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Fixed-Structure $\mathscr{H}_2$ Controller Design for Polytopic Systems Via LMIs

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SUMMARY

In this paper a new approach for fixed-structure H_2 controller design in terms of solutions to a set of linear matrix inequalities are given. Both discrete- and continuous-time single-input single-output (SISO) time-invariant systems are considered. Then the results are extended to systems with polytopic uncertainty. The presented methods are based on an inner convex approximation of the non-convex set of fixed-structure H_2 controllers. The designed procedures initialized either with a stable polynomial or with a stabilizing controller. An iterative procedure for robust controller design is given that converges to a suboptimal solution. The monotonic decreasing of the upper bound on the H_2 norm is established theoretically for both nominal and robust controller design. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: Fixed-order control; H_2 performance; polytopic systems; Linear matrix inequality

1. INTRODUCTION

Fixed-structure controller design is a challenging problem in theory and practice. A fixed-structure controller design problem arises when simplicity, hardware limitations, or reliability in the implementation of a controller are considered as important issues. Moreover, the desired closed-loop performance may enforce a predefined structure for the to-be-designed controller. It is well known that fixed-order controller design in the nominal case, without parametric uncertainty, leads to either a non-convex rank constraint or bilinear matrix inequalities (BMIs) which are computationally intractable. Some researchers have tried to solve these non-convex or BMIs problems to find the local optimal controllers, see e.g. [1], [2] and [3]. Several iterative methods for reduced-order controller design have been proposed over recent years; see, for instance, [4] and [5], and references therein. In [4], the fixed-order controller synthesis problem is formulated as a regular SDP program with additional nonlinear equality constraints. A nonsmooth optimization technique to solve fixed-structure controller synthesis is developed in [5]. The provided gradient-based method converges to

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a local minimum and must be initialized with a stabilizing controller. In [6] an LMI approach to the multiobjective synthesis of linear output-feedback controllers is presented thanks to the linearizing change of variable. The extension of these methods to systems with parametric uncertainty is not evident.

Polytopic representation is a general way of describing the lack of knowledge on the physical system parameters. This kind of uncertainty covers the interval parametric uncertainty [7], ellipsoidal parametric uncertainty [8], as well as multi-model systems. Fixed-structure controller design for polytopic systems becomes more complicated. An LMI based convex optimization problem for robust pole placement with sensitivity function shaping in H_2 norm, using a fixed-order controller, is proposed in [9]. In this paper, the LMI-based method for the fixed-order H_2 controller design is combined with pole placement. In [10] an approach for the fixed-order H_2 controller design is presented. The proposed method is based on the positive polynomial matrices concept. In these papers, a convex set of stabilizing controllers is parameterized such that the closed-loop characteristic polynomial divided by a so-called central polynomial is a strictly positive real (SPR) transfer function. This convex set is an inner approximation of the non-convex set of all fixed-order stabilizing controllers and the quality of this approximation is related to the choice of the central polynomial (SPR-maker). In [11], the constrained dynamic output feedback H_2 control for polytopic systems is investigated. A numerical cross decomposition algorithm is developed and applied for the design of a fixed-order strictly proper controller.The problem of dynamic output-feedback control design for polytopic systems is considered in [12]. The approach is based on BMI optimization initiated from a robust state feedback controller. The robust static output feedback controller synthesis for polytopic systems is considered in [13], [14], [15], [16], [17], [18] and more recently in [19], [20] and [21].

In the present paper, we have provided a new method for dynamic fixed-structure H_2 controller design using some LMIs. Although, it is easy to see that these LMI conditions contain the KYP lemma constraints, the SPRness concept of transfer functions has not been used explicitly for their derivation. The convergence property of the proposed approach to a local minimum is established. The quality of the obtained suboptimal solution depends on an initial central polynomial. The initial central polynomial can be a stable polynomial or the closed-loop characteristic polynomial computed for an initial stabilizing controller. Then, the results are extended to fixed-structure H_2 controller design for polytopic systems in terms of solutions to a set of LMIs. The presented design methods are the extensions of the approach in [22]. Since, the nonconvex fixed-order controller design problem, in this paper, is reformulated as a convex optimization problem, thus, an inherent conservatism exists. The efficiency of our proposed approach is to introduce an extra degree of freedom in the design formulation to reduce the conservatism thanks to the concept of central polynomials. We have shown in the context of the paper, in the case that the closed loop characteristic polynomial computed with an optimal controller is chosen as the central polynomial, the desired controller is a feasible solution of the proposed LMI conditions. However, this optimum central polynomial is unknown at the beginning. To overcome this difficulty, a recursive procedure is proposed to improve the choice of the central polynomial and to ensure the monotonic decreasing of the upper bound on the H_2 norm of the uncertain closed-loop transfer function. Therefore, the proposed approach for the fixed-order controller design suffers from less conservatism with respect to the approaches in which the bilinear terms in decision variables are omitted to obtain

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LMI conditions in expense of some conservatism. One of the features of the proposed method for the discrete-time case is that it can be employed for the uncertain biproper closed-loop transfer functions, contrary to the presented methods in [16] ,[17], [21], [9] and [10]. Note that this property can be utilized for the design of a dynamic biproper controller where a strictly proper controller may cause a high level of conservatism (see example of Section 4.1). Another contribution of the paper is to employ the parameter-dependent SPR-makers introduced in [23] and [24] for the fixed-order H_2 controller design, unlike the provided methods in [9] and [10] which are based on a common SPR-maker for all the systems in the model set. Moreover, note that the presented approach can be employed for both discrete- and continuous-time systems, contrary to the most of the existing approaches.

The rest of the paper is structured as follows. Controller design for a nominal system is investigated in Section 2. The extension of the proposed approach to fixed-structure controller design for polytopic systems is presented in Section 3. Section 4 is devoted to simulation examples. Finally, some conclusions are drawn in the last section.

The notation is fairly standard. $\mathbb{R}^{n\times m}$is the set of $n\times m$ real matrices. I_n is an $n\times n$ identity matrix. $0_{n\times m}$ and 0_n are $n\times m$ and $n\times n$ zero matrices, respectively. The subscript for the dimension may be dropped if the sizes of matrices are clear from the context. M^T is the transpose of a matrix M. $P=P^T>0$ (≥ 0) means that P is positive (semi)definite. The state space realization of a transfer matrix $G(z)$ is shown as follows:

$$G(z)\leftrightarrow\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right].$$

2. CONTROLLER DESIGN FOR NOMINAL SYSTEMS

First, the design method for discrete-time systems is presented. The continuous-time counterpart is given in Appendix B. Consider the transfer function of a discrete-time linear time-invariant SISO system

$$G(z,\bar{\theta})=\frac{\theta_0 z^p+\theta_1 z^{p-1}+\cdots+\theta_p}{z^q+\theta_{p+1}+\cdots+\theta_{r-1}} \tag{1}$$

where $\bar{\theta}=\left[\begin{array}{cccc}\theta_0 & \theta_1 & \cdots & \theta_{r-1}\end{array}\right]^T\in\mathbb{R}^r$ is a vector that parameterizes G. We consider a standard negative feedback configuration. The goal is to design a fixed-structure controller

$$K(z)=\frac{x_0 z^m+x_1 z^{m-1}+\cdots+x_m}{z^m+y_1 z^{m-1}+\cdots+y_m}, \tag{2}$$

such that:

- the closed-loop system is internally stable
- the closed-loop system achieves the H_2 performance $\left\|H(z,\bar{\theta})\right\|_2^2<\gamma$.

Where $H(z,\bar{\theta})$ can be any of the weighted closed-loop transfer functions. We consider

$$H(z,\bar{\theta})=\frac{S(z,\bar{\theta})}{L(z,\bar{\theta})}=\frac{\bar{s}_n z^n+\bar{s}_{n-1}z^{n-1}+\cdots+\bar{s}_1 z+\bar{s}_0}{\bar{l}_n z^n+\bar{l}_{n-1}z^{n-1}+\cdots+\bar{l}_1 z+\bar{l}_0}, \tag{3}$$

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let $\kappa = \begin{pmatrix} 1 & x_0 & \cdots & x_m & y_1 & \cdots & y_m \end{pmatrix}^T \in \mathbb{R}^v$, then $H(z,\bar{\theta})$ can be parameterized as

$$H(z,\bar{\theta}) = \frac{\begin{pmatrix} \psi_{1s} & \psi_{2s} & \cdots & \psi_{vs} \end{pmatrix}\kappa}{\begin{pmatrix} \psi_{1l} & \psi_{2l} & \cdots & \psi_{vl} \end{pmatrix}\kappa}, \tag{4}$$

where, $\psi_{1s},\cdots,\psi_{vs},\psi_{1l},\cdots,\psi_{vl}$ are known polynomials dependent only on $\bar{\theta}$ and the known weighting filter.

The following lemma gives the necessary and sufficient conditions for evaluating the H_2 performance.

Lemma 1

([25]) Consider a nominal SISO discrete-time transfer function $H(z)$ with the state space realization (A_0,B_0,C_0,D_0). Then $\|H(z)\|_2^2 < \gamma$ if and only if there exists $Q = Q^T > 0$ such that

$$\begin{pmatrix} A_0^T Q A_0 - Q & A_0^T Q B_0 \\ B_0^T Q A_0 & B_0^T Q B_0 - 1 \end{pmatrix} < 0, \tag{5}$$

$$\begin{pmatrix} Q & 0 & C_0^T \\ 0 & 1 & D_0^T \\ C_0 & D_0 & \gamma \end{pmatrix} > 0. \tag{6}$$

Since the controller parameters appear in the state space matrix A_0, therefore condition (5) is not an LMI with respect to the controller parameters and cannot be used for the controller design. In the sequel, a synthesis method is provided based on a convex approximation of conditions in (5) and (6).

The problem addressed here is to provide LMI conditions for fixed-structure H_2 controller design. Consider the state space realization (A_0,B_0,C_0,D_0) for the transfer function $H(z,\bar{\theta})$ given in (3). Suppose that a Schur stable polynomial

$$E(z) = e_n z^n + e_{n-1} z^{n-1} + \cdots + e_1 z + e_0$$

is given with the same order as $L(z,\bar{\theta})$.

Let (A,B,C,D) be the state space realization of the following transfer matrix

$$\begin{pmatrix} \psi_{1s} \\ \vdots \\ \psi_{vs} \\ \psi_{1l} \\ \vdots \\ \psi_{vl} \end{pmatrix} E^{-1} \leftrightarrow \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{7}$$

It is worthwhile to remind that matrices A, B, C and D are known matrices. Obviously, it is easy to see that we have

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$$\begin{pmatrix} \frac{S}{E} \\ \frac{L}{E} \end{pmatrix} \leftrightarrow \left[\begin{array}{c:c} A & B \\ \hdashline \begin{pmatrix} C_s \\ C_l \end{pmatrix} & \begin{pmatrix} D_s \\ D_l \end{pmatrix} \end{array}\right], \tag{8}$$

where

$$C_s = \kappa^T \begin{pmatrix} I_v & 0 \end{pmatrix} C, \quad D_s = \kappa^T \begin{pmatrix} I_v & 0 \end{pmatrix} D, \tag{9}$$

$$C_l = \kappa^T \begin{pmatrix} 0 & I_v \end{pmatrix} C, \quad D_l = \kappa^T \begin{pmatrix} 0 & I_v \end{pmatrix} D. \tag{10}$$

The following theorem can be used for fixed-structure controller design.

Theorem 1

Given a Schur stable polynomial E(z), consider a SISO discrete-time transfer function $H(z,\bar{\theta})$, given in (3). Then $\|H(z,\bar{\theta})\|_2^2 < \gamma$ if there exists $P = P^T > 0$ such that

$$Con1(H,P,E) \triangleq \begin{pmatrix} A^TPA - P & A^TPB - C_l^T \\ B^TPA - C_l & B^TPB - (D_l + D_l^T)/2 \end{pmatrix} < 0, \tag{11}$$

$$Con2(H,P,E) \triangleq \begin{pmatrix} P & 0 & C_s^T & C_l^T \\ 0 & (D_l + D_l^T)/2 & 0 & -D_l^T \\ C_s & 0 & \gamma(D_l + D_l^T)/2 & D_s^T \\ C_l & -D_l & D_s & D_l + D_l^T \end{pmatrix} > 0 \tag{12}$$

where $(A, B, \begin{pmatrix} C_s \\ C_l \end{pmatrix}, \begin{pmatrix} D_s \\ D_l \end{pmatrix})$ is the state space realization of the transfer matrix $\begin{pmatrix} \frac{S}{E} & \frac{L}{E} \end{pmatrix}^T$.

Proof

See Appendix A. □

Clearly, if we consider the state space realization given by (8)-(10), conditions (11) and (12) will be LMIs with respect to the controller parameters and may be used for controller synthesis. To minimize the upper bound on the H_2 norm, in the case that D_l is dependent on the controller parameters, the smallest feasible γ is obtained by a bisection algorithm.

It is worthwhile to mention that, based on the KYP lemma [26], (11) is a sufficient condition for the SPRness of the transfer function $L(z,\bar{\theta})/E(z)$. This implies that the stable polynomial $E(z)$ is an SPR-maker for the denominator of the transfer function $H(z,\bar{\theta})$. Take advantage of an SPR-maker polynomial to obtain a convex approximation of the non-convex fixed-structure controller design problem is a well-known approach for the H_∞ controller design, see e.g. [27], [9], [28]. However, to the best of our knowledge this strategy has not been employed for the H_2 controller design yet. The main feature of the condition (11) is that the controller parameters do not appear in matrix A, therefore, it can be used for the controller design unlike (5).

Choice of the central polynomial $E(z)$ is the main source of conservatism for fixed-structure H_2 controller design. However, the upper bound γ on the H_2 norm may be monotonically decreased by some iterations. Suppose that in iteration $i-1$, with a central polynomial $E_{i-1}(z)$, a controller K_{i-1} is resulted from conditions (11) and (12), with $\gamma = \gamma_{i-1}$. Now, for the next iteration, consider

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$E_i(z) = L(z,\bar{\theta})|_{K=K_{i-1}}$.With this SPR-maker, we have

$$\begin{pmatrix} \frac{S|_{K=K_{i-1}}}{E_i} \\ \frac{L|_{K=K_{i-1}}}{E_i} \end{pmatrix} \leftrightarrow \left[\begin{array}{c:c} A_i & B_i \\ \hdashline \begin{pmatrix} C_i \\ 0 \end{pmatrix} & \begin{pmatrix} D_i \\ 1 \end{pmatrix} \end{array}\right]. \tag{13}$$

Employing the above state space realization, condition (11) can be written as

$$\begin{pmatrix} A_i^T P A_i - P & A_i^T P B_i \\ B_i^T P A_i & B_i^T P B_i - 1 \end{pmatrix} < 0, \tag{14}$$

Moreover, by employing twice the Schur complement formula, it is easy to see that the LMI constraint (12) with the state space realization (13) is equivalent to

$$\begin{pmatrix} P & 0 & C_i^T \\ 0 & 1 & D_i^T \\ C_i & D_i & \gamma \end{pmatrix} > 0. \tag{15}$$

Conditions (14) and (15), based on Lemma 1, imply that the controller K_{i-1} is a feasible solution for the LMIs of Theorem 1 with the SPR-maker $E(z) = L(z,\bar{\theta})|_{K=K_{i-1}}$. Therefore, in the i-th iteration, with the central polynomial $E_i(z) = L(z,\bar{\theta})|_{K=K_{i-1}}$, the resulted upper bound γ_i is equal or less than γ_{i-1}. Note that in the proposed procedure, the iterations can be continued until the difference between γ_i and γ_{i-1} is insignificant or below a threshold value. Therefore, this iterative approach may generate a monotonically decreasing sequence of the upper bound on the H_2 norm.

3. CONTROLLER DESIGN FOR POLYTOPIC SYSTEMS

In this section, the fixed-structure H_2 controller design problem is investigated for discrete-time polytopic systems. The related results for continuous-time systems is provided in Appendix B. Consider an uncertain system with the transfer function $G(z,\tilde{\theta})$. The parameter vector $\tilde{\theta}$ belongs to the following polytope with q vertices:

$$pol \triangleq co\{\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_q\}$$

where $co\{\cdot\}$ stands for the convex hull of a set.

The goal is to design a controller $K(z)$, given by (2), such that the uncertain closed-loop system achieves the H_2 performance $\|H(z,\tilde{\theta})\|_2^2 < \gamma$ for all the parameter vectors $\tilde{\theta} \in pol$. Where,

$$H(z,\tilde{\theta}) = \frac{S(z,\tilde{\theta})}{L(z,\tilde{\theta})} = \frac{\tilde{s}_n z^n + \tilde{s}_{n-1} z^{n-1} + \cdots + \tilde{s}_1 z + \tilde{s}_0}{\tilde{l}_n z^n + \tilde{l}_{n-1} z^{n-1} + \cdots + \tilde{l}_1 z + \tilde{l}_0}. \tag{16}$$

In the following, the results of previous section is extended to systems with polytopic uncertainty.

Proposition 1

Given a Schur stable polynomial $E(z)$, a fixed-structure controller $K(z)$, given by (2), stabilizes the uncertain closed-loop system and the H_2 performance $\|H(z,\tilde{\theta})\|_2^2 < \gamma$ is satisfied for all $\tilde{\theta} \in pol$, if

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there exist symmetric matrices $P_i = P_i^T > 0$ such that for $i = 1, \cdots, q$

$$Con1(H_i, P_i, E) < 0, \tag{17}$$

$$Con2(H_i, P_i, E) > 0, \tag{18}$$

Where, $H_i \triangleq H(z, \bar{\theta}_i)$ and

$$\begin{pmatrix} \frac{S(z,\bar{\theta}_i)}{E(z)} \\ \frac{L(z,\bar{\theta}_i)}{E(z)} \end{pmatrix} \leftrightarrow \left[\begin{array}{c:c} A & B \\ \hdashline \begin{pmatrix} C_{si} \\ C_{li} \end{pmatrix} & \begin{pmatrix} D_{si} \\ D_{li} \end{pmatrix} \end{array}\right], \tag{19}$$

is the controllable canonical form realization, for $i = 1, \cdots, q$.

Under a mild assumption that $\tilde{\theta}$ do not appear in $\tilde{s}_n$ and $\tilde{l}_n$, it is possible to reduce the conservatism of the proposed approach in Proposition 1, using a set of central polynomials instead of a common one. It can be easily seen that with this assumption, we can consider

$$D_s = D_{s1} = \cdots = D_{sq},$$

$$D_l = D_{l1} = \cdots = D_{lq}.$$

For example, suppose that $H(z, \tilde{\theta})$ is considered as one of the uncertain closed-loop transfer functions $1/(1 + G(\tilde{\theta})K)$ or $K/(1 + G(\tilde{\theta})K)$ or $G(\tilde{\theta})K/(1 + G(\tilde{\theta})K)$. In the case that either $G(\tilde{\theta})$ or K are strictly proper, the mentioned assumption would be satisfied.

To proceed, we need the following lemmas.

Lemma 2

([29]) Let I, Φ, and Σ be matrices of appropriate dimensions. Then, the following two statements are equivalent.

1. $\begin{bmatrix} I \\ \Phi \end{bmatrix}^T \Sigma \begin{bmatrix} I \\ \Phi \end{bmatrix} > 0.$
2. There exists a matrix Q such that $\Sigma + \begin{bmatrix} \Phi^T \\ -I \end{bmatrix} Q^T + Q \begin{bmatrix} \Phi & -I \end{bmatrix} > 0.$

Proof

This lemma is a particular case of the elimination lemma. □

Lemma 3

Let X_i for $i = 1, \cdots, q$ and Y be matrices of appropriate dimensions. Then the following statements are equivalent.

1. $\exists \varepsilon \in \mathbb{R}$ such that for all $i = 1, \cdots, q$: $\quad X_i + \varepsilon Y Y^T > 0.$
2. For all $i = 1, \cdots, q$, $\exists \varepsilon_i \in \mathbb{R}$ such that: $\quad X_i + \varepsilon_i Y Y^T > 0.$
3. Suppose that $Y^{\perp} Y = 0$ then for all $i = 1, \cdots, q$: $\quad Y^{\perp} X_i Y^{\perp T} > 0.$

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Proof

First, we show that statements (1) and (2) are equivalent. Suppose that (2) is satisfied, we define

$$\varepsilon = \max_i \varepsilon_i, \tag{20}$$

therefore,

$$(\varepsilon - \varepsilon_i) Y Y^T \geq 0.$$

Adding this inequality to the condition in statement (2), statement (1) is obtained. Now, suppose that inequality in statement (1) is satisfied, defining $\varepsilon_i = \varepsilon$ for all i leads to statement (2). Therefore statements (1) and (2) are equivalent. Additionally, Statements (3) is equivalent to statement (2) based on a direct application of the Finsler's lemma. This ends the proof. □

Using Lemma 2, condition (11) is equivalent to the existence of a matrix $Q \in \mathbb{R}^{(2n+1)\times n}$ such that

$$\left[\begin{array}{cc:c} P & C_l^T & 0 \\ C_l & (D_l + D_l^T)/2 & \\ \hdashline \multicolumn{2}{c:}{0} & -P \end{array}\right] + \begin{bmatrix} A^T \\ B^T \\ \hdashline -I \end{bmatrix} Q^T + Q \left[\begin{array}{cc:c} A & B & -I \end{array}\right] > 0$$

Theorem 2

Given a set of central polynomials $\{E_1(z), E_2(z), \cdots, E_q(z)\}$, a fixed-structure controller $K(z)$, given by (2), stabilizes the uncertain closed-loop system and the H_2 performance

$$\left\| H(z,\tilde{\theta}) \right\|_2^2 = \left\| \frac{S(z,\tilde{\theta})}{L(z,\tilde{\theta})} \right\|_2^2 < \gamma, \qquad \forall \tilde{\theta} \in pol \tag{21}$$

is satisfied, if there exist symmetric matrices $P_i = P_i^T > 0$ and a matrix $Q \in \mathbb{R}^{(2n+1)\times n}$ such that for $i = 1, \cdots, q$

$$\left[\begin{array}{cc:c} P_i & C_{li}^T & 0 \\ C_{li} & (D_l + D_l^T)/2 & \\ \hdashline \multicolumn{2}{c:}{0} & -P_i \end{array}\right] + \begin{bmatrix} A_i^T \\ B^T \\ \hdashline -I \end{bmatrix} Q^T + Q \left[\begin{array}{cc:c} A_i & B & -I \end{array}\right] > 0, \tag{22}$$

$$\begin{pmatrix} P_i & 0 & C_{si}^T & C_{li}^T \\ 0 & (D_l + D_l^T)/2 & 0 & -D_l^T \\ C_{si} & 0 & \gamma (D_l + D_l^T)/2 & D_s^T \\ C_{li} & -D_l & D_s & D_l + D_l^T \end{pmatrix} > 0, \tag{23}$$

where,

$$\begin{pmatrix} \frac{S(z,\tilde{\theta}_i)}{E_i(z)} \\ \frac{L(z,\tilde{\theta}_i)}{E_i(z)} \end{pmatrix} \leftrightarrow \left[\begin{array}{c:c} A_i & B \\ \hdashline \begin{pmatrix} C_{si} \\ C_{li} \end{pmatrix} & \begin{pmatrix} D_s \\ D_l \end{pmatrix} \end{array}\right], \tag{24}$$

is the controllable canonical form realization, for $i = 1, \cdots, q$. It is assumed that $\tilde{\theta}$ do not appear in $\tilde{s}_n$ and $\tilde{l}_n$.

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Proof
Consider an uncertain system $G(z,\tilde{\theta})$ in the polytopic set $\tilde{\theta} \in pol$. Therefore, we can consider $\tilde{\theta} = \sum_{i=1}^{q} \alpha_i \bar{\theta}_i$. Note that for any closed-loop transfer function $H(z,\tilde{\theta})$ the coefficients of the polynomials S and L depend affinely on the parameter vector $\tilde{\theta}$. Let $\tilde{u}_m$ to be any arbitrary coefficient of S or L that can be written as $\tilde{u}_m = u_{mf} + u_{mp}\tilde{\theta}$. Since

$$\tilde{\theta} = \sum_{i=1}^{q} \alpha_i \bar{\theta}_i, \qquad \sum_{i=1}^{q} \alpha_i = 1$$

Therefore, we can write

$$\tilde{u}_m = \sum_{i=1}^{q} \alpha_i \tilde{u}_m|_{\tilde{\theta}=\bar{\theta}_i},$$

this implies that

$$S(z,\tilde{\theta}) = \sum_{i=1}^{q} \alpha_i S(z,\bar{\theta}_i), \quad L(z,\tilde{\theta}) = \sum_{i=1}^{q} \alpha_i L(z,\bar{\theta}_i).$$

Thus, the transfer function $H(z,\tilde{\theta})$ can be written as

$$H(z,\tilde{\theta}) = \frac{S(z,\tilde{\theta})}{L(z,\tilde{\theta})} = \frac{\alpha_1 S(z,\bar{\theta}_1) + \cdots + \alpha_q S(z,\bar{\theta}_q)}{\alpha_1 L(z,\bar{\theta}_1) + \cdots + \alpha_q L(z,\bar{\theta}_q)}.$$

Now consider a parameter dependent central polynomial $E(z,\alpha) = \sum_{i=1}^{q} \alpha_i E_i(z)$. Therefore, we have the following controllable canonical form realization

$$\begin{pmatrix} \frac{S(z,\tilde{\theta})}{E(z,\alpha)} \\ \frac{L(z,\tilde{\theta})}{E(z,\alpha)} \end{pmatrix} \leftrightarrow \left[\begin{array}{c:c} \sum_{i=1}^{q} \alpha_i A_i & B \\ \hdashline \begin{pmatrix} \sum_{i=1}^{q} \alpha_i C_{si} \\ \sum_{i=1}^{q} \alpha_i C_{li} \end{pmatrix} & \begin{pmatrix} D_s \\ D_l \end{pmatrix} \end{array}\right], \tag{25}$$

where, A_i, C_{si}, C_{li}, B, D_s and D_l are given by (24). Note that to obtain the above realization, it is assumed that $\tilde{\theta}$ do not appear in $\tilde{s}_n$ and $\tilde{l}_n$.

Now, suppose that conditions (22) and (23) are satisfied for all $i = 1,\cdots,q$. This yields

$$Con1(H(\tilde{\theta}), \sum_{i=1}^{q} \alpha_i P_i, E(\alpha)) < 0,$$

$$Con2(H(\tilde{\theta}), \sum_{i=1}^{q} \alpha_i P_i, E(\alpha)) > 0.$$

Where, $A = \sum_{i=1}^{q} \alpha_i A_i$, $C_s = \sum_{i=1}^{q} \alpha_i C_{si}$ and $C_l = \sum_{i=1}^{q} \alpha_i C_{li}$. Therefore, in virtue of Theorem 1, we have $\left\|H(z,\tilde{\theta})\right\|_2^2 < \gamma$ for $\forall \tilde{\theta} \in pol$. □

Controller parameters appear linearly in matrices C_{si}, C_{li}, D_s and D_l. Therefore the above conditions are LMIs with respect to the controller parameters.

Corollary 1
If a common central polynomial is considered for all the vertices, i.e. $E(z) = E_1(z) = E_2(z) = \cdots = E_q(z)$, Then the conditions in Theorem 2 would be equivalent to those of Proposition 1.

Proof

In the case of a common central polynomial $E(z)$, we have $A \triangleq A_1 = \cdots = A_q$ for the controllable canonical form realizations. Now, by considering the following structure for Q

$$Q = \frac{\alpha}{2}\left[\begin{array}{cc:c} A & B & -I \end{array}\right]^T,$$

where α is a scalar decision variable, condition (22) can be written as

$$\left[\begin{array}{cc:c} P_i & C_{li}^T & 0 \\ C_{li} & (D_l + D_l^T)/2 & \\ \hdashline 0 & & -P_i \end{array}\right] + \alpha \left[\begin{array}{c} A^T \\ B^T \\ \hdashline -I \end{array}\right] \left[\begin{array}{cc:c} A & B & -I \end{array}\right] > 0, \tag{26}$$

Now, in virtue of Lemma 3, the above condition is equivalent to

$$\begin{pmatrix} A^T P_i A - P_i & A^T P_i B - C_{li}^T \\ B^T P_i A - C_{li} & B^T P_i B - (D_l + D_l^T)/2 \end{pmatrix} < 0, \tag{27}$$

for $i = 1, \cdots, q$. Since we have $D_l = D_{li}$ for all $i = 1, \cdots, q$ the above condition is equivalent to (17). Moreover, $D_s = D_{si}$ for all $i = 1, \cdots, q$, thus, it is easy to see that condition (23) is also equivalent to (18). This concludes the proof. □

This means that employing conditions (17)-(18) to design a robust controller, may seem the easiest way of extension the results for the polytopic systems, is a special case of the proposed approach in Theorem 2.

One way to obtain a set of central polynomials is to employ a stabilizing controller K_c (without any specific performance) and then consider the central polynomials as follows:

$$E_i(z) = L(z, \tilde{\theta})|_{K=K_c, \tilde{\theta}=\bar{\theta}_i}. \tag{28}$$

Subsequently, a procedure is presented for choosing the central polynomials for the controller design. The main feature of the procedure is that the upper bound on the H_2 norm of the desired transfer function would be monotonically decreased.

3.1. *A procedure for the controller design*

Step 1: Suppose that there is a stabilizing controller K_c. Then, a set of central polynomials may be computed by (28). Now, using the results of Theorem 2, we solve the following optimization problem:

$$\min_{x_0, \cdots, x_m, y_1, \cdots, y_m, Q, P_1, \cdots, P_q, \gamma} \quad \gamma \tag{29}$$

Subject to:

LMI conditions given by (22) and (23).

This way, a robust controller and a matrix Q is obtained.

Note that the stabilizing controller may be designed using conditions (17)-(18) by employing a fixed Schur stable polynomial $E(z)$, or may be obtained by any other available approaches.

Step 2: Now consider the following unknown initial controller

$$K_c(z) = \frac{x_{0c}z^m + x_{1c}z^{m-1} + \cdots + x_{mc}}{z^m + y_{1c}z^{m-1} + \cdots + y_{mc}}, \tag{30}$$

which results in a set of unknown central polynomials based on (28) with controller parameters as decision variables. Using the same matrix Q resulted in Step 1, the following optimization problem is solved.

$$\min_{x_0,\cdots,x_m,y_1,\cdots,y_m,x_{0c},\cdots,x_{mc},y_{1c},\cdots,y_{mc},P_1,\cdots,P_q,\gamma} \quad \gamma \tag{31}$$

Subject to:

LMI conditions given by (22) and (23).

Note that if the initial controller parameters appear in the coefficient of the term z^n in the central polynomials, we can easily consider a strictly proper structure for the initial controller to preserve the convexity.

Now , some iterations between Step 1 and Step 2 may improve the performance of the resulted robust controller. Take into account that in the iterations, we use the controller $K_c(z)$ obtained in Step 2 as the stabilizing controller in Step 1. The initial stabilizing controller $K_c(z)$ and the obtained controller in Step 1 are always feasible solutions for the optimization problem in Step 2 with the same γ. Therefore, it is easy to see that these iterations cause monotonic decreasing of the upper bound on the H_2 norm.

4. NUMERICAL EXAMPLES

This section provides some examples to demonstrate the effectiveness of the proposed approach. Both nominal and robust controller design methods are investigated. In this section, the optimization problems are solved by YALMIP ([30]) interface for the LMI solver SDPT3 ([31]).

4.1. Controller design for a nominal system

Consider the following system

$$G(z) = \frac{z - 0.2}{z^3 - 1.2z^2 - 3.55z + 8.18}, \tag{32}$$

with three unstable poles. The goal is to design a fixed-structure stabilizing controller, with the minimum upper bound over the H_2 norm of the following weighted closed-loop transfer function

$$H(z) = W(z)\frac{K(z)}{1 + G(z)K(z)}, \tag{33}$$

where, $W(z)$ is given as follows:

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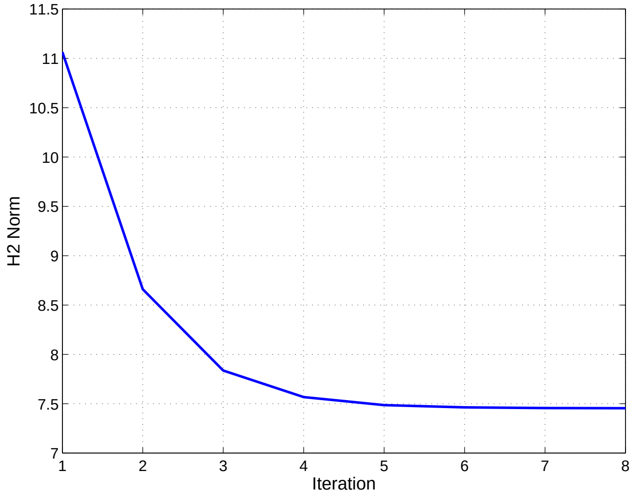

Figure 1. H_2 norm of the weighted transfer function $H(z)$ versus the central polynomial updates.

$$W(z) = \frac{W_n(z)}{W_d(z)} = \frac{0.9204z^2 - 1.7270z + 0.8097}{35(z^2 - 1.9623z + 0.9626)}, \quad (34)$$

which is a low-pass weighting filter to minimize the control input energy for low frequency signals.

Before dealing with the fixed-order controller design, we illustrate one of the advantages of our proposed approach. That is, the monotonic decreasing of the H_2 norm bound. First, using the command *h2syn*, the following strictly proper optimal full-order controller is designed such that $\|H(z)\|_2 = 7.4538$.

$$K_1(z) = \frac{-0.7646z^4 + 10.95z^3 - 52.64z^2 + 74.56z - 32.11}{z^5 - 1.544z^4 + 4.1z^3 - 8.172z^2 + 5.402z - 0.7836} \quad (35)$$

Then, using Theorem 1, we design a controller with the same structure (fifth-order strictly proper) as the controller $K_1(z)$. Generally, central polynomial is chosen such that to contain the denominator of the weighting function and it should be of the same order of the denominator of the weighted closed-loop transfer function. Based on these rules of thumb, we consider an initial central polynomial $E(z) = W_d(z)(z-0.5)^8$ where the zeros at $z = 0.5$ in the central polynomial are chosen arbitrarily. Figure 1 shows that the proposed method converges rapidly to the same norm bound $\|H(z)\|_2 = 7.4538$.

Now, consider the design of a fifth-order proper controller. Based on Theorem 1, the following controller is resulted.

$$K_2(z) = \frac{4.075z^5 - 18.62z^4 + 30.87z^3 - 24.02z^2 + 8.841z - 1.157}{z^5 - 2.764z^4 + 2.95z^3 - 1.501z^2 + 0.3449z - 0.02819},$$

where, $\|H(z)\|_2 = 2.0146$ is much less than that of the strictly proper controller $K_1(z)$.

It is worthwhile to mention that the designed controller $K_1(z)$ has two unstable poles. Therefore, traditional order reduction methods are not able to provide a controller with order less than 2.

Now, consider the design of a first-order controller. Using Theorem 1, after 5 iterations of the central polynomial updates, the upper bound $\gamma = 2.2431$ is obtained with the following controller:

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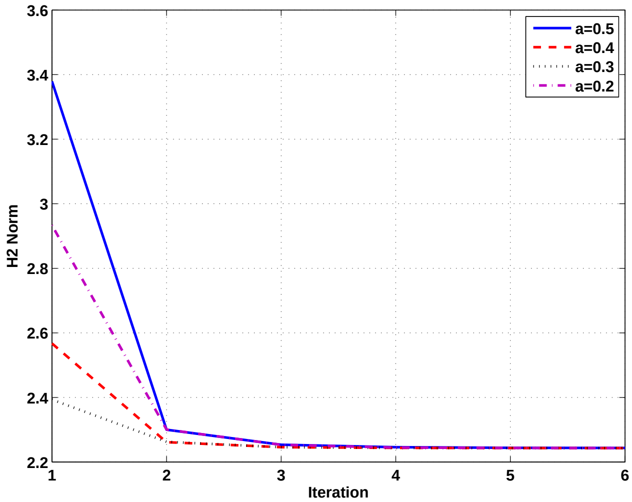

Figure 2. The monotonic decreasing of the H_2 norm bound for (33) with first order controllers designed with different initial central polynomials

$$K_1(z) = \frac{4.249z - 8.299}{z - 0.1828}. \tag{36}$$

In order to reveal the impact of the initial central polynomial on the result, the design procedure has been carried out with different initial central polynomials. We consider these central polynomials as $E(z) = W_d(z)(z-a)^4$ for $a = 0.2, 0.3, 0.4, 0.5$. Figure 2 shows the monotonic decreasing of the norm bound and convergence to a suboptimal solution for these central polynomials.

4.2. *Comparison with the existing methods*

In the following, we try to show the effectiveness of our proposed method by means of comparison with the other existing approaches. Consider the design of 1st and 2nd order controllers for the system given in 4.1. We investigate two LMI-based approaches presented in [9] and [10]. Both of them are constructed based on the concept of the central polynomials. To have a fair comparison, we consider the central polynomials as $E(z) = W(z)(z-0.3)^n$ where n is selected suitably. It should be mentioned that these approaches are not applicable for systems with a biproper weighted closed-loop transfer function. Therefore, to apply them on our example, we have two alternatives. The first one is to consider a strictly proper structure for the to-be-designed controller. In this case, for both of them, a controller with order less than three would not be found. Alternatively, we can append a low-pass filter, e.g. $0.99/(z-0.01)$, into the weighting function $W(z)$ to have a strictly proper weighted closed-loop transfer function. Note that this filter has a very low impact on the magnitude of the weighting function. This way, 1st and 2nd order controllers may be designed. The H_2 norm of the transfer function $W(z)K(z)/(1+K(z)G(z))$ is given in Table I, where $K(z)$ is the designed controller with the different approaches. It is worthwhile to mention that our proposed approach is applicable for biproper transfer functions directly. Moreover, it results better fixed-order controllers at least for this example.

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Approach	Proposed Method	[10]	[9]
1st order controller	2.2971	2.4371	2.6635
2nd order controller	2.2688	2.4293	2.6569

Table I. The H_2 norm of the weighted closed-loop transfer function obtained by different approaches.

4.3. Controller design for a polytopic system

Consider the same third order system as in [9] which is affected by the polytopic (interval) uncertainty.

$$G(z) = \frac{z+\theta_0}{z^3+\theta_1 z^2+\theta_2 z+\theta_3} \tag{37}$$

with, $\theta_0 = -0.2$, $\theta_1 = -1.2$, $\theta_2 = 0.5$ and $\theta_3 = -0.1$. It is assumed that all the parameters are uncertain up to $\pm 12\%$ of their nominal values. Therefore, the parametric uncertainty is in the form of a polytope (hypercube) with $2^4 = 16$ vertices.

The goal is to design a 2nd order controller which includes an integrator and results the minimum upper bound γ for the weighted H_2 norm of the output sensitivity function

$$H(z) = W(z)\frac{1}{1+G(z)K(z)}$$

where $W(z)$ is

$$W(z) = \frac{W_n(z)}{W_d(z)} = \frac{0.4902(z^2-1.0432z+0.3263)}{z^2-1.232z+0.268}.$$

At first, considering a common central polynomial $E(z) = W_d(z)(z-0.1)^5$ for all the vertices and using the conditions (17) and (18) results in a second-order controller with the upper bound γ on the H_2 norm equal to $\gamma = 1.2973^2$.

Now, based on the proposed procedure in Section 3 and by using the above designed controller as the stabilizing controller in Step 1, after a few number of iterations the following controller is obtained:

$$K(z) = \frac{0.39677z^2-0.19009z-0.14077}{(z-1)(z+0.7758)},$$

for this controller the upper bound on the H_2 norm equals to $\gamma = 0.5527^2$.

In order to evaluate the designed controller, HIFOO is employed. HIFOO is a Matlab package designed for fixed-order controller synthesis, using nonsmooth nonconvex optimization techniques [32]. HIFOO is utilized for fixed-order controller design for continuous-time plants with multimodel uncertainty. Using the bilinear transformation, this package has been used for the controller design for the vertices of our discrete-time polytopic system. Note that the designed controller does not guarantee the obtained performance for the whole polytope. But the highest H_2 norm of the closed-loop transfer functions related to the vertices provides a lower bound for the worst-case performance over the whole polytope.

The output of HIFOO may differ on different runs since the initialization is done randomly. Therefore HIFOO was run 10 times for designing a 2nd order controller and the minimum lower bound for the worst-case performance is $\gamma = 0.6306^2$. Consequently, it is easy to see that at least for

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this example the designed controller with our proposed approach provides a better result and more important that it guarantees the performance for the whole polytope and not only for the vertices.

5. CONCLUSIONS

We have presented LMI-based conditions for fixed-structure H_2 controller design. The results are extended to the case of fixed-structure H_2 controller design for polytopic systems in terms of solutions to an LMI-based optimization problem. The conditions can be easily extended for the control design for linear parameter varying (LPV) systems. It is worthwhile to mention that the proposed approach for H_2 controller design can be combined with the existing H_∞ fixed-order controller design methods to have a mixed H_2/H_∞ approach. Numerical examples clearly demonstrated the effectiveness and the advantages of the proposed approach.

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A. PROOF OF THEOREM 1

Consider (A_c, B_c, C_c, D_c) to be the controllable canonical form realization of the transfer function $H(z,\theta) = \frac{S(z,\theta)}{L(z,\theta)}$, given by

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -l_0 l_n^{-1} & -l_1 l_n^{-1} & -l_2 l_n^{-1} & \cdots & -l_{n-1} l_n^{-1} \end{bmatrix} B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{38}$$
$$C_c = l_n^{-1} \begin{bmatrix} s_0 - D_c l_0 & s_1 - D_c l_1 & \cdots & s_{n-1} - D_c l_{n-1} \end{bmatrix},$$
$$D_c = s_n l_n^{-1}.$$

Additionally, consider the similarity transformation with matrix T which converts the state space model

$$(A, B, \begin{pmatrix} C_s \\ C_l \end{pmatrix}, \begin{pmatrix} D_s \\ D_l \end{pmatrix})$$

to a controllable canonical form realization

$$
\begin{aligned}
\bar{A} = TAT^{-1} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -e_0e_n^{-1} & -e_1e_n^{-1} & -e_2e_n^{-1} & \cdots & -e_{n-1}e_n^{-1} \end{bmatrix} \\
\bar{B} = TB &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T, \\
\begin{pmatrix} \bar{C}_s \\ \bar{C}_l \end{pmatrix} &= \begin{pmatrix} C_s \\ C_l \end{pmatrix} T^{-1} \\
&= e_n^{-1} \begin{pmatrix} s_0 - D_s e_0 & s_1 - D_s e_1 & \cdots & s_{n-1} - D_s e_{n-1} \\ l_0 - D_l e_0 & l_1 - D_l e_1 & \cdots & l_{n-1} - D_l e_{n-1} \end{pmatrix} \\
\begin{pmatrix} \bar{D}_s \\ \bar{D}_l \end{pmatrix} &= \begin{pmatrix} D_s \\ D_l \end{pmatrix} = \begin{pmatrix} s_n e_n^{-1} \\ l_n e_n^{-1} \end{pmatrix}.
\end{aligned}
$$

It is easy to see that

$$
C_c = D_l^{-1}(C_s - D_c C_l)T^{-1}, \tag{39}
$$

$$
D_c = D_s D_l^{-1}. \tag{40}
$$

Employing Schur Complement Formula, the inequality constraint (11) can be written as

$$
\begin{pmatrix} -P & -C_l^T & 0 \\ -C_l & -D_l - D_l^T & D_l^T \\ 0 & D_l & -(D_l + D_l^T)/2 \end{pmatrix} + \begin{pmatrix} A^T \\ B^T \\ 0 \end{pmatrix} P \begin{pmatrix} A & B & 0 \end{pmatrix} < 0. \tag{41}
$$

We pre- and post-multiply (41) by matrix

$$
\left(\begin{array}{c:c:c} I & \begin{array}{c} e_0e_n^{-1} - l_0l_n^{-1} \\ e_1e_n^{-1} - l_1l_n^{-1} \\ \vdots \\ e_{n-1}e_n^{-1} - l_{n-1}l_n^{-1} \end{array} & 0 \\ \hdashline 0 & 1 & 1 \end{array} \right) \begin{pmatrix} T^{-T} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{42}
$$

and its transpose, respectively. We obtain

$$\begin{pmatrix} A_c^T Q A_c - Q & A_c^T Q B_c \\ B_c^T Q A_c & B_c^T Q B_c - 1 \end{pmatrix} < 0, \tag{43}$$

where,

$$Q = 2(D_l + D_l^T)^{-1} T^{-T} P T^{-1}. \tag{44}$$

Since the matrix given by (42) is a full row rank matrix, LMI (43) holds if the LMI (11) holds.

Now, We pre- and post-multiply condition (12) by the full row rank matrix

$$\left(\frac{D_l + D_l^T}{2} \right)^{-\frac{1}{2}} \begin{pmatrix} T^{-T} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -D_s D_l^{-1} \end{pmatrix} \tag{45}$$

and its transpose, respectively. Taking into account (39) and (40), we obtain the following condition

$$\begin{pmatrix} Q & 0 & C_c^T \\ 0 & 1 & D_c^T \\ C_c & D_c & \gamma \end{pmatrix} > 0. \tag{46}$$

Now, using Lemma 1 and based on conditions (46) and (43), we conclude that the satisfactions of LMIs (11) and (12) imply the H_2 performance on the transfer function $H(z, \bar{\theta})$.

B. LMI CONDITIONS FOR CONTINUOUS-TIME CASE

In this Appendix, the continuous-time counterpart for the LMI conditions of Theorems 1 and 2 are given. The proofs are similar to those of the discrete-time case and have been omitted for the sake of brevity.

For a SISO continuous-time transfer function, the LMI conditions of Theorem 1 can be replaced by the following conditions.

$$\begin{pmatrix} A^T P + PA & PB - C_l^T \\ B^T P - C_l & -(D_l + D_l^T)/2 \end{pmatrix} < 0, \tag{47}$$

$$\begin{pmatrix} P & C_s^T \\ C_s & \gamma(D_l + D_l^T)/2 \end{pmatrix} > 0. \tag{48}$$

Recall that in H_2 norm for continuous-time systems, the transfer function should be strictly proper for the H_2 performance to be well-defined. This means that D_s should be equal to zero. Moreover, in this case the central polynomial E is considered as a Hurwitz stable polynomial.

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Additionally, for the controller design for continuous-time polytopic systems, the LMI conditions of Theorem 2 can be replaced by

$$\begin{pmatrix} 0 & C_{li}^T & -P_i \\ C_{li} & (D_l + D_l^T)/2 & 0 \\ -P_i & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_i^T \\ B^T \\ -I \end{pmatrix} Q^T + Q \begin{pmatrix} A_i & B & -I \end{pmatrix} > 0, \tag{49}$$

$$\begin{pmatrix} P_i & C_{si}^T \\ C_{si} & \gamma(D_l + D_l^T)/2 \end{pmatrix} > 0. \tag{50}$$

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