N}$ such that:

$$\| (\underline{F}(0), E(0)) \|_{\dot{H}^k_A} \ = \ M_k \ , \tag{12}$$

then there exists a unique global solution to the field equations (4)–(5) with this initial data, and furthermore one has that the following inductive norm bounds *hold:*

$$\begin{aligned} \| F \|_{\dot{H}_A^{\frac{n-4}{2}}} \ &\leqslant \ C\varepsilon_0 \ , \\ \| F \|_{\dot{H}_A^k} \ &\leqslant \ C(M_{\frac{n-4}{2}}, \ldots, M_{k-1})\, M_k \ . \end{aligned}$$

In particular, in this case F remains smooth (in the gauge covariant sense) and bounded for all times.

Remark 1.2. As alluded to above, we will more specifically prove the existence of a global (in space and time) spatial Coulomb gauge such that the coefficient functions of the curvature F, as well as the Christoffel symbols (gauge potentials) of the connection D are in the classical Sobolev spaces $\dot{H}^s$, and such that they satisfy appropriate angularly and spatially microlocalized Strichartz estimates. We have elected to eliminate a discussion of this from the statement from the main theorem in favor of the simpler geometric language so that the reader can at a first glance gain an idea of the content of our result without being confronted with too many technical details.

2. Some Basic Notation

We list here some of the basic conventions used in this work, as well as some constants which will be needed in the sequel. We use the usual notation $a \lesssim b$, to denote that $a \leqslant C \cdot b$ for some (possibly large) constant C which may change from line to line. Likewise we write $a \ll b$ to mean $a \leqslant C^{-1} \cdot b$ for some *large* constant C. In general, C will denote a large constant, but at times we will also call C a connection. The difference should be clear from context. Overall, we will have use for a family of small constants, which satisfy the hierarchy:

$$0 \ < \ \varepsilon_0, \epsilon_0 \ \ll \ \widetilde{\epsilon_0} \ \ll \ \mathcal{E} \ \ll \ \gamma \ \ll \ \delta \ \ll \ 1 \ . \tag{13}$$

3. Some gauge-theoretic preliminaries

In this paper, we are working with a compact semi-simple group Lie G. However, all of our calculations will be carried out in a somewhat larger context. Firstly, we will assume that G is embedded as a subgroup of matrices of some (possibly) larger orthogonal group $O(m)$. In particular, we can identify the Lie algebra $\mathfrak{g}$ with an appropriate sub-algebra of $\mathfrak{o}(m)$. This allows us to perform all of our calculations on a specific collection of matrices. Since our main computation involves complex valued integral operators, we will further need to work in the complexified algebra $\mathbb{C} \otimes \mathfrak{o}(m)$. The Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ extends easily to this context to yield the bilinear form:

$$\langle A, B \rangle \ = \ -\operatorname{trace}(AB^*) \ . \tag{14}$$

Notice that this is a positive definite form when restricted to the real vector space $\mathfrak{o}(m)$, and is a sesquilinear form on the corresponding complexified algebra $\mathbb{C} \otimes \mathfrak{o}(m)$. More importantly, $\langle \cdot, \cdot \rangle$ is $Ad(O(m))$ invariant, and in fact the more general identity holds:

$$\langle g_1^{-1} A h_1, g_2^{-1} B h_2 \rangle \ = \ \langle g_2 g_1^{-1} A h_1 h_2^{-1}, B \rangle \ , \tag{15}$$

for $A, B \in \mathbb{C}\otimes\mathfrak{o}(m)$ and $g_i, h_i \in O(m)$. It will be crucial for us to have an extension of these last two properties to the context of all complex matrices $M(m \times m)$. To do this we treat these matrices as vectors in $\mathbb{C}^{m^2}$ and form the general matrix inner product:

$$\langle A, B\rangle \ = \ \sum_{i,j} a_{ij}\overline{b}_{ij} \ . \tag{16}$$

Notice that this restricts to (14) on the subalgebra $\mathbb{C}\otimes\mathfrak{g}$. Also, it is easy to see that the general adjoint formula (15) continues to hold in this context. This will be of fundamental importance in the sequel. In general, we will use the notation $\| A \|^2 = \langle A, A\rangle$ to denote the action of this norm on any matrix. Also, notice that one has the isometric identity:

$$\| gA \| \ = \ \| A \| \ , \qquad\qquad g \ \in \ O(m) \ . \tag{17}$$

These are all very simple algebraic identities, but our method is incredibly sensitive to them and would collapse entirely if they did not hold.

In the context of matrices, we may compute the action of the connection D on sections F to V as follows:

$$D_X F \ = \ X^\alpha\big(\nabla_\alpha(F) + [A_\alpha, F]\big) \ ,$$

Here, the gauge potentials $\{A_\alpha\}$ are $\mathfrak{g}$-valued, and are defined via the equation:

$$D_\alpha \mathbf{1}_V \ = \ [A_\alpha, \mathbf{1}_V] \ ,$$

where $\mathbf{1}_V$ denotes some chosen orthonormal frame in V, and we are abusively writing $F = F\mathbf{1}_V$. In shorthand notation, we write:

$$D \ = \ d + A \ , \tag{18}$$

where d is the usual exterior derivative on matrix valued functions. Likewise, in this notation we have the well known identity for the curvature of D:

$$F^A \ = \ dA + [A, A] \ . \tag{19}$$

In this last formula, we use the superscript notation to emphasize the fact that the curvature is *not* gauge invariant, but transforms according to the $Ad(G)$ action:

$$F \ \rightsquigarrow \ gFg^{-1} \ ,$$

whenever one performs the change of frame $\mathbf{1}_V \rightsquigarrow g\mathbf{1}_V g^{-1}$. As is well known, the potentials $\{A_\alpha\}$ themselves do not transform according to $Ad(G)$, but instead take on an affine group of transformations:

$$B \ = \ gAg^{-1} + g\,dg^{-1} \ , \tag{20}$$

where $\{B_\alpha\}$ represents the connection D in the frame $g\mathbf{1}_V g^{-1}$. In particular, the difference of two connections obeys the $Ad(G)$ structure, a fact we will have use for in a moment. For instance, any connection $\{C_\alpha\}$ with $F^C = 0$ obeys $Ad(G)$. Furthermore, as is the basic fact of gauge theory, such connections always lead to a globally[3] integrable ODE:

$$dg \ = \ g\,C \ ,$$

[3]Of course this ODE is non-linear, but in the present context it also satisfies the conservation law $gg^\dagger = I$.

where the solution g belongs to G. Thus, we may identify flat connections C with infinitesimal gauge transformations, and it is easy to see that every gauge transformation (20) leads to a flat connection which we may define as $C = g^{-1}dg$. This completes our discussion of elementary gauge theory.

It will also be necessary for us to make use of the basic facts from (non-gauge-covariant) Hodge theory. Even though the connections we work with in this paper are on the full space-time $\mathcal{M}^n$, our use of Hodge theory will always be restricted to time slices $\{t\} \times \mathbb{R}^n$. In particular we use the general notation d, d^* for the exterior derivative and its adjoint acting on $\mathfrak{g}$ (and more generally $M(m \times m)$) valued differential forms on $\mathbb{R}^n$. To emphasize this restriction, we will use Latin indices when computing these operators. For example:

$$(dA)_{ij} \ = \ \nabla_{\{i}A_{j\}} \ , \qquad\qquad (dF)_{ijk} \ = \ \nabla_{[i}F_{jk]} \ ,$$

where $\{\ldots\}$ and $[\ldots]$ denote symmetric and antisymmetric cyclic summing respectively. Also, the adjoint here is taken with respect to the Killing form (14). In particular, we have the Hodge Laplacean:

$$\Delta \ = \ -(dd^* + d^*d) \ , \tag{21}$$

which in our context is simply the usual scalar Laplacean acting component-wise on matrices. Finally, we have the Hodge decomposition which we write as $A = A^{df} + A^{cf}$ where:

$$\begin{aligned} A^{df} \ &= \ -dd^*\Delta^{-1}A \ , \\ A^{cf} \ &= \ -d^*d\,\Delta^{-1}A \ . \end{aligned}$$

This decomposition is bounded on L^p spaces for $1 < p < \infty$ as the operators involved are SIO's. Also, since these operators are all real, this decomposition respects the Lie algebra structure of $\mathfrak{g}$ inside of $\mathbb{C} \otimes \mathfrak{o}(m)$.

The last topic we cover here is the basic underpinning of much of analysis in the context of compact gauge groups. This is the remarkable Uhlenbeck lemma, which allows one to "straighten out" a connection as long as its curvature satisfies appropriate bounds. The important thing for us is that these bounds are precisely at the level of the critical regularity $\dot{H}_A^{\frac{n-2}{2}}$. This result is:

Lemma 3.1 (Classical Uhlenbeck lemma)**.** *Let $D^A = d + A$ be a connection with compact (matrix) group on $\mathbb{R}^n$. Then there is a constant ϵ_0 which only depends on n such that if the curvature F^A of D^A satisfies the bound:*

$$\| F^A \|_{L^{\frac{n}{2}}} \ \leqslant \ \epsilon_0 \ ,$$

then D^A is gauge equivalent to a connection $D^B = d + B$ where the potentials $\{B_i\}$ satisfy the condition:

$$d^*B \ = \ 0.$$

In the sequel, it will be useful for us to have a somewhat more refined version of Lemma 3.1 which does not make reference to the size of the curvature, but rather to the size of the connection $\{A_\alpha\}$ itself in a critical norm which does *not* involve derivatives. This will allow us to prove certain connections exist more directly.

Furthermore, since the basic formulas used in the proof of this result will be important in constructing our parametrix, it will set the pace for much of what follows. Finally, we mention here that our proof is a bit different from that of [13] in that it does not rely on any implicit function theorem type arguments, and is instead completely explicit being based on a simple Picard iteration.

Lemma 3.2 (Uhlenbeck lemma for small L^n perturbations of Coulomb potentials with small $L^{\frac{n}{2}}$ curvature.)**.** *Let $D = d + A$ be a connection on $\mathbb{R}^n \times V$ with compact (matrix) gauge group G. Then there exists a constant ϵ_0 such that if:*

- $\| F^A \|_{L^{\frac{n}{2}}} \ \leqslant \ \epsilon_0$,
- *$d + A$ is gauge equivalent to $d + B$ with $d^* B = 0$* ,

then for every connection $d + \widetilde{A}$ such that:

$$\| \widetilde{A} - A \|_{L^n} \ \leqslant \ \epsilon_0 \ , \tag{22}$$

there exists a gauge equivalent connection $d + \widetilde{B}$ such that $d^* \widetilde{B} = 0$.

Remark 3.3. Before continuing with proof, let us remark here that Lemma 3.2 is in fact more general that the classical Uhlenbeck Lemma. Specifically, 3.2 easily implies 3.1 with smallness condition $\epsilon_0/2$ (where ϵ_0 is determined by Lemma 3.2) through a straightforward induction procedure which we outline now.

First of all, from Lemma 3.2 we see that the set of all connections $d + A$ with curvature such that:

$$\| F^A \|_{L^{\frac{n}{2}}} \ \leqslant \ \frac{\epsilon_0}{2} \ , \tag{23}$$

and such that $d + A$ is equivalent to $d + B$ with $d^* B = 0$ is an open set in the intersection of L^n with the set determined by (23) (in the sense of distributions). Therefore, if the conclusion of Lemma (3.1) were to be violated, it must then be the case that there is a smallest number r^* such that the sphere of radius r^* contains a connection $d + A$ with the property that it *cannot* be put in the Coulomb gauge, even though the bound (23) is valid for this connection. Now, consider the set of connections $d + \lambda A$ where $0 < (1 - \lambda) \ll 1$. A quick calculation shows that these have curvature:

$$F^{\lambda A} \ = \ \lambda F^A + \lambda(\lambda - 1)[A, A] \ .$$

Choose λ such that:

$$(1 - \lambda) \ \leqslant \ (1 + r^*)^{-2} \cdot \frac{\epsilon_0}{2} \ .$$

By the triangle and Hölders inequality, and the definition of r^*, we have that:

$$\| F^{\lambda A} \|_{L^{\frac{n}{2}}} \ \leqslant \ \epsilon_0 \ .$$

Therefore, by the minimality of r^* we have that $d + \lambda A$ can be Coulomb gauged. Again, by the definition of λ, we have that:

$$d + A \ = \ d + \lambda A + \widetilde{A} \ ,$$

where we have the bound:

$$\| \widetilde{A} \|_{L^n} \ \leqslant \ \epsilon_0 \ .$$

Therefore, by an application of Lemma 3.2 we have that $d + A$ can be put in the Coulomb gauge. This contradicts the minimality of r^* as was to be shown.

Proof of Lemma 3.2. It suffices to show that $d + \widetilde{A}$ is gauge equivalent to $d + \widetilde{B}$, with $d^*\widetilde{B} = 0$, provided that:

$$\| \widetilde{A} \|_{L^n} \leqslant C\epsilon_0 \ , \tag{24}$$

with ϵ_0 chosen sufficiently small, where C is some fixed constant. To see this, notice that the smallness condition (22) is gauge invariant because the difference of two connections transforms according to the $Ad(G)$ action which fixes the Killing form used to compute $\| \cdot \|_{L^n}$. Therefore, we may assume from the start that the original connection A is in the Coulomb gauge. That is, it satisfies the div-curl system:

$$\begin{aligned} dA \ &= \ F^A - [A, A] \ , \\ d^*A \ &= \ 0 \ . \end{aligned}$$

Using the fact that $\| F^A \|_{L^n}$ is sufficiently small in conjunction with some standard[4] elliptic estimates, we have that:

$$\| A \|_{L^n} \ \lesssim \ \| F^A \|_{L^{\frac{n}{2}}} \ \leqslant \ \epsilon_0 \ . \tag{25}$$

In particular we may assume that $\| A \|_{L^n} \leqslant \frac{C}{2}\epsilon_0$ in (22), hence the condition (24).

We now construct by hand the gauge transformation:

$$dg \ = \ g\widetilde{A} - \widetilde{B}g \ , \tag{26}$$

with $d^*\widetilde{B} = 0$. This will be done by constructing the infinitesimal gauge transformation $C = g^{-1}dg$. A quick calculation shows that this must satisfy the following div-curl system:

$$dC \ = \ -[C, C] \ , \tag{27a}$$

$$d^*C \ = \ d^*\widetilde{A} + [\widetilde{A}, C] \ . \tag{27b}$$

Unfortunately, the system (27) cannot be solved constructively, say through an iteration scheme. This is because implicit in its structure is the compatibility condition $d^*[C, C] = 0$, which gets destroyed through (at least the usual) Picard iteration. This could be side-stepped by using an implicit function theorem type argument, but since we prefer to do things explicitly we proceed as follows: We first write the system (27) in terms of integral equations:

$$C^{df} \ = \ \frac{d^*}{\Delta}[C, C] \ , \tag{28a}$$

$$C^{cf} \ = \ \frac{d}{\Delta}\left(-d^*\widetilde{A} - [\widetilde{A}, C]\right) \ . \tag{28b}$$

Here $C = C^{df} + C^{cf}$ denotes the Hodge decomposition of the matrix valued one-form C. A solution to system (28) can now be constructed from scratch via Picard iteration starting with $C^{(0)} = 0$. The condition (24) and the embeddings:

$$\begin{aligned} \nabla^2\Delta^{-1} \ &: \ L^n \ \hookrightarrow \ L^n \ , \\ \nabla\Delta^{-1} \ &: \ L^{\frac{n}{2}} \ \hookrightarrow \ L^n \ , \end{aligned}$$

[4] We will use this kind of argument many times in the sequel. For the reader who is unfamiliar with it, see the next paragraph where we prove these in a slightly different context.

guarantee convergence to a solution. Furthermore, because it is true for each iterate, one has the bounds:

$$\| C \|_{L^n} \lesssim \| \widetilde{A} \|_{L^n} \lesssim \epsilon_0. \tag{29}$$

Also, since each iterate belongs pointwise to $\mathfrak{g}$, the solution does also due to the fact that $\mathfrak{g}$ is a linear (and hence closed) subspace of the matrices $M(m \times m)$. We now need to show that this C is indeed a solution to the original system (27). That is, we need to establish the identity:

$$d\, d^* \Delta^{-1} [C, C] \ = \ - [C, C] \ . \tag{30}$$

Notice that this does not follow *algebraically* from the form of the integral system (28), because it is not a-priori clear that in fact $d[C, C] = 0$. However, this is the case, which is a consequence of the following a-priori *estimate* for solutions to (28):

$$\| d\, d^* \Delta^{-1} [C, C] + [C, C] \|_{L^{\frac{n}{2}}} \ \lesssim \ \| C \|_{L^n} \cdot \| d\, d^* \Delta^{-1} [C, C] + [C, C] \|_{L^{\frac{n}{2}}} \ . \tag{31}$$

Notice that (29) and (31) taken together immediately imply the identity (30).

In order to show (31), we first use the Hodge Laplacean (21) to write:

$$d\, d^* \Delta^{-1} [C, C] + [C, C] \ = \ - d^* \Delta^{-1} \left(d[C, C] \right) \ .$$

Next, we compute that:

$$\begin{aligned} (d[C, C])_{ijk} \ &= \ \nabla_{[i} [C_j, C_{k]}] \ , \\ &= \ [\nabla_{[i} C_j, C_{k]}] \ - \ [\nabla_{[i} C_k, C_{j]}] \ , \\ &= \ - [C_{[i}, (dC)_{jk]}] \ . \end{aligned}$$

Therefore, using this last identity in conjunction with fractional integration, and using the identity from line (28a) above, we have that:

$$\begin{aligned} \| d\, d^* \Delta^{-1} [C, C] + [C, C] \|_{L^{\frac{n}{2}}} \ &= \ \| d^* \Delta^{-1} [C, dC] \|_{L^{\frac{n}{2}}} \ , \\ &\lesssim \ \| [C, dC] \|_{L^{\frac{n}{3}}} \ , \\ &\leqslant \ \| \big[C, (dd^* \Delta^{-1} [C, C] + [C, C]) \big] \|_{L^{\frac{n}{3}}} \ + \ \| \big[C, [C, C] \big] \|_{L^{\frac{n}{3}}} \ , \\ &\leqslant \ 2 \| C \|_{L^n} \cdot \| dd^* \Delta^{-1} [C, C] + [C, C] \|_{L^{\frac{n}{2}}} \ . \end{aligned}$$

Notice that the last inequality here follows simply from the Jacobi identity $\big[C, [C, C] \big] = 0$.

To wrap things up, we only need to establish the existence of g on line (26) above with $d^* \widetilde{B} = 0$. By design we have that $F^C = 0$, so we may integrate the equation:

$$dg \ = \ gC \ ,$$

with initial conditions $g(0) = I$ on all of $\mathbb{R}^n$.Defining now:

$$\widetilde{B} \ = \ g \widetilde{A} g^{-1} + g\, dg^{-1} \ ,$$

we have that:

$$\begin{aligned} -\,d^*\widetilde{B} \ &= \ D_i^{\widetilde{B}}\widetilde{B}^i \ , \\ &= \ g\,D_i^{\widetilde{A}}(g^{-1}\widetilde{B}^i g)\,g^{-1} \ , \\ &= \ g\,D_i^{\widetilde{A}}(\widetilde{A}^i - C^i)\,g^{-1} \ , \\ &= \ g\Big(-\,d^*\widetilde{A} + d^*\,C - [\widetilde{A}, C]\Big)\,g^{-1} \ , \\ &= 0, \end{aligned}$$

as was to be shown. This completes the proof of Lemma 3.2. □

4. Some analytic preliminaries

We record here some useful formulas, mostly from elementary harmonic analysis, which will be used many times in the sequel. Firstly, we define the Fourier transform on $\mathbb{C}\otimes\mathfrak{o}(m)$, which is merely the usual scalar Fourier transform acting componentwise on matrices:

$$\widehat{A}(\xi) \ = \ \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} \ A(x) \ dx \ . \tag{32}$$

The Plancherel theorem with respect to the Killing form (14) reads:

$$\int_{\mathbb{R}^n_x} \langle A, B\rangle \ dx \ = \int_{\mathbb{R}^n_\xi} \langle \widehat{A}, \widehat{B}\rangle \ d\xi \ .$$

This follows simply from definition of the inner product (16). While the constructions we make in the sequel are almost explicitly based on the spatial transform (32), it will in certain places be convenient for us to work with the space-time Fourier transform:

$$\widehat{A}(\tau,\xi) \ = \ \int_{\mathbb{R}^{n+1}} e^{-2\pi i(t\tau + x\cdot\xi)} \ A(t,x) \ dt\,dx \ .$$

In the sequel, we will have much use for dyadic frequency decompositions with respect to the spatial variable. For the most part, we will use a fairly loose and heuristic notation for this operation. This will help us to avoid having to come up with different symbols for multipliers which are basically the same. First of all, we let $\chi(\xi)$ denote some smooth bump function adapted to the unit frequency annulus $\{2^{-a} \leqslant |\xi| \leqslant 2^a\}$, where $1 \leqslant a$ is some constant used to define χ which may change from line to line. For a dyadic number $\mu \in \{2^i \ \big| \ i \in \mathbb{Z}\}$, we define the rescaled cutoffs:

$$\chi_\mu(\xi) \ = \ \chi(\mu^{-1}\xi) \ ,$$

and the associated Fourier multipliers $\widehat{P_\mu A} = \chi_\mu \widehat{A}$. The two main facts we will need about these multipliers is the Bernstein inequality:

$$\|\, P_\mu A \,\|_{L^p} \ \lesssim \ \mu^{n(\frac{1}{q}-\frac{1}{p})} \ \|\, A \,\|_{L^q} \ , \tag{33}$$

which holds for all $1 \leqslant q \leqslant p \leqslant \infty$, and the Littlewood-Paley equivalence:

$$\|\, (\sum_\mu |P_\mu A|^2)^{\frac{1}{2}} \,\|_{L^p} \ \sim \ \|\, A \,\|_{L^p} \ , \tag{34}$$

which holds under the restriction $1 < p < \infty$. All of the norms above can be taken with respect to (14).

There are two simple analysis lemmas involving derivatives and multipliers which will come in useful in the sequel. The first of these is the low frequency (operator) commutator estimate:

$$\| [A, P_1] \cdot F \|_{L^p} \ \lesssim \ \| \nabla_x A \|_{L^q} \cdot \| F \|_{L^r} \ , \tag{35}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ (see [8]). The second is the homogeneous paraproduct estimate:

$$\| \nabla_x^k (A \cdot F) \|_{L^p} \ \lesssim \ \| \nabla_x^k A \|_{L^{q_1}} \cdot \| F \|_{L^{r_1}} + \| A \|_{L^{q_2}} \cdot \| \nabla_x^k F \|_{L^{r_2}} \ , \tag{36}$$

for $1 < p, q_i, r_i < \infty$, $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1}$, and $\frac{1}{p} = \frac{1}{q_2} + \frac{1}{r_2}$ whenever $0 < k$. This estimate is true even for non-integer $0 \leqslant k$ by a simple Littlewood-Paley argument. We note here that we only use it the integer case, and there it is only employed as a convenience. For a proof of this, see e.g. Chapter 2 of [12].

We would now like to set up a system to formalize many of the dyadic estimates which will appear in this paper. This is most easily done using the language of Besov spaces. Since we have a specific purpose for these in mind, we introduce the following notation:

$$\| A \|^2_{\dot{B}_2^{p,(q,s)}} \ = \ \sum_{\mu} \ \mu^{2s - 2n(\frac{1}{q} - \frac{1}{p})} \| P_\mu A \|^2_{L^p} \ , \tag{37}$$

This notation may seem a bit mysterious at first, but the thing to keep in mind here is that the first index p in some sense controls the decay, while the second double index (q, s) controls the scaling, which is the same as $\dot{W}^{q,s}$ (homogeneous L^q Sobolev space). In general, the second index will be fixed, so we will strive to have p as low as possible (see Remark 4.2 below). This notation has the following simple significance: $\dot{B}_2^{p,(q,s)}$ is the ℓ^2 Besov space of Lebesgue index p which *contains* the standard Besov space $\dot{B}_2^{q,s}$ defined by:

$$\| A \|^2_{\dot{B}_2^{q,s}} \ = \ \sum_{\mu} \ \mu^{2s} \ \| P_\mu A \|^2_{L^q} \ .$$

This identification is a direct consequence of the Bernstein embedding (33). In general, one has the inclusions:

$$\dot{B}_2^{p_1,(q,s)} \ \subseteq \ \dot{B}_2^{p_2,(q,s)} \ , \qquad\qquad q \leqslant p_1 \leqslant p_2 \leqslant \infty \ . \tag{38}$$

Furthermore, a quick application of the Littlewood-Paley identity (34) gives the Lebesgue space inclusion:

$$\dot{B}_2^{p,\left(q, n(\frac{1}{q} - \frac{1}{p})\right)} \ \subseteq \ L^p \ , \qquad\qquad 2 \leqslant p < \infty \ . \tag{39}$$

The reason we prefer to use this more involved notation, instead of the usual Besov norm convention is that ours allows one to tell at first glance which norms are critical, which is particularly useful in a scale invariant problem like the one of this paper. Specifically, the norms $\dot{B}_2^{p,(2,\frac{n-2}{2})}$ will play a prominent role in what follows.

It will also be necessary for us to employ the ℓ^1 summing version of the norm (37), which we label by $\dot{B}_1^{p,(q,s)}$. This will essentially be used for one purpose only, and that is that the L^∞ endpoint of (39) is true for this space:

$$\dot{B}_1^{\infty,(q,\frac{n}{q})} \subseteq L^\infty , \qquad\qquad 1 \leqslant q \leqslant \infty . \tag{40}$$

Besov spaces are particularly well behaved with respect to the action of Riesz operators, which is exactly why we use them. In general, we define the operator $|D_x|^{-\sigma}$ to be the Fourier multiplier with symbol $|\xi|^{-\sigma}$. The basic embedding we will use in the sequel is the following:

Lemma 4.1. *One has the following bilinear estimate for Besov spaces for $0 \leqslant \sigma$:*

$$|D_x|^{-\sigma} \ : \ \dot{B}_2^{p,(2,s_1)} \cdot \dot{B}_2^{q,(2,s_2)} \ \hookrightarrow \ \dot{B}_1^{r,(2,s_3)} , \tag{41}$$

where the indices $1 \leqslant p,q,r \leqslant \infty$ and σ, s_i satisfy the following conditions:

$$s_3 \ = \ s_1 + s_2 + \sigma - \frac{n}{2} , \qquad\qquad \textit{(scaling)} , \tag{42}$$

$$\sigma + \frac{n}{2} - s_3 \ < \ n(\frac{1}{p} + \frac{1}{q}) , \qquad\qquad \textit{(High} \times \textit{High)} , \tag{43}$$

$$s_1 \ < \ \frac{n}{2} + \min\{n(\frac{1}{q} - \frac{1}{r}) \ , \ 0\} , \qquad\qquad \textit{(Low} \times \textit{High)} , \tag{44}$$

$$s_2 \ < \ \frac{n}{2} + \min\{n(\frac{1}{p} - \frac{1}{r}) \ , \ 0\} , \qquad\qquad \textit{(High} \times \textit{Low)} , \tag{45}$$

$$\frac{1}{r} \ \leqslant \ \frac{1}{p} + \frac{1}{q} , \qquad\qquad \textit{(Lebesgue)} . \tag{46}$$

Remark 4.2. As will become apparent in the proof, it is possible to show frequency localized versions of the embedding (41) such that not all of the conditions (43)–(45) need to be satisfied. Indeed, we will show the following two frequency localized "improvements" are possible:

$$|D_x|^{-\sigma} \ : \ P_{\bullet \ll \lambda}(\dot{B}_2^{p,(2,s_1)}) \cdot P_\lambda(\dot{B}_2^{q,(2,s_2)}) \ \hookrightarrow \ P_\lambda(\dot{B}_1^{r,(2,s_3)}) , \tag{47}$$

$$|D_x|^{-\sigma} \ : \ P_\lambda(\dot{B}_2^{p,(2,s_1)}) \cdot P_\lambda(\dot{B}_2^{q,(2,s_2)}) \ \hookrightarrow \ \Big(\frac{\mu}{\lambda}\Big)^\delta P_\mu(\dot{B}_1^{r,(2,s_3)}) , \tag{48}$$

where $\delta = n(\frac{1}{p}+\frac{1}{q})+s_3-\sigma-\frac{n}{2}$ in estimate (48). Estimate (47) holds whenever (42), (44), and (46) are satisfied. The second estimate (48) is valid whenever we have (42), (43), and (46). In particular, notice that for larger σ this estimate requires lower values of p,q. This fact will have an immense bearing on the estimates we prove in the sequel, and seems to be one of the most difficult factors in lowering the dimension of the overall argument from $n = 6$ (apart from even more difficult things such as null-form estimates).

Proof of estimate (41). The proof is a simple matter of the standard technique of trichotomy. That is, we start with two test matrices A and C, and we run a frequency decomposition on the product:

$$A \cdot C \ = \ \sum_{\lambda,\mu_i} P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \ .$$

Setting now:

$$\gamma \ = \ \min\Big\{ \ \frac{n}{2} - s_1 \ , \ \frac{n}{2} - s_2 \ , \ n(\frac{1}{p}+\frac{1}{q}) + s_3 - \sigma - \frac{n}{2} \ , \ \frac{n}{2} + n(\frac{1}{q}-\frac{1}{r}) - s_1 \ , \ \frac{n}{2} + n(\frac{1}{p}-\frac{1}{r}) - s_2 \ \Big\} \ ,$$

we have from the conditions (43)–(45) that $0 < \gamma$. To prove (41) it suffices to show that:

$$\sum_{\substack{\mu_1 \ : \\ \mu_1 \ll \mu_2 \\ \lambda \sim \mu_2}} \lambda^{s_3 - n(\frac{1}{2}-\frac{1}{r}) - \sigma} \ \| \, P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^r} \ \lesssim$$

$$\sum_{\substack{\mu_1 \ : \\ \mu_1 \ll \mu_2 \\ \lambda \sim \mu_2}} \left(\frac{\mu_1}{\mu_2} \right)^\gamma \ \| \, P_{\mu_1} A \, \|_{\dot{B}^{p,(2,s_1)}} \cdot \| \, P_{\mu_2} C \, \|_{\dot{B}^{q,(2,s_2)}} \ ,$$

$$\sum_{\substack{\mu_2 \ : \\ \mu_2 \ll \mu_1 \\ \lambda \sim \mu_1}} \lambda^{s_3 - n(\frac{1}{2}-\frac{1}{r}) - \sigma} \ \| \, P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^r} \ \lesssim$$

$$\sum_{\substack{\mu_2 \ : \\ \mu_2 \ll \mu_1 \\ \lambda \sim \mu_1}} \left(\frac{\mu_2}{\mu_1} \right)^\gamma \ \| \, P_{\mu_1} A \, \|_{\dot{B}^{p,(2,s_1)}} \cdot \| \, P_{\mu_2} C \, \|_{\dot{C}^{q,(2,s_2)}} \ ,$$

$$\sum_{\substack{\lambda \ : \\ \mu_2 \sim \mu_1 \\ \lambda \lesssim \mu_i}} \lambda^{s_3 - n(\frac{1}{2}-\frac{1}{r}) - \sigma} \ \| \, P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^r} \ \lesssim \ \| \, P_{\mu_1} A \, \|_{\dot{B}^{p,(2,s_1)}} \cdot \| \, P_{\mu_2} C \, \|_{\dot{B}^{q,(2,s_2)}} \ ,$$

That (41) follows from these three estimates is a simple consequence of Young's inequality and Cauchy-Schwartz respectively. These estimates, in turn, are all a consequence of the single fixed frequency bound:

$$\lambda^{s_3 - n(\frac{1}{2}-\frac{1}{r}) - \sigma} \ \| \, P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^r} \ \lesssim \tag{49}$$
$$\left(\frac{\lambda}{\max\{\mu_i\}} \right)^\gamma \cdot \min\{ \left(\frac{\mu_1}{\mu_2} \right)^\gamma , \left(\frac{\mu_2}{\mu_1} \right)^\gamma \} \cdot \| \, P_{\mu_1} A \, \|_{\dot{B}^{p,(2,s_1)}} \cdot \| \, P_{\mu_2} C \, \|_{\dot{B}^{q,(2,s_2)}} \ .$$

The proof of (49) is a simple matter of Hölders and Bernstein's inequalities, and counting weights. There are three cases corresponding to the three summing estimates above. In the first case, we assume that $\lambda \ \lesssim \ \mu_1 \sim \mu_2$. Since (49) is scale invariant, we may assume in this case that both $\mu_i \sim 1$. Using now Hölders inequality which is permissible by (46), followed by the Bernstein inequality, we have that:

$$\| \, P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^r} \ \lesssim \ \lambda^{n(\frac{1}{p}+\frac{1}{q}-\frac{1}{r})} \ \| \, P_{\mu_1} A \, \|_{L^p} \cdot \| \, P_{\mu_2} C \, \|_{L^q} \ .$$

Multiplying this last estimate by the weight $\lambda^{s_3 - n(\frac{1}{2}-\frac{1}{r}) - \sigma}$ we arrive at the bound:

$$\text{(L.H.S.)(49)} \ \lesssim \ \lambda^{n(\frac{1}{p}+\frac{1}{q}) + s_3 - \sigma - \frac{n}{2}} \ \| \, P_{\mu_1} A \, \|_{L^p} \cdot \| \, P_{\mu_2} C \, \|_{L^q} \ .$$

Then (49) follows in this case from the definition of γ and the fact that $\mu_i \sim 1$. The other two cases, which correspond to $\mu_1 \ll \mu_2$ or vice versa are similar, so it suffices to consider the first. In this case we rescale to $\mu_2 \sim \lambda \sim 1$. In the case where $r < q$ we set $\frac{1}{\tilde{p}} = \frac{1}{r} - \frac{1}{q}$, and we again use Hölder and Bernstein to estimate:

$$\| P_\lambda (P_{\mu_1} A \cdot P_{\mu_2} C) \|_{L^r} \lesssim \mu_1^{n(\frac{1}{p} - \frac{1}{\tilde{p}})} \| P_{\mu_1} A \|_{L^p} \cdot \| P_{\mu_2} C \|_{L^q} .$$

If it is the case that $q \leqslant r$, then we simply estimate:

$$\| P_\lambda (P_{\mu_1} A \cdot P_{\mu_2} C) \|_{L^r} \lesssim \mu_1^{\frac{n}{p}} \| P_{\mu_1} A \|_{L^p} \cdot \| P_{\mu_2} C \|_{L^q} .$$

In either case, the claim (49) follows from the definition of γ. This completes the proof of (41). $\square$

Before continuing on, let us note here a slight refinement of the Besov norms (37) and the embedding (41). This involves taking into account functions which live at frequency $\lesssim 1$. If we let $\langle D_x \rangle$ denote the multiplier with symbol $(1 + |\xi|^2)^{\frac{1}{2}}$, then we form the low frequency spaces:

$$\| A \|_{\dot{B}^{p,(q,s)}_{2,10n}} = \| \langle D_x \rangle^{10n} A \|_{\dot{B}^{p,(q,s)}_{2}} , \tag{50}$$

with a similar definition for the ℓ^1 version $\dot{B}^{p,(q,s)}_{1,10n}$. By a straightforward adaptation of the previous argument, it is easy to see that the embedding (41) is equally valid for these low frequency spaces. We leave the details to the reader.

It will also be necessary for us to perform various dyadic decompositions with respect to the angular frequency variable. For each fixed direction ω in the frequency plane $\mathbb{R}^n_\xi$, we decompose the unit sphere $\mathbb{S}^{n-1}_\xi$ into dyadic conical regions:

$$\mathcal{R}(\omega, \theta) = \{\eta \in \mathbb{S}^{n-1}_\xi \bigm| \angle(\omega, \eta) \sim \theta\} , \tag{51}$$

where $\theta \in \{\frac{\pi}{2} \cdot 2^i \bigm| i \in \mathbb{Z}, i \leqslant 0\}$. Here we will not bother to fix the constant in the $\sim$ notation used to define the regions (51), but we will let it change from line to line as we have done for the spatial multipliers above. We also define a smooth partition of unity adapted to these regions, which we label by b^ω_θ. These can always be chosen (e.g. by defining them on a larger sphere and then rescaling) so that they satisfy the differential bounds:

$$|(\omega \cdot \nabla_\xi)^k_\omega \, p_1 b^\omega_\theta| \lesssim 1 , \qquad |(\omega^\perp \cdot \nabla_\xi)^k \, p_1 b^\omega_\theta| \lesssim \theta^{-k} ,$$

where the implicit constants depend on k but are uniform in θ. In particular, if we define the multipliers $\widehat{{}^\omega\Pi_\theta A} = b^\omega_\theta \widehat{A}$, then the operators ${}^\omega\Pi_\theta P_\mu$ are bounded on all L^p spaces uniformly in μ and θ. In fact, the following refinement of the inequality (33) holds, which we also call Bernstein:

$$\| {}^\omega\Pi_\theta P_\mu A \|_{L^p} \lesssim \mu^{n(\frac{1}{q} - \frac{1}{p})} \theta^{(n-1)(\frac{1}{q} - \frac{1}{p})} \| A \|_{L^q} . \tag{52}$$

In all of the above inequalities, we have kept ω as a fixed directional value. However, it will also be necessary for us to have an account of how our multipliers depend on this parameter. In particular, we will need to have bounds for the operators $\nabla_\omega {}^\omega\Pi_\theta$. This is easily achieved by differentiating the associated multiplier. In fact, one has the bounds for fixed ξ:

$$|\nabla^k_\omega b^\omega_\theta| \lesssim \theta^{-k} . \tag{53}$$

The way we shall express this bound in calculations is through the following heuristic operator identity:

$$\nabla_\omega^k\, {}^\omega\Pi_\theta \ \approx\ \theta^{-k}\, {}^\omega\Pi_\theta \ , \tag{54}$$

which we shall take to mean that *the left hand side satisfies all L^p space bounds as the right hand side*. Notice that this relation has a preferred direction (left⇒right). In practice, this means that we have the bound (52) for the operator on the left hand side of (54) with the added factor of θ^{-k}.

Finally, let us end this section by making the following conventions. Firstly, it will be convenient for us at times to write $P_\mu A = A_\mu$ for a localized object. This should not be confused with the μ^{th} component of A in the case that it is a one-form. This should usually be clear from context. Secondly, it will be necessary for us to ensure that certain of our multipliers have real symbol so that they respect the subalgebra $\mathfrak{g}(m) \subseteq M(m\times m)$. This will be done by taking their real part which simply symmetrizes their (real) symbols. In particular, we will denote this by:

$$\Re({}^\omega\Pi_\theta) \ =\ {}^\omega\overline{\Pi}_\theta \ .$$

Secondly, we use the following bulleted notation for the sum of various cutoffs over a given range:

$$P_{\bullet<c} \ =\ \sum_{\mu<c} P_\mu \ , \qquad\qquad {}^\omega\Pi_{\bullet<c} \ =\ \sum_{\theta<c} {}^\omega\Pi_\theta \ ,$$

etc. We will also use the notation $A_{\bullet<c}$ etc. for these operators applied to tensors. Finally, we will set aside a special notation here for cutting off on angles sectors whose width depends on the frequency:

$$ {}^\omega\overline{\Pi}^{(\sigma)} \ =\ \sum_\mu \ {}^\omega\overline{\Pi}_{\mu^\sigma<\bullet}\, P_\mu \ . \tag{55}$$

Notice that this multiplier does not satisfy good bounds of the form (53). However, it can be dealt with using the Littlewood-Paley equivalence (34) if there is a little extra room left to sum over fixed angular dyadics. This ends our description of the basic analysis we will use in this paper.

5. Gauge construction for the initial data; Reduction to a second order system and the main a-priori estimate

We now begin our proof of the main theorem 1.1. As we have already mentioned, one of the central components of the proof is to construct a stable set of “elliptic” coordinates on the bundle V. The way we will do this is to construct this set of coordinates on the $t = 0$ slice $\mathbb{R}^n \times \mathfrak{g}$. We will then show that this frame propagates as the system evolves by solving an auxiliary set of equations for the gauge potentials which respects this frame automatically. The regularity of this system will be provided in the usual translation invariant Sobolev spaces. We then show that this solution is in fact a true solution to the system of equations (4)–(5) by employing a bootstrapping procedure which is similar to that used in the proof of Lemma 3.2. The desired covariant regularity which is contained in the statement of Theorem 1.1 will be provided by a comparison principle. These constructions are all local in time, and are more or less standard. We have included them here for the convenience of the reader, the sake of completeness, and the fact that some of the formulas we develop here will be central to what we do in later sections.

With the local theory established, the global conclusion of Theorem 1.1 will then be a consequence of a certain a-priori estimate on the (usual Sobolev) energy of solutions to (4)–(5) in the gauge we construct. Our task will then be to show that this a-priori estimate is true for all solutions to yet another system auxiliary equations, this time for the curvature. This can be considered to be the main a-priori estimate of the paper. The proof of this estimate turns out to be quite involved, and will occupy the rest of the paper. In the next section, we will prove the main a-priori estimate itself with the help of a certain family of microlocalized space-time (Strichartz) estimates for solutions to second order covariant wave equations on bundles with connections satisfying estimates consistent with our bootstrapping assumptions. The breakdown here is based on the Smith-Tataru (see [10]) $\mathcal{E}$-parametrix idea, which allows one to reduce the needed Strichartz estimates to proving them for a suitable family of frequency localized fundamental solutions. Our rendition of this is essentially equivalent to that contained in the paper [8]. Finally, in the remaining sections of the paper, we develop the linear theory. This is by far the most involved portion of the present work, and requires the construction of some fairly sophisticated oscillatory integrals and microlocal function spaces. This material can be read without reference to the non-linear problem, as long as one is familiar with the algebraic and analytic assumptions we make on the geometry (frequency localized connection). While these come from the non-linear problem, they are of course a bit more general.

5.1. **Construction of the initial frame, and the comparison principle.** The first thing we do here is to put the initial connection $\underline{D}$ into the Coulomb gauge. Via the Uhlenbeck lemma (3.1), we simply need to show that:

$$\| \underline{F} \|_{L^{\frac{n}{2}}} \ \leqslant \ C\varepsilon_0 \ , \tag{56}$$

for fixed C, and ϵ_0 the sufficiently small parameter from line (11) (which should not be confused with the small constant from Lemma 3.1 above). This L^p bound follows immediately from the gauge covariant Sobolev embedding (for n even):

$$\dot{H}_A^{\frac{n-4}{2}} \ \hookrightarrow \ L^{\frac{n}{2}} \ , \tag{57}$$

which in turn follows from repeated application of the usual Sobolev embeddings and the Kato estimate (which follows immediately from (1) and Cauchy-Schwatrz):

$$\big| d|F| \big| \ \leqslant \ |\underline{D}F| \ , \tag{58}$$

where F is any section to $\mathcal{M} \times \mathfrak{g}$ and the absolute norm $|\cdot|$ is taken with respect to the Killing inner product (14).

We may now assume that we are dealing with an initial data set:

$$(\underline{F}(0), \underline{D}(0), E(0)) \ , \tag{59}$$

for the system which is such that connection $\underline{D}(0) = d + \underline{A}(0)$ satisfied the div-curl system:

$$d\underline{A}(0) + [\underline{A}(0), \underline{A}(0)] \ = \ \underline{F}(0) \ , \qquad\qquad d^*\underline{A}(0) \ = \ 0 \ , \tag{60}$$

and such that the compatibility condition:

$$\underline{D}(0)^i E_i(0) \ = \ 0 \ , \tag{61}$$

is satisfied. From the system (60) and the Riesz operator embeddings:

$$\nabla_x \Delta^{-1} \ : \ L^{\frac{n}{2}} \ \hookrightarrow \ L^n \ , \tag{62}$$

$$\nabla_x^2 \Delta^{-1} \ : \ L^{\frac{n}{2}} \ \hookrightarrow \ L^{\frac{n}{2}} \ , \tag{63}$$

and the estimate (56), we have the bounds:

$$\| \nabla_x \underline{A}(0) \|_{L^{\frac{n}{2}}} \ , \ \| \underline{A}(0) \|_{L^n} \ \lesssim \ \varepsilon_0 \ . \tag{64}$$

We will now use this last bound to show that the initial data set (59) is in fact in the classical Sobolev spaces $\dot{H}^k$. This is a consequence of the following:

Lemma 5.1 (Comparison principle for Sobolev norms on $\mathbb{R}^n$)**.** *Let $\underline{D} = d + \underline{A}$ be a connection on $\mathbb{R}^n$, with n even, such that one has the curvature bounds:*

$$\| \underline{F} \|_{\dot{H}_A^{\frac{n-4}{2}}} \ \leqslant \ \epsilon_0 \ , \tag{65}$$

$$\| \underline{F} \|_{\dot{H}_A^k} \ \leqslant \ M_k \ , \tag{66}$$

for $\frac{n-4}{2} < k$. Suppose also that D is in the gauge $d^*\underline{A} = 0$. Then we have the critical classical Sobolev bounds:

$$\| \underline{F} \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ C\epsilon_0 \ , \tag{67}$$

$$\| \underline{A} \|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant \ C\epsilon_0 \ . \tag{68}$$

Furthermore, if G is any $\mathfrak{g}$ valued function, then we have the following inductive comparison of norms:

$$\begin{aligned} C^{-1}(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \ \| G \|_{H^{[k^*,k]}} \ &\leqslant \ \| G \|_{H_A^{[k^*,k]}} \ , \\ &\leqslant \ C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \ \| G \|_{H^{[k^*,k]}} \ , \end{aligned} \tag{69}$$

where the index k^* is such that $\frac{n-4}{2} \leqslant k^* < n$, and where we have set:

$$\| G \|_{H_A^{[k^*,k]}}^2 \ = \ \sum_{k^* \leqslant m \leqslant k} \| \underline{D}^m G \|_{L^2}^2 \ ,$$

to be the interval gauge-covariant Sobolev space. We use an analogous definition for the space $H^{[k^*,k]}$. We also have the non-inductive equivalence between $\nabla_x \underline{A}$ and $\underline{F}$:

$$N_k^{-1} \| \underline{A} \|_{\dot{H}^k} \ \leqslant \ \| \underline{F} \|_{\dot{H}^{k-1}} \ \leqslant \ N_k \| \underline{A} \|_{\dot{H}^k} \ , \tag{70}$$

where N_k, $\frac{n-2}{2} \leqslant k$, is a set of constants which depends only on the dimension and not *on the constant ϵ_0 once it is sufficiently small. In particular, combining all of this, we have the following classical Sobolev bounds on the pair $(\underline{A}, \underline{F})$:*

$$\| \underline{F} \|_{\dot{H}^k} \ \leqslant \ C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \, M_k \ , \tag{71}$$

$$\| \underline{A} \|_{\dot{H}^{k+1}} \ \leqslant \ C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \, M_k \ . \tag{72}$$

for $\frac{n-4}{2} < k$.

Proof of Lemma 5.1. The proof will be accomplished via a series of inductions. In what follows, we will assume the estimate (70), whose proof follows from simple analysis of the elliptic system (60) in Besov spaces of the kind $\dot{B}_2^{p,(2,s)}$. We will perform many reductions like this in the sequel so we leave this one to the reader.

The first step is to prove the critical comparison estimate (67). Note that the potential bounds (68) follow from this and (70). The inductive hypothesis that we make here is that:

$$\| \nabla_x^l \underline{D}^m \underline{F} \|_{L^{\frac{n}{k}}} \ \lesssim \ \epsilon_0 \ , \tag{73}$$

for $k = l + m + 2 \leqslant \frac{n}{2}$ whenever $0 \leqslant l \leqslant l_0$. Notice that this hypothesis is verified for $l_0 = 0$ on account of the assumption (65) and by applying the Kato estimate (58) in conjunction with integer Sobolev embeddings. Notice also that by applying Riesz operator estimates to the elliptic system (60), and using the product estimate (36) along with Sobolev embeddings we have the bounds:

$$\begin{aligned} \| \nabla_x^{l+1} \underline{A} \|_{L^{\frac{n}{k}}} \ &\lesssim \ \| \nabla_x^l \underline{F} \|_{L^{\frac{n}{k}}} + \| \nabla_x^l ([\underline{A}, \underline{A}]) \|_{L^{\frac{n}{k}}} \ , \\ &\lesssim \ \| \nabla_x^l \underline{F} \|_{L^{\frac{n}{k}}} + \| \nabla_x^l \underline{A} \|_{L^{\frac{n}{k-1}}} \cdot \| \underline{A} \|_{L^n} \ , \\ &\lesssim \ \| \nabla_x^l \underline{F} \|_{L^{\frac{n}{k}}} + \epsilon_0 \cdot \| \nabla_x^{l+1} \underline{A} \|_{L^{\frac{n}{k}}} \ . \end{aligned}$$

Therefore, the inductive hypothesis (73) may be assumed to also contain the estimate:

$$\| \nabla_x^{l+1} \underline{A} \|_{L^{\frac{n}{k}}} \ \lesssim \ \epsilon_0 \ , \tag{74}$$

for $k = l + 2 \leqslant \frac{n}{2}$ and $l \leqslant l_0$. To show that (73) holds for all $l \leqslant l_0 + 1$, we start with $l \leqslant l_0$ and we compute using (36) and Sobolev embeddings that:

$$\begin{aligned} &\| \nabla_x^{l+1} \underline{D}^{m-1} \underline{F} \|_{L^{\frac{n}{k}}} \ , \\ \lesssim\ &\| \nabla_x^l \underline{D}^m \underline{F} \|_{L^{\frac{n}{k}}} + \| \nabla_x^l ([\underline{A}, \underline{D}^{m-1} \underline{F}]) \|_{L^{\frac{n}{k}}} \ , \\ \lesssim\ &\epsilon_0 + \| \nabla_x^l \underline{A} \|_{L^{\frac{n}{l+1}}} \cdot \| \underline{D}^{m-1} \underline{F} \|_{L^{\frac{n}{k-l-1}}} + \| \underline{A} \|_{L^n} \cdot \| \nabla_x^l \underline{D}^{m-1} \underline{F} \|_{L^{\frac{n}{k-1}}} \ , \\ \lesssim\ &\epsilon_0 + \epsilon_0 \cdot \| \nabla_x^{l+1} \underline{D}^{m-1} \underline{F} \|_{L^{\frac{n}{k}}} \ . \end{aligned}$$

This inductively establishes (73) and hence proves (67).

We now show (69). We first deal with the leftmost inequality. Our inductive hypothesis is now that:

$$\| \nabla_x^l \underline{D}^m G \|_{L^2} \ \lesssim \ C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \ \| G \|_{H_A^{[k^*,k]}} \ , \tag{75}$$

where $l + m = k_0$ for $k_0 = k$ or $k_0 = k^*$, and for all $l \leqslant l_0$. To compute $\nabla_x^{l+1} \underline{D}^{m-1} G$ in terms of this, we need to split into cases depending on whether or not $l + 1 < \frac{n}{2}$ or not. In the former case we compute that:

$$\begin{aligned} &\| \nabla_x^{l+1} \underline{D}^{m-1} G \|_{L^2} \ , \\ \lesssim\ &\| \nabla_x^l \underline{D}^m G \|_{L^2} + \| \nabla_x^l ([\underline{A}, \underline{D}^{m-1} G]) \|_{L^2} \ , \\ \lesssim\ &C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \ \| G \|_{H_A^{[k^*,k]}} + \| \nabla_x^l \underline{A} \|_{L^{\frac{n}{l+1}}} \cdot \| \underline{D}^{m-1} G \|_{L^{\frac{2n}{n-2l-2}}} \\ &\qquad + \| \underline{A} \|_{L^n} \cdot \| \nabla_x^l \underline{D}^{m-1} G \|_{L^{\frac{2n}{n-2}}} \ , \\ \lesssim\ &C(M_{\frac{n-4}{2}}, \ldots, M_{k-1}) \ \| G \|_{H_A^{[k^*,k]}} + \epsilon_0 \cdot \| \nabla_x^{l+1} \underline{D}^{m-1} G \|_{L^2} \ . \end{aligned}$$

In the case where $\frac{n}{2}-1\leqslant l$ we have the inequality:

$$\begin{aligned}
&\| \nabla_x^{l+1}\underline{D}^{m-1}G\|_{L^2} \ , \\
\lesssim\ & C(M_{\frac{n-4}{2}},\ldots,M_{k-1})\ \| G\|_{H_A^{[k^*,k]}} + \| \nabla_x^l \underline{A}\|_{L^{\frac{2n}{n-2}}}\cdot \| \underline{D}^{m-1}G\|_{L^n} \\
&\qquad + \| \underline{A}\|_{L^n}\cdot \| \nabla_x^l\underline{D}^{m-1}G\|_{L^{\frac{2n}{n-2}}} \ , \\
\lesssim\ & C(M_{\frac{n-4}{2}},\ldots,M_{k-1})\ \| G\|_{H_A^{[k^*,k]}} + \| \nabla_x^{l+1}\underline{A}\|_{L^2}\cdot \| \underline{D}^{\frac{n-2}{2}+m-1}G\|_{L^2} \\
&\qquad + \epsilon_0\cdot \| \nabla_x^{l+1}\underline{D}^{m-1}G\|_{L^2} \ .
\end{aligned}$$

Notice that this last line above used the $L^2\hookrightarrow L^n$ gauge covariant Sobolev embedding. To bound the second term on this line, notice that since $\frac{n}{2}-1\leqslant l$ and we must assume that $1\leqslant m$ for the induction to make sense, we have the bound $k^*\leqslant \frac{n-2}{2}+m-1\leqslant k$. This allows us to bound:

$$\| \underline{D}^{\frac{n-2}{2}+m-1}G\|_{L^2} \ \leqslant\ \| G\|_{H_A^{[k^*,k]}} \ .$$

Furthermore, by placing all of these calculations within an induction on the value of k itself, and using the bound (70) while noting that $l\leqslant k-1$ we may assume the bound:

$$\| \nabla_x^{l+1}\underline{A}\|_{L^2} \ \lesssim\ \| \nabla_x^l\underline{F}\|_{L^2} \ \lesssim\ C(M_{\frac{n-4}{2}},\ldots,M_{k-1}) \ .$$

This completes our inductive proof of (75) above.

The proof of the second inequality on line (69) follows from reasoning similar as that used to prove (75) inductively. We leave it to the reader to set up the inductive hypothesis for this case and work out the details. This completes our proof of Lemma 5.1. □

Using Lemma 5.1 and the assumed bounds (11)–(12), we may assume that our initial data (59) is such that:

$$\| (\underline{F}(0),E(0))\|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant\ \widetilde{\epsilon}_0 \ , \tag{76}$$

$$\| \underline{A}(0)\|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant\ \widetilde{\epsilon}_0 \ , \tag{77}$$

$$\| (\underline{F}(0),E(0))\|_{\dot{H}^k} \ \leqslant\ \widetilde{M}_k \ , \tag{78}$$

$$\| \underline{A}(0)\|_{\dot{H}^{k+1}} \ \leqslant\ \widetilde{M}_k \ , \tag{79}$$

where $\frac{n-4}{2}<k$, and the $\widetilde{M}_k$ depend on the M_k in some inductive way, and we also have that $\widetilde{\epsilon}_0\leqslant C\varepsilon_0$ for some constant C which depends only on the dimension. Here M_k and ε_0 refer to the constants introduced in the statement of Theorem 1.1.

We now decompose the initial field strength $\{E_i(0)\}$ in a way that will be consistent with the evolution of the system (4)–(5). This will be convenient for discussing the Cauchy problem. Our first step is to define the following elliptic quantity:

$$\Delta a_0 \ =\ -[a_i,\nabla^i a_0] + [a^i,E_i] \ . \tag{80}$$

where for convenience we have labeled $\{a_i\}=\{\underline{A}_i(0)\}$. We then define the auxiliary set of quantities:

$$\dot{a}_i \ =\ E_i + \nabla_i a_0 - [a_0,a_i] \ . \tag{81}$$

Notice that as an immediate consequence of the constraint equation (61), the form of (80), and the Coulomb condition $d^*a = 0$, we have the secondary Coulomb condition:

$$\nabla^i \dot{a}_i \ = \ 0 \ .$$

This will turn out to be important in a moment. Now, from the definition of the quantities (80) and (81), the already established bounds (76)–(79), and several rounds of Sobolev embeddings, we have the following differential bounds on the quantities $\{\dot{a}_i\}$:

$$\| \dot{a} \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ \widetilde{\epsilon}_0 \ , \tag{82}$$

$$\| \dot{a} \|_{\dot{H}^k} \ \leqslant \ \widetilde{M}_k \ , \tag{83}$$

for $\frac{n-4}{2} < k$ (after a possible slight redefinition of the constants $\widetilde{\epsilon}_0, \widetilde{M}_k$ by multiplication by some fixed dimensional constant). We now define a *Coulomb admissible* initial data set to be a collection $(\underline{F}, \{a_i\}, \{\dot{a}_i\})$ such that:

$$da + [a,a] \ = \ \underline{F} \ , \qquad d^*a \ = \ 0 \ , \qquad d^*\dot{a} \ = \ 0 \ . \tag{84}$$

Notice that $\underline{F}$ is uniquely determined by the $\{a_i\}$, therefore we do not need to include it in the definition of initial data. We define the *Coulomb-Cauchy problem* to be the task of finding a space-time connection $D = d + A$ such that it satisfies the set of equations:

$$D^\beta F_{\alpha\beta} \ = \ 0 \ , \tag{85a}$$

$$dA + [A,A] \ = \ F \ , \tag{85b}$$

$$d^*\underline{A} \ = \ 0 \ , \tag{85c}$$

and such that at time $t = 0$ we have that:

$$\underline{A}(0) \ = \ a \ , \qquad \partial_t \underline{A}(0) \ = \ \dot{a} \ . \tag{86}$$

We remark briefly here that solving the problem (84)–(86) provides a solution to the original Yang Mills system (4)–(5) with Cauchy data (59) as long as we define the collection $\{\dot{a}\}$ according to the equations (80)–(81). All we need to do to prove this assertion it to show that:

$$F_{0i}(0) \ = \ E_i \ .$$

Our proof of this follows the same bootstrapping philosophy used to show the equivalence (30) in the proof of Lemma 3.2. The claim will follow once from equation (81) we can establish that:

$$A_0(0) \ = \ a_0 \ ,$$

where a_0 is defined by (80). Now, from the system of equations (85) we have that the quantity A_0 is elliptically determined by the equation:

$$\Delta_{\underline{A}} A_0 \ = \ [A_i, \nabla_t A^i] \ , \tag{87}$$

where $\Delta_{\underline{A}} = D^i D_i$ is the gauge covariant Laplacean. Furthermore, by using equation (81) as the definition of E_i, and substituting this into equation (80), we have that the quantity a_0 is elliptically determined by the equation:

$$\Delta_a a_0 \ = \ [a_i, \dot{a}^i] \ . \tag{88}$$

By subtracting (88) from (87) at time $t = 0$ we have that:

$$\Delta_a(A_0(0) - a_0) \ = \ 0 \ .$$

Uniqueness now comes from the Sobolev type estimate:

$$\| B \|_{L^n} \ \lesssim \ \| \Delta_a B \|_{L^{\frac{n}{3}}} \ ,$$

which follows from the smallness condition (77) and the usual Sobolev estimates. The details of the proof are left to the reader.

Keeping the equivalence we have just established in mind, and the first inequality contained in the comparison estimates (69) and (70), we have reduced the demonstration of Theorem 1.1 to showing the following non-gauge covariant global regularity theorem:

Theorem 5.2 (Global regularity in the Coulomb gauge)**.** *Let the number of spatial dimensions be $6 \leqslant n$. Then there exists a set of constants $\widetilde{\epsilon}_0$ and C, C_k, $\frac{n-2}{2} \leqslant k$ such that if $(\underline{F}, \{a_i\}, \{\dot{a}_i\})$ is a Coulomb admissible initial data set such that is satisfies the bounds:*

$$\| \underline{F} \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ \widetilde{\epsilon}_0 \ , \qquad \| \underline{F} \|_{\dot{H}^k} \ \leqslant \ \widetilde{M}_k \ , \tag{89a}$$

$$\| a \|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant \ \widetilde{\epsilon}_0 \ , \qquad \| \dot{a} \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ \widetilde{\epsilon}_0 \ , \tag{89b}$$

$$\| a \|_{\dot{H}^k} \ \leqslant \ \widetilde{M}_{k-1} \ , \qquad \| \dot{a} \|_{\dot{H}^{k-1}} \ \leqslant \ \widetilde{M}_{k-1} \ , \tag{89c}$$

then if $\widetilde{\epsilon}_0$ is sufficiently small there exists a unique global solution $\{A_\alpha\}$ to the system (85) *with this initial data. Furthermore, this solution obeys the following differential estimates:*

$$\| A \|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant \ C\widetilde{\epsilon}_0 \ , \qquad \| \partial_t A \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ C\widetilde{\epsilon}_0, \tag{90a}$$

$$\| A \|_{\dot{H}^k} \ \leqslant \ C_{k-1}\widetilde{M}_{k-1} \ , \qquad \| \partial_t A \|_{\dot{H}^{k-1}} \ \leqslant \ C_{k-1}\widetilde{M}_{k-1} \ , \tag{90b}$$

5.2. **Local existence in the Coulomb gauge.** Our goal here is to reduce the proof of Theorem (5.2) to a certain a-priori estimate involving the energies of the field strength F. This amounts to proving a local existence theorem for the system (84)–(86). The proof of this will allow us to set up a system of equations for the coulomb potentials $\{A_\alpha\}$ which will be of central importance in the sequel. We will show that:

Proposition 5.3 (Local existence in the Coulomb gauge)**.** *Let the number of spatial dimensions be $6 \leqslant n$. Then for every set of constants C, C_k, $\frac{n-2}{2} \leqslant k$, there exists an $\widetilde{\epsilon}_0$ which only depends on C with the following property: If $(\{a_i\}, \{\dot{a}_i\})$ is any set of Coulomb admissible initial data such that:*

$$\| a \|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant \ C\widetilde{\epsilon}_0 \ , \qquad \| \dot{a} \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant \ C\widetilde{\epsilon}_0 \ , \tag{91}$$

$$\| a \|_{\dot{H}^k} \ \leqslant \ C_{k-1}\widetilde{M}_{k-1} \ , \qquad \| \dot{a} \|_{\dot{H}^{k-1}} \ \leqslant \ C_{k-1}\widetilde{M}_{k-1} \ , \tag{92}$$

then for $\widetilde{\epsilon}_0$ sufficiently small there exists a time $0 < T^*$, which only depends on the quantities $C\widetilde{\epsilon}_0, C_{\frac{n}{2}}\widetilde{M}_{\frac{n}{2}}, C_{\frac{n+2}{2}}\widetilde{M}_{\frac{n+2}{2}}$ such that there exists a unique local solution

$\{A_\alpha\}$ *to the system* (84)–(86) *with this set of initial data. Furthermore, on the time interval* $[0,T^*]$ *one has the following norm bounds on the collection* $\{A_\alpha\}$*:*

$$\sup_{0\leqslant t\leqslant T^*} \| A(t) \|_{\dot{H}^{\frac{n-2}{2}}} \ \leqslant\ 2C\widetilde{\epsilon}_0 \ , \tag{93}$$

$$\sup_{0\leqslant t\leqslant T^*} \| \partial_t A\,(t) \|_{\dot{H}^{\frac{n-4}{2}}} \ \leqslant\ 2C\widetilde{\epsilon}_0 \ , \tag{94}$$

$$\sup_{0\leqslant t\leqslant T^*} \| A(t) \|_{\dot{H}^{k}} \ \leqslant\ 2C_k\widetilde{M}_k \ , \tag{95}$$

$$\sup_{0\leqslant t\leqslant T^*} \| \partial_t A\,(t) \|_{\dot{H}^{k-1}} \ \leqslant\ 2C_{k-1}\widetilde{M}_{k-1} \ . \tag{96}$$

Proof of Proposition 5.3. The proof will be reduced to the standard procedure of energy estimates and Sobolev embeddings. Since we are assuming that we the initial data has enough smoothness to cover L^∞, this is more or less trivial. We start by plugging (85b) directly into (85a). After an application of the gauge condition $d^*\underline{A} = 0$ this yields a general second order system of equations which we write as:

$$\Box A_\beta \ =\ -\partial_\beta\partial_t A_0 - [\partial_t A_0, A_\beta] - [A_\alpha, \partial^\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}] \ . \tag{97}$$

To split this into a hyperbolic-elliptic system, we decompose the set of equations (97) into its spatial and temporal parts, and apply the Leray projection:

$$\mathcal{P} \ =\ -\frac{(\mathrm{curl})^2}{\Delta} \ =\ \big(I - \nabla\frac{(\mathrm{div})}{\Delta}\big) \ ,$$

to the resulting spatial equation. After some rearrangement of the elliptic equation this yields the coupled system:

$$\Box A_i \ =\ \mathcal{P}\big(-\ [\partial_t A_0, A_i] - [A_\alpha, \partial^\alpha A_i] - [A^\alpha, F_{\alpha i}]\big) \ , \tag{98a}$$

$$\Delta A_0 \ =\ -[A_i, \partial^i A_0] + [A^i, F_{0i}] \ . \tag{98b}$$

The above system of equations can be solved locally in time with the bounds (93)–(93) through a Picard iteration scheme. We leave this as an exercise for the reader. Notice that the projection $\mathcal{P}$ can be removed in energy estimates because it is an order zero operator. Notice also that even though the smallness of the time interval $[0,T^*]$ will not make up for estimates involving the elliptic equation (98b), the critical smallness assumption (91) allows one to obtain the bootstrapping estimates (93)–(93) if one uses Littlewood-Paley decompositions and paraproducts to make sure at least one factor in the non-linearity on the right hand side of (98b) goes in a critical space. This same comment goes for bounding terms on the right hand side of (98a) in energy estimates when one is bootstrapping the higher norm constants $C_k\widetilde{M}_k$ for $\frac{n+2}{2} < k$. Again, the smallness in time makes up for the size of the first few constants $C\widetilde{\epsilon}_0, C_{\frac{n}{2}}\widetilde{M}_{\frac{n}{2}}, C_{\frac{n+2}{2}}\widetilde{M}_{\frac{n+2}{2}}$.

Having now produced a local solution to the system (98) with the desired properties, we have shown the conclusion of Proposition (5.3) once we show that the spatial potentials which solve (98a) are in fact solutions to the spatial portion of the original second order equation (97). This will be show through our general strategy of coming up with a quantity which yields a critical elliptic bootstrapping estimate which will force it to be zero. This time, the desired quantity turns out to be related to the conservation of electric charge for the Yang-Mills equations. We first write the spatial portion of the non-linearity on the right hand side of (97) as a vector:

$$\mathcal{N}_i \ = \ -\partial_i\partial_t A_0 - [\partial_t A_0, A_i] - [A_\alpha, \partial^\alpha A_i] - [A^\alpha, F_{\alpha i}] \ . \tag{99}$$

We would like to show that the equations (98) force $(I-\mathcal{P})\mathcal{N} = 0$. We compute that:

$$(I-\mathcal{P})\mathcal{N} \ = \ \nabla\Delta^{-1}\left(-\partial_t\Delta A_0 - \partial^i\partial^\alpha[A_\alpha, A_i] - \partial^i[A^\alpha, F_{\alpha i}]\right) \ .$$

Now, using the equation (98) to compute $\partial_t\Delta A_0$, this last line becomes:

$$\begin{aligned}(I-\mathcal{P})\mathcal{N} \ &= \ \nabla\Delta^{-1}\left(-\partial^\beta\partial^\alpha[A_\alpha, A_\beta] - \partial^\beta[A^\alpha, F_{\alpha\beta}]\right) \ , \\ &= \ -\nabla\Delta^{-1}\partial^\beta[A^\alpha, F_{\alpha\beta}] \ .\end{aligned}$$

where the equality of the second line follows on account of skew symmetry. We now isolate the interesting portion of the term on the right hand side of the last line above and use the Jacobi identity to compute that:

$$\begin{aligned}\partial^\beta[A^\alpha, F_{\alpha\beta}] \ &= \ \frac{1}{2}[(dA)^{\alpha\beta}, F_{\alpha\beta}] + [A^\alpha, \partial^\beta F_{\alpha\beta}] \ , \\ &= \ \frac{1}{2}\big[[A^\alpha, A^\beta], F_{\alpha\beta}\big] - \big[A^\alpha, [A^\beta, F_{\alpha\beta}]\big] + [A^\alpha, D^\beta F_{\alpha\beta}] \ , \\ &= \ [A^\alpha, D^\beta F_{\alpha\beta}] \ .\end{aligned}$$

Now, again using equation (98b) we have that $D^\beta F_{0\beta} = 0$. Furthermore, from equation (98a) we also have the identity:

$$D^\beta F_{i\beta} \ = \ -(I-\mathcal{P})_i\mathcal{N} \ .$$

Combining all of this, we have the following equality:

$$(I-\mathcal{P})\mathcal{N} \ = \ \nabla\Delta^{-1}[A^i, (I-\mathcal{P})_i\mathcal{N}] \ . \tag{100}$$

Finally, from the form of (99) and the already established estimates (93)–(96) as well as the boundedness properties of the operator $(1-\mathcal{P})$ we have that:

$$\|\, (I-\mathcal{P})\mathcal{N}(t)\,\|_{L^{\frac{n}{3}}} \ < \ \infty \ ,$$

for all times $t \in [0, T^*]$. However, from the smallness bound (93), the identity (100), and a Sobolev embedding we also have the fixed time bound:

$$\begin{aligned}\|\, (I-\mathcal{P})\mathcal{N}\,\|_{L^{\frac{n}{3}}} \ &\lesssim \ \|\, [A^i, (1-\mathcal{P})_i\mathcal{N}]\,\|_{L^{\frac{n}{4}}} \ , \\ &\leqslant \ \|\, A\,\|_{L^n}\cdot\|\, (I-\mathcal{P})\mathcal{N}\,\|_{L^{\frac{n}{3}}} \ , \\ &\lesssim \ \widetilde{\epsilon}_0\ \|\, (I-\mathcal{P})\mathcal{N}\,\|_{L^{\frac{n}{3}}} \ .\end{aligned}$$

Therefore, for $\widetilde{\epsilon}_0$ sufficiently small we see that we must have $(I-\mathcal{P})\mathcal{N} = 0$ as was to be shown. This completes the proof that the solution to (98) is a solution to the general system (97), and therefore ends our proof of Proposition 5.3. □

5.3. **The second order curvature equation and the main a-priori estimate.** Through a repeated application of the local existence theorem 5.3, we may reduce the proof of the global existence theorem 5.2 to showing a-priori that any solution to the Coulomb system (84)–(86) which exists on a time interval $[0, T^*]$ (possibly large!), and such that it obeys the *both* the initial data bounds (89a)–(89c), as well as the evolution bounds (93)–(96), in fact obeys the improved evolution bounds (90a)–(90b).

Now, it turns out that the system of equations (98) is by itself not so well adapted[5] to the proof of such an a-priori estimate. This stems from the fact that these equations are not covariant. This manifests itself in the projection operator $\mathcal{P}$. If one were to try to write the hyperbolic system of equations (98a) in terms of covariant wave operator $\Box_A$ and a source term, the projection operator which is non-local would end up causing problems in various commutator terms. The way around this is to not only consider the system (98), but to also work directly with the curvature in the equations (85a)–(85b). This is possible because we are not attempting to set up an iteration scheme, but are instead merely trying to prove an a-priori estimate, so we may safely assume that the quantities we work with satisfy any equation which results from the system (85). We will in fact use several such elliptic and hyperbolic equations. As a very rough description of this kind of philosophy, the reader may find it useful to keep in mind the following schematic:

$$\begin{aligned}
\text{Weak control of the connection} \implies & \text{ Improved control of the curvature} \ , \\
\implies & \text{ Improved control of the connection} \ , \\
\implies & \text{ Weak control of the connection for longer times} \ .
\end{aligned}$$

To provide the improved control on the curvature, we will employ a second order equation for it. To derive this, we write the Bianchi identities (85b) in the form (5) and then contract this expression with the covariant derivative D. This yields the equations:

$$\begin{aligned}
0 \ &= \ D^\gamma(D_\alpha F_{\beta\gamma} + D_\gamma F_{\alpha\beta} + D_\beta F_{\gamma\alpha}) \ , \\
&= \ \Box_A F_{\alpha\beta} + [F^\gamma{}_\alpha, F_{\beta\gamma}] + [F^\gamma{}_\beta, F_{\gamma\alpha}] \ , \\
&= \ \Box_A F_{\alpha\beta} - 2[F_{\alpha\gamma}, F_\beta{}^\gamma] \ . \qquad (101)
\end{aligned}$$

In addition to (101) and the system (98), it will also be useful for us to employ a secondary elliptic equation. This will be for the quantity $\partial_t A_0$:

$$\partial_t A_0 \ = \ \Delta^{-1}\nabla^i(-[A_i, \partial_t A_0] + [A_0, \partial_t A_i] + [A^\alpha, F_{i\alpha}]) \ . \qquad (102)$$

This equation follows immediately from differentiating the equation (98b) with respect to time, and then applying the conservation law $\nabla^\alpha[A^\beta, F_{\alpha\beta}] = 0$ to the resulting expression. We are now ready to state our main a-priori estimate:

Theorem 5.4 (Main a-priori estimate for the curvature of the Coulomb system (84)–(86))**.** *Let the space-time connection $D = d + A$ on $\mathbb{R}^{(n+1)}$, where $6 \leqslant n$, be given such that it satisfies the following system of equations on some finite time*

[5] Strictly speaking, this is not entirely true. This can be seen from the fact that if one looks at the localized commutator $[\Box_A, \mathcal{P}]P_\lambda$, where the connection $\{A_\alpha\}$ is assumed to be of much lower frequency than λ, then this is essentially a "derivative falls on low" interaction which can be handled with the available Strichartz estimates in $5 \leqslant n$ dimensions. We have elected instead to follow a formulation of the YM system which is based on the curvature because of its conceptual appeal. However, in lower dimensions, it may be best to work directly with the connection $\{A_\alpha\}$, in part to help mitigate bad *High* $\times$ *High* $\Rightarrow$ *Low* frequency interactions which come from the quadratic term on the right hand side of (101).

interval $[0,T^*]$*:*

$$\Box_A F_{\alpha\beta} = 2[F_{\alpha\gamma}, {F_\beta}^\gamma] \ , \tag{103a}$$

$$dA + [A,A] = F \ , \tag{103b}$$

$$d^*\underline{A} = 0 \ , \tag{103c}$$

$$\Box A_i = \mathcal{P}\big(- [\partial_t A_0, A_i] - [A_\alpha, \partial^\alpha A_i] - [A^\alpha, F_{\alpha i}]\big) \ , \tag{103d}$$

$$\Delta A_0 = \partial^i [A_0, A_i] + [A^i, F_{0i}] \ , \tag{103e}$$

$$\Delta(\partial_t A_0) = \partial^i\big(- [A_i, (\partial_t A_0)] + [A_0, \partial_t A_i] - [A^\alpha, F_{i\alpha}]\big) \ . \tag{103f}$$

Here we have split $\{A_\alpha\} = (A_0, \{\underline{A}_i\})$*. Let there also be given a set of fixed constants* L, N, L_k, N_k *for the indices* $\frac{n-2}{2} \leqslant k$*, such that at time* $t=0$ *we have the initial bounds:*

$$\| F(0) \|_{\dot{H}^{\frac{n-4}{2}}} \leqslant \widetilde{\epsilon}_0 \ , \qquad \| \partial_t F(0) \|_{\dot{H}^{\frac{n-6}{2}}} \leqslant L\widetilde{\epsilon}_0 \ , \tag{104}$$

$$\| F(0) \|_{\dot{H}^k} \leqslant \widetilde{M}_k \ , \qquad \| \partial_t F(0) \|_{\dot{H}^{k-1}} \leqslant L_k \widetilde{M}_k \ . \tag{105}$$

Then if $\widetilde{\epsilon}_0$ *is chosen as to be sufficiently small on line* (104) *above, there exists a collection constants* C, C_k*, which only depend on the dimension and the collection* L, N, L_k, N_k *but* not *on* $\widetilde{\epsilon}_0$ *(once it is small enough) or the collection* $\widetilde{M}_k$*, such that if at later times we have the bounds:*

$$\sup_{0\leqslant t\leqslant T^*} \| \underline{A}(t) \|_{\dot{H}^{\frac{n-2}{2}}} \leqslant 2NC\widetilde{\epsilon}_0 \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t \underline{A}(t) \|_{\dot{H}^{\frac{n-4}{2}}} \leqslant 2NC\widetilde{\epsilon}_0 \ , \tag{106}$$

$$\sup_{0\leqslant t\leqslant T^*} \| F(t) \|_{\dot{H}^{\frac{n-4}{2}}} \leqslant 2NC\widetilde{\epsilon}_0 \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t F(t) \|_{\dot{H}^{\frac{n-6}{2}}} \leqslant 2NC\widetilde{\epsilon}_0 \ , \tag{107}$$

$$\sup_{0\leqslant t\leqslant T^*} \| \underline{A}(t) \|_{\dot{H}^k} < \infty \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t \underline{A}(t) \|_{\dot{H}^{k-1}} < \infty \ , \tag{108}$$

$$\sup_{0\leqslant t\leqslant T^*} \| F(t) \|_{\dot{H}^k} < \infty \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t F(t) \|_{\dot{H}^{k-1}} < \infty \ , \tag{109}$$

the following set of stronger bounds holds:

$$\sup_{0\leqslant t\leqslant T^*} \| F(t) \|_{\dot{H}^{\frac{n-4}{2}}} \leqslant N^{-1}C\widetilde{\epsilon}_0 \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t F(t) \|_{\dot{H}^{\frac{n-6}{2}}} \leqslant N^{-1}C\widetilde{\epsilon}_0 \ , \tag{110}$$

$$\sup_{0\leqslant t\leqslant T^*} \| F(t) \|_{\dot{H}^k} \leqslant N_k^{-1}C_k\widetilde{M}_k \ , \qquad \sup_{0\leqslant t\leqslant T^*} \| \partial_t F(t) \|_{\dot{H}^{k-1}} \leqslant N_k^{-1}C_k\widetilde{M}_k \ . \tag{111}$$

Remark 5.5. The bounds involving (109) and (111) express the fact that the control we provide here is at the critical level. That is, bounds on the higher norms are completely irrelevant in the bootstrapping procedure, except for the fact that they are finite. The only place where we need higher norms to accomplish anything here is in the local existence theorem 5.3. The way we will prove Theorem 5.4 is by first establishing control at the critical level through a bootstrapping argument. The control of the higher norms will then be provided through an a-priori estimate who's proof is essentially identical to that of the critical bootstrapping bound, and will therefore be left to the reader.

Remark 5.6. The reader my find it useful to have a brief description of the various constants appearing in Proposition 5.3 and Theorem 5.4. The constants L, L_k, N, N_k are input into the a-priori machine, and these are meant to cover the transition to and from estimates involving the connection and curvature. The set L, L_k is only needed to deal with the initial data. This is necessary because we must have an account of bounds involving the quantities $\partial_t F$. The other constants N, N_k govern comparison type estimates similar to (70). The constants C, C_k are byproducts of the proof of the a-priori estimate itself. These will very much depend on the L, L_k, N, N_k, but are *independent* of $\widetilde{\epsilon}_0$ when it is small enough. Finally, the main adjusting parameter $\widetilde{\epsilon}_0$ has two important roles. First and foremost, it is needed to prove the a-priori estimate itself. However, it has a second purpose which is also crucial, and that is to keep the dependence of C, C_k on L, L_k, N, N_k from creating a feedback loop. Specifically, we need our various comparison estimates to have constants which *do not* depend on the large constants C, C_k. Since the critical energy of the curvature can grow by a factor of C, we will need the extra influence of $\widetilde{\epsilon}_0$ to make sure this does not cycle back to L, L_k, N, N_k.

Proof that Theorem 5.4 and Proposition 5.3 together imply Theorem 5.2. The proof here is more or less straightforward and will be largely left to the reader. Everything relies on two sets of estimates. The first has to do with showing that the initial data bounds (89a)–(89c) imply the initial control assumed in (104)–(105). This is just a matter of bounding the time derivatives $\partial_t F$, and is why we have included the set of auxiliary constants L, L_k. Using now the field equations (4)–(5) (we have not included them in the system (103), but we may assume they hold), we have the general schematic identity at time $t = 0$:

$$\partial_t F\,(0) \ = \ \nabla_x F\,(0) + [a, F(0)] \ , \tag{112}$$

where we have generically set $a = (a_0, \{a_i\})$. Therefore, to establish the control (104)–(105), we only need to prove the estimates:

$$\|\,[a, F(0)]\,\|_{\dot{H}^{\frac{n-6}{2}}} \ \lesssim \ \widetilde{\epsilon}_0 \ , \qquad\qquad \|\,[a, F(0)]\,\|_{\dot{H}^{k-1}} \ \lesssim \ \widetilde{M}_k \ , \tag{113}$$

assuming that the bounds (89a)–(89c) hold. Notice that while these initial norms do not contain estimates on the quantities $E_i = F_{0i}(0)$, we originally had bounds on this from lines (76)–(79) above. Also, any estimates on a_0 which are needed in this process can be provided, for instance, through the equation (80). Since the proof of estimate (113) is a straightforward paraproduct type bound, similar to what was done in the proof of Lemma 5.1 above, we leave it to the interested reader (see below for more details).

The second set of estimates we need to prove here has to do with the relationship between the later time norms (106)–(111) and the ones (93)–(96) contained in the proof of the local existence proposition. Since our global regularity proof is by iteration of this latter result, we need to first show that the weak control (93)–(96) implies the bootstrapping assumption (106)–(109). This assertion is trivial for norms involving the potentials (106) and (108), as well as the larger norms (109) just by applying the definition of curvature. Therefore, we only need to see that (93)–(94) implies the bounds (107). We first establish the desired bounds for the undifferentiated term F. For the spatial curvature and potentials $(\underline{F}, \underline{A})$, this is just the comparison principle form line (70), and we can assume that the constants N, N_k are large enough to cover that case. To deal with potentials involving time derivatives of $\underline{A}$ or the temporal potential A_0 we have the following general calculation:

$$\begin{aligned} \| F \|_{\dot{H}^{\frac{n-4}{2}}} &\leqslant \| dA \|_{\dot{H}^{\frac{n-4}{2}}} + \| [A, A] \|_{\dot{H}^{\frac{n-4}{2}}} \ , \\ &\lesssim \| A \|_{\dot{H}^{\frac{n-2}{2}}} + \| A \|^2_{\dot{H}^{\frac{n-2}{2}}} \ , \end{aligned}$$

where the quadratic term follows from paraproduct decompositions, Hölders inequality, and Sobolev embeddings as in the proof of (70). The desired result now follows from the smallness of $\widetilde{\epsilon}_0$ and the fact that we may assume the constant C in line (93) does not depend on it. To establish the estimate for the quantity $\partial_t F$, we use the later time version of the identity (112), as well as the estimate which is responsible for the first estimate on line (113) above, which is:

$$\| [A, F] \|_{\dot{H}^{\frac{n-6}{2}}} \lesssim \| A \|_{\dot{H}^{\frac{n-2}{2}}} \cdot \| F \|_{\dot{H}^{\frac{n-4}{2}}} \ .$$

By again assume that the constant $\widetilde{\epsilon}_0$ is sufficiently small with respect to C we have the desired bound.

The final thing we need to do here is to show that the improved bounds (110)–(110) imply the assumed estimates of the local existence theorem (91)–(92). This is again a comparison estimate either identical or similar to (70). Note that we only need to bound the spatial portion of the potentials $\{A_\alpha\}$ and their time derivatives. The undifferentiated terms can be bounded directly by (70) because we may assume that the constant $\widetilde{\epsilon}_0$ on line (110) is small enough that the critical estimate (65) holds. To deal with the time differentiated potentials $\partial_t \underline{A}$, one can simply differentiate the Hodge system (103b)–(103c) with respect to time and then apply essentially the same proof as was used to produce (70). The details of this are left to the ambitious reader. □

6. Proof of the Main Bootstrapping Estimate

We are now ready to begin our proof of the main a-priori estimates (110)–(110). In order to do this, we will need to bootstrap in a function space which is much stronger than the energy type spaces of Theorem 5.4. This will cost us another bootstrapping procedure, but this will be easy to set up because it will be clear the extra norms we create have good bounds on some vary small initial time interval due to the fact that we are assuming the higher energy boundedness (109) and that these norms involve integrations in time. All of the norms we construct here will be of Strichartz type, with an ℓ^2 Besov structure in the spatial variable. It will also be necessary for us to include an angular square sum structure in these estimates. This may seem a bit odd at first because we will not need such norms directly in our proof of Theorem 5.4. The extra norms will instead be used to give the fine control which is needed to handle the linear part of the problem. At each fixed frequency, we form the square-sum norms:

$$\| A \|_{SL^p} \ = \ \sup_{\theta \lesssim 1} \left(\sum_{\substack{\phi\ : \\ \omega_0 \in \Gamma_\phi}} \| {}^{\omega_0}\Pi_\theta A \|_{L^p}^2 \right)^{\frac{1}{2}} \ , \tag{114}$$

where Γ_ϕ is taken to be a (uniformly) finitely overlapping set of spherical caps such that $\mathbb{S}^{n-1} = \cup_\phi \Gamma_\phi$, each of which has size $\sim \theta$ and constructed such a way that one has the bounds:

$$\left(\sum_{\substack{\phi\ : \\ \omega_0 \in \Gamma_\phi}} \| {}^{\omega_0}\Pi_\theta A \|_{L^2} \right)^{\frac{1}{2}} \ \lesssim \ \| A \|_{L^2} \ ,$$

independent of the size of θ. Here we take the condition $\omega_0 \in \Gamma_\phi$ to mean that the variable ω_0 is essentially in the center of that spherical cap Γ_ϕ. The exact placement is not essential. Notice that by construction, these norms are contained in the usual L^p spaces because we can assume that one set of angular sectors we are summing over contains the whole sphere.

Next, using the same prescription that defined the Besov spaces (37), we define the angular square sum Besov spaces to be:

$$\| A \|_{S\dot{B}_2^{p,(q,s)}} \ = \ \left(\sum_\lambda \lambda^{2s - 2n(\frac{1}{q} - \frac{1}{p})} \ \| P_\lambda A \|_{SL^p}^2 \right)^{\frac{1}{2}} \ . \tag{115}$$

We define the main dispersive component of the function spaces we will be working with. These are L^2_t based Strichartz spaces, built on the norms (115) and (37). These are all defined on a finite time interval $[0, T^*]$, which will for the most part be left implicit:

$$\| A \|_{\dot{Z}^s} \ = \ \| A \|_{L^2_t(\dot{B}_2^{\frac{2(n-1)}{n-3},(2,s+\frac{1}{2})})[0,T^*]} \ , \tag{116}$$

$$\| A \|_{S\dot{Z}^s} \ = \ \| A \|_{L^2_t(S\dot{B}_2^{\frac{2(n-1)}{n-3},(2,s+\frac{1}{2})})[0,T^*]} \ . \tag{117}$$

To gain some intuition about these spaces, notice that they all scale like $L^\infty(\dot{H}^s)$ under the change of variables (8). Therefore, they all scale like solutions to the wave equations with $\dot{H}^s$ initial data. Indeed, these spaces are consistent with the available range of Strichartz estimates for the usual scalar wave equation, and it will be our goal to show that one has bounds on the norm (117) for solutions of the covariant wave operator on the left hand side of (103).

To form the overall spaces we will bootstrap in, we intersect the above space-time norms with the energy type spaces used in the statement of the main a-priori estimate (5.4):

$$\dot{X}^s \ = \ L^\infty[0,T^*](\dot{H}^s) \cap S\dot{Z}^s \ , \tag{118}$$

$$\dot{Y}^s \ = \ L^\infty[0,T^*](\dot{H}^s) \cap \dot{Z}^s \ . \tag{119}$$

It will also be necessary for us to estimate time derivatives in the above spaces. Since differentiation will decrease the scaling by one unit, we use the spaces:

$$\| A \|_{\dot{X}^s \times \partial_t^{-1}(\dot{X}^{s-1})} \ = \ \| A \|_{\dot{X}^s} + \| \partial_t A \|_{\dot{X}^{s-1}} \ ,$$

with an analogous definition for $\dot{Y}^s \times \partial_t^{-1}(\dot{Y}^{s-1})$.

6.1. **Proof of the Critical Bootstrapping Estimate.** We are now ready to prove the critical component of Theorem (5.4) (we will now change notation from $\widetilde{\epsilon}_0$ back to ϵ_0):

Proposition 6.1 (Critical bootstrapping estimate in the $\dot{X}^s$ spaces)**.** *Let the dimension be* $6 \leqslant n$*. Let the collection* (F, A) *be a space-time connection curvature pair which obeys the general smoothness conditions* (108)–(109)*, and which satisfies the system of equations* (103)*. Let* L, N *be given constants such that one has the initial bounds:*

$$\| F(0) \|_{\dot{H}^{\frac{n-4}{2}}} + \| \partial_t F\,(0) \|_{\dot{H}^{\frac{n-6}{2}}} \ \leqslant \ L\epsilon_0 \ . \tag{120}$$

Then there exists a constant C *which depends only on* L, N *and the dimension such that if one has the bootstrapping bounds on a time interval* $[0, T^*]$*:*

$$\sup_{0 \leqslant t \leqslant T^*} \| (\underline{A}, \partial_t \underline{A})(t) \|_{\dot{H}^{\frac{n-2}{2}} \times \dot{H}^{\frac{n-4}{2}}} \ \leqslant \ 2NC\epsilon_0 \ , \tag{121}$$

$$\| F \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ 2NC\epsilon_0 \ , \tag{122}$$

then for ϵ_0 *sufficiently small, we have that the following improved bounds on the same time interval* $[0, T^*]$*:*

$$\| F \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ N^{-1}C\epsilon_0 \ . \tag{123}$$

The proof of Proposition 6.1 will be accomplished through the standard use of Littlewood-Paley paraproduct decompositions, and the application of space-time estimates. All of the linear bounds we will need are provided by the following, which is the main technical result of this work:

Theorem 6.2 (Gauge covariant angular square-sum Strichartz estimates for Yang-Mills connections)**.** *Let the number of dimensions be such that* $6 \leqslant n$*, and let* $d + \underline{\widetilde{A}}$ *be a space-time connection defined* defined on all of Minkowski space $\mathcal{M}^{n+1}$ *such that it satisfies the conditions:*

$$\underline{\widetilde{A}}_0 \ = \ 0 \qquad \textit{(Temporal Gauge)} \ , \tag{124a}$$

$$d^* \underline{\widetilde{A}} \ = \ 0 \qquad \textit{(Coulomb Gauge)} \ , \tag{124b}$$

$$P_{|\xi| \ll |\tau|}(\underline{\widetilde{A}}) \ = \ 0 \qquad \textit{(Space-time frequency localization)} \ , \tag{124c}$$

$$\| \underline{\widetilde{A}} \|_{\dot{X}^{\frac{n-2}{2}}} \ \leqslant \ \mathcal{E} \qquad \textit{(Space-time estimate)} \ , \tag{124d}$$

$$\Box \underline{\widetilde{A}} \ = \ \widetilde{\mathcal{P}}([B, H]) \qquad \textit{(Structure equation)} \ , \tag{124e}$$

$$\| (B, H) \|_{\dot{Y}^{\frac{n-2}{2}} \times \dot{Y}^{\frac{n-4}{2}}} \ \leqslant \ \mathcal{E} \qquad \textit{(Structure estimates)} \ , \tag{124f}$$

where (B,H) is an auxiliary set of $\mathfrak{g}$ valued functions defined on all of $\mathcal{M}^{n+1}$. The symbol $\widetilde{\mathcal{P}}$ denotes a composition of the Leray projection $\mathcal{P}$ with some frequency cutoff function which is bounded on all mixed Lebesgue-Besov spaces of the type $L^p(\dot{B}_2^{p,(2,s)})$. We assume also that the connection $d+\underline{\widetilde{A}}$ satisfies the general smoothness bounds:

$$\sup_{-T^*\leqslant t\leqslant T^*} \| \underline{\widetilde{A}}(t) \|_{\dot{H}^k} \ < \ C \ , \qquad\qquad \frac{n-2}{2} \ < \ k \ , \tag{125}$$

for each fixed time T^*. Let now F be any other $\mathfrak{g}$ valued function which satisfies the inhomogeneous equation:

$$\Box_{\underline{\widetilde{A}}} F \ = \ G \ , \tag{126}$$

with Cauchy data:

$$F(0) \ = \ f \ , \qquad\qquad \partial_t F(0) \ = \ \dot{f} \ . \tag{127}$$

Then if the constant $\mathcal{E}$ in lines (124d) *and* (124f) *above is sufficiently small, one has the following family of space-time estimates:*

$$\| F \|_{\dot{X}^{\frac{n-4}{2}}\times\partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \lesssim \ \| (f,\dot{f}) \|_{\dot{H}^{\frac{n-4}{2}}\times\dot{H}^{\frac{n-6}{2}}} + \| G \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \ . \tag{128}$$

Remark 6.3. In the above Theorem, the Strichartz estimates have a preferred scaling. This is consistent with the application we have in mind. In general, it is not possible to prove estimates of the type (128) for higher Sobolev indices without assuming that the connection $\underline{\widetilde{A}}$ itself has more regularity. In the case where $\underline{\widetilde{A}}$ does have better regularity, a proof similar to that given after Proposition 7.1 below can be used to show estimates for those higher norms.

Proof of Proposition 6.1. The proof requires another bootstrapping argument. This will be done on subintervals $[0,T^{**}]\subseteq[0,T^*]$. Using the initial bounds (120) and the general smoothness assumption (109) we may assume that for $T^{**}\ll 1$ we have the estimate (122). Therefore, it suffices to prove that (122) implies (123) on all subintervals $[0,T^{**}]$. But this is just the same as proving Proposition 6.1 itself since T^* is arbitrary.

The proof will be accomplished in a series of steps. Our first goal will be to derive $\dot{X}^s$ and $\dot{Z}^s$ type bounds for the connection $d+A$. We will then split this connection into a sum of two pieces $d+\widetilde{A}+\widetilde{\widetilde{A}}$ where the potentials $\widetilde{A}$ satisfy the criteria of Theorem 6.2, and the remainder term $\widetilde{\widetilde{A}}$ obeys better the better $L^1(L^\infty)$ space-time estimate. This is enough to be able to write the equation (103a) schematically as:

$$\Box_{\widetilde{A}} F \ = \ [\nabla\widetilde{\widetilde{A}},F] + [\widetilde{\widetilde{A}},\nabla F] + \big[\widetilde{A},[\widetilde{\widetilde{A}},F]\big] + \big[\widetilde{\widetilde{A}},[\widetilde{\widetilde{A}},F]\big] + [F,F] \ . \tag{129}$$

One is then in a position where Theorem 6.2 can be applied directly, and we only need to choose our constant C depending on L,N and the constant which appears on line (128). The key thing is that the dangerous term $[\widetilde{\widetilde{A}},\nabla F]$ can safely be put in $L^1(\dot{H}^{\frac{n-6}{2}})$ using the improve pace-time estimate for $\widetilde{\widetilde{A}}$ and just the energy estimate for F. Throughout the proof we will use the usual splitting $\{A_\alpha\} = (A_0,\underline{A})$ of

$d + A$ into its temporal and spatial components.

• $\dot{X}^{\frac{n-2}{2}}$ **estimates for** $\{\underline{A}_i\}$**.** Here we write $\underline{F}$ for the spatial components of the field strength and use the Hodge system (103b)–(103c) to write schematically:

$$\underline{A} \ = \ \nabla_x \Delta^{-1}(-\underline{F} + [\underline{A}, \underline{A}]) \ . \tag{130}$$

As a preliminary first step, we will show that the potentials $\{\underline{A}_i\}$ can be estimated in $\dot{Y}^{\frac{n-2}{2}}$ with bounds comparable to $NC\epsilon_0$. Now, it is not too difficult to see directly from the definition that:

$$\nabla_x \Delta^{-1} \ : \ \dot{Y}^{\frac{n-4}{2}} \ \hookrightarrow \ \dot{Y}^{\frac{n-2}{2}} \ .$$

Next, notice that we have the bilinear estimate:

$$\nabla_x \Delta^{-1} \ : \ L^\infty(\dot{H}^{\frac{n-2}{2}}) \cdot \dot{Y}^{\frac{n-2}{2}} \ \hookrightarrow \ \dot{Y}^{\frac{n-2}{2}} \ , \tag{131}$$

which follows integrating the bound (41). Note that in this case, the range restrictions (42)–(46) are easily satisfied. Therefore, using the critical bounds (121) as well as the general smoothness criteria (108) (so that in particular we may assume the $\dot{Y}^{\frac{n-2}{2}}$ norm of $\{\underline{A}_i\}$ is finite) we see we may absorb the quadratic term on the right hand side of (130) onto the left in the desired estimates.

Our task is now to show the more restrictive $\dot{X}^{\frac{n-2}{2}}$ estimates for the potentials $\{\underline{A}_i\}$. Again from the definition, it is not hard to see that we have the embedding:

$$\nabla_x \Delta^{-1} \ : \ \dot{X}^{\frac{n-4}{2}} \ \hookrightarrow \ \dot{X}^{\frac{n-2}{2}} \ .$$

Therefore, keeping in mind the $\dot{Y}^{\frac{n-2}{2}}$ bounds just proved, we see that is suffices to be able to show the bilinear estimate:

$$\nabla_x \Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}} \cdot \dot{Y}^{\frac{n-2}{2}} \ \hookrightarrow \ \dot{X}^{\frac{n-2}{2}} \ . \tag{132}$$

The main issue here is, of course, to be able to include the angular square sum structure. This turns out to be very simple. Notice first that by orthogonality and the general nesting (38) we have the inclusion (on any finite time interval $[0, T^*]$):

$$L^\infty(\dot{H}^{\frac{n-2}{2}}) \cap L^2(\dot{H}^{\frac{n-1}{2}}) \ \subseteq \ \dot{X}^{\frac{n-2}{2}} \ .$$

Therefore, to conclude (132) we see that it suffices to be able to show the set of bilinear estimates:

$$\nabla_x \Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}} \cdot \dot{Y}^{\frac{n-2}{2}} \ \hookrightarrow \ L^\infty(\dot{H}^{\frac{n-2}{2}}) \ , \tag{133}$$

$$\nabla_x \Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}} \cdot \dot{Y}^{\frac{n-2}{2}} \ \hookrightarrow \ L^2(\dot{H}^{\frac{n-1}{2}}) \ . \tag{134}$$

The first of these embedding follows easily from:

$$\nabla_x \Delta^{-1} \ : \ L^\infty(\dot{H}^{\frac{n-2}{2}}) \cdot L^\infty(\dot{H}^{\frac{n-2}{2}}) \ \hookrightarrow \ L^\infty(\dot{H}^{\frac{n-2}{2}}) \ ,$$

which in turn follows directly from (41). The second estimate (134) above is more bilinear in nature. It follows from applying trichotomy and then summing the following two fixed frequency bilinear inclusions:

(135)
$$\nabla_x\Delta^{-1} \ : \ P_{\bullet\ll\lambda}\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})\big)\cdot P_\lambda\big(L^\infty(\dot{H}^{\frac{n-2}{2}})\big) \ \hookrightarrow \ P_\lambda\big(L^2(\dot{H}^{\frac{n-1}{2}})\big) \ .$$

(136)
$$\nabla_x\Delta^{-1} \ : \ P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})\big)\cdot P_\lambda\big(L^\infty(\dot{H}^{\frac{n-2}{2}})\big) \ \hookrightarrow \ \Big(\frac{\mu}{\lambda}\Big)^\delta P_\mu\big(L^2(\dot{H}^{\frac{n-1}{2}})\big) \ ,$$

where we have set $\delta = 2(\frac{n-2}{n-1}) - \frac{3}{2}$ to be the "gap" constant. The estimates (135)–(136) follow directly from the frequency localized bounds (47)–(48). Note that in this case, the various positivity conditions are satisfied.

- $\dot{Y}^{\frac{n-2}{2}}\times\dot{Y}^{\frac{n-4}{2}}$ **bounds for the pair** $(A_0,\partial_t A_0)$**.** Our first step here is to deal with the variable A_0. We integrate equation (103e) and write it schematically as:

$$A_0 \ = \ \Delta^{-1}(\nabla_x[A_0,\underline{A}] + [\underline{A},F]) \ . \tag{137}$$

The desired estimate now follows by constructing A_0 from scratch by iteration, using the already established estimates and bilinear embedding (131) and the following:

$$\Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}}\cdot\dot{Y}^{\frac{n-4}{2}} \ \hookrightarrow \ \dot{Y}^{\frac{n-2}{2}} \ . \tag{138}$$

This last embedding follows in turn from the pair of estimates:

$$\Delta^{-1} \ : \ L^\infty(\dot{H}^{\frac{n-2}{2}})\cdot L^\infty(\dot{H}^{\frac{n-4}{2}}) \ \hookrightarrow \ L^\infty(\dot{H}^{\frac{n-2}{2}}) \ ,$$
$$\Delta^{-1} \ : \ L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})\cdot L^\infty(\dot{H}^{\frac{n-4}{2}}) \ \hookrightarrow \ L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}) \ .$$

Both of these are easy consequences of (41) and we leave the numerology to the reader.

To establish the $\dot{Y}^{\frac{n-4}{2}}$ bound for $\partial_t A_0$, we can use the equation (103f) to treat it as a separate variable. In that equation we have quantities of the form $\partial_t\underline{A}$. We can use the curvature equation (103b) to swap this for spatial derivatives as follows:

$$\partial_t\underline{A} \ = \ \nabla_x A_0 - [A_0,\underline{A}] + F \ . \tag{139}$$

This allows us to write schematically:

$$(\partial_t A_0) \ = \ \nabla_x\Delta^{-1}\big([A,(\partial_t A_0)] + [A,\nabla_x A] + \big[A,[A,A]\big] + [A,F]\big) \ , \tag{140}$$

where A now denotes any of the full set of potentials $\{A_\alpha\}$ which we have estimated in the space $\dot{Y}^{\frac{n-2}{2}}$. We may now iterate the equation (140) in the space $\dot{Y}^{\frac{n-4}{2}}$ to constructively obtain the desired bounds using the bilinear embedding:

$$\nabla_x\Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}}\cdot\dot{Y}^{\frac{n-4}{2}} \ \hookrightarrow \ \dot{Y}^{\frac{n-4}{2}} \ .$$

which follows from differentiating (138) above. Notice that the needed inclusion $[A,A]\hookrightarrow\dot{Y}^{\frac{n-4}{2}}$ follows, for instance, from differentiating the embedding (131).

• **Splitting the spatial potentials.** Our next goal is to split the spatial potentials $\{\underline{A}_i\}$ into a sum of two pieces which are each more easily managed. This will be done using the "structure" equation (103d). Using the formula (139) to get rid of terms of the form $\partial_t \underline{A}$ on the right hand side of this equation, and using the various $\dot{Y}^s$ space embeddings we have just shown (on the time interval $[0, T^*]$), we may write this equation in the schematic form:

$$\Box \underline{A} \ = \ \mathcal{P}([B, H]) \ , \tag{141}$$

where the quantities (B, H) obey the estimate:

$$\| (B, H) \|_{\dot{Y}^{\frac{n-2}{2}} \times \dot{Y}^{\frac{n-4}{2}}} \ \lesssim \ NC\epsilon_0 \ ,$$

where the implicit constant in the above inequality comes from the estimates just shown. Using Duhamel's principle and (sharp) time cutoffs, we now extend (141) to all possible times. This is done simply by writing:

$$\underline{A}(t) \ = \ \underline{A}^{(0)}(t) + \int_0^t \ \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} \ \mathcal{P}([B, H])(s) \cdot \chi_{[0,T^*]}(s) \ ds \ , \tag{142}$$

where $\underline{A}^{(0)}$ denotes to propagation of $\big(\underline{A}(0), \partial_t \underline{A}\,(0)\big)$ as a solution to the free scalar wave equation. Also, here $\chi_{[0,T^*]}$ denotes the indicator function of the time interval $[0, T^*]$. This implies that we have the condition:

$$\Box \underline{A}\ (t) \ = \ 0 \ , \qquad\qquad t \ < \ 0 \ , \qquad\qquad T^* \ < \ t \ .$$

Now, from the bootstrapping assumption (121) we have the pair of bounds:

$$\| \big(\underline{A}(0), \partial_t \underline{A}\,(0)\big) \|_{\dot{H}^{\frac{n-2}{2}} \times \dot{H}^{\frac{n-4}{2}}} \ \leqslant \ NC\epsilon_0 \ ,$$
$$\| \big(\underline{A}(T^*), \partial_t \underline{A}\,(T^*)\big) \|_{\dot{H}^{\frac{n-2}{2}} \times \dot{H}^{\frac{n-4}{2}}} \ \leqslant \ NC\epsilon_0 \ .$$

Therefore, using the bounds we have just shown in conjunction with the usual Strichartz estimates for the wave equation, we have that this extension of the potentials $\{\underline{A}_i\}$ satisfies the bounds:

$$\| \underline{A} \|_{\dot{X}^{\frac{n-2}{2}}} \ \lesssim \ NC\epsilon_0 \ .$$

Notice that the angular square function structure inherent in the $\dot{X}^s$ norms is provided automatically by the fact that the usual wave equation commutes with the angular cutoffs ${}^\omega\Pi_\theta$.

Our next step to introduce the space–time frequency cutoff $S_{|\tau| \lesssim |\xi|}$, which cuts off smoothly on the region $|\tau| \lesssim |\xi|$. That is, the compound multipliers $P_\lambda S_{|\tau| \lesssim |\xi|}$ all have L^1 kernels with uniform bounds. We denote by $S_{|\xi| \ll |\tau|} = I - S_{|\tau| \lesssim |\xi|}$. Our decomposition of $\{\underline{A}_i\}$ is now given by the formula:

$$\widetilde{\underline{A}} \ = \ S_{|\tau| \lesssim |\xi|} \underline{A} \ , \qquad\qquad \widetilde{\widetilde{\underline{A}}} \ = \ S_{|\xi| \ll |\tau|} \underline{A} \ .$$

We now need to show that both the potential sets $\{\widetilde{\underline{A}_i}\}$ and $\{\widetilde{\widetilde{\underline{A}_i}}\}$ obey good $\dot{X}^{\frac{n-2}{2}}$ estimates. Since the original collection of extended potentials does, we only need to prove this assertion for one of these sets. This is most easily shown for the collection $\{\widetilde{\underline{A}_i}\}$. As we have already mentioned, the cutoffs $P_\lambda S_{|\tau| \lesssim |\xi|}$ are bounded on all mixed Lebesgue spaces. Therefore, the entire multiplier $S_{|\tau| \lesssim |\xi|}$ is bounded on any mixed Lebesgue-Besov space of the type $L^q(\dot{B}^{p,(2,s)})$. This implies that this multiplier is in fact bounded on the $\dot{X}^s$ spaces, which is enough to support our claim.

Finally, we would like to prove two fixed frequency multiplier estimates which will be useful in the sequel when dealing with the two sets of potentials $\{\widetilde{\underline{A}_i}\}$ and $\{\widetilde{\widetilde{\underline{A}_i}}\}$. The first is:

$$\| \partial_t P_\lambda S_{|\tau|\lesssim|\xi|} A \|_{L^p} \lesssim \lambda \, \| A \|_{L^p} \qquad\qquad 1 \leqslant p \leqslant \infty . \tag{143}$$

This is easily demonstrated by rescaling to frequency $\lambda = 1$ and using the L^1 bound on the convolution kernel of $\partial_t P_\lambda S_{|\tau|\lesssim|\xi|}$. Combining this with the remarks made above, we see that we have the estimate:

$$\| \partial_t \widetilde{\underline{A}} \|_{\dot{X}^{\frac{n-4}{2}}} \lesssim NC\epsilon_0 .$$

In particular, from everything we have shown, the potential set $\{\widetilde{\underline{A}_i}\}$ satisfies all of the requirements (124) of Theorem 6.2 when ϵ_0 is sufficiently small.

The second fixed frequency multiplier bound that will be of use shortly is the space–time estimate:

$$\| \varXi^{-1} P_\lambda S_{|\xi|\ll|\tau|} A \|_{L^q(L^p)} \lesssim \lambda^{-2} \, \| A \|_{L^q(L^p)} . \tag{144}$$

Here $\varXi$ is the multiplier with symbol $\varXi(\tau,\xi) = \tau^2 - |\xi|^2$. To prove this, we employ a family of Littlewood-Paley space-time cutoffs which we denote by S_μ. By this we mean that the space-time frequency support of these is supported where $|\tau| + |\xi| \sim \mu$. As usual, these are all chosen so as to have uniform L^1 bounds on their convolution kernels. Using the support restrictions of the $S_{|\xi|\ll|\tau|}$ multiplier, we have the formula:

$$P_\lambda S_{|\xi|\ll|\tau|} A = \sum_{\substack{\mu\, : \\ \lambda\lesssim\mu}} P_\lambda S_\mu S_{|\xi|\ll|\tau|} A .$$

Therefore, by dyadic summing and the boundedness of the multiplier P_λ, to prove (144) it suffices to be able to show that:

$$\| \varXi^{-1} S_{|\xi|\ll|\tau|} S_\mu A \|_{L^q(L^p)} \lesssim \mu^2 \, \| A \|_{L^q(L^p)} .$$

This last bound follows easily from rescaling to frequency $\mu = 1$ and the appropriate differential bounds on the symbol of $\varXi^{-1} S_{|\xi|\ll|\tau|}$ which we leave to the reader.

- $L^1(L^\infty)$ **bounds for the connection** $\{\widetilde{\widetilde{A}}_\alpha\} = (A_0, \{\widetilde{\widetilde{\underline{A}}}\})$**.** Our goal here is to show the ℓ^1 type Besov estimate:

$$\| (A_0, \{\widetilde{\widetilde{\underline{A}}}\}) \|_{L^1(\dot{B}_1^{\infty,(2,\frac{n}{2})})} \lesssim NC\epsilon_0 . \tag{145}$$

By repeatedly using the estimate (144), we have that the multiplier $\varXi^{-1}\Delta S_{|\xi|\ll|\tau|}$ is bounded on the space $L^1(\dot{B}_1^{\infty,(2,\frac{n}{2})})$. Furthermore, from all of the estimates we have shown above, and by distributing the derivative in the first term on the right hand side of (137), we see that the right hand side of the schematics (137) and (141) are equivalent. Therefore, we have the following heuristic schematic for the potentials $\{\widetilde{\widetilde{A}}_\alpha\}$:

$$\widetilde{\widetilde{A}} = \Delta^{-1}([B, H]) ,$$

where the pair (B,H) enjoys the bounds:

$$\| (B,H) \|_{\dot{Y}^{\frac{n-2}{2}} \times \dot{Y}^{\frac{n-4}{2}}} \lesssim NC\epsilon_0 \ .$$

The bound (145) now follows from the bilinear estimate:

$$\Delta^{-1} \ : \ \dot{Y}^{\frac{n-2}{2}} \cdot \dot{Y}^{\frac{n-4}{2}} \hookrightarrow L^1(\dot{B}_1^{\infty,(2,\frac{n}{2})}) \ .$$

This in turn follows from the product estimate:

$$\Delta^{-1} \ : \ L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}) \cdot L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}) \hookrightarrow L^1(\dot{B}_1^{\infty,(2,\frac{n}{2})}) \ .$$

This last estimate follows at once from (41). The check on the conditions (42)–(46) is left to the reader.

• **Improving the curvature.** This is the final part of the proof of Proposition 6.1. Recalling the schematic (129) and using the Strichartz estimates (128), our goal here is to show the following four bounds:

$$\| [\nabla \widetilde{\widetilde{A}}, F] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim N^2 C^2 \epsilon_0^2 \ , \tag{146}$$

$$\| [\widetilde{\widetilde{A}}, \nabla F] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim N^2 C^2 \epsilon_0^2 \ , \tag{147}$$

$$\| [\widetilde{A}, [\widetilde{\widetilde{A}}, F]] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim N^2 C^2 \epsilon_0^2 \ , \tag{148}$$

$$\| [\widetilde{\widetilde{A}}, [\widetilde{\widetilde{A}}, F]] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim N^2 C^2 \epsilon_0^2 \ , \tag{149}$$

$$\| [F, F] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim N^2 C^2 \epsilon_0^2 \ . \tag{150}$$

For ϵ_0 sufficiently small, this will be enough for us to conclude the improved bootstrapping estimates (123) by choosing C to be such that $\frac{1}{2}N^{-1}C$ is equal to the constant appearing on the right hand side of estimate (128). This works because the implicit constants which appear in (146)–(150) above have only been manufactured in the estimates of this proof, and can all be chosen to be independent of N and C if ϵ_0 is chosen small enough.

To prove these bounds, first notice that the estimates (146) and (148)–(150) are essentially identical. This follows from the equivalence (in terms of $\dot{Y}^s$ spaces) $\nabla_x \widetilde{\widetilde{A}} \approx F$. We also have the equivalences $[\widetilde{A}, \widetilde{\widetilde{A}}] \approx F$ and $[\widetilde{\widetilde{A}}, \widetilde{\widetilde{A}}] \approx F$. These are given by the inclusion:

$$\dot{Y}^{\frac{n-2}{2}} \cdot \dot{Y}^{\frac{n-2}{2}} \subseteq \dot{Y}^{\frac{n-4}{2}} \ . \tag{151}$$

This is easily demonstrated, as we have already mentioned, by differentiating the inclusion (131) and using the boundedness of $\nabla^2 \Delta^{-1}$ on the various $\dot{Y}^s$ component spaces. Therefore, to prove (146) and (148)–(150) we only need to know that:

$$L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}) \cdot L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}) \hookrightarrow L^1(\dot{H}^{\frac{n-6}{2}}) \ . \tag{152}$$

This is yet again a consequence of our general Besov calculus (41), and we leave the various additions to the reader.

Our final task here is to prove the estimate (147). This needs to be frequency decomposed using a trichotomy. Specifically, we have the following set of fixed frequency estimates in the three cases (note that in the first two estimates below the square summing needs to be done *inside* the time integral):

$$P_{\bullet\ll\lambda}\big(L^1(\dot{B}^{\infty,(n,\frac{n}{2})})\big)\cdot P_\lambda\big(L^\infty(\dot{H}^{\frac{n-6}{2}})\big) \ \hookrightarrow\ P_\lambda\big(L^1(\dot{H}^{\frac{n-6}{2}})\big)\ , \tag{153}$$

$$P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}\big)\cdot P_{\bullet\ll\lambda}\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})\big) \ \hookrightarrow\ P_\lambda\big(L^1(\dot{H}^{\frac{n-6}{2}})\big)\ , \tag{154}$$

$$P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}\big)\cdot P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})\big) \ \hookrightarrow\ \Big(\frac{\mu}{\lambda}\Big)^\delta P_\mu\big(L^1(\dot{H}^{\frac{n-6}{2}})\big)\ , \tag{155}$$

where the quantity δ in the last estimate (155) above can be computed to be $\delta = n(\frac{n-3}{n-2}) - 3$. The estimate (153) follows from inspection. The latter two estimates (154)–(155) follow from (47)–(47) of Remark 4.2. This completes the proof of Proposition 6.1. $\square$

7. Reduction to Approximate Half-Wave Operators

This is a preliminary technical section where we reduce the proof of the Strichartz estimates (128) to a more easily managed form. This material more or less standard, and we again follow closely what was done in [8]. Our first step here is to reduce the proof of Theorem 6.2 to the following:

Proposition 7.1 (Existence of a fixed frequency parametrix)**.** *Let the number of dimensions be $6\leqslant n$, and let $d+\underline{A}_{\bullet\ll\lambda}$ be a connection which satisfies the conditions* (124)*. In addition assume that we have the frequency localization condition:*

$$P_{\lambda\lesssim\bullet}(\underline{A}_{\bullet\ll\lambda}) \ = \ 0\ , \tag{156}$$

where $P_{\lambda\lesssim\bullet}$ is a frequency cutoff on the region where $2^{-10a}\lambda\leqslant|\xi|$, where $1\leqslant a$ is some fixed parameter. Then if the constant $\mathcal{E}$ on lines (124d) *and* (124f) *is sufficiently small, there exists a family of approximate propagation operators $W^\lambda_{\underline{A}_{\bullet\ll\lambda}}(s)$ (or just W^λ_s for short) such that if (f_λ,g_λ) is any set of λ–frequency initial data with Fourier support in the region $2^{-a}\lambda\leqslant|\xi|\leqslant 2^a\lambda$, the following estimates hold:*

$$\|\,W^\lambda_s(f_\lambda,g_\lambda)\,\|_{\dot{X}^0\times\partial_t^{-1}(\dot{X}^{-1})} \ \lesssim\ E^{\frac12}(f_\lambda,g_\lambda)\ , \tag{157a}$$

$$\|\,W^\lambda_s(f_\lambda,g_\lambda)(s) - f_\lambda\,\|_{L^2} \ \lesssim\ \mathcal{E}^{\frac12}\,E^{\frac12}(f_\lambda,g_\lambda)\ , \tag{157b}$$

$$\|\,\partial_t W^\lambda_s(f_\lambda,g_\lambda)(s) - g_\lambda\,\|_{L^2} \ \lesssim\ \lambda\,\mathcal{E}^{\frac12}\,E^{\frac12}(f_\lambda,g_\lambda)\ , \tag{157c}$$

$$\|\,\Box_{\underline{A}_{\bullet\ll\lambda}} W^\lambda_s(f_\lambda,g_\lambda)\,\|_{L^1(L^2)} \ \lesssim\ \lambda\,\mathcal{E}\,E^{\frac12}(f_\lambda,g_\lambda)\ . \tag{157d}$$

Here we have set $E(f_\lambda,g_\lambda)$ to the L^2 normalized energy:

$$E(f_\lambda,g_\lambda) \ = \ \|\,f_\lambda\,\|^2_{L^2} + \lambda^{-2}\,\|\,g_\lambda\,\|^2_{L^2}\ .$$

Finally, we have that the frequency support of the parametrix is contained in the set $2^{-2a}\lambda\leqslant|\xi|\leqslant 2^{2a}\lambda$, where a is as above.

Proof that Proposition 7.1 implies Theorem 6.2. The first step here is to reduce the estimate (128) to the case where $G \equiv 0$. This is done in the usual way via Duhamel's principle. We define the true propagation operator $U_s(t)$ via the formulas:

$$U_s(s)(f,g) \ = \ f \ , \qquad\qquad \partial_t U_s(s)(f,g) \ = \ g \ ,$$

and:

$$\Box_{\underline{A}} U_s(f,g) \ = \ 0 \ ,$$

We then have that:

$$F(t) \ = \ U_0(t)(f,\dot{f}) \ + \ \int_0^t \ U_s(t)(0,G(s)) \ ds \ , \tag{158}$$

solves the problem (126)–(127). In particular, by Minkowski's triangle inequality we easily have that:

$$\| \int_0^t \ U_s(t)(0,G(s)) \ ds \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ \int_0^\infty \ \| \, U_s(0,G(s)) \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ ds \ .$$

Therefore, we are trying to show:

$$\| \, U_s(f,g) \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ C \, \| \, (f,g) \, \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} \ , \tag{159}$$

for any pair of functions (f,g) and any initial time s. Since it is easy to see that the conditions (124) are translation invariant, it suffices to show this estimates for $s = 0$.

The estimate (159) will be shown using a bootstrapping procedure. This will be done inside of the compact intervals $[0,T^*]$. What we will do is to first assume that (159) is true for all $0 \leqslant s \leqslant T^*$ on all time intervals of the form $[0,s]$ and $[s,T^*]$, where the constant on the left hand side of (159) is replaced by $2C$. Our goal is then to improve the constant by proving the desired bound (159) on the time subintervals of $[0,T^*]$. Once this is accomplished, we can easily extend the bound (159) to all subintervals of a slightly larger time interval $[0,T^*+\gamma]$, where the constant $0 < \gamma \ll 1$ is determined by the bound (125). This is provided by the usual local existence theory based on energy and L^∞ estimates. Once this is done, the bootstrapping closes. Notice again that, by using the local existence theory and the bound (125), we may begin the argument for some very small time interval $[0,\gamma]$.

We are now assuming that (159) holds on our time interval $[0,T^*]$ with constant $2C$ which we will decide on in a moment. We are working with a solution:

$$\Box_{\underline{A}} F \ = \ 0 \ , \tag{160}$$

where the connection $d + \underline{A}$ satisfies (124), and where we have the initial data:

$$F(0) \ = \ f, \qquad\qquad \partial_t F\,(0) \ = \ g \ . \tag{161}$$

We now split this initial data into a sum frequency localized pieces:

$$f \ = \ \sum_\lambda \ P_\lambda(f) \ = \ \sum_\lambda \ f_\lambda \ ,$$
$$g \ = \ \sum_\lambda \ P_\lambda(g) \ = \ \sum_\lambda \ g_\lambda \ ,$$

and then repeatedly use Proposition 7.1 to construct an approximate solution to (160)–(161) as follows:

$$\widetilde{F} \ = \ \sum_\lambda \ \widetilde{F}_\lambda \ = \ \sum_\lambda \ W_0^\lambda(f_\lambda, g_\lambda) \ .$$

By summing over the parametrix estimate (157a) we automatically have that:

$$\| \, \widetilde{F} \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ \frac{1}{2} C \ \| \, (f,g) \, \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} \ ,$$

where C is some fixed constant. We choose this to be our definition of the constant on the right hand side of (159). Thus, our goal is to conclude that:

$$\| \, F - \widetilde{F} \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \leqslant \ \frac{1}{2} C \ \| \, (f,g) \, \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} \ . \tag{162}$$

To do this, we use the Duhamel formula (158) to express everything in terms of the operators $U_s(t)$:

$$F(t) - \widetilde{F}(t) \ = \ U_0(t)\big(f - \widetilde{F}(0), g - \partial_t \widetilde{F}\,(0)\big) - \int_0^t \ U_s(t)\big(0, \Box_{\underline{A}} \widetilde{F}\,(s)\big) \ ds \ .$$

By combining the assumed estimate (159) and the approximation bounds (157b)–(157c), we have that:

$$\| \, U_0\big(f - \widetilde{F}(0), g - \partial_t \widetilde{F}\,(0)\big) \, \|_{\dot{X}^{\frac{n-4}{2}} \times \partial_t^{-1}(\dot{X}^{\frac{n-6}{2}})} \ \lesssim \ C\mathcal{E}^{\frac{1}{2}} \, \| \, (f,g) \, \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} \ .$$

Therefore, by using Minkowski's triangle inequality and again using the bootstrapping assumption (159), we see that in order to conclude (162) we only need to show the following remainder estimate on the time interval $[0, T^*]$:

$$\| \, \Box_{\underline{A}} \widetilde{F} \, \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \ \lesssim \ C\mathcal{E} \, \| \, (f,g) \, \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} \ . \tag{163}$$

To show the estimate (163), we use a family of frequency cutoffs:

$$I \ = \ P_{\bullet \ll \lambda} + P_{\lambda \lesssim \bullet} \ ,$$

for each scale λ such that they all have L^1 kernels with uniform bounds, and such that the cutoff $P_{\bullet \ll \lambda}$ is consistent with the definition of $d + \underline{A}_{\,\bullet \ll \lambda}$ in the statement of Proposition 7.1. This allows us to schematically write:

$$\begin{aligned} \Box_{\underline{A}} \widetilde{F} \ = \ \sum_\lambda \Big(& \Box_{\underline{A}_{\,\bullet \ll \lambda}} \widetilde{F}_\lambda + [\nabla_x \underline{A}_{\,\lambda \lesssim \bullet}, \widetilde{F}_\lambda] + [\underline{A}_{\,\lambda \lesssim \bullet}, \nabla_x \widetilde{F}_\lambda] \\ & + \ \big[[\underline{A}_{\,\bullet \ll \lambda}, \underline{A}_{\,\lambda \lesssim \bullet}], \widetilde{F}_\lambda\big] + \big[[\underline{A}_{\,\lambda \lesssim \bullet}, \underline{A}_{\,\lambda \lesssim \bullet}], \widetilde{F}_\lambda\big] \Big) \ . \end{aligned} \tag{164}$$

The bound (163) for the term $\sum_\lambda \Box_{\underline{A}_{\,\bullet \ll \lambda}} \widetilde{F}_\lambda$ is a direct consequence of repeatedly applying the estimate (157d) while using the fact that each term in this sum is supported in frequency where $|\xi| \sim \lambda$ to gain the orthogonality needed to obtain bounds in terms of the pair (f, g). Therefore, we are reduced to showing the following family of error estimates:

$$\sum_{\lambda} \| [\nabla_x \underline{A}_{\lambda \lesssim \bullet}, \widetilde{F}_\lambda] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \mathcal{E} \| (f,g) \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} , \tag{165}$$

$$\sum_{\lambda} \| [\underline{A}_{\lambda \lesssim \bullet}, \nabla_x \widetilde{F}_\lambda] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \mathcal{E} \| (f,g) \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} , \tag{166}$$

$$\| \sum_{\lambda} \big[[\underline{A}_{\bullet \ll \lambda}, \underline{A}_{\lambda \lesssim \bullet}], \widetilde{F}_\lambda \big] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \mathcal{E} \| (f,g) \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} , \tag{167}$$

$$\| \sum_{\lambda} \big[[\underline{A}_{\lambda \lesssim \bullet}, \underline{A}_{\lambda \lesssim \bullet}], \widetilde{F}_\lambda \big] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \mathcal{E} \| (f,g) \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} . \tag{168}$$

These estimates are all very similar to each other, and to estimates we have already proved in the last section, in particular (146)–(150). To prove the first estimate (165) above, we further decompose the left hand side into frequencies and use the triangle inequality to bound:

$$\text{(L.H.S.)(165)} \leqslant \sum_{\substack{\lambda, \mu : \\ \lambda \lesssim \mu}} \| [\nabla_x P_\mu(\underline{A}), \widetilde{F}_\lambda] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} .$$

Thus, by Young's inequality, it suffices to show the following family of fixed frequency estimates:

$$\| [\nabla_x P_\mu(\underline{A}), \widetilde{F}_\lambda] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \left(\frac{\lambda}{\mu}\right)^{\delta} \| P_\mu(\underline{A}) \|_{\dot{Z}^{\frac{n-2}{2}}} \cdot \| (f_\lambda, g_\lambda) \|_{\dot{H}^{\frac{n-4}{2}} \times \dot{H}^{\frac{n-6}{2}}} ,$$

where we have set $\delta = \frac{3}{2} - \frac{n}{n-1}$. Notice that we have used the $\dot{Z}^{\frac{n-2}{2}}$ norm for the $\{\underline{A}_i\}$ on the right hand side. This allows us to reconstruct norms through square-summing. For $\lambda \sim \mu$ this estimate is nothing but a fixed frequency version of the estimate (152) above, so it suffices to consider case $\lambda \ll \mu$. Using the simple inclusion $\nabla \dot{X}^{\frac{n-2}{2}} \subseteq \dot{X}^{\frac{n-4}{2}}$, this is a consequence of the fixed frequency embedding:

$$P_\mu\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})\big) \cdot P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})\big) \hookrightarrow \left(\frac{\lambda}{\mu}\right)^{\delta} L^1(\dot{H}^{\frac{n-6}{2}}) , \tag{169}$$

which follows at once from the fixed frequency estimate (49) which helps to generate the general estimate (41). Notice that the proof of the second estimate (166) above is very similar to what we have just done. In fact, there is more room because the derivative is on the low frequency term. We leave the details to the reader.

It remains to prove the two estimates (167)–(168). Since these follow from essentially identical reasoning, we concentrate on proving the second of these estimates. This one in fact requires a bit more work than the fist because it has more frequency overlap. Applying a trichotomy to the product, we see that it suffices to be able to show the following three estimates:

$$\int_0^{T^*} \Big(\sum_{\lambda} \Big(\sum_{\substack{\mu\, : \\ \mu \ll \lambda}} \| \big[P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}]), \widetilde{F}_\lambda\big](s) \|_{\dot{H}^{\frac{n-6}{2}}}\Big)^2\Big)^{\frac{1}{2}}\, ds \lesssim \|\underline{A}\|^2_{\dot{X}^{\frac{n-2}{2}}} \cdot \|(f,g)\|_{\dot{H}^{\frac{n-4}{2}}\times \dot{H}^{\frac{n-6}{2}}} \ , \tag{170}$$

$$\int_0^{T^*} \Big(\sum_{\mu} \Big(\sum_{\substack{\lambda\, : \\ \lambda \ll \mu}} \| \big[P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}]), \widetilde{F}_\lambda\big](s) \|_{L^1(\dot{H}^{\frac{n-6}{2}})}\Big)^2\Big)^{\frac{1}{2}}\, ds \lesssim \|\underline{A}\|^2_{\dot{X}^{\frac{n-2}{2}}} \cdot \|(f,g)\|_{\dot{H}^{\frac{n-4}{2}}\times \dot{H}^{\frac{n-6}{2}}} \ , \tag{171}$$

$$\sum_{\substack{\lambda,\mu\, : \\ \lambda\sim\mu}} \| \big[P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}]), \widetilde{F}_\lambda\big] \|_{L^1(\dot{H}^{\frac{n-6}{2}})} \lesssim \|\underline{A}\|^2_{\dot{X}^{\frac{n-2}{2}}} \cdot \|(f,g)\|_{\dot{H}^{\frac{n-4}{2}}\times \dot{H}^{\frac{n-6}{2}}} \ . \tag{172}$$

The first two estimates (170)–(171) follow from first fixing time and then proving the fixed frequency estimate:

$$\begin{aligned} &\| \big[P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}]), \widetilde{F}_\lambda\big](s) \|_{\dot{H}^{\frac{n-6}{2}}} \\ \lesssim\ & \min_{\pm} \Big(\frac{\lambda}{\mu}\Big)^{\pm\delta} \| P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}])(s) \|_{\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \cdot \| \widetilde{F}_\lambda(s) \|_{\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \ , \end{aligned}$$

where δ is the same constant from estimate (169). Indeed, this last line follows from the non-time integrated version of that estimate. Applying Young's inequality to this, integrating in time and applying Cauchy-Schwartz, using the parametrix bound (157a), the product embedding (151), and the fact that for each fixed value of λ the multipliers $P_{\bullet\ll\lambda}$ and $P_{\lambda\lesssim\bullet}$ are bounded on the $\dot{X}^s$ spaces we arrive at the desired pair of estimates.

It remains for us to prove the last estimate (172) above. After another application of the embedding (152) and a Cauchy-Schwartz, followed by the parametrix estimate (157a), we are left with showing the bound:

$$\Big(\sum_{\substack{\lambda,\mu\, : \\ \lambda\sim\mu}} \| \big[P_\mu([\underline{A}_{\lambda\lesssim\bullet}, \underline{A}_{\lambda\lesssim\bullet}]) \|^2_{L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})}\Big)^{\frac{1}{2}} \lesssim \|\underline{A}\|^2_{\dot{X}^{\frac{n-2}{2}}} \ .$$

This last estimate follows from applying a further trichotomy, and then using Young's inequality after reduction to the various fixed frequency versions of the product estimate (151) which are provided by the general fixed frequency estimates (47)–(47). We leave the details to the diligent reader. This completes the proof of our reduction of Theorem 6.2 to Proposition 7.1. □

The final thing we will do in this section is to make one further reduction of the Strichartz estimates (128). This involves the following proposition:

Proposition 7.2 (Existence of approximate half-wave parametrices)**.** *Let the number of dimensions be* $6 \leqslant n$*, and let* $d + \underline{A}_{\bullet\ll 1}$ *be a connection which satisfies the conditions* (124) *as well as the frequency localization condition* (156) *for* $\lambda = 1$*. Then* *there exists pair of evolution operators $\Phi^{\pm}(\widehat{f})(t)$ from $L^2(\mathbb{R}^n_\xi)$ to $L^2(\mathbb{R}^n_x)$ such that the fixed time adjoints $(\Phi^{\pm}(t))^*$ are always supported in the region $2^{-a} \leqslant |\xi| \leqslant 2^a$ for some fixed $1 \leqslant a$, and such that they obey the following estimates:*

$$\| \Big(P_1\Phi^{\pm}(\widehat{f}), \Phi^{\pm}(\widehat{f}) \Big) \|_{\dot{X}^0 \times L^2_x} \ \lesssim \ \| \widehat{f} \|_{L^2_\xi} \ , \tag{173a}$$

$$\| \nabla_x \Phi^{\pm}(\widehat{f}) \|_{L^2_t(L_x^{\frac{2(n-1)}{n-3}})} \ \lesssim \ \| \widehat{f} \|_{L^2_\xi} \ , \tag{173b}$$

$$\| \partial_t P_1 \Phi^{\pm}(\widehat{f}) \mp P_1\Phi^{\pm}(2\pi i|\xi|\widehat{f}) \|_{\dot{X}^0} \ \lesssim \ \mathcal{E} \ \| \widehat{f} \|_{L^2_\xi} \ , \tag{173c}$$

$$\| \Phi^{\pm}(0)\big((2\pi|\xi|)^\alpha (\Phi^{\pm}(0))^*\big)g - (-\Delta)^{\frac{\alpha}{2}} P_1(g) \|_{L^2_x} \ \lesssim \ \mathcal{E}^{\frac{1}{2}} \ \| g \|_{L^2_x} \ , \tag{173d}$$

$$\| \Box_{\underline{A}_{\bullet \ll 1}} \Phi^{\pm}(\widehat{f}) \|_{L^1_t(L^2_x)} \ \lesssim \ \mathcal{E} \ \| \widehat{f} \|_{L^2_\xi} \ . \tag{173e}$$

Proof that Proposition 7.2 implies Proposition 7.1. This is a simple matter, and we explain it briefly. Notice first that it suffices to prove Proposition 7.1 on the scale $\lambda = 1$ because everything in sight is scale invariant. We now let (f_1, g_1) be any pair of unit frequency initial data, and we define the approximate unit frequency wave propagator:

$$\begin{aligned} W^1_0(f_1,g_1)(t) \ = \ & P_1\Big(\ \frac{1}{2}\Phi^+(t)(\Phi^+(0))^* f_1 + \frac{1}{2}\Phi^-(t)(\Phi^-(0))^* f_1 \\ & + \ \Phi^+(t)\big(\frac{1}{4\pi i|\xi|}(\Phi^+(0))^*\big)g_1 - \Phi^-(t)\big(\frac{1}{4\pi i|\xi|}(\Phi^-(0))^*\big)g_1 \ \Big) \ . \end{aligned} \tag{174}$$

Here P_1 is defined to be the cutoff on line (173d) which is also chosen large enough such that $P_1(f_1, g_1) = (f_1, g_1)$. From the boundedness of the P_1 multiplier, the estimates (173a) and (173c), the frequency support of the adjoints, and the dualized $L^2_x \to L^2_\xi$ estimate contained in (173a), we easily have that the operator (174) obeys the estimate (157a). Next, notice that by applying (173d) with $\alpha = 0$ and $\alpha = -1$, and using the unit frequency condition which implies the boundedness of $(-\Delta)^{-\frac{1}{2}}$, we have the estimate (157b). Furthermore, by using estimate (173c) in conjunction with (173d), where this time we use the indices $\alpha = 0$ and $\alpha = 1$, and using the boundedness of $(-\Delta)^{\frac{1}{2}}$ at unit frequency, we have the second accuracy estimate (157c). Therefore, it remains to show that we have the error estimate (157d). By the estimate (173e) and by again making use of the dual $L^2_x \to L^2_\xi$ adjoint bound, we are reduced to proving (operator) commutator bounds of the type:

$$\| [\Box_{\underline{A}_{\bullet \ll 1}}, P_1]\Phi^{\pm}(\widehat{h}) \|_{L^1_t(L^2_x)} \ \lesssim \ \mathcal{E} \ \| \widehat{h} \|_{L^2_\xi} \ .$$

Using the commutator estimate (35) in conjunction with the parametrix bounds (173a)–(173b) (this is where the extra bound on the gradient comes in), this reduces to showing the two bounds:

$$\| \nabla_x \underline{A}_{\bullet \ll 1} \|_{L^2_t(L^{n-1}_x)} \ \lesssim \ \| \underline{A}_{\bullet \ll 1} \|_{\dot{X}^{\frac{n-2}{2}}} \ , \tag{175}$$

$$\| \nabla_x [\underline{A}_{\bullet \ll 1}, \underline{A}_{\bullet \ll 1}] \|_{L^1_t(L^\infty_x)} \ \lesssim \ \| \underline{A}_{\bullet \ll 1} \|^2_{\dot{X}^{\frac{n-2}{2}}} \ . \tag{176}$$

The first estimate follows easily from integrating the following Besov and low frequency Besov nestings:

$$P_{\bullet \lesssim 1}(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}) \ \subseteq \ \dot{B}^{n-1,(2,n(\frac{n-3}{2(n-1)}))} \ \subseteq \ L^{n-1} \ .$$

The second estimate follows as easily from first distributing the derivative and then integrating the two low frequency nestings:

$$P_{\bullet\lesssim 1}(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})})\ ,\ P_{\bullet\lesssim 1}(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})\ \subseteq\ B_1^{\infty,(2,\frac{n}{2})}\ \subseteq\ L^\infty\ .$$

This completes the proof that Proposition 7.2 implies Proposition 7.1. $\Box$

8. Construction of the half wave operators

We now begin construction of our approximate solutions $\Phi^\pm$ to the reduced covariant wave equation $\Box_{\underline{A}_{\bullet\ll 1}}$. This will be accomplished by integrating over a collection of gauge transformations designed to eliminate the highest order effect of troublesome term $\underline{A}^\alpha_{\bullet\ll 1}\nabla_\alpha$. In order to understand what such a gauge transformation should be, we begin with a simple calculation. We consider the covariant wave equation $\Box_{{}^\omega\! A}$, where the connection ${}^\omega D = d + {}^\omega A$ will be determined in a moment, acting on a vector valued plane wave $e^{2\pi i\lambda\, {}^\omega u_\pm}\widehat{f}$. Here $\widehat{f}$ is a constant complex valued matrix in $\mathbb{C}\otimes\mathfrak{o}(m)$, and the ${}^\omega u^\pm$ are the standard plane wave optical functions:

$${}^\omega u^+ \ =\ t+\omega\cdot x\ , \qquad\qquad {}^\omega u^- \ =\ -t+\omega\cdot x\ .$$

In particular, $\nabla^\alpha({}^\omega u^\pm)\ =\ ({}^\omega L^\mp)^\alpha$, where the ${}^\omega L^\pm$ are the associated null hypersurface generators:

$${}^\omega L^+ \ =\ \nabla_t+\omega\cdot\nabla_x\ , \qquad\qquad {}^\omega L^- \ =\ -\nabla_t+\omega\cdot\nabla_x\ .$$

With these identifications, we easily have the calculation:

$$\Box_{{}^\omega\! A}(e^{2\pi i\lambda\, {}^\omega u_\pm}\widehat{f})\ =\ e^{2\pi i\lambda\, {}^\omega u_\pm}\cdot\Big(4\pi i\lambda\big[{}^\omega A({}^\omega L^\mp),\widehat{f}\big]+D^{{}^\omega\! A}_\alpha\big[{}^\omega A^\alpha,\widehat{f}\big]\Big)\ . \tag{177}$$

Using the heuristic[6] that terms of the form $\nabla({}^\omega A)$ and $[{}^\omega A,{}^\omega A]$ are lower order, and splitting the potentials $\{{}^\omega A_\alpha\}$ into the sets $\{{}^\omega A^\pm_\alpha\}$ associated with the optical functions ${}^\omega u_\pm$ (resp.), we see that in order eliminate the highest order term on the right hand side of (177) would need to assume this connection is in the backward (resp. forward) *ω-null-gauge*:

$${}^\omega A^+({}^\omega L^-)\ =\ 0\ , \qquad\qquad {}^\omega A^-({}^\omega L^+)\ =\ 0\ . \tag{178}$$

Of course, it is not possible to assume that a given *fixed* connection will simultaneously be in the null-gauge for every direction ω. However, it is more or less clear that since these gauges are of Crönstrom type, it is always possible to transform a given connection so that it is in the null-gauge for a fixed direction. This motivates the following form of an approximate solution to $\Box_{\underline{A}_{\bullet\ll 1}}$:

$$\Phi^\pm(\widehat{f})\ =\ \int_{\mathbb{R}^n} e^{2\pi i\lambda{}^\omega u^\pm}\ {}^\omega g_\pm^{-1}\,\widehat{f}(\lambda\omega)\,{}^\omega g_\pm\ \chi_{(\frac{1}{2},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\ , \tag{179}$$

where $\chi_{(\frac{1}{2},2)}$ is a smooth bump function such that $\chi_{(\frac{1}{2},2)}\equiv 1$ on the interval $[2^{-1},2]$ and such that $\chi_{(\frac{1}{2},2)}\equiv 0$ outside of $[4^{-1},4]$ (the variable width assumption of Proposition 7.2 can be achieved with similar bump functions). Here, the gauge transformation:

$${}^\omega B^\pm\ =\ {}^\omega g_\pm\underline{A}_{\bullet\ll 1}({}^\omega g_\pm^{-1})+{}^\omega g_\pm\, d({}^\omega g_\pm^{-1})\ , \tag{180}$$

---

[6]For those who are familiar with this kind of problem, this is precisely the elimination of the famous *Low* $\times$ *High* frequency interaction $A^\alpha\nabla_\alpha\Phi$.

will be chosen so that ${}^{\omega}B^{\pm}$ approximately satisfies (178). It seems that there are in fact many choices of how to do this, although the naive choice of letting ${}^{\omega}B^{\pm}$ satisfy (178) directly by solving the appropriate transport equations[7] leads to group elements with poor regularity properties. Therefore, the procedure for arriving at the correct choice deserves some motivation.

The heart of the matter is two-fold. First and foremost, we need to come up with a construction that gives us explicit formulas so that we may perform certain standard calculations on the integral (179). In particular, we will need to perform integration by parts with respect to the variable ω. Since G is assumed to be non-abelian, and since we will not be able to localize things to a neighborhood of any fixed point on the group[8], this is actually a non-trivial matter. For example, it is not possible to do this directly through a use of the exponential map because we would run into trouble with conjugate points.

Secondly, we will need to replace the transport equation which defines the naive pure null-gauge transformation, with something that has more "elliptic" features. That such a choice is possible is, strangely enough, determined by the fact that the connection $\{\underline{A}_{\bullet\ll 1}\}$ is not arbitrary, but instead evolves according to a hyperbolic equation. This is taken into account by condition (124e). This kind of structure seems to be ubiquitous in geometric wave equations, both semi and quasi-linear, and the observation that it makes the crucial difference goes back to work of Klainerman-Rodnianski on quasi-linear wave equations [5]. The particular form we will use it in here is almost identical to that of [8], but since everything we do is non-abelian, the derivation will seem bit different at first.

The first observation we use is that just like the Crönstrom gauge, the null-gauge allows one to recover the potentials directly from the curvature. However, since we aim to derive an (sub)-elliptic equation, we do not do this by simply integrating along null directions. Instead, we write:

$$ {}^{\omega}L^{\mp}\,{}^{\omega}B^{\pm}_{\alpha} \ = \ F^{{}^{\omega}B^{\pm}}({}^{\omega}L^{\mp}, \partial_{\alpha}) \ . \tag{181} $$

Making now the approximate assumption that the $\{{}^{\omega}B^{\pm}\}$ are simply a solution to the scalar wave equation $\Box = \nabla_{\alpha}\nabla^{\alpha}$, which we write as:

$$ \Box \ = \ {}^{\omega}L^{\pm}\,{}^{\omega}L^{\mp} \ + \ \Delta_{\omega^{\perp}} \ , \tag{182} $$

the identity (181) can be written in the integral form:

$$ {}^{\omega}B^{\pm}_{\alpha} \ = \ -\,{}^{\omega}L^{\pm}\Delta^{-1}_{\omega^{\perp}} \ F^{{}^{\omega}B^{\pm}}({}^{\omega}L^{\mp}, \partial_{\alpha}) \ . \tag{183} $$

Here $\Delta_{\omega^{\perp}} = \Delta - \nabla^2_{\omega}$ is simply the Laplacean on the plane perpendicular to the ω direction in $\mathbb{R}^n$. We would now like to make (183) our "choice" for the gauge transformed connection on the right hand side of (180). For example, even though it was based on the approximate assumption the $\{{}^{\omega}B^{\pm}\}$ satisfy the scalar wave equation, it still respects the null-gauge (178) simply by the skew-symmetry property of the

[7] This would end up being the usual a frequency based Hadamard parametrix for the operator $\Box_{\underline{A}_{\bullet\ll 1}}$.

[8] This is an artifact of the critical nature of the problem. Specifically, the group elements have the heuristic form ${}^{\omega}g = \exp(\nabla^{-1}\,{}^{\omega}A)$. Since we do not have L^{∞} control on $\nabla^{-1}\,{}^{\omega}A$ we cannot localize its image.

curvature. Unfortunately, (183) has several undesirable features. Firstly, we would like an expression which involves the curvature of $\{\underline{A}_{\bullet\ll 1}\}$, not the curvature $F^{{}^\omega B^\pm}$. Secondly, the sub-Laplacean on the right hand side of this expression needs to be smoothed out in some way so that its dependence on the angular variable ω is not so rough.

To get around the first of these problems, we simply pretend that the various differential operators on the right hand side of (183) are gauge covariant. Assuming this and then conjugating both sides of that expression by ${}^\omega g_\pm$ and moving these group elements past the differential operators on the right, and throwing away quadratic terms from the curvature while assuming that the reduced connection $\underline{A}_{\bullet\ll 1}$ satisfies the usual homogeneous wave equation, we are left with the approximate identities:

$$\begin{aligned} {}^\omega g_\pm^{-1}\, {}^\omega B_\alpha^\pm\, {}^\omega g_\pm \ &\approx\ -\,{}^\omega L^\pm \Delta_{\omega^\perp}^{-1}\ F^{\underline{A}_{\bullet\ll 1}}({}^\omega L^\mp, \partial_\alpha)\ ,\\ &\approx\ (\underline{A}_{\bullet\ll 1})_\alpha + \nabla_\alpha\, {}^\omega L^\pm \Delta_{\omega^\perp}^{-1} \underline{A}_{\bullet\ll 1}({}^\omega L^\mp)\ . \end{aligned}$$

To get around the second problem, we mollify the angular variable of the second term on the right hand side of this last expression. Doing this and looking back on the definition (180), we see that we would like our group elements to be such that:

$$(184)\qquad {}^\omega g_\pm^{-1} d({}^\omega g_\pm)\ \approx\ -\,{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x\, {}^\omega L^\pm \Delta_{\omega^\perp}^{-1} \underline{A}_{\bullet\ll 1}(\partial_\omega)\ .$$

Here we have set:

$$(185)\qquad 0\ <\ \gamma\ \ll\ \delta\ \ll\ 1\ ,$$

where γ is our small all purpose constant from line (13) above. Now the problem is, of course, that right hand side of the above formula does not in general represent a flat connection. However, as one can see immediately, *its curvature is small* in some sense because it is a quadratic expression. At this point, the problem now looks essentially like what happens for wave-maps[9] (see e.g. [11] and [9]). In particular, it is clear that the right way to define the group elements ${}^\omega g_\pm$ so that the approximate formula (184) holds is to flatten out the right hand side of that expression as much as possible by using the potential version 3.2 of the Uhlenbeck lemma. Therefore, what we need to do is to show the fixed time estimate:

$$(186)\qquad \|\, {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x\, {}^\omega L^\pm \Delta_{\omega^\perp}^{-1}\ \underline{A}_{\bullet\ll 1}(\partial_\omega)\,\|_{L^n}\ \lesssim\ \mathcal{E}\ ,$$

and then assume that $\mathcal{E}$ is chosen small enough to that we may use it as the constant in (22). Because of its utility in the sequel, we will in fact prove the more general estimate:

$$(187)\qquad \|\, {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x\, {}^\omega L^\pm \Delta_{\omega^\perp}^{-1}\ \underline{A}_{\bullet\ll 1}(\partial_\omega)\,\|_{\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})}}\ \lesssim\ \mathcal{E}\ ,$$

where p_γ is a dimension dependent Lebesgue index which we set to:

$$(188)\qquad p_\gamma\ =\ \frac{2(n-1)}{n-3-2\gamma}\ .$$

[9]It is very much our philosophy here that this problem is essentially equivalent to wave-maps after a microlocalization. Of course, as the reader will see, this microlocalization is quite costly and introduces many objects that are not present in the original wave-maps problem.

Here $0 < \gamma \ll 1$ is again the all-purpose constant which we have fixed in section 2 to be small enough so that it is compatible with its use here. Notice that (187) implies the estimate (186) thanks to the embedding (39) and the fact that for γ sufficiently small there is plenty of room in the inequality $p_\gamma < n$.

Now, because the norm $\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}$ is ℓ^2 based, by orthogonality and the $L^\infty(L^2)$ estimate contained in the bootstrapping assumption (124d), we see that in order to conclude (187) it is enough to show the fixed frequency estimate (note that there are no high frequencies here):

$$\| \nabla_x {}^\omega L^\pm \Delta^{-1}_{\omega^\perp} (\underline{A}_{\bullet\ll 1})_\mu(\partial_\omega) \|_{L^{p_\gamma}} \ \lesssim \ \mu^{n(\frac{1}{2}-\frac{1}{p_\gamma})} \| (\underline{A}_{\bullet\ll 1})_\mu \|_{L^2} \ .$$

Decomposing the spatial frequency variable into fixed dyadic angular sectors spread from the direction ω: $P_\mu = \sum_\theta {}^\omega\Pi_\theta P_\mu$, this estimate further reduces (after dyadic summing) to being able being able to prove that:

$$\| {}^\omega\Pi_\theta \nabla_x {}^\omega L^\pm \Delta^{-1}_{\omega^\perp} (\underline{A}_{\bullet\ll 1})_\mu(\partial_\omega) \|_{L^{p_\gamma}} \ \lesssim \ \theta^\gamma \mu^{n(\frac{1}{2}-\frac{1}{p_\gamma})} \| (\underline{A}_{\bullet\ll 1})_\mu \|_{L^2} \ . \tag{189}$$

We are now almost at the point where we can apply the angular Bernstein inequality (52) directly, because in the current localized setting we have the symbol bounds:

$${}^\omega\Pi_\theta \nabla_x {}^\omega L^\pm \Delta^{-1}_{\omega^\perp} P_\mu \ \approx \ \theta^{-2} P_\mu \ , \tag{190}$$

where we are enforcing the heuristic notation introduced on line (54). However, since Bernstein only nets us a savings of:

$$\theta^{(n-1)(\frac{1}{2}-\frac{1}{p_\gamma})} \ = \ \theta^{1+\gamma} \ ,$$

in this context, we need to be a bit more careful in order to gain an extra power of θ. This is provided by the fact that the potentials $\{\underline{A}_{\bullet\ll 1}\}$ are in the Coulomb gauge. Notice that if say, $\frac{1}{10} < \theta$ there is nothing to worry about and we have estimate (189) without any problem. On the other-hand, if it is the case that $\theta < \frac{1}{10}$, then we can use the fact that ${}^\omega\Pi_\theta \nabla^{-1}_\omega$ is elliptic (in terms of symbol bounds) in conjunction with the gauge condition $d^*\underline{A}_{\bullet\ll 1} = 0$ to write:

$${}^\omega\Pi_\theta \underline{A}_{\bullet\ll 1}(\partial_\omega) \ = \ \nabla^{-1}_\omega {}^\omega\Pi_\theta \, \not{d}^* \underline{\not{A}}_{\bullet\ll 1} \ \approx \ \theta \ {}^\omega\Pi_\theta \underline{\not{A}}_{\bullet\ll 1} \ . \tag{191}$$

Here $\{\underline{\not{A}}_{\bullet\ll 1}\}$ the induced connection (angular portion) on the hyperplane $\mathcal{H}_{\omega^\perp}$ perpendicular to ω, and $\not{d}^*$ is the associated divergence. We note here that this identity will turn out to be very useful and will be used many times throughout the sequel. With these extra savings in mind, an application of Bernstein now directly yields the desired estimate (189).

We have now constructed the infinitesimal group elements ${}^\omega g_\pm$ in equations (180), which is explicitly defined by the formulas (27) in Lemma 3.2 applied to the connection:

$${}^\omega\underline{A}^\pm \ = \ - {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x {}^\omega L^\pm \Delta^{-1}_{\omega^\perp} \underline{A}_{\bullet\ll 1}(\partial_\omega) \ . \tag{192}$$

This has the pleasant effect that we will never need to explicitly refer to the connection $\{{}^\omega B^\pm\}$ in line (180). We can calculate the conjugated right hand side of that expression to be:

$${}^\omega g^{-1}_\pm {}^\omega B^\pm {}^\omega g_\pm \ = \ \underline{A}_{\bullet\ll 1} - {}^\omega C^\pm \ , \tag{193}$$

where we have set:

$$({194})\qquad {}^\omega g_\pm^{-1} d({}^\omega g_\pm) \ = \ {}^\omega C^\pm \ .$$

Using the formulas (27), we have the following expressions for the spatial components $\{{}^\omega \underline{C}^\pm\}$:

$$({195a})\qquad ({}^\omega \underline{C}^\pm)^{df} \ = \ d^* \, \Delta^{-1}[{}^\omega \underline{C}^\pm, {}^\omega \underline{C}^\pm] \ ,$$

$$({195b})\qquad ({}^\omega \underline{C}^\pm)^{cf} \ = \ {}^\omega \underline{A}^\pm \ - \ \nabla_x \Delta^{-1}[{}^\omega \underline{A}^\pm, {}^\omega \underline{C}^\pm] \ .$$

In order to compute a formula for the temporal potential ${}^\omega C_0^\pm$, we simply use the fact that $F^{{}^\omega C^\pm} = 0$ and the formula (195b) which together imply (by computing $d^* E^{{}^\omega C^\pm}$):

$$({196})\qquad {}^\omega C_0^\pm \ = \ {}^\omega A_0^\pm \ - \ \nabla_t \Delta^{-1}[{}^\omega \underline{A}^\pm, {}^\omega \underline{C}^\pm] \ - \ d^* \Delta^{-1}[{}^\omega C_0^\pm, {}^\omega \underline{C}^\pm] \ ,$$

where we have:

$$ {}^\omega A_0^\pm \ = \ - {}^\omega \overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_t {}^\omega L^\pm \Delta_{\omega^\perp}^{-1} \underline{A}_{\bullet \ll 1}(\partial_\omega) \ .$$

We remark here that the importance of the system of equations (195a)–(196) is that they give the following decomposition of the infinitesimal gauge transformation $\{{}^\omega C^\pm\}$:

$$({197})\qquad {}^\omega C^\pm \ = \ - \nabla_{t,x} \, {}^\omega \overline{\Pi}^{(\frac{1}{2}-\delta)} {}^\omega L^\pm \Delta_{\omega^\perp}^{-1} \, \underline{A}_{\bullet \ll 1}(\partial_\omega) \ + \ \ \{\text{Quadratic Error}\} \ \ .$$

The linear term in the above expression is enough to kill off the worst error term when differentiating the parametrix (179). It should be noted that this linear term is precisely what one gets more directly in the abelian case studied in [8]. We should also point out here that the quadratic error on the right hand side of (197) above is *much* more delicate than the quadratic error resulting form the cancelation involving the linear term in this expression. In order to control this, we will need the full force of the orthogonality properties of our parametrix, which are contained in the bootstrapping assumption (124d), as well as some rather technical function spaces and multilinear estimates which we will develop in Section 11.

To close out this section, we apply the truncated covariant wave operator $\Box_{\underline{A}_{\bullet \ll 1}}$ to the parametrix (179) and record the various error terms which result. We gather this together in the following proposition:

Proposition 8.1 (Error terms for the differentiated parametrix)**.** *Consider the parametrix* $\Phi^{\pm}(\widehat{f})$ *defined by the formula* (179)*, with infinitesimal gauge transformations given by equations* (195a)–(196)*. Then one has the identity:*

(198)

$$\begin{aligned}
&\Box_{\underline{A}_{\bullet\ll 1}}\Phi^{\pm}(\widehat{f})\\
= 4\pi i &\int_{\mathbb{R}^n} e^{2\pi i\lambda{}^{\omega}u^{\pm}}\ \big[\underline{A}_{\bullet\ll 1}({}^{\omega}L^{\mp}) - {}^{\omega}C^{\pm}({}^{\omega}L^{\mp})\ ,\ {}^{\omega}g_{\pm}^{-1}\,\widehat{f}(\lambda\omega)\,{}^{\omega}g_{\pm}\big]\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n}d\lambda d\omega\\
- &\int_{\mathbb{R}^n} e^{2\pi i\lambda{}^{\omega}u^{\pm}}\ \big[D_{\alpha}^{\underline{A}_{\bullet\ll 1}}\big({}^{\omega}C^{\pm}\big)^{\alpha}\ ,\ {}^{\omega}g_{\pm}^{-1}\,\widehat{f}(\lambda\omega)\,{}^{\omega}g_{\pm}\big]\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\\
+ &\int_{\mathbb{R}^n} e^{2\pi i\lambda{}^{\omega}u^{\pm}}\ \Big[\underline{A}^{\alpha}_{\bullet\ll 1} - ({}^{\omega}C^{\pm})^{\alpha}\ ,\ \big[(\underline{A}_{\bullet\ll 1})_{\alpha} - {}^{\omega}C^{\pm}_{\alpha}\ ,\ {}^{\omega}g_{\pm}^{-1}\,\widehat{f}(\lambda\omega)\,{}^{\omega}g_{\pm}\big]\Big]\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\ .
\end{aligned}$$

Remark 8.2. The worst error term in the expression (198) is of course the "derivative fall on high" term which is the first on the right hand side. However, using the structure equation (124e), this takes the form:

$$\begin{aligned}
(199)\qquad &\underline{A}_{\bullet\ll 1}({}^{\omega}L^{\mp}) - {}^{\omega}C^{\pm}({}^{\omega}L^{\mp})\ ,\\
&= \underline{A}_{\bullet\ll 1}(\partial_{\omega})\ +\ {}^{\omega}\Pi^{(\frac{1}{2}-\delta)}{}^{\omega}L^{\mp}{}^{\omega}L^{\pm}\Delta^{-1}_{\omega^{\perp}}\,\underline{A}_{\bullet\ll 1}(\partial_{\omega})\ +\ \{\text{Quadratic Error}\}\ ,\\
&= (I - {}^{\omega}\Pi^{(\frac{1}{2}-\delta)})\underline{A}_{\bullet\ll 1}(\partial_{\omega})\ +\ \{\text{Quadratic Error}\}\ .
\end{aligned}$$

The key observation now is that since the operator $(I - {}^{\omega}\Pi^{(\frac{1}{2}-\delta)})$ cuts off on such a small angular sector with respect to the spatial frequency, an application of Bernstein's inequality gains enough extra *spatial* derivatives to put this term in the mixed Lebesgue space $L^2(L^{n-1})$. Furthermore, the quadratic error term which is left over involves enough bilinear interactions to go in $L^1(L^{\infty})$. So in this sense, as we have mentions before, the problem reduces to something which is reminiscent of wave-maps. Of course, there is a somewhat heavy price to pay for this "renormalization", which is that it must take place under an integral sign. Finally, it is worth pointing out that this top order cancelation is completely analogous to what happens in the abelian case [8].

Proof of the error identity (198)*.* The proof is a simple consequence of using gauge transformations in conjunction with the identity (177). Applying the truncated covariant wave operator, and differentiating under the integral sign, we see that:

$$
\begin{aligned}
&\Box_{\underline{A}_{\bullet\ll 1}}\Phi^{\pm}_{f} \ ,\\
=& \int_{\mathbb{R}^n} \Box_{\underline{A}_{\bullet\ll 1}}\Big(e^{2\pi i\lambda {}^\omega u^\pm}\ {}^\omega g_\pm^{-1}\,\widehat{f}(\lambda\omega)\,{}^\omega g_\pm\Big)\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega \ ,\\
=& \int_{\mathbb{R}^n} {}^\omega g_\pm^{-1}\ \Box_{{}^\omega B^\pm}\Big(e^{2\pi i\lambda {}^\omega u^\pm}\ \widehat{f}(\lambda\omega)\Big)\ {}^\omega g_\pm\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega \ ,\\
=& \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ {}^\omega g_\pm^{-1}\Big(4\pi i\lambda\big[{}^\omega B^\pm({}^\omega L^\mp),\widehat{f}\big] + D_\alpha^{{}^\omega B^\pm}\big[{}^\omega B^{\pm\alpha},\widehat{f}\big]\Big)\,{}^\omega g_\pm\ \chi_{(2^{-1},2)}(\lambda)\ \lambda^{n-1}\ d\lambda d\omega \ ,\\
=& \ 4\pi i \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ \big[{}^\omega g_\pm^{-1}\,{}^\omega B^\pm({}^\omega L^\mp)\,{}^\omega g_\pm\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega\\
&\qquad + \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ D_\alpha^{\underline{A}_{\bullet\ll 1}}\big[{}^\omega g_\pm^{-1}\,{}^\omega B^{\pm\alpha}\,{}^\omega g_\pm\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega \ ,\\
=& \ 4\pi i \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ \big[\underline{A}_{\bullet\ll 1}({}^\omega L^\mp) - {}^\omega C^\pm({}^\omega L^\mp)\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega\\
&\qquad - \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ \big[\nabla_\alpha({}^\omega C^\pm)^\alpha\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega\\
&\qquad + \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ \big[(\underline{A}_{\bullet\ll 1})_\alpha - {}^\omega C^\pm_\alpha\ ,\ \nabla^\alpha\ \big({}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big)\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega\\
&\qquad + \int_{\mathbb{R}^n} e^{2\pi i\lambda\, {}^\omega u_\pm}\ \big[\ (\underline{A}_{\bullet\ll 1})^\alpha\ ,\ [(\underline{A}_{\bullet\ll 1})_\alpha - {}^\omega C^\pm_\alpha\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm]\big]\ \lambda^{n-1}\ \chi_{(2^{-1},2)}(\lambda)\ d\lambda d\omega\\
=& \ (\text{L.H.S.})(198)\ .
\end{aligned}
$$

Notice that the equality on the last line follows from:

$$
\nabla_\alpha\ \big({}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\big)\ =\ \big[\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ ,\ {}^\omega C^\pm_\alpha\ \big]\ ,
$$

which is a consequence of line (194) above, followed by the Jacobi identity:

$$
\begin{aligned}
&\big[\ (\underline{A}_{\bullet\ll 1})^\alpha - ({}^\omega C^\pm)^\alpha\ ,\ [\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ ,\ {}^\omega C^\pm_\alpha\ ]\ \big]\ ,\\
=&\ -\big[\ {}^\omega C^\pm_\alpha\ ,\ [\ (\underline{A}_{\bullet\ll 1})^\alpha - ({}^\omega C^\pm)^\alpha\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ ]\ \big]\\
&\qquad - \big[\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ ,\ [\ {}^\omega C^\pm_\alpha\ ,\ (\underline{A}_{\bullet\ll 1})^\alpha - ({}^\omega C^\pm)^\alpha\ ]\ \big]\ ,\\
=&\ -\big[\ {}^\omega C^\pm_\alpha\ ,\ [\ (\underline{A}_{\bullet\ll 1})^\alpha - ({}^\omega C^\pm)^\alpha\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ ]\ \big]\\
&\qquad - \big[\ [\ (\underline{A}_{\bullet\ll 1})_\alpha\ ,\ ({}^\omega C^\pm)^\alpha\ ]\ ,\ {}^\omega g_\pm^{-1}\widehat{f}\ {}^\omega g_\pm\ \big]\ .
\end{aligned}
$$

This completes the proof of (198). □

9. Fixed Time L^2 Estimates for the Parametrix

We now begin our proof of the estimates (173) for the integral operator (179) introduced in the last section. Here we cover bounds which are of non-differentiated energy type. Specifically, we will show the undifferentiated energy estimate $L^\infty(L^2)$ estimate contained in (173a), as well as the multiplier-approximation bound (173d). Both of these will follow from the same set of estimates. At a heuristic level, they are not much more involved that a standard TT^* argument followed by some integration by parts, although the details turn out to be a bit involved. Things will be computed more or less directly by an appeal to the explicit equations (195a)–(196), taking a little bit of care to use them properly. This will be done by considering them as "path lifting" formulas from Minkowski space $\mathcal{M}^n$ to the compact group G. This allows us to employ an integral form of the intermediate value theorem from elementary calculus which is valid in the context of Lie groups. It turns out that this identity can be differentiated as many times as necessary with respect to the angular variable, although this fact is provided through a surprisingly delicate bootstrapping argument. Here the unitarity of the group is needed in a crucial way to keep everything from collapsing. Once the bootstrapping is complete, the estimates themselves will be proved using a "trace-Bernstein" type inequality that we construct by hand using various multipliers. Once the integration by parts portion of things is taken care of, we will close the L^2 estimate by showing that the "non-smooth" remainder kernel has small amplitudes after integration in the angular frequency variable. This involves some fairly technical bilinear estimates because the necessary othogonality arguments are difficult to pass through Hodge systems. The details of these procedures are as follows.

Throughout this section we will replace the specific cutoff function $\chi_{(\frac{1}{2},2)}$ appearing in the definition of parametrix (179) with an arbitrary smooth scalar bump function $\chi(\xi)$ that we may assume to be supported in the frequency annulus $\{4^{-1} < |\xi| < 4\}$. At fixed time t_0, we define the operator $T(\widehat{f}) = \Phi(\widehat{f})(t_0)$, where we have suppressed the $\pm$ notation because it will be irrelevant for what we do here. Our first goal is the prove the bound:

$$\| T(\widehat{f}) \|_{L^2} \lesssim \| \widehat{f} \|_{L^2} . \tag{200}$$

Squaring this, it suffices to show that (here f has no relation to $\widehat{f}$ and simply represents a function of the physical-space variables):

$$\| TT^*(f) \|_{L^2} \lesssim \| f \|_{L^2} , \tag{201}$$

where the adjoint T^* is taken with respect to the Killing form (14). A quick calculation of the kernel of this operator shows that:

$$K^{TT^*}(x,y) = \int_{\mathbb{R}^n} e^{2\pi i (x-y)\cdot\xi} \ {}^\omega g^{-1}(x) {}^\omega g(y) \big[\bullet \big] \ {}^\omega g^{-1}(y) {}^\omega g(x) \ \chi(\xi) \ d\xi , \tag{202}$$

where we use the $[\bullet]$ notation to emphasize the fact that this operator acts via conjugation. Our task is now to show the estimates:

$$\| K^{TT^*} \|_{L^\infty_y(L^1_x)} , \ \| K^{TT^*} \|_{L^\infty_x(L^1_y)} \lesssim 1 . \tag{203}$$

Since K^{TT^*} is essentially symmetric in (x,y), we may concentrate on the second such estimate.

To proceed, we first decompose physical space into the dyadic regions:

$$\mathcal{D}_\sigma = \{ |x-y| \sim \sigma \ \big| \ \sigma = 2^i \ , \ i \in \mathbb{N} \} . \tag{204}$$

We then decompose the kernel TT^* kernel into the dyadic sum:

$$K^{TT^*} = \sum_\sigma \chi_{\mathcal{D}_\sigma} K^{TT^*} = \sum_\sigma K^{TT^*}_\sigma .$$

By dyadic summing, to show (203) it suffices to be able to show the single estimate:

$$\| K^{TT^*}_\sigma \|_{L^\infty_y(L^1_x)} \lesssim \sigma^{-\gamma} , \tag{205}$$

where $0 < \gamma \ll 1$ now represents a small savings in physical space decay. Now (205) would be easy to show if we had the absolute decay estimate:

$$|K_\sigma^{TT^*}(x,y)| \ \lesssim \ |x-y|^{-(n+\gamma)} \ ,$$

and this is almost true. Unfortunately, there is a regularity problem due to the degeneracy of the sub-Laplacean $\Delta_{\omega^\perp}$ used in the connection (195) which provides the group elements ${}^\omega g$. This forces us to write the kernel $K_\sigma^{TT^*}$ as a sum of two terms:

$$K_\sigma^{TT^*} \ = \ \widetilde{K}_\sigma^{TT^*} \ + \ \mathcal{R}_\sigma^{TT^*} \ . \tag{206}$$

We will then prove that both:

$$|\widetilde{K}_\sigma^{TT^*}(x,y)| \ \lesssim \ |x-y|^{-(n+\gamma)} \ , \tag{207}$$

$$\| \, \mathcal{R}_\sigma^{TT^*} \, \|_{L_y^\infty(L_x^1)} \ \lesssim \ \sigma^{-\gamma} \ . \tag{208}$$

To define the splitting (206), we factor the group elements ${}^\omega g$ into a product of *smooth* and *small* parts. This is completely analogous to the procedure used in [8], but since things are non-abelian (and hence non-linear) here, the estimates required are quite a bit more involved. What we will do is construct another gauge transformation $\widetilde{{}^\omega g}$, which is based on a further smoothing of the connection (192). This will produce a group element which can be treated as a standard symbol. To this end, we define the scale mollified connection:

$$\widetilde{{}^\omega \underline{A}^{(\sigma)}} \ = \ - \, {}^\omega\overline{\Pi}_{\sigma^{-1+\gamma}<\bullet} {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x {}^\omega L \ \Delta_{\omega^\perp}^{-1} \, \underline{A}_{\,\bullet\ll 1}(\partial_\omega) \ , \tag{209}$$

where γ is, again, the small dimensional constant from line (185). Again, we have dropped the $\pm$ notation because it is irrelevant. Following the proof of (186), and using the fact that the multipliers ${}^\omega\overline{\Pi}_{\sigma^{-1+\gamma}<\bullet}$ are bounded on frequency localized Lebesgue spaces, we may apply Lemma 3.2 to the connection $\{\widetilde{{}^\omega \underline{A}^{(\sigma)}}\}$. This produces a group element $\widetilde{{}^\omega g}$, which is defined by the infinitesimal generator:

$$\widetilde{{}^\omega g}^{\,-1} d(\widetilde{{}^\omega g}) \ = \ \widetilde{{}^\omega \underline{C}} \ . \tag{210}$$

Furthermore, this generator is itself defined via the Hodge system:

$$(\widetilde{{}^\omega \underline{C}})^{df} \ = \ d^* \, \Delta^{-1} [\widetilde{{}^\omega \underline{C}}, \widetilde{{}^\omega \underline{C}}] \ , \tag{211a}$$

$$(\widetilde{{}^\omega \underline{C}})^{cf} \ = \ \widetilde{{}^\omega \underline{A}^{(\sigma)}} \ - \ \nabla_x \Delta^{-1} [\widetilde{{}^\omega \underline{A}^{(\sigma)}}, \widetilde{{}^\omega \underline{C}}] \ . \tag{211b}$$

Using this new group element $\widetilde{{}^\omega g}$, we define the remainder group element ${}^\omega h$ via the product:

$${}^\omega g \ = \ {}^\omega h \, \widetilde{{}^\omega g} \ . \tag{212}$$

To compute the infinitesimal generator of ${}^\omega h$, we first use the identity:

$$\begin{aligned} d({}^\omega h) \ &= \ d({}^\omega g) \widetilde{{}^\omega g}^{\,-1} + g \, d(\widetilde{{}^\omega g}^{\,-1}) \ , \\ &= \ {}^\omega h \ \widetilde{{}^\omega g} \left({}^\omega \underline{C} - \widetilde{{}^\omega \underline{C}} \right) \widetilde{{}^\omega g}^{\,-1} \ . \end{aligned} \tag{213}$$

This leads us to define the difference connection:

$$\widetilde{\widetilde{{}^\omega \underline{C}}} \ = \ {}^\omega \underline{C} - \widetilde{{}^\omega \underline{C}} \ . \tag{214}$$

A quick calculation using the systems (195) and (211) shows that this new connection can be pinned down via the Hodge system:

$$(\widetilde{\widetilde{{}^\omega\underline{C}}})^{df} \ = \ d^* \Delta^{-1}\big([{}^\omega\widetilde{\underline{C}}, \widetilde{\widetilde{{}^\omega\underline{C}}}] \ + [\widetilde{\widetilde{{}^\omega\underline{C}}}, \widetilde{{}^\omega\underline{C}}] \ \big) \ , \tag{215a}$$

$$(\widetilde{\widetilde{{}^\omega\underline{C}}})^{cf} \ = \ \widetilde{{}^\omega\underline{A}} - \widetilde{{}^\omega\underline{A}^{(\sigma)}} \ - \ \nabla_x \Delta^{-1}\Big([\widetilde{{}^\omega\underline{A}} - \widetilde{{}^\omega\underline{A}^{(\sigma)}}, \widetilde{{}^\omega\underline{C}}] \ + \ [\widetilde{{}^\omega\underline{A}^{(\sigma)}}, \widetilde{\widetilde{{}^\omega\underline{C}}}]\Big) \ , \tag{215b}$$

where a simple computation shows that:

$$ {}^\omega\underline{A} - \widetilde{{}^\omega\underline{A}^{(\sigma)}} \ = \ - {}^\omega\overline{\Pi}_{\bullet\leqslant\sigma^{-1+\gamma}} {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_x {}^\omega L \ \Delta_{\omega^\perp}^{-1} \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \ , \tag{216}$$

We now define the decomposition (206) along the following decompositions of the group element products in the kernel (202):

$$ {}^\omega g^{-1}(x){}^\omega g(y) \ = \ \widetilde{{}^\omega g}^{-1}(x)\widetilde{{}^\omega g}(y) + \widetilde{{}^\omega g}^{-1}(x) \left({}^\omega h^{-1}(x){}^\omega h(y) - I\right) \widetilde{{}^\omega g}(y) \ , \tag{217}$$

$$ {}^\omega g^{-1}(y){}^\omega g(x) \ = \ \widetilde{{}^\omega g}^{-1}(y)\widetilde{{}^\omega g}(x) + \widetilde{{}^\omega g}^{-1}(y) \left({}^\omega h^{-1}(y){}^\omega h(x) - I\right) \widetilde{{}^\omega g}(x) \ . \tag{218}$$

Accordingly, we define:

$$ \widetilde{K}^{TT^*}(x,y) \ = \ \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \widetilde{{}^\omega g}^{-1}(x)\widetilde{{}^\omega g}(y)\big[\,\bullet\,\big] \ \widetilde{{}^\omega g}^{-1}(y)\widetilde{{}^\omega g}(x) \ \chi(\xi) \ d\xi \ , \tag{219}$$

and then define $\mathcal{R}_\sigma^{TT^*}$ according to the formula (206). The idea now is that while one can only perform integration by parts in the kernel (219) above, the group element ${}^\omega h^{-1}(x){}^\omega h(y)$ and its inverse, which must be contained as at least one factor in the remainder, are so close to the identity matrix that the resulting difference expression can be estimated without use of the oscillations which take place under the integral sign.

We now begin our proof of the estimate (207). To do this, we simply integrate by parts as may times as necessary with respect to the variable ξ in order to pick up the needed point-wise decay. Doing this, we see that in order to draw our conclusion, it suffices to show the following symbol bounds for $1 \leqslant k$:

$$\chi_{\mathcal{D}_\sigma} \ \| \nabla_\xi^k\big(\widetilde{{}^\omega g}^{-1}(x)\widetilde{{}^\omega g}(y)\big) \| \ \lesssim \ \mathcal{E}\cdot\sigma^{k(1-\gamma)} \ , \tag{220}$$

$$\chi_{\mathcal{D}_\sigma} \ \| \nabla_\xi^k\big(\widetilde{{}^\omega g}^{-1}(y)\widetilde{{}^\omega g}(x)\big) \| \ \lesssim \ \mathcal{E}\cdot\sigma^{k(1-\gamma)} \ . \tag{221}$$

In fact, we shall prove the following more general bounds, which contain (220)–(221) as a special case, and which will be useful in the sequel:

Proposition 9.1 (Symbol bounds for the smoothed amplitudes $\widetilde{{}^\omega g}^{-1}(t,x)\widetilde{{}^\omega g}(s,y)$ and $\widetilde{{}^\omega g}^{-1}(s,y)\widetilde{{}^\omega g}(t,x)$)**.** *Let the group elements $\widetilde{{}^\omega g}$ be defined infinitesimally by the Hodge system* (211)*, where the parameter $\sigma^{-1+\gamma}$ is replaced by M^{-1}, where M lies in the range:*

$$(|t-s| + |x-y|)^{\frac{1}{2}} \ \leqslant \ M \ \leqslant |t-s| + |x-y|. \tag{222}$$

Then for any integer $1 \leqslant k$, one has the following symbol bounds assuming that the bootstrapping constant $\mathcal{E}$ from line (124d) *is chosen sufficiently small (with respect* *to each fixed k):*

$$\| \nabla_\xi^k \big(\widetilde{{}^\omega g}^{-1}(t,x) \widetilde{{}^\omega g}(s,y) \big) \| \ \lesssim \ \mathcal{E}\cdot M^k \ , \tag{223}$$

$$\| \nabla_\xi^k \big(\widetilde{{}^\omega g}^{-1}(s,y) \widetilde{{}^\omega g}(t,x) \big) \| \ \lesssim \ \mathcal{E}\cdot M^k \ . \tag{224}$$

Here the ∇_ξ^k notation is shorthand for all k^{th} order partial derivatives involving the variable ξ, and $\|\cdot\|$ is the standard matrix vector-norm from line (16)*. The implicit constants on the right hand side depend on k, but are uniform in the parameter M for each fixed k.*

Proof of the estimates (223)–(224). It suffices for us to prove the first bound (223), as the second follows from virtually identical reasoning. The goal is to reduce this via an ODE bootstrapping type argument to an associated estimate involving the connection $\{\widetilde{{}^\omega C}\}$. This associated estimate will then be proved by another bootstrapping argument in certain mixed Lebesgue-Besov spaces naturally associated with the ODE problem from the first step. The goal of the second bootstrapping will be to reduce things to proving the Besov estimates for the connection $\{\underline{A}_{\bullet\ll 1}\}$ which appears as the linear term on the right hand side of the Hodge system (211a).

Before proceeding, we first make a preliminary reduction on the product $\widetilde{{}^\omega g}^{-1}(t,x)\widetilde{{}^\omega g}(s,y)$. We would like be set up as to only have to handle products which involve the same space or same time variables. This is easily accomplished via the product decomposition:

$$\widetilde{{}^\omega g}^{-1}(t,x)\widetilde{{}^\omega g}(s,y) \ = \ \widetilde{{}^\omega g}^{-1}(t,x)\widetilde{{}^\omega g}(t,y)\cdot \widetilde{{}^\omega g}^{-1}(t,y)\widetilde{{}^\omega g}(s,y) \ . \tag{225}$$

It is clear that if we can produce the bounds (223) for each of the terms on the right hand side of (225) separately, then by the product rule for derivatives we have the estimate (223) for the full term. Since they require slightly different arguments, we will proceed separately for each of these two factors.

Our first task is to prove the bound (223) for the spatial product $\widetilde{{}^\omega g}^{-1}(t,x)\widetilde{{}^\omega g}(t,y)$. This will be done inductively with respect to the value of k. Since we will proceed via a bootstrapping type procedure, we first assume that we can prove the desired bounds over small intervals and then try to use this knowledge to extend things to longer intervals. To do this, we differentiate the product $\widetilde{{}^\omega g}^{-1}(t,\ell)\widetilde{{}^\omega g}(t,y)$, where $[y,\ell]$ is some shorter line segment inside of $[y,x]$, with respect to the operators $(M^{-1}\nabla_\xi)^k$. This yields the equation:

$$\begin{aligned}(M^{-1}\nabla_\xi)^k\big(\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(y)\big) \ &= \\ &\sum_{i=0}^{k-1}(M^{-1}\nabla_\xi)^{k-i}\big(\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(x_1)\big)\cdot(M^{-1}\nabla_\xi)^{i}\big(\widetilde{{}^\omega g}^{-1}(x_1)\widetilde{{}^\omega g}(y)\big) \\ &\qquad + \ \ \big(\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(x_1)\big)\cdot(M^{-1}\nabla_\xi)^{k}\big(\widetilde{{}^\omega g}^{-1}(x_1)\widetilde{{}^\omega g}(y)\big) \ .\end{aligned} \tag{226}$$

In the above identity, we have dropped the dependence on time as it no longer has any bearing on how we proceed. Also $[x_1,\ell]$ denotes an even smaller interval embedded in the overall bootstrapping line segment $[y,\ell]$. We will let this smaller segment go to zero. Before doing this, we collect the last term on the right hand side of (226) onto the left, apply the matrix norm (16) and the reverse triangle inequality, and use the isometric identity (17) to arrive at the bound:

$$\Big| \| (M^{-1}\nabla_\xi)^k \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(y) \big) \| - \| (M^{-1}\nabla_\xi)^k \big(\widetilde{{}^\omega g}^{-1}(x_1) \widetilde{{}^\omega g}(y) \big) \| \Big| \leqslant \sum_{i=0}^{k-1} \| (M^{-1}\nabla_\xi)^{k-i} \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(x_1) \big) \| \cdot \| (M^{-1}\nabla_\xi)^{i} \big(\widetilde{{}^\omega g}^{-1}(x_1) \widetilde{{}^\omega g}(y) \big) \| \ . \tag{227}$$

We now divide both sides of this last expression by the small interval length $|x_1 - \ell|$ and let the resulting expression go the limit $x_1 \to \ell$. To compute this, we only need to handle the expressions:

$$\lim_{x_1 \to \ell} \ |x_1 - \ell|^{-1} \cdot \| (M^{-1}\nabla_\xi)^{k-i} \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(x_1) \big) \| \ , \tag{228}$$

where we have the important restriction $1 \leqslant k - i$. We do this by using the fact that the gauge equation (210) gives us an explicit realization of the product $\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(x_1)$ as an integral over the interval $[x_1, \ell]$:

$$\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(x_1) \ = \ \int_{x_1}^{\ell} \widetilde{{}^\omega g}^{-1}(x_1)\widetilde{{}^\omega g}(s) \ \widetilde{{}^\omega \underline{C}}_{\alpha(\ell)}(s) \ ds \ + \ I \ . \tag{229}$$

Here the $\alpha(\ell)$ index denotes the component of the connection $\{{}^\omega \underline{C}\}$ in the direction of the line segment $[y, x]$. Plugging this last expression into the limit (228) and using the fundamental theorem of calculus on the resulting identity we arrive at the simple equation:

$$(228) \ = \ \| (M^{-1}\nabla_\xi)^{k-i} \Big(\widetilde{{}^\omega \underline{C}}_{\alpha(\ell)}(\ell) \Big) \| \ . \tag{230}$$

Notice that the identity matrix on line (229) drops out because of the condition $1 \leqslant k - i$, and that all terms where the derivatives fall on the group elements are zero because when $x_1 = \ell$ these are again just derivatives of the identity matrix I. Now, substituting (230) into the limiting version of (227) we have the differential inequality:

$$\Big| \| (M^{-1}\nabla_\xi)^k \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(y) \big) \|' \Big| \leqslant \sum_{i=0}^{k-1} \| (M^{-1}\nabla_\xi)^{k-i} \Big(\widetilde{{}^\omega \underline{C}}_{\alpha(\ell)}(\ell) \Big) \| \cdot \| (M^{-1}\nabla_\xi)^{i} \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(y) \big) \| \ . \tag{231}$$

Assuming now that we have proved the inductive bound:

$$\sup_{0 \leqslant i \leqslant k-1} \| (M^{-1}\nabla_\xi)^{i} \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(y) \big) \| \ \lesssim \ 1 \ ,$$

which is easy when $k - 1 = 0$ on account of the compactness of the group $O(m)$, we see that by integrating the expression $\| (M^{-1}\nabla_\xi)^k \big(\widetilde{{}^\omega g}^{-1}(\ell) \widetilde{{}^\omega g}(y) \big) \|'$ the proof of (220) at the k^{th} step boils down to being able to establish the line integral estimate:

$$\sum_{i=0}^{k-1} \int_y^x \| (M^{-1}\nabla_\xi)^{k-i} \Big(\widetilde{{}^\omega \underline{C}}_{\alpha(\ell)}(\ell) \Big) \| \ d\ell \ \lesssim \ \mathcal{E} \ . \tag{232}$$

The reason this bound will be possible is that we have taken care to make sure that there is always at least one copy of the operator $(M^{-1}\nabla_\xi)$ in each of the above integrals, and it is the presence of the extra factor M^{-1} in conjunction with the range restriction (222) that will be enough to provide the needed integrability. In fact, using the condition that $M^{-1} \leqslant |x-y|^{-\frac{1}{2}}$ and the Cauchy-Schwartz inequality, we see that it suffices to be able to prove the bound:

$$\sum_{i=0}^{k-1} \int_y^x \| (M^{-1}\nabla_\xi)^i \nabla_\xi \Big(\widetilde{{}^\omega\underline{C}}_{\alpha(\ell)}(\ell) \Big) \|^2 \, d\ell \lesssim \mathcal{E}^2 . \tag{233}$$

This last integral can now be bounded in terms of energy type estimates once one applies the $L^\infty \to L^2$ trace theorem to it. However, because of the various angular degeneracies involved in the potentials $\{\nabla_\xi \widetilde{{}^\omega\underline{C}}\}$, it will be necessary for us to use a more refined "trace-Bernstein" type inequality. Furthermore, since the connection $\{\widetilde{{}^\omega\underline{C}}\}$ is only defined implicitly via the Hodge system (211), it will be necessary for us to prove estimate (233) via a bootstrapping argument in mixed Lebesgue spaces. What we will do is to show the following somewhat more restrictive estimate which yields (233) as a consequence:

Lemma 9.2. *Let the connection $\{\widetilde{{}^\omega\underline{C}}\}$ be defined via the Hodge system* (211)*:*

$$(\widetilde{{}^\omega\underline{C}})^{df} = d^* \Delta^{-1}[\widetilde{{}^\omega\underline{C}}, \widetilde{{}^\omega\underline{C}}] , \tag{234a}$$

$$(\widetilde{{}^\omega\underline{C}})^{cf} = \widetilde{{}^\omega\underline{A}^{(M)}} - \nabla_x \Delta^{-1}[\widetilde{{}^\omega\underline{A}^{(M)}}, \widetilde{{}^\omega\underline{C}}] . \tag{234b}$$

where we have set:

$$\widetilde{{}^\omega\underline{A}^{(M)}} = -\nabla_x \, {}^\omega\overline{\Pi}_{M^{-1}<\bullet} \, {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \, {}^\omega L \, \Delta_{\omega^\perp}^{-1} \, \underline{A}_{\bullet\ll 1}(\partial_\omega) . \tag{235}$$

Furthermore, the parameter M^{-1} which lies in the range (222) *(although this is not essential). Then the following mixed Lebesgue space estimates of Besov type hold:*

$$\sum_{i=0}^{k-1} \sum_\mu \| (M^{-1}\nabla_\xi)^i \nabla_\xi \, P_\mu (\widetilde{{}^\omega\underline{C}}) \|_{L^\infty_{\ell^\perp}(L^2_\ell)} \lesssim \mathcal{E} . \tag{236}$$

Proof of estimate (236)*.* Things will be a bit easier if we prove the following more restrictive estimate:

$$\sum_{i=0}^{k-1} \sum_\mu \mu^{-\gamma}(1+\mu)^n \| (M^{-1}\nabla_\xi)^i \nabla_\xi \, P_\mu (\widetilde{{}^\omega\underline{C}}) \|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \mathcal{E} . \tag{237}$$

That (236) is a consequence of (237) is a simple matter applying the Minkowski inequality for mixed Lebesgue spaces and the fact that the weights in (237) are clearly more restrictive. Now, the proof of this second estimate is essentially no more complicated than using the Bernstein inequality in the hyperplane plane $\mathbb{R}^{n-1}_{\ell^\perp}$ to turn things into the energy estimate contained in the bootstrapping norm (124d). To see this, we begin our proof of (237) by first establishing this bound for the reduced Coulomb potentials $\{\widetilde{{}^\omega\underline{A}^{(M)}}\}$.

We are now trying to prove that:

$$\sum_{j=0,1} \sum_{i=0}^{k-1} \sum_\mu \mu^{-\gamma}(1+\mu)^n \| (M^{-1}\nabla_\xi)^i \nabla_\xi^j \, P_\mu (\widetilde{{}^\omega\underline{A}^{(M)}}) \|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \mathcal{E} . \tag{238}$$

For each fixed frequency in the above sum, we decompose things into all frequencies corresponding to the $\mathbb{R}^{n-1}_{\ell^\perp}$ plane, as well as all possible dyadic angular sectors spread from the ω (fixed) direction:

$$P_\mu \ = \ \sum_{\substack{\theta,\lambda\ :\\ \lambda\leqslant\mu}} {}^\omega\Pi_\theta Q_\lambda P_\mu \ ,$$

where Q_λ is an $(n-1)$ dimensional fixed frequency multiplier which is defined in analogy with P_λ. Freezing all frequencies, our goal will be to show the following estimate:

$$\| \, (M^{-1}\nabla_\xi)^i \ \nabla_\xi^j \ {}^\omega\Pi_\theta Q_\lambda P_\mu\big(\widetilde{{}^\omega\underline{A}^{(M)}}\big) \, \|_{L^2_\ell(L^\infty_{\ell^\perp})} \ \lesssim \ \theta^\gamma \left(\frac{\lambda}{\mu}\right)^\gamma \mu^{2\gamma}\cdot\mathcal{E} \ . \tag{239}$$

By adding in the weights $\mu^{-\gamma}(1+\mu)^n$, using the fact that the potentials $\{\widetilde{{}^\omega\underline{A}^{(M)}}\}$ are truncated to frequencies $\mu\ll 1$, and dyadic summing, the fixed frequency estimate (239) implies (238) with room to spare. To deal with all of the ξ derivatives, notice that we have the following heuristic multipliers bounds:

$$(M^{-1}\nabla_\xi)^i \ \nabla_\xi^j \ {}^\omega\Pi_\theta Q_\lambda P_\mu\big(\widetilde{{}^\omega\underline{A}^{(M)}}\big) \ \approx \ \theta^{-2} \ {}^\omega\Pi_\theta {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} Q_\lambda P_\mu\big(\underline{A}_{\bullet\ll 1}\big) \ , \tag{240}$$

where we are enforcing the notation introduced on line (54). That is, the left hand side of the above identity satisfies *all* mixed Lebesgue space bounds as the right hand side with the same constants. Notice that this bound uses the extra Coulomb savings introduced on line (191) above to kill off one power of θ^{-1} from the degenerate Laplacean $\Delta_{\omega^\perp}$. The other power of θ^{-1} on the right hand side of (240) comes from the operator ∇_ξ which has no smoothing factor of M^{-1}. This is precisely what one pays for passing from the L^1 integral (232) to the more manageable L^2 integral (233). Finally, it is important to point out that although we have not emphasized it, the multipliers Q_λ depend on ω, but the fact that $\lambda\ll\theta$ implies that the multiplier product on the left hand side of (240) is zero prevents the derivatives of Q_λ with respect to ξ from costing more than derivatives of ${}^\omega\Pi_\theta$ (alternatively, we could have applied the Q_λ multipliers on the outside of the ∇_ξ^k operators, because differentiation will not change the support of the various multipliers).

Now, to use the Bernstein inequality on the $\mathbb{R}^{n-1}_{\ell^\perp}$ plane, we simply note that one has the multiplier identity:

$${}^\omega\Pi_\theta Q_\lambda P_\mu \ = \ {}^{\omega||\ell^\perp}B_{(\mu\theta)} {}^\omega\Pi_\theta Q_\lambda P_\mu \ , \tag{241}$$

where ${}^{\omega||\ell^\perp}B_{(\mu\theta)}$ a block type cutoff in the $\mathbb{R}^{n-1}_{\ell^\perp}$ frequency plane of dimensions $1\times(\mu\theta)\times\ldots\times(\mu\theta)$ which has its long side centered along the projection[10] of the unit vector ω onto the $\mathbb{R}^{n-1}_{\ell^\perp}$ (frequency) plane. The crucial fact about the geometry of the multiplier (241) is that is has support contained in a box of size $\lambda\times(\mu\theta)\times\ldots\times(\mu\theta)$ in the $\mathbb{R}^{n-1}_\xi$ (frequency) plane. Using now the identities (240) and (241), as well as the $n-1$ dimensional Bernstein inequality, we see that we

[10]In the case that ω lies perfectly in the ℓ direction, we will just be wasteful and choose any direction.

may estimate:

$$\| (M^{-1}\nabla_\xi)^i \, \nabla_\xi^j \, {}^\omega\Pi_\theta Q_\lambda P_\mu \big(\widetilde{{}^\omega\underline{A}^{(M)}} \big) \|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \theta^{-2}\cdot\lambda^{\frac{1}{2}}(\mu\theta)^{\frac{n-2}{2}} \ \| P_\mu\big(\underline{A}_{\bullet\ll 1}\big)\|_{L^2} \ . \tag{242}$$

To deal with the weights on the right hand side, we use the truncation condition that $\mu^{\frac{1}{2}-\delta}\leqslant\theta$, as well as the fact that $\lambda\leqslant\mu$ to conclude the bound:

$$\theta^{-2}\cdot\lambda^{\frac{1}{2}}(\mu\theta)^{\frac{n-2}{2}} \ \leqslant\ \mu^{\frac{n-2}{2}}\cdot\theta^\gamma\left(\frac{\lambda}{\mu}\right)^\gamma\mu^{2\gamma}\ .$$

Substituting this into the right hand side of estimate (242) and using the $L^\infty(L^2)$ bound contained in the bootstrapping estimate (124d), we have achieved the desired result (239).

It is now our task to use (239) and the Hodge system (234) to pass to the more general estimate (236). In order to do this, it will be necessary for us to first prove some *critical* estimates for the potentials $\{\widetilde{{}^\omega\underline{C}}\}$. These will then be used as a reference point in certain bilinear estimates involving the space used to define estimate (236). While we're at it, this will also give us a chance to prove some estimates which will be used many times in the sequel. What we will show is that:

$$\| (M^{-1}\nabla_\xi)^k \ {}^\omega\underline{C}\,\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}} \ \lesssim\ \mathcal{E}\ , \tag{243}$$

$$\| (M^{-1}\nabla_\xi)^k\nabla_t {}^\omega\underline{C}\,\|_{\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n}} \ \lesssim\ \mathcal{E}\ , \tag{244}$$

where p_γ is exponent defined on line (188) above. Both of the bounds (243)–(244) will easily follow via our general Besov embedding (41) once we have established them for the linear term on the right hand side of the Hodge system (234). That is, we fist establish that:

$$\| (M^{-1}\nabla_\xi)^k \ \widetilde{{}^\omega\underline{A}^{(M)}}\,\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}} \ \lesssim\ \mathcal{E}\ , \tag{245}$$

$$\| (M^{-1}\nabla_\xi)^k\nabla_t \ \widetilde{{}^\omega\underline{A}^{(M)}}\,\|_{\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n}} \ \lesssim\ \mathcal{E}\ , \tag{246}$$

These follow from immediately from the steps used to prove (187) above, and the following heuristic identity which follows our convention established on line (54):

$$\nabla_\xi^k\left({}^\omega\Pi_\theta P_\mu \ \widetilde{{}^\omega\underline{A}^{(M)}}\right) \ \approx\ \theta^{-k}\ {}^\omega\Pi_\theta{}^\omega\overline{\Pi}_{M^{-1}<\bullet}\ P_\mu\,\nabla_x\ {}^\omega L\ \Delta^{-1}_{\omega^\perp}\,\underline{A}_{\bullet\ll 1}(\partial_\omega)\ , \tag{247}$$

$$\nabla_\xi^k\left({}^\omega\Pi_\theta P_\mu\,\nabla_t\ \widetilde{{}^\omega\underline{A}^{(M)}}\right) \ \approx\ \mu\theta^{-k}\ {}^\omega\Pi_\theta{}^\omega\overline{\Pi}_{M^{-1}<\bullet}\ P_\mu\,\nabla_x\ {}^\omega L\ \Delta^{-1}_{\omega^\perp}\,\underline{A}_{\bullet\ll 1}(\partial_\omega)\ , \tag{248}$$

Notice that the space-time frequency localization (124c) allows us to trade the ∇_t with the factor of μ on the second line above.

We now prove the estimates (243)–(244) by proceeding inductively on the value of k. If $k=0$ the first estimate (243) holds because one can solve the system (234) via Picard iteration in the space $\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}$ thanks to the bilinear embedding (41) which furnishes the embedding:

$$\nabla_x \Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ \hookrightarrow \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ . \tag{249}$$

The key thing to point out here is that for γ sufficiently small, and in dimensions $6 \leqslant n$ we have the bound $p_\gamma < n$, which is all that is needed to satisfy the gap condition (43) in this case. The other conditions of Lemma 4.1 are also easily seen to be satisfied for this set of indices.

To establish (243) for $0 < k$, we simply differentiate the system (211) k times with respect to the operator $(M^{-1}\nabla_\xi)$. Doing this yields the linearized set of equations:

$$(M^{-1}\nabla_\xi)^k (\widetilde{{}^\omega\underline{C}})^{df} \ = \ \sum_{j=0}^{k} \ d^* \, \Delta^{-1}\Big[(M^{-1}\nabla_\xi)^{k-j} \ \widetilde{{}^\omega\underline{C}} \ , \ (M^{-1}\nabla_\xi)^j \ \widetilde{{}^\omega\underline{C}}\Big] \ , \tag{250}$$

$$\begin{aligned} (M^{-1}\nabla_\xi)^k (\widetilde{{}^\omega\underline{C}})^{cf} \ &= \ (M^{-1}\nabla_\xi)^k \ \widetilde{{}^\omega\underline{A}^{(M)}} \ - \\ &\sum_{j=0}^{k} \ \nabla_x \Delta^{-1}\Big[(M^{-1}\nabla_\xi)^{k-j} \ \widetilde{{}^\omega\underline{A}^{(M)}} \ , \ (M^{-1}\nabla_\xi)^j \ \widetilde{{}^\omega\underline{C}}\Big] \ , \end{aligned} \tag{251}$$

which can again be solved in the Besov space $\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}$ by using the already established estimate (245) for the linear term, in conjunction with the (inductive) hypothesis that estimate (243) holds for $k-1$, and absorbing the highest derivative (involving $(M^{-1}\nabla_\xi)$ falling on $\widetilde{{}^\omega\underline{C}}$) term to the left hand side. All of this is permissible by referring to the embedding (249).

To prove the second estimate (244) above, we first apply the time derivative ∇_t to both sides of the system (250)–(251) above. The resulting system of equations, which we will not write down, can easily be solved in the derivative critical Besov space $\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n}$ by again using an induction on k, the already established estimate (246) for the linear term, and the following bilinear Besov estimate which is again a special case of (41):

$$\nabla_x \Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n} \ \hookrightarrow \ \dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n} \ . \tag{252}$$

Notice that (252) is permissible because for γ sufficiently small, we have the condition $p_\gamma < \frac{2n}{3}$ in dimensions $6 \leqslant n$ which is necessary to get around the gap condition (43). The other conditions of (41) are easily satisfied for this choice of indices.

Armed with estimates (238) and (243), we now move back to the proof of estimate (236). We set the norm in that latter bound equal to:

$$\| \, A \, \|_{\mathcal{N}_1^{-\gamma,2,\infty}} \ = \ \sum_{\mu} \ \mu^{-\gamma}(1+\mu)^n \ \| \, P_\mu(A) \, \|_{L^2_\ell(L^\infty_{\ell^\perp})} \ .$$

By differentiating the system (234) with respect to the operators $(M^{-1}\nabla_\xi)^k\nabla_\xi$, we see that the claim will now follow once we can prove the bilinear Riesz operator bound:

$$\nabla_x \Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \mathcal{N}_1^{-\gamma,2,\infty} \ \hookrightarrow \ \mathcal{N}_1^{-\gamma,2,\infty} \ . \tag{253}$$

We now let A and C be any two elements of the two spaces on the left hand side of (253). By applying the trichotomy, we see that it suffices to be able to prove the three estimates:

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\ll\mu_2\\ \lambda\sim\mu_2}} \lambda^{-\gamma}(1+\lambda)^n \ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \|A\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\cdot\|C\|_{\mathcal{N}_1^{-\gamma,2,\infty}} , \tag{254}$$

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_2\ll\mu_1\\ \lambda\sim\mu_1}} \lambda^{-\gamma}(1+\lambda)^n \ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \|A\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\cdot\|C\|_{\mathcal{N}_1^{-\gamma,2,\infty}} , \tag{255}$$

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\sim\mu_2\\ \lambda\lesssim\mu_1,\mu_2}} \lambda^{-\gamma}(1+\lambda)^n \ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_\ell(L^\infty_{\ell^\perp})} \lesssim \|A\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\cdot\|C\|_{\mathcal{N}_1^{-\gamma,2,\infty}} . \tag{256}$$

The proofs of (254)–(255) are very simple, and follow from essentially the same principle. First of all, we use the fact that the kernel of the fixed frequency operator $\lambda\cdot\nabla_x\Delta^{-1}P_\lambda$ is in L^1 with norm independent of λ. Thus, it is bounded on all mixed Lebesgue spaces. This, used in conjunction with the estimate:

$$\sum_{\substack{\mu_1\ :\\ \mu_1\lesssim\lambda}} \lambda^{-1}\ \|P_{\mu_1}A\|_{L^\infty} \lesssim \|A\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}} , \tag{257}$$

which follows easily from Bernstein's inequality and dyadic summing, is enough for us to conclude the first estimate (254). To conclude the second estimate, (255), we simply employ the fixed frequency version of (257) and then estimate:

$$\begin{aligned}
\sum_{\substack{\lambda,\mu_2\ :\\ \mu_2\lesssim\lambda}} \lambda^{-\gamma}\ \|P_{\mu_2}C\|_{L^2_\ell(L^\infty_{\ell^\perp})} &= \sum_{\substack{\lambda,\mu_2\ :\\ \mu_2\lesssim\lambda}} \left(\frac{\mu_2}{\lambda}\right)^\gamma \mu_2^{-\gamma}\|P_{\mu_2}C\|_{L^2_\ell(L^\infty_{\ell^\perp})} ,\\
&= \sum_{\mu_2} \mu_2^{-\gamma}\ \|P_{\mu_2}C\|_{L^2_\ell(L^\infty_{\ell^\perp})} \cdot \sum_{\substack{\lambda\ :\\ \mu_2\lesssim\lambda}} \left(\frac{\mu_2}{\lambda}\right)^\gamma ,\\
&\lesssim \|C\|_{\mathcal{N}_1^{-\gamma,2,\infty}} .
\end{aligned}$$

We now move to prove the last estimate (256). This is only slightly more complicated than what we have already done. Here we will only bother to estimate things for $\lambda\lesssim 1$. The case where $1\ll\lambda$ is much easier due to the extra smoothing in the norms we are working with and is left to the reader. As a first step, we freeze all frequencies and decompose $P_\lambda = \sum_{\substack{\sigma\ :\\ \sigma\leqslant\lambda}} Q_\sigma P_\lambda$, where Q_σ is again the fixed frequency cutoff in the $\mathbb{R}^{n-1}_{\ell^\perp}$ frequency plane. Using Bernstein, this allows us to estimate:

$$\lambda^{-\gamma} \, \| \, \nabla_x \Delta^{-1} P_\lambda \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^2_\ell (L^\infty_{\ell^\perp})} \ , \tag{258}$$
$$\lesssim \sum_{\substack{\sigma \, : \\ \sigma \leqslant \lambda}} \lambda^{-1-\gamma} \, \| \, Q_\sigma \left(P_{\mu_1} A \cdot P_{\mu_2} C \right) \|_{L^2_\ell (L^\infty_{\ell^\perp})} \ ,$$
$$\lesssim \sum_{\substack{\sigma \, : \\ \sigma \leqslant \lambda}} \lambda^{-1-\gamma} \sigma^{\frac{n-1}{p_\gamma}} \, \| \, P_{\mu_1} A \cdot P_{\mu_2} C \, \|_{L^2_\ell (L^{p_\gamma}_{\ell^\perp})} \ ,$$
$$\lesssim \lambda^{\frac{n-1}{p_\gamma} - 1 - \gamma} \, \| \, P_{\mu_1} A \cdot P_{\mu_2} C \, \|_{L^2_\ell (L^{p_\gamma}_{\ell^\perp})} \ .$$

By using Hölders inequality and rearranging weights, and using the index bound:

$$1 + 2\gamma < \frac{n-1}{p_\gamma} \ ,$$

which follows on account of the fact that $\gamma \ll 1$ and we are in $n \leqslant 6$ dimensions, we see that we have the fixed frequency estimate:

$$(258) \ \lesssim \ \left(\frac{\lambda}{\mu_1} \right)^\gamma \mu_1^{\frac{n-1}{p_\gamma} - 1} \| \, P_{\mu_1} A \, \|_{L^\infty_\ell (L^{p_\gamma}_{\ell^\perp})} \cdot \mu_2^{-\gamma} \| \, P_{\mu_2} C \, \|_{L^2_\ell (L^\infty_{\ell^\perp})} \ . \tag{259}$$

We are done once we deal with the first factor on the right hand side of the above inequality. To do this, we run a multiplier decomposition $P_{\mu_1} = \sum_{\substack{\sigma \, : \\ \sigma \leqslant \mu_2}} \widetilde{Q}_\sigma P_{\mu_1}$, where this time $\widetilde{Q}_\sigma$ is a fixed frequency cutoff on the $\mathbb{R}_\ell$ line. Using Minkowski's integral inequality and Bernstein on the real line, we estimate:

$$\begin{aligned}
\mu_1^{\frac{n-1}{p_\gamma} - 1} \| \, P_{\mu_1} A \, \|_{L^\infty_\ell (L^{p_\gamma}_{\ell^\perp})} \ &\lesssim \ \sum_{\substack{\sigma \, : \\ \sigma \leqslant \mu_2}} \mu_1^{\frac{n-1}{p_\gamma} - 1} \| \, \widetilde{Q}_\sigma P_{\mu_1} A \, \|_{L^{p_\gamma}_{\ell^\perp} (L^\infty_\ell)} \ , \\
&\lesssim \ \mu_1^{\frac{n}{p_\gamma} - 1} \| \, P_{\mu_1} A \, \|_{L^{p_\gamma}} \ , \\
&= \ \| \, P_{\mu_1} A \, \|_{\dot{B}_2^{p_\gamma, (2, \frac{n-2}{2})}} \ .
\end{aligned}$$

Plugging this last bound into the right hand side of (259) and summing over all $\lambda \lesssim \mu_1, \mu_2$ yields the bound (256) as was to be shown. This completes the proof of the bilinear estimate (253) and hence the proof of (236). $\square$

To wrap things up here, we need to prove the symbol bounds (223)–(224) for the second factor in the product (225). By repeating the steps which started on line (226) and culminated in the differential inequality (231) for this term, we arrive at the temporal differential inequality:

$$\begin{aligned}
&\left| \| \, (M^{-1} \nabla_\xi)^k \left({}^{\omega}\!\widetilde{g}^{-1}(\ell) {}^{\omega}\!\widetilde{g}(s) \right) \|' \right| \ \leqslant \\
&\qquad \sum_{i=0}^{k-1} \| \, (M^{-1} \nabla_\xi)^{k-i} \left({}^{\omega}\!\widetilde{C}_0(\ell) \right) \| \cdot \| \, (M^{-1} \nabla_\xi)^i \left({}^{\omega}\!\widetilde{g}^{-1}(\ell) {}^{\omega}\!\widetilde{g}(s) \right) \| \ ,
\end{aligned} \tag{260}$$

where this time ℓ denotes a single variable which lies in the range $s \leqslant \ell \leqslant t$, and ${}^{\omega}\!\widetilde{C}_0$ is the temporal potential which is defined via the equation:

$${}^{\omega}\!\widetilde{g}^{-1} \nabla_t ({}^{\omega}\!\widetilde{g}) \ = \ {}^{\omega}\!\widetilde{C}_0 \ .$$

Via integration in time of the quantity $\|\,(M^{-1}\nabla_\xi)^k\big(\widetilde{{}^\omega g}^{-1}(\ell)\widetilde{{}^\omega g}(s)\big)\,\|'$, and keeping in mind the derivation of the temporal potential equation (196) above, we see that to prove the estimates (223) for the product $\widetilde{{}^\omega g}^{-1}(t)\widetilde{{}^\omega g}(s)$ it suffices to show (the same estimate works to establish (223)):

Lemma 9.3. *Let the temporal potential $\widetilde{C}_0$ be defined via the elliptic equation:*

$$\widetilde{{}^\omega C}_0 \;=\; \widetilde{{}^\omega A_0^{(M)}} \;-\; \nabla_t\Delta^{-1}[\widetilde{{}^\omega \underline{A}^{(M)}},\widetilde{{}^\omega \underline{C}}] \;-\; d^*\Delta^{-1}[\widetilde{{}^\omega C}_0,\widetilde{{}^\omega \underline{C}}]\ , \tag{261}$$

where the spatial connection $\{\widetilde{{}^\omega \underline{C}}\}$ is defined via the Hodge system (234)*, and where we have set:*

$$\widetilde{{}^\omega A_0^{(M)}} \;=\; -\,\nabla_t\ {}^\omega\overline{\Pi}_{M^{-1}<\bullet}\,{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ {}^\omega L\ \Delta^{-1}_{\omega^\perp}\,\underline{A}_{\bullet\ll 1}(\partial_\omega)\ , \tag{262}$$

The parameter M^{-1} which lies in the range (222) *(although this is not essential). Then the following mixed Lebesgue space estimates of Besov type hold:*

$$\sum_{i=1}^k\sum_\mu\ \|\,(M^{-1}\nabla_\xi)^i\ P_\mu\big(\widetilde{{}^\omega C}_0\big)\,\|_{L^\infty_x(L^1_t[s,t])}\ \lesssim\ \mathcal{E}\ . \tag{263}$$

Proof of the estimate (264). As with the proof of (236) above, it will be convenient to prove the somewhat more restrictive estimate:

$$\sum_{i=1}^k\sum_\mu\ \mu^{-\gamma}(1+\mu)^n\ \|\,(M^{-1}\nabla_\xi)^i\ P_\mu\big(\widetilde{{}^\omega C}_0\big)\,\|_{L^1_t[s,t](L^\infty_x)}\ \lesssim\ \mathcal{E}\ . \tag{264}$$

This will again be done by essentially proving that this estimate is true for the potentials $\{\widetilde{{}^\omega A^{(M)}}\}$ contained in the right hand side of (261), and then transferring that knowledge to $\widetilde{{}^\omega C}_0$ through that elliptic equation. A little care needs to be taken in this regard due to the effect of bad $High\times High\Rightarrow Low$ frequency interactions coming from the Δ^{-1} in the second term on the right hand side of (261) which sits by itself because the time derivative must be distributed. This all needs to be tempered against the fact that we need to recover enough in the low frequencies to apply $L^2(L^q)$ Strichartz estimates to integrate over the line segment $[s,t]$. The Lebesgue exponent which is important in this regard is the following:

$$q_\gamma\ =\ \frac{2(n-1)}{n-5-2\gamma}\ . \tag{265}$$

The significance of q_γ is that it is the smallest Lebesgue exponent such that one can recover an extra angular weight of θ^{-1} via running Bernstein from the Strichartz endpoint and still have an extra factor of θ^γ to spare for dyadic summing.

We proceed with our proof of (264) by first establishing a fixed time estimate. Notice that the Besov norm $\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}$ embeds into the spatial norm on the left hand side of (264). Thus, our first step will be to establish the following fixed time estimate for $1\leqslant k$:

(266)

$$\|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega C}_0(t_0)\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}}\ \lesssim\ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \widetilde{{}^\omega A^{(M)}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}}\ ,$$

where $\{\widetilde{{}^\omega A^{(M)}}\}$ is the full space-time connection defined on lines (235) and (262). The important thing about estimate (266) is that we retain at least one copy of the operator $(M^{-1}\nabla_\xi)$ on the right hand side so that we may pass to an L^2_t integral via Cauchy-Schwartz. Our proof of (266) require that the bootstrapping constant $\mathcal{E}$ from line (124d) is sufficiently small. Based on previous work, our task here is largely finished. Our first step here will be to differentiate the equation (264) as many times as necessary with respect to the operators $(M^{-1}\nabla_\xi)$. Doing this and distributing the time derivative in the second term on the right hand side yields the equation:

$$\begin{aligned}(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega C}_0 \ &= \ (M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega A_0^{(M)}}\\ &- \ \sum_{i=0}^k \ \Big(\Delta^{-1}[(M^{-1}\nabla_\xi)^{k-i}\nabla_t\widetilde{{}^\omega \underline{A}^{(M)}}, (M^{-1}\nabla_\xi)^i\widetilde{{}^\omega\underline{C}}]\\ &+ \ \Delta^{-1}[(M^{-1}\nabla_\xi)^{k-i}\widetilde{{}^\omega\underline{A}^{(M)}}, (M^{-1}\nabla_\xi)^i\nabla_t\widetilde{{}^\omega\underline{C}}]\Big)\\ &- \ \sum_{i=0}^k \ d^*\Delta[(M^{-1}\nabla_\xi)^{k-i}\widetilde{{}^\omega C}_0, (M^{-1}\nabla_\xi)^i\widetilde{{}^\omega\underline{C}}]\ ,\\ &= \ T_1\ +\ T_2\ +\ T_3\ .\end{aligned}\tag{267}$$

Our second step is to prove the intermediate estimate:

$$\begin{aligned}&\|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega C}_0(t_0)\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}} \ \lesssim\\ &\|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega A^{(M)}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}} \ + \ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}}\\ &\qquad + \ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \nabla_t\widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n}}\ .\end{aligned}\tag{268}$$

This in turn is a consequence of the three estimates:

$$\|\,T_1\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}} \ \lesssim \ \|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega A^{(M)}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}}\ ,\tag{269}$$

$$\begin{aligned}\|\,T_2\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}} \ &\lesssim \ \|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega A^{(M)}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}}\\ &+ \ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}}\\ &+ \ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \nabla_t\widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n}}\ ,\end{aligned}\tag{270}$$

$$\begin{aligned}\|\,T_3\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}} \ &\lesssim \ \|\,(M^{-1}\nabla_\xi)^k\ \widetilde{{}^\omega C}_0(t_0)\,\|_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n}}\cdot\|\,\widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\\ &+ \ \sum_{i=1}^k\ \|\,(M^{-1}\nabla_\xi)^i\ \widetilde{{}^\omega\underline{C}}(t_0)\,\|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}}\ .\end{aligned}\tag{271}$$

Notice that these all combine to give (268) because the estimate (243) in conjunction with the assumption that $\mathcal{E}$ is sufficiently small allows one to absorb the first term on the right hand side of (271) into the left hand side. The proof of the first estimate, (269) above is a trivial consequence of the Besov nesting (38), and the fact that we allow for an extra power of μ^γ to sum over the low frequencies to turn the ℓ^2 sum into and ℓ^1 sum.

The proof of (270) is the most involved, and is why we have been forced to work with the exponent q_γ. There are several cases to consider, depending on wether or not the time derivative falls on the term containing at least one copy of the operator $(M^{-1}\nabla_\xi)$. An inspection of the structure of the T_2 term shows that these can all be taken into account through an application of the already established estimates (243)–(244) and (245)–(246), and application of the bilinear embeddings:

$$\Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n} \ \hookrightarrow \ \dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n} \ , \tag{272}$$

$$\Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n} \cdot \dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n} \ \hookrightarrow \ \dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n} \ . \tag{273}$$

A quick calculation shows that one has the needed gap bound (by condition (43)):

$$2+\gamma \ < \ n(\frac{1}{p_\gamma}+\frac{1}{q_\gamma}) \ ,$$

for γ sufficiently small when the dimension satisfies the bound $6\leqslant n$. For example, when $n=6$ we have that $p_\gamma = \frac{10}{3}+\epsilon$ and $q_\gamma = 10+\epsilon$ where $\epsilon\to 0$ as $\gamma\to 0$. Notice that there is not a whole lot of room in this. The other condition in (42)–(46) are easily verified in the above estimates. Notice that for the term in T_2 where all the $(M^{-1}\nabla_\xi)$ derivatives, as well as the time derivative ∇_t fall on the linear potentials $\{\widetilde{{}^\omega\underline{A}^{(M)}}\}$, we use (243) and the first embedding (272) above, together with the easy bound (which follows from the truncation (124c)):

$$\| \, (M^{-1}\nabla_\xi)^k \nabla_t \widetilde{{}^\omega\underline{A}^{(M)}} \, \|_{\dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n}} \ \lesssim \ \| \, (M^{-1}\nabla_\xi)^k \ \widetilde{{}^\omega\underline{A}^{(M)}} \, \|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}} \ , \tag{274}$$

to conclude (270) for that portion of things.

To conclude our second main step in the proof of (266), we need to establish the estimate (271). To tie things down, we first need to know that $\widetilde{{}^\omega C}_0$ satisfies a critical estimate similar to (243). This is:

$$\| \, (M^{-1}\nabla_\xi)^i \widetilde{{}^\omega C}_0 \, \|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}} \ \lesssim \ \mathcal{E} \ . \tag{275}$$

This in turn is provided through applying the already established estimates (243)–(243) and (246) to the equation (261) with the help of the bilinear estimate (249) and the following embedding which follows as yet another special case of our general bound (41):

$$\Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n} \cdot \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ \hookrightarrow \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{1,10n} \ . \tag{276}$$

Armed with the estimate (275), we can proceed to prove (271) by applying the following bilinear estimates to the various terms contained in T_3:

$$\nabla_x\Delta \ : \ \dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n} \cdot \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ \hookrightarrow \ \dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{1,10n} \ , \tag{277}$$

$$\nabla_x\Delta \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n} \ \hookrightarrow \ \dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{1,10n} \ . \tag{278}$$

We use the second embedding (278) in conjunction with the nesting (38) to derive the second term on the right hand side of (271). We note that in estimates (277)–(278), it is a simple matter to check the validity of the conditions (42)–(46). We leave this as an exercise for the reader.

To complete this portion of the proof, we need to establish the implication (268)⇒(266). This will be done once we can show that (keeping in mind the bounds of the form (274)):

(279)

$$\sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ {}^{\omega}\widetilde{\underline{C}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}} \lesssim \sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ \widetilde{{}^{\omega}A^{(M)}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}} ,$$

(280)

$$\sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ \nabla_t {}^{\omega}\widetilde{\underline{C}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n}} \lesssim \sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ \nabla_t \widetilde{{}^{\omega}A^{(M)}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n}}$$
$$+ \sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ \widetilde{{}^{\omega}A^{(M)}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}} + \sum_{i=1}^{k} \| (M^{-1}\nabla_\xi)^i \ {}^{\omega}\widetilde{\underline{C}}(t_0) \|_{\dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n}} .$$

The estimate (279) is a simple consequence of applying the embedding (278) to the differentiated Hodge system (250)–(250), while using the already established critical estimates (243) and (245) to tie things down. To prove the second estimate (280) above, we apply the time derivative ∇_t to the system (250)–(250), and then employ the embeddings:

$$\nabla_x\Delta \ : \ \dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n} \cdot \dot{B}^{q_\gamma,(2,\frac{n}{2}-\gamma)}_{2,10n} \hookrightarrow \dot{B}^{\infty,(2,\frac{n-2}{2}-\gamma)}_{1,10n} , \tag{281}$$

$$\nabla_x\Delta \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{2,10n} \hookrightarrow \dot{B}^{q_\gamma,(2,\frac{n-2}{2}-\gamma)}_{1,10n} . \tag{282}$$

Notice that these estimates have the same small amount of room as (272)–(273) above when measuring the gap condition (43) for this set of exponents. Using (281)–(282) in conjunction with the already established estimates (243)–(244) and (245)–(246), we may conclude (280) when the bootstrapping constant $\mathcal{E}$ is sufficiently small.

We have now established the estimate (266). Integrating this in time, and applying a Cauchy-Schwartz with respect to the time integration and using the condition $|t-s|^{\frac{1}{2}} \leqslant M$, we have the estimate:

$$(\text{L.H.S.})(264) \lesssim \sum_{i=0}^{k-1} \| (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ \widetilde{{}^{\omega}A^{(M)}} \|_{L^2_t\left(\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} .$$

Therefore, to conclude the estimate (263), we simply need to prove the bound:

$$\sum_{i=0}^{k-1} \| (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ \widetilde{{}^{\omega}A^{(M)}} \|_{L^2_t\left(\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} \lesssim \mathcal{E} . \tag{283}$$

At a heuristic level, this estimate is true because there is enough room in the norm $L^2_t\big(\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}\big)$ verses the bootstrapping norm (124d) to save precisely $\frac{1}{2}-2\gamma$

derivatives. This, used in conjunction with the truncation condition coming form the operator ${}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}$, is enough to absorb the extra angular factor θ^{-1} produced by the unsmoothed derivative ∇_ξ. All throughout the calculation, the exponent q_γ is high enough that the intrinsic angular singularity contained in the potentials $\{\widetilde{{}^{\omega}A^{(M)}}\}$ can be recovered by an application of Bernstein's inequality to the end-point Strichartz spatial exponent $L^{\frac{2(n-1)}{n-3}}$. We now spell out briefly the details of this procedure:

Freezing now the frequency and the number of $(M^{-1}\nabla_\xi)$ derivatives on the right hand side of (283), and using the bootstrapping condition (124d), we see that it suffices to show the bound (note that (283) is already in square function form):

$$\| \, (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ P_\mu(\widetilde{{}^{\omega}A^{(M)}}) \, \|_{L^2_t\left(\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} \ \lesssim \ \| \, P_\mu(\underline{A}_{\bullet\ll 1}) \, \|_{L^2\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}_{2,10n}\right)} \ .$$

In fact, after a further localization in the angle, we will show that:

$$\| \, (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ {}^{\omega}\Pi_\theta \ P_\mu(\widetilde{{}^{\omega}A^{(M)}}) \, \|_{L^2_t\left(\dot{B}^{q_\gamma,(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} \ \lesssim \ \theta^\gamma \ \| \, P_\mu(\underline{A}_{\bullet\ll 1}) \, \|_{L^2\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}_{2,10n}\right)} \ .$$

By an application of the Bernstein inequality, and recalling the definition (265) of the exponent q_γ, this last estimate is a consequence of being able to show that:

$$\| \, (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ {}^{\omega}\Pi_\theta \ P_\mu(\widetilde{{}^{\omega}A^{(M)}}) \, \|_{L^2_t\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} \ \lesssim \ \theta^{-1} \ \| \, P_\mu(\underline{A}_{\bullet\ll 1}) \, \|_{L^2\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}_{2,10n}\right)} \ . \tag{284}$$

Using now the heuristic operator bound (190) in conjunction with the Coulomb savings (191) and the heuristic symbol type bounds (247), we have the following heuristic identity which follows our strict convention (54):

$$(M^{-1}\nabla_\xi)^i \ \nabla_\xi \ {}^{\omega}\Pi_\theta \ P_\mu(\widetilde{{}^{\omega}A^{(M)}}) \ \approx \ \theta^{-2} \ P_\mu(\underline{A}_{\bullet\ll 1}) \ .$$

Plugging this last bound into the left hand side of (284), and rearranging the Besov weights, we have the estimate:

$$\| \, (M^{-1}\nabla_\xi)^i \ \nabla_\xi \ {}^{\omega}\Pi_\theta \ P_\mu(\widetilde{{}^{\omega}A^{(M)}}) \, \|_{L^2_t\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n}{2}-2\gamma)}_{2,10n}\right)} \ \lesssim \ \left(\frac{\mu^{\frac{1}{2}-2\gamma}}{\theta}\right) \theta^{-1} \ \| \, P_\mu(\underline{A}_{\bullet\ll 1}) \, \|_{L^2\left(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}_{2,10n}\right)} \ .$$

The truncation condition that $\mu^{\frac{1}{2}-\delta} \leqslant \theta$ (note that $\mu \lesssim 1$) now guarantees that we have the bound:

$$\left(\frac{\mu^{\frac{1}{2}-2\gamma}}{\theta}\right) \ \lesssim \ 1 \ ,$$

when γ is sufficiently small compared to δ. This completes the proof of (284), and hence (283), which in turn finishes our proof of estimate (264). □

Having now established the symbol bounds (223)–(223) separately for each of the two terms in the product (225). By using the Leibniz rule for derivatives, these together imply the bounds (223)–(223) for the full product on the left hand side of (225). This completes our proof of Proposition 9.1. □

We now proceed to prove the second main estimate (208) for remainder kernel in the splitting (206). This involves a sum of kernels, each of which according to the identities (217)–(218) has at least one copy of the terms ${}^\omega h^{-1}(x){}^\omega h(y) - I$ and ${}^\omega h(x){}^\omega h^{-1}(y) - I$. Therefore, without loss of generality, we may assume that we are trying to prove the estimate:

$$\int_{\mathbb{R}^n_x} \chi_{\mathcal{D}_\sigma}(x) \, \| \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ {}^\omega G(x,y)\chi(\xi) \, d\xi \, \| \, dx \ \lesssim \ \sigma^{-\gamma} \ . \tag{285}$$

where we have set:

$${}^\omega G(x,y) \ = \ {}^\omega g^{-1}(x) \left({}^\omega h^{-1}(x){}^\omega h(y) - I\right) {}^\omega g(y) \big[\, \bullet \, \big] \ {}^\omega g^{-1}(y){}^\omega g(x) \ .$$

We note here that the corresponding estimates for the other terms in $\mathcal{R}^{TT^*}_\sigma$ are similar and are left to the reader.

To prove (285), we use following angular cutoff functions to split:

$${}^\omega G \ = \ \chi_{|\cos(\theta_{\xi,x-y})| \geqslant |x-y|^{-1+\gamma}} \, {}^\omega G \ + \ \chi_{|\cos(\theta_{\xi,x-y})| < |x-y|^{-1+\gamma}} \, {}^\omega G \ .$$

Therefore, using the triangle and Minkowski inequalities, we see that it suffices to prove the pair of bounds:

(286)

$$\int_{\mathbb{R}^n_x} \chi_{\mathcal{D}_\sigma}(x) \, \| \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \chi_{|\cos(\theta_{\xi,x-y})| \geqslant |x-y|^{-1+\gamma}} {}^\omega G(x,y)\chi(\xi) \, d\xi \, \| \, dx \ \lesssim \ \sigma^{-\gamma} \ ,$$

(287)

$$\int_{\mathbb{R}^n_x} \chi_{\mathcal{D}_\sigma}(x) \, \| \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \chi_{|\cos(\theta_{\xi,x-y})| < |x-y|^{-1+\gamma}} {}^\omega G(x,y)\chi(\xi) \, d\xi \, \| \, dx \ \lesssim \ \sigma^{-\gamma} \ .$$

The proof of the first estimate, (286), is a simple matter of integrating by parts as many times as necessary with respect to the weighted radial derivative $\frac{1}{2\pi i|x-y|\cos(\theta_{\xi,x-y})}\partial_{|\xi|}$, taking account of the fact that ${}^\omega G$ is independent of the variable $|\xi|$. Assuming that $|x-y| \sim \sigma$ is sufficiently large, we will eventually have that:

$$\Big|\big(\frac{1}{2\pi i|x-y|\cos(\theta_{\xi,x-y})}\partial_{|\xi|}\big)^k \chi(\xi)\Big| \ \lesssim \ \sigma^{-n-\gamma} \ , \tag{288}$$

at which point we may stop the integration by parts and put absolute value signs around the remaining integral. The right hand side of (286) will then follow as a direct consequence of (288) and the simple bounds:

$$\int_{\mathbb{R}^n_x} \chi_{\mathcal{D}_\sigma}(x) \, dx \ \lesssim \ \sigma^n \ ,$$

$$\sup_{x,\omega} \, \| \, {}^\omega G(x,y) \, \| \ \lesssim \ 1 \ .$$

To conclude the proof of estimate (285), we need to show the second estimate (287) above. At this point, we have stripped things down to where oscillations under the integral sign are no longer of any use, so we simply strive to estimate the absolute value of the integrand. Here the smallness of the function ${}^\omega G(x,y)$ is essential. To make use of this, we rearrange the order in the absolute integral and use Hölders inequality to bound:

$$\text{(L.H.S.)(287)} \lesssim \int_{\mathbb{S}^{n-1}} \sup_{x\in\mathcal{D}_\sigma} \| {}^\omega G(x,y) \| \, d\omega \cdot \sup_\omega \int_{\mathbb{R}^n_x} \chi_{|\cos(\theta_{\xi,x-y})|<|x-y|^{-1+\gamma}}(x) \ \chi_{\mathcal{D}_\sigma}(x) \ dx \ . \tag{289}$$

To bound the second integral on the right hand side of the above product, we translate by the vector y and then apply a rotation to reduce the bound we wish to show to the following:

$$\int_{|x|\sim\sigma} \chi_{|\cos(\theta_{(1,0),x})|<|x|^{-1+\gamma}}(x) \ dx \ \lesssim \ \sigma^{n-1+\gamma} \ . \tag{290}$$

The validity of (290) follows trivially from the fact that if we split $x = (x_1, x')$, we have the bounds $|x_1| \lesssim \sigma^\gamma$ and $|x'| \lesssim \sigma$ over the range of integration thanks to the angular cutoff and the identity:

$$\cos(\theta_{(1,0),x}) \ = \ \frac{x_1}{|x|} \ .$$

Thus, keeping in mind the bound (290), we see from estimate (289) that the proof of (287) follows from a Cauchy-Schwartz on the sphere $\mathbb{S}^{n-1}$ and the following integrated bounds:

$$\left(\int_{\mathbb{S}^{n-1}} \sup_{x\in\mathcal{D}_\sigma} \| {}^\omega G(x,y) \|^2 \ d\omega \right)^{\frac{1}{2}} \ \lesssim \ \sigma^{1-n-2\gamma} \ . \tag{291}$$

Due to its use in the next section, we will in fact show the following more general set of estimates which includes (291) as a special case:

Proposition 9.4 (Estimates for integrated remainder group elements)**.** *Let the group elements* ${}^\omega h$ *be defined infinitesimally via the equations* (213)–(214) *and the Hodge system* (215)*, where the parameter* $\sigma^{-1+\gamma}$ *is replaced by* M^{-1}*. Then upon integration, one has the following bounds:*

$$\left(\int_{\mathbb{S}^{n-1}} \sup_{|x-y|\sim N} \| {}^\omega h^{-1}(t,x) {}^\omega h(s,y) - I \|^2 \ d\omega \right)^{\frac{1}{2}} \ \lesssim \ \mathcal{E}\,(1 + |t-s| + N) \cdot M^{-n-\delta} \ , \tag{292}$$

$$\left(\int_{\mathbb{S}^{n-1}} \sup_{|x-y|\sim N} \| {}^\omega h(t,x) {}^\omega h^{-1}(s,y) - I \|^2 \ d\omega \right)^{\frac{1}{2}} \ \lesssim \ \mathcal{E}\,(1 + |t-s| + N) \cdot M^{-n-\delta} \ , \tag{293}$$

where $\mathcal{E}$ *is the bootstrapping constant from line* (124d)*. The above estimates are uniform in the value of* M *when it is sufficiently large.*

Proof of the estimates (292)–(293)*.* As will become apparent to the reader, it suffices to show the first bound (292), as the second follows from essentially identical reasoning. Our first step here is to disentangle the products, and to work exclusively with either spatially separated or temporally separated products. This is accomplished via the following simple algebraic identity:

$$
\begin{aligned}
(294)\quad {}^{\omega}h^{-1}(t,x){}^{\omega}h(s,y) - I \ &= \ {}^{\omega}h^{-1}(t,x){}^{\omega}h(s,x) - I \\
&\quad + \ {}^{\omega}h^{-1}(t,x){}^{\omega}h(s,x)\cdot\Big({}^{\omega}h^{-1}(s,x){}^{\omega}h(s,y) - I\Big) \ .
\end{aligned}
$$

Working now, for the moment, with the second term in this last expression substituted into the estimate (292) we expand:

$$
{}^{\omega}h^{-1}(s,x){}^{\omega}h(s,y) - I \ = \ \int_x^y \ {}^{\omega}h^{-1}(s,x)\partial_\ell({}^{\omega}h(s,\ell))\ d\ell \ .
$$

Integrating this last expression in L^2_ω we are reduced to proving the following:

Lemma 9.5. *Let the (spatial) connection $\{\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}}\}$ be defined via the Hodge system:*

$$
(295a)\qquad (\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}})^{df} \ = \ d^*\,\Delta^{-1}\big([\widetilde{{}^{\omega}\underline{C}},\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}}] \ + [\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}},\widetilde{{}^{\omega}\underline{C}}]\ \big)\ ,
$$

$$
(295b)\qquad (\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}})^{cf} \ = \ \widetilde{\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}} \ - \ \nabla_x\Delta^{-1}\Big([\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}},\widetilde{{}^{\omega}\underline{C}}] \ + \ [\widetilde{{}^{\omega}\underline{A}^{(M)}},\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}}]\Big)\ ,
$$

where we have set:

$$
(296)\qquad \widetilde{\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}} \ = \ -\nabla_x\ {}^{\omega}\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}\ {}^{\omega}L\ \Delta^{-1}_{\omega^\perp}\ \underline{A}_{\bullet\ll 1}(\partial_\omega)\ ,
$$

and where the spatial connections $\{\widetilde{{}^{\omega}\underline{A}^{(M)}}\}$ and $\{\widetilde{{}^{\omega}\underline{C}}\}$ are defined on the lines (234) *and* (235) *above. Then one has the following integrated estimate uniform in the parameter M:*

$$
(297)\qquad \Big(\int_{\mathbb{S}^{n-1}}\ \sup_x \|\,\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}}(x)\,\|^2\ d\omega\Big)^{\frac{1}{2}} \ \lesssim \ \mathcal{E}\cdot M^{-n-\delta}\ .
$$

Proof of the estimate (297). Our strategy here is similar to the previous lemmas. We first prove things for the linear term in (295b), and then use the critical embeddings (245) and (243) to transfer things to the connection $\{\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{C}}}}\}$ via the Hodge system (295).

Our first step then is to show that:

$$
(298)\qquad \Big(\int_{\mathbb{S}^{n-1}}\ \sup_x \|\,\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}}(x)\,\|^2\ d\omega\Big)^{\frac{1}{2}} \ \lesssim \ \mathcal{E}\cdot M^{-n-\delta}\ .
$$

In fact, we will show the following somewhat stronger estimate which will easily imply (298), and which is more robust with respect to Hodge systems:

$$
(299)\qquad \Big(\int_{\mathbb{S}^{n-1}}\ \|\,\widetilde{\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}}\,\|^2_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}}\ d\omega\Big)^{\frac{1}{2}} \ \lesssim \ \mathcal{E}\cdot M^{-n-\delta}\ .
$$

This last estimate is a simple matter of using Bernstein's inequality and orthogonality which will net us the factor M^{1-n}, followed by the condition that $\mu^{\frac{1}{2}+\delta}\lesssim M^{-1-\delta}$

at each fixed frequency thanks to the ${}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}$ multiplier which nets us the remaining powers of M^{-1}. The implementation is as follows: We first decompose things into the sum over all frequencies $\mu \lesssim 1$ and angles $\theta \lesssim 1$:

$$\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \ = \ \sum_{\substack{\theta,\mu \ : \\ \mu\lesssim 1}} {}^\omega\Pi_\theta P_\mu \widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \ .$$

Keeping in mind the spatial frequency truncation of $\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}$, and by the square sum definition of the Besov norms, the triangle inequality, and dyadic summing, we see that it suffices to show the following fixed frequency estimate:

$$\Big(\int_{\mathbb{S}^{n-1}} \sup_x \| \, {}^\omega\Pi_\theta P_\mu \widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}(x) \, \|^2 \ d\omega\Big)^{\frac{1}{2}} \ \lesssim \ \mu^\gamma\theta^\gamma \cdot \mathcal{E} \cdot M^{-n-\delta} \ . \tag{300}$$

For each fixed ω, we use Bernstein's inequality and the equivalence:

$${}^\omega\Pi_\theta P_\mu \widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \ \approx \ \theta^{-1}{}^\omega\Pi_\theta {}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} P_\mu\big(\underline{A}_{\,\bullet\ll 1}\big) \ ,$$

to compute that:

$$\begin{aligned}
&\sup_x \ \| \, {}^\omega\Pi_\theta P_\mu \widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}(x) \, \| \ , \\
\lesssim \ & \theta^{-1}\mu\cdot\theta^{\frac{n-1}{2}} \cdot \| \, {}^\omega\Pi_\theta {}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} P_\mu\big(\underline{A}_{\,\bullet\ll 1}\big) \, \|_{\dot{H}_x^{\frac{n-2}{2}}} \ , \\
\lesssim \ & \mu^\gamma\theta^\gamma\cdot M^{-\frac{n+1}{2}-\delta} \, \| \, {}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}} P_\mu\big(\underline{A}_{\,\bullet\ll 1}\big) \, \|_{\dot{H}_x^{\frac{n-2}{2}}} \ .
\end{aligned}$$

Notice that this last line follows from the truncation condition $\mu^{1-2\delta} \lesssim \theta^2$ as well as the small constant bounds $\gamma \ll \delta$. The proof of (300) is now a result of the following simple calculation involving Plancherel:

$$\begin{aligned}
&\Big(\int_{\mathbb{S}^{n-1}} \| \, {}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}} P_\mu\big(\underline{A}_{\,\bullet\ll 1}\big) \, \|^2_{\dot{H}_x^{\frac{n-2}{2}}} \ d\omega\Big)^{\frac{1}{2}} \ , \\
\lesssim \ & \Big(\int_{\mathbb{R}^n_\xi}\int_{\mathbb{S}^{n-1}} \| \, (b^\omega_{\bullet\leqslant M^{-1}} + b^{-\omega}_{\bullet\leqslant M^{-1}}) \, |\xi|^{\frac{n-2}{2}} p_\mu \widehat{\underline{A}_{\,\bullet\ll 1}}(\xi) \, \|^2 \ d\omega d\xi\Big)^{\frac{1}{2}} \ , \\
\lesssim \ & M^{-\frac{n-1}{2}} \, \| \, P_\mu\big(\underline{A}_{\,\bullet\ll 1}\big) \, \|_{\dot{H}_x^{\frac{n-2}{2}}} \ , \\
\lesssim \ & \mathcal{E}\cdot M^{-\frac{n-1}{2}} \ .
\end{aligned} \tag{301}$$

To finish the proof of (297), we simply need to pass the estimate (299) onto the set of spatial potentials $\{\widetilde{\widetilde{{}^\omega\underline{C}}}\}$. To do this, we set up auxiliary spaces $L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})$ and $L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})$. From the estimates (245) and (243) we immediately have that:

$$\| \widetilde{{}^{\omega}\underline{A}^{(M)}} \|_{L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})} \lesssim \mathcal{E} \ , \tag{302}$$

$$\| \widetilde{{}^{\omega}\underline{C}} \|_{L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})} \lesssim \mathcal{E} \ , \tag{303}$$

where the index p_γ is the exponent from the line (188) above. The desired result now follows from the bilinear estimate:

$$\nabla_x \Delta^{-1} \ : \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \cdot L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \ . \tag{304}$$

This is a simple consequence of the condition $p_\gamma < n$ which allows us to fulfill the condition (43) of the general embedding (41). The result follows from integrating this bound in L^2_ω. $\square$

We now turn our attention to proving the bound (292) for the temporally separated product which is the first term on the right hand side of equation (294) above. Expand the integrand here an the derivative of another integral over time line, we have that:

$$ {}^{\omega}h^{-1}(t,x){}^{\omega}h(s,x) - I \ = \ \int_s^t \ {}^{\omega}h^{-1}(t,x)\partial_t({}^{\omega}h(\ell,x)) \ d\ell \ .$$

After integrating in L^2_ω the right hand side of this last expression, we see that we are reduced to proving that:

Lemma 9.6. *Let the quantity* $\widetilde{\widetilde{{}^{\omega}C}}_0$ *be defined implicitly via the elliptic equation:*

$$\begin{aligned} \widetilde{\widetilde{{}^{\omega}C}}_0 \ = \ \widetilde{\widetilde{{}^{\omega}A^{(M)}_0}} \ - \ \nabla_t\Delta^{-1}\Big([\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}, \widetilde{{}^{\omega}\underline{C}}] + [\widetilde{{}^{\omega}\underline{A}^{(M)}}, \widetilde{\widetilde{{}^{\omega}\underline{C}}}] \Big) \\ - \ d^*\Delta^{-1}\Big([\widetilde{\widetilde{{}^{\omega}C}}_0, \widetilde{{}^{\omega}\underline{C}}] + \ [\widetilde{{}^{\omega}C}_0, \widetilde{\widetilde{{}^{\omega}\underline{C}}}] \Big) \ , \end{aligned} \tag{305}$$

where we have set:

$$\widetilde{\widetilde{{}^{\omega}A^{(M)}_0}} \ = \ -\nabla_t \ {}^{\omega}\overline{\Pi}_{\bullet\leqslant M^{-1}} {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)} \ {}^{\omega}L \ \Delta^{-1}_{\omega^\perp} \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \ , \tag{306}$$

and where the connections $\{\widetilde{{}^{\omega}\underline{A}^{(M)}}\}$ *and* $\{\widetilde{{}^{\omega}\underline{C}}\}$ *are as in Lemma 9.5, and where the quantity* $\widetilde{{}^{\omega}A^{(M)}_0}$ *is defined on line* (262) *above. Then one has the following integrated estimate uniform in the parameter* M*:*

$$\Big(\int_{\mathbb{S}^{n-1}} \sup_x \| \widetilde{\widetilde{{}^{\omega}C}}_0(x) \|^2 \ d\omega \Big)^{\frac{1}{2}} \ \lesssim \ \mathcal{E} \cdot M^{-n-\delta} \ . \tag{307}$$

Proof of the estimate (307). As in the proof of the previous Lemma, our goal here is to first prove the $L^2_\omega(\dot{B}^{\infty,(n,\frac{n}{2}-\gamma)}_{2,10n})$ improvement of this claim for the terms on the right hand side of the equation (305) which do not involve the variable $\widetilde{\widetilde{{}^{\omega}C}}_0$. The desired bound can then be achieved via iteration or bootstrapping using the bilinear estimate (304) and the estimate (303) to deal with the term involving $\widetilde{\widetilde{{}^{\omega}C}}_0$

on the right hand side of (305). Therefore, we are trying to show the following three estimates:

$$\left(\int_{\mathbb{S}^{n-1}} \| \widetilde{\widetilde{\widetilde{{}^\omega A_0^{(M)}}}} \|^2_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}} \, d\omega\right)^{\frac{1}{2}} \lesssim \mathcal{E}\cdot M^{-n-\delta} \ , \tag{308}$$

$$\left(\int_{\mathbb{S}^{n-1}} \| d^*\Delta^{-1} \, [\widetilde{{}^\omega C}_0, \widetilde{\widetilde{{}^\omega \underline{C}}}] \|^2_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}} \, d\omega\right)^{\frac{1}{2}} \lesssim \mathcal{E}\cdot M^{-n-\delta} \ , \tag{309}$$

(310)

$$\left(\int_{\mathbb{S}^{n-1}} \| \frac{\nabla_t}{\Delta}\Big(\widetilde{\widetilde{[{}^\omega \underline{A}^{(M)}, \widetilde{{}^\omega \underline{C}}]}} + [\widetilde{{}^\omega \underline{A}^{(M)}}, \widetilde{\widetilde{{}^\omega \underline{C}}}]\Big) \|^2_{\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}} \, d\omega\right)^{\frac{1}{2}} \lesssim \mathcal{E}\cdot M^{-n-\delta} \ .$$

The proof of the first estimate (308) is essentially identical to that of (299) above, once one takes into account the truncation condition (124c). The proof of the second estimate (309) follows from the $L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})$ bound proved for the potentials $\{\widetilde{\widetilde{{}^\omega \underline{C}}}\}$ proved in the previous lemma, the bilinear estimate (304), and the following:

$$\| \widetilde{{}^\omega C}_0 \|_{L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})} \lesssim \mathcal{E} \ ,$$

which is a direct consequence of (275) above.

Therefore, it remains for us to prove the last bound (310). Unfortunately, this does not follow directly from the procedure we have been using so far. The trouble is that the time derivative ∇_t will in general not cancel with the Laplacean, and it is not possible to prove a bilinear estimate which is morally of the form $\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_2\cdot\dot{B}^{\infty,(2,\frac{n}{2})}_2 \subseteq \Delta\dot{B}^{\infty,(2,\frac{n}{2})}_2$ due to bad $High\times High$ frequency interactions in dimension $n=6$. The only way around this seems to be to do something which is quite a bit more involved. The way we will prove (310) is in a series of steps designed to reduce things to a term which, in some sense, represents the central difficulty. This last term will be dealt with using a scale of non-isotropic spaces which are similar to the ones employed in the proof of Lemma 9.2 above. The argument we will present here is largely ad-hoc, and there are many variations. We will proceed by proving certain estimates which may be cut out at this stage of the overall paper, but will turn out to be useful in the sequel.

The first step we make here is to recall that, although we have been suppressing it, there is additional polarity information in the definition of the connections $d+{}^\omega A$ (see (192)). This comes from the choice of null vector-field ${}^\omega L^{\pm}$. For convenience, we will use here an implicitly defined notation which we call ${}^\omega \underline{L}$, to denote the *opposite* vector-field for any given choice of polarization. That is, we always have the formula:

$$\Box \ = \ {}^\omega\underline{L}{}^\omega L + \Delta_{\omega^\perp} \ . \tag{311}$$

Now, for a given choice of polarization, we can always write $\pm\nabla_t = {}^\omega L^{\pm} \mp \omega\cdot\nabla_x$. Therefore, modulo proving estimates which are identical to those Lemma 9.5 above, and distributing the ${}^\omega\underline{L}$ derivative, we that the proof of (310) can be reduced to the proof of the following three bilinear estimates:

$$\| \Delta^{-1}[{}^\omega\underline{L}\widetilde{{}^\omega\underline{A}^{(M)}}, \widetilde{\widetilde{{}^\omega\underline{C}}}] \|_{L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \lesssim \mathcal{E}\cdot M^{-n-\delta} , \tag{312}$$

$$\| \Delta^{-1}[\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}, {}^\omega\underline{L}\widetilde{{}^\omega\underline{C}}] \|_{L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \lesssim \mathcal{E}\cdot M^{-n-\delta} , \tag{313}$$

$$\| \Delta^{-1}\Big([{}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}, \widetilde{{}^\omega\underline{C}}] + [\widetilde{{}^\omega\underline{A}^{(M)}}, {}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{C}}}]\Big) \|_{L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \lesssim \mathcal{E}\cdot M^{-n-\delta} . \tag{314}$$

Our first step is to prove the estimates (312)–(313). To do this, we introduce the auxiliary index:

$$r_\gamma \ = \ \frac{2n(n-1)}{(n-2)(n+1)-3\gamma n} . \tag{315}$$

We now show that one has the following improvements over the estimates (246), (244):

$$\| {}^\omega\underline{L}\widetilde{{}^\omega\underline{A}^{(M)}} \|_{L^\infty_\omega(\dot{B}^{r_\gamma,(2,\frac{n-4}{2})}_{2,10n})} \lesssim \mathcal{E} , \tag{316}$$

$$\| {}^\omega\underline{L}\widetilde{{}^\omega\underline{C}} \|_{L^\infty_\omega(\dot{B}^{r_\gamma,(2,\frac{n-4}{2})}_{2,10n})} \lesssim \mathcal{E} . \tag{317}$$

With the help of (316)–(317),the proof of the estimates (312)–(313) follows from the $L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})$ proved in the previous Lemma, and the following bilinear embedding:

$$\Delta^{-1} \ : \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})\cdot L^\infty_\omega(\dot{B}^{r_\gamma,(2,\frac{n-4}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) . \tag{318}$$

Notice that the validity of this last estimate follows from the condition (43), because for $0<\gamma\ll 1$ we have the index bounds:

$$2+\gamma \ < \ \frac{n}{r_\gamma} ,$$

which follows easily from the definitions of r_γ. The proof of (316)–(317) we proceed by first showing (316), and then using the Hodge system (234) to show that (316) $\Rightarrow$ (317).

We are now trying to show (316). We use the identity (311) and the definition (235) and the structure equation (124e) to compute that:

$$\begin{aligned} {}^\omega\underline{L}\widetilde{{}^\omega\underline{A}^{(M)}} \ =& \ \nabla_x\, {}^\omega\overline{\Pi}_{M^{-1}<\bullet}\, {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \underline{A}_{\bullet\ll 1}(\partial_\omega) \\ & - \ \nabla_x\, {}^\omega\overline{\Pi}_{M^{-1}<\bullet}\, {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega) . \end{aligned} \tag{319}$$

The estimate (316) for the first term on the right hand side of this last expression is a trivial consequence the $\dot{H}^{\frac{n-4}{2}}$ bound (at fixed time) for that term which is provided through the energy type norm contained in the bootstrapping assumption (124d), and the Besov nesting (38). Therefore, we strive to bound the second term on the right hand side of (319) above. To do this, we first decompose things in to a sum over all possible angles spread from the ω direction and write:

$$\nabla_x\ {}^\omega\overline{\Pi}_{M^{-1}<\bullet}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)$$
$$= \sum_\theta \nabla_x\ {}^\omega\Pi_\theta\ {}^\omega\overline{\Pi}_{M^{-1}<\bullet}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)\ .$$

For each angularly localized piece in this last expression, we may make use of the Coulomb savings (191) to show the following heuristic multiplier bound (again making use of our convention explained below (54) above):

$$\nabla_x\ {}^\omega\Pi_\theta\ {}^\omega\overline{\Pi}_{M^{-1}<\bullet}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,P_\mu\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)\ \approx\ (\mu\theta)^{-1}\ {}^\omega\Pi_\theta\ P_\mu P_{\bullet\ll 1}\big([B,H]\big)\ .$$

Therefore, dropping the small frequency multiplier, our goal is to show the following fixed angle estimate:

$$\|\,{}^\omega\Pi_\theta|D_x|^{-1}\big([B,H]\big)\,\|_{\dot{B}_2^{r_\gamma,(2,\frac{n-4}{2})}}\ \lesssim\ \theta^{1+\gamma}\cdot\mathcal{E}\ .$$

Using Bernstein's inequality on each fixed dyadic block in the Besov nesting (38), and making use of a small numerical calculation which we leave to the reader, one finds that this last estimate is a consequence of the following non-localized Besov space estimate:

$$\|\,|D_x|^{-1}\big([B,H]\big)\,\|_{\dot{B}_2^{\frac{2n}{n+2-\gamma},(2,\frac{n-4}{2})}}\ \lesssim\ \mathcal{E}\ .$$

This last bound is now a consequence of the bootstrapping structure estimate (124f) and the following bilinear embedding:

$$|D_x|^{-1}\ :\ \dot{B}_2^{2,(2,\frac{n-2}{2})}\cdot\dot{B}_2^{2,(2,\frac{n-4}{2})}\ \hookrightarrow\ \dot{B}_2^{\frac{2n}{n+2-\gamma},(2,\frac{n-4}{2})}\ .$$

Notice that the reason we are forced to work with the relatively high space $L^{\frac{2n}{n+2-\gamma}}$ is because of $Low\times High$ frequency interactions. This is why we are forced to work in the less aesthetic space L^{r_γ} above instead of L^2. This completes the proof of (316).

Our next step is to establish the implication (316) $\Rightarrow$ (317). This follows immediately from differentiation of the Hodge system (234) with respect to the ${}^\omega\underline{L}$ vector-field, and then using the following bilinear estimate to bootstrap:

$$\nabla_x\Delta^{-1}\ :\ \dot{B}_{2,10n}^{r_\gamma,(2,\frac{n-4}{2})}\cdot\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-6}{2})}\ \hookrightarrow\ \dot{B}_{2,10n}^{r_\gamma,(2,\frac{n-4}{2})}\ .$$

We leave it to the reader to check that the various conditions of estimate (41) are satisfied in this case.

It remains for us to show the bound (314). To make this a bit easier, we employ the skew symmetry of the Lie brackets in that expression to write it as:

$$\Delta^{-1}\Big([{}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}},\widetilde{{}^\omega\underline{C}}]+[\widetilde{{}^\omega\underline{A}^{(M)}},{}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{C}}}]\Big)$$
$$=\ \Delta^{-1}\Big([{}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}},\widetilde{{}^\omega\underline{C}}-\widetilde{{}^\omega\underline{A}^{(M)}}]+[\widetilde{{}^\omega\underline{A}^{(M)}},{}^\omega\underline{L}(\widetilde{\widetilde{{}^\omega\underline{C}}}-\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}})]\Big)\ .$$

From this we see that the proof of (314) will follow once we can establish the three separate estimates:

$$\| \Delta^{-1}\Big([{}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}, \widetilde{{}^{\omega}\underline{C}} - \widetilde{{}^{\omega}\underline{A}^{(M)}}] \Big) \|_{L^2_{\omega}(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \ \lesssim\ \mathcal{E}\cdot M^{-n-\delta} \ , \tag{320}$$

$$\| \Delta^{-1}\Big([\widetilde{{}^{\omega}\underline{A}^{(M)}}, {}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{C}}}] \Big) \|_{L^2_{\omega}(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \ \lesssim\ \mathcal{E}\cdot M^{-n-\delta} \ , \tag{321}$$

$$\| \Delta^{-1}\Big([\widetilde{{}^{\omega}\underline{A}^{(M)}}, {}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}] \Big) \|_{L^2_{\omega}(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n})} \ \lesssim\ \mathcal{E}\cdot M^{-n-\delta} \ . \tag{322}$$

To prove the first estimate (320) above, we make use of the fact that $\widetilde{{}^{\omega}\underline{C}} - \widetilde{{}^{\omega}\underline{A}^{(M)}}$ obeys a better bound than either term in that expression does individually:

$$\| \widetilde{{}^{\omega}\underline{C}} - \widetilde{{}^{\omega}\underline{A}^{(M)}} \|_{\dot{B}^{s_\gamma,(2,\frac{n-2}{2})}_{2,10n}} \ \lesssim\ \mathcal{E} \ , \tag{323}$$

where we have set the index s_γ to be:

$$s_\gamma \ = \ \frac{np_\gamma}{n+p_\gamma} + \gamma \ .$$

The proof of (323) follows immediately from the entirely quadratic structure of the terms in the expression $\widetilde{{}^{\omega}\underline{C}} - \widetilde{{}^{\omega}\underline{A}^{(M)}}$, in conjunction with following bilinear estimate whose proof is a simple consequence of the definition of the p_γ indices, (188), and the general embedding (41):

$$\nabla_x \Delta^{-1} \ : \ \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \cdot \dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ \hookrightarrow \ \dot{B}^{s_\gamma,(2,\frac{n-2}{2})}_{2,10n} \ .$$

Furthermore, by taking the ${}^{\omega}\underline{L}$ derivative of the potentials in the estimate (300), and making use of the truncation condition (124c), we easily have the following:

$$\| {}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}} \|_{\dot{B}^{\infty,(2,\frac{n-2}{2}-\gamma)}_{2,10n}} \ \lesssim\ \mathcal{E}\cdot M^{-n-\delta} \ . \tag{324}$$

The proof of (320) now follows from combining estimates (323) and (324) in to the following bilinear embedding whose validity follows easily from (41) and the condition $2+\gamma < \frac{n}{s_\gamma}$ (say for $n=6$ or higher):

$$\Delta^{-1} \ : \ L^2_{\omega}(\dot{B}^{\infty,(2,\frac{n-2}{2}-\gamma)}_{2,10n}) \cdot L^{\infty}_{\omega}(\dot{B}^{s_\gamma,(2,\frac{n-2}{2})}_{2,10n}) \ \hookrightarrow \ L^2_{\omega}(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \ . \tag{325}$$

We have now come to the point where the current techniques reach an impasse. Notice that while the terms ${}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{C}}}$ and ${}^{\omega}\underline{L}\,\widetilde{\widetilde{{}^{\omega}\underline{A}^{(M)}}}$ do seem to have a better structure at first glance via the equation (319), it is surprisingly difficult to pass this into integrated estimates of the form (300). This is because while the linear term on the right hand side of (319) is quite nice, the only saving grace of the quadratic term in that expression is that it can go in lower spatial L^p space, which is not particularly useful when half of the needed savings in the estimate (300) comes form orthogonality (meaning that anything below L^2 gets wasted). A way to get rid of this problem is to employ non-isotropic spaces. Specifically, we define the norm:

$$\| A \|_{{}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}} \ = \ \sum_{\mu} \ \mu^{-\frac{1}{2}-\gamma}(1+\mu)^{10n} \ \| P_\mu(A) \|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})} \ .$$

Our goal is now to show the following estimate which represents a more manageable form of the differentiated version of (299):

$$\| {}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \|_{L^2_\omega({}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n})} \ \lesssim \ \mathcal{E}\cdot M^{-n-\delta} \ . \tag{326}$$

Having done this, our next goal will be to pass an estimates of this form on to the non-linear potential $\widetilde{\widetilde{{}^\omega\underline{C}}}$. For reasons which will become apparent in a moment, it is more convenient to state this estimate for the following sum of spaces:

$$\| {}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{C}}} \|_{L^2_\omega({}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n})+L^2_\omega(\dot{B}^{n,(2,\frac{n-2}{2}-\gamma)}_{2,10n})} \ \lesssim \ \mathcal{E}\cdot M^{-n-\delta} \ . \tag{327}$$

Once this is accomplished, the proof of (321)–(321) will follow from the two bilinear estimates:

$$\Delta^{-1} \ : \ L^\infty_\omega\big(P_{\bullet\ll 1}\Delta^{-\frac{1}{2}}_{\omega^\perp}\dot{H}^{\frac{n-4}{2}}\big)\cdot L^2_\omega({}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \ , \tag{328}$$

$$\Delta^{-1} \ : \ L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\cdot L^2_\omega(\dot{B}^{n,(2,\frac{n-4}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \ . \tag{329}$$

Here the space in the first term in the product on the left hand side of (328) above is given by the norm:

$$\| A \|_{\Delta^{-\frac{1}{2}}_{\omega^\perp}\dot{H}^{\frac{n-4}{2}}} \ = \ \| \Delta^{\frac{1}{2}}_{\omega^\perp} A \|_{\dot{H}^{\frac{n-4}{2}}} \ .$$

That the set of potentials $\{\widetilde{{}^\omega\underline{A}^{(M)}}\}$ is in this space with norm $\lesssim \mathcal{E}$ follows from the explicit formula (235) and the Coulomb gauge savings (191). Having now outlined the general strategy, we move to the proofs of the individual estimates.

To prove (327) we use the spatial truncation condition (156), the triangle inequality, and dyadic summing to reduce things to the following single frequency estimate:

$$\left(\int_{\mathbb{S}^{n-1}} \| P_\mu({}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}) \|^2_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})} \ d\omega\right)^{\frac{1}{2}} \ \lesssim \ \mu^{\frac{1}{2}+2\gamma} \ \mathcal{E}\cdot M^{-n-\delta} \ . \tag{330}$$

Now freeze ω and run a Littlewood-Paley decomposition in the $\mathbb{R}^{n-1}_{\omega^\perp}$ frequency plane:

$$\begin{aligned} P_\mu({}^\omega\underline{L}\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}) \ &= \ \sum_{\substack{\lambda\ :\\ \lambda\lesssim M^{-1}\mu}} \nabla_x {}^\omega\underline{L}{}^\omega L\Delta^{-1}_{\omega^\perp}{}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}Q_\lambda P_\mu\big(\underline{A}_{\bullet\ll 1}\big)(\partial_\omega) \ , \\ &\approx \ M^{-1}\sum_{\substack{\lambda\ :\\ \lambda\lesssim M^{-1}\mu}} \mu^3\lambda^{-2} \ {}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}Q_\lambda P_\mu\big(\underline{A}_{\bullet\ll 1}\big) \ , \end{aligned} \tag{331}$$

where the last line follows from the truncation condition (124c) and our heuristic multiplier convention. Notice that the sum restriction in these formulas comes because of the presence of the cutoff ${}^\omega\overline{\Pi}_{\bullet\leqslant M^{-1}}$. The extra M^{-1} factor comes from this same angular cutoff and the Coulomb gauge savings (191). Working now with the right hand side of (331), we use Bernstein's inequality and dyadic summing to compute that:

$$
\begin{aligned}
&\| P_\mu \big({}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \big) \|_{L^2_{\omega||}(L^\infty_{\omega\perp})} \ , \\
&\lesssim M^{-1} \sum_{\substack{\lambda\ : \\ \lambda \lesssim\ M^{-1}\mu}} \mu^3 \lambda^{-2} \, \| Q_\lambda {}^\omega\overline{\Pi}_{\bullet \leqslant M^{-1}} {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} P_\mu \big(\underline{A}_{\,\bullet \ll 1} \big) \|_{L^2_{\omega||}(L^\infty_{\omega\perp})} \ , \\
&\lesssim M^{-\frac{n-3}{2}} \mu^{\frac{3}{2}} \ \| {}^\omega\overline{\Pi}_{\bullet \leqslant M^{-1}} {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \ P_\mu \big(\underline{A}_{\,\bullet \ll 1} \big) \|_{\dot{H}_x^{\frac{n-2}{2}}} \ , \\
&\lesssim M^{-\frac{n+1}{2}-\delta} \mu^{\frac{1}{2}+2\gamma} \ \| {}^\omega\overline{\Pi}_{\bullet \leqslant M^{-1}} {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)} \ \underline{A}_{\,\bullet \ll 1} \|_{\dot{H}_x^{\frac{n-2}{2}}} \ .
\end{aligned}
$$

Integrating now this last line in L^2_ω, and using the orthogonality computation which began on line (301) above we have achieved (330) as was to be shown.

Our goal is now to pass the estimate (326) on to potentials $\{{}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{C}}}\}$ modulo terms which are in the more regular space $L^2_\omega(\dot{B}^{n,(2,\frac{n-2}{2}-\gamma)}_{2,10n})$. To do this, we differentiate the system (295) with respect to the vector-field ${}^\omega\underline{L}$, and write it heuristically as:

$$
\begin{aligned}
{}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{C}}} \ = \ & {}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}} \ + \ \nabla_x \Delta^{-1} \Big([{}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}, \widetilde{{}^\omega\underline{C}}] + [\widetilde{\widetilde{{}^\omega\underline{A}^{(M)}}}, {}^\omega\underline{L}\,\widetilde{{}^\omega\underline{C}}] \\
& + \ [{}^\omega\underline{L}\,\widetilde{{}^\omega\underline{A}^{(M)}}, \widetilde{\widetilde{{}^\omega\underline{C}}}] + [\widetilde{{}^\omega\underline{A}^{(M)}}, {}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{C}}}] \ + \ [{}^\omega\underline{L}\,\widetilde{{}^\omega\underline{C}}, \widetilde{\widetilde{{}^\omega\underline{C}}}] \ + \ [\widetilde{{}^\omega\underline{C}}, {}^\omega\underline{L}\,\widetilde{\widetilde{{}^\omega\underline{C}}}] \Big) \ .
\end{aligned}
$$

Therefore, the desired bound will follow from a bootstrapping argument the estimates (326), (302)–(303), and (316)–(317) with the help of the following three bilinear estimates:

$$
\nabla_x \Delta^{-1} \ : \ L^2_\omega(\dot{B}^{\infty,(2,\frac{n}{2}-\gamma)}_{2,10n}) \cdot L^\infty_\omega(\dot{B}^{r_\gamma,(2,\frac{n-4}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{n,(2,\frac{n-4}{2})}_{2,10n}) \ , \tag{332}
$$

$$
\nabla_x \Delta^{-1} \ : \ L^2_\omega(\dot{B}^{n,(2,\frac{n-2}{2}-\gamma)}_{2,10n}) \cdot L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega(\dot{B}^{n,(2,\frac{n-4}{2})}_{2,10n}) \ , \tag{333}
$$

$$
\nabla_x \Delta^{-1} \ : \ L^2_\omega({}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}) \cdot L^\infty_\omega(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}) \ \hookrightarrow \ L^2_\omega({}^\omega\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}) \ . \tag{334}
$$

The estimates (332)–(333) are again an integrated form of the general Besov embedding (41), and we leave it to the reader to check that the indices p_γ, r_γ are in the right range to satisfy the conditions (43)–(46). It remains for us to prove the inclusion (334). We do this now. Let A and C be two test matrices. By performing a trichotomy, we see that it suffices to prove the following three frequency localized summation estimates for fixed values of ω:

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\ll\mu_2\\ \lambda\sim\mu_2}} \lambda^{-\frac{1}{2}-\gamma}(1+\lambda)^{10n}\ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\ \lesssim \tag{335}$$
$$\|A\|_{{}^{\omega}\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}}\cdot\|C\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\ ,$$

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_2\ll\mu_1\\ \lambda\sim\mu_1}} \lambda^{-\frac{1}{2}-\gamma}(1+\lambda)^{10n}\ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\ \lesssim \tag{336}$$
$$\|A\|_{{}^{\omega}\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}}\cdot\|C\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\ ,$$

$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\sim\mu_2\\ \lambda\lesssim\mu_1,\mu_2}} \lambda^{-\frac{1}{2}-\gamma}(1+\lambda)^{10n}\ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\ \lesssim \tag{337}$$
$$\|A\|_{{}^{\omega}\mathcal{N}^{-\frac{1}{2}-\gamma,2,\infty}_{1,10n}}\cdot\|C\|_{\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n}}\ .$$

The proof of (335)–(337) is essentially identical to the proof of the three estimates (254)–(256) we have shown earlier, although the proof of the last estimate (337) requires a slightly more delicate argument due to the presence of additional low frequency weights. We leave (335)–(336) to the reader. To show the last estimate above, we follow the proof of (256) which begins on line (258), although we do so without throwing away the P_λ multiplier so soon. This leaves us with the fixed frequency estimate, which we expand out into all frequencies in the $\omega^{||}$ variable, calling the appropriate multipliers $\widetilde{Q}_\sigma$:

$$\begin{aligned}
&\lambda^{-\frac{1}{2}-\gamma}\ \|\nabla_x\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^2_\ell(L^\infty_{\ell^\perp})}\ ,\\
\lesssim\ &\lambda^{\frac{n-1}{p_\gamma}-\frac{3}{2}-\gamma}\ \|P_\lambda(P_{\mu_1}A\cdot P_{\mu_2}C)\|_{L^2_\ell(L^{p_\gamma}_{\ell^\perp})}\ ,\\
\lesssim\ &\lambda^{\frac{n-1}{p_\gamma}-\frac{3}{2}-\gamma}\ \sum_{\sigma\lesssim\lambda}\ \|\widetilde{Q}_\sigma(P_{\mu_1}A\cdot P_{\mu_2}C)\|_{L^2_\ell(L^{p_\gamma}_{\ell^\perp})}\ .
\end{aligned}\tag{338}$$

Now we use the fact that the multiplier $\widetilde{Q}_\sigma$ only acts in the $\omega^{||}$ variable. In that variable its action can be written in terms of a kernel $K^{\widetilde{Q}_\sigma}$ which has uniform $L^1_{\omega^{||}}$ norm (in terms of σ) and has amplitude $\sim\sigma$. Therefore, via Young's and then Hölder's inequality, and a little dyadic summing, this allows us to bound:

$$\begin{aligned}
\text{(L.H.S.)(338)}\ \lesssim\ &\lambda^{\frac{n-1}{p_\gamma}-\frac{3}{2}-\gamma}\ \sum_{\sigma\lesssim\lambda}\ \|\left(|K^{\widetilde{Q}_\sigma}| * \|(P_{\mu_1}A\cdot P_{\mu_2}C)\|_{L^{p_\gamma}_{\omega^\perp}}\right)\|_{L^2_{\omega^{||}}}\ ,\\
\lesssim\ &\lambda^{\frac{n-1}{p_\gamma}-\frac{3}{2}-\gamma}\ \sum_{\sigma\lesssim\lambda}\ \sigma^{\frac{1}{p_\gamma}}\ \|P_{\mu_1}A\cdot P_{\mu_2}C\|_{L^{\frac{2p_\gamma}{2+p_\gamma}}_{\omega^{||}}(L^{p_\gamma}_{\omega^\perp})}\ ,\\
\lesssim\ &\lambda^{\frac{n}{p_\gamma}-\frac{3}{2}-\gamma}\ \|P_{\mu_1}A\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\cdot\|P_{\mu_2}C\|_{L^{p_\gamma}}\ ,\\
\lesssim\ &\left(\frac{\lambda}{\mu_1}\right)^{\frac{n}{p_\gamma}-\frac{3}{2}-\gamma}\ \mu_1^{-\frac{1}{2}-\gamma}\|P_{\mu_1}A\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\cdot\mu_2^{\frac{n}{p_\gamma}-1}\|P_{\mu_2}C\|_{L^{p_\gamma}}\ .
\end{aligned}$$

This last line and the condition $0<\frac{n}{p_\gamma}-\frac{3}{2}-\gamma$ allows us to safely make the sum on the left hand side of (337) and then proceed via Cauchy-Schwartz to arrive at the desired bound. This completes our proof of the bilinear estimate (334).

The last thing we need to do here is to prove the two final estimates (328) and (329). The second of these is of course simply an integrated version of the general estimate (41). Therefore we concentrate on proving the first. To do this, we proceed as we did in the proof of estimate (334) and run a trichotomy on a product of test matrices $A \cdot B$. This leaves us with establishing the three estimates (forgetting about the extra high frequency weights which are not central):

(339)
$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\ll\mu_2\\ \lambda\sim\mu_2}} \lambda^{-\gamma}\, \| \Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^\infty} \ \lesssim\ \|A\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot{H}^{\frac{n-4}{2}}}\cdot \|C\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \ ,$$

(340)
$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_2\ll\mu_1\\ \lambda\sim\mu_1}} \lambda^{-\gamma}\, \| \Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^\infty} \ \lesssim\ \|A\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot{H}^{\frac{n-4}{2}}}\cdot \|C\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \ ,$$

(341)
$$\sum_{\substack{\lambda,\mu_i\ :\\ \mu_1\sim\mu_2\\ \lambda\lesssim\mu_1,\mu_2}} \lambda^{-\gamma}\, \| \Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^\infty} \ \lesssim\ \|A\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot{H}^{\frac{n-4}{2}}}\cdot \|C\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \ .$$

The proofs of the two $Low\times High$ interaction estimates, (339)–(340), are both similar and very simple. They follow from the pair of L^∞ estimates:

$$\|P_{\mu_1}(A)\|_{L^\infty}\ \lesssim\ \mu_1\ \|P_{\mu_1}(A)\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot{H}^{\frac{n-4}{2}}} \ , \tag{342}$$

$$\|P_{\mu_2}(C)\|_{L^\infty}\ \lesssim\ \mu_2^{1+\gamma}\ \|P_{\mu_2}(C)\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \ . \tag{343}$$

The proof of (342) follows easily from the kind of angular decomposition and Bernstein inequality tricks used to prove estimate (187) above. To prove the second estimate (343), we let $\widetilde{Q}_\sigma$ again denote a family of frequency cutoffs in the $\mathbb{R}_{\omega^{||}}$ variable and we compute via Bernstein:

$$\begin{aligned}
\|P_{\mu_2}(C)\|_{L^\infty}\ &\lesssim\ \sum_{\sigma\lesssim\mu_1}\ \|\widetilde{Q}_\sigma P_{\mu_2}(C)\|_{L^\infty_{\omega^\perp}(L^\infty_{\omega^{||}})} \ ,\\
&\lesssim\ \sum_{\sigma\lesssim\mu_1}\ \sigma^{\frac{1}{2}}\ \|P_{\mu_2}(C)\|_{L^\infty_{\omega^\perp}(L^2_{\omega^{||}})} \ ,\\
&\lesssim\ \mu_2^{1+\gamma}\ \|P_{\mu_2}(C)\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \ .
\end{aligned}$$

Using now (342)–(343) we have the pair of fixed frequency bounds:

$$\begin{aligned}
&\lambda^{-\gamma}\ \|\Delta^{-1}P_\lambda\left(P_{\mu_1}A\cdot P_{\mu_2}C\right)\|_{L^\infty}\\
\lesssim\ &\left(\frac{\mu_1}{\mu_2}\right)\ \|P_{\mu_1}(A)\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot{H}^{\frac{n-4}{2}}}\cdot\|P_{\mu_2}(C)\|_{{}^{\omega}\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}} \qquad \mu_1\ \ll\ \mu_2 \ ,
\end{aligned}$$

and:

$$\lambda^{-\gamma}\,\|\,\Delta^{-1}P_\lambda\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^\infty}$$
$$\lesssim\ \Big(\frac{\mu_2}{\mu_1}\Big)^{1+\gamma}\ \|\,P_{\mu_1}(A)\,\|_{\Delta_{\omega^\perp}^{-\frac{1}{2}}\dot H^{\frac{n-4}{2}}}\cdot\|\,P_{\mu_2}(C)\,\|_{{}^\omega\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}}\qquad \mu_2\ \ll\ \mu_1\ .$$

These may easily be summed over the respective ranges on the left hand side of (339)–(340) to yield the desired bounds.

Our final task here is to establish the $High\times High$ frequency interaction estimate (341). This is where the non-isotropic spaces really shine. In what follows, we let Q_{σ_1} denote a frequency cutoff in the $\mathbb{R}^{n-1}_{\omega^\perp}$ frequency plane, and $\widetilde{Q}_{\sigma_2}$ a cutoff in the orthogonal direction. We compute that:

$$\begin{aligned}
&\lambda^{-\gamma}\,\|\,\Delta^{-1}P_\lambda\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^\infty}\ ,\\
\lesssim\ &\lambda^{-2-\gamma}\sum_{\substack{\sigma_i\,:\\ \sigma_i\lesssim\lambda}}\|\,\widetilde{Q}_{\sigma_2}Q_{\sigma_1}\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^\infty_{\omega^\perp}(L^\infty_{\omega^{||}})}\ ,\\
\lesssim\ &\lambda^{-1-\gamma}\sum_{\substack{\sigma_1\,:\\ \sigma_1\lesssim\lambda}}\|\,Q_{\sigma_1}\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^\infty_{\omega^\perp}(L^1_{\omega^{||}})}\ ,\\
\lesssim\ &\lambda^{-1-\gamma}\sum_{\substack{\sigma_1\,:\\ \sigma_1\lesssim\lambda}}\|\,Q_{\sigma_1}\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^1_{\omega^{||}}(L^\infty_{\omega^\perp})}\ ,\\
\lesssim\ &\lambda^{-1-\gamma}\sum_{\substack{\sigma_1\,:\\ \sigma_1\lesssim\lambda}}\sigma_1^{\frac{n-3}{2}}\|\,Q_{\sigma_1}\,(P_{\mu_1}A\cdot P_{\mu_2}C)\,\|_{L^1_{\omega^{||}}(L^{\frac{2(n-1)}{n-3}}_{\omega^\perp})}\ ,\\
\lesssim\ &\lambda^{\frac{n-5}{2}-\gamma}\,\|\,P_{\mu_1}(A)\,\|_{L^2_{\omega^{||}}(L^{\frac{2(n-1)}{n-3}}_{\omega^\perp})}\cdot\|\,P_{\mu_2}(C)\,\|_{L^2_{\omega^{||}}(L^\infty_{\omega^\perp})}\ ,\\
\lesssim\ &\Big(\frac{\lambda}{\mu_1}\Big)^{\frac{n-5}{2}-\gamma}\ \|\,\Delta^{\frac{1}{2}}_{\omega^\perp}P_{\mu_1}(A)\,\|_{\dot H_x^{\frac{n-4}{2}}}\cdot\|\,P_{\mu_2}(C)\,\|_{{}^\omega\mathcal{N}_{1,10n}^{-\frac{1}{2}-\gamma,2,\infty}}\ .
\end{aligned}$$

Notice that the last line above follows from the $\dot H^1$ Sobolev embedding in the $\mathbb{R}^{n-1}_{\omega^\perp}$ plane. This estimate can now be safely summed using the condition that $6\leqslant n$ to sum the dyadics, and then using Cauchy-Schwartz to sum over the frequency localized pieces. This completes our proof of the bilinear estimate (334), and hence our proof of the integrated bound (307). $\square$

Having now established the proof of both the integrated bounds (297)–(307), we have proved the integrated group element bounds (292)–(293). This ends our proof of Proposition 9.4. $\square$

9.1. **Proof of the Accuracy estimate** (173d)**.** We will now give a short proof of the multiplier equivalence bound (173d). This will follow almost directly from the estimates we have already shown. We compute the kernel of the operator

$\Phi(0)\big((2\pi|\xi|)^\alpha(\Phi(0))^*\big) - (-\Delta)^{\frac{\alpha}{2}}P_1$ to be (again suppressing $\pm$ notations):

$$K^\alpha(x,y) \ = \ \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ {}^\omega g^{-1}(x){}^\omega g(y)\big[\bullet\big] \ {}^\omega g^{-1}(y){}^\omega g(x) \ \chi^\alpha(\xi) \ d\xi - \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \big[\bullet\big] \ \chi^\alpha(\xi) \ d\xi \ , \tag{344}$$

where $\chi^\alpha(\xi) = (2\pi|\xi|)^\alpha\chi_{(-\frac{1}{2},2)}(\xi)$. Notice that this cutoff function satisfies the general requirements of the generic bump function χ used throughout this section. In particular, there exist constants C_k which depend only on α and the original $\chi_{(-\frac{1}{2},2)}$ such that:

$$\int_{\mathbb{R}^n} |\nabla_\xi^k \chi^\alpha(\xi)| \ d\xi \ \leqslant \ C_k \ . \tag{345}$$

We now decompose the kernel $K^\alpha = \sum_\sigma K^\alpha_\sigma$ according to the dyadic physical space decomposition (204). For each fixed value of the small constant $\mathcal{E}$ on line (173d) we write this sum in terms of two pieces, a "close" part and a "far" part:

$$\begin{aligned} K^\alpha \ &= \ K^\alpha_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}} + K^\alpha_{\mathcal{E}^{-\frac{1}{2(n+1)}}<\bullet} \ , \\ &= \sum_{\substack{\sigma \ : \\ \sigma\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}}} K^\alpha_\sigma \ + \sum_{\substack{\sigma \ : \\ \mathcal{E}^{-\frac{1}{2(n+1)}}<\sigma}} K^\alpha_\sigma \ . \end{aligned} \tag{346}$$

To estimate the near portion of things, we do a little algebraic manipulation and write the kernel as:

$$\begin{aligned} K^\alpha_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}} \ = \ \chi_{\mathcal{D}_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}}} \Big(\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \big({}^\omega g^{-1}(x){}^\omega g(y) - I\big)\big[\bullet\big] \ {}^\omega g^{-1}(y){}^\omega g(x) \ \chi^\alpha(\xi) \ d\xi \\ + \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \big[\bullet\big] \ \big({}^\omega g^{-1}(y){}^\omega g(x) - I\big) \ \chi^\alpha(\xi) \ d\xi\Big) \ . \end{aligned}$$

By a direct application of the pair of integrated bounds (292)–(293) (with $M\sim 1$) this last expression gives us the absolute kernel bound:

$$|K^\alpha_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}}(x,y)| \ \lesssim \ \mathcal{E}\cdot(1+|x-y|)\chi_{\mathcal{D}_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}}}(|x-y|) \ .$$

By integrating the right hand side of this last inequality we easily arrive at the pair of Schur-test bounds:

$$\| \, K^\alpha_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}} \, \|_{L^\infty_y(L^1_x)} \ , \ \| \, K^\alpha_{\bullet\leqslant \mathcal{E}^{-\frac{1}{2(n+1)}}} \, \|_{L^\infty_x(L^1_y)} \ \lesssim \ \mathcal{E}^{\frac{1}{2}} \ . \tag{347}$$

To estimate the second kernel on the right hand side of (346), we do things separately for each term in the sum (344). For the second term which does not contain the group elements, a simple application of the estimate (345) and integration by parts shows that one has the absolute bounds:

$$\begin{aligned} &\big|\chi_{\mathcal{D}_{\mathcal{E}^{-\frac{1}{2(n+1)}}<\bullet}}(|x-y|)\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} \ \big[\bullet\big] \ \chi^\alpha(\xi) \ d\xi\big| \ , \\ &\lesssim \chi_{\mathcal{D}_{\mathcal{E}^{-\frac{1}{2(n+1)}}<\bullet}}(|x-y|)\cdot(1+|x-y|)^{-2(n+1)} \ , \\ &\lesssim \mathcal{E}^{\frac{1}{2}}\cdot(1+|x-y|)^{-(n+1)} \ . \end{aligned} \tag{348}$$

This easily yields Schur-test bounds of the form (347). Therefore, it remains to prove these bounds for the first integral expression on the right hand side (344) after it has been cut off in the far region $\mathcal{E}^{-\frac{1}{2(n+1)}} < |x-y|$. This follows at once from writing this kernel as a sum over various dyadic regions, and using the symbol bounds (220)–(221) as well as the reduction to the integrated estimates (292)–(293). The key thing to notice here is that there is only one place where we do not pick up the factor of $\mathcal{E}$ in the resulting estimates. That is in the integration by parts argument when the derivatives ∇_ξ^k *all* fall on the cutoff function χ^α. In that case we can simply use the compactness of the group elements and proceed in a way that is analogous to the computation which started on line (348) above. This completes our proof of the general multiplier approximation estimate (173d).

10. The Dispersive Estimate

In this section, we complete our proof of the non-microlocalized version of the Strichartz estimates contained in (173a). Using the abstract machinery of [4], these will follow once we can show that the parametrix (179) satisfies a dispersive estimate. If at fixed time t we write that operator as:

$$T(t)(\widehat{f}) \ = \ \Phi(t)(\widehat{f}) \ ,$$

where we have suppressed the $\pm$ notation, then we seek to prove the bound (where f has nothing to do with the original $\widehat{f}$, but just represents a function of the physical space variables):

$$\| T(t)T^*(s)f \|_{L^\infty_x} \ \lesssim \ (1+|t-s|)^{-\frac{n-1}{2}} \ \| f \|_{L^1_x} \ . \tag{349}$$

Now, a calculation similar that used to produce (202) shows that the kernel of the above operator can be computed to be:

$$\begin{aligned} K^{TT^*}(t,s;x,y) \ = \ \\ \int_{\mathbb{R}^n} e^{2\pi i\big((t-s)|\xi|+(x-y)\cdot\xi\big)} \ {}^\omega g^{-1}(t,x){}^\omega g(s,y)\big[\, \bullet \,\big] \ {}^\omega g^{-1}(s,y){}^\omega g(t,x) \ \chi(\xi) \ d\xi \ . \end{aligned} \tag{350}$$

Therefore, as is usually the case, we see that it suffices to show the fixed time uniform bound:

$$\| K^{TT^*}(t,s;\cdot,\cdot) \|_{L^\infty_{x,y}} \ \lesssim \ (1+|t-s|)^{-\frac{n-1}{2}} \ . \tag{351}$$

The proof of (351) turns out to be a straightforward consequence of the bounds established in the previous section. The strategy we follow here is almost identical. We first decompose the K^{TT*} kernel into a sum of two pieces:

$$K^{TT^*}_\sigma \ = \ \widetilde{K}^{TT^*} \ + \ \mathcal{R}^{TT^*} \ ,$$

for which we'll show the bound (351) individually. The $\widetilde{K}^{TT^*}$ kernel will be smooth enough that we can use a standard stationary phase computation on it. The remainder kernel $\mathcal{R}^{TT^*}$ will be small in absolute value without using any sophisticated integration by parts (although, as in the previous section, there will be some use for oscillations in this term also). As in the previous section, the definition of $\widetilde{K}^{TT^*}$ will depend on a physical space scale, in this case the vale of $(1+|t-s|+|x-y|)$. This will again be effected by the choice of an auxiliary gauge transformation $\widetilde{{}^\omega g}$. This time we define $\widetilde{{}^\omega g}$ to be the transformation into the Coulomb gauge of the smoothed out potential:

$$\widetilde{{}^\omega \underline{A}^{(M)}} \ = \ -\,{}^\omega\overline{\Pi}_{M^{-1}<\bullet}{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\nabla_x{}^\omega L \ \Delta^{-1}_{\omega^\perp} \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \ , \tag{352}$$

where we define the scale M to be such that:

$$M \ = \ (1 + |t-s| + |x-y|)^{\frac{1}{2}} \ .$$

As before, we use the splitting (217)–(218) to compute:

$$\widetilde{K}^{TT^*}(t,s;x,y) \ = \tag{353}$$
$$\int_{\mathbb{R}^n} e^{2\pi i\big((t-s)|\xi| + (x-y)\cdot\xi\big)} \ {}^{\omega}\!\widetilde{g}^{-1}(t,x)\,{}^{\omega}\!\widetilde{g}(s,y)\big[\,\bullet\,\big] \ {}^{\omega}\!\widetilde{g}^{-1}(s,y)\,{}^{\omega}\!\widetilde{g}(t,x) \ \chi(\xi) \ d\xi \ .$$

Our first step here is to notice that it suffices to show (351) for the kernel (353) under the condition that $|x-y| > \frac{1}{2}(1+|t-s|)$, for if this were not the case then we could simply integrate by parts as many times as necessary with respect to the variable $\lambda = |\xi|$ in the expression (353) and easily achieve (351). Therefore, we will now show that:

$$\| \, \widetilde{K}^{TT^*}(t,s;x,y) \, \| \ \lesssim \ |x-y|^{-\frac{n-1}{2}} \ . \tag{354}$$

We now factor the phase in (353) as:

$$e^{2\pi i\big((t-s)|\xi| + (x-y)\cdot\xi\big)} \ = \ e^{2\pi i (t-s)\lambda} \, e^{2\pi i \lambda \, |x-y| \ \cos(\Theta_{x-y,\omega})} \ ,$$

where we are using the frequency polar coordinates $\xi = \lambda\omega$. Integrating first on the sphere $\mathbb{S}^{n-1}$, we see that to conclude (354) it is enough to show that:

$$\| \int_{\mathbb{S}^{n-1}} e^{2\pi i \lambda \, |x-y| \ \cos(\Theta_{x-y,\omega})} \ {}^{\omega}\!\widetilde{g}^{-1}(t,x)\,{}^{\omega}\!\widetilde{g}(s,y)\big[\,\bullet\,\big] \ {}^{\omega}\!\widetilde{g}^{-1}(s,y)\,{}^{\omega}\!\widetilde{g}(t,x) \ d\omega \, \| \tag{355}$$
$$\lesssim \ |x-y|^{-\frac{n-1}{2}} \ .$$

This last estimate will follow easily from the Morse lemma and the already established symbol bounds (223)–(224). To implement this, we first cut off the above integral into small neighborhoods of stationary points of the pase and a remainder. We do this with the smooth partition of unity:

$$1 \ = \ \chi_{|1-\cos(\Theta_{x-y,\omega})| < \frac{1}{8}} + \chi_{|1+\cos(\Theta_{x-y,\omega})| < \frac{1}{8}} + \widetilde{\chi} \ .$$

The cutoff $\widetilde{\chi}$ cuts off on the region where $\cos(\Theta_{x-y,\omega})$ is bounded away from ± 1, and there we have the gradient estimate:

$$c \ < \ |\nabla_\omega \cos(\Theta_{x-y,\omega})| \ ,$$

for a sufficiently small constant c. Using this, and integrating by parts $n-1$ times while using the symbol bounds (223)–(224), we easily have that:

$$\| \int_{\mathbb{S}^{n-1}} e^{2\pi i \lambda \, |x-y| \ \cos(\Theta_{x-y,\omega})} \ {}^{\omega}\!\widetilde{g}^{-1}(t,x)\,{}^{\omega}\!\widetilde{g}(s,y)\big[\,\bullet\,\big] \ {}^{\omega}\!\widetilde{g}^{-1}(s,y)\,{}^{\omega}\!\widetilde{g}(t,x) \ \widetilde{\chi}(\omega) \ d\omega \, \|$$
$$\lesssim \ |x-y|^{-\frac{n-1}{2}} \ .$$

This proves (355) because we may assume that $\frac{1}{4} < \lambda$. Our goal is now to prove the localized estimate:

$$\| \int_{\mathbb{S}^{n-1}} e^{2\pi i \lambda \, |x-y| \ \cos(\Theta_{x-y,\omega})} \ {}^{\omega}\!\widetilde{g}^{-1}(t,x)\,{}^{\omega}\!\widetilde{g}(s,y)\big[\,\bullet\,\big]$$
$${}^{\omega}\!\widetilde{g}^{-1}(s,y)\,{}^{\omega}\!\widetilde{g}(t,x) \ \widetilde{\chi}_{|1-\cos(\Theta_{x-y,\omega})| < \frac{1}{8}}(\omega) \ d\omega \, \| \quad \lesssim \quad |x-y|^{-\frac{n-1}{2}} \ .$$

It will become clear that the corresponding estimate for the region where $|1+\cos(\Theta_{x-y,\omega})| < \frac{1}{8}$ follows from identical calculations.

Now, the angular function $\cos(\Theta_{x-y,\omega})$ has a single non-degenerate critical point in a neighborhood of the unit vector $(x-y)/|x-y|$ with index $n-1$. Therefore, by the Morse lemma there exists a diffeomorphism $\theta = \varphi(\omega)$ in a neighborhood of this point such that:

$$1-\cos(\Theta_{x-y,\omega}) \ = \ \theta_1^2 + \ldots + \theta_{n-1}^2 \ .$$

By making this change of variables, we see that we are trying to prove that:

$$\begin{aligned}|||\int_{\mathbb{R}^{n-1}} \ & e^{2\pi i\lambda\,|x-y|\,|\theta|^2} \ {}^{\varphi^{-1}(\theta)}\widetilde{g}^{-1}(t,x)^{\varphi^{-1}(\theta)}\widetilde{g}(s,y)\big[\,\bullet\,\big] \\ & {}^{\varphi^{-1}(\theta)}\widetilde{g}^{-1}(s,y)^{\varphi^{-1}(\theta)}\widetilde{g}(t,x) \ \chi(\theta) \ J_{\varphi^{-1}}(\theta) \ d\theta||| \quad \lesssim \quad |x-y|^{-\frac{n-1}{2}} \ .\end{aligned}$$

Here $J_{\varphi^{-1}}$ denotes the Jacobian matrix of φ^{-1}, and χ is some smooth function which is supported where $|\theta| \leqslant 1$. Making now the simple change of variables $\sqrt{\lambda|x-y|}\theta = \theta'$, it suffices to be able to show that:

(356)
$$\begin{aligned}|||\int_{\mathbb{R}^{n-1}} \ e^{2\pi i\,|\theta'|^2} \ {}^{\widetilde{\varphi}(\theta')}\widetilde{g}^{-1}(t,x)^{\widetilde{\varphi}(\theta')}\widetilde{g}(s,y)\big[\,\bullet\,\big] \ {}^{\widetilde{\varphi}(\theta')}\widetilde{g}^{-1}(s,y)^{\widetilde{\varphi}(\theta')}\widetilde{g}(t,x) \ \widetilde{J}(\theta') \ d\theta'||| \\ \lesssim \quad 1 \ .\end{aligned}$$

Here $\widetilde{J}(\theta')$ denotes a smooth function with (large) compact support and uniform gradient bounds:

$$|\nabla_{\theta'}^k \widetilde{J}| \ \lesssim \ 1 \ .$$

Furthermore, the function $\widetilde{\varphi}(\theta')$ obeys the gradient bounds:

$$|\nabla_{\theta'}^k \widetilde{\varphi}| \ \lesssim \ |x-y|^{-\frac{k}{2}} \ .$$

Combining this last estimate with the symbol bounds (223)–(224) and the truncation condition $M = |x-y|^{\frac{1}{2}}$, we have the uniform gradient estimates:

$$\begin{aligned}\|\,\nabla_{\theta'}^k\big({}^{\widetilde{\varphi}}\widetilde{g}^{-1}(t,x)^{\widetilde{\varphi}}\widetilde{g}(s,y)\big)\,\| \ &\lesssim \ 1 \ , \\ \|\,\nabla_{\theta'}^k\big({}^{\widetilde{\varphi}}\widetilde{g}^{-1}(s,y)^{\widetilde{\varphi}}\widetilde{g}(t,x)\big)\,\| \ &\lesssim \ 1 \ .\end{aligned}$$

Using these bounds, we can prove the bound (356) by treating the quantity on the left hand side as a Fresnel integral and performing n integrations by parts in the region where $1 < |\theta'|$.

To complete our proof of (351) we need to show that:

$$\|\,\mathcal{R}^{TT^*}(t,s;\cdot,\cdot)\,\|_{L^\infty_{x,y}} \ \lesssim \ (1+|t-s|)^{-\frac{n-1}{2}} \ , \tag{357}$$

where $\mathcal{R}^{TT^*}$ is the kernel which is defined by subtracting (353) from (350). Using the splitting (217)–(218) we see that this has at least one factor involving the expressions ${}^{\omega}h^{-1}(x){}^{\omega}h(y) - I$ or ${}^{\omega}h(x){}^{\omega}h^{-1}(y) - I$ under the integral sign. There are several such combination, but we will choose to estimate only one such term and leave the others to reader as they can be treated analogously. Therefore, we may without loss of generality assume that we are trying to prove the bound:

$$\| \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{2\pi i\lambda\big((t-s)+(x-y)\cdot\omega\big)}\ {}^\omega G(t,x;s,y)\ \chi(\lambda)\ \lambda^{n-1}\ d\lambda d\omega \| \lesssim (1+|t-s|)^{-\frac{n-1}{2}} , \tag{358}$$

where we have set:

$${}^\omega G(t,x;s,y) \ = \ {}^\omega g^{-1}(t,x) \left({}^\omega h^{-1}(t,x){}^\omega h(s,y) - I\right) {}^\omega g(s,y)\big[\,\bullet\,\big]\ {}^\omega g^{-1}(s,y){}^\omega g(t,x) \ .$$

As in the proof of (351) above for the smoothed out kernel $\widetilde{K}^{TT^*}$, we may without loss of generality assume that we trying to prove (358) in the region where $|x-y| > \frac{1}{2}(1+|t-s|)$ because otherwise we may integrate as many times as necessary with respect to the radial frequency variable to pick up the desired decay.

To proceed further, we will first decompose the range of frequency integration into a small set and a remainder where we can again integrate by parts with respect to λ. This is accomplished by using the angular partition of unity:

$$1 \ = \ \chi_{|\frac{t-s}{|x-y|}+\cos(\Theta_{x-y,\omega})|>|x-y|^{\gamma-1}} + \chi_{|\frac{t-s}{|x-y|}+\cos(\Theta_{x-y,\omega})|\leqslant|x-y|^{\gamma-1}} \ .$$

To deal with the bound (358) for the first cutoff function above, we need to show that:

$$\begin{aligned} |||\int_{\mathbb{S}^{n-1}} &{}^\omega G(t,x;s,y)\ d\omega \\ \cdot &\int_0^\infty e^{2\pi i\lambda\big((t-s)+(x-y)\cdot\omega\big)}\ \chi_{|\frac{t-s}{|x-y|}+\cos(\Theta_{x-y,\omega})|>|x-y|^{\gamma-1}}\ \chi(\lambda)\ \lambda^{n-1}\ d\lambda||| \\ &\lesssim \ |x-y|^{-\frac{n-1}{2}} \ . \end{aligned}$$

This bound follows easily from radial integration by parts in the inner integral, followed by the simple compactness estimate:

$$\int_{\mathbb{S}^{n-1}} \|\, {}^\omega G(t,x;s,y) \,\|\ d\omega \ \lesssim \ 1 \ ,$$

which is of course uniform in the variables $(t,x;s,y)$.

To wrap things up here, we need to show the absolute estimate:

$$\int_{\mathbb{R}^n} \|\, {}^\omega G(t,x;s,y) \,\|\ \chi_{|\frac{t-s}{|x-y|}+\cos(\Theta_{x-y,\omega})|\leqslant|x-y|^{\gamma-1}}\ \chi(\xi)\ d\xi \ \lesssim \ |x-y|^{-\frac{n-1}{2}} \ .$$

After a Cauchy-Schwartz, this will follow once we can establish that both:

$$\left(\int_{\mathbb{R}^n} \chi_{|\frac{t-s}{|x-y|}+\cos(\Theta_{x-y,\omega})|\leqslant|x-y|^{\gamma-1}}\ \chi(\xi)\ d\xi\right)^{\frac{1}{2}} \ \lesssim \ |x-y|^{\frac{1}{2}(\gamma-1)} \ , \tag{359}$$

$$\left(\int_{\mathbb{S}^{n-1}} \|\, {}^\omega G(t,x;s,y) \,\|^2\ d\omega\right)^{\frac{1}{2}} \ \lesssim \ |x-y|^{-\frac{1}{2}(n-2+\gamma)} \ . \tag{360}$$

The first estimate, (359) follows from elementary bounds. Notice first that after a rotation, it suffices to assume that the vector $x-y$ lies along the $(1,0)$ direction.

Then the cutoff function is supported in the region where:

$$\frac{\xi_1}{|\xi|} \ = \ -\frac{t-s}{|x-y|} + O(|x-y|^{\gamma-1}) \ ,$$

which is a conical set about the ξ_1-axis of volume no greater than a constant times $|x-y|^{\gamma-1}$ in the region where $|\xi| \lesssim 1$. The second estimate (360) above we have already shown. It is a special case of the bound (292) which was proved in the previous section. This completes our proof of (357), and hence our demonstration of the dispersive estimate (351).

11. The Decomposable Function Spaces: Proof of the Square-Sum and Differentiated Strichartz Estimates

We now introduce a piece of machinery which will be of central importance for the remainder of the paper. This is a suitable reinterpretation of the important "decomposable function" criterion from the work [8]. In our context, we set up the general situation as follows: Suppose we are given an $M(m \times m)$ valued Fourier integral operator:

$$\Phi(\hat{f})(t,x) \ = \ \int_{\mathbb{R}^n} e^{2\pi i \psi(t,x;\xi)} e^{2\pi i x\cdot\xi} \ g_1(t,x;\xi) \ \widehat{f}(\xi) \ g_2(t,x;\xi) \ d\xi \ , \tag{361}$$

where the g_i are arbitrary matrix valued functions, such that this operator satisfies certain mixed Lebesgue space mapping properties (uniform in y_0):

$$\| \, \Phi_{y_0}(\hat{f}) \, \|_{L^{q_1}(L^{r_1})} \ \lesssim \ \| \, \widehat{f} \, \|_{L^2} \ , \tag{362}$$

where Φ_{y_0} is the same operator as (361) but with phase ψ replaced by $\psi_{y_0} = \psi(t, x-y_0;\xi)$. Suppose now that we are given a matrix valued function $C(t,x;\omega)$ which only depends on the angular variable $\omega = \xi/|\xi|$ in frequency. We would like to prove estimates for the coupled operator (we only discuss left multiplication here, the case of right multiplication is analogous):

$$\widetilde{\Phi}(\hat{f})(t,x) \ = \ \int_{\mathbb{R}^n} e^{2\pi i \psi(t,x;\xi)} e^{2\pi i x\cdot\xi} \ C(t,x;\omega)\cdot g_1(t,x;\xi) \ \widehat{f}(\xi) \ g_2(t,x;\xi) \ d\xi \ . \tag{363}$$

These should be done in a way that the decay properties of the function $C(t,x;\omega)$ can be used to improve the range of the estimates (362). A robust way for doing this in the linear case has been worked out in the paper of Rodnianski–Tao [8]. The answer is to fix an angular scale, say θ, and then to form the norm ("classical" decomposable norm):

$$\| \, C \, \|^2_{D^{cl}_\theta\left(L^{q_2}_t(L^{r_2}_x)\right)} \ = \ \sum_{k=0}^{10n} \theta^{-n+1} \int_{\mathbb{S}^{n-1}_\omega} \| \, (\theta\nabla_\xi)^k \ C \, \|^2_{L^{q_2}_t(L^{r_2}_x)} \ d\omega \ . \tag{364}$$

By decomposing the frequency variable in (363) into angular sectors of size $\sim \theta$, a straightforward computation then shows that one has the estimate:

$$\| \, \widetilde{\Phi}(\hat{f}) \, \|_{L^q(L^r)} \ \lesssim \ \| \, C \, \|_{D^{cl}_\theta\left(L^{q_2}_t(L^{r_2}_x)\right)} \cdot \| \, \widehat{f} \, \|_{L^2} \ , \tag{365}$$

whenever estimate (362) holds with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$.

There are two problems which occur when trying to apply (364) in the present context. The first is that this norm is for a single scale, which causes problems in products where many different scales interact with each other. The other problem, which is conceptually much more serious, is that the estimate (364) contains the highly singular factor of $\theta^{-\frac{n-1}{2}}$, which needs to be eliminated with a delicate orthogonality argument, the kind which is not preserved in this problem for a variety of reasons (non-linear Hodge systems, a covariant wave equation that does not commute with angular cutoffs, etc). However, with only a slight reworking the basic idea behind (364) can be shown to be surprisingly robust. First of all, for a fixed scale we replace (364) with a square function norm which has the same effect, and which will be very easy to verify in the present context. Since we will be using multiple scales in a moment, we introduce the *solid* angular cutoff functions $\overline{b^\phi}_\theta(\omega)$ (not to be confused with the *hollow* multipliers $b^\omega_\theta(\xi)$ introduced in Section 4), such that:

$$\overline{b^\phi}_\theta(\omega) \ \equiv \ 1 \ , \tag{366}$$

when $\omega \in \Gamma_\phi$, for the angular sector Γ_ϕ which we interpret as a cap in a finitely overlapping collection on the sphere $\mathbb{S}^{n-1}_\omega = \cup_\phi \Gamma_\phi$. Here the scale is determined by the condition $|\Gamma_\phi| \sim \theta$. On this scale, we replace (364) with the norm:

$$\| C \|_{D_\theta\left(L^{q_2}_t(L^{r_2}_x)\right)} \ = \ \Big\| \Big(\sum_{k=0}^{10n} \sum_{\phi} \sup_{\omega} \ \| \overline{b^\phi}_\theta \ (\theta\nabla_\xi)^k \ C \|^2_{L^{r_2}_x} \Big)^{\frac{1}{2}} \Big\|_{L^{q_2}_t} \ . \tag{367}$$

It is not difficult to see that by decomposing the integral on the right hand side of (364) into fine and course scales, and applying Hölder's on the fine (continuous) scales, that the Rodnianski-Tao norm (364) with the time integral on the outside is bounded by the norm (367). Furthermore, it is easy to see from the proof given in [8] that having the time integral on the outside does not effect the bound (365) so long as the index q_1 implicitly appearing in this bound is such that $2 \leqslant q_1$. This allows one to use Minkowski's inequality to pull the square sum on the parametrix through the time integral. For us this index condition will always hold because we are working with Strichartz type norms. We leave it to the reader to work out the details of these claims.

We now form an ℓ^1 Banach space based on incorporating the norms (367) over all dyadic angular scales $\theta \lesssim 1$. The elements of this space we denote by $\{C\} = \{C^{(\theta)}\}$, and we define its norm $\ell^1(D_\theta)$ norm as:

$$\| \{C\} \|_{\ell^1\left(D_\theta(L^{q_2}_t(L^{r_2}_x))\right)} \ = \ \sum_\theta \ \| C^{(\theta)} \|_{D_\theta\left(L^{q_2}_t(L^{r_2}_x)\right)} \ . \tag{368}$$

There is also the forgetful map from the space $\ell^1(D_\theta)$ to functions which define as:

$$C \ \rightsquigarrow \ \sum_\theta \ C^{(\theta)} \ , \tag{369}$$

and we will in practice abusively identify $\{C\}$ with C via the map (369). The main point is that given any function C, there may be a *variety* of ways which we embed C in the space $\ell^1(D_\theta)$, and it is up to the structure of the application to decide how this should be done. Of course, given the square function norms (114) we are working with our choice here is somewhat canonical.

Now, if we consider the C in (369) as embedded in the integral (363), we easily have the estimate:

$$\|\widetilde{\Phi}(\widehat{f})\|_{L^q(L^r)} \lesssim \|\{C\}\|_{\ell^1\left(D_\theta(L_t^{q_2}(L_x^{r_2}))\right)} \cdot \|\widehat{f}\|_{L^2} \ . \tag{370}$$

We also form spatial Besov versions of the norm (370), which we denote as $\ell^1 D_\theta(L^q(\dot{B}_2^{r,(2,s)}))$. This leads us to the basic notation of this section:

Definition 11.1. For a given matrix valued function, we say it is in the *decomposable space* $D(L^q(\dot{B}_2^{r,(2,s)}))$ if the following norm is finite:

$$\|C\|_{D(L^q(\dot{B}_2^{r,(2,s)}))} = \inf_{C=\sum_\theta C^{(\theta)}} \Big\{\sum_\theta \|C^{(\theta)}\|_{D_\theta(L^q(\dot{B}_2^{r,(2,s)}))}\Big\} \ . \tag{371}$$

We also define the low frequency analog of these spaces, which we denote by $D(L^q(\dot{B}_{2,10n}^{r,(2,s)}))$, similarly.

We remark here that it is easy to see that the norm (371) leads to a Banach space. This will be important in a moment. Also, it is easy to show that the various Besov-Lebesgue space inclusions (38)–(40) hold for these spaces if we define $D(L^p)$ analogously to (371). This is a simple consequence of the fact that the Littlewood-Paley theory commutes with the derivatives ∇_ξ^k. We now show that this space satisfies the expected range of bilinear Riesz operator estimates:

Lemma 11.2 (A decomposable Besov calculus)**.** *Let the indices* $0 \leqslant \sigma$*,* $1 \leqslant q_i, r_i \leqslant \infty$*, and* s_i *be given. Then one has the following family of bilinear estimates:*

$$|D_x|^{-\sigma} \ : \ D\big(L_t^{q_1}(\dot{B}_2^{r_1,(2,s_1)})\big) \cdot D\big(L_t^{q_2}(\dot{B}_2^{r_2,(2,s_2)})\big) \hookrightarrow D\big(L_t^{q_3}(\dot{B}_1^{r_3,(2,s_3)})\big) \ , \tag{372}$$

where the various indices satisfy the conditions:

$$s_3 = s_1 + s_2 + \sigma - \frac{n}{2} \ , \tag{373}$$

$$\sigma + \frac{n}{2} - s_3 < n(\frac{1}{r_1} + \frac{1}{r_2}) \ , \tag{374}$$

$$s_1 < \frac{n}{2} + \min\{n(\frac{1}{r_2} - \frac{1}{r_3}) \ , \ 0\} \ , \tag{375}$$

$$s_2 < \frac{n}{2} + \min\{(\frac{1}{r_1} - \frac{1}{r_3}) \ , \ 0\} \ , \tag{376}$$

$$\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} \ , \tag{377}$$

$$\frac{1}{r_3} \leqslant \frac{1}{r_1} + \frac{1}{r_2} \ . \tag{378}$$

Proof of estimate (372). The proof of (372) is largely a triviality given that it is true for the norms $L_t^{q_1}(\dot{B}_2^{r_1,(2,s_1)})$ without the decomposable structure. First of all, notice that from the definition 11.1 it suffices to establish things with the norms $D\big(L_t^{q_i}(\dot{B}_2^{r_i,(2,s_i)})\big)$ replaced by their vector generalizations $\ell^1 D_\theta\big(L_t^{q_i}(\dot{B}_2^{r_i,(2,s_i)})\big)$.

This follows at once from working with two test matrices A and B and decomposing them into sums:

$$A \ = \ \sum_\theta \ A^{(\theta)} \ , \qquad\qquad C \ = \ \sum_\theta \ C^{(\theta)} \ ,$$

where the $\{A\}$ and $\{C\}$ collections have norms no greater than twice that of A and C respectively.

Suppose now that we are given two test elements $\{A^{(\theta)}\}$ and $\{C^{(\theta)}\}$. The we write their product under the map (369) as:

$$\begin{aligned} A\cdot C \ &= \ \sum_{\theta_1,\theta_2} \ A^{(\theta_1)}\cdot C^{(\theta_2)} \ , \\ &= \ \sum_\theta \ \Big(\sum_{\substack{\theta_1 \ : \\ \theta<\theta_1}} \ A^{(\theta_1)}\cdot C^{(\theta)} \ + \ \sum_{\substack{\theta_2 \ : \\ \theta\leqslant\theta_2}} \ A^{(\theta)}\cdot C^{(\theta_2)} \Big) \ , \\ &= \ \sum_\theta \ T_1^{(\theta)} \ + \ T_2^{(\theta)} \ . \end{aligned} \tag{379}$$

Freezing the scale θ, we will prove the following two estimates:

$$\| \, |D_x|^{-\sigma} \ T_1^{(\theta)} \, \|_{D_\theta\left(L_t^{q_3}(\dot{B}_2^{r_3,(2,s_3)})\right)} \ \lesssim \ \| \, \{A\} \, \|_{\ell^1 D_\theta\left(L_t^{q_1}\dot{B}_2^{r_1,(2,s_1)}\right)} \cdot \| \, C^{(\theta)} \, \|_{D_\theta\left(L_t^{q_2}\dot{B}_2^{r_2,(2,s_2)}\right)} \ , \tag{380}$$

$$\| \, |D_x|^{-\sigma} \ T_2^{(\theta)} \, \|_{D_\theta\left(L_t^{q_3}(\dot{B}_2^{r_3,(2,s_3)})\right)} \ \lesssim \ \| \, A^{(\theta)} \, \|_{D_\theta\left(L_t^{q_1}\dot{B}_2^{r_1,(2,s_1)}\right)} \cdot \| \, \{C\} \, \|_{\ell^1 D_\theta\left(L_t^{q_2}\dot{B}_2^{r_2,(2,s_2)}\right)} \ . \tag{381}$$

We will only concentrate on (380), as the second estimate above follows from virtually identical reasoning. Expanding out the sum in $T_1^{(\theta)}$, it suffices to show:

$$\begin{aligned} \| \, |D_x|^{-\sigma} \ (A^{(\theta_1)}\cdot C^{(\theta)}) \, \|_{D_\theta\left(L_t^{q_3}(\dot{B}_2^{r_3,(2,s_3)})\right)} \ &\lesssim \\ \| \, A^{(\theta_1)} \, \|_{D_{\theta_1}\left(L_t^{q_1}\dot{B}_2^{r_1,(2,s_1)}\right)} &\cdot \| \, C^{(\theta)} \, \|_{D_\theta\left(L_t^{q_2}\dot{B}_2^{r_2,(2,s_2)}\right)} \ , \end{aligned} \tag{382}$$

where we have the condition $\theta\leqslant\theta_1$. We now compute the norm on the right hand side of this last equation. For the remainder of the proof we fix the time variable. This can then be dealt with at the end by integrating in time and using Hölders inequality because all of the action in the norms (367) takes place under the time integral. To proceed, we first fix the angular sector Γ_ϕ and the number of $(\theta\nabla_\xi)$ derivatives to compute that:

$$\begin{aligned} &\sup_\omega \ \| \, \overline{b^\phi}_\theta \ (\theta\nabla_\xi)^k |D_x|^{-\sigma} \ (A^{(\theta_1)}\cdot C^{(\theta)}) \, \|_{\dot{B}_2^{r_3,(2,s_3)}} \ , \\ \lesssim \ &\sum_{i=0}^k \ \sum_{\substack{\phi_1 \ : \\ \Gamma_{\phi_1}\subseteq 10\Gamma_\phi}} \ \sup_\omega \ \| \, |D_x|^{-\sigma} \ \Big(\overline{b^{\phi_1}}_\theta (\theta\nabla_\xi)^{k-i} A^{(\theta_1)} \ \cdot \ \overline{b^\phi}_\theta (\theta\nabla_\xi)^i C^{(\theta)} \Big) \, \|_{\dot{B}_2^{r_3,(2,s_3)}} \ , \\ \lesssim \ &\sum_{i=0}^k \ \sum_{\substack{\phi_1 \ : \\ \Gamma_{\phi_1}\subseteq 10\Gamma_\phi}} \ \sup_\omega \ \| \, \overline{b^{\phi_1}}_\theta (\theta\nabla_\xi)^{k-i} A^{(\theta_1)} \, \|_{\dot{B}_2^{r_1,(2,s_1)}} \ \cdot \ \sup_\omega \ \| \, \overline{b^\phi}_\theta (\theta\nabla_\xi)^i C^{(\theta)} \, \|_{\dot{B}_2^{r_2,(2,s_2)}} \ . \end{aligned}$$

Square summing this last expression over angular sectors, and adding over all $0 \leqslant k \leqslant 10n$ we arrive at the estimate:

$$\| \, |D_x|^{-\sigma} \, (A^{(\theta_1)} \cdot C^{(\theta)}) \, \|_{D_\theta\left(\dot{B}_2^{r_3,(2,s_3)}\right)} \lesssim \sup_{\phi} \sum_{k=0}^{10n} \sup_{\omega} \| \overline{b^\phi}_\theta (\theta \nabla_\xi)^k A^{(\theta_1)} \|_{\dot{B}_2^{r_1,(2,s_1)}} \cdot \| C^{(\theta)} \|_{D_\theta\left(\dot{B}_2^{r_2,(2,s_2)}\right)} \ .$$

We can now conclude (382) on account of the condition $\theta \leqslant \theta_1$ which implies the trivial bound:

$$\begin{aligned} &\sup_{\phi} \sum_{k=0}^{10n} \sup_{\omega} \| \overline{b^\phi}_\theta (\theta \nabla_\xi)^k A^{(\theta_1)} \|_{\dot{B}_2^{r_1,(2,s_1)}} \ , \\ \lesssim\ &\sup_{\phi'} \sum_{k=0}^{10n} \sup_{\omega} \| \overline{b^{\phi'}}_{\theta_1} (\theta_1 \nabla_\xi)^k A^{(\theta_1)} \|_{\dot{B}_2^{r_1,(2,s_1)}} \ , \\ \lesssim\ &\| A^{(\theta_1)} \|_{D_{\theta_1}\left(\dot{B}_2^{r_1,(2,s_1)}\right)} \ . \end{aligned}$$

This completes our proof of the estimate (372). □

We now establish the link which relates the norms (371) to the ones we are using in this paper:

Lemma 11.3 (Core decomposable estimates for the potentials $\{{}^\omega A^\pm\}$ and $\{{}^\omega C^\pm\}$)**.** *Let the sets of potentials $\{{}^\omega A^\pm\}$ and $\{{}^\omega C^\pm\}$ be defined as on lines* (192), (195), *and* (196) *above. Then one has the following family of decomposable bounds:*

(383) $$\| {}^\omega A^\pm \|_{D\left(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\right)} \lesssim \mathcal{E} \ , \qquad \| {}^\omega A^\pm \|_{D\left(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\right)} \lesssim \mathcal{E} \ ,$$

(384) $$\| \nabla_t {}^\omega \underline{A}^\pm \|_{D\left(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-4}{2})})\right)} \lesssim \mathcal{E} \ , \quad \| \nabla_t {}^\omega \underline{A}^\pm \|_{D\left(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-3}{2})})\right)} \lesssim \mathcal{E} \ ,$$

(385) $$\| {}^\omega C^\pm \|_{D\left(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\right)} \lesssim \mathcal{E} \ , \qquad \| {}^\omega C^\pm \|_{D\left(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\right)} \lesssim \mathcal{E} \ ,$$

(386) $$\| \nabla_t {}^\omega \underline{C}^\pm \|_{D\left(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-4}{2})})\right)} \lesssim \mathcal{E} \ , \quad \| \nabla_t {}^\omega \underline{C}^\pm \|_{D\left(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-3}{2})})\right)} \lesssim \mathcal{E} \ ,$$

where p_γ and q_γ are the dimensional constants from lines (188) *and* (265) *above. Furthermore, one has the following improved null-differentiated space-time bounds:*

(387) $$\| \left({}^\omega L^\mp {}^\omega \underline{A}^\pm \ , \nabla_t \Delta^{-\frac{1}{2}} {}^\omega L^\mp {}^\omega \underline{A}^\pm \right) \|_{D\left(L_t^2(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-3}{2})})\right)} \lesssim \mathcal{E} \ ,$$

(388) $$\| \left({}^\omega L^\mp {}^\omega \underline{C}^\pm \ , \nabla_t \Delta^{-\frac{1}{2}} {}^\omega L^\mp {}^\omega \underline{C}^\pm \right) \|_{D\left(L_t^2(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-3}{2})})\right)} \lesssim \mathcal{E} \ .$$

In all of these estimates, the small constant $\mathcal{E}$ is the same as on lines (124d) *and* (124f) *above.*

Proof of the estimates (383)–(388). With the current setup, the proof of these bounds is very simple and repeats many of things we have already done in previous sections. Starting with the estimates (383)–(383), we see that using the truncation condition (124c) it suffices to prove the first collection, as the time differentiated versions will follow from these with little fuss. We now follow essentially the same steps used to prove estimates (187) and (283). The only difference here is that we incorporate the square function norms contained in the $\dot{X}^{\frac{n-2}{2}}$ spaces. In what follows, we will in fact only prove the space-time estimate which is the second bound on the right hand side of (383) above. The first bound on this line follows from similar reasoning and is left to the reader. The first step is to define the scale decomposition (we now ignore $\pm$ notation for the remainder of the proof):

$$ {}^{\omega}A \ = \ \sum_{\theta} \ {}^{\omega}A^{(\theta)} \ = \ \sum_{\theta} \ {}^{\omega}\Pi_{\theta} \ {}^{\omega}A \ . $$

Our goal is now to prove the following fixed time bounds which can easily be summed over and then integrated to achieve the desired goal:

$$ \| \, {}^{\omega}\Pi_{\theta} {}^{\omega}A \, (t) \, \|_{D_{\theta}\left(\dot{B}^{q_{\gamma},(2,\frac{n-1}{2})}_{2,10n}\right)} \ \lesssim \ \theta^{\gamma} \ \| \, \underline{A}_{\bullet \ll 1} \, (t) \, \|_{S\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \ . $$

By using the square function structure contained in the definition of the various Besov and decomposable Besov norms and taking into account the low frequency truncation of the potentials $\{{}^{\omega}A\}$ and $\{\underline{A}_{\bullet \ll 1}\}$, the proof of this last estimate reduces to the fixed frequency bound:

$$ \| \, {}^{\omega}\Pi_{\theta} P_{\mu} {}^{\omega}A \, (t) \, \|_{D_{\theta}\left(\dot{B}^{q_{\gamma},(2,\frac{n-1}{2})}_{2}\right)} \ \lesssim \ \theta^{\gamma} \ \| \, P_{\mu} \underline{A}_{\bullet \ll 1} \, (t) \, \|_{S\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \ . $$

Expanding now the decomposable norm on the left hand side of this last inequality, we see that the proof reduces to showing the square function bounds:

$$ \sum_{k=0}^{10n} \sum_{\phi} \sup_{\omega} \ \| \, \overline{b^{\phi}}_{\theta}(\omega) \ (\theta \nabla_{\xi})^k \ {}^{\omega}\Pi_{\theta} P_{\mu} {}^{\omega}A \, (t) \, \|^2_{\dot{B}_2^{q_{\gamma},(2,\frac{n-1}{2})}} \tag{389} $$
$$ \lesssim \ \theta^{2\gamma} \sum_{\substack{\phi \, : \\ \omega_0 \in \Gamma_{\phi}}} \ \| \, \widetilde{{}^{\omega_0}\Pi}_{\theta} P_{\mu} \underline{A}_{\bullet \ll 1} \, (t) \, \|^2_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \ , $$

where $\widetilde{{}^{\omega}\Pi}_{\theta}$ is a fixed thickening of the multiplier ${}^{\omega}\Pi_{\theta}$ such that one has the general quasi-idempotence bound:

$$ \sup_{\omega} \| \, \overline{b^{\phi}}_{\theta}(\omega) \, \widetilde{\widetilde{{}^{\omega}\Pi}}_{\theta} A \, \|_{L^q} \ \lesssim \ \| \, \widetilde{{}^{\omega_0}\Pi}_{\theta} A \, \|_{L^q} \ , \qquad\qquad \omega_0 \in \Gamma_{\phi} \ , \tag{390} $$

where $\widetilde{\widetilde{{}^{\omega}\Pi}}_{\theta}$ is any multiplier with frequency support contained in the frequency support of ${}^{\omega}\Pi_{\theta}$ whose convolution kernel satisfies comparable L^1 bounds. Here the statement that $\omega_0 \in \Gamma_{\phi}$ is taken to mean that ω_0 is in the center of the cap Γ_{ϕ}, the very same notion we used in the definition of the square function norms (114) above. Using now the general bound (390) as well as the heuristic multiplier identity:

$$ (\theta \nabla_{\xi})^k \ {}^{\omega}\Pi_{\theta} P_{\mu} {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)} \nabla_{t,x} {}^{\omega}L \, \Delta^{-1}_{\omega^{\perp}} \underline{A}_{\bullet \ll 1}(\partial_{\omega}) \, (t) \ \approx \ \theta^{-1} \ {}^{\omega}\Pi_{\theta} P_{\mu} \underline{A}_{\bullet \ll 1} \, (t) \ , $$

we have the bound:

$$ (\text{L.H.S})(389) \ \lesssim \ \sum_{\substack{\phi \, : \\ \omega_0 \in \Gamma_{\phi}}} \ \theta^{-2} \ \| \, \widetilde{{}^{\omega_0}\Pi}_{\theta} P_{\mu} \underline{A}_{\bullet \ll 1} \, (t) \, \|^2_{\dot{B}_2^{q_{\gamma},(2,\frac{n-1}{2})}} \ . $$

The estimate (389) now follows from the Bernstein nested-Besov inclusion:

$$ \widetilde{{}^{\omega_0}\Pi}_{\theta}(\dot{B}_2^{q_{\gamma},(2,\frac{n-1}{2})}) \ \subseteq \ \theta^{1+\gamma} \ \widetilde{{}^{\omega_0}\Pi}_{\theta}(\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}) \ . $$

Our next goal is to pass the estimates (383)–(384) on to the non-linear set of potentials $({}^\omega C_0, \{{}^\omega \underline{C}\})$. Since it is a-priori not clear that these functions have finite $D(L^q(\dot{B}_2^{r,(2,s)}))$ norms, we construct the bounds from scratch by running a contraction mapping argument in these spaces on the Picard iterates of the systems (195) and (196). To guarantee convergence of the resulting sequences, we make use specific instances of the general embedding (372). Our general strategy here is the following. We first establish the non-time differentiated estimates (385) for the spatial potentials $\{{}^\omega \underline{C}\}$. Then, assuming the non-time differentiated versions of the improved estimates (387)–(388) (whose proof relies only on the previously established bounds) we prove the time-differentiated estimates (386). Having established these, we then prove the estimates (385) for the temporal potential ${}^\omega C_0$. Our next order of business is to prove the non-time differentiated versions of the improved null-differentiated bounds (387)–(388). Finally, armed with all of this, we show the version of the estimates (387)–(388) which contain the extra time derivatives. In what follows, we will only list out the various bilinear estimates which yield the desired bounds. Since these are almost identical to many of the estimates we have dealt with in the past sections, we leave the verification of the numerology to the reader.

To prove the non-time differentiated versions of (385) for the collection $\{{}^\omega \underline{C}\}$ we use the pair of bounds:

(391)
$$\nabla_x \Delta^{-1} \ : \ D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \cdot D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \ \hookrightarrow \ D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \ ,$$
(392)
$$\nabla_x \Delta^{-1} \ : \ D\big(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big) \cdot D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \ \hookrightarrow \ D\big(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big) \ .$$

To establish the first bound on line (386) we first differentiate the Hodge system (195) with respect to time and then apply the embedding:
(393)
$$\nabla_x \Delta^{-1} \ : \ D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-4}{2})})\big) \cdot D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \ \hookrightarrow \ D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-4}{2})})\big) \ .$$

To prove the time integrated bound which is the second on line (386) we decompose the vector-field ∇_t into $\pm{}^\omega \underline{L} \mp \omega\cdot\nabla_x$ just as we did starting on line (311) above. Then, modulo estimates of the form (392), and assuming that we have shown the non-time differentiated versions of (387)–(388) we may reduce things to the embedding:
(394)
$$\nabla_x \Delta^{-1} \ : \ D\big(L_t^2(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-3}{2})})\big) \cdot D\big(L_t^\infty(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-2}{2})})\big) \ \hookrightarrow \ D\big(L_t^2(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-3}{2})})\big) \ .$$

Our next step is to prove the estimates (385) for the temporal potential ${}^\omega C_0$. By an inspection of the elliptic equation (196), we see that modulo embeddings of the form (391)–(392) and the bounds we have already shown, we only need to establish things for the term $\nabla_t\Delta^{-1}([{}^\omega \underline{A}, {}^\omega \underline{C}])$. Again expanding the time derivative as $\pm{}^\omega \underline{L} \mp \omega\cdot\nabla_x$ and distributing the ${}^\omega \underline{L}$ derivative via the Leibniz rule, we are reduced knowing the following (which just represent another form of the embeddings (393)–(394)):

(395)
$$\Delta^{-1} \ : \ D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n})\big)\cdot D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\big) \ \hookrightarrow \ D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\big) \ .$$
(396)
$$\Delta^{-1} \ : \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n})\big)\cdot D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\big) \ \hookrightarrow \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-1}{2})}_{2,10n})\big) \ .$$

Finally, we wish to show the improved bounds (387)–(388). We work recursively here. First, we assume that the non-time differentiated versions of these estimates are valid. By the truncation condition (124c), we see that the proof of the estimate (387) with the extra operator $\partial_t\Delta^{-\frac{1}{2}}$ follows from the proof of this estimate without that operator. Thus, our aim is to establish the estimate (388) in the presence of the extra $\partial_t\Delta^{-\frac{1}{2}}$ derivatives. Applying this operator to the ${}^\omega\underline{L}$ differentiated Hodge system (195), we see that things can be handled with the help of the two bilinear inclusions:

(397)
$$\nabla_x\Delta^{-\frac{3}{2}} \ : \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-5}{2})}_{2,10n})\big)\cdot D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\big) \ \hookrightarrow \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n})\big) \ .$$
(398)
$$\nabla_x\Delta^{-\frac{3}{2}} \ : \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n})\big)\cdot D\big(L^\infty_t(\dot{B}^{p_\gamma,(2,\frac{n-4}{2})}_{2,10n})\big) \ \hookrightarrow \ D\big(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n})\big) \ .$$

Notice that the numerology in these last two estimates is a bit tight in $High\times High$ frequency regime. In particular, the condition (374) only has room of about 1/30 in $n=6$ dimensions. The next item on the stack for us is the estimates (387)–(388) without the extra derivative $\nabla_t\Delta^{-1}$. Assuming for the moment that this is true for (387), we see that the proof of (388) in this case follows easily from ${}^\omega\underline{L}$ differentiating the Hodge system (195) and applying a less singular version of the estimate (397). Therefore, we are now at the point where everything has been reduced to the proof of the first estimate (387). To do this we apply the following instance of the identity (319):

$$(399)\qquad {}^\omega\underline{L}{}^\omega\underline{A} \ = \ \nabla_x\ {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \underline{A}_{\bullet\ll 1}(\partial_\omega) \ - \ \nabla_x\ {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega) \ .$$

The estimate (387) for the first term on the right hand side of (399) is very simple and left to the reader. It follows from steps similar to the proof we gave above of the estimates (383). Notice that there are no singular angular factors here so there is a lot of room in this estimate if one takes into account the extra Coulomb savings (191).

We are now trying to prove (387) for the second term on the right hand side of (399) which we decompose into angular scales as:
(400)
$$\nabla_x\ {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega) \ = \ \sum_\theta\ \nabla_x\ {}^\omega\Pi_\theta{}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta^{-1}_{\omega^\perp}\,\widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega) \ .$$

Applying now the triangle inequality to the norm (371) at each fixed time, we are trying to bound the expression:

$$(401)\quad \| \nabla_x\, {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta_{\omega^\perp}^{-1}\, \widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)\,(t)\,\|_{D\big(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n}\big)}\ ,$$

$$= \sum_\theta \Big(\sum_{k=0}^{10n} \sum_\phi \sup_\omega \| \overline{b^\phi}_\theta (\theta\nabla_\xi)^k\, \nabla_x\, {}^{\omega}\Pi_\theta {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta_{\omega^\perp}^{-1}\, \widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)\,(t)\,\|^2_{\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,10n}} \Big)^{\frac{1}{2}} .$$

For each fixed value of θ, and for each fixed spatial frequency μ we have the following heuristic multiplier bound one the Coulomb savings are taken into account:

$$(\theta\nabla_\xi)^k\, \nabla_x\, {}^{\omega}\Pi_\theta {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}\ \Delta_{\omega^\perp}^{-1}\, P_\mu\, \widetilde{\mathcal{P}}\big([B,H]\big)(\partial_\omega)\,(t)\ \approx\ (\mu\theta)^{-1}\, {}^{\omega}\Pi_\theta P_\mu P_{\bullet\ll 1}\big([B,H]\big)\,(t)\ .$$

Taking this into account, and using the same multiplier reductions used to prove (389) above, we have the inequality:

$$\begin{aligned}
(\text{L.H.S.})(401)\ &\lesssim\ \sum_\theta \Big(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_{\bullet\ll 1}\big([B,H]\big)\,(t)\,\|^2_{\dot{B}^{p_\gamma,(2,\frac{n-5}{2})}_{2,10n}} \Big)^{\frac{1}{2}}\ ,\\
&\lesssim\ \sum_\theta \theta^\gamma \Big(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta \big([B,H]\big)\,(t)\,\|^2_{\dot{B}^{2,(2,\frac{n-5}{2})}_{2}} \Big)^{\frac{1}{2}}\ ,\\
&\lesssim\ \| \big([B,H]\big)\,(t)\,\|_{\dot{B}^{2,(2,\frac{n-3}{2})}_{2}}\ . \qquad (402)
\end{aligned}$$

This last set of inequalities results from the localized Besov inclusion:

$$\widetilde{{}^{\omega_0}\Pi}_\theta(\dot{B}^{p_\gamma,(2,\frac{n-5}{2})}_2)\ \subseteq\ \theta^{1+\gamma}\ \widetilde{{}^{\omega_0}\Pi}_\theta(\dot{B}^{2,(2,\frac{n-5}{2})}_2)\ ,$$

an orthogonality argument, and dyadic summing. Integrating the bound (402) L^2 in time, we see that our proof of (387) is reduced to showing the following:

$$\| [B,H] \|_{L^2_t(\dot{H}^{\frac{n-5}{2}})}\ \lesssim\ \mathcal{E}\ .$$

Keeping in mind the bootstrapping estimates (124f), we see that this last line is simply a more singular version of the embedding (134) shown above. In the $Low\times High$ case the proof follows from (135). In the $High\times Low$ case there is even more room and one can again use something similar to (135). In the $High\times High$ case we use the embedding:

$$P_\lambda\big(L^2(\dot{B}^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})\big)\cdot P_\lambda\big(L^\infty(\dot{H}^{\frac{n-4}{2}})\big)\ \hookrightarrow\ \Big(\frac{\mu}{\lambda}\Big)^\sigma P_\mu\big(L^2(\dot{H}^{\frac{n-5}{2}})\big)\ ,$$

where $\sigma = n(\frac{n-2}{n-1}) - \frac{5}{2}$. This last bound follows from the general frequency localized embedding (48). Note that in dimensions $6\leqslant n$ we have the necessary condition $0<\sigma$. This completes our proof of the estimate (387) and therefore our demonstration of Lemma 11.3. $\square$

11.1. **Proof of the Square Sum Strichartz Estimates.** We now come to what is perhaps the linchpin of our argument. These are the square sum structure estimates contained in (173a). With the current machinery in hand, these will be quite easy to establish. At the heart of things is whether the angular multipliers ${}^{\omega}\Pi_\theta$ "commute with the dynamics" of the covariant wave operator $\Box_{\underline{A}_{\bullet\ll 1}}$. At a quick first glance using Duhamel's principle, this seems to be connected with whether one can control the commutator $[{}^{\omega}\Pi_\theta, \Box_{\underline{A}_{\bullet\ll 1}}]$. Unfortunately, it is not too difficult to see that one runs into serious difficulties as soon as $\theta \ll 1$. This is not the end of the story however, because it turns out that modulo a very nice error term, one can control the commutator with the "integrated" form the equations $[{}^{\omega}\Pi_\theta, \Phi]$. This shows one of the deep advantages to working with the parametrix as opposed to dealing directly with the equations themselves[11]. We proceed as follows.

Our first step is to fix a scale θ and run a cap decomposition $\mathbb{S}^{n-1} = \cup_\phi \Gamma_\phi$. The next thing we do is to decompose the parametrix $\Phi(\widehat{f})$ into a sum of three pieces:

$$\begin{aligned}
\Phi(\widehat{f}) \ &= \ \int_{\mathbb{R}^n} e^{2\pi i\lambda {}^{\omega}u} \ {}^{\omega}g^{-1}_{\bullet\ll\theta} \, \widehat{f}(\lambda\omega) \, {}^{\omega}g_{\bullet\ll\theta} \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \\
&\qquad + \ \int_{\mathbb{R}^n} e^{2\pi i\lambda {}^{\omega}u} \ {}^{\omega}g^{-1}_{\bullet\ll\theta} \, \widehat{f}(\lambda\omega) \, {}^{\omega}g_{\theta\lesssim\bullet} \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \\
&\qquad\qquad + \ \int_{\mathbb{R}^n} e^{2\pi i\lambda {}^{\omega}u} \ {}^{\omega}g^{-1}_{\theta\lesssim\bullet} \, \widehat{f}(\lambda\omega) \, {}^{\omega}g \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \ , \\
&= \ I_1 \ + \ I_2 \ + \ I_3 \ .
\end{aligned}$$

Here:

$$ {}^{\omega}g \ = \ {}^{\omega}g_{\bullet\ll\theta} + {}^{\omega}g_{\theta\lesssim\bullet} \ = \ P_{\bullet\ll\theta}({}^{\omega}g) + P_{\theta\lesssim\bullet}({}^{\omega}g) \ , $$

is a low-high frequency decomposition of the group element ${}^{\omega}g$. We define the decomposition for ${}^{\omega}g^{-1}$ similarly. Our goal is now to prove the following three estimates:

$$ \sum_{\substack{\phi\ : \\ \omega_0\in\Gamma_\phi}} \| \, {}^{\omega_0}\Pi_\theta P_1(I_1) \, \|^2_{L^2(L^{\frac{2(n-1)}{n-3}})} \ \lesssim \ \| \widehat{f} \, \|^2_{L^2} \ , \tag{403} $$

$$ \sum_{\substack{\phi\ : \\ \omega_0\in\Gamma_\phi}} \| \, {}^{\omega_0}\Pi_\theta P_1(I_2) \, \|^2_{L^2(L^{\frac{2(n-1)}{n-3}})} \ \lesssim \ \| \widehat{f} \, \|^2_{L^2} \ , \tag{404} $$

$$ \sum_{\substack{\phi\ : \\ \omega_0\in\Gamma_\phi}} \| \, {}^{\omega_0}\Pi_\theta P_1(I_3) \, \|^2_{L^2(L^{\frac{2(n-1)}{n-3}})} \ \lesssim \ \| \widehat{f} \, \|^2_{L^2} \ . \tag{405} $$

The proof of the first bound (403) follows easily from the plain endpoint Strichartz estimate we have already established. To see this, first notice that for a fixed angle

[11]This also seems to have far reaching philosophical consequences for how one should proceed in lower dimensions. Specifically, it seems to suggest that the correct "covariant" $X^{s,\theta}$ space should be defined in terms of the parametrix and *not* in terms of the symbol of the covariant equation.

one has the identity:

$$\begin{aligned} &{}^{\omega_0}\Pi_\theta P_1 \int_{\mathbb{R}^n} e^{2\pi i\lambda^\omega u}\ {}^\omega g^{-1}_{\bullet\ll\theta}\ \widehat{f}(\lambda\omega)\,{}^\omega g_{\bullet\ll\theta}\ \chi_{(\frac{1}{2},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega \\ &\quad = \ {}^{\omega_0}\Pi_\theta P_1 \int_{\mathbb{R}^n} e^{2\pi i\lambda^\omega u}\ {}^\omega g^{-1}_{\bullet\ll\theta}\ \overline{b^{\phi'}}_\theta(\omega)\widehat{f}(\lambda\omega)\,{}^\omega g_{\bullet\ll\theta}\ \chi_{(\frac{1}{2},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\ , \end{aligned}$$

where $\Gamma_\phi \subset 2\Gamma_{\phi'}$ is some fixed thickening of the spherical cap that $\omega_0 \in \Gamma_\phi$. That this is the case follows easily from the fact that the Fourier transform of the function:

$$e^{2\pi i x\cdot\xi}\ {}^\omega g^{-1}_{\bullet\ll\theta}(x)\ \widehat{f}(\lambda\omega)\,{}^\omega g_{\bullet\ll\theta}(x)\ ,$$

is a tempered distribution with support contained in an $O(c\theta)$ neighborhood of the point ξ for some small constant c, uniform in the value of θ. Using now the boundedness of the multiplier ${}^{\omega_0}\Pi_\theta P_1$, we only need to establish that the truncated parametrix I_1 obeys the endpoint Strichartz estimate. We reduce this claim further by writing this integral in the form:

$$I_1 \ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K^{P_{\bullet\ll\theta}}(w)K^{P_{\bullet\ll\theta}}(y)\int_{\mathbb{R}^n} e^{2\pi i\lambda^\omega u}\ {}^\omega g^{-1}_w\ \widehat{f}(\lambda\omega)\,{}^\omega g_y\ \chi_{(\frac{1}{2},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\ dwdy\ ,$$

where ${}^\omega g^{-1}_w(x) = {}^\omega g^{-1}(x-w)$ and ${}^\omega g_y(x) = {}^\omega g(x-y)$ denote the translated group elements. Using the fact that the convolution kernel $K^{P_{\bullet\ll\theta}}$ has $O(1)$ L^1 norm uniform in the value of θ, we are left with establishing the L^2 and dispersive estimates of the previous sections for more general kernels of the form:

$$\Phi_{g_1,g_2}(\widehat{f}) \ = \int_{\mathbb{R}^n} e^{2\pi i\lambda^\omega u}\ {}^\omega g_1^{-1}\ \widehat{f}(\lambda\omega)\,{}^\omega g_2\ \chi_{(\frac{1}{2},2)}(\lambda)\ \lambda^{n-1}d\lambda d\omega\ , \tag{406}$$

where ${}^\omega g_1$ and ${}^\omega g_2$ are unrelated group elements which are generated from Hodge systems and connections of the form (192)–(196), which satisfy the general requirements (124) and (156) for $\lambda = 1$. This indeed turns out to be the case, and the key observation is that by using the identity (15), all of the TT^* arguments go through just as they did in previous sections.

It remains for us to prove the bounds (404)–(405). These are essentially identical to each other so we concentrate on the proof of the first of these, leaving the other one to the reader. By an application of Bernstein's inequality and orthogonality, we see that it suffices for us to show the estimate:

$$\|\,\theta\ I_2\,\|_{L^2(L^2)}\ \lesssim\ \|\,\widehat{f}\,\|_{L^2}\ . \tag{407}$$

At a heuristic level, this estimate is true because one has the identity $\theta\ {}^\omega g_{\theta\lesssim\bullet} \approx \nabla_x{}^\omega g = g^\omega\underline{C}$. And we see that in this case things would follow easily from the $D\big(L^2(L^\infty)\big)$ contained in the estimates (385). To implement this in a rigorous way, we derive the following elliptic equation for ${}^\omega g_{\theta\lesssim\bullet}$ based on the formulas (194):

$$\begin{aligned} {}^\omega g_{\theta\lesssim\bullet} \ &= \ \nabla^i\Delta^{-1}P_{\theta\lesssim\bullet}({}^\omega g^\omega\underline{C}_i)\ , \\ &= \ \sum_{\substack{\lambda:\\ \theta\lesssim\lambda}} \nabla^i\Delta^{-1}P_\lambda({}^\omega g^\omega\underline{C}_i)\ . \end{aligned}$$

If denote the (vector) kernel of operator $\nabla_x\Delta^{-1}P_\lambda$ by $K^{\nabla\Delta^{-1}}_\lambda$, then we have the uniform L^1 bounds:

$$\|\,K^{\nabla\Delta^{-1}}_\lambda\,\|_{L^1}\ \lesssim\ \lambda^{-1}\ .$$

Using this and taking into account the previous reductions used in the proof of estimate (403) above we easily arrive at the bound:

$$\| \theta \ I_2 \|_{L^2(L^2)} \lesssim \sum_{\substack{\lambda : \\ \theta \lesssim \lambda}} \left(\frac{\theta}{\lambda}\right) \sup_{w,y} \| \widetilde{I}_{w,y} \|_{L^2(L^2)} \ ,$$
$$\lesssim \sup_{w,y} \| \widetilde{I}_{w,y} \|_{L^2(L^2)} \ ,$$

where $\widetilde{I}_{w,y}$ is the family of translated kernels:

$$\widetilde{I}_{w,y} = \int_{\mathbb{R}^n} e^{2\pi i \lambda {}^\omega u} \ {}^\omega g_w^{-1} \ \widehat{f}(\lambda\omega) \, {}^\omega g_y {}^\omega \underline{C}_y \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \ dwdy \ , \tag{408}$$

where we have also now set ${}^\omega\underline{C}_y(x) = {}^\omega\underline{C}(x-y)$. Using the decomposable estimate (370), we now have that:

$$\| \widetilde{I}_{w,y} \|_{L^2(L^2)} \lesssim \| {}^\omega\underline{C}_y \|_{D\left(L^2(L^\infty)\right)} \cdot \| I_{w,y} \|_{L^\infty(L^2)} \ ,$$

where the integral $I_{w,y}$ is the same as $\widetilde{I}_{w,y}$ but with the matrix ${}^\omega\underline{C}_y$ removed. Using now the nesting:

$$D\big(L^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big) \subseteq D\big(L^2(L^\infty)\big) \ , \tag{409}$$

the estimate (385), and the remarks made above about general kernels of the form (406), we have the pair of estimates:

$$\| {}^\omega\underline{C}_y \|_{D\left(L^2(L^\infty)\right)} \lesssim \mathcal{E} \ , \qquad \| I_{w,y} \|_{L^\infty(L^2)} \lesssim \| \widehat{f} \|_{L^2} \ ,$$

uniform in the values of w, y. This is enough to prove the estimate (404). This completes our proof of the square sum Strichartz estimates contained in (173a).

11.2. **Proof of the Differentiated Strichartz Estimates** (173b)–(173c)**.** To wrap things up for this overall section, we prove the estimates (173b)–(173c). This will follows easily from the general list of decomposable estimates contained in Lemma 11.3. In what follows, we will only bother to prove the time differentiated estimate (173c). The proof of the gradient estimate (173b) follows from identical reasoning and is left to the reader (in fact one only need apply the plain Strichartz estimates shown in previous sections followed by a $D\big(L^\infty(L^\infty)\big)$ estimate for the spatial potentials $\{{}^\omega\underline{C}\}$). Time differentiating the parametrix $\Phi^\pm(\widehat{f})$ we see that:

$$\begin{aligned}
\nabla_t \ \Phi^\pm(\widehat{f}) \ &= \int_{\mathbb{R}^n} (\pm 2\pi i \, |\xi|) \, e^{2\pi i \lambda {}^\omega u^\pm} \ {}^\omega g_\pm^{-1} \ \widehat{f}(\lambda\omega) \, {}^\omega g_\pm \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \\
&\quad + \int_{\mathbb{R}^n} e^{2\pi i \lambda {}^\omega u^\pm} \ \big[{}^\omega g_\pm^{-1} \ \widehat{f}(\lambda\omega) \, {}^\omega g_\pm \ , \ {}^\omega C_0^\pm \big] \ \chi_{(\frac{1}{2},2)}(\lambda) \ \lambda^{n-1} d\lambda d\omega \ , \\
&= \Phi^\pm(\pm 2\pi i \, |\xi| \, \widehat{f}) \ + \ \widetilde{I} \ .
\end{aligned}$$

Therefore, our task is to show the pair of estimates:

$$\| P_1 \widetilde{I}_1 \|_{L^2(SL^{\frac{2(n-1)}{n-3}})} \lesssim \mathcal{E} \cdot \| \widehat{f} \|_{L^2} \ , \tag{410}$$

$$\| P_1 \widetilde{I}_1 \|_{L^\infty(L^2)} \lesssim \mathcal{E} \cdot \| \widehat{f} \|_{L^2} \ . \tag{411}$$

The estimate (410) follows from essentially identical reasoning to that employed in the proof of estimates (404)–(405)above. The main point is to drop to $L^2(L^2)$ via Bernstein, and then use the $D\big(L^2(L^\infty)\big)$ estimate for the potential ${}^\omega C_0$ contained on line (385) above. The proof of the second estimate (411) above follows easily from the $D\big(L^\infty(L^\infty)\big)$ estimate for ${}^\omega C_0$ contained on line (385) above. Specifically, one has the nesting:

$$D\big(L^\infty(\dot{B}^{p_\gamma,(2,\frac{n-2}{2})}_{2,10n})\big) \ \subseteq \ D\big(L^\infty(L^\infty)\big) \ .$$

This completes our demonstration of (173b)–(173c) and ends this section.

12. Completion of the proof: Controlling the $L^1(L^2)$ Norm of the Differentiated Parametrix

Our final task here is to prove the estimate (173e) which guarantees that our parametrix is a good approximation the covariant wave equation $\Box_{\underline{A}_{\bullet\ll 1}}$. This essentially boils down to applying the estimates (383)–(388) to the various error terms listed on the right hand side of equation (198) above. We will prove the desired estimates for each of these terms separately.

Decomposing the term $\underline{A}_{\bullet\ll 1}({}^\omega L^\mp) - {}^\omega C^\pm({}^\omega L^\mp)$. This represents the worst error term which comes out of our approximation, as well as the main "renormalization" which the parametrix creates. In what follows we will eliminate the $\pm$ notation on favor of the ${}^\omega\underline{L}$ notation introduced on line (311) above. Using this convention, a short computation involving the formulas (195)–(196) and the structure equation (124e) yields the identity:

$$\begin{aligned} &\underline{A}_{\bullet\ll 1}({}^\omega\underline{L}) - {}^\omega C({}^\omega\underline{L}) \\ &= \ (I - {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)})\underline{A}_{\bullet\ll 1}(\partial_\omega) \ + \ {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\Delta^{-1}_{\omega^\perp}\widetilde{\mathcal{P}}([B,H])(\partial_\omega) \\ &\qquad + \ {}^\omega\underline{L}\Delta^{-1}([{}^\omega\underline{A}, {}^\omega\underline{C}]) \ - \ d^*\Delta^{-1}[{}^\omega\underline{C}, {}^\omega C({}^\omega\underline{L})] \ , \\ &= \ T_1 \ + \ T_2 \ + \ T_3 \ + \ T_4 \ . \end{aligned} \tag{412}$$

Our goal is prove the following four estimates:

$$\|\, T_1 \,\|_{D\big(L^2(L^{n-1})\big)} \ \lesssim \ \mathcal{E} \ , \qquad\qquad \|\, T_2 \,\|_{D\big(L^1(L^\infty)\big)} \ \lesssim \ \mathcal{E} \ , \tag{413}$$

$$\|\, T_3 \,\|_{D\big(L^1(L^\infty)\big)} \ \lesssim \ \mathcal{E} \ , \qquad\qquad \|\, T_4 \,\|_{D\big(L^1(L^\infty)\big)} \ \lesssim \ \mathcal{E} \ . \tag{414}$$

To prove the first estimate on line (413), we see from the decomposable version of the Besov nesting (40) that is suffices to prove the following:

$$\|\, (I - {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)})\underline{A}_{\bullet\ll 1}(\partial_\omega) \,\|_{D\big(L^2(\dot{B}_2^{n-1,(2,\frac{n(n-3)}{2(n-1)})})\big)} \ \lesssim \ \mathcal{E} \ .$$

By the square sum nature of the Besov and decomposable norms, and keeping in mind the Besov version of the endpoint Strichartz estimate contained in the bootstrapping estimate (124d), we see that it suffices to prove this estimate at

fixed frequency. Thus, we are trying to prove that:
(415)
$$\| (I - {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)})P_\mu \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \|_{D\left(L^2(L^{n-1})\right)} \ \lesssim \ \| P_\mu \underline{A}_{\bullet\ll 1} \|_{L^2(S\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})} \ .$$

Decomposing the term on the left hand side of this expression into all dyadic angular regions spread from the direction ω this is further reduced to showing that:

$$\| {}^{\omega}\Pi_\theta(I - {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)})P_\mu \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \|_{D_\theta\left(L^2(L^{n-1})\right)} \ \lesssim \ \theta^\gamma \| P_\mu \underline{A}_{\bullet\ll 1} \|_{L^2(S\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})} \ .$$

Notice that we are only trying to show this for values $\theta \lesssim \ \mu^{\frac{1}{2}-\delta}$. Further computing the term on the left hand side of this last expression, and applying the heuristic multiplier bound (also using the Coulomb savings (191)):

$$(\theta\nabla_\xi^k){}^{\omega}\Pi_\theta(I - {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)})P_\mu \ \underline{A}_{\bullet\ll 1}(\partial_\omega) \ \approx \ \theta \ {}^{\omega}\Pi_\theta P_\mu \ \underline{A}_{\bullet\ll 1} \ .$$

Plugging this into the definition of the norm $D_\theta\big(L^2(L^{n-1})\big)$, using the multiplier-sum reductions employed in the proof of the inequality (389), and reverting back to Besov notation we have the inequality sequence involving Bernstein's inequality (52) and a simple index manipulation:

$$\begin{aligned}
(\text{L.H.S.})(415) \ &\lesssim \ \theta \ \Bigg(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\mu \underline{A}_{\bullet\ll 1} \|^2_{L^2(\dot{B}_2^{n-1,(2,\frac{n(n-3)}{2(n-1)})})}\Bigg)^{\frac{1}{2}} \ , \\
&\lesssim \ \theta^{\frac{n-3}{2}} \Bigg(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\mu \underline{A}_{\bullet\ll 1} \|^2_{L^2(\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n(n-3)}{2(n-1)})})}\Bigg)^{\frac{1}{2}} \ , \\
&\lesssim \ \theta^{\frac{n-3}{2}}\mu^{\frac{-n-1}{2(n-1)}} \Bigg(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\mu \underline{A}_{\bullet\ll 1} \|^2_{L^2(\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})})}\Bigg)^{\frac{1}{2}} \ .
\end{aligned}$$

Estimate (415) now follows from the fact that:

$$\theta^{\frac{n-3}{2}}\mu^{\frac{-n-1}{2(n-1)}} \ \lesssim \ \theta^\gamma \ ,$$

which is a consequence of the truncation condition $\theta \lesssim \ \mu^{\frac{1}{2}-\delta}$ and the fact that $6 \leqslant n$, and the fact that we have chosen δ, γ according to (185). This ends our proof of the first estimate on line (413).

Our next step is to prove the second estimate on line (413) above. We will show the somewhat more regular estimate:

$$\| {}^{\omega}\overline{\Pi}^{(\frac{1}{2}-\delta)}\Delta^{-1}_{\omega^\perp}\widetilde{\mathcal{P}}([B,H])(\partial_\omega) \|_{D\left(L^1(\dot{B}_1^{\infty,(n,\frac{n}{2})})\right)} \ \lesssim \ \mathcal{E} \ . \tag{416}$$

Decomposing the term inside the norm on the left hand side of this last inequality into dyadic angular scales, applying the definition of the fixed scale decomposable norms $D_\theta\big(L^1(\dot{B}_1^{\infty,(n,\frac{n}{2})})\big)$, using the (fixed time) fixed frequency heuristic multiplier bound (which again takes into account the savings (191)):

$$(\theta\nabla_\xi)^k \ {}^\omega\Pi_\theta {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\Delta_{\omega^\perp}^{-1} P_\lambda \ \widetilde{\mathcal{P}}([B,H])(\partial_\omega) \ \approx \ \theta^{-1}\lambda^{-2} \ {}^\omega\Pi_\theta P_\lambda([B,H]) \ ,$$

expanding the resulting expression into a trichotomy, applying the multiplier square sum reduction used previously in the proof of estimate (389) above, and keeping in mind the bootstrapping structure estimates (124f), we see that the estimate (416) reduces to the demonstration of the following three fixed time bounds:

(417)

$$\sum_{\substack{\lambda,\mu_i \ : \\ \mu_1\ll\mu_2\sim\lambda}} \lambda^{-2} \Big(\sum_{\substack{\phi \ : \\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\lambda([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^\infty}^2 \Big)^{\frac{1}{2}}$$
$$\lesssim \ \theta^{1+\gamma} \ \| B(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \cdot \| H(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \ ,$$

(418)

$$\sum_{\substack{\lambda,\mu_i \ : \\ \mu_2\ll\mu_1\sim\lambda}} \lambda^{-2} \Big(\sum_{\substack{\phi \ : \\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\lambda([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^\infty}^2 \Big)^{\frac{1}{2}}$$
$$\lesssim \ \theta^{1+\gamma} \ \| B(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \cdot \| H(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \ ,$$

(419)

$$\sum_{\substack{\lambda,\mu_i \ : \\ \lambda\lesssim\mu_1\sim\mu_2}} \lambda^{-2} \Big(\sum_{\substack{\phi \ : \\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\lambda([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^\infty}^2 \Big)^{\frac{1}{2}}$$
$$\lesssim \ \theta^{1+\gamma} \ \| B(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \cdot \| H(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \ .$$

We begin with the proof of the first estimate (417). This is the most singular of the three. Fixing all of the spatial frequencies on the left hand side of this bound, we see that by an application of Young's inequality, it suffices to prove the following refinement:

$$\lambda^{-2} \Big(\sum_{\substack{\phi \ : \\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi}_\theta P_\lambda([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^\infty}^2 \Big)^{\frac{1}{2}}$$
$$\lesssim \ \Big(\frac{\mu_1}{\mu_2}\Big)^\gamma \theta^{1+\gamma} \ \| P_{\mu_1}(B)(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}} \cdot \| P_{\mu_2}(H)(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}} \ . \tag{420}$$

This bound is scale invariant, so we may assume that $1 = \lambda \sim \mu_2$. To aid in the demonstration, we introduce the auxiliary index:

$$\widetilde{r}_\gamma \ = \ \frac{2n(n-1)}{n^2-2n-1-2(n-1)\gamma} \ .$$

Notice that this has been chosen precisely so that one has the identity:

$$\gamma \ = \ \frac{1}{2} + n(\frac{n-3}{2(n-1)} - \frac{1}{\widetilde{r}_\gamma}) \ ,$$

so that ultimately we can make a reference to the fixed frequency bound (49). The problem here is that we have $2 < \widetilde{r}_\gamma$ (in any dimension), so we are going to run into orthogonality issues in the square-sum on the left hand side of (420). This will end up costing some extra powers of θ^{-1}, but luckily the Bernstein inequality will more than make up for this. Applying Bernstein to each term in the sum on the left hand side of (420) we arrive at the bound:

$$\text{(L.H.S.)(420)} \ \lesssim\ \theta^{\frac{n-1}{\tilde{r}_\gamma}} \Big(\sum_{\substack{\phi\,:\\ \omega_0\in\Gamma_\phi}} \| \widetilde{{}^{\omega_0}\Pi_\theta} P_1([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^{\tilde{r}_\gamma}}^2 \Big)^{\frac{1}{2}} \ . \tag{421}$$

To get rid of the square-sum on the right hand side of this last expression, we introduce the following map from $L^p(\mathbb{R}^n)$ to $\ell^2(L^p(\mathbb{R}^n))$:

$$\mathcal{T}^\theta(A) \ =\ \Big(\widetilde{{}^{\omega_1}\Pi_\theta} P_1(A), \ldots, \widetilde{{}^{\omega_N}\Pi_\theta} P_1(A) \Big) \ ,$$

where $(\omega_1, \ldots, \omega_N)$ is some ordering of the Γ_ϕ spherical cap "base-points". Notice that there are $N \sim \theta^{1-n}$ of these. By orthogonality, and using the uniform boundedness of the multipliers ${}^\omega\Pi_\theta P_1$ on L^∞ we have the pair of estimates:

$$\begin{aligned} \| \mathcal{T}^\theta(A) \|_{\ell^2(L^2)} \ &\lesssim\ \| P_1(A) \|_{L^2} \ , \\ \| \mathcal{T}^\theta(A) \|_{\ell^2(L^\infty)} \ &\lesssim\ \theta^{\frac{1-n}{2}} \ \| P_1(A) \|_{L^\infty} \ . \end{aligned}$$

By interpolating these to bounds in the pair of spaces $(\ell^2(L^2), \ell^2(L^\infty))$ and (L^2, L^∞) (see [1]), we have the bound:

$$\| \mathcal{T}^\theta(A) \|_{\ell^2(L^{\tilde{r}_\gamma})} \ \lesssim\ \theta^{(1-n)(\frac{1}{2} - \frac{1}{\tilde{r}_\gamma})} \ \| P_1(A) \|_{L^{\tilde{r}_\gamma}} \ .$$

Plugging this last estimate into the right hand side of (421) above, and finally applying generic fixed frequency estimate (49) we have that:

$$\begin{aligned} &\text{(L.H.S.)(420)} \ , \\ \lesssim\ & \theta^{(n-1)(\frac{2}{r_\gamma} - \frac{1}{2})} \ \| P_1([P_{\mu_1}(B)(t), P_{\mu_2}(H)(t)]) \|_{L^{\tilde{r}_\gamma}} \ , \\ \lesssim\ & \left(\frac{\mu_1}{\mu_2}\right)^\gamma \theta^{(n-1)(\frac{2}{r_\gamma} - \frac{1}{2})} \ \| P_{\mu_1}(B)(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3}, (2, \frac{n-1}{2})}} \cdot \| P_{\mu_2}(H)(t) \|_{\dot{B}_2^{\frac{2(n-1)}{n-3}, (2, \frac{n-3}{2})}} \ . \end{aligned}$$

The estimate (420) now follows from the bound:

$$\theta^{(n-1)(\frac{2}{r_\gamma} - \frac{1}{2})} \ \lesssim\ \theta^{1+\gamma} \ ,$$

which holds in dimensions $6 \leqslant n$. We leave the verification of this to the reader. This ends our demonstration of the *Low* $\times$ *High* frequency estimate (417). Notice that the second estimate (418) is simply a less singular version of this. In fact, repeating the above procedure, we see that in that case there is an extra factor of $(\frac{\mu_2}{\mu_1})$ in the analog of the fixed frequency bound (420).

We have now reduced the proof of the second estimate on line (413) to the *High* $\times$ *High* interaction bound (419). By applying the $L^\infty \to L^2$ version of Bernstein, using orthogonality, and then applying the general bound (48), we have the fixed frequency estimate:

$$\lambda^{-2}\Big(\sum_{\substack{\phi\ :\\ \omega_0\in\Gamma_\phi}} \|\,\widetilde{{}^{\omega_0}\Pi}_\theta P_\lambda([P_{\mu_1}(B)(t),P_{\mu_2}(H)(t)])\,\|_{L^\infty}^2\Big)^{\frac{1}{2}}\ ,$$
$$\lesssim \theta^{\frac{n-1}{2}}\lambda^{\frac{n-4}{2}}\ \|\,P_\lambda([P_{\mu_1}(B)(t),P_{\mu_2}(H)(t)])\,\|_{L^2}\ ,$$
$$\lesssim \Big(\frac{\lambda}{\mu_1}\Big)^{\sigma}\theta^{\frac{n-1}{2}}\ \|\,P_{\mu_1}(B)(t)\,\|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-1}{2})}}\cdot\|\,P_{\mu_2}(H)(t)\,\|_{\dot{B}_2^{\frac{2(n-1)}{n-3},(2,\frac{n-3}{2})}}\ ,$$

where $0<\sigma=n(\frac{n-3}{n-1})-2$. By summing this last line and then applying Cauchy-Schwartz, we easily arrive at the bound (419).

To finish this subsection, we only need to prove the two estimates on line (414) above. To show the first estimate involving the T_3 term, we simply expand the ${}^\omega\underline{L}$ derivative into the product via the Leibniz rule, and then use the decomposable bounds (383) and (385) and (387)–(388) in conjunction with the following instance of the bilinear decomposable estimate (372):

$$\Delta^{-1}\ :\ D\big(L_t^2(\dot{B}_{2,10n}^{p_\gamma,(2,\frac{n-3}{2})})\big)\cdot D\big(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big)\ \hookrightarrow\ D\big(L_t^1(\dot{B}_1^{\infty,(2,\frac{n}{2})})\big)\ .$$

To show the second bound on line (414), we again use the estimates (383) and (385), this time in conjunction with:

$$\nabla_x\Delta^{-1}\ :\ D\big(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big)\cdot D\big(L_t^2(\dot{B}_{2,10n}^{q_\gamma,(2,\frac{n-1}{2})})\big)\ \hookrightarrow\ D\big(L_t^1(\dot{B}_1^{\infty,(2,\frac{n}{2})})\big)\ .$$

This completes our decomposable estimates for the error term $\underline{A}_{\bullet\ll 1}({}^\omega L^\mp)-{}^\omega C^\pm({}^\omega L^\mp)$.

Decomposing the term $D_\alpha^{\underline{A}_{\bullet\ll1}}\big({}^\omega C^\pm\big)^\alpha$. Again dropping the $\pm$ notation and using the equations (195)–(196) and the identity (311) as well as the structure equation (124e), we can write this as:

$$\begin{aligned} D_\alpha^{\underline{A}_{\bullet\ll1}}({}^\omega C)^\alpha\ =\ &-\ {}^\omega\overline{\Pi}^{(\frac{1}{2}-\delta)}\,{}^\omega L\,\Delta_{\omega^\perp}^{-1}\,\widetilde{\mathcal{P}}([B,H])(\partial_\omega)\\ &\qquad+\ \ (\pm{}^\omega\underline{L}\mp\omega\cdot\nabla_x)\nabla_t\Delta^{-1}[{}^\omega\underline{A},{}^\omega\underline{C}]\ +\ \nabla_t d^*\Delta^{-1}[{}^\omega C_0,{}^\omega\underline{C}]\\ &\qquad\qquad-\ [{}^\omega\underline{A},{}^\omega\underline{C}]\ +\ [\underline{A}_{\bullet\ll1},{}^\omega\underline{C}]\ ,\\ =\ &\widetilde{T}_2\ +\ \widetilde{T}_3\ +\ \widetilde{T}_4\ +\ \widetilde{T}_5\ +\ \widetilde{T}_6\ . \end{aligned}$$

We will show that all of these terms obey the estimate:

$$\|\,\widetilde{T}_k\,\|_{D\big(L^1(L^\infty)\big)}\ \lesssim\ \mathcal{E}\ ,\qquad\qquad 2\ \leqslant\ k\ \leqslant\ 6\ . \tag{422}$$

Notice that, for the most part, the terms $\widetilde{T}_k$ represent less singular versions of the T_k on line (412) above. In fact, they can all be treated using similar embeddings by simply wasting one derivative. Specifically, the estimate (422) for the first term $\widetilde{T}_2$ follows directly from (416) above once one takes into account the presence of the truncation (124c) inherent in the projection $\widetilde{\mathcal{P}}$. To prove the estimate (422) for the portion of term $\widetilde{T}_3$ containing the ∇_t derivative, we use the same embedding employed in the proof of the estimate for T_3 on line (414) above. This follows because one can distribute the time derivative and simply waste smoothness in the estimates (384), (386), and (387)–(388). Specifically, by taking advantage of the low frequency behavior of these estimates, we have the bounds:

$$\| \nabla_t {}^{\omega}\underline{A} \|_{D\left(L^2_t(\dot{B}^{q_\gamma,(2,\frac{n-1}{2})}_{2,9n})\right)} \lesssim \mathcal{E} , \qquad \| \nabla_t {}^{\omega}\underline{C} \|_{D\left(L^2_t(\dot{B}^{q_\gamma,(2,\frac{n-1}{2})}_{2,9n})\right)} \lesssim \mathcal{E} , \tag{423}$$

$$\| \nabla_t {}^{\omega}\underline{L}{}^{\omega}\underline{A} \|_{D\left(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,9n})\right)} \lesssim \mathcal{E} , \quad \| \nabla_t {}^{\omega}\underline{L}{}^{\omega}\underline{C} \|_{D\left(L^2_t(\dot{B}^{p_\gamma,(2,\frac{n-3}{2})}_{2,9n})\right)} \lesssim \mathcal{E} . \tag{424}$$

Using a similar strategy, we can prove the estimate (422) for the portion of $\widetilde{T}_3$ containing the $\omega \cdot \nabla_x$ derivative (notice that the functions ω_i are trivially decomposable) as well as the term $\widetilde{T}_4$ in the same way as we showed (414) for the term T_4 above. All we need to do is to show the estimate:

$$\| \nabla_t {}^{\omega}C_0 \|_{D\left(L^2_t(\dot{B}^{q_\gamma,(2,\frac{n-1}{2})}_{2,9n})\right)} \lesssim \mathcal{E} .$$

This follows in the same way we proved the undifferentiated estimate (385) for ${}^{\omega}C_0$ above, but instead of using the undifferentiated versions of (383), (385), and (387)–(388), we simply use (423)–(424). Finally, notice that the proof of the estimate (422) for the terms $\widetilde{T}_5$ and $\widetilde{T}_6$ above follows by simply multiplying (decompose twice!) the $D\big(L^2(L^\infty)\big)$ estimate which is implied by the bounds (383) and (385) above. This completes our decomposition of the second error term on the right hand side of (198) above.

Decomposing the term $\Big[\underline{A}^{\alpha}_{\bullet\ll 1} - ({}^{\omega}C^{\pm})^{\alpha} \ , \ \big[(\underline{A}_{\bullet\ll 1})_{\alpha} - {}^{\omega}C^{\pm}_{\alpha} \ , \ \bullet \ \big]\Big]$**.** Here we again use the norm $D\big(L^1(L^\infty)\big)$, which we can achieve as a product of $D\big(L^2(L^\infty)\big)$ estimates, again making an appeal to (383) and (385) above.

This completes our proof of the approximation estimate (173e) and thus, at last, the proof of Proposition 7.2 which allows us to close the bootstrapping begun in Proposition (6.1). FS.

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