

# LEFT-INDUCED MODEL STRUCTURES AND DIAGRAM CATEGORIES

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ABSTRACT. We prove existence results for and verify certain elementary properties of left-induced model structures, of which the injective model structure on a diagram category is an important example. We refine our existence results and prove additional properties for the injective model structure. To conclude, we investigate the fibrant generation of (generalized) Reedy categories. In passing, we also consider the cofibrant generation, cellular presentation, and small object argument for Reedy diagrams.

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## 1. INTRODUCTION

Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category and  $\mathbf{C}$  be a bicomplete category. Given a pair of adjoint functors

$$\mathbf{M} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{C},$$

there are well known conditions, such as [10, Theorems 11.3.1 and 11.3.2], under which there is a model structure on  $\mathbf{C}$ , which we call the *right-induced model category structure*, with  $R^{-1}(\mathcal{W})$ ,  $R^{-1}(\mathcal{F})$  as weak equivalences and fibrations, respectively.

In this paper we study the dual situation, where one has a pair of adjoint functors

$$\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$$

and wants to know when is there a model structure on  $\mathbf{C}$  with  $L^{-1}(\mathcal{W})$ ,  $L^{-1}(\mathcal{C})$  as weak equivalences and cofibrations, respectively. We call this a *left-induced model structure*. Note that if the left-induced model structure exists, then the adjunction above is a Quillen pair with respect to the left-induced structure on  $\mathbf{C}$  and the given model structure on  $\mathbf{M}$ .

The conditions on our existence results for left-induced model structures (Corollaries 2.18 and 2.19 and Theorem 2.21) are expressed in terms of either *fibrant generation* or a *Postnikov presentation* of the model category  $\mathbf{M}$ . We say that  $\mathbf{M}$  is fibrantly generated by a pair  $(\mathcal{X}, \mathcal{Z})$  of classes of morphisms, called the *generating fibrations* and *generating acyclic fibrations*, if the acyclic cofibrations (respectively, cofibrations) of  $\mathbf{M}$  are exactly the morphisms that have the left lifting property with respect to  $\mathcal{X}$  (respectively,  $\mathcal{Z}$ ). If, in addition, each fibration (respectively, acyclic fibration) in  $\mathbf{M}$  is a retract of the limit of a tower of morphisms built by pullback of elements of  $\mathcal{X}$  (respectively,  $\mathcal{Z}$ ), then  $(\mathcal{X}, \mathcal{Z})$  is a Postnikov presentation.

Working with fibrant generation and Postnikov presentations is a more delicate affair than the more familiar dual case. If a cocomplete category  $\mathbf{M}$  and a set of maps  $\mathcal{J}$  satisfy certain set-theoretical “smallness” conditions, then by a procedure called the *small object argument* it is possible to construct a functorial factorization whose right factor has the right lifting property with respect to  $\mathcal{J}$  and whose left factor is a *relative  $\mathcal{J}$ -cell complex*: a colimit of sequences of pushouts of maps in  $\mathcal{J}$ ; see Definition 2.6 or [10, Proposition 11.2.1]. As a consequence of this construction, for any model category  $\mathbf{M}$  that is cofibrantly generated by a pair  $(\mathcal{J}, \mathcal{I})$  of sets of maps satisfying the smallness conditions, the cofibrations (respectively, acyclic cofibrations) are retracts of relative  $\mathcal{J}$ -cell complexes (respectively, relative  $\mathcal{I}$ -cell complexes): the small object argument implies that the pair  $(\mathcal{J}, \mathcal{I})$  defines a *cellular presentation* (see Proposition 2.7).

No such general result holds in the dual case of fibrant generation or Postnikov presentations: the *cosmall object argument*, defined by dualizing the colimit constructions of the small object argument, requires “cosmallness” conditions, which are never satisfied in practice. For this reason, the terminology we introduce in Section 2.1 separates the lifting properties, the cellular presentation, and the smallness conditions that are normally unified by the adjective “cofibrantly generated” (see Remarks 2.5 and 2.8). Our motivation for persevering despite the technical difficulties presented by the theory of Postnikov presentations and fibrant generation is that left-induced model structures include interesting examples that were

previously unknown, e.g., [7, Theorem 2.10], [9, Theorem 6.2]) and Theorem 3.8 in this paper.

We are particularly interested in studying the injective model structure on diagram categories  $M^D$ , where  $M$  is a model category and  $D$  is a small category. The injective model structure is left-induced from the “pointwise” model structure on the category of  $\text{Ob } D$ -indexed diagrams in  $M$ . Applying Corollary 2.18, we obtain an existence result for injective model structures on diagram categories  $M^D$ , when the cofibrations of  $M$  are exactly the monomorphisms, and  $M$  admits a (possibly trivial) Postnikov presentation (Theorem 4.19). In contrast with the standard existence result, our theorem does not require the model structure on  $M$  to be combinatorial.

As a complement to our analysis of injective model structures, we show that if  $R$  is a Reedy category or, more generally, a dualizable generalized Reedy category, and  $M$  is a fibrantly generated model category, then  $M^R$ , endowed with its (generalized) Reedy model structure, is also fibrantly generated (Theorems 5.9 and 5.22). Moreover, if  $M$  has a Postnikov presentation, then so does  $M^R$  (Theorem 5.11 and Remark 5.23).

We also establish certain elementary properties of left-induced model structures in general (Lemmas 2.16, 2.22, and 2.23) and of injective model category structures in particular (Lemmas 4.14, 4.21, and 4.22). However, certain obvious and relevant questions about Postnikov presentations and fibrant generation remain open. For example, if  $M$  is a model category with Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , and  $C$  admits the left-induced model structure with respect to the adjunction above, then  $C$  is fibrantly generated by  $(R\mathcal{X}, R\mathcal{Z})$  by Lemma 2.16. On the other hand, we do not know of reasonable conditions that imply that  $(R\mathcal{X}, R\mathcal{Z})$  is actually a Postnikov presentation of  $C$ , as the set-theoretical difficulties mentioned above appear to present obstacles to obtaining a result in this direction. We hope to answer this question in the near future.

Other extensions of this work are possible. For example, the injective model structure on a “generalized diagram category” constructed from a diagram of model categories (as studied by [12], [2], and [6]) will be fibrantly generated if each of the model categories in the diagram is.

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## 2. LEFT-INDUCED MODEL STRUCTURES

In this section, we prove a number of existence results for left-induced model structures. Our results are formulated in terms of either fibrant generation or Postnikov presentations of model categories, notions that we recall below. We conclude by establishing elementary results concerning properness and enrichment of left-induced model structures.

*Notation 2.1.* Throughout this section, if  $\mathcal{X}$  is a class of morphisms in a category, then  $\widehat{\mathcal{X}}$  denotes its closure under retracts.

### 2.1. Fibrant generation and Postnikov presentations of model categories.

We begin this section by recalling the elegant formulation of the definition of model categories in terms of weak factorization systems due to Joyal and Tierney. We then express the definitions of fibrant generation and of Postnikov presentations in this framework, dualizing the more familiar notions.

*Notation 2.2.* Let  $f$  and  $g$  be morphisms in a category  $\mathbf{C}$ . If for every commutative diagram in  $\mathbf{C}$

$$\begin{array}{ccc} \cdot & \xrightarrow{a} & \cdot \\ f \downarrow & \nearrow c & \downarrow g \\ \cdot & \xrightarrow{b} & \cdot \end{array}$$

the dotted lift  $c$  exists, i.e.,  $gc = b$  and  $cf = a$ , then we write  $f \boxdot g$ .

If  $\mathcal{X}$  is a class of morphisms in a category  $\mathbf{C}$ , then

$$\boxdot\mathcal{X} = \{f \in \text{Mor } \mathbf{C} \mid f \boxdot x \quad \forall x \in \mathcal{X}\},$$

and

$$\mathcal{X}^\boxdot = \{f \in \text{Mor } \mathbf{C} \mid x \boxdot f \quad \forall x \in \mathcal{X}\}.$$

**Definition 2.3.** A *weak factorization system* on a category  $\mathbf{C}$  consists of a pair  $(\mathcal{L}, \mathcal{R})$  of classes of morphisms in  $\mathbf{C}$  such that

- any morphism in  $\mathbf{C}$  can be factored as a morphism in  $\mathcal{L}$  followed by a morphism in  $\mathcal{R}$ , and
- $\mathcal{L} = \boxdot\mathcal{R}$  and  $\mathcal{R} = \mathcal{L}^\boxdot$ .

A weak factorization system  $(\mathcal{L}, \mathcal{R})$  is *cofibrantly generated* if there is a class of morphisms  $\mathcal{J}$  such that  $\mathcal{R} = \mathcal{J}^\boxdot$  and thus  $\mathcal{L} = \boxdot(\mathcal{J}^\boxdot)$ . It is *fibrantly generated* if there is a class of morphisms  $\mathcal{X}$  such that  $\mathcal{L} = \boxdot\mathcal{X}$  and thus  $\mathcal{R} = (\boxdot\mathcal{X})^\boxdot$ .

The definition of a model category can be compactly formulated in terms of weak factorization systems [13, 7.8], [18, 14.2.1].

**Definition 2.4.** A *model category* consists of a bicomplete category  $\mathbf{M}$ , together with three classes of morphism  $\mathcal{F}$ ,  $\mathcal{C}$ , and  $\mathcal{W}$ , called *fibrations*, *cofibrations*, and *weak equivalences*, such that

- $\mathcal{W}$  contains the identities and satisfies “2-out-of-3,” i.e., given two composable morphisms  $f$  and  $g$  in  $\mathcal{W}$ , if two of  $f$ ,  $g$ , and  $gf$  are in  $\mathcal{W}$ , so is the third; and
- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are both weak factorization systems.

A model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is *cofibrantly generated* by a pair of classes of morphisms  $(\mathcal{J}, \mathcal{J})$  if the weak factorization systems  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are cofibrantly generated by  $\mathcal{J}$  and  $\mathcal{J}$ , respectively. The elements of  $\mathcal{J}$  and  $\mathcal{J}$  are then *generating cofibrations* and *generating acyclic cofibrations*.

A model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is *fibrantly generated* by a pair of classes of morphisms  $(\mathcal{X}, \mathcal{Z})$  if the weak factorization systems  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  are fibrantly generated by  $\mathcal{X}$  and  $\mathcal{Z}$ , respectively. The elements of  $\mathcal{X}$  and  $\mathcal{Z}$  are *generating fibrations* and *generating acyclic fibrations*.

*Remark 2.5.* Our notion of cofibrant generation is **weaker** than the usual definition (see, for example, [10, Definition 11.1.2]) in two ways. First, we allow the generators to be a class of maps; observe that any model category is (trivially) cofibrantly

generated by its cofibrations and acyclic cofibrations. The reason for this convention is that past work has shown that even trivial Postnikov presentations can have non-trivial applications; see [9, Theorem 6.2].

Second, we wish to introduce separate terminology for the usual smallness conditions involving the category  $\mathbf{M}$  and the classes  $\mathcal{J}$  and  $\mathcal{J}$  that permit the use of the small object argument and its consequent characterizations of the cofibrations and acyclic cofibrations. This refined terminology is necessary because the dual conditions, appropriate to the case of fibrant generation, are not satisfied in practice.

Frequently (see Proposition 2.7), the cofibrations and acyclic cofibrations in a cofibrantly generated model category can be characterized in terms of the following construction.

**Definition 2.6.** Let  $\mathcal{J}$  be a class of morphisms in a cocomplete category  $\mathbf{C}$ . Let  $Y: \lambda \rightarrow \mathbf{C}$  be a functor, where  $\lambda$  is an ordinal. If for all  $\beta < \lambda$ , there is a pushout

$$\begin{array}{ccc} X_{\beta+1} & \xrightarrow{k_{\beta} \in \mathcal{C}} & Y_{\beta} \\ i_{\beta+1} \in \mathcal{J} \downarrow & & \downarrow \ulcorner \\ X'_{\beta+1} & \longrightarrow & Y_{\beta+1} \end{array}$$

and  $Y_{\gamma} := \operatorname{colim}_{\beta < \gamma} Y_{\beta}$  for all limit ordinals  $\gamma < \lambda$ , then the composition of the sequence

$$Y_0 \rightarrow \operatorname{colim}_{\lambda} Y_{\beta},$$

is a *relative  $\mathcal{J}$ -cell complex*. The class of all relative  $\mathcal{J}$ -cell complexes is denoted  $\operatorname{Cell}_{\mathcal{J}}$ .

A *cellular presentation* of a weak factorization system  $(\mathcal{L}, \mathcal{R})$  in a cocomplete category  $\mathbf{C}$  consists of a class  $\mathcal{J}$  of morphisms such that  $\mathcal{L} = \widehat{\operatorname{Cell}_{\mathcal{J}}}$ . This condition implies that  $\mathcal{R} = \mathcal{J}^{\square}$ , so cellularly presented weak factorization systems are also cofibrantly generated.

A *cellular presentation* of a model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  consists of a pair of classes of morphisms  $(\mathcal{J}, \mathcal{J})$  that are cellular presentations of the weak factorization systems  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ , respectively.

Any cellular presentation  $(\mathcal{J}, \mathcal{J})$  defines generating cofibrations and acyclic cofibrations for the model category  $\mathbf{M}$ . Under certain hypotheses, the small object argument (see [10, Proposition 10.5.16] or [18, Proposition 15.1.11]) implies the converse. We say that  $\mathbf{M}$ ,  $\mathcal{J}$ , and  $\mathcal{J}$  *permit the small object argument* if  $\mathcal{J}$  and  $\mathcal{J}$  are sets and a ‘‘smallness’’ condition is satisfied. If  $\mathbf{M}$  is locally presentable, then the smallness condition is automatically satisfied by any generating sets  $\mathcal{J}$  and  $\mathcal{J}$ . If  $\mathbf{M}$  is not locally presentable, there are more general notions of smallness that may be satisfied by particular sets  $\mathcal{J}$  and  $\mathcal{J}$ . We refer to [10, Definition 10.4.1] or [18, Definition 15.1.7] for the definition of smallness.

**Proposition 2.7** ([10, Proposition 11.2.1]). *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category that is cofibrantly generated by a pair of sets of morphisms  $(\mathcal{J}, \mathcal{J})$  and permits the small object argument, then  $(\mathcal{J}, \mathcal{J})$  is a cellular presentation of  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ .*

*Remark 2.8.* A model category is cofibrantly generated in the sense of [10, Definition 11.1.2] if and only if it satisfies the hypotheses of Proposition 2.7.

We recall from [7] the following definition, which dualizes Definition 2.6.

**Definition 2.9.** Let  $\mathcal{X}$  be a class of morphisms in a complete category  $\mathbf{C}$ . Let  $Y: \lambda^{\text{op}} \rightarrow \mathbf{C}$  be a functor, where  $\lambda$  is an ordinal. If for all  $\beta < \lambda$ , there is a pullback

$$\begin{array}{ccc} Y^{\beta+1} & \longrightarrow & X'^{\beta+1} \\ \downarrow & \lrcorner & \downarrow x^{\beta+1} \in \mathcal{X} \\ Y^\beta & \xrightarrow[k^\beta \in \mathbf{C}]{} & X^{\beta+1} \end{array}$$

and  $Y^\gamma := \lim_{\beta < \gamma} Y^\beta$  for all limit ordinals  $\gamma < \lambda$ , then the composition of the tower

$$\lim_{\lambda^{\text{op}}} Y^\beta \rightarrow Y^0,$$

is an  $\mathcal{X}$ -Postnikov tower. The class of all  $\mathcal{X}$ -Postnikov towers is denoted  $\text{Post}_{\mathcal{X}}$ .

A *Postnikov presentation* of a weak factorization system  $(\mathcal{L}, \mathcal{R})$  in a complete category  $\mathbf{C}$  consists of a class  $\mathcal{X}$  of morphisms such that  $\mathcal{R} = \widehat{\text{Post}}_{\mathcal{X}}$ . This implies that  $\mathcal{L} = \square \mathcal{X}$ ; a weak factorization system with a Postnikov presentation is also fibrantly generated.

A *Postnikov presentation* of a model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  consists of a pair of classes of morphisms  $(\mathcal{X}, \mathcal{Z})$  that are Postnikov presentations of the weak factorization systems  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ , respectively.

*Remark 2.10.* Every model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  admits a trivial Postnikov presentation  $(\mathcal{F}, \mathcal{F} \cap \mathcal{W})$ . On the other hand, knowledge of a particularly small and tractable Postnikov presentation, such as that discussed in Section 3, may simplify certain computations and verifications of properties of the model category, in a manner analogous to the well known, dual case of particularly nice cellular presentations.

*Remark 2.11.* If  $(\mathcal{X}, \mathcal{Z})$  is a Postnikov presentation for  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ , then  $\mathcal{C} = \square \mathcal{X}$  and  $\mathcal{C} \cap \mathcal{W} = \square \mathcal{Z}$ , whence  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$ . However, if a pair of sets  $(\mathcal{X}, \mathcal{Z})$  fibrantly generates  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ , there are no practical conditions that guarantee that the triple  $\mathbf{M}$ ,  $\mathcal{X}$ , and  $\mathcal{Z}$  “permit the cosmall object argument.” In the absence of an explicit “Postnikov-like” construction of factorizations, we cannot conclude that  $\mathcal{F} = \widehat{\text{Post}}_{\mathcal{X}}$  and  $\mathcal{F} \cap \mathcal{W} = \widehat{\text{Post}}_{\mathcal{Z}}$ .

*Remark 2.12.* One way in which Postnikov presentations of model categories, even trivial ones, can be useful is in enabling us to construct “explicit” fibrant replacements, as (retracts of) Postnikov towers, with particularly simple layers, as in [9, Section 7]. For example, given a Postnikov presentation for a model structure on a diagram category, one could construct explicit models for homotopy limits.

**2.2. Existence of left-induced model structures.** We now apply fibrant generation and Postnikov presentations to prove the existence of left-induced model structures, which are defined as follows.

**Definition 2.13.** Let  $\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$  be an adjoint pair of functors, where

$(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category, and  $\mathbf{C}$  is a bicomplete category. If the triple of classes of morphisms in  $\mathbf{C}$

$$\left( (L^{-1}(\mathcal{C} \cap \mathcal{W}))^\square, L^{-1}(\mathcal{C}), L^{-1}(\mathcal{W}) \right)$$

satisfies the axioms of a model category, then it is a *left-induced model structure* on  $\mathbf{C}$ .

*Remark 2.14.* If  $\mathbf{C}$  admits a model structure left-induced from that of  $\mathbf{M}$  via an adjunction as in the definition above, then  $L \dashv R$  is a Quillen pair with respect to the left-induced model structure on  $\mathbf{C}$  and the given model structure on  $\mathbf{M}$ .

*Remark 2.15.* A typical instance in which one is interested in the existence of a left-induced model structure is that of the forgetful/cofree adjunction associated to a comonad  $\mathbb{K}$  on model category  $\mathbf{M}$ ,

$$\mathbf{M}_{\mathbb{K}} \begin{array}{c} \xrightarrow{U_{\mathbb{K}}} \\ \perp \\ \xleftarrow{F_{\mathbb{K}}} \end{array} \mathbf{M} ,$$

where  $\mathbf{M}_{\mathbb{K}}$  is category of  $\mathbb{K}$ -coalgebras,  $F_{\mathbb{K}}$  is the cofree  $\mathbb{K}$ -coalgebra functor, and  $U_{\mathbb{K}}$  is the forgetful functor. In [9], the authors provide conditions under which the left-induced model structure exists for such an adjunction.

**Lemma 2.16.** *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$ . Any model structure on a bicomplete category  $\mathbf{C}$  that is left-induced by an adjoint pair*

$$\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$$

*is fibrantly generated by  $(R(\mathcal{X}), R(\mathcal{Z}))$ .*

*Proof.* The proof proceeds by a simple game of adjunction:

$$f \in L^{-1}(\mathcal{C}) \iff L(f) \in \mathcal{C} = \square\mathcal{X} \iff f \in \square R(\mathcal{X}),$$

and similarly for  $\mathcal{Z}$ . □

General existence results for left-induced model structures can be formulated in terms of the relation between the adjunction and either fibrant generation or Postnikov presentations, as a consequence of the following theorem. The statement below is a slight variant of that in [7], in that retracts are allowed in the desired factorizations; the proof in [7] works in this more general setting as well.

**Theorem 2.17** ([7, Theorem 5.11]). *Let  $\mathbf{C}$  be a bicomplete category, and let  $\mathcal{W}'$ ,  $\mathcal{C}'$ ,  $\mathcal{X}'$ , and  $\mathcal{Z}'$  be classes of morphisms in  $\mathbf{C}$  satisfying the following conditions.*

- (a) *The class  $\mathcal{W}'$  satisfies “2-out-of-3” and contains all identities.*
- (b) *The classes  $\mathcal{W}'$  and  $\mathcal{C}'$  are closed under retracts.*
- (c)  *$\mathcal{Z}' \subseteq \mathcal{C}'^{\square}$  and  $\mathcal{X}' \subseteq (\mathcal{C}' \cap \mathcal{W}')^{\square}$ .*
- (d)  *$\mathcal{Z}' \subseteq \mathcal{W}'$ .*
- (e) *For all  $f \in \text{Mor } \mathbf{C}$ , there exist*
  - (i)  *$i \in \mathcal{C}'$  and  $p \in \widehat{\mathcal{Z}}'$  such that  $f = pi$ ;*
  - (ii)  *$j \in \mathcal{C}' \cap \mathcal{W}'$  and  $q \in \widehat{\mathcal{X}}'$  such that  $f = qj$ .*

*Then  $(\mathbf{C}, \widehat{\mathcal{X}}', \mathcal{C}', \mathcal{W}')$  is a model category that is fibrantly generated by  $(\mathcal{X}', \mathcal{Z}')$ .*

*Sketch of proof.* Conditions (b), (c), and (e) imply that  $(\mathcal{C}' \cap \mathcal{W}', \widehat{\mathcal{X}}')$  and  $(\mathcal{C}', \widehat{\mathcal{Z}}')$  are weak factorization systems. Conditions (a) and (d) imply that  $\widehat{\mathcal{Z}}' = \widehat{\mathcal{X}}' \cap \mathcal{W}'$ . Hence, the conditions of Definition 2.4 are satisfied. □

In [7], Theorem 2.17 was applied to prove the existence of left-induced model structure in the presence of a Postnikov presentation.

**Corollary 2.18** ([7, Corollary 5.15]). *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category with Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ . Let  $\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$  be a pair of adjoint functors, where  $\mathbf{C}$  is a bicomplete category. If*

$$(a) \ L(\text{Post}_{R(\mathcal{Z})}) \subseteq \mathcal{W},$$

and for all  $f \in \text{Mor } \mathbf{C}$  there exist

$$(b) \ i \in L^{-1}(\mathcal{C}) \text{ and } p \in \widehat{\text{Post}}_{R(\mathcal{Z})} \text{ such that } f = pi, \text{ and}$$

$$(c) \ j \in L^{-1}(\mathcal{C} \cap \mathcal{W}) \text{ and } q \in \widehat{\text{Post}}_{R(\mathcal{X})} \text{ such that } f = qj,$$

then the model structure on  $\mathbf{C}$  left-induced by the adjunction  $L \dashv R$  exists and admits a Postnikov presentation  $(R(\mathcal{X}), R(\mathcal{Z}))$ .

This corollary has been applied in [7] and [9] to prove the existence of explicit left-induced model structures. A very similar proof to that in [7] enables us to establish an existence result in the presence of the weaker hypothesis of fibrant generation as well.

**Corollary 2.19.** *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$ .*

Let  $\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$  be a pair of adjoint functors, where  $\mathbf{C}$  is a bicomplete category. If

$$(a) \ L\left(\left(\square R(\mathcal{Z})\right)^\square\right) \subseteq \mathcal{W},$$

and for all  $f \in \text{Mor } \mathbf{C}$  there exist

$$(b) \ i \in L^{-1}(\mathcal{C}) \text{ and } p \in \left(\square R(\mathcal{Z})\right)^\square \text{ such that } f = pi, \text{ and}$$

$$(c) \ j \in L^{-1}(\mathcal{C} \cap \mathcal{W}) \text{ and } q \in \left(\square R(\mathcal{X})\right)^\square \text{ such that } f = qj,$$

then the model structure on  $\mathbf{C}$  left-induced by the adjunction  $L \dashv R$  exists and is fibrantly generated by  $(R(\mathcal{X}), R(\mathcal{Z}))$ .

*Proof.* Apply Theorem 2.17 to  $\mathcal{W}' = L^{-1}(\mathcal{W})$ ,  $\mathcal{C}' = L^{-1}(\mathcal{C})$ ,  $\mathcal{X}' = \left(\square R(\mathcal{X})\right)^\square$ , and  $\mathcal{Z}' = \left(\square R(\mathcal{Z})\right)^\square$ .  $\square$

*Remark 2.20.* In both of the corollaries above, we see that condition (a) is the “glue” binding together the two potential weak factorization systems given by conditions (b) and (c), thus ensuring that we indeed obtain a model category. In practice, the explicit description of Postnikov towers as limits of towers of pullbacks provides us with tools for proving condition (a) of Corollary 2.18 that are not available in the more general case of Corollary 2.19.

Together with a recent result of Makkai and Rosický [17, Theorem 3.2], Corollary 2.19 implies the following existence result for left-induced model structures, when  $\mathbf{M}$  and  $\mathbf{C}$  are locally presentable. Recall that in a locally presentable category, any generating set of morphisms gives a cellular presentation of its weak factorization system.



**Theorem 2.21.** Let  $\mathcal{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathcal{M}$  be an adjoint pair of functors, where  $\mathcal{C}$  is a locally presentable category and  $(\mathcal{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a locally presentable model category that is cofibrantly generated by a pair of sets of morphisms  $(\mathcal{J}, \mathcal{I})$  and fibrantly generated by a pair of classes of morphisms  $(\mathcal{X}, \mathcal{Z})$ . If  $L\left(\left(\square R(\mathcal{Z})\right)^\square\right) \subseteq \mathcal{W}$ , then the left-induced model structure on  $\mathcal{C}$  exists and is both cofibrantly and fibrantly generated.

*Proof.* Under the hypotheses above, [17, Theorem 3.2] implies that

$$\left(R(\mathcal{X}), \left(\square R(\mathcal{Z})\right)^\square\right) \quad \text{and} \quad \left(R(\mathcal{Z}), \left(\square R(\mathcal{X})\right)^\square\right)$$

are weak factorization systems (cf. [17, Remark 3.8]), whence conditions (b) and (c) of Corollary 2.19 hold. Since condition (a) holds by hypothesis, we can conclude.  $\square$

**2.3. Elementary properties of left-induced structures.** In this section, we show that left-induced model structures inherit certain properness and enrichment properties.

Recall that a model structure is *left proper* if the pushout of a weak equivalence along a cofibration is always a weak equivalence.

**Lemma 2.22.** Let  $(\mathcal{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category, and let  $\mathcal{C}$  be a bicomplete category. Suppose that  $\mathcal{C}$  admits a model structure that is left-induced from  $\mathcal{M}$  via the adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathcal{M}.$$

If  $\mathcal{M}$  is left proper, then  $\mathcal{C}$  is left proper, too.

*Proof.* Consider a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \bar{g} \\ C & \xrightarrow{\bar{f}} & B \sqcup_A C \end{array}$$

in  $\mathcal{C}$ , where  $g \in L^{-1}(\mathcal{W})$  and  $f \in L^{-1}(\mathcal{C})$ . Since left adjoints commute with colimits, applying  $L$  gives rise to a pushout in  $\mathcal{M}$

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ Lg \downarrow & & \downarrow L\bar{g} \\ LC & \xrightarrow{L\bar{f}} & L(B \sqcup_A C) \cong LB \sqcup_{LA} LC, \end{array}$$

where  $Lf \in \mathcal{C}$  and  $Lg \in \mathcal{W}$ . Since  $\mathcal{M}$  is left proper, the morphism

$$L\bar{g}: LB \rightarrow LB \sqcup_{LA} LC \cong L(B \sqcup_A C)$$

is a weak equivalence in  $\mathcal{M}$ , whence  $\bar{g} \in L^{-1}(\mathcal{W})$ , so that  $\mathcal{C}$  is left proper, as desired.  $\square$

Extra assumptions are required for the transfer of right-properness. In Section 4.4, we prove a right properness result for diagram categories.

We refer the reader to Appendix A for background on enriched model categories.

**Lemma 2.23.** *Let  $(V, \wedge, \mathbb{I})$  be a closed symmetric monoidal model category, and let  $(M, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a  $V$ -model category. Suppose that  $\mathcal{C}$  admits a model structure that is left-induced from  $M$  via the adjunction*

$$\mathcal{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} M.$$

*If  $\mathcal{C}$  is a tensored and cotensored  $V$ -category and if  $L \dashv R$  is a  $V$ -adjunction, then  $\mathcal{C}$  is a  $V$ -model category.*

*Proof.* We need to show that the SM7 axiom and the unit axiom from Definition A.1 hold in  $\mathcal{C}$ .

Let  $i: X \rightarrow Y$  be a cofibration in  $\mathcal{C}$ , i.e.,  $i \in L^{-1}(\mathcal{C})$ , and let  $j: V \rightarrow W$  be a cofibration in  $V$ . We must show that the map

$$j \widehat{\otimes} i: V \otimes Y \sqcup_{V \otimes X} W \otimes X \rightarrow W \otimes Y$$

is a cofibration in  $\mathcal{C}$ , which is a weak equivalence if either  $j$  or  $i$  is a weak equivalence. Observe that

$$\begin{aligned} L(V \otimes Y \sqcup_{V \otimes X} W \otimes X) &\cong L(V \otimes Y) \sqcup_{L(V \otimes X)} L(W \otimes X) \\ &\cong V \otimes LY \sqcup_{V \otimes LX} W \otimes LX, \end{aligned}$$

where the first isomorphism holds because  $L$  commutes with colimits and the second isomorphism holds because  $L \dashv R$  is a  $V$ -adjunction (see Definition A.3).

Up to isomorphism we can therefore identify the morphism  $L(j \widehat{\otimes} i)$  with

$$j \widehat{\otimes} Li: V \otimes LY \sqcup_{V \otimes LX} W \otimes LX \rightarrow W \otimes LY.$$

Since  $Li \in \mathcal{C}$ , and  $M$  satisfies SM7, it follows that  $j \widehat{\otimes} Li \in \mathcal{C}$ , whence  $j \widehat{\otimes} i$  is a cofibration in  $\mathcal{C}$ , as desired. If, in addition,  $Li$  or  $j$  is a weak equivalence, then by SM7 in  $M$ , the map  $j \widehat{\otimes} Li \in \mathcal{C} \cap \mathcal{W}$ , whence  $j \widehat{\otimes} i$  is an acyclic cofibration in  $\mathcal{C}$ .

To prove the unit axiom in  $\mathcal{C}$ , we need to check that for any cofibrant object  $C$  in  $\mathcal{C}$ , the morphism  $q \otimes \text{Id}_C: Q\mathbb{I} \otimes C \rightarrow \mathbb{I} \otimes C$  is a weak equivalence in  $\mathcal{C}$ , where  $q: Q\mathbb{I} \rightarrow \mathbb{I}$  denotes cofibrant replacement. By the definition of the left-induced model structure, this is equivalent to asking that the map  $L(q \otimes \text{Id}_C): L(Q\mathbb{I} \otimes C) \rightarrow L(\mathbb{I} \otimes C)$  be a weak equivalence in  $M$ . Since  $L(Q\mathbb{I} \otimes C) \cong Q\mathbb{I} \otimes LC$  and  $L(\mathbb{I} \otimes C) \cong \mathbb{I} \otimes LC$ ,  $L(q \otimes \text{Id}_C)$  is isomorphic to

$$q \otimes \text{Id}_{LC}: Q\mathbb{I} \otimes LC \rightarrow \mathbb{I} \otimes LC.$$

Because  $L$  preserves cofibrations and initial objects,  $LC$  is a cofibrant object in  $M$ . Since the unit axiom holds in  $M$ , we conclude that  $q \otimes \text{Id}_{LC}$  is a weak equivalence in  $M$ ; therefore,  $q \otimes \text{Id}_C$  is a weak equivalence in the left-induced model structure on  $\mathcal{C}$ .  $\square$

In Section 4.4 we apply Lemma 2.23 to establish the existence of enriched model structure on certain diagram categories.

## 3. EXAMPLES

In this section we provide examples to illustrate the theory of section 2. We first give an explicit Postnikov presentation of the category of non-negatively graded chain complexes over a field, then apply Theorem 2.21 to prove the existence of a left-induced model category structure on the category of  $H$ -comodule algebras over a field, where  $H$  is a chain Hopf algebra of finite type.

**3.1. An explicit Postnikov presentation.** Let  $\mathbb{k}$  be a field, and let  $\mathbf{Ch}$  be the category of non-negatively graded chain complexes of  $\mathbb{k}$ -vector spaces. This category admits a model structure whose weak equivalences are quasi-isomorphisms, whose cofibrations are degreewise monomorphisms, and whose fibrations are degreewise surjective in positive degrees [5]. Our aim in this section is to show that this model structure admits a Postnikov presentation, which provides an elementary and non-obvious characterization of the fibrations and acyclic fibrations.

Write  $D^n$  for the  $n$ -disc and  $S^n$  for the  $n$ -sphere defined by

$$(D^n)_\ell = \begin{cases} \mathbb{k} : \ell = n, n-1 \\ 0 : \text{else.} \end{cases} \quad (S^n)_\ell = \begin{cases} \mathbb{k} : \ell = n \\ 0 : \text{else.} \end{cases}$$

It is productive to think of  $S^n$  as  $K(\mathbb{k}, n)$  and  $D^n$  as based paths in the Eilenberg Mac Lane space  $K(\mathbb{k}, n-1)$ ; a chain map  $X \rightarrow D^n$  is an element of  $X_{n-1}^*$  and a chain map  $X \rightarrow S^n$  is an element of  $Z^n X^*$ , where  $X^*$  denotes the dual chain complex.

Define

$$\mathcal{X} = \{p_n : D^n \rightarrow S^n\}_{n \geq 1} \cup \{p_0 : 0 \rightarrow S^0\} \quad \text{and} \quad \mathcal{Z} = \{q_n : D^n \rightarrow 0\}_{n \geq 1}.$$

**Theorem 3.1.** *The pair  $(\mathcal{X}, \mathcal{Z})$  defines a Postnikov presentation for the model category  $\mathbf{Ch}$ .*

The proof of Theorem 3.1 will be given by the following two lemmas, which show that the two constituent weak factorization systems have Postnikov presentations. We adopt different strategies for each proof. For the acyclic fibrations, we construct a functorial  $\mathcal{Z}$ -Postnikov presentation in the proof of Lemma 3.2. For the fibrations, we instead define an ad-hoc  $\mathcal{X}$ -Postnikov presentation for each fibration in the proof of Lemma 3.3. The advantage of this approach is that the Postnikov tower we construct is “smaller” (e.g., in dimension of the constituent vector spaces) than would be possible with a functorial construction.

**Lemma 3.2.** *The set  $\mathcal{Z}$  defines a Postnikov presentation for the acyclic fibrations in  $\mathbf{Ch}$ .*

*Proof.* A generic chain map  $f : X \rightarrow Y$  factors as

$$X \xrightarrow{(f, \text{ev})} Y \times \prod_n \prod_{X_n^*} D^{n+1} \xrightarrow{\text{proj}} Y.$$

The right factor is a pullback of products of maps  $q_n$ , which may be re-expressed as a limit of a tower of pullbacks of maps in  $\mathcal{Z}$ . Hence the right factor lies in  $\mathbf{Post}_{\mathcal{Z}}$ , and moreover the left factor is a degreewise monomorphism. By the retract argument, if  $f$  is an acyclic fibration, then  $f$  is a retract of the displayed right factor. Hence, the acyclic fibrations admit a Postnikov presentation.  $\square$

**Lemma 3.3.** *The set  $\mathcal{X}$  defines a Postnikov presentation for the fibrations in  $\mathbf{Ch}$ .*

*Proof.* Consider a fibration  $\pi: E \rightarrow B$ . We have  $B_0 \cong \pi_0(E_0) \oplus \overline{B}_0$ . Pick a basis  $\overline{B}_0 \cong \mathbb{k}[\beta_0]$  for the cokernel of  $p_0$ , and then define  $E^0$  to be the pullback

$$\begin{array}{ccc} E^0 & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \Pi_{\beta_0} p_0 \\ B & \xrightarrow{\text{proj}_{\overline{B}_0}} & \prod_{\beta_0} S^0 \end{array}$$

That is, we form  $E^0$  by killing the cokernel of  $\pi$  in degree zero. There is a canonical map  $\pi^0: E \rightarrow E^0$  that is surjective in degree zero because  $(E^0)_0 \cong \pi_0(E_0)$ .

Now we have  $E_0 \cong \ker \pi_0 \oplus \overline{E}_0$ . By the first isomorphism theorem  $\overline{E}_0 \cong \pi_0(E_0)$ . Write  $\ker \pi_0 \cong \mathbb{k}[\alpha_0]$ . Define

$$\begin{array}{ccccc} E^1 & \longrightarrow & \prod_{\alpha_0} S^0 & \longrightarrow & \prod_{\alpha_0} D^1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ E^0 & \longrightarrow & 0 & \longrightarrow & \prod_{\alpha_0} S^1 \end{array}$$

by successive pullbacks. There is a natural map  $\pi^1: E \rightarrow E^1$ , and it is an isomorphism in degree zero. The displayed pullbacks use maps  $S^0 \rightarrow 0$  to attach the kernel of  $\pi_0$  to  $E^0$ .

Note  $(E^1)_1 \cong (E^0)_1 \cong B_1$  so  $\pi^1_1: E_1 \rightarrow (E^1)_1$  is surjective. To factor this map through a map that is an isomorphism in degree one, we just need to attach the kernel. The procedure will be similar to the one just described.

Write  $E_1 \cong \ker(\pi^1)_1 \oplus \overline{E}_1$ , where  $\ker(\pi^1)_1 \cong \mathbb{k}[\alpha_1]$ . Each element of  $\ker(\pi^1)_1$  is a cycle because  $\ker(\pi^1)_0 = 0$  by construction. So, as above, we can attach this kernel by constructing the pullback

$$\begin{array}{ccccc} E^2 & \longrightarrow & \prod_{\alpha_1} S^1 & \longrightarrow & \prod_{\alpha_1} D^2 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ E^1 & \longrightarrow & 0 & \longrightarrow & \prod_{\alpha_1} S^2 \end{array}$$

The natural map  $\pi^2: E \rightarrow E^2$  is an isomorphism in degree zero and in degree one:  $(E^2)_0 \cong (E^1)_0$  so  $\pi^2_0$  is an iso, and  $(E^2)_1 = (E^1)_0 \oplus \prod_{\alpha_1} S^1 \cong E_1$ , so  $\pi^2_1$  is an isomorphism.

The Postnikov presentation for  $\pi$  is constructed by iterating this last step, constructing a chain complex  $E^{n+1}$  and map  $\pi^{n+1}: E \rightarrow E^{n+1}$  that is an isomorphism in degrees zero through  $n$  by attaching the kernel of  $(\pi^n)_n$  to  $E^n$ .  $\square$

**3.2. An application of Theorem 2.21.** In this section  $\mathbf{Ch}$  refers again to the category of non-negatively graded chain complexes over a field  $\mathbb{k}$ . Let  $(\mathbf{Ch}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  denote the projective model category structure on  $\mathbf{Ch}$ , as above. It is well known that  $(\mathbf{Ch}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a combinatorial model category [26, 3.7]. Since we are working over a field, all objects in  $\mathbf{Ch}$  are both fibrant and cofibrant.

The theorem below, the proof of which relies strongly on Theorem 2.21, plays an important role in Karpova's study of homotopic Hopf-Galois extensions of differential graded algebras in [14]. The mathematical concepts that are in play are the following.

**Definition 3.4.** Let  $(\mathcal{V}, \otimes, \mathbb{I})$  be a symmetric monoidal category. Let  $H$  be a bimonoid in  $\mathcal{V}$ . An  $H$ -comodule algebra is an  $H$ -comodule in the category of monoids in  $\mathcal{V}$ .

*Remark 3.5.* For every bimonoid  $H$ , there is an adjunction

$$\mathrm{Alg}^H \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{-\otimes H} \end{array} \mathrm{Alg} ,$$

where  $\mathrm{Alg}$  denotes the category of monoids in  $\mathcal{V}$  and  $\mathrm{Alg}^H$  the category of  $H$ -comodule algebras, and  $U$  is the forgetful functor. Restricting to augmented monoids, we obtain an adjunction

$$\mathrm{Alg}_*^H \begin{array}{c} \xrightarrow{U} \\ \perp \\ \xleftarrow{-\otimes H} \end{array} \mathrm{Alg}_* .$$

Note that  $\mathrm{Alg}^H$  (respectively,  $\mathrm{Alg}_*^H$ ) is the category of coalgebras for the comonad

$$\mathbb{K}_H = (- \otimes H, - \otimes \delta, - \otimes \varepsilon)$$

acting on  $\mathrm{Alg}$  (respectively,  $\mathrm{Alg}_*$ ), where  $\delta$  and  $\varepsilon$  are the comultiplication and counit of  $H$ , respectively.

*Remark 3.6.* For use in the proof below, we recall the elementary fact that for any comonad  $\mathbb{K}$  acting on a cocomplete category  $\mathcal{C}$ , the category  $\mathcal{C}_{\mathbb{K}}$  of  $\mathbb{K}$ -coalgebras is cocomplete, with colimits created in  $\mathcal{C}$ .

*Remark 3.7.* The following observation, combining results due to Schwede and Shipley and to Adámek and Rosický, is also useful in the proof below. Let  $(\mathcal{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a combinatorial model category with a set  $\mathcal{J}$  of generating acyclic cofibrations. Let  $\mathbb{T} = (T, \mu, \eta)$  be a monad acting on  $\mathcal{M}$ . Suppose that  $T$  preserves filtered colimits, and  $U^{\mathbb{T}}(\mathrm{Cell}_{F^{\mathbb{T}}\mathcal{J}}) \subseteq \mathcal{W}$ . By [25, Lemma 2.3],  $\mathcal{M}^{\mathbb{T}}$  admits a cellularly presented, right-induced model category structure. On the other hand, as proved in [1, Theorem 2.78], since  $\mathcal{M}$  is locally presentable, and  $\mathbb{T}$  preserves filtered colimits,  $\mathcal{M}^{\mathbb{T}}$  is also locally presentable. It follows that the free  $\mathbb{T}$ -algebra/forgetful-adjunction

$$\mathcal{M} \begin{array}{c} \xrightarrow{F^{\mathbb{T}}} \\ \perp \\ \xleftarrow{U^{\mathbb{T}}} \end{array} \mathcal{M}^{\mathbb{T}}$$

right-induces a combinatorial model category structure on the category  $\mathcal{M}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras.

**Theorem 3.8.** *If  $\mathbb{k}$  is a field, and  $H$  is a differential graded  $\mathbb{k}$ -Hopf algebra that is finite dimensional in each degree, then the adjunction  $U \dashv (- \otimes H)$  left-induces a combinatorial model category structure on the category  $\mathrm{Alg}_*^H$  of augmented right  $H$ -comodule algebras.*

*Proof.* As proved in [25, Theorem 4.1], the hypotheses of Remark 3.7 are satisfied by the free associative algebra monad  $\mathbb{T}_{\mathrm{Alg}} = (T, \mu, \eta)$  acting on  $(\mathrm{Ch}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ . There is therefore a combinatorial model category structure on the category of associative chain algebras, inducing a combinatorial model structure  $(\mathcal{F}_{\mathrm{Alg}}, \mathcal{C}_{\mathrm{Alg}}, \mathcal{W}_{\mathrm{Alg}})$  on the category of augmented, associative chain algebras  $\mathrm{Alg}_*$ , which is also trivially fibrantly generated by  $(\mathcal{F}_{\mathrm{Alg}}, \mathcal{F}_{\mathrm{Alg}} \cap \mathcal{W}_{\mathrm{Alg}})$ .

Since limits in  $\mathbf{Alg}_*$  are created in  $\mathbf{Ch}$ , the proof of [9, Lemma 6.8] implies that all limits in  $\mathbf{Alg}_*^H$  exist and are created in  $\mathbf{Ch}$  as well, since  $H$  is finite dimensional in each degree. By Remark 3.6, we deduce that  $\mathbf{Alg}_*^H$  is bicomplete.

Theorem 2.21 now implies that, in order to conclude, it suffices to show that

$$U\left(\left(\mathbb{Q}((\mathcal{F}_{\mathbf{Alg}} \cap \mathcal{W}_{\mathbf{Alg}}) \otimes H)\right)^{\mathbb{Q}}\right) \subseteq \mathcal{W}_{\mathbf{Alg}},$$

which is equivalent, by the usual adjunction argument, to proving that

$$(U^{-1}\mathcal{C}_{\mathbf{Alg}})^{\mathbb{Q}} \subseteq U^{-1}\mathcal{W}_{\mathbf{Alg}},$$

where the elements of  $\mathcal{W}_{\mathbf{Alg}}$  are quasi-isomorphisms of associative chain algebras.

We begin by showing that every element of  $(U^{-1}\mathcal{C}_{\mathbf{Alg}})^{\mathbb{Q}}$  is a surjection. Let  $\mathbf{Bar}$  denote the reduced bar construction functor, which associates a connected, coassociative chain coalgebra to any object in  $\mathbf{Alg}_*$ , and let  $\Omega$  denote its adjoint, the reduced cobar construction functor [21, §10.3]. By [21, Corollary 10.5.4], every component

$$\varepsilon_A : \Omega \mathbf{Bar} A \rightarrow A$$

of the counit of the adjunction is a quasi-isomorphism. We now prove that if  $(A, \rho)$  is an  $H$ -comodule algebra, then the  $H$ -coaction lifts to a coaction  $\tilde{\rho}$  on  $\Omega \mathbf{Bar} A$ , so that  $\varepsilon_A$  is a morphism of  $H$ -comodule algebras.

The coaction  $\tilde{\rho}$  is equal to the composite

$$\begin{array}{ccccc} \Omega \mathbf{Bar} A & \xrightarrow{\Omega \mathbf{Bar} \rho} & \Omega \mathbf{Bar}(A \otimes H) & \xrightarrow{\varphi_{AW}} & \Omega(\mathbf{Bar} A \otimes \mathbf{Bar} H) \\ & & & & \downarrow q \\ & & & & \Omega \mathbf{Bar} A \otimes \Omega \mathbf{Bar} H \\ & \searrow \tilde{\rho} & & & \downarrow \Omega \mathbf{Bar} A \otimes \varepsilon_H \\ & & & & \Omega \mathbf{Bar} A \otimes H, \end{array}$$

where  $\varphi_{AW}$  is the natural quasi-isomorphism of chain algebras realizing the strongly homotopy coalgebra map structure of the Alexander-Whitney map

$$\mathbf{Bar}(A \otimes H) \rightarrow \mathbf{Bar} A \otimes \mathbf{Bar} H$$

[8, Theorem A.11], and  $q$  is Milgram's natural quasi-isomorphism of chain algebras [19].

Next we explicitly define  $\tilde{\rho}$  on elements. For any  $a \in A$ , write  $\rho(a) = a_k \otimes h^k(a)$ , using the Einstein summation convention, i.e., we sum over any index that is both a subscript and a superscript, as  $k$  is here. Note that since  $\rho$  is a morphism of algebras, for all  $a_1, \dots, a_n \in A$ ,

$$(3.1) \quad (a_1 \cdots a_n)_k \otimes h^k(a_1 \cdots a_n) = \pm (a_1)_{k_1} \cdots (a_n)_{k_n} \otimes h^{k_1}(a_1) \cdots h^{k_n}(a_n),$$

where the sign is given by the Koszul rule, i.e., a sign  $(-1)^{pq}$  is introduced every time an element of degree  $p$  commutes past an element of degree  $q$ . Moreover, since  $\rho$  is a chain map,

$$(3.2) \quad (da)_k \otimes h^k(da) = da_k \otimes h^k(a) + (-1)^{\deg a_k} a_k \otimes dh^k(a)$$

for all  $a \in A$ , where  $d$  denotes the differentials on both  $A$  and  $H$ .

Let  $s$  denote the endofunctor on graded vector spaces specified by  $(sV)_n = V_{n-1}$ , with inverse functor  $s^{-1}$ . Since the chain algebra  $\Omega \mathbf{Bar} A$  is free as an algebra on

$s^{-1}(\text{Bar } A)_{>0}$ , of which a typical element is  $s^{-1}(sa_1|\cdots|sa_n)$ , where  $a_1, \dots, a_n \in A$ , it suffices to define  $\tilde{\rho}$  on such elements, then to extend to a morphism of algebras.

Define  $\tilde{\rho} : \Omega \text{Bar } A \rightarrow \Omega \text{Bar } A \otimes H$  by setting

$$\tilde{\rho}(s^{-1}(sa_1|\cdots|sa_n)) = \pm s^{-1}(s(a_1)_{k_1}|\cdots|s(a_n)_{k_n}) \otimes h^{k_1}(a_1) \cdots h^{k_n}(a_n),$$

where this time the sign also takes into account that  $s$  and  $s^{-1}$  are operators of degrees 1 and  $-1$ , respectively. Since  $(\rho \otimes H)\rho = (A \otimes \delta)\rho$ , where  $\delta$  is the comultiplication on  $H$ , it follows that  $(\tilde{\rho} \otimes H)\tilde{\rho} = (A \otimes \delta)\tilde{\rho}$ . It is easy to check that  $\tilde{\rho}$  is a chain map as well, using the formulas in [21, §10.3] for the bar and cobar differentials and identities (3.1) and (3.2). Finally, since  $\varepsilon_A : \Omega \text{Bar } A \rightarrow A$  is specified by

$$\varepsilon_A(s^{-1}(sa_1|\cdots|sa_n)) = \begin{cases} a_1 & : n = 1 \\ 0 & : n > 1, \end{cases}$$

it is clear that  $\varepsilon_A$  is a morphism of  $H$ -comodule algebras, with respect to  $\rho$  and  $\tilde{\rho}$ .

Let  $p : (A', \rho') \rightarrow (A, \rho)$  be an element of  $(U^{-1}\mathcal{C}_{\text{Alg}})^{\square}$ . Consider the following commuting diagram

$$\begin{array}{ccc} \mathbb{k} & \longrightarrow & (A', \rho') \\ \downarrow & \dashrightarrow^{\tilde{\varepsilon}} & \downarrow p \\ (\Omega \text{Bar } A, \tilde{\rho}) & \xrightarrow{\varepsilon_A} & (A, \rho), \end{array}$$

where  $\mathbb{k}$  is the initial and terminal object of  $\text{Alg}_*$ . The lefthand vertical arrow is an element of  $U^{-1}(\mathcal{C}_{\text{Alg}})$ , since  $\Omega \text{Bar } A$  is cofibrant in  $\text{Alg}_*$ ; note that it is important here that we are working over a field, so that the chain complexes underlying  $A$  and  $\text{Bar } A$  are cofibrant. The dotted lift  $\tilde{\varepsilon}$  therefore exists. Because  $\varepsilon_A$  is surjective,  $p$  must be as well, since  $p\tilde{\varepsilon} = \varepsilon_A$ .

We next show that if  $p : (A, \rho) \rightarrow \mathbb{k}$  is an element of  $(U^{-1}\mathcal{C}_{\text{Alg}})^{\square}$ , then  $p$  is a quasi-isomorphism. Let  $\text{frAlg}_*^H$  denote the full subcategory of  $\text{Alg}_*^H$ , the objects of which are such that the underlying algebra is free associative. Define an endofunctor  $\text{Cone}$  on  $\text{frAlg}_*^H$  as follows. For every  $(TV, d, \rho) \in \text{Ob frAlg}_*^H$ ,

$$\text{Cone}(TV, d, \rho) = (T(V \oplus sV), D, \hat{\rho}),$$

where  $D|_V = d$  and  $D(sv) = -s(dv) + v$  for all  $v \in V$ , while the algebra morphism  $\hat{\rho}$  is specified on generators  $v \in V$  by  $\hat{\rho}(v) = \rho(v)$  and

$$\hat{\rho}(sv) = \sum_i sv_{i1}|v_{i2}|\cdots|v_{in_i} \otimes h_i,$$

if  $\rho(v) = \sum_i v_{i1}|v_{i2}|\cdots|v_{in_i} \otimes h_i$ . It is easy to check that  $(\hat{\rho} \otimes H)\hat{\rho} = (A \otimes \delta)\hat{\rho}$ , since  $(\rho \otimes H)\rho = (A \otimes \delta)\rho$ , and that  $\hat{\rho}$  is a chain map, since  $\rho$  is a chain map. Note that  $(TV, d, \rho)$  is a sub  $H$ -comodule algebra of  $\text{Cone}(TV, d, \rho)$  and that the inclusion

$$\iota : (TV, d, \rho) \hookrightarrow \text{Cone}(TV, d, \rho)$$

is always an element of  $U^{-1}(\mathcal{C}_{\text{Alg}})$ , since we are working over a field. Observe moreover that the unique morphism  $e : \text{Cone}(TV, d, \rho) \rightarrow \mathbb{k}$  is always a quasi-isomorphism.

Consider the commuting diagram

$$\begin{array}{ccc} (\Omega \text{ Bar } A, \tilde{\rho}) & \xrightarrow{\varepsilon_A} & (A, \rho) \\ \downarrow \iota & \nearrow \widehat{e} & \downarrow p \\ \text{Cone}(\Omega \text{ Bar } A, \tilde{\rho}) & \xrightarrow{e} & \mathbb{k}. \end{array}$$

Since  $\iota \in U^{-1}(\mathcal{C}_{\text{Alg}})$ , the dotted lift  $\widehat{e}$  exists. Because  $\widehat{e}\iota = \varepsilon_A$  and  $p\widehat{e} = e$ , both of which are quasi-isomorphisms, it follows by “2-out-of-6” that  $\iota$ ,  $\widehat{e}$  and, in particular,  $p$  are all quasi-isomorphisms.

We are finally ready to prove that every element of  $(U^{-1}\mathcal{C}_{\text{Alg}})^{\square}$  is in fact a quasi-isomorphism. Let  $p : (A', \rho') \rightarrow (A, \rho)$  be an element of  $(U^{-1}\mathcal{C}_{\text{Alg}})^{\square}$ , and take the pullback in  $\text{Alg}_*^H$

$$\begin{array}{ccc} (A'', \rho'') & \longrightarrow & (A', \rho') \\ \downarrow \bar{p} & \lrcorner & \downarrow p \\ \mathbb{k} & \xrightarrow{\eta} & (A, \rho), \end{array}$$

where  $\eta$  is the unit of the algebra  $A$ . Observe that  $\bar{p} \in (U^{-1}\mathcal{C}_{\text{Alg}})^{\square}$ , since any class of morphisms characterized by a right lifting property is closed under pullback. By the previous step in the proof,  $\bar{p}$  is a quasi-isomorphism. Moreover,  $A'' \cong \mathbb{k} \oplus \ker p$ , as chain complexes, because limits in  $\text{Alg}_*^H$  are created in  $\text{Ch}$ . It follows that  $\ker p$  is acyclic, and thus  $p$  is a quasi-isomorphism, since there is a short exact sequence of chain complexes

$$0 \rightarrow \ker p \hookrightarrow A' \xrightarrow{p} A \rightarrow 0. \quad \square$$

*Remark 3.9.* We will show in a forthcoming article that Theorem 3.8 actually holds over any commutative ring. Our proof depends on a generalization of Theorem 2.21 to the context of enriched cofibrant generation and on working with relative model structures.

#### 4. INJECTIVE MODEL STRUCTURES

Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category, and let  $\mathbf{D}$  be a small category. Recall that the *projective model structure* on the diagram category  $\mathbf{M}^{\mathbf{D}}$ , if it exists, is a model structure in which the fibrations and weak equivalences are determined objectwise, i.e.,  $\tau : \Phi \rightarrow \Psi$  is a fibration (respectively, weak equivalence) if and only if  $\tau_d : \Phi(d) \rightarrow \Psi(d)$  is a fibration (respectively, weak equivalence) in  $\mathbf{M}$  for every  $d \in \text{Ob } \mathbf{D}$ . Dually, the *injective model structure* on  $\mathbf{M}^{\mathbf{D}}$ , if it exists, is a model structure in which the cofibrations and weak equivalences are determined objectwise. An (acyclic) fibration in the injective model category structure is called an *injective (acyclic) fibration*.

If  $\mathbf{M}$  is cofibrantly generated by a pair of sets of maps that permit the small object argument, then  $\mathbf{M}^{\mathbf{D}}$  admits a projective model structure for any small category  $\mathbf{D}$ . The proof constructs an explicit cellular presentation for the projective model structure. In this section, we apply the notions of Postnikov presentation and fibrant generation to studying the following problem.

**Problem 4.1.** *Let  $\mathbf{D}$  be a small category, and let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category. When does  $\mathbf{M}^{\mathbf{D}}$  admit the injective model structure?*



These are already well-known existence results for injective model structures in the literature. Most significantly, if  $\mathbf{M}$  is a combinatorial model category, then  $\mathbf{M}^{\mathbf{D}}$  admits the injective model structure, for all small categories  $\mathbf{D}$ ; see [16, Proposition A.2.8.2]. Moreover, Bergner and Rezk showed in [4, Proposition 3.15] that if  $\mathbf{D}$  is an elegant Reedy category, and  $\mathbf{M}$  is the category of simplicial presheaves on some other small category, endowed with the injective model structure, then  $\mathbf{M}^{\mathbf{D}^{\text{op}}}$  admits the injective model structure, which turns out to be the same as the Reedy model structure. Finally, if  $\mathbf{D}$  is an inverse category, then  $\mathbf{M}^{\mathbf{D}}$  always admits the injective model structure; see for example [11, Theorem 5.1.3]. Again, the injective model structure coincides with the Reedy model structure in this case.

We show here that Postnikov presentations and fibrant generation provide us with new, useful tools for proving the existence of injective model structures, which, in turn, also admit Postnikov presentations or are fibrantly generated.

**4.1. Diagrams as algebras.** As a warm up to constructing and studying injective model structures on diagram categories, we take a new look at one existence proof for projective model structures.

Let  $\mathbf{C}$  be any bicomplete category, and let  $\mathbf{D}$  be a small category, with discrete subcategory  $\mathbf{D}_\delta$ . Let  $\iota_{\mathbf{D}} : \mathbf{D}_\delta \hookrightarrow \mathbf{D}$  denote the inclusion functor. Let

$$\text{Lan}_{\iota_{\mathbf{D}}} : \mathbf{C}^{\mathbf{D}_\delta} \rightarrow \mathbf{C}^{\mathbf{D}}$$

denote the left adjoint of the *restriction functor*

$$\iota_{\mathbf{D}}^* : \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}^{\mathbf{D}_\delta},$$

which is given by precomposing with  $\iota_{\mathbf{D}}$ . The image under  $\text{Lan}_{\iota_{\mathbf{D}}}$  of any functor  $\Phi : \mathbf{D}_\delta \rightarrow \mathbf{C}$  is its left Kan extension along  $\iota_{\mathbf{D}}$ , which is the same as its associated *copower construction*, i.e.,

$$\text{Lan}_{\iota_{\mathbf{D}}}(\Phi)(d) = \coprod_{d' \in \text{Ob } \mathbf{D}_\delta} \prod_{f: d' \rightarrow d} \Phi(d').$$

The next result, certainly well known to category theorists, is quite useful for studying model structures on diagram categories.

**Lemma 4.2.** *Let  $\mathbb{T}_{\mathbf{D}}$  denote the monad associated to the adjunction*

$$\mathbf{C}^{\mathbf{D}_\delta} \begin{array}{c} \xrightarrow{\text{Lan}_{\iota_{\mathbf{D}}}} \\ \perp \\ \xleftarrow{\iota_{\mathbf{D}}^*} \end{array} \mathbf{C}^{\mathbf{D}}.$$

*The category  $(\mathbf{C}^{\mathbf{D}_\delta})^{\mathbb{T}_{\mathbf{D}}}$  of  $\mathbb{T}_{\mathbf{D}}$ -algebras in  $\mathbf{C}^{\mathbf{D}_\delta}$  is isomorphic to  $\mathbf{C}^{\mathbf{D}}$ .*

*Proof.* The isomorphism can be proved by inspection or by application of Beck's monadicity theorem.  $\square$

To illustrate the utility of this algebraic description of diagram categories and to motivate considering the coalgebraic description below, we now show that [25, Lemma 2.3] and Lemma 4.2 together imply easily the well-known result that every category of diagrams taking values in a cofibrantly generated model category that permits the small object argument admits the projective model structure, which is also cofibrantly generated, has a cellular presentation, and permits the small object argument [10, Theorem 11.6.1].

**Definition 4.3.** Let  $\mathbf{C}$  be a cocomplete category and  $\mathbf{D}$  a small category. The copower induces a bifunctor  $- \otimes -: \mathbf{C} \times \mathbf{Set}^{\mathbf{D}} \rightarrow \mathbf{C}^{\mathbf{D}}$  defined by

$$X \otimes \mathbf{D}(d, -) := \coprod_{\mathbf{D}(d, -)} X,$$

for any  $X \in \text{Ob } \mathbf{C}$  and  $d \in \text{Ob } \mathbf{D}$ . For any class  $\mathcal{X}$  of morphisms in  $\mathbf{C}$  and any small category  $\mathbf{D}$ , let

$$\mathcal{X} \otimes \mathbf{D} = \{f \otimes \mathbf{D}(d, -) \mid f \in \mathcal{X}, d \in \text{Ob } \mathbf{D}\}.$$

*Remark 4.4.* Note that

$$\mathcal{X} \otimes \mathbf{D} = \text{Lan}_{\iota_{\mathbf{D}}}(\mathcal{X} \otimes \mathbf{D}_{\delta}).$$

**Proposition 4.5.** *Let  $\mathbf{D}$  be a small category. If  $\mathbf{M}$  is a cofibrantly generated model category with a set  $\mathcal{J}$  of generating cofibrations and a set  $\mathcal{J}$  of generating acyclic cofibrations that permit the small object argument, then  $\mathbf{M}^{\mathbf{D}}$  admits the projective model structure, which has a cellular presentation given by the sets  $\mathcal{J} \otimes \mathbf{D}$  of generating cofibrations and  $\mathcal{J} \otimes \mathbf{D}$  of generating acyclic cofibrations.*

*Proof.* The proof consists of a straightforward application of [25, Lemma 2.3]. The underlying functor of the monad  $\mathbb{T}_{\mathbf{D}}$  is  $\iota_{\mathbf{D}}^* \circ \text{Lan}_{\iota_{\mathbf{D}}}$ , which commutes with all colimits since both  $\text{Lan}_{\iota_{\mathbf{D}}}$  and  $\iota_{\mathbf{D}}^*$  are left adjoints. The smallness condition on  $\mathcal{J} \otimes \mathbf{D}$  and  $\mathcal{J} \otimes \mathbf{D}$  follows from the smallness condition on  $\mathcal{J}$  and  $\mathcal{J}$ .

To conclude we need only to show that every relative  $(\mathcal{J} \otimes \mathbf{D})$ -cell complex (called a regular  $(\mathcal{J} \otimes \mathbf{D}_{\delta})_{\mathbb{T}_{\mathbf{D}}}$ -cofibration in [25]) is an objectwise weak equivalence, which is a consequence of the following observation.

Let  $\tau: \Phi \rightarrow \Psi$  be any objectwise acyclic cofibration in  $\mathbf{M}^{\mathbf{D}_{\delta}}$ . For every morphism  $\sigma: \text{Lan}_{\iota_{\mathbf{D}}}(\Phi) \rightarrow \Omega$  in  $\mathbf{M}^{\mathbf{D}}$ , the morphism

$$\hat{\tau}: \Omega \rightarrow \Omega \coprod_{\text{Lan}(\Phi)} \text{Lan}_{\iota_{\mathbf{D}}}(\Psi)$$

induced by pushing  $\text{Lan}_{\iota_{\mathbf{D}}}(\tau)$  along  $\sigma$  is also an objectwise acyclic cofibration. To check this, evaluate the pushout of functors at an object  $d$  of  $\mathbf{D}$ , obtaining a pushout diagram

$$\begin{array}{ccc} \coprod_{d' \in \text{Ob } \mathbf{D}} \coprod_{f: d' \rightarrow d} \Phi(d') & \xrightarrow{\sigma_d} & \Omega(d) \\ \downarrow \coprod_{d' \in \text{Ob } \mathbf{D}} \coprod_{f: d' \rightarrow d} \tau_{d'} & & \downarrow \hat{\tau}_d \\ \coprod_{d' \in \text{Ob } \mathbf{D}} \coprod_{f: d' \rightarrow d} \Phi(d') & \xrightarrow{\hat{\sigma}_d} & (\Omega \coprod_{\text{Lan}_{\iota_{\mathbf{D}}}(\Phi)} \text{Lan}(\Psi))(d) \end{array}$$

in  $\mathbf{M}$ . The left-hand vertical arrow is an acyclic cofibration, since it is a coproduct of acyclic cofibrations. It follows that  $\hat{\tau}_d$  is also an acyclic cofibration, as acyclic cofibrations are preserved under pushout. Composites of objectwise acyclic cofibrations in  $\mathbf{M}^{\mathbf{D}}$  are clearly objectwise acyclic cofibrations, so we conclude that every relative  $(\mathcal{J} \otimes \mathbf{D})$ -cell complex is an objectwise weak equivalence.

Finally, it is clear by adjunction that  $(\mathcal{J} \otimes \mathbf{D})^{\square}$  and  $(\mathcal{J} \otimes \mathbf{D})^{\square}$  are the classes of objectwise acyclic fibrations and objectwise fibrations in  $\mathbf{M}^{\mathbf{D}}$ . By Proposition 2.7, the pair  $(\mathcal{J} \otimes \mathbf{D}, \mathcal{J} \otimes \mathbf{D})$  defines a cellular presentation for the projective model structure.  $\square$

The adjunction of Lemma 4.2 and construction of Definition 4.3 enable us to show that injective (acyclic) fibrations are objectwise (acyclic) fibrations. This result will

surprise no one familiar with injective model structures, but proofs in the literature (e.g., [16, Remark A.2.8.5]) frequently assume more restrictive hypotheses on the model category  $\mathbf{M}$ .

**Lemma 4.6.** *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category, and let  $\mathbf{D}$  be a small category. Injective (acyclic) fibrations in  $\mathbf{M}^{\mathbf{D}}$  are objectwise (acyclic) fibrations.*

*Proof.* Let  $\tau: \Phi \rightarrow \Psi$  be an injective fibration, and let  $\tau_d$  denote its component at  $d \in \text{Ob } \mathbf{D}$ . We must show that  $i \boxtimes \tau_d$  for any acyclic cofibration  $i$  in  $\mathbf{M}$ . By adjunction, this is the case if and only if  $i \otimes \mathbf{D}(d, -) \boxtimes \tau$ . The map  $i \otimes \mathbf{D}(d, -)$  is an objectwise acyclic cofibration by inspection: its components are coproducts of the map  $i$ . Hence, this lifting problem has a solution, and we conclude. For injective acyclic fibrations, the proof is similar, but one considers instead lifts against any cofibration  $i$ ; alternatively, observe that injective acyclic fibrations are both injective fibrations and objectwise weak equivalences.  $\square$

**4.2. Diagrams as coalgebras.** The category-theoretic framework in the previous section dualizes in an obvious manner. Let

$$\text{Ran}_{\iota_{\mathbf{D}}}: \mathbf{C}^{\mathbf{D}^{\delta}} \rightarrow \mathbf{C}^{\mathbf{D}}$$

denote the right adjoint of  $\iota_{\mathbf{D}}^*: \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}^{\mathbf{D}^{\delta}}$ . The image under  $\text{Ran}_{\iota_{\mathbf{D}}}$  of any functor  $\Phi: \mathbf{D}^{\delta} \rightarrow \mathbf{C}$  is its right Kan extension along  $\iota_{\mathbf{D}}$ , which is the same as its associated *power construction*, i.e.,

$$\text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(d) = \prod_{d' \in \text{Ob } \mathbf{D}} \prod_{f: d \rightarrow d'} \Phi(d').$$

Observe moreover that for any morphism  $g: d \rightarrow e$  in  $\mathbf{D}$ , the induced morphism in  $\mathbf{C}$

$$\text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(g): \text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(d) \rightarrow \text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(e)$$

is defined by specifying that for every  $d' \in \text{Ob } \mathbf{D}$  and every  $f: e \rightarrow d'$  the composite

$$\text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(d) \xrightarrow{\text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(g)} \text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(e) \xrightarrow{\text{proj}_f: e \rightarrow d'} \Phi(d')$$

is equal to the projection

$$\text{Ran}_{\iota_{\mathbf{D}}}(\Phi)(d) \xrightarrow{\text{proj}_{fg: d \rightarrow d'}} \Phi(d').$$

*Remark 4.7.* Note moreover that for any  $d \in \text{Ob } \mathbf{D}$  and any  $\Phi: \mathbf{D} \rightarrow \mathbf{C}$ , the  $d$ -component of the  $\Phi$ -component of the unit map  $\eta$  of the  $\iota_{\mathbf{D}}^* \dashv \text{Ran}_{\iota_{\mathbf{D}}}$  adjunction is given by

$$(\eta_{\Phi})_d = (\Phi(f))_{d', f: d \rightarrow d'}: \Phi(d) \rightarrow \prod_{d' \in \text{Ob } \mathbf{D}} \prod_{f: d \rightarrow d'} \Phi(d'),$$

i.e.,  $\text{proj}_{f: d \rightarrow d'} \circ (\eta_{\Phi})_d = \Phi(f): \Phi(d) \rightarrow \Phi(d')$ . In particular,  $\eta_{\Phi}$  is an objectwise monomorphism for all  $\Phi: \mathbf{D} \rightarrow \mathbf{C}$ .

Lemma 4.2 dualizes to the following result.

**Lemma 4.8.** *Let  $\mathbb{K}_{\mathbf{D}}$  denote the comonad associated to the adjunction*

$$\mathbf{C}^{\mathbf{D}} \begin{array}{c} \xrightarrow{\iota_{\mathbf{D}}^*} \\ \perp \\ \xleftarrow{\text{Ran}_{\iota_{\mathbf{D}}}} \end{array} \mathbf{C}^{\mathbf{D}^{\delta}}.$$

*The category  $(\mathbf{C}^{\mathbf{D}^{\delta}})_{\mathbb{K}_{\mathbf{D}}}$  of  $\mathbb{K}_{\mathbf{D}}$ -coalgebras in  $\mathbf{C}^{\mathbf{D}^{\delta}}$  is isomorphic to  $\mathbf{C}^{\mathbf{D}}$ .*

*Remark 4.9.* Suppose that  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category and that  $\mathbf{D}$  is a small category. If there exists a model structure on  $\mathbf{M}^{\mathbf{D}}$  left-induced by the adjunction  $\iota_{\mathbf{D}}^* \dashv \text{Ran}_{\iota_{\mathbf{D}}}$  of the lemma above, then it is necessarily the injective model structure.

To prove the existence of injective model structures on diagram categories, one could attempt to exploit this comonadic description of diagram categories to prove a result dual to Proposition 4.5. To do so, we begin by dualizing the construction in Definition 4.3.

**Definition 4.10.** Let  $\mathbf{C}$  be a complete category and  $\mathbf{D}$  a small category. The power induces a bifunctor  $- \pitchfork - : \mathbf{C} \times (\mathbf{Set}^{\mathbf{D}^{\text{op}}})^{\text{op}} \rightarrow \mathbf{C}^{\mathbf{D}}$  defined by

$$X \pitchfork \mathbf{D}(-, d) := \prod_{\mathbf{D}(-, d)} X,$$

for any  $X \in \text{Ob } \mathbf{C}$  and  $d \in \text{Ob } \mathbf{D}$ . For any class  $\mathcal{X}$  of morphisms in  $\mathbf{C}$  and any small category  $\mathbf{D}$ , let

$$\mathcal{X} \pitchfork \mathbf{D} = \{p \pitchfork \mathbf{D}(-, d) \mid p \in \mathcal{X}, d \in \text{Ob } \mathbf{D}\}.$$

*Remark 4.11.* Note that

$$\mathcal{X} \pitchfork \mathbf{D} = \text{Ran}_{\iota_{\mathbf{D}}}(\mathcal{X} \pitchfork \mathbf{D}_{\delta}).$$

*Remark 4.12.* A typical element of  $\mathcal{X} \pitchfork \mathbf{D}$  corresponding to  $p: E \rightarrow B$  in  $\mathcal{X}$  and  $d \in \text{Ob } \mathbf{D}$  is a natural transformation

$$p \pitchfork \mathbf{D}(-, d): E \pitchfork \mathbf{D}(-, d) \rightarrow B \pitchfork \mathbf{D}(-, d)$$

whose component at  $d' \in \text{Ob } \mathbf{D}$  is

$$\prod_{\mathbf{D}(d', d)} p: \prod_{\mathbf{D}(d', d)} E \rightarrow \prod_{\mathbf{D}(d', d)} B.$$

*Remark 4.13.* It is easy to see that if  $\mathcal{X}$  fibrantly generates (respectively, is a Postnikov presentation for) a weak factorization system  $(\mathcal{L}, \mathcal{R})$ , then  $\mathcal{X} \pitchfork \mathbf{D}_{\delta}$  fibrantly generates (respectively, is a Postnikov presentation for) the induced objectwise weak factorization system on  $\mathbf{C}^{\mathbf{D}_{\delta}} \cong \prod_{d \in \text{Ob } \mathbf{D}} \mathbf{C}$ .

The remark above implies that the next result is a special case of Lemma 2.16.

**Lemma 4.14.** *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$ , and let  $\mathbf{D}$  be a small category. If  $\mathbf{M}^{\mathbf{D}}$  admits the injective model structure, then it is fibrantly generated by  $(\mathcal{X} \pitchfork \mathbf{D}, \mathcal{Z} \pitchfork \mathbf{D})$ .*

We do not have a general proof that if a model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  has a Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , and the injective model category structure on  $\mathbf{M}^{\mathbf{D}}$  exists, then  $(\mathcal{X} \pitchfork \mathbf{D}, \mathcal{Z} \pitchfork \mathbf{D})$  is a Postnikov presentation of the injective model category structure. However the injective model structures covered by the existence results of Section 4.3 do have induced Postnikov presentations.

**Lemma 4.15.** *Let  $\mathbf{D}$  be a small category. If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category with Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , then every element of  $\text{Post}_{\mathcal{X} \pitchfork \mathbf{D}}$  (respectively,  $\text{Post}_{\mathcal{Z} \pitchfork \mathbf{D}}$ ) is an objectwise fibration (respectively, objectwise acyclic fibration).*

*Proof.* By the explicit description given in Remark 4.12, it is obvious that elements of  $\mathcal{X} \pitchfork \mathbf{D}$  are objectwise fibrations. Since limits in  $\mathbf{M}^{\mathbf{D}}$  are computed objectwise, the component at  $d \in \text{Ob } \mathbf{D}$  of a  $(\mathcal{X} \pitchfork \mathbf{D})$ -Postnikov tower is a tower in  $\mathbf{M}$  built

from the  $d$ -components of maps in  $X \pitchfork D$ , which we know are fibrations in  $M$ . Because fibrations are characterized by a right lifting property, retracts of towers of pullbacks of products of such maps are again fibrations in  $M$ . The proof for  $Z$  is identical to that for  $X$ , simply replacing fibrations with acyclic fibrations.  $\square$

**4.3. Existence of injective model structures.** In this section, we exploit Postnikov presentations, in order to provide conditions under which injective model structures exist.

*Remark 4.16.* The main theorem of [9], which provides conditions for left transfer of model structure to categories of coalgebras over comonads, can unfortunately not be applied directly to solve Problem 4.1, as a crucial axiom from [9] (Axiom (K6)) does not hold in this context.

Our partial solutions to Problem 4.1 rely on Corollary 2.18. We first show that it is enough to prove that the two desired types of factorization actually exist.

**Proposition 4.17.** *Let  $D$  be a small category, and let  $(M, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category with Postnikov presentation  $(X, Z)$ . If it is possible to factor any map in  $M^D$  as*

- (a) *an objectwise cofibration followed by a map in  $\widehat{\text{Post}}_{Z \pitchfork D}$ , and as*
- (b) *an objectwise acyclic cofibration followed by a map in  $\widehat{\text{Post}}_{X \pitchfork D}$ ,*

*then  $M^D$  admits the injective model structure, with Postnikov presentation  $(X \pitchfork D, Z \pitchfork D)$ .*

*Proof.* We need only to check that condition (a) of Corollary 2.18 holds for the adjunction

$$\begin{array}{ccc} M^D & \xrightarrow{\iota_D^*} & M^{D_\delta} \\ & \perp & \\ & \xleftarrow{\text{Ran}_{\iota_D}} & \end{array}$$

with respect to the induced Postnikov presentation  $(X \pitchfork D_\delta, Z \pitchfork D_\delta)$  of the objectwise model structure on  $M^{D_\delta}$ .

By Lemma 4.15, every element of  $\widehat{\text{Post}}_{Z \pitchfork D}$  is an acyclic fibration. Since acyclic fibrations are, in particular, weak equivalences, we conclude that  $(Z \pitchfork D)$ -Postnikov towers are objectwise weak equivalences, as desired.  $\square$

*Remark 4.18.* The proof above illustrates the utility of Postnikov presentations. Suppose that we knew only that  $(M, \mathcal{F}, \mathcal{C}, \mathcal{W})$  was fibrantly generated by  $(X, Z)$ , and that any map in  $M^D$  could be factored as

- (a) an objectwise cofibration followed by a map in  $(\square(Z \pitchfork D))^{\square}$ , and as
- (b) an objectwise acyclic cofibration followed by a map in  $(\square(X \pitchfork D))^{\square}$ .

According to Corollary 2.19, to conclude that the injective model structure on  $M^D$  exists, we must show

$$\iota_D^* \left( (\square(Z \pitchfork D))^{\square} \right) \subseteq \mathcal{W},$$

which is not as obvious as the fact that  $\iota_D^*(\widehat{\text{Post}}_{Z \pitchfork D}) \subseteq \mathcal{W}$  and which may not even be true in general.

We now show that if the cofibrations of  $M$  are exactly the monomorphisms, then one of the required factorization axioms always holds.

**Theorem 4.19.** *Let  $\mathcal{D}$  be a small category, and let  $(\mathcal{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category with Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$  such that the cofibrations in  $\mathcal{M}$  are exactly the monomorphisms. If for all morphisms  $\tau: \Phi \rightarrow \Psi$  in  $\mathcal{M}^{\mathcal{D}}$  there exists an objectwise acyclic cofibration  $\iota$  and a map  $\rho \in \widehat{\text{Post}}_{\mathcal{X} \pitchfork \mathcal{D}}$  such that  $\tau = \rho \iota$ , then  $\mathcal{M}^{\mathcal{D}}$  admits an injective model structure with Postnikov presentation  $(\mathcal{X} \pitchfork \mathcal{D}, \mathcal{Z} \pitchfork \mathcal{D})$ .*

*Remark 4.20.* The following general properties of monomorphisms, which are easy to check, are used in the proof of this theorem.

- (1) If  $\{i_\alpha: A_\alpha \rightarrow B_\alpha \mid \alpha \in J\}$  is a collection of monomorphisms in a category  $\mathcal{C}$  that admits products, then

$$\prod_{\alpha \in J} i_\alpha: \prod_{\alpha \in J} A_\alpha \rightarrow \prod_{\alpha \in J} B_\alpha$$

is also a monomorphism in  $\mathcal{C}$ .

- (2) If  $i: A \rightarrow B$  is a monomorphism in a category  $\mathcal{C}$  admitting pullbacks, then for every commuting diagram in  $\mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & & \downarrow g \\ B & \xrightarrow{h} & D, \end{array}$$

the induced morphism  $(i, f): A \rightarrow B \times_D C$  is also a monomorphism.

*Proof of Theorem 4.19.* By Proposition 4.17, we need only to check that condition (b) of Corollary 2.18 holds for the adjunction

$$\begin{array}{ccc} \mathcal{M}^{\mathcal{D}} & \xrightarrow{\iota_{\mathcal{D}}^*} & \mathcal{M}^{\mathcal{D}_\delta} \\ & \perp & \\ & \xleftarrow{\text{Ran}_{\iota_{\mathcal{D}}}} & \end{array}$$

with respect to the induced Postnikov presentation  $(\mathcal{X} \pitchfork \mathcal{D}_\delta, \mathcal{Z} \pitchfork \mathcal{D}_\delta)$  of  $\mathcal{M}^{\mathcal{D}_\delta}$  of the objectwise model structure on  $\mathcal{M}^{\mathcal{D}_\delta}$ .

Let  $\tau: \Phi \rightarrow \Psi$  be a morphism in  $\mathcal{M}^{\mathcal{D}}$ . Let  $*$  denote the constant diagram at the terminal object. Since  $(\mathcal{X} \pitchfork \mathcal{D}_\delta, \mathcal{Z} \pitchfork \mathcal{D}_\delta)$  is a Postnikov presentation for  $\mathcal{M}^{\mathcal{D}_\delta}$ , the unique morphism  $\iota_{\mathcal{D}}^* \Phi \rightarrow *$  in  $\mathcal{M}^{\mathcal{D}_\delta}$  admits a factorization

$$\begin{array}{ccc} \iota_{\mathcal{D}}^* \Phi & \xrightarrow{!} & * \\ & \searrow \theta & \nearrow \rho \\ & \Phi' & \end{array}$$

where  $\theta$  is an objectwise cofibration, i.e., an objectwise monomorphism, and  $\rho \in \text{Post}_{\mathcal{Z} \pitchfork \mathcal{D}_\delta}$ . Taking transposes, we obtain a commuting diagram in  $\mathcal{M}^{\mathcal{D}}$

$$\begin{array}{ccc} \Phi & \xrightarrow{!} & \text{Ran}_{\iota_{\mathcal{D}}} * = * \\ \eta_\Phi \swarrow & & \nearrow \text{Ran}_{\iota_{\mathcal{D}}}(\rho) \\ \text{Ran}_{\iota_{\mathcal{D}}}(\iota_{\mathcal{D}}^* \Phi) & \xrightarrow{\text{Ran}_{\iota_{\mathcal{D}}}(\theta)} & \text{Ran}_{\iota_{\mathcal{D}}}(\Phi') \end{array}$$

where

- $\eta_\Phi$  is an objectwise monomorphism by Remark 4.7,
- $\text{Ran}_{\iota_D}(\theta)$  is an objectwise monomorphism by Remark 4.20(1), using the explicit description of  $\text{Ran}_{\iota_D}$  above, and
- $\text{Ran}_{\iota_D}(\rho) \in \text{Ran}_{\iota_D}(\text{Post}_{\mathcal{Z}, \mathfrak{h}D_\delta}) \subseteq \text{Post}_{\mathcal{Z}, \mathfrak{h}D}$ .

The inclusion in the last point follows from the definition of Postnikov towers, the fact that  $\text{Ran}_{\iota_D}$  is a right adjoint, and Remark 4.11. It follows that the unique morphism  $\Phi \rightarrow *$  factors as an objectwise monomorphism,  $\theta^\sharp$ , followed by an element of  $\text{Post}_{\mathcal{Z}, \mathfrak{h}D}$ , namely  $\text{Ran}_{\iota_D}(\rho)$ .

To conclude, consider the following commuting diagram in  $\mathcal{M}^D$ .

$$\begin{array}{ccc}
 \Phi & \xrightarrow{\tau} & \Psi \\
 & \searrow^{(\theta^\sharp, \tau)} & \nearrow_{\text{proj}_2} \\
 & & \text{Ran}_{\iota_D}(\Phi') \times \Psi
 \end{array}$$

By Remark 4.20(2), the natural transformation  $(\theta^\sharp, \tau)$  is an objectwise monomorphism. Moreover,  $\text{proj}_2 \in \text{Post}_{\mathcal{Z}, \mathfrak{h}D}$ , as it is obtained by pulling back  $\text{Ran}_{\iota_D}(\rho)$  along the unique morphism  $\Psi \rightarrow *$ , and  $\text{Post}_{\mathcal{Z}, \mathfrak{h}D}$  is closed under pullbacks. The factorization  $\tau = \text{proj}_2 \circ (\theta^\sharp, \tau)$  satisfies condition (b) of Corollary 2.18.  $\square$

**4.4. Properties of injective model structures.** As seen in Section 2.3, left properness is inherited by all left-induced model structures. We show now that right properness is inherited as well for the injective model category structure on a diagram category  $\mathcal{M}^D$ .

Recall that a model structure is *right proper* if the pullback of a weak equivalence along a fibration is always a weak equivalence.

**Lemma 4.21.** *Let  $(\mathcal{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a right proper model category with Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , let  $D$  be a small category, and suppose that  $\mathcal{M}^D$  admits the injective model structure. If  $\mathcal{M}$  is right proper, then so is  $\mathcal{M}^D$ .*

*Proof.* Since limits in  $\mathcal{M}^D$  are calculated objectwise, to prove right properness of the injective model category structure on  $\mathcal{M}^D$ , it is enough to show that every injective fibration in  $\mathcal{M}^D$  is an objectwise fibration. This was done in Lemma 4.6.  $\square$

Let  $\mathcal{V}$  be a closed symmetric monoidal category. Recall that if  $\mathcal{M}$  is a  $\mathcal{V}$ -category that is tensored and cotensored over  $\mathcal{V}$ , and  $D$  is any small category, the diagram category  $\mathcal{M}^D$  inherits a natural  $\mathcal{V}$ -enrichment and is moreover tensored and cotensored over  $\mathcal{V}$  (cf. Appendix A).

**Lemma 4.22.** *Let  $(\mathcal{V}, \wedge, \mathbb{I})$  be a closed symmetric monoidal model category. Let  $\mathcal{M}$  be a model category, and let  $D$  be a small category. Suppose that  $\mathcal{M}^D$  admits the injective model structure. If  $\mathcal{M}$  is a  $\mathcal{V}$ -model category, then so is  $\mathcal{M}^D$ .*

*Remark 4.23.* In the case where  $\mathcal{V}$  is an excellent model category ([16, Definition A.3.2.16]) and  $\mathcal{M}$  is combinatorial, Lurie proved in [16, Proposition A.3.3.2] that the injective model structure on  $\mathcal{M}^D$  exists and is a  $\mathcal{V}$ -model structure, for all small categories  $D$ .

*Proof.* By Lemma 2.23, it is enough to show that the adjunction  $\iota_D^* \dashv \text{Ran}_{\iota_D}$  is a  $\mathcal{V}$ -adjunction, which is obvious, since  $\iota_D^*$  is clearly a tensor  $\mathcal{V}$ -functor (cf. Definition A.3 and Section A.2).  $\square$

## 5. (GENERALIZED) REEDY DIAGRAM CATEGORIES AND FIBRANT GENERATION

In this section, we show that if  $\mathbf{R}$  is a (generalized) Reedy category, and  $\mathbf{M}$  is a fibrantly generated model category, then the diagram category  $\mathbf{M}^{\mathbf{R}}$ , endowed with its Reedy model structure, is also fibrantly generated (Theorems 5.9 and 5.22). Moreover, if  $\mathbf{R}$  is a (classical) Reedy category and if  $\mathbf{M}$  admits a Postnikov presentation, then so does  $\mathbf{M}^{\mathbf{R}}$  (Theorem 5.11). A similar result is true for generalized Reedy categories, though we neglect to give full details of the proof here (Remark 5.23).

**5.1. The case of Reedy categories.** We begin by recalling the definition of a Reedy category, as formulated in [11].

**Definition 5.1.** A *Reedy category* is a small category  $\mathbf{R}$  together with a *degree function*  $d : \text{Ob } \mathbf{R} \rightarrow \mathbb{N}$  and two wide subcategories  $\mathbf{R}^+$  and  $\mathbf{R}^-$  satisfying the following axioms.

- (1) Non-identity morphisms of  $\mathbf{R}^+$  strictly raise degree.
- (2) Non-identity morphisms of  $\mathbf{R}^-$  strictly lower degree.
- (3) Every morphism in  $\mathbf{R}$  factors uniquely as a morphism in  $\mathbf{R}^-$  followed by a morphism in  $\mathbf{R}^+$ .

*Example 5.2.* An inverse or a direct category is an example of a Reedy category. In the former case, all morphisms are degree-decreasing, and in the latter case all morphisms are degree-increasing.

The following constructions play an essential role in the definition of the Reedy model structure.

**Definition 5.3.** Let  $\mathbf{R}$  be a Reedy category, and let  $\mathbf{C}$  be a bicomplete category. The  *$r$ -th latching object* of an object  $\Phi$  in  $\mathbf{C}^{\mathbf{R}}$  is

$$L_r\Phi = \text{colim}(\mathbf{R}_{<\text{deg}(r)}^+ / r \xrightarrow{U} \mathbf{R} \xrightarrow{\Phi} \mathbf{C}),$$

and the  *$r$ -th matching object* is

$$M_r\Phi = \text{lim}(r / \mathbf{R}_{<\text{deg}(r)}^- \xrightarrow{U} \mathbf{R} \xrightarrow{\Phi} \mathbf{C}),$$

where  $\mathbf{R}_{<\text{deg}(r)}^+ / r$  is the slice category over  $r$  with objects restricted to degree less than the degree of  $r$  and morphisms in  $\mathbf{R}^+$ , and  $r / \mathbf{R}_{<\text{deg}(r)}^-$  is defined dually, and  $U$  denotes a forgetful functor from a slice category.

*Remark 5.4.* For every  $r \in \text{Ob } \mathbf{R}$  and every morphism  $\tau : \Phi \rightarrow \Psi$ , there are natural morphisms in  $\mathbf{C}$ : the *relative latching map*

$$\ell_r(\tau) : \Phi(r) \sqcup_{L_r\Phi} L_r\Psi \rightarrow \Psi(r)$$

and the *relative matching map*

$$m_r(\tau) : \Phi(r) \rightarrow M_r\Phi \times_{M_r\Psi} \Psi(r).$$

**Theorem 5.5** ([11, Theorem 5.2.5]). *Let  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  be a model category and  $\mathbf{R}$  a Reedy category. Then  $\mathbf{M}^{\mathbf{R}}$  can be equipped with a model structure (called the Reedy model structure) where a morphism  $\tau : \Phi \rightarrow \Psi$  is*

- a weak equivalence if and only if  $\tau_r \in \mathcal{W}$  for every  $r \in \text{Ob } \mathbf{R}$ ;
- a cofibration if and only if  $\ell_r(\tau) \in \mathcal{C}$  for every  $r \in \text{Ob } \mathbf{R}$ ; and
- a fibration if and only if  $m_r(\tau) \in \mathcal{F}$  for every  $r \in \text{Ob } \mathbf{R}$ .



Diagram categories equipped with a Reedy model structure inherit cofibrant generation from the target category. The generating (acyclic) cofibrations are defined by a “relative” version of the construction given in Definition 4.3.

**Definition 5.6.** Let  $r$  be an object of a Reedy category  $\mathbf{R}$ . Define  $\kappa_r: \partial\mathbf{R}(r, -) \hookrightarrow \mathbf{R}(r, -)$  to be the subfunctor of the representable functor consisting of all morphisms with domain  $r$  that factor through an object of degree less than  $\deg(r)$ . Given a map  $i: A \rightarrow B$  in a cocomplete category  $\mathbf{C}$ , let

$$i\widehat{\otimes}\kappa_r: A \otimes \mathbf{R}(r, -) \sqcup_{A \otimes \partial\mathbf{R}(r, -)} B \otimes \partial\mathbf{R}(r, -) \rightarrow B \otimes \mathbf{R}(r, -)$$

be the map in  $\mathbf{C}^{\mathbf{R}}$  defined by the “pushout-product” construction. For any set of maps  $\mathcal{J}$  in  $\mathbf{C}$ , write

$$\mathcal{J}\widehat{\otimes}\mathbf{R} = \{i\widehat{\otimes}\kappa_r \mid i \in \mathcal{J}, r \in \text{Ob } \mathbf{R}\}.$$

**Theorem 5.7** ([10, Proposition 15.6.24]). *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category that is cofibrantly generated (in the sense of Definition 2.4) by  $(\mathcal{J}, \mathcal{I})$ , and  $\mathbf{R}$  is a Reedy category, then  $\mathbf{M}^{\mathbf{R}}$ , endowed with the Reedy model structure, is cofibrantly generated (in the same sense) by  $(\mathcal{J}\widehat{\otimes}\mathbf{R}, \mathcal{I}\widehat{\otimes}\mathbf{R})$ .*

*Remark 5.8.* Moreover, if the pair  $(\mathcal{J}, \mathcal{I})$  defines a cellular presentation for the model category  $\mathbf{M}$ , then  $(\mathcal{J}\widehat{\otimes}\mathbf{R}, \mathcal{I}\widehat{\otimes}\mathbf{R})$  defines a cellular presentation for the Reedy model structure (see [24, Proposition 7.7]). We prove the dual form of this result in Theorem 5.11.

Note that it is not automatically true under these hypotheses that  $\mathbf{M}^{\mathbf{R}}$ ,  $\mathcal{J}\widehat{\otimes}\mathbf{R}$ , and  $\mathcal{I}\widehat{\otimes}\mathbf{R}$  permit the small object argument: additional smallness conditions on the codomains of  $\mathcal{J}$  and  $\mathcal{I}$  are required (see [10, Proposition 15.6.27]). However, this is irrelevant to the existence of cellular presentations for the cofibrations and acyclic cofibrations in the Reedy model structure.

We show that the dual result to this theorem holds as well: Reedy model categories also inherit fibrant generation from the target category.

**Theorem 5.9.** *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category that is fibrantly generated by sets  $(\mathcal{X}, \mathcal{Z})$ , then the Reedy model structure on  $\mathbf{M}^{\mathbf{R}}$  is fibrantly generated.*

*Proof.* The dual of a Reedy category is again Reedy, with  $(\mathbf{R}^{\text{op}})^+ = (\mathbf{R}^-)^{\text{op}}$  and  $(\mathbf{R}^{\text{op}})^- = (\mathbf{R}^+)^{\text{op}}$ . Similarly,  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$  if and only if  $(\mathbf{M}^{\text{op}}, \mathcal{C}^{\text{op}}, \mathcal{F}^{\text{op}}, \mathcal{W}^{\text{op}})$  is a model category weakly cofibrantly generated by  $(\mathcal{X}^{\text{op}}, \mathcal{Z}^{\text{op}})$ . Moreover,  $(\mathbf{M}^{\mathbf{R}})^{\text{op}} \cong (\mathbf{M}^{\text{op}})^{\mathbf{R}^{\text{op}}}$ .

Applying Theorem 5.7 to  $\mathbf{M}^{\text{op}}$ ,  $\mathbf{R}^{\text{op}}$ , and  $(\mathcal{X}^{\text{op}}, \mathcal{Z}^{\text{op}})$ , we see that  $(\mathbf{M}^{\text{op}})^{\mathbf{R}^{\text{op}}}$  is weakly cofibrantly generated by sets  $(\mathcal{X}^{\text{op}}\widehat{\otimes}\mathbf{R}^{\text{op}}, \mathcal{Z}^{\text{op}}\widehat{\otimes}\mathbf{R}^{\text{op}})$ . Dualizing one more time, we conclude that  $\mathbf{M}^{\mathbf{R}}$  is fibrantly generated by  $((\mathcal{X}^{\text{op}}\widehat{\otimes}\mathbf{R}^{\text{op}})^{\text{op}}, (\mathcal{Z}^{\text{op}}\widehat{\otimes}\mathbf{R}^{\text{op}})^{\text{op}})$ .  $\square$

*Remark 5.10.* Let us unpack the meaning of the set  $(\mathcal{X}^{\text{op}}\widehat{\otimes}\mathbf{R}^{\text{op}})^{\text{op}}$  of generating fibrations. The “pushout product” construction of Definition 5.6 in  $\mathbf{C}^{\text{op}}$  dualizes to a “pullback cotensor” construction in  $\mathbf{C}$  in the same way that Definition 4.10 dualizes Definition 4.3. Substituting the Reedy category  $\mathbf{R}^{\text{op}}$  for  $\mathbf{R}$ , the maps  $\kappa_r$  of definition 5.6 dualize to  $\kappa^r: \partial\mathbf{R}(-, r) \hookrightarrow \mathbf{R}(-, r)$ , where  $\partial\mathbf{R}(-, r)$  denotes the subfunctor consisting of all morphisms with codomain  $r$  that factor through an object of degree less than  $\deg(r)$ . Given  $p: E \rightarrow B$  in a complete category  $\mathbf{C}$ , let

$$p\widehat{\pitchfork}\kappa^r: E \pitchfork \mathbf{R}(-, r) \rightarrow E \pitchfork \partial\mathbf{R}(-, r) \times_{B\pitchfork\partial\mathbf{R}(-, r)} B \pitchfork \mathbf{R}(-, r).$$

Then for a set of maps  $\mathcal{X}$  in  $\mathbf{C}$ , we write

$$\mathcal{X}\widehat{\mathfrak{h}}\mathbf{R} = \{p\widehat{\mathfrak{h}}\kappa^r \mid p \in \mathcal{X}, r \in \text{Ob } \mathbf{R}\} = (\mathcal{X}^{\text{op}} \widehat{\otimes} \mathbf{R}^{\text{op}})^{\text{op}}.$$

**Theorem 5.11.** *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a model category that has a Postnikov presentation  $(\mathcal{X}, \mathcal{Z})$ , then the Reedy model structure on  $\mathbf{M}^{\mathbf{R}}$  has a Postnikov presentation  $(\mathcal{X}\widehat{\mathfrak{h}}\mathbf{R}, \mathcal{Z}\widehat{\mathfrak{h}}\mathbf{R})$ .*

*Proof.* Any morphism  $\tau: \Phi \rightarrow \Psi$  has a canonical presentation as a relative cell complex with cells  $\ell_r(\tau) \widehat{\otimes} \kappa_r$  and also a canonical presentation as Postnikov tower with layers  $m_r(\tau) \widehat{\mathfrak{h}}\kappa^r$  by [24, Proposition 6.3] and its dual. If  $\tau$  is a Reedy (acyclic) fibration, then each of its relative matching maps  $m_r(\tau)$  is an (acyclic) fibration. By hypothesis, we know that each  $m_r(\tau)$  is a retract of an  $\mathcal{X}$ - (respectively,  $\mathcal{Z}$ -) Postnikov tower. The Postnikov presentations commute with the  $\widehat{\mathfrak{h}}$  construction by the dual of [24, Lemma 5.6], completing the proof.  $\square$

**5.2. The case of generalized Reedy categories.** Generalized Reedy categories were introduced in [3]. A (classical) Reedy category has no non-identity automorphisms; the idea of this generalization is to allow non-trivial degree-preserving automorphisms.

**Definition 5.12.** A *generalized Reedy category* is a small category  $\mathbf{R}$  together with a degree function  $d: \text{Ob } \mathbf{R} \rightarrow \mathbb{N}$  and two wide subcategories  $\mathbf{R}^+$  and  $\mathbf{R}^-$  satisfying the following axioms.

- Non-invertible morphisms of  $\mathbf{R}^+$  strictly raise degree.
- Non-invertible morphisms of  $\mathbf{R}^-$  strictly lower degree.
- Isomorphisms in  $\mathbf{R}$  preserve degree.
- $\mathbf{R}^+ \cap \mathbf{R}^- = \text{Iso}(\mathbf{R})$ .
- Every morphism in  $\mathbf{R}$  factors uniquely up to isomorphism as a morphism in  $\mathbf{R}^-$  followed by a morphism in  $\mathbf{R}^+$ .
- If  $\theta f = f$  for  $\theta \in \text{Iso}(\mathbf{R})$  and  $f \in \mathbf{R}^-$ , then  $\theta$  is an identity.

A *dualizable* generalized Reedy category is a generalized Reedy category satisfying the additional axiom:

- If  $f\theta = f$  for  $\theta \in \text{Iso}(\mathbf{R})$  and  $f \in \mathbf{R}^+$ , then  $\theta$  is an identity.

*Remark 5.13.* If  $\mathbf{R}$  is a dualizable generalized Reedy category, then  $\mathbf{R}^{\text{op}}$  is as well. Most known examples of generalized Reedy categories are dualizable.

*Examples 5.14.* Examples of dualizable generalized Reedy categories include all groupoids, the category of finite sets, orbit categories of finite groups, and the tree category  $\Omega$ , presheaves on which is the category of dendroidal sets [20].

**Definition 5.15.** Let  $\mathbf{R}$  be a generalized Reedy category. A model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is  *$\mathbf{R}$ -projective* if  $\mathbf{M}^{\text{Aut}(r)}$  admits the projective model structure for every object  $r \in \mathbf{R}$ .

*Example 5.16.* If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is cofibrantly generated and permits the small object argument, then it is  $\mathbf{R}$ -projective for any generalized Reedy category  $\mathbf{R}$  by Proposition 4.5 or [10, 11.6.1].

**Theorem 5.17** ([3, Theorem 1.6]). *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is an  $\mathbf{R}$ -projective model category, and  $\mathbf{R}$  is a generalized Reedy category, then  $\mathbf{M}^{\mathbf{R}}$  admits a model structure where a map  $\tau: \Phi \rightarrow \Psi$  is a*

- *weak equivalence if and only if for every  $r \in \text{Ob } \mathbf{R}$ ,  $\tau_r : \Phi(r) \rightarrow \Psi(r)$  is a weak equivalence in the projective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ ;*
- *cofibration if and only if for every  $r \in \text{Ob } \mathbf{R}$ , the relative latching map  $\ell_r(\tau)$  is a cofibration in the projective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ ;*
- *fibration if and only if for every  $r \in \text{Ob } \mathbf{R}$ , the relative matching map  $m_r(\tau)$  is a fibration in the projective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ .*

**Theorem 5.18.** *If  $\mathbf{R}$  is a generalized Reedy category and  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is cofibrantly generated (in the sense of Definition 2.4) by  $(\mathcal{J}, \mathcal{J})$ , then  $\mathbf{M}^{\mathbf{R}}$ , equipped with a model structure of Theorem 5.17, is cofibrantly generated (in the same sense) by  $(\mathcal{J} \widehat{\otimes} \mathbf{R}, \mathcal{J} \widehat{\otimes} \mathbf{R})$ .*

*Proof.* Consider a morphism  $\tau$  in  $\mathbf{M}^{\mathbf{R}}$ . By adjunction  $(\mathcal{J} \widehat{\otimes} \kappa_r) \lrcorner \tau$  if and only if  $\mathcal{J} \lrcorner m_r(\tau)$ , where the relative matching map for  $\tau$  is defined as in Remark 5.4 (see [22] for more details). This lifting problem has a solution if and only if  $m_r(\tau)$  is an acyclic fibration in  $\mathbf{M}$ , which is the case if and only if it is an acyclic fibration in the projective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ . Thus,  $(\mathcal{J} \widehat{\otimes} \mathbf{R})^{\lrcorner}$  is the class of Reedy acyclic fibrations. The same argument works for  $\mathcal{J} \widehat{\otimes} \mathbf{R}$ .  $\square$

We now show that the generalized Reedy model structure is fibrantly generated, provided  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a fibrantly generated model category, and  $\mathbf{R}$  is dualizable. The model structure from the Theorem 5.17 is not self dual, so we must construct a “dual” version of it, using the injective model structure on an automorphism category of any object in  $\mathbf{R}$ .

**Definition 5.19.** Let  $\mathbf{R}$  be a generalized Reedy category. A model category  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is  *$\mathbf{R}$ -injective* if  $\mathbf{M}^{\text{Aut}(r)}$  admits the injective model structure for every object  $r \in \mathbf{R}$ .

*Example 5.20.* If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is a combinatorial model category, then it is an  $\mathbf{R}$ -injective model category for any generalized Reedy category  $\mathbf{R}$  [16, Proposition A.2.8.2].

**Theorem 5.21.** *If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is an  $\mathbf{R}$ -injective model category, and  $\mathbf{R}$  is a dualizable generalized Reedy category, then  $\mathbf{M}^{\mathbf{R}}$  can be equipped with a model structure where a map  $\tau : \Phi \rightarrow \Psi$  is a*

- *weak equivalence if and only if for every  $r \in \text{Ob } \mathbf{R}$ ,  $\tau_r : \Phi(r) \rightarrow \Psi(r)$  is a weak equivalence in the injective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ ;*
- *cofibration if and only if for every  $r \in \text{Ob } \mathbf{R}$ , the relative latching map  $\ell_r(\tau)$  is a cofibration in the injective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ ;*
- *fibration if and only if for every  $r \in \text{Ob } \mathbf{R}$ , the relative matching map  $m_r(\tau)$  is a fibration in the injective model structure on  $\mathbf{M}^{\text{Aut}(r)}$ .*

*Proof.* If  $(\mathbf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is  $\mathbf{R}$ -injective, then  $(\mathbf{M}^{\text{op}}, \mathcal{C}^{\text{op}}, \mathcal{F}^{\text{op}}, \mathcal{W}^{\text{op}})$  is  $\mathbf{R}^{\text{op}}$ -projective. We can therefore apply Theorem 5.17 to  $(\mathbf{M}^{\text{op}}, \mathcal{C}^{\text{op}}, \mathcal{F}^{\text{op}}, \mathcal{W}^{\text{op}})$  and  $\mathbf{R}^{\text{op}}$ . Dualizing the resulting model structure on  $(\mathbf{M}^{\text{op}})^{\mathbf{R}^{\text{op}}} \cong (\mathbf{M}^{\mathbf{R}})^{\text{op}}$  gives rise to the desired model structure on  $\mathbf{M}^{\mathbf{R}}$ , since relative latching maps and relative matching maps are mutually dual.  $\square$

As was the case in the proof of Theorem 5.9, we can prove the following theorem by dualizing Theorem 5.18.

**Theorem 5.22.** *If  $\mathbb{R}$  is a dualizable generalized Reedy category, and  $(\mathbb{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$  is an  $\mathbb{R}$ -injective model category that is fibrantly generated by  $(\mathcal{X}, \mathcal{Z})$ , then the model structure on  $\mathbb{M}^{\mathbb{R}}$  of Theorem 5.21 is fibrantly generated by  $(\widehat{\mathcal{X}} \hat{\cap} \mathbb{R}, \widehat{\mathcal{Z}} \hat{\cap} \mathbb{R})$ .*

*Remark 5.23.* Theorem 5.11 can be extended to generalized Reedy categories and the model structure of Theorem 5.21 because analogous results to [24, Proposition 6.3] are true in this context. The proofs however are not entirely straightforward. See [22].

## APPENDIX A. ENRICHED MODEL CATEGORIES

We recall from e.g., [15] and [23, Chapters 3, 10], those elements of the theory of enriched categories and the compatibility with model structure that are essential for understanding the proof of Lemma 2.23. We then describe the particular case of diagrams in an enriched category, which we need for Lemma 4.22.

### A.1. Enriched categories, functors, adjunctions.

**Definition A.1.** Let  $(\mathbb{V}, \wedge, \mathbb{I})$  denote a closed symmetric monoidal model category.

- A category  $\mathbb{M}$  is *enriched over  $\mathbb{V}$*  if there exists a bifunctor

$$\mathrm{Map}_{\mathbb{M}}: \mathbb{M}^{\mathrm{op}} \times \mathbb{M} \rightarrow \mathbb{V}: (X, Y) \mapsto \mathrm{Map}_{\mathbb{M}}(X, Y)$$

together with collections of morphisms in  $\mathbb{V}$

$$c_{X,Y,Z}: \mathrm{Map}_{\mathbb{M}}(Y, Z) \wedge \mathrm{Map}_{\mathbb{M}}(X, Y) \rightarrow \mathrm{Map}_{\mathbb{M}}(X, Z)$$

for all  $X, Y, Z \in \mathbb{M}$ , called *compositions*, and

$$i_X: \mathbb{I} \rightarrow \mathrm{Map}_{\mathbb{M}}(X, X)$$

for all  $X \in \mathbb{M}$ , called *units*, such that the composition is associative and unital.

- A  $\mathbb{V}$ -enriched category  $\mathbb{M}$  is *tensored over  $\mathbb{V}$*  if there is a bifunctor

$$- \otimes -: \mathbb{V} \times \mathbb{M} \rightarrow \mathbb{M}: (V, X) \mapsto V \otimes X$$

with natural isomorphisms  $\mathrm{Map}_{\mathbb{M}}(V \otimes X, Y) \cong \mathrm{Hom}_{\mathbb{V}}(V, \mathrm{Map}_{\mathbb{M}}(X, Y))$ , for all  $X, Y \in \mathbb{M}$  and  $V \in \mathbb{V}$ .

- A  $\mathbb{V}$ -enriched category  $\mathbb{M}$  is *cotensored over  $\mathbb{V}$*  if there is a bifunctor

$$(-)^{(-)}: \mathbb{V}^{\mathrm{op}} \times \mathbb{M} \rightarrow \mathbb{M}: (V, X) \mapsto X^V,$$

with natural isomorphisms  $\mathrm{Map}_{\mathbb{M}}(X, Y^V) \cong \mathrm{Hom}_{\mathbb{V}}(V, \mathrm{Map}_{\mathbb{M}}(X, Y))$ , for all  $X, Y \in \mathbb{M}$  and  $V \in \mathbb{V}$ . Note that if  $\mathbb{M}$  is tensored and cotensored over  $\mathbb{V}$ , then for all  $V \in \mathbb{V}$ , the functors  $- \otimes V: \mathbb{M} \rightarrow \mathbb{M}$  and  $(-)^V: \mathbb{M} \rightarrow \mathbb{M}$  form an adjunction.

- If  $\mathbb{V}$  is a monoidal model category and  $\mathbb{M}$  is a model category that is tensored over  $\mathbb{V}$ , then  $\mathbb{M}$  *satisfies SM7* if

$$j \widehat{\otimes} i: V \otimes Y \sqcup_{V \otimes X} W \otimes X \rightarrow W \otimes Y$$

is cofibration whenever  $i: X \rightarrow Y$  is a cofibration in  $\mathbb{M}$  and  $j$  is a cofibration in  $\mathbb{V}$ . If  $i$  or  $j$  is also a weak equivalence, then  $j \widehat{\otimes} i$  must also be a weak equivalence.

- The *unit axiom* holds in  $\mathbb{M}$  if, for any cofibrant object  $X$  in  $\mathbb{M}$ , the map  $Q \mathbb{I} \otimes X \xrightarrow{q \otimes \mathrm{Id}} \mathbb{I} \otimes X$  is a weak equivalence in  $\mathbb{M}$ , where  $q: Q(-) \rightarrow (-)$  denotes the cofibrant replacement functor and natural transformation.

If a model category  $\mathbf{M}$  satisfies all properties of Definition A.1 with respect to a closed symmetric monoidal model category  $(\mathbf{V}, \wedge, \mathbb{I})$ , then  $\mathbf{M}$  is a  $\mathbf{V}$ -*model category*.

**Definition A.2.** Let  $\mathbf{M}$  and  $\mathbf{M}'$  be  $\mathbf{V}$ -enriched categories. A  $\mathbf{V}$ -*functor*  $F: \mathbf{M} \rightarrow \mathbf{M}'$  consists of a function

$$F: \text{Ob } \mathbf{M} \rightarrow \text{Ob } \mathbf{M}': X \mapsto FX$$

and a collection of morphisms in  $\mathbf{V}$

$$F_{X,Y}: \text{Map}_{\mathbf{M}}(X, Y) \rightarrow \text{Map}_{\mathbf{M}'}(FX, FY)$$

for all  $X, Y \in \mathbf{M}$ , called the *components*, such that

$$c \circ (F_{Y,Z} \wedge F_{X,Y}) = F_{X,Z} \circ c \text{ and } F_{X,X} \circ i_X = i_{FX}$$

for all  $X, Y, Z \in \mathbf{M}$ .

**Definition A.3.** Let  $(\mathbf{V}, \wedge, \mathbb{I})$  be a closed symmetric monoidal category. An adjunction

$\mathbf{C} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathbf{M}$  between two  $\mathbf{V}$ -categories is a  $\mathbf{V}$ -*adjunction* if it satisfies one of the following equivalent conditions.

- (1) The left adjoint  $L$  is a *tensor  $\mathbf{V}$ -functor*, i.e., there are natural isomorphisms  $L(V \otimes X) \cong V \otimes LX$ , for all  $V \in \text{Ob } \mathbf{V}$  and  $X \in \text{Ob } \mathbf{C}$  that are associative and unital with respect to the monoidal structure on  $\mathbf{V}$ .
- (2) The right adjoint  $R$  is a *cotensor  $\mathbf{V}$ -functor*, i.e., there are natural isomorphisms  $(RY)^V \cong R(Y^V)$ , for all  $V \in \text{Ob } \mathbf{V}$  and  $Y \in \text{Ob } \mathbf{M}$ , associative and unital with respect to the monoidal structure on  $\mathbf{V}$ .
- (3) The adjunction is  $\mathbf{V}$ -*enriched*, i.e., there are natural isomorphisms

$$\text{Map}_{\mathbf{M}}(LX, Y) \cong \text{Map}_{\mathbf{C}}(X, RY),$$

for all  $X \in \text{Ob } \mathbf{C}$  and  $Y \in \text{Ob } \mathbf{M}$ .

**A.2. Enriched diagram categories.** Let  $\mathbf{V}$  be a complete closed symmetric monoidal category. If  $\mathbf{M}$  is a complete and cocomplete, tensored and cotensored  $\mathbf{V}$ -enriched category, and  $\mathbf{D}$  is a small category, then the diagram category,  $\mathbf{M}^{\mathbf{D}}$  is naturally  $\mathbf{V}$ -enriched, as well as tensored and cotensored over  $\mathbf{V}$ .

**Definition A.4.** Let  $\Phi, \Psi: \mathbf{D} \rightarrow \mathbf{M}$  be functors. The *mapping object*  $\text{Map}_{\mathbf{M}^{\mathbf{D}}}(\Phi, \Psi)$  is the end of the bifunctor  $\text{Map}_{\mathbf{M}}(\Phi(-), \Psi(-)): \mathbf{D}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{V}$ , i.e.,

$$\text{Map}_{\mathbf{M}^{\mathbf{D}}}(\Phi, \Psi) = \int_{d \in \mathbf{D}} \text{Map}_{\mathbf{M}}(\Phi(d), \Psi(d)).$$

This end exists, since  $\mathbf{D}$  is small and  $\mathbf{V}$  is complete.

More precisely, the object  $\text{Map}_{\mathbf{M}^{\mathbf{D}}}(\Phi, \Psi)$  is constructed as an equalizer

$$\text{equal} \left( \prod_{d \in \mathbf{D}} \text{Map}_{\mathbf{M}}(\Phi(d), \Psi(d)) \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \prod_{d, d' \in \mathbf{D}} \prod_{\alpha \in \mathbf{D}(d, d')} \text{Map}_{\mathbf{M}}(\Phi(d), \Psi(d')) \right)$$

in  $\mathbf{V}$ .

Tensors and cotensors for  $\mathbf{M}^{\mathbf{D}}$  over  $\mathbf{V}$  are defined objectwise, i.e.,

$$(V \otimes \Phi)(d) = V \otimes \Phi(d) \quad \text{and} \quad (\Phi^V)(d) = \Phi(d)^V$$

for all  $\Phi \in \text{Ob } \mathbf{M}^{\mathbf{D}}$ ,  $V \in \text{Ob } \mathbf{V}$ , and  $d \in \text{Ob } \mathbf{D}$ .

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