Theory of the scrape-off layer width in inner-wall limited tokamak plasmas

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Abstract

We develop a predictive theory applicable to the scrape-off layer (SOL) of inner-wall limited plasmas. Using the non-linear flattening of the pressure profile as a saturation mechanism for resistive ballooning modes, we are able to demonstrate and quantify the increase of the SOL width with plasma size, connection length, plasma \( \beta \), and collisionality. Individual aspects of the theory, such as saturation physics, parallel dynamics, and system size scaling, are tested and verified using non-linear, 3D flux-driven SOL turbulence simulations. Altogether, very good agreement between theory and simulation is found.

Keywords: SOL width, turbulence, tokamak, ballooning modes, limited plasmas

(Some figures may appear in colour only in the online journal)

1. Introduction

While determining the scrape-off layer (SOL) width and understanding the transport mechanisms involved in SOL profile formation are crucial issues for ITER and all future tokamak devices, a complete understanding of the subject is still needed. In recent years, for instance, it was discovered that the L-mode scaling used to calculate the ITER heat-flux width and design the inner wall [1] did not apply to the start-up phase of the discharge, when the plasma is wall-limited. Many inner-wall limited experiments were then carried out to understand how the SOL width varies with the plasma parameters [2–8].

The goal of the present paper is to address the turbulent dynamics of circular, inner-wall limited SOL plasmas, such as the proposed ITER start-up scenario, and establish a theory-based predictive capability in this simplest configuration. Our investigations concentrate on the characteristic pressure gradient length \( L_p = -p/\nabla p \), which regulates the steady-state heat load on the wall. Building on recent theoretical investigations [9, 10], and complementing the initial comparison with experimental data presented in a recent letter [11], we provide a generalized analytical framework that allows us to understand the scaling of the SOL width as a function of its operational parameters. The predictions of our model are in remarkable agreement with a large simulation scan, presented here, covering a wide range of SOL parameters.

The SOL width (i.e. \( L_p \)) results from a competition between plasma outflow from the core, perpendicular transport driven by turbulent structures, and parallel losses at the end of the magnetic field lines that are determined by sheath physics. In order to address these effects, we use a global, drift-reduced Braginskii fluid model [12] in combination with a proper set of boundary conditions at the magnetic pre-sheath entrance [13]. We carry out flux-driven, global 3D simulations in circular geometry with a toroidal limiter on the equatorial plane at the high-field side. The quasi-steady-state profiles result from turbulence driven by flute-like, large amplitude meso-scale structures such as those experimentally observed.

Our main results can be summarized as follows. The saturation model proposed, in combination with a linear stability code, yields \( L_p \) within 10% of the fully non-linear computation using the SOL operational parameters only as an input. Noticing that the resistive ballooning mode (RBM) is the instability driving turbulence in the simulations, it is shown that the steady state \( L_p \) (normalized by \( \rho_s \)) can be estimated using the following scaling:

\[
L_p = \left[ \frac{2\pi \rho_s \alpha d (1 - \alpha)^{1/2}}{q} \right]^{-1/2},
\]

for \( \alpha \) below the ideal ballooning threshold. The dimensionless parameters regulating the SOL width are the safety factor \( q = (r/R)(B_0/B_i)_e \); the collisional parameter \( \alpha d = 2^{-1/4} \sqrt{\rho_s L_p^{1/4}}/(\pi q) \); the ideal ballooning stability parameter \( \alpha = q^2 \beta R/L_p \); and the normalized ion-sound gyroradius \( \rho_s = \rho_i/R \). (\( r \) and \( R \) are the tokamak minor and major radii, \( v = e^2 n R/(m_i c \sigma_i) \) is the normalized Spitzer resistivity, \( \beta = 2\mu_0 \rho_i B^2 \) is the ratio of kinetic to magnetic pressure, and \( \sigma_i \) is the Spitzer conductivity. Also, note that the typical definition of \( \rho_s \) uses the minor radius \( a \) instead of \( R \).) Our analysis of the simulation data reveals very good agreement with our analytical theory. At fixed \( \rho_s \), it is shown that the variation of the SOL width is determined by the dimensionless parameters \( \alpha d/q \) and \( \alpha \). We also analyse in...
detail the SOL width dependence on $\rho_*$, discussing deviations from the RBM estimate. Finally, it is remarked that, while there have been previous attempts at understanding the SOL width scaling using 2D models [14–17], our results reveal the importance of 3D effects such as parallel resistivity and electromagnetic fluctuations in determining $L_p$ in the SOL.

This paper is organized as follows. In section 2, the physical model used to study SOL turbulence is discussed. Then, in section 3, a theory predicting the SOL width as a function of the dimensionless parameters is derived. Section 4 describes the results of SOL non-linear turbulence simulations, which serve as a verification of our theory. We discuss and disentangle, in particular, saturation physics and the effects of parallel dynamics and system size. Section 5 presents a summary of the findings of this work.

2. Model

In the tokamak SOL, it is of particular interest to understand temperature and density profile formation, which occurs as a power balance between particle and heat injection from the plasma core, perpendicular transport driven by turbulence, and parallel losses at the sheaths where the magnetic field lines intersect the vacuum vessel. The excitation of turbulent modes can result from a combination of unfavourable magnetic field curvature, pressure gradients, and electron adiabaticity breaking either by resistivity or electron inertia effects, which can be enhanced by electromagnetic effects. The resulting turbulent dynamics is characterized by the presence of meso-scale structures, i.e. large amplitude structures with a significant radial extension.

SOL turbulence investigations must therefore be global, and flux driven; both micro (up to $\rho_*)$ and macroscopic ($L_p$) length scales must be resolved, and the turbulent fluctuations cannot be formally separated from the background. Particle trapping is negligible since $\nu_\parallel \gg 1$ in the SOL of limited plasmas, while finite Larmor radius effects are small since $\nu_\parallel \rho_* \sim 0.1$ for the typical dominant modes in the non-linear stage. Since the plasma is relatively cold, a fluid model can capture the essential physical ingredients of this system.

For the present study, we use a cold-ion drift-reduced model [12], which can be derived from the Braginskii two-fluid equations [18] by imposing the orderings $d/dt \ll \omega_{ci}$, $k_\perp \gg k_i$, and $T_i \ll T_e$. The drift-reduced equations, in normalized units, read as follows:

$$\frac{\partial n}{\partial t} = -\rho_*^{-1}\left[\phi, n\right] - \nabla \cdot (nv_{||}) + 2\left[\hat{C}(p_e) - n\hat{C}(\phi)\right] + D_n\nabla^2 n + S_n$$

$$\frac{\partial \omega}{\partial t} = -\rho_*^{-1}\left[\phi, \omega\right] - v_e\nabla \cdot \omega + 2\left[\hat{C}(p_e) - n\hat{C}(\phi)\right] + \nabla \cdot (\nabla j)$$

$$\frac{\partial \chi}{\partial t} = -\rho_*^{-1}\left[\phi, \chi\right] - v_e\nabla \cdot \chi + D_\omega\nabla^2 \omega$$

$$\frac{\partial v_{||}}{\partial t} = -\rho_*^{-1}\left[\phi, v_{||}\right] - v_e\nabla \cdot v_{||} + D_{v_{||}}\nabla^2 v_{||}$$

$$\frac{\partial T_e}{\partial t} = -\rho_*^{-1}\left[\phi, T_e\right] - v_e\nabla \cdot T_e + 4\left[\hat{C}(T_e) + \frac{T_e}{n}\hat{C}(n) - \hat{C}(\phi)\right]$$

$$\frac{2}{3}\frac{\partial T_e}{\partial t} - \frac{0.71}{n}\nabla \cdot v_{||j} + \frac{D_T}{n}\nabla^2 T_e + S_{T_e},$$

where $\omega = \nabla^2 \phi$ is the vorticity and equation (3) has been simplified using the Boussinesq approximation $\nabla \cdot (nd\nabla \phi) \approx nd\nabla^2 \phi$. The quantity $\chi = [v_{||e} + m_i\rho_0/2m_e]$ represents a combination of inertial and induction effects in the Ohm’s law, $j_i = n(v_{||e} - v_{||e\phi})$ is the parallel current, $v = e^2nR/(\mu_0\rho_0\phi)$ is the normalized Spitzer resistivity, and $\rho_0 = 2\mu_0\sigma T_e/B_0^2$ is the reference beta ($\vec{n}$ and $\vec{T}_e$ are, respectively, the reference electron density and temperature, and $\vec{c}_e = \sqrt{T_e/m_e}$). Here, $\psi = -b_0 \cdot A_1$ is the parallel component of the magnetic vector potential given by $\nabla \times \vec{A}$.

The following normalizations are used in the drift-reduced equations: $t = \hat{t}/(R/\vec{c}_e)$, $\nabla \parallel = \hat{\rho}_i\nabla \parallel$, $\nabla \perp = R\nabla \perp$, $v_{||} = v_{||e}/\vec{c}_e$, $n = n/\bar{n}$, $T_e = \bar{T}_e/\bar{T}_e$, $\phi = e\phi/\vec{c}_e$, and, $\psi = [2\vec{c}_e m_e/(e\vec{\rho}_0)]$. Here, the hildes denote quantities in MKS physical units, and the bars denote reference quantities defined in terms of the normalized density $\bar{n}$ and temperature $\bar{T}_e$. All variables are expressed in their adimensional form unless specified otherwise.

Plasma outflow from the closed flux-surface region is mimicked using density and temperature sources, respectively, $S_n$ and $S_{T_e}$. The $G_e$ and $G_i$ terms represent the gyroviscous part of the pressure tensor (see [12]). Small perpendicular diffusion terms of the form $D_n\nabla^2 f$ are added mostly to allow the numerical solution of the system. In addition, $[f, g] = b_0 \cdot (\nabla f \times \nabla g)$ is the Poisson bracket, while $C(f) = \langle b_0/2\rangle(\nabla \times \phi)'$ is the curvature operator. The parallel gradient includes the effect of perpendicular electromagnetic perturbations, and is defined as $\nabla f = b_0 \cdot \nabla f + \phi \nabla \rho_0^{-1}[\psi, f]/2$.

We consider a SOL model in circular geometry with a toroidal limiter set at the high-field-side equatorial mid-plane. The coordinate system used is $(\bar{\theta}, \bar{r}, \bar{\varphi})$, right-handed—$\bar{r}$ is the radial coordinate, with $\bar{r} = 0$ set at the last closed flux surface, $\bar{\theta}$ is the poloidal angle, and $\bar{\varphi}$ is the toroidal angle. Under these assumptions, the curvature operator reduces to $\hat{C}(f) = (\sin \bar{\theta})\bar{\theta}_f + (\cos \bar{\theta} + \sin \bar{\theta})\bar{\theta}_f$ and the Poisson bracket is defined as $[f, g] = a^{-1}(\partial_\theta f \delta_\varphi g - \delta_\varphi f \partial_\theta g)$ ($a = (\bar{a} + \bar{r}q'/q$ is the magnetic shear).

The plasma interfaces with the vacuum vessel through a magnetized pre-sheath where the fluid drift approximation breaks down. The validity of the drift-reduced model, therefore, formally extends until the magnetic pre-sheath entrance, where we apply an appropriate set of boundary conditions [13]:

$$v_{||} = \pm c_s$$

$$v_{||e} = \pm c_s \exp(A - \phi/T_e)$$

$$\omega = -\cos^2\left(\frac{r}{qR}\right)\left[\frac{\partial v_{||e}}{\partial \bar{\theta}}\right]^2 \pm c_s \frac{\partial^2 v_{||e}}{\partial \bar{\theta}^2} a^{-2}$$
\[ \psi = 0 \] (10)
\[ \frac{\partial n}{\partial \theta} = \frac{n}{c_s} \frac{\partial v_{ij}}{\partial \theta} \] (11)
\[ \frac{\partial \phi}{\partial \theta} = \pm c_s \frac{\partial v_{ij}}{\partial \theta} \] (12)
\[ \frac{\partial T_e}{\partial \theta} = 0, \] (13)

where \( \Lambda \approx 3 \).

Since our study discusses plasma size and quantities such as \( \nu, q \) and \( \beta \), which determine parallel dynamics, it is useful to discuss how these parameters enter the drift-reduced Braginskii model. The simulated plasma size is incorporated into the equations through the \( E \times B \) flow advective terms arising from the total time derivative, which are varied through the dimensionless parameter \( \rho_s \). In our simulations, decreasing \( \rho_s \) is equivalent to increasing the toroidal magnetic field, decreasing the temperature, or increasing the machine size. In addition, the plasma size enters the equations through the normalization of the collisionality \( \nu \).

Parallel dynamics involve, on one hand, sonic flows that carry the plasma towards the plasma sheaths, and, on the other, a coupling between the vorticity equation \( \nabla \cdot j = 0 \) and Ohm’s law (equations (3) and (4)). The parallel flows affect mainly the bulk of the plasma and carry bulk density and heat towards the limiter where the magnetic field lines terminate (equations (2) and (6)). Since the major radius is used to normalize the parallel scale length, its value does not appear explicitly in the parallel loss terms \( \sim \nabla^2 \left( \nu \rho_{\parallel} \right) \).

The parallel electron motion, which is coupled to the vorticity equation, is responsible for adiabaticity breaking, allowing different instabilities to grow. The effects included in the Ohm’s law are the plasma resistivity, electron inertia, electromagnetic induction, and diamagnetic stabilization. This leads to our model being able to describe the resistive and inertial branches of drift waves and ballooning modes, and, in addition, the ideal ballooning mode. In the regime of interest for typical limited discharges (\( \beta \approx 10^{-4}, 0 < \beta \ll 2 \), \( \nu \approx 0.005-0.05, 3 < q < 10 \)), resistivity is the most important destabilization mechanism and mode growth is fed by unfavourable curvature [19].

The plasma size and parallel dynamics are instrumental in understanding the SOL width, since they affect the amplitude and dominant wavelength of the turbulent modes in the non-linear quasi-steady-state phase. Our theoretical understanding of these effects is developed below in section 3, and a large set of non-linear simulation results testing our theory are presented in section 4.

### 3. Theory of the SOL width

In a recent study [9], it has been shown that the magnitude of the turbulent fluxes in the SOL can be predicted using the gradient-removal mechanism, i.e. the local non-linear flattening of the pressure profile caused by the turbulent structures. The gradient-removal model has been used to explain experimental observations of \( L_p \) in a number of tokamaks [11], the transition between different unstable modes depending on the plasma parameters [19], the effects of finite aspect ratio [20], and the transition between the electrostatic and the ideal ballooning unstable regimes [10]. A short summary of the model follows below.

In our non-linear, flux-driven simulations, it is observed that sheared flows are unable to significantly affect the turbulence levels. Turbulent saturation, in fact, occurs when the linear drive from the background gradient is locally exhausted by the pressure non-linearity. Starting from this hypothesis, it follows that the amplitude of the turbulence can be estimated as \( p_{\parallel}/p_{\parallel} \approx \sigma_{r}/L_p \). The radial extension of the mode, \( \sigma_{r} \approx L_p \rho(\rho_{\parallel}/k_\theta) \), is obtained from a non-local linear theory [21, 22]. Then, the leading order contribution of a continuity equation leads to an estimate of the turbulent \( E \times B \) flux, \( \Gamma = p_{\parallel} \gamma/\kappa_\theta \), where \( \gamma \) is the linear growth rate of the instability that dominates the non-linear dynamics. Power balance between perpendicular turbulent transport, \( \partial_t \Gamma \approx \Gamma/L_p \approx p_{\parallel} \gamma/(\kappa_\theta L_p) \), and the parallel losses at the sheath, \( \nabla_{\parallel} \phi_{\parallel} \approx p_{\parallel} \sigma_{r}/q \), results in an estimate of the profile length

\[ L_p \approx c_{\rho_{\parallel}} \left( \frac{\gamma}{k_\theta} \right)_{\max}. \] (14)

In the following, we will assume \( c_{\rho_{\parallel}} = 1 \), which corresponds to normalizing \( T_e \) to its background value. Furthermore, it is assumed that the flux is driven by a single mode that maximizes \( \gamma/k_\theta \), i.e. we assume that the non-linear phase is dominated by the poloidal mode that leads to the flattest pressure profile.

The steady state \( L_p \) can be predicted provided that the linear growth rate of the transport-driving mode is known. SOL turbulence in limited plasmas has been addressed with 3D electromagnetic simulations, finding that RBMs dominate the plasma dynamics [10, 23]. In the absence of poloidal periodicity, RBMs are dominant over non-linearly driven drift waves [23], and linearly unstable drift waves are damped by the magnetic shear [24]. Moreover, it is possible to estimate the non-linear regime instability using linear calculations and an estimate of the radial flux driven by each instability [19]. This calculation confirms the importance of RBMs in the SOL of limited plasmas.

We henceforth concentrate on the dynamics of RBMs and how they affect the SOL width. Using equations (2)–(6) together with equation (14), it is possible to show that the dimensionless parameters regulating the SOL width are \( q, \rho_{\perp}, \sigma = q^2 \beta R/L_p \), and \( \kappa_\theta = 2^{-7/4}q^{-1/2}q^{-1/2}(\alpha_{\rho_{\parallel}} L^{-1/4})/(\sigma q) \). The computation yields a dimensionless scaling, which predicts fully non-linear simulation results reasonably well. In order to obtain this result, we first simplify the two-fluid system (equations (2)–(6)) considerably. These simplifications are supported aposteriori by the fact that the simple model captures the principal ingredients of the non-linear steady-state. Starting from equations (2)–(6), we neglect: the ion parallel motion equation; compressibility effects and parallel couplings in the density and temperature equations; electron inertia and diamagnetic effects in Ohm’s law; all diffusion and gyroviscous terms; and, finally, electromagnetic perturbations are ignored everywhere except for the left-hand-side of Ohm’s law. The density and temperature equations are added to obtain an equation for the total pressure. The resulting system of equations is, essentially, a reduced MHD model describing resistive and ideal ballooning modes:

\[ \frac{\partial p}{\partial t} = -\rho_{\perp}^{-1} \left[ \phi, p \right] \] (15)
adiabatic electron response and has a destabilizing effect on \( \gamma/k_0 \). In deriving equation (18) we have assumed \( k_1 \sim 1/q, k_\perp \sim k_0 \), and the curvature operator is assumed to take the simple form \( \hat{C}(f) = \partial_\theta f \). Magnetic shear effects are neglected, since they have a weak influence on the RBM growth rate in the regime of interest [24].

Assuming \( \alpha < 1 \) (the ideal ballooning branch is neglected) and \( \gamma^2 \ll \gamma_0^2 \), the gradient-removal flux estimate, \( \Gamma = \gamma / (\gamma / k_0)_{\text{max},} \) can be obtained analytically. This is equivalent to a low \( \beta \), low-frequency regime for RBMs. We start by solving the equation \( \partial_\theta \psi / \gamma_0 = 0 \), which yields \( k_0^2 = 3 \sqrt{2} k_0^2 / \sqrt{2} \approx 0.93 k_0^2 \), with \( k_0^2 = (1 - \alpha) q^{-1} \gamma_0^{-1} \approx (1 - \alpha) \gamma_0^{-1} / \gamma_0^2 \). Note that the modes dominating the non-linear state have a wavelength that is intermediate between marginal stability and the strongly unstable regime. Therefore, it appears that it is not possible to simplify equation (18) using \( k_0 \) as an expansion parameter. The saturated growth rate is obtained by substituting \( k_0 \) into the solution of equation (18), which gives \( \gamma = \gamma_0 / \sqrt{3} \approx 0.57 \gamma_0 \). To obtain \( L_p \), we use equation (14) together with the estimates \( \gamma = \gamma_0 \) and \( k_\parallel = k_0 \), which yields a dimensionless scaling:

\[
L_p = \left[ \frac{2\pi \rho_0 \alpha_d (1 - \alpha)^{1/2} / q}{\gamma} \right]^{1/2}.
\]

Equation (19) clearly identifies the dimensionless parameters describing the scaling of \( L_p \): \( \rho_0 \), which describes the system size scaling; \( \alpha_d / q \), which includes collisional effects and the connection length; and \( \gamma_0 \), which is due to electromagnetic effects. The parameters act as follows: \( \rho_0 \) modifies the linear drive through \( \gamma_0 \); increasing \( \alpha_d / q \) is equivalent to increasing the conductivity or decreasing the connection length, which inhibits mode growth at low \( k_0 \); while \( \alpha \) enhances the non-adiabatic electron response and has a destabilizing effect on the electrostatic resistive branch.

This expression for \( L_p \) is equivalent to the dimensionless scaling derived in [11] (with \( \rho_0 \) being equivalent to the parameter \( \tilde{R}^{-1} \) introduced therein). Additionally, note that both \( \alpha_d \) and \( \alpha \) include factors of \( L_p \)—therefore, equation (19) is an implicit scaling. In the low beta case, \( \alpha \ll 1 \), it is possible to obtain an explicit scaling as a function of the GBS input parameters \( \rho_0, q, \) and \( \nu \). The expression for the SOL width is

\[
L_p = 2^{3/7} \rho_0^{-3/7} \gamma^{8/7} k_0^{2/7}.
\]

This simple theory of the SOL width, which is applicable in inner-wall limited discharges, has been fully verified against non-linear simulations describing SOL profile formation. More specifically, we have explored the effects of changing plasma size, resistivity, plasma \( \beta \), and connection length separately. The non-linear simulation results are described below.

4. Non-linear simulations

We have carried out an extensive simulation campaign aiming to understand saturation physics, parallel dynamics, and plasma size effects in inner-wall limited plasmas. Global, flux-driven, non-linear simulations of the SOL dynamics are carried out with GBS [12], a numerical implementation of the global drift-reduced Braginskii model (equations (2)–(6) with boundary conditions (7)–(13)). GBS was originally developed to study turbulence in basic plasma physics experiments, and is fully validated against TORPEX probe measurements (e.g. [22–25]). Since 2011, GBS is also capable of carrying out flux-driven simulations of the tokamak SOL in limited configuration. The plasma dynamics are evolved within an annulus in the open magnetic field line region of the plasma vessel. Entire flux surfaces, up to the limiter, are included in the simulation domain. We use a simple circular geometry with a toroidal limiter on the high-field side mid-plane, with constant \( q \) and constant \( \delta \).

In the simulated SOL dynamics, there is no separation between fluctuations and background profiles, and no length scale separation is imposed. Plasma sources, which mimic the plasma outflow from the core, increase the pressure gradient until linearly unstable modes appear, driving turbulence that leads to perpendicular transport. Over a longer period, a non-linear quasi-steady turbulent state is naturally achieved as a power balance between plasma injection, turbulent transport, and parallel losses at the plasma sheaths.

We simulate SOL plasmas where RBMs are expected to dominate transport, and we attempt to maximize the range of the dimensionless parameter space probed. We use \( q = [3, 4, 6], \nu = [0.01, 0.1, 1], m_i/m_e = 200 \) and \( \beta_0 = 0–3 \times 10^{-5} \). The normalized plasma sizes used were \( \rho_0 = [500, 1000, 2000]^{-1} \), with an aspect ratio \( R/a = 4 \). In the plasma size scan we used \( \delta = 0.1 \). Note that, since GBS simulations are global, the simulation domain size must be increased with \( \rho_0^{-1} \), resulting in significant use of high-performance computing resources. The pressure profiles observed in the turbulent steady-state typically have the form \( p \sim \exp(-r/L_p) \) with \( L_p \) ranging from 25 to 150.

In the subsections below we deal specifically with the different aspects of our SOL width theory (section 3). The saturation model is investigated in the context of non-linear simulations in section 4.1. Then, in section 4.2, the theory-based scaling is directly compared against a scaling obtained from the simulation results. The parallel dynamics effects (\( q, \alpha, \) and \( \alpha_d \)) and the system size scaling effects (\( \rho_0 \)) are studied in sections 4.3 and 4.4, respectively.

4.1. Saturation mechanism

Here, we present a verification of the gradient-removal model using the simulation database described above. To that effect, we implement an iterative scheme to solve equation (14). With the SOL dimensionless operational parameters \( \rho_0, q, \nu, \beta_0, \) and \( \delta \) fixed, the linear growth rate \( \gamma \) is computed using a drift-Braginskii linear solver [24] and \( L_p \) is varied iteratively until equation (14) is satisfied. It is remarked that \( \gamma \) is a function of the plasma parameters \( L_p, k_0, \rho_0, q, \nu, \beta_0, \delta \), while the obtained result is \( L_p \) as a function of \( \rho_0, q, \nu, \beta_0, \) and \( \delta \).
The result of the computation is the value of \( L_p \) that satisfies the power balance between perpendicular transport and parallel losses at the field line ends, for a set of SOL operational parameters.

This procedure was carried out for all the simulations described in this paper. The results of the verification exercise are shown in figure 1. The abscissa shows the theoretical prediction provided by equation (14) while the ordinate shows the GBS fully non-linear result. Overall, very good agreement is found throughout the entire parameter range, with a coefficient of determination \( R^2 \approx 0.93 \). By providing an explanation of the power balance, gradient-removal theory gives an interpretation for the different physical effects involved in setting the SOL width.

The saturation mechanism, when applied to the RBM, yields the dimensionless scaling given by equation (19). We have verified this scaling against non-linear simulations with \( \alpha \lesssim 0.5 \), which is necessary in order to avoid ideal instability. This low \( \beta \) regime is, in fact, the experimentally relevant scenario for inner-wall limited discharges. The results of the comparison are shown in figure 2, where the abscissa provides the analytical scaling estimate, while the ordinate provides the fully non-linear \( L_p \). There is relatively good agreement between theory and simulations for a large range of parameters, with a coefficient of determination for the fit of \( R^2 \approx 0.72 \).

Finally, since the gradient-removal computation also yields the expected non-linear value of \( k_o \), it is worth commenting on this issue. We have found that the peak \( k_o \) of the non-linear simulations increases with \( \alpha_d \) and decreases with \( \gamma_b \). This is exactly what is expected from the RBM theory, since \( k_o \propto \alpha_d/\gamma_b \). However, the non-linear turbulent spectra are rather wide, with a full-width half-maximum \( \Delta k_o \sim k_o \). Therefore, a detailed quantitative comparison between theory and simulations (e.g. such as figure 1) is not possible.

### 4.2. Comparison with non-linear regression analysis of simulation data

In addition to the theoretical scaling, equation (19), it is also possible to obtain a dimensionless SOL width scaling by carrying out a least-squares fit on the simulation data. We have carried out this exercise, which also serves as a verification of our theory. All simulations in the scan are included, provided that they reside below the ideal stability threshold.

The least-squares fit based on the GBS parameters \( q, \rho_s, \alpha_d, \) and \( \alpha \) gives

\[
L_p = 0.42 q^{0.55} \rho_s^{0.53} \alpha_d^{0.32} (1 - \alpha)^{-0.24}.
\]  

(21)

The coefficient of determination resulting from this fit is \( R^2 \approx 0.94 \). Note that the exponents and even the numerical constant of equation (21) are very similar to the ones found in equation (19). With the exception of the \( \alpha_d \) (collisional) dependence, all exponents are within 10% of the theoretical results.

On the other hand, the fit based on the GBS parameters \( q, \rho_s, \nu \) and \( \beta_0 \) is

\[
L_p = 1.00 q^{0.98} \rho_s^{0.46} \nu^{0.17} \beta_0^{0.0},
\]  

(22)

which then leads to an MKSA scaling of the form

\[
L_p \sim q^{0.98} R^{0.61} B^{-0.56},
\]  

(23)

and with weak dependences on density and temperature, i.e. essentially the same scaling found in our recent letter [11] and reported in equation (20) using only theoretical arguments.

We have investigated the difference between the collisional stability dependences found in the theory (\( L_p \sim \nu^{-2/7} \)) and in the simulations. We employ the quasi-linear, self-consistent computation of \( L_p \) described in section 4.1 at fixed \( R = 500, \ q = 4, \) and varying \( \nu \). The result is that the \( \nu \) exponent is affected (a) by the use of an increased electron mass in the GBS simulations and (b) by the assumption of full non-adiabaticity in equation (17). Therefore, the assumptions used to deduce the RBM scaling are not fully satisfied in our simulations.

### 4.3. Parallel dynamics

Here, we investigate the effects of the dimensionless parameters describing the SOL parallel dynamics. Our objective is to build a SOL dimensionless parameter space describing collisional and finite \( \beta \) effects, verifying that \( \alpha \) and \( \alpha_d/q \) are indeed the relevant dimensionless parameters. The SOL width \( L_p \) and the dimensionless parameters suggested by equation (19), \( \alpha_d \) and \( \alpha \), are computed using equation (14) together with the linear growth rate provided by the drift-Braginskii linear solver [24]. The following parameter
space was explored: \( \rho_s = 500^{-1}, \hat{s} = 0, q = 3, 4, 6, \nu = 0.01-1, \beta_0 < 10^{-2} \). The resulting dimensionless parameter spaces for \( q = 3, 4, \) and \( 6 \) are displayed in figure 3, where we show contours of equal \( L_p \) with non-linear simulation results superimposed showing good agreement. It is observed that the dimensionless parameter space covered is different in each case, as both \( \alpha \) and \( \alpha_d \) depend on \( q \). High \( q \) values naturally result in a wider SOL through a combination of increased resistive \( (\alpha_d) \) and electromagnetic \( (\alpha) \) effects. The calculation confirms the interpretation obtained starting from equation (18), i.e. \( L_p \) increases in the regime where we predicted destabilization of resistive modes from low \( \alpha_d \) (a combination of high resistivity and long connection length), and increasing \( \alpha \) (increased non-adiabatic electron response through electromagnetic perturbations).

Based on these results, and provided that the SOL dynamics are given by gradient-removal saturated RBMs, we now establish a unified dimensionless parameter space that is valid for any value of \( q \). Equation (19) suggests that the dimensionless parameter space is given by \( \alpha \) and \( \alpha_d/q \). In figure 4, we plot the \( L_p \) contours for \( q = 4 \) as a function of these two parameters. We have superimposed the GBS simulation data obtained for all the simulations with \( \rho^* = 500, \) for all of the \( q \) values. The theoretical model yields a good prediction of the values of \( L_p \) obtained in non-linear simulations, with \( L_p \) often being predicted within \( 5\rho_s \). Note that the relevant dimensionless parameter describing resistive and connection length effects is therefore \( \alpha_d/q \) and not \( \alpha_d \) itself.

These dimensionless parameters, which we propose for the SOL of limited discharges, may not apply to the SOL of diverted discharges. In Alcator C-Mod, for example, it has been shown that the relevant dimensionless parameters for the near SOL are \( \alpha_d \) and \( \alpha \) [29, 30]. In particular, it was demonstrated that \( L_p \) decreases with decreasing \( \alpha_d \) and, furthermore, it was observed that \( \alpha_d \) and \( \alpha \) both decrease as the Greenwald density limit is approached.

4.4. Plasma size scaling

The variation of \( L_p \) with system size is one of the greatest uncertainties that must be resolved as we approach the era of ITER (\( \rho_s \sim 10^{-4} \), which is impractical to simulate at the present time). We have carried out a set of simulations that address the system size scaling of the SOL width in limited plasmas such as the proposed ITER start-up scenario. In order to understand \( \rho_s \) effects, we have chosen to decrease \( \rho_s \) while leaving the normalized collisionality \( v \) and the injected power per unit volume unchanged. This is equivalent to increasing the strength of the magnetic field at fixed temperature and system size.

We start with the parameters \( \rho_s = 500^{-1}, q = 4, v = 0.01, 0.1, 1, \hat{s} = 0, 1, 2, \) and increase the plasma size to \( \rho_s = 1000^{-1} \) and \( \rho_s = 2000^{-1} \). We consider the electrostatic limit, \( \beta \ll 1 \). The smallest simulated plasma size (\( \rho_s = 500^{-1} \)) was used above to explore the parallel dynamics effects; the second size, \( \rho_s = 1000^{-1} \), is roughly equivalent to the CASTOR tokamak [31]; while \( \rho_s = 2000^{-1} \) has similar physical parameters as the SOL of TCV in limited

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**Figure 3.** Gradient length is shown as a function of \( \alpha \) and \( \alpha_d \) for \( q = 3, 4 \) and \( 6 \). The coloured contours indicate \( L_p \) predicted by equation (14), while each symbol represents a non-linear simulation. Triangles represent simulations with \( q = 3 \), squares simulations with \( q = 4 \), and diamonds simulations with \( q = 6 \).

**Figure 4.** Unified dimensionless electromagnetic parameter space built using gradient-removal theory in combination with the fluid linear system. Here \( L_p \) is shown as a function of \( \alpha \) and \( \alpha_d/q \). The coloured contours indicate \( L_p \) predicted by equation (14), while each symbol represents a non-linear simulation. Triangles represent simulations with \( q = 3 \), squares simulations with \( q = 4 \), and diamonds simulations with \( q = 6 \).
increases as indicated as a black line on the low-field-side equatorial mid-plane, in accordance to the prediction of the analytical model varying $q$ and bottom rows, respectively. Our diagnostic shows that the SOL of limited discharges has an experimental width of a red bar on the equatorial mid-plane. The profile lengths are RBMs with the resistive response slightly weakened by diamagnetic effects. In addition, we have calculated the cross-coherence function which decreases as $\hat{q}$ increases. The decreased phase lag and the correlation observed between $\phi$ and $p_e$ perturbations are a consequence of the adiabatic electron pressure response in the Ohm’s law. This effect is not taken into account in equation (18). Altogether, the mode appears to be of the ballooning kind with the resistivity providing the destabilization mechanism. We therefore confirm the importance of RBMs in limited SOL plasmas at experimentally relevant parameters. It is noted, however, that diamagnetic effects do play a role in the dynamics, in particular by weakening the non-adiabatic parallel electron response.

Finally, we estimate how $L_p$ scales with $\rho_*$ at fixed normalized collisionality $v$. Figure 7 shows $L_p$ from GBS simulations as a function of $\rho_*$. The gradient-removal estimates of $L_p$, obtained by solving equation (14) as a function of the SOL parameters, are superimposed as lines. Once again, the saturation theory provides a reasonably good prediction of the non-linear quasi-steady-state $L_p$. The gradient-removal estimate for $v = 0.01$ follows the size scaling $L_p \sim \rho_*^{-3/7}$, while our analytical theory gives a slightly different scaling, $L_p \sim \rho_*^{-3/7} \approx \rho_*^{-0.43}$. A more detailed quasi-linear analysis has revealed that the difference between the scalings originates from a combination of effects. On the one hand, the use of $m_i/m_e = 200$ slightly exaggerates the $\rho_*$ dependence in the gradient-removal solution. On the other hand, the neglect of compressibility terms and (especially) the adiabatic electron response in the Ohm’s law in deriving equation (18) weakens the $\rho_*$ dependence with respect to the full model. This finding supports our statement that the dominant non-linear modes are RBMs with the resistive response slightly weakened by diamagnetic effects.

5. Summary and conclusions

In conclusion, we have developed and verified a theory for the SOL width of inner-wall limited plasmas. The dimensionless parameters regulating the SOL width have been identified and the effects of parallel dynamics and plasma size have been explored with the aid of large scale numerical simulations. The GBS code was used to explore a large portion of the dimensionless parameter space, including experimentally relevant physical parameters.

First, we studied the mechanism responsible for setting the amplitude of the turbulent structures. It was found that sheared flows are rather weak in our simulations. Consequently, the turbulent amplitude is limited by the local non-linear flattening of the pressure profile. From the pressure continuity equation (a power-balance relation), we can compute the expected $L_p$ as a function of the SOL operational parameters only. Overall, excellent agreement between theory and simulation is observed for all the simulations in our large database.

Second, we have clarified the importance of parallel dynamics in setting the SOL width. Assuming that RBMs drive the perpendicular transport, and applying the gradient-removal theory, it was shown that, in limited plasmas, $L_p$ is governed by the parameters $\alpha_d/q$ and $\alpha$, in good agreement with GBS simulation results. These dimensionless parameters.
Figure 6. Phase difference (left) and cross-correlation (right) between $\phi$ and $p_\rho$ perturbations are shown for GBS simulations with $q = 4$, $\nu = 0.01$, $\rho_\star = 2000^{-1}$, $\tilde{s} = 0, 1, 2$ (top, centre, and bottom rows, respectively). The phase diagram has been renormalized using the power spectrum of the $\phi$ fluctuations. The cross-correlation diagram involves the distribution functions of the perturbations rescaled using their standard deviation ($\sigma$).

Figure 7. The pressure gradient length $L_p$ calculated from GBS simulations with $q = 4$, $\nu = 0.01$, $\rho_\star = 500, 1000, 2000$ are shown as triangles ($\nu = 0.01$), circles ($\nu = 0.1$) and squares ($\nu = 1$). We superimpose the predictions of the gradient-removal theory, shown as lines. The profile length at $\nu = 0.01$ scales like $L_p \sim \rho_\star^{-0.57}$, while the analytical theory (equation (19)) predicted $L_p \sim \rho_\star^{-3/7}$. We note that the electromagnetic parameter space found (e.g. figure 4) is different respect to what has been observed for diverted L-mode discharges in C-Mod. In this configuration, LaBombard et al [29, 30] found that the relevant dimensionless parameters are $\alpha_d$ and $\alpha$. It is conjectured that a different saturation mechanism could be at play in diverted shots. In particular, the connection length (the safety factor $q$) becomes very large near the X-point, and in such cases we expect Kelvin–Helmholtz modes to become important in the turbulent saturation process [9].

We also explored the effects of increasing the 'plasma size', more specifically, decreasing $\rho_\star$ towards experimentally relevant levels. Strictly speaking, the scan carried out in this paper is equivalent to keeping the physical plasma size, temperature, and normalized collisionality constant, while varying the toroidal magnetic field. The GBS simulations have parameters equivalent to the TCV SOL at one-quarter, one-half, and full toroidal magnetic field strength. The pressure gradient length found in the GBS simulations was found to approximately scale with size like $\rho_\star^{-0.57}$, while our analytical theory predicted a slightly weaker scaling $L_p \sim \rho_\star^{-3/7}$. The difference in exponents is due to a combination of effects (compressibility effects and adiabatic electron response) which are neglected in the derivation of equation (18). We have also confirmed that RBMs are relevant in the non-linear turbulent state at realistic SOL parameters. Our scaling, as shown in our recent letter [11], predicts $L_p \approx 6–10$ cm with $q = 6–8$ during the start-up phase of ITER.

Finally, it is suggested that, in order to reproduce our $\alpha_d$ scaling using 2D turbulence simulations of SOL transport, the closure of the parallel current in the vorticity equation could be modified to include a resistive damping term. Using such approach, however, it may be very challenging to model electromagnetic effects—this is particularly true in diverted configurations, where electromagnetic fluctuations are expected to be important [29, 30]. Overall, our study once again remarks the importance of retaining 3D effects, a global domain, and a full power-balance in understanding SOL.

\begin{equation}
L_p = \left[2\pi \rho_\star \alpha_d (1 - \alpha_d)^{1/2} / q\right]^{-1/2}
\end{equation}
profile formation—the properties of the turbulent structures, and consequently the resulting profiles, are strongly affected by the parallel dynamics, with the result that the SOL width scales favourably with plasma size.

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